

RELATIONS AND FUNCTIONS

1.1 Overview

1.1.1 Relation

A relation R from a non-empty set A to a non empty set B is a subset of the Cartesian product $A \times B$. The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R . The set of all second elements in a relation R from a set A to a set B is called the range of the relation R . The whole set B is called the codomain of the relation R . Note that range is always a subset of codomain.

1.1.2 Types of Relations

A relation R in a set A is subset of $A \times A$. Thus empty set ϕ and $A \times A$ are two extreme relations.

- (i) A relation R in a set A is called empty relation, if no element of A is related to any element of A , i.e., $R = \phi \subset A \times A$.
- (ii) A relation R in a set A is called universal relation, if each element of A is related to every element of A , i.e., $R = A \times A$.
- (iii) A relation R in A is said to be reflexive if aRa for all $a \in A$, R is symmetric if $aRb \Rightarrow bRa$, $\forall a, b \in A$ and it is said to be transitive if aRb and $bRc \Rightarrow aRc$ $\forall a, b, c \in A$. Any relation which is reflexive, symmetric and transitive is called an equivalence relation.

 **Note:** An important property of an equivalence relation is that it divides the set into pairwise disjoint subsets called equivalent classes whose collection is called a partition of the set. Note that the union of all equivalence classes gives the whole set.

1.1.3 Types of Functions

- (i) A function $f: X \rightarrow Y$ is defined to be one-one (or injective), if the images of distinct elements of X under f are distinct, i.e.,

$$x_1, x_2 \in X, f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$
- (ii) A function $f: X \rightarrow Y$ is said to be onto (or surjective), if every element of Y is the image of some element of X under f , i.e., for every $y \in Y$ there exists an element $x \in X$ such that $f(x) = y$.

- (iii) A function $f: X \rightarrow Y$ is said to be one-one and onto (or bijective), if f is both one-one and onto.

1.1.4 Composition of Functions

- (i) Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions. Then, the composition of f and g , denoted by $g \circ f$, is defined as the function $g \circ f: A \rightarrow C$ given by

$$g \circ f(x) = g(f(x)), \quad \forall x \in A.$$

- (ii) If $f: A \rightarrow B$ and $g: B \rightarrow C$ are one-one, then $g \circ f: A \rightarrow C$ is also one-one
 (iii) If $f: A \rightarrow B$ and $g: B \rightarrow C$ are onto, then $g \circ f: A \rightarrow C$ is also onto.

However, converse of above stated results (ii) and (iii) need not be true. Moreover, we have the following results in this direction.

- (iv) Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be the given functions such that $g \circ f$ is one-one. Then f is one-one.
 (v) Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be the given functions such that $g \circ f$ is onto. Then g is onto.

1.1.5 Invertible Function

- (i) A function $f: X \rightarrow Y$ is defined to be invertible, if there exists a function $g: Y \rightarrow X$ such that $g \circ f = I_x$ and $f \circ g = I_y$. The function g is called the inverse of f and is denoted by f^{-1} .
 (ii) A function $f: X \rightarrow Y$ is invertible if and only if f is a bijective function.
 (iii) If $f: X \rightarrow Y$, $g: Y \rightarrow Z$ and $h: Z \rightarrow S$ are functions, then $h \circ (g \circ f) = (h \circ g) \circ f$.
 (iv) Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be two invertible functions. Then $g \circ f$ is also invertible with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

1.1.6 Binary Operations

- (i) A binary operation $*$ on a set A is a function $*$: $A \times A \rightarrow A$. We denote $*(a, b)$ by $a * b$.
 (ii) A binary operation $*$ on the set X is called commutative, if $a * b = b * a$ for every $a, b \in X$.
 (iii) A binary operation $*$: $A \times A \rightarrow A$ is said to be associative if $(a * b) * c = a * (b * c)$, for every $a, b, c \in A$.
 (iv) Given a binary operation $*$: $A \times A \rightarrow A$, an element $e \in A$, if it exists, is called identity for the operation $*$, if $a * e = a = e * a$, $\forall a \in A$.

- (v) Given a binary operation $*$: $A \times A \rightarrow A$, with the identity element e in A , an element $a \in A$, is said to be invertible with respect to the operation $*$, if there exists an element b in A such that $a * b = e = b * a$ and b is called the inverse of a and is denoted by a^{-1} .

1.2 Solved Examples

Short Answer (S.A.)

