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In this chapter, we will study some of the basic concepts about vectors, various operations on vectors, and their algebraic and geometric properties.

VECTOR ALGEBRA

|TOPIC 1|

Basic Concepts

CHAPTER CHECKLIST

- Basic Concepts
- Vector Joining Two Points
- Scalar Product of Two Vectors
- Vector (or Cross) Product of Two Vectors

SCALAR QUANTITIES OR SCALARS

Those quantities which have magnitude but no direction, are called scalar quantities or scalars. **e.g.** The height of Ram may be 1.6 m. Here, height involves only one value (magnitude), which is a real number. Some scalar quantities are length, mass, time, distance, speed, area, volume, temperature, work, density, voltage, resistance, etc. A scalar quantity is represented by a real number alongwith a suitable unit.

VECTOR QUANTITIES OR VECTORS

Those quantities which have magnitude as well as direction are called vector quantities or vectors. **e.g.** A football player hit the ball to give a pass to another player of his team. Hence, he apply a quantity (called force) which involves muscular strength (magnitude) and direction in which another player is positioned. Some vector quantities are force, displacement, velocity, acceleration, weight, momentum, electric field intensity, etc.

EXAMPLE |1| Classify the following measures as scalars and vectors.

(i) 40 watt (ii) $20\ m/s^2$

(iii) $100\ m^2$ (iv) 2 m North-West

Sol. (i) 40 watt represents power, which is a scalar.

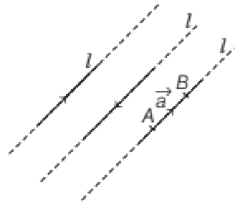
(ii) $20\ m/s^2$ represents acceleration, which is a vector.

(iii) $100\ m^2$ represents an area, which is a scalar.

(iv) 2 m North-West has magnitude as well as direction, so it is a vector.

Representation of a Vector

Any straight line 'l' in plane or three dimensional space can be given two directions by means of arrowheads. A line with one of these directions prescribed is called a **directed line** and if we restrict the line l to the line segment AB, then a magnitude is prescribed on the line l with one of the two directions, so we obtain a **directed line segment**.



Thus, a directed line segment has magnitude as well as direction, so it is called vector denoted as $\overrightarrow{AB}$ (or simply as $\vec{a}$) and read as vector 'AB' (or vector 'a').

Here, the point A from where the vector $\overrightarrow{AB}$ starts is called its **initial point** and the point B where it ends is called its **terminal point**. Generally, vectors are denoted by $\vec{a}, \vec{b}, \vec{c}$, etc.

Note The arrow indicates the direction of the vector.

MAGNITUDE (OR LENGTH) OF A VECTOR

The length of the vector $\overrightarrow{AB}$ or $\vec{a}$ is called the magnitude of $\overrightarrow{AB}$ or $\vec{a}$ and it is represented by $|\overrightarrow{AB}|$ or $|\vec{a}|$ or a. Since, the length is never negative, the notation $|\overrightarrow{AB}| < 0$ has no meaning.

In other words, the distance between **initial point** and **terminal point** of a vector is called its length or magnitude.

Types of Vectors

There are following types of vectors as given below

(i) **Zero or Null Vector** A vector whose magnitude is zero, i.e. whose initial and terminal points coincide, is called a null vector or zero vector. It is denoted by $\vec{0}$ and it cannot be assigned a definite direction, as its magnitude is zero. Otherwise, it may be regarded as having any direction. Thus, $\overrightarrow{AA}$, $\overrightarrow{BB}$ represent the zero vectors.

(ii) **Unit Vector** A vector whose magnitude is one unit (i.e. unity) is called a unit vector. The unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$ and read as 'a cap'.

(iii) **Coinitial Vectors** Two or more vectors having the same initial point are called coinitial vectors.

(iv) **Collinear or Parallel Vectors** Two or more vectors are said to be collinear, if they are parallel to same line, irrespective of their magnitudes and directions.

(v) **Equal Vectors** Two vectors are said to be **equal**, if they have same magnitude and direction regardless of the positions of their initial points. Symbolically if $\vec{a}$ and $\vec{b}$ are equal, then it is written as $\vec{a} = \vec{b}$.

