

Chapter 1 Relations and Functions

EXERCISE 1.1

Question 1:

Determine whether each of the following relations are reflexive, symmetric and transitive.

- (i) Relation R in the set $A = \{1, 2, 3, \dots, 13, 14\}$ defined as

$$R = \{(x, y) : 3x - y = 0\}$$

- (ii) Relation R in the set of N natural numbers defined as

$$R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$$

- (iii) Relation R in the set $A = \{1, 2, 3, 4, 5, 6\}$ defined as

$$R = \{(x, y) : y \text{ is divisible by } x\}$$

- (iv) Relation R in the set of Z integers defined as

$$R = \{(x, y) : x - y \text{ is an integer}\}$$

- (v) Relation R in the set of human beings in a town at a particular time given by

(a) $R = \{(x, y) : x \text{ and } y \text{ work at the same place}\}$

(b) $R = \{(x, y) : x \text{ and } y \text{ live in the same locality}\}$

(c) $R = \{(x, y) : x \text{ is exactly 7cm taller than } y\}$

(d) $R = \{(x, y) : x \text{ is wife of } y\}$

(e) $R = \{(x, y) : x \text{ is father of } y\}$

Solution:

- (i) $R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$

R is not reflexive because $(1, 1), (2, 2), \dots$ and $(14, 14) \notin R$.

R is not symmetric because $(1, 3) \in R$, but $(3, 1) \notin R$. [since $3(3) \neq 0$].

R is not transitive because $(1, 3), (3, 9) \in R$, but $(1, 9) \notin R$. [since $3(1) - 9 \neq 0$].

Hence, R is neither reflexive nor symmetric nor transitive.

- (ii) $R = \{(1, 6), (2, 7), (3, 8)\}$

R is not reflexive because $(1, 1) \notin R$.

R is not symmetric because $(1, 6) \in R$ but $(6, 1) \notin R$.

R is not transitive because there isn't any ordered pair in R such that $(x, y), (y, z) \in R$, so $(x, z) \notin R$.

Hence, R is neither reflexive nor symmetric nor transitive.

- (iii) $R = \{(x, y) : y \text{ is divisible by } x\}$

We know that any number other than 0 is divisible by itself.

Thus, $(x, x) \in R$

So, R is reflexive.

$(2, 4) \in R$ [because 4 is divisible by 2]

But $(4, 2) \notin R$ [since 2 is not divisible by 4]

So, R is not symmetric.

Let (x, y) and $(y, z) \in R$. So, y is divisible by x and z is divisible by y .

So, z is divisible by $x \Rightarrow (x, z) \in R$

So, R is transitive.

So, R is reflexive and transitive but not symmetric.

(iv) $R = \{(x, y) : x - y \text{ is an integer}\}$

For $x \in \mathbb{Z}$, $(x, x) \in R$ because $x - x = 0$ is an integer.

So, R is reflexive.

For, $x, y \in \mathbb{Z}$, if $x, y \in R$, then $x - y$ is an integer $\Rightarrow (y - x)$ is an integer.

So, $(y, x) \in R$

So, R is symmetric.

Let (x, y) and $(y, z) \in R$, where $x, y, z \in \mathbb{Z}$.

$\Rightarrow (x - y)$ and $(y - z)$ are integers.

$\Rightarrow x - z = (x - y) + (y - z)$ is an integer.

So, R is transitive.

So, R is reflexive, symmetric and transitive.

(v)

a) $R = \{(x, y) : x \text{ and } y \text{ work at the same place}\}$

R is reflexive because $(x, x) \in R$

R is symmetric because ,

If $(x, y) \in R$, then x and y work at the same place and y and x also work at the same place. $(y, x) \in R$.

R is transitive because,

Let $(x, y), (y, z) \in R$

x and y work at the same place and y and z work at the same place.

Then, x and z also works at the same place. $(x, z) \in R$.

Hence, R is reflexive, symmetric and transitive.

b) $R = \{(x, y) : x \text{ and } y \text{ live in the same locality}\}$

R is reflexive because $(x, x) \in R$

R is symmetric because,

If $(x, y) \in R$, then x and y live in the same locality and y and x also live in the same locality $(y, x) \in R$.

R is transitive because,

Let $(x, y), (y, z) \in R$

x and y live in the same locality and y and z live in the same locality.

Then x and z also live in the same locality. $(x, z) \in R$.

Hence, R is reflexive, symmetric and transitive.

c) $R = \{(x, y) : x \text{ is exactly } 7\text{cm taller than } y\}$

R is not reflexive because $(x, x) \notin R$.

R is not symmetric because,

If $(x, y) \in R$, then x is exactly 7cm taller than y and y is clearly not taller than x .
 $(y, x) \notin R$.

R is not transitive because,

Let $(x, y), (y, z) \in R$

x is exactly 7cm taller than y and y is exactly 7cm taller than z .

Then x is exactly 14cm taller than z . $(x, z) \notin R$

Hence, R is neither reflexive nor symmetric nor transitive.

d) $R = \{(x, y) : x \text{ is wife of } y\}$

R is not reflexive because $(x, x) \notin R$.

R is not symmetric because,

Let $(x, y) \in R$, x is the wife of y and y is not the wife of x . $(y, x) \notin R$.

R is not transitive because,

Let $(x, y), (y, z) \in R$

x is wife of y and y is wife of z , which is not possible.

$(x, z) \notin R$.

Hence, R is neither reflexive nor symmetric nor transitive.

e) $R = \{(x, y) : x \text{ is father of } y\}$

R is not reflexive because $(x, x) \notin R$.

R is not symmetric because,

Let $(x, y) \in R$, x is the father of y and y is not the father of x . $(y, x) \notin R$.

R is not transitive because,

Let $(x, y), (y, z) \in R$

x is father of y and y is father of z , x is not father of z . $(x, z) \notin R$.

Hence, R is neither reflexive nor symmetric nor transitive.

Question 2:

Show that the relation R in the set R of real numbers, defined as $R = \{(a, b) : a \leq b^2\}$ is neither reflexive nor symmetric nor transitive.

Solution:

$$R = \{(a, b) : a \leq b^2\}$$

$$\left(\frac{1}{2}, \frac{1}{2}\right) \notin R \quad \text{because} \quad \frac{1}{2} > \left(\frac{1}{2}\right)^2$$

R is not reflexive.

$$(1, 4) \in R \text{ as } 1 < 4. \text{ But } 4 \text{ is not less than } 1^2.$$

$$(4, 1) \notin R$$

R is not symmetric.

$$(3, 2)(2, 1.5) \in R \quad [\text{Because } 3 < 2^2=4 \text{ and } 2 < (1.5)^2=2.25]$$

$$3 > (1.5)^2 = 2.25$$

$$\therefore (3, 1.5) \notin R$$

R is not transitive.

R is neither reflexive nor symmetric nor transitive.

Question 3:

Check whether the relation R defined in the set $\{1, 2, 3, 4, 5, 6\}$ as $R = \{(a, b) : b = a + 1\}$ is reflexive, symmetric or transitive.

Solution:

$$A = \{1, 2, 3, 4, 5, 6\}$$

$$R = \{(a, b) : b = a + 1\}$$

$$R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$$

$$(a, a) \notin R, a \in A$$

$$(1, 1), (2, 2), (3, 3), (4, 4), (5, 5) \notin R$$

R is not reflexive.

$$(1, 2) \in R, \text{ but } (2, 1) \notin R$$

R is not symmetric.

$$(1,2), (2,3) \in R$$

$$(1,3) \notin R$$

R is not transitive.

R is neither reflective nor symmetric nor transitive.

Question 4:

Show that the relation R in R defined as $R = \{(a,b) : a \leq b\}$ is reflexive and transitive, but not symmetric.

Solution:

$$R = \{(a,b) : a \leq b\}$$

$$(a,a) \in R$$

R is reflexive.

$$(2,4) \in R \text{ (as } 2 < 4)$$

$$(4,2) \notin R \text{ (as } 4 > 2)$$

R is not symmetric.

$$(a,b), (b,c) \in R$$

$$a \leq b \text{ and } b \leq c$$

$$\Rightarrow a \leq c$$

$$\Rightarrow (a,c) \in R$$

R is transitive.

R is reflexive and transitive but not symmetric.

Question 5:

Check whether the relation R in R defined as $R = \{(a,b) : a \leq b^3\}$ is reflexive, symmetric or transitive.

Solution:

$$R = \{(a,b) : a \leq b^3\}$$

$$\left(\frac{1}{2}, \frac{1}{2}\right) \notin R, \text{ since } \frac{1}{2} > \left(\frac{1}{2}\right)^3$$

R is not reflexive.

$$(1,2) \in R \text{ (as } 1 < 2^3 = 8 \text{)}$$

$$(2,1) \notin R \text{ (as } 2^3 > 1 = 8 \text{)}$$

R is not symmetric.

$$\left(3, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{6}{5}\right) \in R, \text{ since } 3 < \left(\frac{3}{2}\right)^3 \text{ and } \frac{2}{3} < \left(\frac{6}{5}\right)^3$$

$$\left(3, \frac{6}{5}\right) \notin R \text{ as } 3 > \left(\frac{6}{5}\right)^3$$

R is not transitive.

R is neither reflexive nor symmetric nor transitive.

Question 6:

Show that the relation R in the set $\{1,2,3\}$ given by $R = \{(1,2), (2,1)\}$ is symmetric but neither reflexive nor transitive.

Solution:

$$A = \{1,2,3\}$$

$$R = \{(1,2), (2,1)\}$$

$$(1,1), (2,2), (3,3) \notin R$$

R is not reflexive.

$$(1,2) \in R \text{ and } (2,1) \in R$$

R is symmetric.

$$(1,2) \in R \text{ and } (2,1) \in R$$

$$(1,1) \notin R$$

R is not transitive.

R is symmetric, but not reflexive or transitive.

Question 7:

Show that the relation R in the set A of all books in a library of a college, given by $R = \{(x,y) : x \text{ and } y \text{ have same number of pages}\}$ is an equivalence relation.

Solution:

$$R = \{(x,y) : x \text{ and } y \text{ have same number of pages}\}$$

R is reflexive since $(x,x) \in R$ as x and x have same number of pages.

R is reflexive.

$$(x, y) \in R$$

x and y have same number of pages and y and x have same number of pages $(y, x) \in R$

R is symmetric.

$$(x, y) \in R, (y, z) \in R$$

x and y have same number of pages, y and z have same number of pages.

Then x and z have same number of pages.

$$(x, z) \in R$$

R is transitive.

R is an equivalence relation.

Question 8:

Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b| \text{ is even}\}$ is an equivalence relation. Show that all the elements of $\{1, 3, 5\}$ are related to each other and all the elements of $\{2, 4\}$ are related to each other. But no element of $\{1, 3, 5\}$ is related to any element of $\{2, 4\}$.

Solution:

$$a \in A$$

$$|a - a| = 0 \text{ (which is even)}$$

R is reflexive.

$$(a, b) \in R$$

$$\Rightarrow |a - b| \text{ [is even]}$$

$$\Rightarrow |-(a - b)| = |b - a| \text{ [is even]}$$

$$(b, a) \in R$$

R is symmetric.

$$(a, b) \in R \text{ and } (b, c) \in R$$

$$\Rightarrow |a - b| \text{ is even and } |b - c| \text{ is even}$$

$$\Rightarrow (a - b) \text{ is even and } (b - c) \text{ is even}$$

$$\Rightarrow (a - c) = (a - b) + (b - c) \text{ is even}$$

$\Rightarrow |a-b|$ is even

$\Rightarrow (a,c) \in R$

R is transitive.

R is an equivalence relation.

All elements of $\{1,3,5\}$ are related to each other because they are all odd. So, the modulus of the difference between any two elements is even.

Similarly, all elements $\{2,4\}$ are related to each other because they are all even.

No element of $\{1,3,5\}$ is related to any elements of $\{2,4\}$ as all elements of $\{1,3,5\}$ are odd and all elements of $\{2,4\}$ are even. So, the modulus of the difference between the two elements will not be even.

Question 9:

Show that each of the relation R in the set $A = \{x \in \mathbb{Z} : 0 \leq x \leq 12\}$, given by

i. $R = \{(a,b) : |a-b| \text{ is a multiple of } 4\}$

ii. $R = \{(a,b) : a = b\}$

Is an equivalence relation. Find the set of all elements related to 1 in each case.