Example 1 Let $A = \{0, 1, 2, 3\}$ and define a relation R on A as follows:

$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}.$$

Is R reflexive? symmetric? transitive?

Solution R is reflexive and symmetric, but not transitive since for $(1, 0) \in R$ and $(0, 3) \in R$ whereas $(1, 3) \notin R$.

Example 2 For the set $A = \{1, 2, 3\}$, define a relation R in the set A as follows:

$$R = \{(1, 1), (2, 2), (3, 3), (1, 3)\}.$$

Write the ordered pairs to be added to R to make it the smallest equivalence relation.

Solution $(3, 1)$ is the single ordered pair which needs to be added to R to make it the smallest equivalence relation.

Example 3 Let R be the equivalence relation in the set $\mathbf{Z}$ of integers given by $R = \{(a, b) : 2 \text{ divides } a - b\}$. Write the equivalence class $[0]$.

Solution $[0] = \{0, \pm 2, \pm 4, \pm 6, \dots\}$

Example 4 Let the function $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = 4x - 1, \forall x \in \mathbf{R}$. Then, show that f is one-one.

Solution For any two elements $x_1, x_2 \in \mathbf{R}$ such that $f(x_1) = f(x_2)$, we have

$$4x_1 - 1 = 4x_2 - 1$$

$$\Rightarrow 4x_1 = 4x_2, \text{ i.e., } x_1 = x_2$$

Hence f is one-one.

Example 5 If $f = \{(5, 2), (6, 3)\}$, $g = \{(2, 5), (3, 6)\}$, write $f \circ g$.

Solution $f \circ g = \{(2, 2), (3, 3)\}$

Example 6 Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $f(x) = 4x - 3 \forall x \in \mathbf{R}$. Then write f^{-1} .

Solution Given that $f(x) = 4x - 3 = y$ (say), then

$$4x = y + 3$$

$$\Rightarrow x = \frac{y+3}{4}$$

$$\text{Hence } f^{-1}(y) = \frac{y+3}{4} \Rightarrow f^{-1}(x) = \frac{y+3}{4}$$

Example 7 Is the binary operation $*$ defined on $\mathbf{Z}$ (set of integer) by $m * n = m - n + mn \quad \forall m, n \in \mathbf{Z}$ commutative?

Solution No. Since for $1, 2 \in \mathbf{Z}$, $1 * 2 = 1 - 2 + 1.2 = 1$ while $2 * 1 = 2 - 1 + 2.1 = 3$ so that $1 * 2 \neq 2 * 1$.

Example 8 If $f = \{(5, 2), (6, 3)\}$ and $g = \{(2, 5), (3, 6)\}$, write the range of f and g .

Solution The range of $f = \{2, 3\}$ and the range of $g = \{5, 6\}$.

Example 9 If $A = \{1, 2, 3\}$ and f, g are relations corresponding to the subset of $A \times A$ indicated against them, which of f, g is a function? Why?

$$f = \{(1, 3), (2, 3), (3, 2)\}$$

$$g = \{(1, 2), (1, 3), (3, 1)\}$$

Solution f is a function since each element of A in the first place in the ordered pairs is related to only one element of A in the second place while g is not a function because 1 is related to more than one element of A , namely, 2 and 3.

Example 10 If $A = \{a, b, c, d\}$ and $f = \{a, b), (b, d), (c, a), (d, c)\}$, show that f is one-one from A onto A . Find f^{-1} .

Solution f is one-one since each element of A is assigned to distinct element of the set A . Also, f is onto since $f(A) = A$. Moreover, $f^{-1} = \{(b, a), (d, b), (a, c), (c, d)\}$.

Example 11 In the set $\mathbf{N}$ of natural numbers, define the binary operation $*$ by $m * n = g.c.d(m, n)$, $m, n \in \mathbf{N}$. Is the operation $*$ commutative and associative?

Solution The operation is clearly commutative since

$$m * n = g.c.d(m, n) = g.c.d(n, m) = n * m \quad \forall m, n \in \mathbf{N}.$$

It is also associative because for $l, m, n \in \mathbf{N}$, we have

$$\begin{aligned} l * (m * n) &= g.c.d(l, g.c.d(m, n)) \\ &= g.c.d.(g.c.d(l, m), n) \\ &= (l * m) * n. \end{aligned}$$

Long Answer (L.A.)