(vi) **Negative of a Vector** A vector whose magnitude is same as that of a given vector but the direction is opposite to that of it, is called negative of the given vector.

e.g. Let $\overrightarrow{AB}$ be a vector, then $-\overrightarrow{AB}$ or $\overrightarrow{BA}$ is its negative vector.

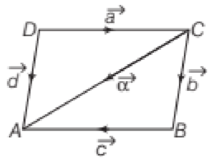
(vii) **Coplanar Vectors** Three or more vectors, which either lie in the same plane or are parallel to the same plane, are called coplanar vectors.

Note

(i) Two vectors are always coplanar.

(ii) If the value of a vector depends only on its magnitude and direction and is independent of its position in the space, it is called **free vector**. Throughout this chapter, we will be dealing with free vectors only.

EXAMPLE |2| In the given parallelogram $ABCD$, identify the vectors are



(i) equal. (ii) collinear but not equal.

(iii) coinitial vector.

Sol. (i) $\vec{b}$ and $\vec{d}$ are equal vectors.

(ii) $\vec{a}$ and $\vec{c}$ are collinear vectors but not equal.

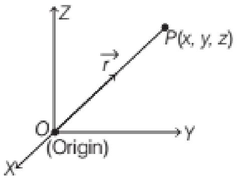
[here, $\vec{a}$ and $\vec{c}$ have opposite directions]

(iii) $\vec{\alpha}$ and $\vec{b}$; $\vec{d}$ and $\vec{a}$ are coinitial vectors.

Position Vector

Let $O(0, 0, 0)$ be the origin and P be a point in space having coordinates (x, y, z) with respect to the origin O. Then, the vector $\overrightarrow{OP}$ or $\vec{r}$ is called the position vector of the point P with respect to O.

Here, O is its initial point and P is its terminal point.



By using the distance formula, the magnitude of $\overrightarrow{OP}$ or $\vec{r}$ is given by

$$|\overrightarrow{OP}| = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

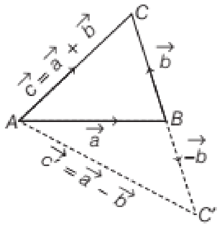
Generally, we denote the position vector of points A, B, C, etc., with respect to the origin O by $\vec{a}, \vec{b}, \vec{c}$, etc., respectively.

ADDITION OF VECTORS

There are two laws of vector addition, which are given below

Triangle Law of Vector Addition

Suppose two vectors $\vec{a}$ and $\vec{b}$ are represented by the two adjacent sides of a triangle such that the initial point of one coincides with the terminal point of the other (say $\overrightarrow{AB}$ and $\overrightarrow{BC}$), then their sum $\vec{c}$ is represented by the third side $\overrightarrow{AC}$ of ΔABC.



Thus, $\vec{a} + \vec{b} = \vec{c}$ or $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

This is known as the triangle law of vector addition.

Now, construct a vector $\overrightarrow{BC'}$ or $\vec{b}'$, such that its magnitude is same as the vector $\overrightarrow{BC}$ and direction opposite to $\overrightarrow{BC}$.

i.e. $\overrightarrow{BC'} = -\overrightarrow{BC}$ or $\vec{b}' = -\vec{b}$

Then, by triangle law, we have

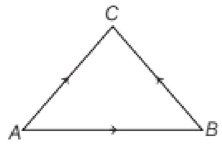
$$\overrightarrow{AC'} = \overrightarrow{AB} + \overrightarrow{BC'}$$
$$= \overrightarrow{AB} + (-\overrightarrow{BC})$$
$$= \overrightarrow{AB} - \overrightarrow{BC} \text{ or } \vec{c}' = \vec{a} - \vec{b}$$

Here, the vector $\overrightarrow{AC'}$ or $\vec{c}'$ is said to be represent the difference of $\vec{a}$ and $\vec{b}$.