Solution:

$$A = \{x \in \mathbb{Z} : 0 \leq x \leq 12\} = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$$

i. $R = \{(a,b) : |a-b| \text{ is a multiple of } 4\}$

$$a \in A, (a,a) \in R \quad [|a-a| = 0 \text{ is a multiple of } 4]$$

R is reflexive.

$$(a,b) \in R \Rightarrow |a-b| \text{ [is a multiple of } 4]$$

$$\Rightarrow |-(a-b)| = |b-a| \text{ [is a multiple of } 4]$$

$$(b,a) \in R$$

R is symmetric.

$$(a,b) \in R \text{ and } (b,c) \in R$$

$$\Rightarrow |a-b| \text{ is a multiple of } 4 \text{ and } |b-c| \text{ is a multiple of } 4$$

$$\Rightarrow (a-b) \text{ is a multiple of } 4 \text{ and } (b-c) \text{ is a multiple of } 4$$

$$\Rightarrow (a-c) = (a-b) + (b-c) \text{ is a multiple of } 4$$

$$\Rightarrow |a-c| \text{ is a multiple of } 4$$

$$\Rightarrow (a, c) \in R$$

R is transitive.

R is an equivalence relation.

The set of elements related to 1 is $\{1, 5, 9\}$ as

$$|1 - 1| = 0 \text{ is a multiple of 4.}$$

$$|5 - 1| = 4 \text{ is a multiple of 4.}$$

$$|9 - 1| = 8 \text{ is a multiple of 4.}$$

ii. $R = \{(a, b) : a = b\}$

$$a \in A, (a, a) \in R \quad [\text{since } a=a]$$

R is reflexive.

$$(a, b) \in R$$

$$\Rightarrow a = b$$

$$\Rightarrow b = a$$

$$\Rightarrow (b, a) \in R$$

R is symmetric.

$$(a, b) \in R \text{ and } (b, c) \in R$$

$$\Rightarrow a = b \text{ and } b = c$$

$$\Rightarrow a = c$$

$$\Rightarrow (a, c) \in R$$

R is transitive.

R is an equivalence relation.

The set of elements related to 1 is $\{1\}$.

Question 10:

Give an example of a relation, which is

- Symmetric but neither reflexive nor transitive.
- Transitive but neither reflexive nor symmetric.
- Reflexive and symmetric but not transitive.
- Reflexive and transitive but not symmetric.
- Symmetric and transitive but not reflexive.

Solution:

-

$$A = \{5, 6, 7\}$$

$$R = \{(5, 6), (6, 5)\}$$

$$(5, 5), (6, 6), (7, 7) \notin R$$

R is not reflexive as $(5, 5), (6, 6), (7, 7) \notin R$

$(5, 6), (6, 5) \in R$ and $(6, 5) \in R$, R is symmetric.

$\Rightarrow (5, 6), (6, 5) \in R$, but $(5, 5) \notin R$

R is not transitive.

Relation R is symmetric but not reflexive or transitive.

ii. $R = \{(a, b) : a < b\}$

$a \in R, (a, a) \notin R$ [since a cannot be less than itself]

R is not reflexive.

$$(1, 2) \in R \text{ (as } 1 < 2)$$

But 2 is not less than 1

$$\therefore (2, 1) \notin R$$

R is not symmetric.

$$(a, b), (b, c) \in R$$

$$\Rightarrow a < b \text{ and } b < c$$

$$\Rightarrow a < c$$

$$\Rightarrow (a, c) \in R$$

R is transitive.

Relation R is transitive but not reflexive and symmetric.

iii. $A = \{4, 6, 8\}$

$$A = \{(4, 4), (6, 6), (8, 8), (4, 6), (6, 8), (8, 6)\}$$

R is reflexive since $a \in A, (a, a) \in R$

R is symmetric since $(a, b) \in R$

$$\Rightarrow (b, a) \in R \text{ for } a, b \in R$$

R is not transitive since $(4, 6), (6, 8) \in R$, but $(4, 8) \notin R$

R is reflexive and symmetric but not transitive.

iv. $R = \{(a, b) : a^3 > b^3\}$

$$(a, a) \in R$$

R is reflexive.

$$(2, 1) \in R$$

$$\text{But } (1, 2) \notin R$$

$\therefore R$ is not symmetric.

$$(a, b), (b, c) \in R$$

$$\Rightarrow a^3 \geq b^3 \text{ and } b^3 < c^3$$

$$\Rightarrow a^3 < c^3$$

$$\Rightarrow (a, c) \in R$$

$\therefore R$ is transitive.

R is reflexive and transitive but not symmetric

v.

$$A = \{1, 3, 5\}$$

Define a Relation R

On A .

$$R : A \rightarrow A$$

$$R = \{(1, 3) (3, 1) (1, 1) (3, 3)\}$$

Relation R is not Reflexive as $(5, 5) \notin R$

Relation R is Symmetric as

$$(1, 3) \in R \Rightarrow (3, 1) \in R$$

Relation R is Transitive

$$(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$$

$$(3, 1) \in R, (1, 1) \in R \Rightarrow (3, 1) \in R$$

Alternative Answer

$$R = (a, b) : a \text{ is brother of } b \text{ \{suppose } a \text{ and } b \text{ are male\}}$$

$Ref \rightarrow a$ is not brother of a

So, $(a, a) \notin R$

Relation R is not Reflexive

Symmetric $\rightarrow a$ is brother of b so

b is brother of a

$$a, b \in R \Rightarrow (b, a) \in R$$

Transitive $\rightarrow a$ is brother of b and

b is brother of c so

a is brother of c

$$(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$$

Question 11:

Show that the relation R in the set A of points in a plane given by

$$R = \{(P, Q) : \text{Distance of the point } P \text{ from the origin is same as the distance of the point } Q \text{ from the origin}\}$$

, is an equivalence relation. Further, show that the set of all points related to a point $P \neq (0,0)$ is the circle passing through P with origin as centre.

Solution:

$$R = \{(P, Q) : \text{Distance of the point } P \text{ from the origin is same as the distance of the point } Q \text{ from the origin}\}$$

Clearly, $(P, P) \in R$

R is reflexive.

$$(P, Q) \in R$$

Clearly R is symmetric.

$$(P, Q), (Q, S) \in R$$

$\Rightarrow$ The distance of P and Q from the origin is the same and also, the distance of Q and S from the origin is the same.

$\Rightarrow$ The distance of P and S from the origin is the same.

$$(P, S) \in R$$

R is transitive.

R is an equivalence relation.

The set of points related to $P \neq (0,0)$ will be those points whose distance from origin is same as distance of P from the origin.

Set of points forms a circle with the centre as origin and this circle passes through P .

Question 12:

Show that the relation R in the set A of all triangles as $R = \{(T_1, T_2) : T_1 \text{ is similar to } T_2\}$, is an equivalence relation. Consider three right angle triangles T_1 with sides 3,4,5, T_2 with sides 5,12,13 and T_3 with sides 6,8,10. Which triangle among T_1, T_2, T_3 are related?

Solution:

$$R = \{(T_1, T_2) : T_1 \text{ is similar to } T_2\}$$

R is reflexive since every triangle is similar to itself.

If $(T_1, T_2) \in R$, then T_1 is similar to T_2 .

T_2 is similar to T_1 .

$$\Rightarrow (T_2, T_1) \in R$$

R is symmetric.

$$(T_1, T_2), (T_2, T_3) \in R$$

is similar to T_2 and T_2 is similar to T_3 .

$\therefore T_1$ is similar to T_3 .

$$\Rightarrow (T_1, T_3) \in R$$

$\therefore R$ is transitive.

$$\frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \left(\frac{1}{2}\right)$$

$\therefore$ Corresponding sides of triangles T_1 and T_3 are in the same ratio.

Triangle T_1 is similar to triangle T_3 .

Hence, T_1 is related to T_3 .

Question 13:

Show that the relation R in the set A of all polygons as $R = \{(P_1, P_2) : P_1 \text{ and } P_2 \text{ have same number of sides}\}$, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5?

Solution:

$$R = \{(P_1, P_2) : P_1 \text{ and } P_2 \text{ have same number of sides}\}$$

$(P_1, P_2) \in R$ as same polygon has same number of sides.

$\therefore R$ is reflexive.

$$(P_1, P_2) \in R$$

$\Rightarrow P_1$ and P_2 have same number of sides.

$\Rightarrow P_2$ and P_1 have same number of sides.

$$\Rightarrow (P_2, P_1) \in R$$

$\therefore R$ is symmetric.

$$(P_1, P_2), (P_2, P_3) \in R$$

$\Rightarrow P_1$ and P_2 have same number of sides.

P_2 and P_3 have same number of sides.

$\Rightarrow P_1$ and P_3 have same number of sides.

$$\Rightarrow (P_1, P_3) \in R$$

$\therefore R$ is transitive.

R is an equivalence relation.

The elements in A related to right-angled triangle (T) with sides 3, 4, 5 are those polygons which have three sides.

Set of all elements in A related to triangle T is the set of all triangles.

Question 14:

Let L be the set of all lines in XY plane and R be the relation in L defined as

$R = \{(L_1, L_2) : L_1 \text{ is parallel to } L_2\}$. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

Solution:

$$R = \{(L_1, L_2) : L_1 \text{ is parallel to } L_2\}$$

R is reflexive as any line L_1 is parallel to itself i.e., $(L_1, L_1) \in R$

If $(L_1, L_2) \in R$, then

$\Rightarrow L_1$ is parallel to L_2 .

$\Rightarrow L_2$ is parallel to L_1 .

$$\Rightarrow (L_2, L_1) \in R$$

$\therefore R$ is symmetric.

$$(L_1, L_2), (L_2, L_3) \in R$$

$$\Rightarrow L_1 \text{ is parallel to } L_2$$

$$\Rightarrow L_2 \text{ is parallel to } L_3$$

$$\therefore L_1 \text{ is parallel to } L_3.$$

$$\Rightarrow (L_1, L_3) \in R$$

$\therefore R$ is transitive.

R is an equivalence relation.

Set of all lines related to the line $y = 2x + 4$ is the set of all lines that are parallel to the line $y = 2x + 4$.

Slope of the line $y = 2x + 4$ is $m = 2$.

Line parallel to the given line is in the form $y = 2x + c$, where $c \in R$.

Set of all lines related to the given line is given by $y = 2x + c$, where $c \in R$.

Question 15:

Let R be the relation in the set $\{1, 2, 3, 4\}$ given by

$$R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}.$$

Choose the correct answer.

- A. R is reflexive and symmetric but not transitive.
- B. R is reflexive and transitive but not symmetric.
- C. R is symmetric and transitive but not reflexive.
- D. R is an equivalence relation.

Solution:

$$R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$$

$$(a, a) \in R \text{ for every } a \in \{1, 2, 3, 4\}$$

$\therefore R$ is reflexive.

$$(1, 2) \in R \text{ but } (2, 1) \notin R$$

$\therefore R$ is not symmetric.

$$(a, b), (b, c) \in R \text{ for all } a, b, c \in \{1, 2, 3, 4\}$$

$\therefore R$ is not transitive.

R is reflexive and transitive but not symmetric.

The correct answer is B.

Question 16:

Let R be the relation in the set N given by $R = \{(a, b) : a = b - 2, b > 6\}$. Choose the correct answer.

- A. $(2, 4) \in R$
- B. $(3, 8) \in R$
- C. $(6, 8) \in R$
- D. $(8, 7) \in R$

Solution:

$$R = \{(a, b) : a = b - 2, b > 6\}$$

Now,

$$b > 6, (2, 4) \notin R$$

$$3 \neq 8 - 2$$

$$\therefore (3, 8) \notin R \text{ and as } 8 \neq 7 - 2$$

$$\therefore (8, 7) \notin R$$

Consider $(6, 8)$

$$8 > 6 \text{ and } 6 = 8 - 2$$

$$\therefore (6, 8) \in R$$

The correct answer is C.

EXERCISE 1.2

Question 1:

Show that the function $f: R_{\bullet} \rightarrow R_{\bullet}$ defined by $f(x) = \frac{1}{x}$ is one –one and onto, where $R_{\bullet}$ is the set of all non –zero real numbers. Is the result true, if the domain $R_{\bullet}$ is replaced by $\mathbb{N}$ with co-domain being same as $R_{\bullet}$?

Solution:

$f: R_{\bullet} \rightarrow R_{\bullet}$ is by $f(x) = \frac{1}{x}$

For one-one:

$x, y \in R_{\bullet}$ such that $f(x) = f(y)$

$$\Rightarrow \frac{1}{x} = \frac{1}{y}$$

$$\Rightarrow x = y$$

$\therefore f$ is one-one.