Example 12 In the set of natural numbers $\mathbf{N}$, define a relation R as follows:
 $\forall n, m \in \mathbf{N}, nRm$ if on division by 5 each of the integers n and m leaves the remainder less than 5, i.e. one of the numbers 0, 1, 2, 3 and 4. Show that R is equivalence relation. Also, obtain the pairwise disjoint subsets determined by R .

Solution R is reflexive since for each $a \in \mathbf{N}$, aRa . R is symmetric since if aRb , then bRa for $a, b \in \mathbf{N}$. Also, R is transitive since for $a, b, c \in \mathbf{N}$, if aRb and bRc , then aRc . Hence R is an equivalence relation in $\mathbf{N}$ which will partition the set $\mathbf{N}$ into the pairwise disjoint subsets. The equivalent classes are as mentioned below:

$$A_0 = \{5, 10, 15, 20, \dots\}$$

$$A_1 = \{1, 6, 11, 16, 21, \dots\}$$

$$A_2 = \{2, 7, 12, 17, 22, \dots\}$$

$$A_3 = \{3, 8, 13, 18, 23, \dots\}$$

$$A_4 = \{4, 9, 14, 19, 24, \dots\}$$

It is evident that the above five sets are pairwise disjoint and

$$A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 = \bigcup_{i=0}^4 A_i = \mathbf{N}.$$

Example 13 Show that the function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = \frac{x}{x^2+1}, \forall x \in \mathbf{R}$, is

neither one-one nor onto.

Solution For $x_1, x_2 \in \mathbf{R}$, consider

$$f(x_1) = f(x_2)$$

$$\Rightarrow \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$$\Rightarrow x_1 x_2^2 + x_1 = x_2 x_1^2 + x_2$$

$$\Rightarrow x_1 x_2 (x_2 - x_1) = x_2 - x_1$$

$$\Rightarrow x_1 = x_2 \text{ or } x_1 x_2 = 1$$

We note that there are point, x_1 and x_2 with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$, for instance, if

we take $x_1 = 2$ and $x_2 = \frac{1}{2}$, then we have $f(x_1) = \frac{2}{5}$ and $f(x_2) = \frac{2}{5}$ but $2 \neq \frac{1}{2}$. Hence

f is not one-one. Also, f is not onto for if so then for $1 \in \mathbf{R} \exists x \in \mathbf{R}$ such that $f(x) = 1$

which gives $\frac{x}{x^2+1}=1$. But there is no such x in the domain $\mathbf{R}$, since the equation $x^2 - x + 1 = 0$ does not give any real value of x .

Example 14 Let $f, g : \mathbf{R} \rightarrow \mathbf{R}$ be two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x \quad \forall x \in \mathbf{R}$. Then, find $f \circ g$ and $g \circ f$.

Solution Here $f(x) = |x| + x$ which can be redefined as

$$f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Similarly, the function g defined by $g(x) = |x| - x$ may be redefined as

$$g(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$

Therefore, $g \circ f$ gets defined as :

For $x \geq 0$, $(g \circ f)(x) = g(f(x)) = g(2x) = 0$

and for $x < 0$, $(g \circ f)(x) = g(f(x)) = g(0) = 0$.

Consequently, we have $(g \circ f)(x) = 0, \quad \forall x \in \mathbf{R}$.

Similarly, $f \circ g$ gets defined as:

For $x \geq 0$, $(f \circ g)(x) = f(g(x)) = f(0) = 0$,

and for $x < 0$, $(f \circ g)(x) = f(g(x)) = f(-2x) = -4x$.

$$\text{i.e. } (f \circ g)(x) = \begin{cases} 0, & x > 0 \\ -4x, & x < 0 \end{cases}$$

Example 15 Let $\mathbf{R}$ be the set of real numbers and $f : \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $f(x) = 4x + 5$. Show that f is invertible and find f^{-1} .