EXAMPLE |3| If A, B and C are the vertices of a ΔABC, then what is the value of $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$? **[Delhi 2011C]**

Sol. By triangle law of vector addition, we get

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{CA} + \overrightarrow{AC}$$
$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{CA} - \overrightarrow{CA} \quad [\because \overrightarrow{AC} = -\overrightarrow{CA}]$$
$$\therefore \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$$

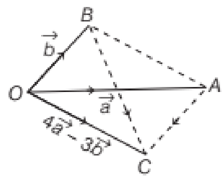
Note When the sides of a triangle are taken in order, it leads to zero resultant as the initial and terminal points get coincided.

EXAMPLE |4| Vectors drawn from the origin O to the points A, B and C are respectively $\vec{a}$, $\vec{b}$ and $4\vec{a} - 3\vec{b}$. Find $\overrightarrow{AC}$ and $\overrightarrow{BC}$.

Sol. We have, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = 4\vec{a} - 3\vec{b}$.

Clearly, from ΔOAC, we get $\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$

[using triangle law of addition]



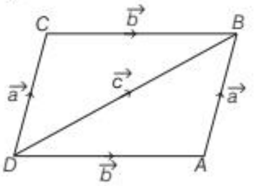
$$\Rightarrow \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$
$$= 4\vec{a} - 3\vec{b} - \vec{a}$$
$$= 3\vec{a} - 3\vec{b} = 3(\vec{a} - \vec{b})$$

and from ΔOBC, we get

$$\overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC} \quad \text{[using triangle law of addition]}$$
$$\Rightarrow \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$
$$= 4\vec{a} - 3\vec{b} - \vec{b}$$
$$= 4\vec{a} - 4\vec{b} = 4(\vec{a} - \vec{b})$$

Parallelogram Law of Vector Addition

Suppose two vectors $\vec{a}$ and $\vec{b}$ are represented by the two adjacent sides of a parallelogram, then their sum $\vec{c}$ is represented by the diagonal of the parallelogram, which is coinitial with the given vectors.



Mathematically, $\vec{c} = \vec{a} + \vec{b}$

This is known as parallelogram law of vector addition.

Note

(i) For applying parallelogram law, vectors should have same initial point.

(ii) From the above figure, using the triangle law of vector addition, we have

$$\overrightarrow{DA} + \overrightarrow{AB} = \overrightarrow{DB} \text{ or } \overrightarrow{DA} + \overrightarrow{DC} = \overrightarrow{DB} \quad [\because \overrightarrow{AB} = \overrightarrow{DC}]$$

which is parallelogram law. Thus, we may say that the two laws of vector addition are equivalent to each other.

PROPERTIES OF VECTOR ADDITION

Suppose $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors, then

(i) Vector addition is **commutative,**

i.e. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(ii) Vector addition is **associative,**

i.e. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

(iii) Existence of **additive identity,**

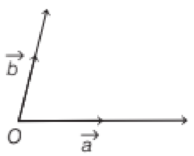
$$\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$$

where $\vec{0}$ is called additive identity.

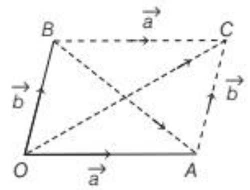
Note The associative property of vector addition enables us to write the sum of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ as $\vec{a} + \vec{b} + \vec{c}$ without using brackets.

EXAMPLE |5| If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors having the same initial point. What are the vectors represented by $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$?

Sol. According to the given condition, we have the following figure



Let us make the following parallelogram.



Clearly, by parallelogram law, we have

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \vec{a} + \vec{b}$$

and $\overrightarrow{BA} = \overrightarrow{BC} + \overrightarrow{CA} = \vec{a} + (-\vec{b}) = \vec{a} - \vec{b}$

Thus, $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ represent the diagonals of the parallelogram.

MULTIPLICATION OF A VECTOR BY A SCALAR

Let λ be a scalar and $\vec{a}$ be a vector, then $\lambda\vec{a}$ is defined as a vector whose magnitude is $|\lambda|$ times the magnitude of $\vec{a}$, i.e. $|\lambda\vec{a}| = |\lambda||\vec{a}|$ and the direction is same or opposite of $\vec{a}$, according as λ is positive or negative.

e.g. When $\lambda = -2$, then $\lambda\vec{a} = -2\vec{a}$, which is a vector having magnitude double to the magnitude of $\vec{a}$ and direction opposite to that of the direction of $\vec{a}$.