For onto:

For $y \in R$, there exists $x = \frac{1}{y} \in R_{\bullet}$ [as $y \neq 0$] such that

$$f(x) = \frac{1}{\left(\frac{1}{y}\right)} = y$$

$\therefore f$ is onto.

Given function f is one-one and onto.

Consider function $g : N \rightarrow R$, defined by $g(x) = \frac{1}{x}$

We have, $g(x_1) = g(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$

$\therefore g$ is one-one.

g is not onto as for $1.2 \in R$, there exist any x in N such that $g(x) = \frac{1}{1.2}$

Function g is one-one but not onto.

Question 2:

Check the injectivity and surjectivity of the following functions:

- i. $f : N \rightarrow N$ given by $f(x) = x^2$
- ii. $f : Z \rightarrow Z$ given by $f(x) = x^2$
- iii. $f : R \rightarrow R$ given by $f(x) = x^2$
- iv. $f : N \rightarrow N$ given by $f(x) = x^3$
- v. $f : Z \rightarrow Z$ given by $f(x) = x^3$

Solution:

- i. For $f : N \rightarrow N$ given by $f(x) = x^2$
 $x, y \in N$
 $f(x) = f(y) \Rightarrow x^2 = y^2 \Rightarrow x = y$
 $\therefore f$ is injective.

$2 \in N$. But, there does not exist any x in N such that $f(x) = x^2 = 2$

$\therefore f$ is not surjective

Function f is injective but not surjective.

ii. $f : Z \rightarrow Z$ given by $f(x) = x^2$

$$f(-1) = f(1) = 1 \text{ but } -1 \neq 1$$

$\therefore f$ is not injective.

$-2 \in Z$ But, there does not exist any $x \in Z$ such that $f(x) = -2 \Rightarrow x^2 = -2$

$\therefore f$ is not surjective.

Function f is neither injective nor surjective.

iii. $f : R \rightarrow R$ given by $f(x) = x^2$

$$f(-1) = f(1) = 1 \text{ but } -1 \neq 1$$

$\therefore f$ is not injective.

$-2 \in Z$ But, there does not exist any $x \in Z$ such that $f(x) = -2 \Rightarrow x^2 = -2$

$\therefore f$ is not surjective.

Function f is neither injective nor surjective.

iv. $f : N \rightarrow N$ given by $f(x) = x^3$

$$x, y \in N$$

$$f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$$

$\therefore f$ is injective.

$2 \in N$. But, there does not exist any x in N such that $f(x) = x^3 = 2$

$\therefore f$ is not surjective

Function f is injective but not surjective.

v. $f : Z \rightarrow Z$ given by $f(x) = x^3$

$$x, y \in Z$$

$$f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$$

$\therefore f$ is injective.

$2 \in Z$. But, there does not exist any x in Z such that $f(x) = x^3 = 2$

$\therefore f$ is not surjective.

Function f is injective but not surjective.

Question 3:

Prove that the greatest integer function $f : R \rightarrow R$ given by $f(x) = [x]$ is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to x .

Solution:

$f : R \rightarrow R$ given by $f(x) = [x]$
 $f(1.2) = [1.2] = 1, f(1.9) = [1.9] = 1$
 $\therefore f(1.2) = f(1.9)$, but $1.2 \neq 1.9$
 $\therefore f$ is not one-one.

Consider $0.7 \in R$

$f(x) = [x]$ is an integer. There does not exist any element $x \in R$ such that $f(x) = 0.7$
 $\therefore f$ is not onto.

The greatest integer function is neither one-one nor onto.

Question 4:

Show that the modulus function $f : R \rightarrow R$ given by $f(x) = |x|$ is neither one-one nor onto, where $|x|$ is x , if x is positive or 0 and $|x|$ is $-x$, if x is negative.

Solution:

$f : R \rightarrow R$ is $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$
 $f(-1) = |-1| = 1$ and $f(1) = |1| = 1$
 $\therefore f(-1) = f(1)$ but $-1 \neq 1$
 $\therefore f$ is not one-one.

Consider $-1 \in R$

$f(x) = |x|$ is non-negative. There exist any element x in domain R such that $f(x) = |x| = -1$
 $\therefore f$ is not onto.

The modulus function is neither one-one nor onto.

Question 5:

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

Show that the signum function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by is neither one-one nor onto.

Solution:

$$f : \mathbb{R} \rightarrow \mathbb{R} \text{ is } f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

$$f(1) = f(2) = 1, \text{ but } 1 \neq 2$$

$\therefore f$ is not one-one.

$f(x)$ takes only 3 values $(1, 0, -1)$ for the element -2 in co-domain

$\mathbb{R}$, there does not exist any x in domain $\mathbb{R}$ such that $f(x) = -2$.

$\therefore f$ is not onto.

The signum function is neither one-one nor onto.

Question 6:

Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B . Show that f is one-one.

Solution:

$$A = \{1, 2, 3\}, B = \{4, 5, 6, 7\}$$

$$f : A \rightarrow B \text{ is defined as } f = \{(1, 4), (2, 5), (3, 6)\}$$

$$\therefore f(1) = 4, f(2) = 5, f(3) = 6$$

It is seen that the images of distinct elements of A under f are distinct.

$\therefore f$ is one-one.

Question 7:

In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

i. $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 3 - 4x$

ii. $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 1 + x^2$

Solution:

i. $f : R \rightarrow R$ defined by $f(x) = 3 - 4x$

$x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$$\Rightarrow 3 - 4x_1 = 3 - 4x_2$$

$$\Rightarrow -4x_1 = -4x_2$$

$$\Rightarrow x_1 = x_2$$

$\therefore f$ is one-one.

For any real number (y) in R , there exists $\frac{3-y}{4}$ in R such that $f\left(\frac{3-y}{4}\right) = 3 - 4\left(\frac{3-y}{4}\right) = y$
 $\therefore f$ is onto.

Hence, f is bijective.

ii. $f : R \rightarrow R$ defined by $f(x) = 1 + x^2$

$x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$$\Rightarrow 1 + x_1^2 = 1 + x_2^2$$

$$\Rightarrow x_1^2 = x_2^2$$

$$\Rightarrow x_1 = \pm x_2$$

$\therefore f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$

Consider $f(1) = f(-1) = 2$

$\therefore f$ is not one-one.

Consider an element -2 in co domain R .

It is seen that $f(x) = 1 + x^2$ is positive for all $x \in R$.

$\therefore f$ is not onto.

Hence, f is neither one-one nor onto.

Question 8:

Let A and B be sets. Show that $f : A \times B \rightarrow B \times A$ such that $(a, b) \mapsto (b, a)$ is a bijective function.

Solution:

$f : A \times B \rightarrow B \times A$ is defined as $(a, b) \mapsto (b, a)$.

$(a_1, b_1), (a_2, b_2) \in A \times B$ such that $f(a_1, b_1) = f(a_2, b_2)$

$$\Rightarrow (b_1, a_1) = (b_2, a_2)$$

$$\Rightarrow b_1 = b_2 \text{ and } a_1 = a_2$$

$$\Rightarrow (a_1, b_1) = (a_2, b_2)$$

$\therefore f$ is one-one.

$(b, a) \in B \times A$ there exist $(a, b) \in A \times B$ such that $f(a, b) = (b, a)$

$\therefore f$ is onto.

f is bijective.

Question 9:

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

Let $f : N \rightarrow N$ be defined as for all $n \in N$. State whether the function f is bijective. Justify your answer.

Solution:

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases} \text{ for all } n \in N.$$

$f : N \rightarrow N$ be defined as

$$f(1) = \frac{1+1}{2} = 1 \text{ and } f(2) = \frac{2}{2} = 1$$

$$f(1) = f(2), \text{ where } 1 \neq 2$$

$\therefore f$ is not one-one.

Consider a natural number n in co domain N .

Case I: n is odd

$\therefore n = 2r + 1$ for some $r \in N$ there exists $4r + 1 \in N$ such that

$$f(4r + 1) = \frac{4r + 1 + 1}{2} = 2r + 1$$

Case II: n is even

$\therefore n = 2r$ for some $r \in N$ there exists $4r \in N$ such that

$$f(4r) = \frac{4r}{2} = 2r$$

$\therefore f$ is onto.

f is not a bijective function.

Question 10:

Let $A = \mathbb{R} - \{3\}$, $B = \mathbb{R} - \{1\}$ and $f : A \rightarrow B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$. Is f one-one and onto? Justify your answer.

Solution:

$A = \mathbb{R} - \{3\}$, $B = \mathbb{R} - \{1\}$ and $f : A \rightarrow B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$

$x, y \in A$ such that $f(x) = f(y)$

$$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$$

$$\Rightarrow (x-2)(y-3) = (y-2)(x-3)$$

$$\Rightarrow xy - 3x - 2y + 6 = xy - 3y - 2x + 6$$

$$\Rightarrow -3x - 2y = -3y - 2x$$

$$\Rightarrow 3x - 2x = 3y - 2y$$

$$\Rightarrow x = y$$

$\therefore f$ is one-one.

Let $y \in B = \mathbb{R} - \{1\}$, then $y \neq 1$

The function f is onto if there exists $x \in A$ such that $f(x) = y$.

Now,

$$f(x) = y$$

$$\Rightarrow \frac{x-2}{x-3} = y$$

$$\Rightarrow x-2 = xy-3y$$

$$\Rightarrow x(1-y) = -3y+2$$

$$\Rightarrow x = \frac{2-3y}{1-y} \in A \quad [y \neq 1]$$

Thus, for any $y \in B$, there exists $\frac{2-3y}{1-y} \in A$ such that

$$f\left(\frac{2-3y}{1-y}\right) = \frac{\left(\frac{2-3y}{1-y}\right)-2}{\left(\frac{2-3y}{1-y}\right)-3} = \frac{2-3y-2+2y}{2-3y-3+3y} = \frac{-y}{-1} = y$$

$\therefore f$ is onto.

Hence, the function is one-one and onto.

Question 11:

Let $f : R \rightarrow R$ defined as $f(x) = x^4$. Choose the correct answer.

- A. f is one-one onto
- B. f is many-one onto
- C. f is one-one but not onto
- D. f is neither one-one nor onto

Solution:

$f : R \rightarrow R$ defined as $f(x) = x^4$

$x, y \in R$ such that $f(x) = f(y)$

$$\Rightarrow x^4 = y^4$$

$$\Rightarrow x = \pm y$$

$\therefore f(x) = f(y)$ does not imply that $x = y$.

For example $f(1) = f(-1) = 1$

$\therefore f$ is not one-one.

Consider an element 2 in co domain R there does not exist any x in domain R such that $f(x) = 2$.

$\therefore f$ is not onto.

Function f is neither one-one nor onto.

The correct answer is D.

Question 12:

Let $f : R \rightarrow R$ defined as $f(x) = 3x$. Choose the correct answer.

- A. f is one-one onto
- B. f is many-one onto
- C. f is one-one but not onto
- D. f is neither one-one nor onto

Solution:

$f : R \rightarrow R$ defined as $f(x) = 3x$

$x, y \in R$ such that $f(x) = f(y)$

$$\Rightarrow 3x = 3y$$

$$\Rightarrow x = y$$

$\therefore f$ is one-one.

For any real number y in co domain $\mathbb{R}$, there exist $\frac{y}{3}$ in $\mathbb{R}$ such that $f\left(\frac{y}{3}\right) = 3\left(\frac{y}{3}\right) = y$

$\therefore f$ is onto.

Hence, function f is one-one and onto.

The correct answer is A.

EXERCISE 1.3

Question 1:

Let $f: \{1, 3, 4\} \rightarrow \{1, 2, 5\}$ and $g: \{1, 2, 5\} \rightarrow \{1, 3\}$ be given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down gof .