Solution Here the function $f : \mathbf{R} \rightarrow \mathbf{R}$ is defined as $f(x) = 4x + 5 = y$ (say). Then

$$4x = y - 5 \quad \text{or} \quad x = \frac{y-5}{4}.$$

This leads to a function $g : \mathbf{R} \rightarrow \mathbf{R}$ defined as

$$g(y) = \frac{y-5}{4}.$$

Therefore, $(g \circ f)(x) = g(f(x)) = g(4x + 5)$

$$= \frac{4x+5-5}{4} = x$$

or

$$g \circ f = I_{\mathbf{R}}$$

Similarly

$$(f \circ g)(y) = f(g(y))$$

$$= f\left(\frac{y-5}{4}\right)$$

$$= 4\left(\frac{y-5}{4}\right) + 5 = y$$

or

$$f \circ g = I_{\mathbf{R}}.$$

Hence f is invertible and $f^{-1} = g$ which is given by

$$f^{-1}(x) = \frac{x-5}{4}$$

Example 16 Let $*$ be a binary operation defined on $\mathbf{Q}$. Find which of the following binary operations are associative

(i) $a * b = a - b$ for $a, b \in \mathbf{Q}$.

(ii) $a * b = \frac{ab}{4}$ for $a, b \in \mathbf{Q}$.

(iii) $a * b = a - b + ab$ for $a, b \in \mathbf{Q}$.

(iv) $a * b = ab^2$ for $a, b \in \mathbf{Q}$.

Solution

(i) $*$ is not associative for if we take $a = 1$, $b = 2$ and $c = 3$, then

$$(a * b) * c = (1 * 2) * 3 = (1 - 2) * 3 = -1 - 3 = -4 \text{ and}$$

$$a * (b * c) = 1 * (2 * 3) = 1 * (2 - 3) = 1 - (-1) = 2.$$

Thus $(a * b) * c \neq a * (b * c)$ and hence $*$ is not associative.

(ii) $*$ is associative since $\mathbf{Q}$ is associative with respect to multiplication.

(iii) $*$ is not associative for if we take $a = 2$, $b = 3$ and $c = 4$, then

$$(a * b) * c = (2 * 3) * 4 = (2 - 3 + 6) * 4 = 5 * 4 = 5 - 4 + 20 = 21, \text{ and}$$

$$a * (b * c) = 2 * (3 * 4) = 2 * (3 - 4 + 12) = 2 * 11 = 2 - 11 + 22 = 13$$

Thus $(a * b) * c \neq a * (b * c)$ and hence $*$ is not associative.

(iv) $*$ is not associative for if we take $a = 1$, $b = 2$ and $c = 3$, then $(a * b) * c = (1 * 2) * 3 = 4 * 3 = 4 \times 9 = 36$ and $a * (b * c) = 1 * (2 * 3) = 1 * 18 = 1 \times 18^2 = 324$.

Thus $(a * b) * c \neq a * (b * c)$ and hence $*$ is not associative.

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 17 to 25.

Example 17 Let R be a relation on the set $\mathbf{N}$ of natural numbers defined by nRm if n divides m . Then R is

- | | |
|-----------------------------|---|
| (A) Reflexive and symmetric | (B) Transitive and symmetric |
| (C) Equivalence | (D) Reflexive, transitive but not symmetric |

Solution The correct choice is (D).

Since n divides n , $\forall n \in \mathbf{N}$, R is reflexive. R is not symmetric since for $3, 6 \in \mathbf{N}$, $3 R 6 \nRightarrow 6 R 3$. R is transitive since for n, m, r whenever n/m and $m/r \Rightarrow n/r$, i.e., n divides m and m divides r , then n will divide r .

Example 18 Let L denote the set of all straight lines in a plane. Let a relation R be defined by lRm if and only if l is perpendicular to $m \forall l, m \in L$. Then R is

- | | |
|----------------|-------------------|
| (A) reflexive | (B) symmetric |
| (C) transitive | (D) none of these |

Solution The correct choice is (B).

Example 19 Let $\mathbf{N}$ be the set of natural numbers and the function $f : \mathbf{N} \rightarrow \mathbf{N}$ be defined by $f(n) = 2n + 3 \forall n \in \mathbf{N}$. Then f is

- | | |
|----------------|-------------------|
| (A) surjective | (B) injective |
| (C) bijective | (D) none of these |

Solution (B) is the correct option.

Example 20 Set A has 3 elements and the set B has 4 elements. Then the number of

injective mappings that can be defined from A to B is

- | | |
|---------|--------|
| (A) 144 | (B) 12 |
| (C) 24 | (D) 64 |

Solution The correct choice is (C). The total number of injective mappings from the set containing 3 elements into the set containing 4 elements is ${}^4P_3 = 4! = 24$.