Note

(i) For any value of λ, the vector $\lambda\vec{a}$ is always collinear to the vector $\vec{a}$.

(ii) The vector $-\vec{a}$ is called the negative (or additive inverse) of $\vec{a}$, as $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$.

(iii) If $\lambda = \frac{1}{|\vec{a}|}$, provided $\vec{a} \neq \vec{0}$ (i.e. $\vec{a}$ is not a null vector)

Then, $|\lambda\vec{a}| = |\lambda||\vec{a}| = \frac{1}{|\vec{a}|} \cdot |\vec{a}| = 1$

So, $\lambda\vec{a}$ represents the unit vector in the direction of $\vec{a}$ and we write it as $\hat{a} = \frac{1}{|\vec{a}|} \cdot \vec{a}$.

(iv) For any scalar k, $k \cdot \vec{0} = \vec{0}$.

PROPERTIES OF MULTIPLICATION OF A VECTOR BY A SCALAR

The addition of vectors and the multiplication of a vector by a scalar, together constitute the following distribution law

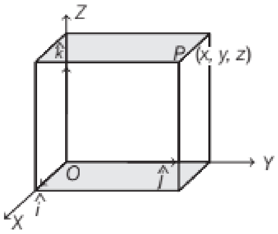
Let $\vec{a}$ and $\vec{b}$ be any two vectors and k and m be any two scalars, then

(i) $k\vec{a} + m\vec{a} = (k+m)\vec{a}$ (ii) $k(\vec{a}+\vec{b}) = k\vec{a} + k\vec{b}$

(iii) $k(m\vec{a}) = km(\vec{a})$

COMPONENTS OF A VECTOR

Let a point P in a space has coordinates (x, y, z) and $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along OX, OY and OZ-axes, respectively. Then, the position vector of P with respect to O is given by $\overrightarrow{OP}$ (or $\vec{r}$) $= x\hat{i} + y\hat{j} + z\hat{k}$.

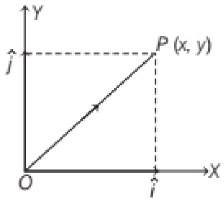


This form of vector $\overrightarrow{OP}$ is called **component form**. Here, x, y and z are called the **scalar components** and $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the **vector components** of $\overrightarrow{OP}$ (or $\vec{r}$) along the respective axes.

Sometimes x, y and z are also termed as **rectangular components.** The length of any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is given by

$$|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}.$$

Note If a point P lie in a plane, say XY-plane and has coordinates (x, y). Then, $\overrightarrow{OP} = x\hat{i} + y\hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors along OX and OY-axes, respectively. Also, $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$.



EXAMPLE |6| If $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, then find its scalar and vector components.

Sol. We have, $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$.

Clearly, the scalar components are 2, 3, 4 and the vector components are $2\hat{i}, 3\hat{j}, 4\hat{k}$.

EXAMPLE |7| Find a vector in the direction of vector $\vec{a} = \hat{i} - 2\hat{j}$ that has magnitude 7 units.

[NCERT Exemplar]

Sol. We have, $\vec{a} = \hat{i} - 2\hat{j}$, then

$$|\vec{a}| = \sqrt{(1)^2 + (-2)^2} = \sqrt{1+4} = \sqrt{5}$$

The unit vector in the direction of the given vector $\vec{a}$ is

$$\hat{a} = \frac{1}{|\vec{a}|}\cdot\vec{a} = \frac{1}{\sqrt{5}}(\hat{i} - 2\hat{j}) = \frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}$$

Now, the vector having magnitude 7 units and in the direction of $\vec{a} = 7\hat{a}$

$$= 7\left(\frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j}\right) = \frac{7}{\sqrt{5}}\hat{i} - \frac{14}{\sqrt{5}}\hat{j}$$

Important Results in Component Form

If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form such that $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then,

(i) the sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$.

(ii) the difference of the vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$.