Solution:

The functions $f: \{1, 3, 4\} \rightarrow \{1, 2, 5\}$ and $g: \{1, 2, 5\} \rightarrow \{1, 3\}$ are $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$

$$gof(1) = g[f(1)] = g(2) = 3 \quad [as\ f(1) = 2\ and\ g(2) = 3]$$

$$gof(3) = g[f(3)] = g(5) = 1 \quad [as\ f(3) = 5\ and\ g(5) = 1]$$

$$gof(4) = g[f(4)] = g(1) = 3 \quad [as\ f(4) = 1\ and\ g(1) = 3]$$

$$\therefore gof = \{(1, 3), (3, 1), (4, 3)\}$$

Question 2:

Let f, g, h be functions from R to R . Show that

$$(f + g)oh = foh + goh$$

$$(f.g)oh = (foh).(goh)$$

Solution:

$$(f + g)oh = foh + goh$$

$$\begin{aligned} LHS &= [(f + g)oh](x) \\ &= (f + g)[h(x)] = f[h(x)] + g[h(x)] \\ &= (foh)(x) + goh(x) \\ &= \{(foh) + (goh)\}(x) = RHS \end{aligned}$$

$$\therefore \{(f + g)oh\}(x) = \{(foh) + (goh)\}(x) \text{ for all } x \in R$$

Hence, $(f + g)oh = foh + goh$

$$(f.g)oh = (foh).(goh)$$

$$\begin{aligned} LHS &= [(f.g)oh](x) \\ &= (f.g)[h(x)] = f[h(x)].g[h(x)] \\ &= (foh)(x).(goh)(x) \\ &= \{(foh).(goh)\}(x) = RHS \end{aligned}$$

$$\therefore [(f.g)oh](x) = \{(foh).(goh)\}(x) \text{ for all } x \in R$$

Hence, $(f.g)oh = (foh).(goh)$

Question 3:

Find gof and fog , if

i. $f(x) = |x|$ and $g(x) = |5x - 2|$

ii. $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

Solution:

i. $f(x) = |x|$ and $g(x) = |5x - 2|$

$$\therefore gof(x) = g(f(x)) = g(|x|) = |5|x| - 2|$$

$$fog(x) = f(g(x)) = f(|5x - 2|) = ||5x - 2|| = |5x - 2|$$

ii. $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

$$\therefore gof(x) = g(f(x)) = g(8x^3) = (8x^3)^{\frac{1}{3}} = 2x$$

$$fog(x) = f(g(x)) = f\left(x^{\frac{1}{3}}\right) = 8\left(x^{\frac{1}{3}}\right)^3 = 8x$$

Question 4:

If $f(x) = \frac{(4x+3)}{(6x-4)}, x \neq \frac{2}{3}$, show that $fof(x) = x$, for all $x \neq \frac{2}{3}$. What is the reverse of f ?

Solution:

$$(fof)(x) = f(f(x)) = f\left(\frac{4x+3}{6x-4}\right)$$

$$= \frac{4\left(\frac{4x+3}{6x-4}\right) + 3}{6\left(\frac{4x+3}{6x-4}\right) - 4} = \frac{16x + 12 + 18x - 12}{24x + 18 - 24x + 16} = \frac{34x}{34} = x$$

$$\therefore fof(x) = x \text{ for all } x \neq \frac{2}{3}$$

$$\Rightarrow fof = 1$$

Hence, the given function f is invertible and the inverse of f is f itself.

Question 5:

State with reason whether the following functions have inverse.

- i. $f : \{1, 2, 3, 4\} \rightarrow \{10\}$ with $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$
- ii. $g : \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$ with $g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$
- iii. $h : \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$ with $h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

Solution:

- i. $f : \{1, 2, 3, 4\} \rightarrow \{10\}$ with $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

f is a many one function as $f(1) = f(2) = f(3) = f(4) = 10$

$\therefore f$ is not one-one.

Function f does not have an inverse.

- ii. $g : \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$ with $g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

g is a many one function as $g(5) = g(7) = 4$

$\therefore g$ is not one-one.

Function g does not have an inverse.

- iii. $h : \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$ with $h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

All distinct elements of the set $\{2, 3, 4, 5\}$ have distinct images under h .

$\therefore h$ is one-one.

h is onto since for every element y of the set $\{7, 9, 11, 13\}$, there exists an element x in the set $\{2, 3, 4, 5\}$, such that $h(x) = y$.

h is a one-one and onto function.

Function h has an inverse.

Question 6:

Show that $f : [-1, 1] \rightarrow R$, given by $f(x) = \frac{x}{(x+2)}$ is one-one. Find the inverse of the function $f : [-1, 1] \rightarrow \text{Range } f$.

(Hint: For $y \in \text{Range } f$, $y = f(x) = \frac{x}{x+2}$, for some x in $[-1, 1]$, i.e., $x = \frac{2y}{(1-y)}$)

Solution:

$f : [-1, 1] \rightarrow R$, given by $f(x) = \frac{x}{(x+2)}$

For one-one

$$f(x) = f(y)$$

$$\Rightarrow \frac{x}{x+2} = \frac{y}{y+2}$$

$$\Rightarrow xy + 2x = xy + 2y$$

$$\Rightarrow 2x = 2y$$

$$\Rightarrow x = y$$

$\therefore f$ is a one-one function.

It is clear that $f : [-1, 1] \rightarrow R$ is onto.

$f : [-1, 1] \rightarrow R$ is one-one and onto and therefore, the inverse of the function $f : [-1, 1] \rightarrow R$ exists.

Let $g : \text{Range } f \rightarrow [-1, 1]$ be the inverse of f .

Let y be an arbitrary element of range f .

Since $f : [-1, 1] \rightarrow \text{Range } f$ is onto, we have:

$$y = f(x) \text{ for some } x \in [-1, 1]$$

$$\Rightarrow y = \frac{x}{x+2}$$

$$\Rightarrow xy + 2y = x$$

$$\Rightarrow x(1-y) = 2y$$

$$\Rightarrow x = \frac{2y}{1-y}, y \neq 1$$

Now, let us define $g : \text{Range } f \rightarrow [-1, 1]$ as

$$g(y) = \frac{2y}{1-y}, y \neq 1$$

Now,

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{x}{x+2}\right) = \frac{2\left(\frac{x}{x+2}\right)}{1 - \frac{x}{x+2}} = \frac{2x}{x+2-x} = \frac{2x}{2} = x$$

$$(f \circ g)(x) = f(g(y)) = f\left(\frac{2y}{1-y}\right) = \frac{\frac{2y}{1-y}}{\frac{2y}{1-y} + 2} = \frac{2y}{2y+2-2y} = \frac{2y}{2} = y$$

$$\therefore g \circ f = I_{[-1,1]} \quad \text{and} \quad f \circ g = I_{\text{Range } f}$$

$$\therefore f^{-1} = g$$

$$\Rightarrow f^{-1}(y) = \frac{2y}{1-y}, y \neq 1$$

Question 7:

Consider $f: R \rightarrow R$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f .

Solution:

$f: R \rightarrow R$ given by $f(x) = 4x + 3$

For one-one

$$f(x) = f(y)$$

$$\Rightarrow 4x + 3 = 4y + 3$$

$$\Rightarrow 4x = 4y$$

$$\Rightarrow x = y$$

$\therefore f$ is a one-one function.

For onto

$$y \in R, \text{ let } y = 4x + 3$$

$$\Rightarrow x = \frac{y-3}{4} \in R$$

Therefore, for any $y \in R$, there exists $x = \frac{y-3}{4} \in R$ such that

$$f(x) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y$$

$\therefore f$ is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define $g: R \rightarrow R$ by $g(x) = \frac{y-3}{4}$

Now,

$$(g \circ f)(x) = g(f(x)) = g(4x+3) = \frac{(4x+3)-3}{4} = x$$

$$(f \circ g)(y) = f(g(y)) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y - 3 + 3 = y$$

$$\therefore g \circ f = f \circ g = I_R$$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \frac{y-3}{4}.$$

Question 8:

Consider $f: R_+ \rightarrow [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with inverse f^{-1} of given f by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

Solution:

$$f: R_+ \rightarrow [4, \infty) \text{ given by } f(x) = x^2 + 4$$

For one-one:

$$\text{Let } f(x) = f(y)$$

$$\Rightarrow x^2 + 4 = y^2 + 4$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow x = y \quad [as \ x \in R]$$

$\therefore f$ is a one-one function.

For onto:

$$\text{For } y \in [4, \infty), \text{ let } y = x^2 + 4$$

$$\Rightarrow x^2 = y - 4 \geq 0 \quad [as \ y \geq 4]$$

$$\Rightarrow x = \sqrt{y-4} \geq 0$$

Therefore, for any $y \in R$, there exists $x = \sqrt{y-4} \in R$ such that

$$f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y - 4 + 4 = y$$

$\therefore f$ is an onto function.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define $g: [4, \infty) \rightarrow R_+$ by

$$g(y) = \sqrt{y-4}$$

$$\text{Now, } g \circ f(x) = g(f(x)) = g(x^2 + 4) = \sqrt{(x^2 + 4) - 4} = \sqrt{x^2} = x$$

$$\text{And } f \circ g(y) = f(g(y)) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = (y-4) + 4 = y$$

$$\therefore g \circ f = f \circ g = I_R$$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \sqrt{y-4}.$$

Question 9:

Consider $f: R_+ \rightarrow [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with

$$f^{-1}(y) = \left(\frac{(\sqrt{y+6}) - 1}{3} \right).$$

Solution:

$$f: R_+ \rightarrow [-5, \infty) \text{ given by } f(x) = 9x^2 + 6x - 5$$

Let y be an arbitrary element of $[-5, \infty)$.

$$\text{Let } y = 9x^2 + 6x - 5$$

$$\Rightarrow y = (3x+1)^2 - 1 - 5$$

$$\Rightarrow y = (3x+1)^2 - 6$$

$$\Rightarrow (3x+1)^2 = y+6$$

$$\Rightarrow 3x+1 = \sqrt{y+6} \quad [as \ y \geq -5 \Rightarrow y+6 > 0]$$

$$\Rightarrow x = \frac{\sqrt{y+6} - 1}{3}$$

$\therefore f$ is onto, thereby range $f = [-5, \infty)$.

$$\text{Let us define } g: [-5, \infty) \rightarrow R_+ \text{ as } g(y) = \frac{\sqrt{y+6} - 1}{3}$$

We have,

$$\begin{aligned}
 (gof)(x) &= g(f(x)) = g(9x^2 + 6x - 5) \\
 &= g((3x+1)^2 - 6) \\
 &= \frac{\sqrt{(3x+1)^2 - 6} + 6 - 1}{3} \\
 &= \frac{3x+1-1}{3} = x
 \end{aligned}$$

And,

$$\begin{aligned}
 (fog)(y) &= f(g(y)) = f\left(\frac{\sqrt{y+6}-1}{3}\right) \\
 &= \left[3\left(\frac{\sqrt{y+6}-1}{3}\right) + 1\right]^2 - 6 \\
 &= (\sqrt{y+6})^2 - 6 = y + 6 - 6 = y
 \end{aligned}$$

$$\therefore gof = I_R \text{ and } fog = I_{[-5, \infty)}$$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \frac{\sqrt{y+6}-1}{3}.$$

Question 10:

Let $f : X \rightarrow Y$ be an invertible function. Show that f has unique inverse.

(Hint: suppose g_1 and g_2 are two inverses of f . Then for all $y \in Y$, $fog_1(y) = I_Y(y) = fog_2(y)$. Use one-one ness of f .)

Solution:

Let $f : X \rightarrow Y$ be an invertible function.

Also suppose f has two inverses (g_1 and g_2)

Then, for all $y \in Y$,

$$fog_1(y) = I_Y(y) = fog_2(y)$$

$$\Rightarrow f(g_1(y)) = f(g_2(y))$$

$$\Rightarrow g_1(y) = g_2(y)$$

$$[f \text{ is invertible} \Rightarrow f \text{ is one-one}]$$

$$\Rightarrow g_1 = g_2$$

$$[g \text{ is one-one}]$$

Hence, f has unique inverse.

Question 11:

Consider $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$ given by $f(1) = a, f(2) = b, f(3) = c$. Find $(f^{-1})^{-1} = f$.

Solution:

Function $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$ given by $f(1) = a, f(2) = b, f(3) = c$

If we define $g : \{a, b, c\} \rightarrow \{1, 2, 3\}$ as $g(a) = 1, g(b) = 2, g(c) = 3$

$$(fog)(a) = f(g(a)) = f(1) = a$$

$$(fog)(b) = f(g(b)) = f(2) = b$$

$$(fog)(c) = f(g(c)) = f(3) = c$$

And,

$$(gof)(1) = g(f(1)) = g(a) = 1$$

$$(gof)(2) = g(f(2)) = g(b) = 2$$

$$(gof)(3) = g(f(3)) = g(c) = 3$$

$$\therefore fog = I_X \quad \text{and} \quad gof = I_Y \quad \text{where } X = \{1, 2, 3\} \text{ and } Y = \{a, b, c\}$$

Thus, the inverse of f exists and $f^{-1} = g$.

$$\therefore f^{-1} : \{a, b, c\} \rightarrow \{1, 2, 3\} \text{ is given by, } f^{-1}(a) = 1, f^{-1}(b) = 2, f^{-1}(c) = 3$$

We need to find the inverse of f^{-1} i.e., inverse of g .