Example 21 Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = \sin x$ and $g: \mathbf{R} \rightarrow \mathbf{R}$ be defined by $g(x) = x^2$, then $f \circ g$ is

- | | |
|------------------|--------------------------|
| (A) $x^2 \sin x$ | (B) $(\sin x)^2$ |
| (C) $\sin x^2$ | (D) $\frac{\sin x}{x^2}$ |

Solution (C) is the correct choice.

Example 22 Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = 3x - 4$. Then $f^{-1}(x)$ is given by

- | | |
|---------------------|-----------------------|
| (A) $\frac{x+4}{3}$ | (B) $\frac{x}{3} - 4$ |
| (C) $3x + 4$ | (D) None of these |

Solution (A) is the correct choice.

Example 23 Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = x^2 + 1$. Then, pre-images of 17 and -3 , respectively, are

- | | |
|-----------------------|----------------------------|
| (A) $\phi, \{4, -4\}$ | (B) $\{3, -3\}, \phi$ |
| (C) $\{4, -4\}, \phi$ | (D) $\{4, -4, \{2, -2\}\}$ |

Solution (C) is the correct choice since for $f^{-1}(17) = x \Rightarrow f(x) = 17$ or $x^2 + 1 = 17 \Rightarrow x = \pm 4$ or $f^{-1}(17) = \{4, -4\}$ and for $f^{-1}(-3) = x \Rightarrow f(x) = -3 \Rightarrow x^2 + 1 = -3 \Rightarrow x^2 = -4$ and hence $f^{-1}(-3) = \phi$.

Example 24 For real numbers x and y , define xRy if and only if $x - y + \sqrt{2}$ is an irrational number. Then the relation R is

- | | |
|----------------|-------------------|
| (A) reflexive | (B) symmetric |
| (C) transitive | (D) none of these |

Solution (A) is the correct choice.

Fill in the blanks in each of the Examples 25 to 30.

Example 25 Consider the set $A = \{1, 2, 3\}$ and R be the smallest equivalence relation on A, then $R =$ _____

Solution $R = \{(1, 1), (2, 2), (3, 3)\}$.

Example 26 The domain of the function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = \sqrt{x^2 - 3x + 2}$ is _____.

Solution Here $x^2 - 3x + 2 \geq 0$
 $\Rightarrow (x - 1)(x - 2) \geq 0$
 $\Rightarrow x \leq 1$ or $x \geq 2$

Hence the domain of $f = (-\infty, 1] \cup [2, \infty)$

Example 27 Consider the set A containing n elements. Then, the total number of injective functions from A onto itself is _____.

Solution $n!$

Example 28 Let $\mathbf{Z}$ be the set of integers and R be the relation defined in $\mathbf{Z}$ such that aRb if $a - b$ is divisible by 3. Then R partitions the set $\mathbf{Z}$ into _____ pairwise disjoint subsets.

Solution Three.

Example 29 Let $\mathbf{R}$ be the set of real numbers and $*$ be the binary operation defined on $\mathbf{R}$ as $a * b = a + b - ab \quad \forall a, b \in \mathbf{R}$. Then, the identity element with respect to the binary operation $*$ is _____.

Solution 0 is the identity element with respect to the binary operation $*$.

State **True** or **False** for the statements in each of the Examples 30 to 34.

Example 30 Consider the set $A = \{1, 2, 3\}$ and the relation $R = \{(1, 2), (1, 3)\}$. R is a transitive relation.

Solution True.

Example 31 Let A be a finite set. Then, each injective function from A into itself is not surjective.

Solution False.

Example 32 For sets A , B and C , let $f: A \rightarrow B$, $g: B \rightarrow C$ be functions such that $g \circ f$ is injective. Then both f and g are injective functions.

Solution False.

Example 33 For sets A , B and C , let $f: A \rightarrow B$, $g: B \rightarrow C$ be functions such that $g \circ f$ is surjective. Then g is surjective

Solution True.

Example 34 Let $\mathbf{N}$ be the set of natural numbers. Then, the binary operation $*$ in $\mathbf{N}$ defined as $a * b = a + b$, $\forall a, b \in \mathbf{N}$ has identity element.

Solution False.

1.3 EXERCISE

Short Answer (S.A.)

1. Let $A = \{a, b, c\}$ and the relation R be defined on A as follows:

$$R = \{(a, a), (b, c), (a, b)\}.$$

Then, write minimum number of ordered pairs to be added in R to make R reflexive and transitive.