(iii) the vectors $\vec{a}$ and $\vec{b}$ are equal, if and only if $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

(iv) the multiplication of vector $\vec{a}$ by any scalar λ is given by $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$.

(v) The vectors $\vec{a}$ and $\vec{b}$ are collinear (or parallel) if and only if there exists a non-zero scalar λ such that $\vec{b} = \lambda\vec{a}$.

$$\Rightarrow (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$\Rightarrow \quad b_1 = a_1\lambda, b_2 = a_2\lambda \text{ and } b_3 = a_3\lambda$$

$$\Rightarrow \quad \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

Thus, the vectors $\vec{a}$ and $\vec{b}$ are collinear (or parallel) iff

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda \quad \text{[non-zero constant]}$$

EXAMPLE |8| If $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 4\hat{j} - 3\hat{k}$, then find $|\vec{a} - 2\vec{b}|$.

Sol. We have, $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ ...(i)

and $\vec{b} = 2\hat{i} - 4\hat{j} - 3\hat{k}$

$\Rightarrow 2\vec{b} = 2(2\hat{i} - 4\hat{j} - 3\hat{k})$

[multiplying by 2 on both sides]

$\Rightarrow 2\vec{b} = 4\hat{i} - 8\hat{j} - 6\hat{k}$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$\vec{a} - 2\vec{b} = (3\hat{i} - 2\hat{j} + \hat{k}) - (4\hat{i} - 8\hat{j} - 6\hat{k})$$
$$= (3-4)\hat{i} + (-2+8)\hat{j} + (1+6)\hat{k}$$

[subtracting corresponding terms]

$$= -\hat{i} + 6\hat{j} + 7\hat{k}$$

Now, $|\vec{a} - 2\vec{b}| = |-\hat{i} + 6\hat{j} + 7\hat{k}|$

$$= \sqrt{(-1)^2 + (6)^2 + (7)^2}$$
$$= \sqrt{1 + 36 + 49} = \sqrt{86}$$

EXAMPLE |9| For what values of a, the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $a\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear? **[Delhi 2011]**

Sol. Let given vectors are $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = a\hat{i} + 6\hat{j} - 8\hat{k}$.

We know that vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear, iff

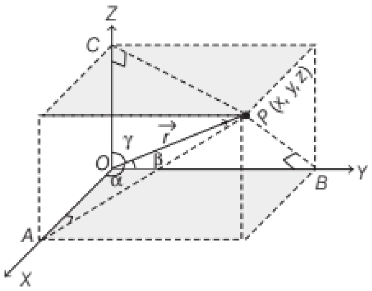
$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3}.$$

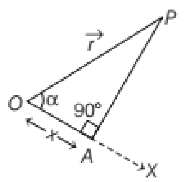
$$\therefore \quad \frac{a}{2} = \frac{6}{-3} = \frac{-8}{4}$$

$$\Rightarrow \quad \frac{a}{2} = -2 \Rightarrow a = -4$$

DIRECTION COSINES AND DIRECTION RATIOS OF A VECTOR

Suppose $\overrightarrow{OP}$ (or $\vec{r}$) is the position vector of a point $P(x, y, z)$ shown in the figure.





Here, the angles α, β and γ made by the vector $\overrightarrow{OP}$ with positive directions of coordinates axes OX, OY and OZ respectively, are called **direction angles**. The cosine values of these angles, i.e. $\cos\alpha, \cos\beta$ and $\cos\gamma$ are known as the **direction cosines** of $\overrightarrow{OP}$.

Generally, they are denoted by l, m and n respectively, i.e. $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.

In the figure, the ΔOAP is a right angled triangle, so we have, $\cos\alpha = \frac{x}{r}$, where $r = |\vec{r}|$.

Similarly, from the right angled ΔOBP and ΔOCP, we have $\cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$. Thus, the coordinates of the point P may also be expressed as $(r\cos\alpha, r\cos\beta, r\cos\gamma)$ or (lr, mr, nr). The numbers lr, mr and nr are proportional to the direction cosines, called **direction ratios** of vector $\vec{r}$ and denoted as x, y and z, respectively.