If we define $h : \{1, 2, 3\} \rightarrow \{a, b, c\}$ as $h(1) = a, h(2) = b, h(3) = c$

$$(goh)(1) = g(h(1)) = g(a) = 1$$

$$(goh)(2) = g(h(2)) = g(b) = 2$$

$$(goh)(3) = g(h(3)) = g(c) = 3$$

And,

$$(hog)(a) = h(g(a)) = h(1) = a$$

$$(hog)(b) = h(g(b)) = h(2) = b$$

$$(hog)(c) = h(g(c)) = h(3) = c$$

$$\therefore goh = I_X \quad \text{and} \quad hog = I_Y \quad \text{where } X = \{1, 2, 3\} \text{ and } Y = \{a, b, c\}$$

Thus, the inverse of g exists and $g^{-1} = h \Rightarrow (f^{-1})^{-1} = h$.

It can be noted that $h = f$.

Hence, $(f^{-1})^{-1} = f$

Question 12:

Let $f : X \rightarrow Y$ be an invertible function. Show that the inverse of f^{-1} is f i.e., $(f^{-1})^{-1} = f$.

Solution:

Let $f : X \rightarrow Y$ be an invertible function.

Then there exists a function $g : Y \rightarrow X$ such that $gof = I_X$ and $fog = I_Y$

Here, $f^{-1} = g$

Now, $gof = I_X$ and $fog = I_Y$

$\Rightarrow f^{-1}of = I_X$ and $fof^{-1} = I_Y$

Hence, $f^{-1} : Y \rightarrow X$ is invertible and f^{-1} is f i.e., $(f^{-1})^{-1} = f$.

Question 13:

If $f : R \rightarrow R$ is given by $f(x) = (3 - x^3)^{\frac{1}{3}}$, then $fof(x)$ is:

- A. $\frac{1}{x^3}$
- B. x^3
- C. x
- D. $(3 - x^3)$

Solution:

$f : R \rightarrow R$ is given by $f(x) = (3 - x^3)^{\frac{1}{3}}$

$f(x) = (3 - x^3)^{\frac{1}{3}}$

$$\therefore fof(x) = f(f(x)) = f\left((3 - x^3)^{\frac{1}{3}}\right) = \left[3 - \left((3 - x^3)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}$$

$$= \left[3 - (3 - x^3)\right]^{\frac{1}{3}} = (x^3)^{\frac{1}{3}} = x$$

$$\therefore fof(x) = x$$

The correct answer is C.

Question 14:

If $f : R - \left\{ -\frac{4}{3} \right\} \rightarrow R$ be a function defined as $f(x) = \frac{4x}{3x+4}$. The inverse of f is the map $g : \text{Range } f \rightarrow R - \left\{ -\frac{4}{3} \right\}$ given by :

A. $g(y) = \frac{3y}{3-4y}$

B. $g(y) = \frac{4y}{4-3y}$

C. $g(y) = \frac{4y}{3-4y}$

D. $g(y) = \frac{3y}{4-3y}$

Solution:

It is given that $f : R - \left\{ -\frac{4}{3} \right\} \rightarrow R$ is defined as $f(x) = \frac{4x}{3x+4}$

Let y be an arbitrary element of $\text{Range } f$.

Then, there exists $x \in R - \left\{ -\frac{4}{3} \right\}$ such that $y = f(x)$.

$$\Rightarrow y = \frac{4x}{3x+4}$$

$$\Rightarrow 3xy + 4y = 4x$$

$$\Rightarrow x(4-3y) = 4y$$

$$\Rightarrow x = \frac{4y}{4-3y}$$

Define $f : R - \left\{ -\frac{4}{3} \right\} \rightarrow R$ as $g(y) = \frac{4y}{4-3y}$

Now,

$$\begin{aligned}
 (g \circ f)(x) &= g(f(x)) = g\left(\frac{4x}{3x+4}\right) \\
 &= \frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)} = \frac{16x}{12x+16-12x} \\
 &= \frac{16x}{16} = x
 \end{aligned}$$

And

$$\begin{aligned}
 (f \circ g)(x) &= f(g(x)) = f\left(\frac{4y}{4-3y}\right) \\
 &= \frac{4\left(\frac{4y}{4-3y}\right)}{3\left(\frac{4y}{4-3y}\right)+4} = \frac{16y}{12y+16-12y} \\
 &= \frac{16y}{16} = y
 \end{aligned}$$

$$\therefore g \circ f = I_{R - \left\{-\frac{4}{3}\right\}} \text{ and } f \circ g = I_{\text{Range } f}$$

Thus, g is the inverse of f i.e., $f^{-1} = g$

Hence, the inverse of f is the map $g : \text{Range } f \rightarrow R - \left\{-\frac{4}{3}\right\}$, which is given by $g(y) = \frac{4y}{4-3y}$.

The correct answer is B.

EXERCISE 1.4

Question 1:

Determine whether or not each of the definition of $*$ given below gives a binary operation. In the event that $*$ is not a binary operation, give justification for this.

- i. On $\mathbf{Z}^+$, define $*$ by $a * b = a - b$
- ii. On $\mathbf{Z}^+$, define $*$ by $a * b = ab$
- iii. On $\mathbf{R}$, define $*$ by $a * b = ab^2$
- iv. On $\mathbf{Z}^+$, define $*$ by $a * b = |a - b|$
- v. On $\mathbf{Z}^+$, define $*$ by $a * b = a$

Solution:

- i. On $\mathbf{Z}^+$, define $*$ by $a * b = a - b$

It is not a binary operation as the image of $(1, 2)$ under $*$ is
 $1 * 2 = 1 - 2$

$$\Rightarrow -1 \notin \mathbf{Z}^+.$$

Therefore, $*$ is not a binary operation.

- ii. On $\mathbf{Z}^+$, define $*$ by $a * b = ab$

It is seen that for each $a, b \in \mathbf{Z}^+$, there is a unique element ab in $\mathbf{Z}^+$.

This means that $*$ carries each pair (a, b) to a unique element $a * b = ab$ in $\mathbf{Z}^+$.

Therefore, $*$ is a binary operation.

- iii. On $\mathbf{R}$, define $*$ by $a * b = ab^2$

It is seen that for each $a, b \in \mathbf{R}$, there is a unique element ab^2 in $\mathbf{R}$. This means that $*$ carries each pair (a, b) to a unique element $a * b = ab^2$ in $\mathbf{R}$.

Therefore, $*$ is a binary operation.

- iv. On $\mathbf{Z}^+$, define $*$ by $a * b = |a - b|$

It is seen that for each $a, b \in \mathbf{Z}^+$, there is a unique element $|a - b|$ in $\mathbf{Z}^+$. This means that $*$ carries each pair (a, b) to a unique element $a * b = |a - b|$ in $\mathbf{Z}^+$. Therefore, $*$ is a binary operation.

- v. On $\mathbf{Z}^+$, define $*$ by $a * b = a$

$*$ carries each pair (a, b) to a unique element in $a * b = a$ in $\mathbf{Z}^+$.

Therefore, $*$ is a binary operation.

Question 2:

For each binary operation $*$ defined below, determine whether $*$ is commutative or associative.

- i. On $\mathbf{Z}^+$, define $a * b = a - b$

- ii. On $\mathbf{Q}$, define $a * b = ab + 1$
- iii. On $\mathbf{Q}$, define $a * b = \frac{ab}{2}$
- iv. On $\mathbf{Z}^+$, define $a * b = 2^{ab}$
- v. On $\mathbf{Z}^+$, define $a * b = a^b$
- vi. On $\mathbf{R} - \{-1\}$, define $a * b = \frac{a}{b+1}$

Solution:

- i. On $\mathbf{Z}^+$, define $a * b = a - b$

It can be observed that $1 * 2 = 1 - 2 = -1$ and $2 * 1 = 2 - 1 = 1$.

$\therefore 1 * 2 \neq 2 * 1$; where $1, 2 \in \mathbf{Z}$

Hence, the operation $*$ is not commutative.

Also,

$$(1 * 2) * 3 = (1 - 2) * 3 = -1 * 3 = -1 - 3 = -4$$

$$1 * (2 * 3) = 1 * (2 - 3) = 1 * -1 = 1 - (-1) = 2$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$$

where $1, 2, 3 \in \mathbf{Z}$

Hence, the operation $*$ is not associative.

- ii. On $\mathbf{Q}$, define $a * b = ab + 1$

$$ab = ba \quad \text{for all } a, b \in \mathbf{Q}$$

$$\Rightarrow ab + 1 = ba + 1 \quad \text{for all } a, b \in \mathbf{Q}$$

$$\Rightarrow a * b = b * a \quad \text{for all } a, b \in \mathbf{Q}$$

Hence, the operation $*$ is commutative.

$$(1 * 2) * 3 = (1 \times 2 + 1) * 3 = 3 * 3 = 3 \times 3 + 1 = 10$$

$$1 * (2 * 3) = 1 * (2 \times 3 + 1) = 1 * 7 = 1 \times 7 + 1 = 8$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$$

where $1, 2, 3 \in \mathbf{Q}$

Hence, the operation $*$ is not associative.

- iii. On $\mathbf{Q}$, define $a * b = \frac{ab}{2}$

$$ab = ba \quad \text{for all } a, b \in \mathbf{Q}$$

$$\Rightarrow \frac{ab}{2} = \frac{ba}{2} \quad \text{for all } a, b \in \mathbf{Q}$$

$$\Rightarrow a * b = b * a \quad \text{for all } a, b \in \mathbf{Q}$$

Hence, the operation $*$ is commutative.

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{\left(\frac{ab}{2}\right)^c}{2} = \frac{abc}{4}$$

And

$$a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4}$$

$$\therefore (a * b) * c = a * (b * c)$$

where $a, b, c \in \mathbf{Q}$

Hence, the operation $*$ is associative.

iv. On $\mathbf{Z}^+$, define $a * b = 2^{ab}$

$$ab = ba \quad \text{for all } a, b \in \mathbf{Z}$$

$$\Rightarrow 2^{ab} = 2^{ba} \quad \text{for all } a, b \in \mathbf{Z}$$

$$\Rightarrow a * b = b * a \quad \text{for all } a, b \in \mathbf{Z}$$

Hence, the operation $*$ is commutative.

$$(1 * 2) * 3 = 2^{1 \times 2} * 3 = 4 * 3 = 2^{4 \times 3} = 2^{12}$$

$$1 * (2 * 3) = 1 * 2^{2 \times 3} = 1 * 2^6 = 1 * 64 = 2^{64}$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$$

where $1, 2, 3 \in \mathbf{Z}^+$

Hence, the operation $*$ is not associative.

v. On $\mathbf{Z}^+$, define $a * b = a^b$

$$1 * 2 = 1^2 = 1$$

$$2 * 1 = 2^1 = 2$$

$$\therefore 1 * 2 \neq 2 * 1$$

where $1, 2, \in \mathbf{Z}^+$

Hence, the operation $*$ is not commutative.

$$(2 * 3) * 4 = 2^3 * 4 = 8 * 4 = 8^4 = 2^{12}$$

$$2 * (3 * 4) = 2 * 3^4 = 2 * 81 = 2^{81}$$

$$\therefore (2 * 3) * 4 \neq 2 * (3 * 4)$$

where $2, 3, 4 \in \mathbf{Z}^+$

Hence, the operation $*$ is not associative.

vi. On $\mathbf{R} - \{-1\}$, define $a * b = \frac{a}{b+1}$

$$1 * 2 = \frac{1}{2+1} = \frac{1}{3}$$

$$2 * 1 = \frac{2}{1+1} = \frac{2}{2} = 1$$

$$\therefore 1 * 2 \neq 2 * 1$$

where $1, 2 \in \mathbf{R} - \{-1\}$

Hence, the operation $*$ is not commutative.

$$(1 * 2) * 3 = \frac{1}{3} * 3 = \frac{\frac{1}{3}}{3+1} = \frac{1}{12}$$

$$1 * (2 * 3) = 1 * \frac{2}{3+1} = 1 * \frac{2}{4} = 1 * \frac{1}{2} = \frac{1}{\frac{1}{2}+1} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$$

where $1, 2, 3 \in \mathbf{R} - \{-1\}$

Hence, the operation $*$ is not associative.

Question 3:

Consider the binary operation $\wedge$ on the set $\{1, 2, 3, 4, 5\}$ defined by $a \wedge b = \min\{a, b\}$. Write the operation table of the operation $\wedge$.

Solution:

The binary operation $\wedge$ on the set $\{1, 2, 3, 4, 5\}$ is defined by $a \wedge b = \min\{a, b\}$ for all $a, b \in \{1, 2, 3, 4, 5\}$.