2. Let D be the domain of the real valued function f defined by $f(x) = \sqrt{25 - x^2}$. Then, write D .
3. Let $f, g : \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$, $\forall x \in \mathbf{R}$, respectively. Then, find $g \circ f$.
4. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $f(x) = 2x - 3$ $\forall x \in \mathbf{R}$. write f^{-1} .
5. If $A = \{a, b, c, d\}$ and the function $f = \{(a, b), (b, d), (c, a), (d, c)\}$, write f^{-1} .
6. If $f : \mathbf{R} \rightarrow \mathbf{R}$ is defined by $f(x) = x^2 - 3x + 2$, write $f(f(x))$.
7. Is $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function? If g is described by $g(x) = \alpha x + \beta$, then what value should be assigned to α and β .
8. Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective.
- (i) $\{(x, y) : x \text{ is a person, } y \text{ is the mother of } x\}$.
- (ii) $\{(a, b) : a \text{ is a person, } b \text{ is an ancestor of } a\}$.
9. If the mappings f and g are given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$, write $f \circ g$.
10. Let $\mathbf{C}$ be the set of complex numbers. Prove that the mapping $f : \mathbf{C} \rightarrow \mathbf{R}$ given by $f(z) = |z|$, $\forall z \in \mathbf{C}$, is neither one-one nor onto.
11. Let the function $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = \cos x$, $\forall x \in \mathbf{R}$. Show that f is neither one-one nor onto.
12. Let $X = \{1, 2, 3\}$ and $Y = \{4, 5\}$. Find whether the following subsets of $X \times Y$ are functions from X to Y or not.
- (i) $f = \{(1, 4), (1, 5), (2, 4), (3, 5)\}$ (ii) $g = \{(1, 4), (2, 4), (3, 4)\}$
- (iii) $h = \{(1, 4), (2, 5), (3, 5)\}$ (iv) $k = \{(1, 4), (2, 5)\}$.
13. If functions $f : A \rightarrow B$ and $g : B \rightarrow A$ satisfy $g \circ f = I_A$, then show that f is one-one and g is onto.

14. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $f(x) = \frac{1}{2 - \cos x}$, $x \in \mathbf{R}$. Then, find the range of f .
15. Let n be a fixed positive integer. Define a relation R in $\mathbf{Z}$ as follows: $a, b \in \mathbf{Z}$, aRb if and only if $a - b$ is divisible by n . Show that R is an equivalence relation.

Long Answer (L.A.)

16. If $A = \{1, 2, 3, 4\}$, define relations on A which have properties of being:
- reflexive, transitive but not symmetric
 - symmetric but neither reflexive nor transitive
 - reflexive, symmetric and transitive.
17. Let R be relation defined on the set of natural number $\mathbf{N}$ as follows:
 $R = \{(x, y): x \in \mathbf{N}, y \in \mathbf{N}, 2x + y = 41\}$. Find the domain and range of the relation R . Also verify whether R is reflexive, symmetric and transitive.
18. Given $A = \{2, 3, 4\}$, $B = \{2, 5, 6, 7\}$. Construct an example of each of the following:
- an injective mapping from A to B
 - a mapping from A to B which is not injective
 - a mapping from B to A .
19. Give an example of a map
- which is one-one but not onto
 - which is not one-one but onto
 - which is neither one-one nor onto.
20. Let $A = \mathbf{R} - \{3\}$, $B = \mathbf{R} - \{1\}$. Let $f: A \rightarrow B$ be defined by $f(x) = \frac{x-2}{x-3}$, $x \in A$. Then show that f is bijective.
21. Let $A = [-1, 1]$. Then, discuss whether the following functions defined on A are one-one, onto or bijective:
- $f(x) = \frac{x}{2}$
 - $g(x) = |x|$
 - $h(x) = x|x|$
 - $k(x) = x^2$
22. Each of the following defines a relation on $\mathbf{N}$:
- x is greater than y , $x, y \in \mathbf{N}$
 - $x + y = 10$, $x, y \in \mathbf{N}$

(iii) $x \cdot y$ is square of an integer $x, y \in \mathbf{N}$

(iv) $x + 4y = 10$ $x, y \in \mathbf{N}$.

Determine which of the above relations are reflexive, symmetric and transitive.