The operation table for the given operation $\wedge$ can be given as:

$\wedge$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Question 4:

Consider a binary operation $*$ on the set $\{1, 2, 3, 4, 5\}$ given by the following multiplication table.

- Compute $(2 * 3) * 4$ and $2 * (3 * 4)$
- Is $*$ commutative?
- Compute $(2 * 3) * (4 * 5)$.
(Hint: Use the following table)

$*$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1

3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Solution:

$$(2 * 3) * 4 = 1 * 4 = 1$$

- i. $2 * (3 * 4) = 2 * 1 = 1$
- ii. For every $a, b \in \{1, 2, 3, 4, 5\}$, we have $a * b = b * a$. Therefore, $*$ is commutative.
- iii. $(2 * 3) * (4 * 5)$
 $(2 * 3) = 1$ and $(4 * 5) = 1$
 $\therefore (2 * 3) * (4 * 5) = 1 * 1 = 1$

Question 5:

Let $*$ ' be the binary operation on the set $\{1, 2, 3, 4, 5\}$ defined by $a *' b = \text{H.C.F. of } a \text{ and } b$. Is the operation $*$ ' same as the operation $*$ defined in Exercise 4 above? Justify your answer.

Solution:

The binary operation on the set $\{1, 2, 3, 4, 5\}$ is defined by $a *' b = \text{H.C.F. of } a \text{ and } b$.

The operation table for the operation $*$ ' can be given as:

$*$ '	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

The operation table for the operations $*$ ' and $*$ are same.
operation $*$ ' is same as operation $*$.

Question 6:

Let $*$ be the binary operation on $\mathbb{N}$ defined by $a * b = \text{L.C.M. of } a \text{ and } b$. Find

- i. $5 * 7, 20 * 16$
- ii. Is $*$ commutative?
- iii. Is $*$ associative?
- iv. Find the identity of $*$ in $\mathbb{N}$
- v. Which elements of $\mathbb{N}$ are invertible for the operation $*$?

Solution:

The binary operation on N is defined by $a * b = \text{L.C.M. of } a \text{ and } b$.

- i. $5 * 7 = \text{L.C.M of } 5 \text{ and } 7 = 35$
 $20 * 16 = \text{LCM of } 20 \text{ and } 16 = 80$
- ii. $\text{L.C.M. of } a \text{ and } b = \text{LCM of } b \text{ and } a \text{ for all } a, b \in N$
 $\therefore a * b = b * a$
Operation $*$ is commutative.
- iii. For $a, b, c \in N$
 $(a * b) * c = (\text{L.C.M. of } a \text{ and } b) * c = \text{L.C.M. of } a, b, c$
 $a * (b * c) = a * (\text{L.C.M. of } b \text{ and } c) = \text{L.C.M. of } a, b, c$
 $\therefore (a * b) * c = a * (b * c)$
Operation $*$ is associative.
- iv. $\text{L.C.M. of } a \text{ and } 1 = a = \text{L.C.M. of } 1 \text{ and } a \text{ for all } a \in N$
 $a * 1 = a = 1 * a \text{ for all } a \in N$
Therefore, 1 is the identity of $*$ in N .
- v. An element a in N is invertible with respect to the operation $*$ if there exists an element b in N , such that $a * b = e = b * a$
 $e = 1$
 $\text{L.C.M. of } a \text{ and } b = 1 = \text{LCM of } b \text{ and } a \text{ possible only when } a \text{ and } b \text{ are equal to } 1.$
 1 is the only invertible element of N with respect to the operation $*$.

Question 7:

Is $*$ defined on the set $\{1, 2, 3, 4, 5\}$ by $a * b = \text{LCM of } a \text{ and } b$ a binary operation? Justify your answer.

Solution:

The operation $*$ on the set $\{1, 2, 3, 4, 5\}$ is defined by $a * b = \text{LCM of } a \text{ and } b$.

The operation table for the operation $*$ can be given as:

*	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	6	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

$$3 * 2 = 2 * 3 = 6 \notin A,$$

$$5 * 2 = 2 * 5 = 10 \notin A,$$

$$3 * 4 = 4 * 3 = 12 \notin A,$$

$$3 * 5 = 5 * 3 = 15 \notin A,$$

$$4 * 5 = 5 * 4 = 20 \notin A$$

The given operation $*$ is not a binary operation.

Question 8:

Let $*$ be the binary operation on N defined by $a * b = \text{H.C.F. of } a \text{ and } b$. Is $*$ commutative? Is $*$ associative? Does there exist identity for this binary operation on N ?

Solution:

The binary operation $*$ on N defined by $a * b = \text{H.C.F. of } a \text{ and } b$.

$$\therefore a * b = b * a$$

Operation $*$ is commutative.

For all $a, b, c \in N$,

$$(a * b) * c = (\text{HCF of } a \text{ and } b) * c = \text{HCF of } a, b, c$$

$$a * (b * c) = a * (\text{HCF of } b \text{ and } c) = \text{HCF of } a, b, c$$

$$\therefore (a * b) * c = a * (b * c)$$

Operation $*$ is associative.

$e \in N$ will be the identity for the operation $*$ if $a * e = a = e * a$ for all $a \in N$. But this relation is not true for any $a \in N$.

Operation $*$ does not have any identity in N .

Question 9:

Let $*$ be the binary operation on Q of rational numbers as follows:

i. $a * b = a - b$

ii. $a * b = a^2 + b^2$

iii. $a * b = a + ab$

iv. $a * b = (a - b)^2$

v. $a + b = \frac{ab}{4}$

vi. $a * b = ab^2$

Find which of the binary operations are commutative and which are associative.

Solution:

- i. On Q , the operation $*$ is defined as $a * b = a - b$

$$\frac{1}{2} * \frac{1}{3} = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{3} = \frac{1}{3}$$

And

$$\frac{1}{3} * \frac{1}{2} = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = \frac{-1}{6}$$

$$\therefore \left(\frac{1}{2} * \frac{1}{3} \right) \neq \left(\frac{1}{3} * \frac{1}{2} \right)$$

where $\frac{1}{2}, \frac{1}{3} \in Q$

Operation $*$ is not commutative.

$$\left(\frac{1}{2} * \frac{1}{3} \right) * \frac{1}{4} = \left(\frac{1}{2} - \frac{1}{3} \right) * \frac{1}{4} = \frac{1}{6} * \frac{1}{4} = \frac{1}{6} - \frac{1}{4} = \frac{2-3}{12} = \frac{-1}{12}$$

$$\frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4} \right) = \frac{1}{2} * \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{1}{2} * \frac{1}{12} = \frac{1}{2} - \frac{1}{12} = \frac{6-1}{12} = \frac{5}{12}$$

$$\therefore \left(\frac{1}{2} * \frac{1}{3} \right) * \frac{1}{4} \neq \frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4} \right)$$

where $\frac{1}{2}, \frac{1}{3}, \frac{1}{4} \in Q$

Operation $*$ is not associative.

- ii. On Q , the operation $*$ is defined as $a * b = a^2 + b^2$

For $a, b \in Q$

$$a * b = a^2 + b^2 = b^2 + a^2 = b * a$$

$$\therefore a * b = b * a$$

Operation $*$ is commutative.

$$(1 * 2) * 3 = (1^2 + 2^2) * 3 = (1 + 4) * 3 = 5 * 3 = 5^2 + 3^2 = 25 + 9 = 34$$

$$1 * (2 * 3) = 1 * (2^2 + 3^2) = 1 * (4 + 9) = 1 * 13 = 1^2 + 13^2 = 1 + 169 = 170$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$$

where $1, 2, 3 \in Q$

Operation $*$ is not associative.

- iii. On Q , the operation $*$ is defined as $a * b = a + ab$

$$1 * 2 = 1 + 1 \times 2 = 1 + 2 = 3$$

$$2 * 1 = 2 + 2 \times 1 = 2 + 2 = 4$$

$$\therefore 1 * 2 \neq 2 * 1$$

where $1, 2 \in Q$

Operation $*$ is not commutative.

$$(1 * 2) * 3 = (1 + 1 \times 2) * 3 = 3 * 3 = 3 + 3 \times 3 = 3 + 9 = 12$$

$$1 * (2 * 3) = 1 * (2 + 2 \times 3) = 1 * 8 = 1 + 1 \times 8 = 1 + 8 = 9$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$$

where $1, 2, 3 \in Q$

Operation $*$ is not associative.

- iv. On Q , the operation $*$ is defined as $a * b = (a - b)^2$

For $a, b \in Q$

$$a * b = (a - b)^2$$

$$b * a = (b - a)^2 = [-(a - b)]^2 = (a - b)^2$$

$$\therefore a * b = b * a$$

Operation $*$ is commutative.

$$(1 * 2) * 3 = (1 - 2)^2 * 3 = (-1)^2 * 3 = 1 * 3 = (1 - 3)^2 = (-2)^2 = 4$$

$$1 * (2 * 3) = 1 * (2 - 3)^2 = 1 * (-1)^2 = 1 * 1 = (1 - 1)^2 = 0$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$$

where $1, 2, 3 \in Q$

Operation $*$ is not associative.

- v. On Q , the operation $*$ is defined as $a * b = \frac{ab}{4}$

For $a, b \in Q$

$$a * b = \frac{ab}{4} = \frac{ba}{4} = b * a$$

$$\therefore a * b = b * a$$

Operation $*$ is commutative.

For $a, b, c \in Q$

$$(a * b) * c = \frac{ab}{4} * c = \frac{\frac{ab}{4} \cdot c}{4} = \frac{abc}{16}$$

$$a * (b * c) = a * \frac{bc}{4} = \frac{a \cdot \frac{bc}{4}}{4} = \frac{abc}{16}$$

$$\therefore (a * b) * c = a * (b * c)$$

where $a, b, c \in Q$

Operation $*$ is associative.

- vi. On Q , the operation $*$ is defined as $a * b = ab^2$

$$\frac{1}{2} * \frac{1}{3} = \frac{1}{2} \cdot \left(\frac{1}{3}\right)^2 = \frac{1}{2} \cdot \frac{1}{9} = \frac{1}{18}$$

$$\frac{1}{3} * \frac{1}{2} = \frac{1}{3} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

$$\therefore \left(\frac{1}{2} * \frac{1}{3}\right) \neq \left(\frac{1}{3} * \frac{1}{2}\right)$$

where $\frac{1}{2}, \frac{1}{3} \in Q$

Operation $*$ is not commutative.

$$\begin{aligned} \left(\frac{1}{2} * \frac{1}{3}\right) * \frac{1}{4} &= \left(\frac{1}{2} \cdot \left(\frac{1}{3}\right)^2\right) * \frac{1}{4} = \frac{1}{18} * \frac{1}{4} = \frac{1}{18} \cdot \left(\frac{1}{4}\right)^2 = \frac{1}{18 \times 16} \\ \frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4}\right) &= \frac{1}{2} * \left(\frac{1}{3} \cdot \left(\frac{1}{4}\right)^2\right) = \frac{1}{2} * \frac{1}{48} = \frac{1}{2} \cdot \left(\frac{1}{48}\right)^2 = \frac{1}{2 \times (48)^2} \\ \therefore \left(\frac{1}{2} * \frac{1}{3}\right) * \frac{1}{4} &\neq \frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4}\right) \quad \text{where } \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \in Q \\ \text{Operation } * &\text{ is not associative.} \end{aligned}$$

Operations defined in (ii), (iv), (v) are commutative and the operation defined in (v) is associative.

Question 10:

Find which of the operations given above has identity.

Solution:

An element $e \in Q$ will be the identity element for the operation $*$ if

$$a * e = a = e * a, \text{ for all } a \in Q$$

$$a * b = \frac{ab}{4}$$

$$\Rightarrow a * e = a$$

$$\Rightarrow \frac{ae}{4} = a$$

$$\Rightarrow e = 4$$

Similarly, it can be checked for $e * a = a$, we get $e = 4$ is the identity.

Question 11:

$A = N \times N$ and $*$ be the binary operation on A defined by $(a, b) * (c, d) = (a + c, b + d)$. Show that $*$ is commutative and associative. Find the identity element for $*$ on A , if any.

Solution:

$A = N \times N$ and $*$ be the binary operation on A defined by

$$(a,b)*(c,d)=(a+c,b+d)$$

$$(a,b)*(c,d) \in A$$

$$a,b,c,d \in N$$

$$(a,b)*(c,d)=(a+c,b+d)$$

$$(c,d)*(a,b)=(c+a,d+b)=(a+c,b+d)$$

$$\therefore (a,b)*(c,d)=(c,d)*(a,b)$$

Operation $*$ is commutative.