23. Let $A = \{1, 2, 3, \dots, 9\}$ and R be the relation in $A \times A$ defined by $(a, b) R (c, d)$ if $a + d = b + c$ for $(a, b), (c, d) \in A \times A$. Prove that R is an equivalence relation and also obtain the equivalent class $[(2, 5)]$.
24. Using the definition, prove that the function $f: A \rightarrow B$ is invertible if and only if f is both one-one and onto.
25. Functions $f, g: \mathbf{R} \rightarrow \mathbf{R}$ are defined, respectively, by $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$, find
- (i) $f \circ g$ (ii) $g \circ f$ (iii) $f \circ f$ (iv) $g \circ g$
26. Let $*$ be the binary operation defined on $\mathbf{Q}$. Find which of the following binary operations are commutative
- (i) $a * b = a - b$ $a, b \in \mathbf{Q}$ (ii) $a * b = a^2 + b^2$ $a, b \in \mathbf{Q}$
- (iii) $a * b = a + ab$ $a, b \in \mathbf{Q}$ (iv) $a * b = (a - b)^2$ $a, b \in \mathbf{Q}$
27. Let $*$ be binary operation defined on $\mathbf{R}$ by $a * b = 1 + ab$, $a, b \in \mathbf{R}$. Then the operation $*$ is
- (i) commutative but not associative
- (ii) associative but not commutative
- (iii) neither commutative nor associative
- (iv) both commutative and associative

Objective Type Questions

Choose the correct answer out of the given four options in each of the Exercises from 28 to 47 (M.C.Q.).

28. Let T be the set of all triangles in the Euclidean plane, and let a relation R on T be defined as aRb if a is congruent to b $a, b \in T$. Then R is
- (A) reflexive but not transitive (B) transitive but not symmetric
- (C) equivalence (D) none of these
29. Consider the non-empty set consisting of children in a family and a relation R defined as aRb if a is brother of b . Then R is
- (A) symmetric but not transitive (B) transitive but not symmetric
- (C) neither symmetric nor transitive (D) both symmetric and transitive

30. The maximum number of equivalence relations on the set $A = \{1, 2, 3\}$ are
 (A) 1 (B) 2
 (C) 3 (D) 5
31. If a relation R on the set $\{1, 2, 3\}$ be defined by $R = \{(1, 2)\}$, then R is
 (A) reflexive (B) transitive
 (C) symmetric (D) none of these
32. Let us define a relation R in $\mathbf{R}$ as aRb if $a \geq b$. Then R is
 (A) an equivalence relation (B) reflexive, transitive but not symmetric
 (C) symmetric, transitive but not reflexive (D) neither transitive nor reflexive but symmetric.
33. Let $A = \{1, 2, 3\}$ and consider the relation
 $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$.
 Then R is
 (A) reflexive but not symmetric (B) reflexive but not transitive
 (C) symmetric and transitive (D) neither symmetric, nor transitive
34. The identity element for the binary operation $*$ defined on $Q \sim \{0\}$ as
 $a * b = \frac{ab}{2} \quad a, b \in Q \sim \{0\}$ is
 (A) 1 (B) 0
 (C) 2 (D) none of these
35. If the set A contains 5 elements and the set B contains 6 elements, then the number of one-one and onto mappings from A to B is
 (A) 720 (B) 120
 (C) 0 (D) none of these
36. Let $A = \{1, 2, 3, \dots, n\}$ and $B = \{a, b\}$. Then the number of surjections from A into B is
 (A) nP_2 (B) $2^n - 2$
 (C) $2^n - 1$ (D) None of these

37. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = \frac{1}{x}$ $x \in \mathbf{R}$. Then f is
- (A) one-one (B) onto
(C) bijective (D) f is not defined
38. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = 3x^2 - 5$ and $g: \mathbf{R} \rightarrow \mathbf{R}$ by $g(x) = \frac{x}{x^2 + 1}$.
Then $g \circ f$ is
- (A) $\frac{3x^2 - 5}{9x^4 - 30x^2 + 26}$ (B) $\frac{3x^2 - 5}{9x^4 - 6x^2 + 26}$
(C) $\frac{3x^2}{x^4 + 2x^2 - 4}$ (D) $\frac{3x^2}{9x^4 + 30x^2 - 2}$
39. Which of the following functions from $\mathbf{Z}$ into $\mathbf{Z}$ are bijections?
- (A) $f(x) = x^3$ (B) $f(x) = x + 2$
(C) $f(x) = 2x + 1$ (D) $f(x) = x^2 + 1$
40. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be the functions defined by $f(x) = x^3 + 5$. Then $f^{-1}(x)$ is
- (A) $(x+5)^{\frac{1}{3}}$ (B) $(x-5)^{\frac{1}{3}}$
(C) $(5-x)^{\frac{1}{3}}$ (D) $5 - x$
41. Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be the bijective functions. Then $(g \circ f)^{-1}$ is
- (A) $f^{-1} \circ g^{-1}$ (B) $f \circ g$
(C) $g^{-1} \circ f^{-1}$ (D) $g \circ f$
42. Let $f: \mathbf{R} - \left\{ \frac{3}{5} \right\} \rightarrow \mathbf{R}$ be defined by $f(x) = \frac{3x+2}{5x-3}$. Then
- (A) $f^{-1}(x) = f(x)$ (B) $f^{-1}(x) = -f(x)$
(C) $(f \circ f)x = -x$ (D) $f^{-1}(x) = \frac{1}{19}f(x)$
43. Let $f: [0, 1] \rightarrow [0, 1]$ be defined by $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$

Then $(f \circ f) x$ is

(A) constant

(B) $1 + x$

(C) x

(D) none of these

44. Let $f : [2, \infty) \rightarrow \mathbf{R}$ be the function defined by $f(x) = x^2 - 4x + 5$, then the range of f is

(A) $\mathbf{R}$

(B) $[1, \infty)$

(C) $[4, \infty)$

(B) $[5, \infty)$

45. Let $f : \mathbf{N} \rightarrow \mathbf{R}$ be the function defined by $f(x) = \frac{2x-1}{2}$ and $g : \mathbf{Q} \rightarrow \mathbf{R}$ be

another function defined by $g(x) = x + 2$. Then $(g \circ f) \frac{3}{2}$ is

(A) 1

(B) 1

(C) $\frac{7}{2}$

(B) none of these

46. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 2x : x > 3 \\ x^2 : 1 < x \leq 3 \\ 3x : x \leq 1 \end{cases}$$

Then $f(-1) + f(2) + f(4)$ is

(A) 9

(B) 14

(C) 5

(D) none of these

47. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be given by $f(x) = \tan x$. Then $f^{-1}(1)$ is

(A) $\frac{\pi}{4}$

(B) $\{n\pi + \frac{\pi}{4} : n \in \mathbf{Z}\}$

(C) does not exist

(D) none of these

Fill in the blanks in each of the Exercises 48 to 52.

48. Let the relation R be defined in $\mathbf{N}$ by aRb if $2a + 3b = 30$. Then $R = \underline{\hspace{2cm}}$.

49. Let the relation R be defined on the set

$A = \{1, 2, 3, 4, 5\}$ by $R = \{(a, b) : |a^2 - b^2| < 8\}$. Then R is given by $\underline{\hspace{2cm}}$.

50. Let $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$. Then $g \circ f = \underline{\hspace{2cm}}$ and $f \circ g = \underline{\hspace{2cm}}$.

51. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = \frac{x}{\sqrt{1+x^2}}$. Then $(f \circ f \circ f)(x) = \text{_____}$
52. If $f(x) = (4 - (x-7)^3)$, then $f^{-1}(x) = \text{_____}$.

State **True** or **False** for the statements in each of the Exercises 53 to 63.

53. Let $R = \{(3, 1), (1, 3), (3, 3)\}$ be a relation defined on the set $A = \{1, 2, 3\}$. Then R is symmetric, transitive but not reflexive.
54. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $f(x) = \sin(3x+2)$ $x \in \mathbf{R}$. Then f is invertible.
55. Every relation which is symmetric and transitive is also reflexive.
56. An integer m is said to be related to another integer n if m is a integral multiple of n . This relation in $\mathbf{Z}$ is reflexive, symmetric and transitive.
57. Let $A = \{0, 1\}$ and $\mathbf{N}$ be the set of natural numbers. Then the mapping $f : \mathbf{N} \rightarrow A$ defined by $f(2n-1) = 0, f(2n) = 1, n \in \mathbf{N}$, is onto.
58. The relation R on the set $A = \{1, 2, 3\}$ defined as $R = \{(1, 1), (1, 2), (2, 1), (3, 3)\}$ is reflexive, symmetric and transitive.
59. The composition of functions is commutative.
60. The composition of functions is associative.
61. Every function is invertible.
62. A binary operation on a set has always the identity element.