Now, let $(a, b), (c, d), (e, f) \in A$

$a, b, c, d, e, f \in N$

$$[(a, b) * (c, d)] * (e, f) = (a + c, b + d) * (e, f) = (a + c + e, b + d + f)$$

$$(a, b) * [(c, d) * (e, f)] = (a, b) * (c + e, d + f) = (a + c + e, b + d + f)$$

$$\therefore [(a, b) * (c, d)] * (e, f) = (a, b) * [(c, d) * (e, f)]$$

Operation $*$ is associative.

An element $e = (e_1, e_2) \in A$ will be an identity element for the operation $*$ if $a + e = a = e * a$ for all $a = (a_1, a_2) \in A$ i.e., $(a_1 + e_1, a_2 + e_2) = (a_1, a_2) = (e_1 + a_1, e_2 + a_2)$, which is not true for any element in A .

Therefore, the operation $*$ does not have any identity element.

Question 12:

State whether the following statements are true or false. Justify.

- For an arbitrary binary operation $*$ on a set N , $a * a = a$ for all $a \in N$.
- If $*$ is a commutative binary operation on N , then $a * (b * c) = (c * b) * a$

Solution:

- Define operation $*$ on a set N as $a * a = a$ for all $a \in N$.

In particular, for $a = 3$,

$$3 * 3 = 9 \neq 3$$

Therefore, statement (i) is false.

- $$\begin{aligned} \text{R.H.S.} &= (c * b) * a \\ &= (b * c) * a \quad [* \text{ is commutative}] \\ &= a * (b * c) \quad [\text{Again, as } * \text{ is commutative}] \\ &= \text{L.H.S.} \\ \therefore a * (b * c) &= (c * b) * a \end{aligned}$$

Therefore, statement (ii) is true.

Question 13:

Consider a binary operation $*$ on N defined as $a * b = a^3 + b^3$. Choose the correct answer.

- Is $*$ both associative and commutative?
- Is $*$ commutative but not associative?
- Is $*$ associative but not commutative?
- Is $*$ neither commutative nor associative?

Solution:

On $\mathbb{N}$, operation $*$ is defined as $a * b = a^3 + b^3$.

For all $a, b \in \mathbb{N}$

$$a * b = a^3 + b^3 = b^3 + a^3 = b * a$$

Operation $*$ is commutative.

$$(1 * 2) * 3 = (1^3 + 2^3) * 3 = (1 + 8) * 3 = 9 * 3 = 9^3 + 3^3 = 729 + 27 = 756$$

$$1 * (2 * 3) = 1 * (2^3 + 3^3) = 1 * (8 + 27) = 1 * 35 = 1^3 + 35^3 = 1 + 42875 = 42876$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$$

associative.

Operation $*$ is not

Therefore, Operation $*$ is commutative, but not associative.

The correct answer is B.

MISCELLANEOUS EXERCISE

Question 1:

Let $f : R \rightarrow R$ be defined as $f(x) = 10x + 7$. Find the function $g : R \rightarrow R$ such that $gof = f \circ g = I_R$.

Solution:

$f : R \rightarrow R$ is defined as $f(x) = 10x + 7$

For one-one:

$$f(x) = f(y) \text{ where } x, y \in R$$

$$\Rightarrow 10x + 7 = 10y + 7$$

$$\Rightarrow x = y$$

$\therefore f$ is one-one.

For onto:

$$y \in R, \text{ Let } y = 10x + 7$$

$$\Rightarrow x = \frac{y-7}{10} \in R$$

For any $y \in R$, there exists $x = \frac{y-7}{10} \in R$ such that

$$f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

$\therefore f$ is onto.

Thus, f is an invertible function.

Let us define $g : R \rightarrow R$ as $g(y) = \frac{y-7}{10}$.

Now,

$$gof(x) = g(f(x)) = g(10x + 7) = \frac{(10x + 7) - 7}{10} = \frac{10x}{10} = x$$

And,

$$f \circ g(y) = f(g(y)) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

$$\therefore gof = I_R \text{ and } f \circ g = I_R$$

Hence, the required function $g : R \rightarrow R$ as $g(y) = \frac{y-7}{10}$.

Question 2:

Let $f : W \rightarrow W$ be defined as $f(n) = n - 1$, if n is odd and $f(n) = n + 1$, if n is even. Show that f is invertible. Find the inverse of f . Here, W is the set of all whole numbers.

Solution:

$f : W \rightarrow W$ is defined as $f(n) = \begin{cases} n - 1, & \text{If } n \text{ is odd} \\ n + 1, & \text{If } n \text{ is even} \end{cases}$

For one-one:

$$f(n) = f(m)$$

If n is odd and m is even, then we will have $n - 1 = m + 1$.

$$\Rightarrow n - m = 2$$

Similarly, the possibility of n being even and m being odd can also be ignored under a similar argument.

$\therefore$ Both n and m must be either odd or even.

Now, if both n and m are odd, then we have:

$$f(n) = f(m)$$

$$\Rightarrow n - 1 = m - 1$$

$$\Rightarrow n = m$$

Again, if both n and m are even, then we have:

$$f(n) = f(m)$$

$$\Rightarrow n + 1 = m + 1$$

$$\Rightarrow n = m$$

$\therefore f$ is one-one.

For onto:

Any odd number $2r + 1$ in co-domain N is the image of $2r$ in domain N and any even number $2r$ in co-domain N is the image of $2r + 1$ in domain N .

$\therefore f$ is onto.

f is an invertible function.

Let us define $g : W \rightarrow W$ as $f(m) = \begin{cases} m - 1, & \text{If } m \text{ is odd} \\ m + 1, & \text{If } m \text{ is even} \end{cases}$

When r is odd

$$g \circ f(n) = g(f(n)) = g(n - 1) = n - 1 + 1 = n$$

When r is even

$$gof(n) = g(f(n)) = g(n+1) = n+1-1 = n$$

When m is odd

$$fog(n) = f(g(m)) = f(m-1) = m-1+1 = m$$

When m is even

$$fog(m) = f(g(m)) = f(m+1) = m+1-1 = m$$

$$\therefore gof = I_W \text{ and } fog = I_W$$

f is invertible and the inverse of f is given by $f^{-1} = g$, which is the same as f .
inverse of f is f itself.

Question 3:

If $f: R \rightarrow R$ be defined as $f(x) = x^2 - 3x + 2$, find $f(f(x))$.

Solution:

$f: R \rightarrow R$ is defined as $f(x) = x^2 - 3x + 2$.

$$\begin{aligned} f(f(x)) &= f(x^2 - 3x + 2) \\ &= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2 \\ &= (x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2) + (-3x^2 + 9x - 6) + 2 \\ &= x^4 - 6x^3 + 10x^2 - 3x \end{aligned}$$

Question 4:

Show that function $f: R \rightarrow \{x \in R: -1 < x < 1\}$ be defined by $f(x) = \frac{x}{1+|x|}$, $x \in R$ is one-one and onto function.

Solution:

$f: R \rightarrow \{x \in R: -1 < x < 1\}$ is defined by $f(x) = \frac{x}{1+|x|}$, $x \in R$.

For one-one:

$$f(x) = f(y) \quad \text{where } x, y \in R$$

$$\Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$$

If x is positive and y is negative,

$$\frac{x}{1+|x|} = \frac{y}{1+|y|}$$

$$\Rightarrow 2xy = x - y$$

Since, x is positive and y is negative,

$$x > y \Rightarrow x - y > 0$$

$2xy$ is negative.

$$2xy \neq x - y$$

Case of x being positive and y being negative, can be ruled out.

$\therefore x$ and y have to be either positive or negative.

If x and y are positive,

$$f(x) = f(y)$$

$$\Rightarrow \frac{x}{1+x} = \frac{y}{1+y}$$

$$\Rightarrow x - xy = y - xy$$

$$\Rightarrow x = y$$

$\therefore f$ is one-one.

For onto:

Let $y \in R$ such that $-1 < y < 1$.

If x is negative, then there exists $x = \frac{y}{1+y} \in R$ such that

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)} = \frac{y}{1+y-y} = y$$

If x is positive, then there exists $x = \frac{y}{1-y} \in R$ such that

$$f(x) = f\left(\frac{y}{1-y}\right) = \frac{\left(\frac{y}{1-y}\right)}{1 + \left|\frac{y}{1-y}\right|} = \frac{\frac{y}{1-y}}{1 + \left(\frac{y}{1-y}\right)} = \frac{y}{1-y+y} = y$$

$\therefore f$ is onto.

Hence, f is one-one and onto.

Question 5:

Show that function $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^3$ is injective.

Solution:

$f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = x^3$

For one-one:

$$f(x) = f(y) \quad \text{where } x, y \in \mathbb{R}$$

$$x^3 = y^3 \dots\dots\dots (1)$$

We need to show that $x = y$

Suppose $x \neq y$, their cubes will also not be equal.

$$\Rightarrow x^3 \neq y^3$$

This will be a contradiction to (1).

$\therefore x = y$. Hence, f is injective.

Question 6:

Give examples of two functions $f: \mathbb{N} \rightarrow \mathbb{Z}$ and $g: \mathbb{Z} \rightarrow \mathbb{Z}$ such that $g \circ f$ is injective but g is not injective.

(Hint: Consider $f(x) = x$ and $g(x) = |x|$)

Solution:

Define $f: \mathbb{N} \rightarrow \mathbb{Z}$ as $f(x) = x$ and $g: \mathbb{Z} \rightarrow \mathbb{Z}$ as $g(x) = |x|$

Let us first show that g is not injective.

$$(-1) = |-1| = 1$$

$$(1) = |1| = 1$$

$$\therefore (-1) = g(1), \text{ but } -1 \neq 1$$

$\therefore g$ is not injective.

$gof : N \rightarrow Z$ is defined as $gof(x) = g(f(x)) = g(x) = |x|$

$x, y \in N$ such that $gof(x) = gof(y)$

$$\Rightarrow |x| = |y|$$

Since $x, y \in N$, both are positive.

$$\therefore |x| = |y|$$

$$\Rightarrow x = y$$

$\therefore gof$ is injective.

Question 7:

Given examples of two functions $f : N \rightarrow N$ and $g : N \rightarrow N$ such that gof is onto but f is not onto.

(Hint: Consider $f(x) = x+1$ and $g(x) = \begin{cases} x-1, & \text{if } x > 1 \\ 1, & \text{if } x = 1 \end{cases}$)

Solution:

Define $f : N \rightarrow Z$ as $f(x) = x+1$ and $g : Z \rightarrow Z$ as $g(x) = \begin{cases} x-1, & \text{if } x > 1 \\ 1, & \text{if } x = 1 \end{cases}$

Let us first show that g is not onto.

Consider element 1 in co-domain N . This element is not an image of any of the elements in domain N .

$\therefore f$ is not onto.

$g : N \rightarrow N$ is defined by

$$gof(x) = g(f(x)) = g(x+1) = x+1-1 = x \quad [x \in N \Rightarrow x+1 > 1]$$

For $y \in N$, there exists $x = y \in N$ such that $gof(x) = y$.

$\therefore gof$ is onto.

Question 8:

Given a non-empty set X , consider $P(X)$ which is the set of all subsets of X .

Define the relation R in $P(X)$ as follows:

For subsets A, B in $P(X)$, ARB if and only if $A \subset B$. Is R an equivalence relation on $P(X)$? Justify your answer.

Solution:

Since every set is a subset of itself, $A \subset A$ for all $A \in P(X)$.

$\therefore R$ is reflexive.

Let $ARB \Rightarrow A \subset B$

This cannot be implied to $B \subset A$.

If $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then it cannot be implied that B is related to A .

$\therefore R$ is not symmetric.

If ARB and BRC , then $A \subset B$ and $B \subset C$.

$\Rightarrow A \subset C$

$\Rightarrow ARC$

$\therefore R$ is transitive.

R is not an equivalence relation as it is not symmetric.

Question 9:

Given a non-empty set X , consider the binary operation $*$: $P(X) \times P(X) \rightarrow P(X)$ given by $A * B = A \cap B \quad \forall A, B$ in $P(X)$ is the power set of X . Show that X is the identity element for this operation and X is the only invertible element in $P(X)$ with respect to the operation $*$.

Solution:

$P(X) \times P(X) \rightarrow P(X)$ given by $A * B = A \cap B \quad \forall A, B$ in $P(X)$

$A \cap X = A = X \cap A$ for all $A \in P(X)$

$\Rightarrow A * X = A = X * A$ for all $A \in P(X)$

X is the identity element for the given binary operation $*$.

An element $A \in P(X)$ is invertible if there exists $B \in P(X)$ such that

$A * B = X = B * A$ [As X is the identity element]

Or

$A \cap B = X = B \cap A$

This case is possible only when $A = X = B$.

X is the only invertible element in $P(X)$ with respect to the given operation $*$.

Question 10:

Find the number of all onto functions from the set $\{1, 2, 3, \dots, n\}$ to itself.

Solution:

Onto functions from the set $\{1, 2, 3, \dots, n\}$ to itself is simply a permutation on n symbols $1, 2, 3, \dots, n$.

Thus, the total number of onto maps from $\{1, 2, 3, \dots, n\}$ to itself is the same as the total number of permutations on n symbols $1, 2, 3, \dots, n$, which is $n!$.

Question 11:

Let $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$. Find F^{-1} of the following functions F from S to T , if it exists.

- $F = \{(a, 3), (b, 2), (c, 1)\}$
- $F = \{(a, 2), (b, 1), (c, 1)\}$

Solution: $S = \{a, b, c\}, T = \{1, 2, 3\}$

- $F : S \rightarrow T$ is defined by $F = \{(a, 3), (b, 2), (c, 1)\}$
 $\Rightarrow F(a) = 3, F(b) = 2, F(c) = 1$

Therefore, $F^{-1} : T \rightarrow S$ is given by $F^{-1} = \{(3, a), (2, b), (1, c)\}$

- $F : S \rightarrow T$ is defined by $F = \{(a, 2), (b, 1), (c, 1)\}$

Since, $F(b) = F(c) = 1$, F is not one-one.

Hence, F is not invertible i.e., F^{-1} does not exist.

Question 12:

Consider the binary operations $*$: $R \times R \rightarrow R$ and $\circ$: $R \times R \rightarrow R$ defined as $a * b = |a - b|$ and $a \circ b = a, \forall a, b \in R$. Show that $*$ is commutative but not associative $\circ$ is associative but not commutative. Further, show that $\forall a, b, c \in R, a * (b \circ c) = (a * b) \circ (a * c)$. [If it is so, we say that the operation $*$ distributes over the operation $\circ$]. Does $\circ$ distribute over $*$? Justify your answer.

Solution:

It is given that $*$: $R \times R \rightarrow R$ and $\circ$: $R \times R \rightarrow R$ defined as $a * b = |a - b|$ and $a \circ b = a, \forall a, b \in R$.

For $a, b \in R$, we have $a * b = |a - b|$ and $b * a = |b - a| = |-(a - b)| = |a - b|$

$$\therefore a * b = b * a$$

$\therefore$ The operation $*$ is commutative.

$$(1 * 2) * 3 = (|1 - 2|) * 3 = 1 * 3 = |1 - 3| = 2$$

$$1 * (2 * 3) = 1 * (|2 - 3|) = 1 * 1 = |1 - 1| = 0$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3) \quad \text{where } 1, 2, 3 \in R$$

$\therefore$ The operation $*$ is not associative.

Now, consider the operation θ :

It can be observed that $1\theta 2 = 1$ and $2\theta 1 = 2$.

$$\therefore 1\theta 2 \neq 2\theta 1 \quad (\text{where } 1, 2 \in R)$$

$\therefore$ The operation θ is not commutative.

Let $a, b, c \in R$. Then, we have:

$$(aob)oc = aoc = a$$

$$ao(boc) = aob = a$$

$$\Rightarrow (aob)oc = ao(boc)$$

$\therefore$ The operation θ is associative.

Now, let $a, b, c \in R$, then we have:

$$a * (boc) = a * b = |a - b|$$

$$(a * b) \theta (a * c) = (|a - b|) \theta (|a - c|) = |a - b|$$

$$\text{Hence, } a * (boc) = (a * b) \theta (a * c)$$

Now,

$$1\theta (2 * 3) = 1\theta (|2 - 3|) = 1\theta 1 = 1$$

$$(1\theta 2) * (1\theta 3) = 1 * 1 = |1 - 1| = 0$$

$$\therefore 1\theta (2 * 3) \neq (1\theta 2) * (1\theta 3) \quad \text{where } 1, 2, 3 \in R$$

$\therefore$ The operation θ does not distribute over $*$.

Question 13:

Given a non - empty set X , let $*$: $P(X) \times P(X) \rightarrow P(X)$ be defined as $A * B = (A - B) \cup (B - A)$, $\forall A, B \in P(X)$. Show that the empty set Φ is the identity for the operation $*$ and all the elements A of $P(X)$ are invertible with $A^{-1} = A$.
(Hint: $(A - \Phi) \cup (\Phi - A) = A$ and $(A - A) \cup (A - A) = A * A = \Phi$).

Solution:

It is given that $*$: $P(X) \times P(X) \rightarrow P(X)$ is defined as $A * B = (A - B) \cup (B - A)$, $\forall A, B \in P(X)$
 $A \in P(X)$ then,

$$A * \Phi = (A - \Phi) \cup (\Phi - A) = A \cup \Phi = A$$

$$\Phi * A = (\Phi - A) \cup (A - \Phi) = \Phi \cup A = A$$

$$\therefore A * \Phi = A = \Phi * A \quad \text{for all } A \in P(X)$$

Φ is the identity for the operation $*$.

Element $A \in P(X)$ will be invertible if there exists $B \in P(X)$ such that
 $A * B = \Phi = B * A$ [As Φ is the identity element]

$$A * A = (A - A) \cup (A - A) = \Phi \cup \Phi = \Phi \quad \text{for all } A \in P(X).$$

All the elements A of $P(X)$ are invertible with $A^{-1} = A$.

Question 14:

Define a binary operation $*$ on the set $\{0, 1, 2, 3, 4, 5\}$ as

$$a + b = \begin{cases} a + b, & \text{if } a + b < 6 \\ a + b - 6 & \text{if } a + b \geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $6 - a$ being the inverse of a .

Solution:

Let $X = \{0, 1, 2, 3, 4, 5\}$

The operation $*$ is defined as
$$a + b = \begin{cases} a + b, & \text{if } a + b < 6 \\ a + b - 6, & \text{if } a + b \geq 6 \end{cases}$$

An element $e \in X$ is the identity element for the operation $*$, if $a * e = a = e * a \quad \forall a \in X$

For $a \in X$,

$$a * 0 = a + 0 = a \quad [a \in X \Rightarrow a + 0 < 6]$$

$$0 * a = 0 + a = a \quad [a \in X \Rightarrow 0 + a < 6]$$

$$\therefore a * 0 = a = 0 * a \quad \forall a \in X$$

Thus, 0 is the identity element for the given operation $*$.

An element $a \in X$ is invertible if there exists $b \in X$ such that $a * b = 0 = b * a$.

$$\text{i.e., } \left\{ \begin{array}{ll} a + b = 0 = b + a, & \text{if } a + b < 6 \\ a + b - 6 = 0 = b + a - 6 & \text{if } a + b \geq 6 \end{array} \right\}$$

$$\Rightarrow a = -b \text{ or } b = 6 - a$$

$$X = \{0, 1, 2, 3, 4, 5\} \text{ and } a, b \in X. \text{ Then } a \neq -b.$$

$$\therefore b = 6 - a \text{ is the inverse of } a \text{ for all } a \in X.$$

Inverse of an element $a \in X$, $a \neq 0$ is $6 - a$ i.e., $a^{-1} = 6 - a$.

Question 15:

Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$ and $f, g: A \rightarrow B$ be functions defined by $x^2 - x$, $x \in A$ and

$$g(x) = 2 \left| x - \frac{1}{2} \right| - 1, x \in A. \text{ Are } f \text{ and } g \text{ equal?}$$

Solution:

It is given that $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$

Also, $f, g: A \rightarrow B$ is defined by $x^2 - x$, $x \in A$ and $g(x) = 2 \left| x - \frac{1}{2} \right| - 1, x \in A$.

$$f(-1) = (-1)^2 - (-1) = 1 + 1 = 2$$

$$g(-1) = 2 \left| (-1) - \frac{1}{2} \right| - 1 = 2 \left(\frac{3}{2} \right) - 1 = 3 - 1 = 2$$

$$\Rightarrow f(-1) = g(-1)$$

$$f(0) = (0)^2 - 0 = 0$$

$$g(0) = 2 \left| 0 - \frac{1}{2} \right| - 1 = 2 \left(\frac{1}{2} \right) - 1 = 1 - 1 = 0$$

$$\Rightarrow f(0) = g(0)$$

$$f(1) = (1)^2 - 1 = 0$$

$$g(1) = 2 \left| 1 - \frac{1}{2} \right| - 1 = 2 \left(\frac{1}{2} \right) - 1 = 1 - 1 = 0$$

$$\Rightarrow f(1) = g(1)$$

$$f(2) = (2)^2 - 2 = 2$$

$$g(2) = 2 \left| 2 - \frac{1}{2} \right| - 1 = 2 \left(\frac{3}{2} \right) - 1 = 3 - 1 = 2$$

$$\Rightarrow f(2) = g(2)$$

$$\therefore f(a) = g(a) \quad \forall a \in A$$

Hence, the functions f and g are equal.

Question 16:

Let $A = \{1, 2, 3\}$. Then number of relations containing $(1, 2)$ and $(1, 3)$ which are reflexive and symmetric but not transitive is,

- A. 1
- B. 2
- C. 3
- D. 4

Solution:

The given set is $A = \{1, 2, 3\}$.

The smallest relation containing $(1, 2)$ and $(1, 3)$ which are reflexive and symmetric but not transitive is given by,

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 1), (3, 1)\}$$

This is because relation R is reflexive as $\{(1, 1), (2, 2), (3, 3)\} \in R$.

Relation R is symmetric as $\{(1, 2), (2, 1)\} \in R$ and $\{(1, 3), (3, 1)\} \in R$.

Relation R is transitive as $\{(3, 1), (1, 2)\} \in R$ but $(3, 2) \notin R$.

Now, if we add any two pairs $(3, 2)$ and $(2, 3)$ (or both) to relation R , then relation R will become transitive.

Hence, the total number of desired relations is one.

The correct answer is A.

Question 17:

Let $A = \{1, 2, 3\}$. Then number of equivalence relations containing $(1, 2)$ is,

- A. 1
- B. 2
- C. 3
- D. 4

Solution:

The given set is $A = \{1, 2, 3\}$.

The smallest equivalence relation containing $(1, 2)$ is given by;

$$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$$

Now, we are left with only four pairs i.e., $(2, 3), (3, 2), (1, 3)$ and $(3, 1)$.

If we add any one pair [say $(2, 3)$] to R_1 , then for symmetry we must add $(3, 2)$. Also, for transitivity we are required to add $(1, 3)$ and $(3, 1)$.

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

This shows that the total number of equivalence relations containing $(1, 2)$ is two.
The correct answer is B.

Question 18:

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the Signum Function defined as $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be the greatest integer function given by $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x . Then does $f \circ g$ and $g \circ f$ coincide in $(0, 1]$?

Solution:

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

It is given that $f: \mathbb{R} \rightarrow \mathbb{R}$ be the Signum Function defined as

Also $g: \mathbb{R} \rightarrow \mathbb{R}$ is defined as $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x .

Now let $x \in (0, 1]$,

$$[x] = 1 \text{ if } x = 1 \text{ and } [x] = 0 \text{ if } 0 < x < 1.$$

$$\therefore fog(x) = f(g(x)) = f([x]) = \begin{cases} f(1), & \text{if } x = 1 \\ f(0), & \text{if } x \in (0,1) \end{cases} = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \in (0,1) \end{cases}$$

$$\begin{aligned} gof(x) &= g(f(x)) \\ &= g(1) \quad [x > 0] \\ &= [1] = 1 \end{aligned}$$

Thus, when $x \in (0,1)$, we have $fog(x) = 0$ and $gof(x) = 1$.

Hence, fog and gof does not coincide in $(0,1]$.

Question 19:

Number of binary operations on the set $\{a,b\}$ are

- A. 10
- B. 16
- C. 20
- D. 8

Solution:

A binary operation $*$ on $\{a,b\}$ is a function from $\{a,b\} \times \{a,b\} \rightarrow \{a,b\}$

i.e., $*$ is a function from $\{(a,a), (a,b), (b,a), (b,b)\} \rightarrow \{a,b\}$

Hence, the total number of binary operations on the set $\{a,b\}$ is $2^4 = 16$.

The correct answer is B.