Note

(i) $l = \frac{x}{r} = \frac{x}{|\vec{r}|}$, $m = \frac{y}{r} = \frac{y}{|\vec{r}|}$ and $n = \frac{z}{r} = \frac{z}{|\vec{r}|}$.

(ii) In general, it may be noted that $l^2 + m^2 + n^2 = 1$ but $x^2 + y^2 + z^2 \neq 1$.

(iii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then a_1, a_2 and a_3 are also called direction ratios of $\vec{a}$.

(iv) If it is given that l, m and n are direction cosines of a vector, then $l\hat{i} + m\hat{j} + n\hat{k} = (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$ is the unit vector in the direction of that vector, where α, β and γ are the angles which the vector makes with X, Y and Z-axes, respectively.

EXAMPLE |10| Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with X-axis, $\frac{\pi}{2}$ with Y-axis and an acute angle θ with Z-axis. **[All India 2014]**

Sol. Given, vector $\vec{a}$ makes an angle $\frac{\pi}{4}$ with X-axis and $\frac{\pi}{2}$ with Y-axis.

So, $l = \cos\frac{\pi}{4}$ and $m = \cos\frac{\pi}{2}$

$\Rightarrow l = \frac{1}{\sqrt{2}}$ and $m = 0$

We know that, $l^2 + m^2 + n^2 = 1$

$$\therefore \quad \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2 + n^2 = 1 \Rightarrow \frac{1}{2} + n^2 = 1$$

$$\Rightarrow \quad n^2 = 1 - \frac{1}{2} \Rightarrow n = \pm\frac{1}{\sqrt{2}}$$

$\Rightarrow n = \frac{1}{\sqrt{2}}$ [$\because \vec{a}$ makes acute angle with Z-axis, so we take a positive sign]

$\therefore \quad \cos\theta = \frac{1}{\sqrt{2}} \quad$ [$\because \theta$ is an acute angle with Z-axis]

Thus, the direction cosines of $\vec{a}$ are $\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$.

Now, vector $\vec{a} = |\vec{a}|\left(\cos\frac{\pi}{4}\hat{i} + \cos\frac{\pi}{2}\hat{j} + \cos\theta\hat{k}\right)$

$= 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\hat{i} + (0)\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right) = 5\hat{i} + 5\hat{k}$

EXAMPLE |11| Write the direction ratios of the vector $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ and hence calculate its direction cosines. **[NCERT]**

Sol. We have, vector $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$

We know that, direction ratios of a vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ are the scalar components x, y and z of the vector.

So, required direction ratios are 1, 1 and − 2.

Now, $|\vec{a}| = |\hat{i} + \hat{j} - 2\hat{k}| = \sqrt{(1)^2 + (1)^2 + (-2)^2}$

$= \sqrt{1+1+4} = \sqrt{6}$

Suppose l, m and n are the direction cosines of given vector $\vec{a}$.

Then, $l = \frac{x}{|\vec{a}|}, m = \frac{y}{|\vec{a}|}$ and $n = \frac{z}{|\vec{a}|}$

$\Rightarrow \quad l = \frac{1}{\sqrt{6}}, \; m = \frac{1}{\sqrt{6}}$ and $n = \frac{-2}{\sqrt{6}}$

Hence, the required direction cosines are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$.

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 The magnitude of the vector $6\hat{i} + 2\hat{j} + 3\hat{k}$ is **[NCERT Exemplar]**

(a) 5 (b) 7
(c) 12 (d) 1

2 The vector in the direction of vector $\hat{i} - 2\hat{j} + 2\hat{k}$ that has magnitude 12 is

(a) $\hat{i} - 2\hat{j} + 2\hat{k}$ (b) $\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$
(c) $4(\hat{i} - 2\hat{j} + 2\hat{k})$ (d) $9(\hat{i} - 2\hat{j} + 2\hat{k})$

3 In triangle ABC, which of the following is not true **[NCERT]**

(a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$ (b) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = 0$
(c) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = 0$ (d) $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = 0$

4 If a and b are two collinear vectors, then which of the following are incorrect **[NCERT]**

(a) $\vec{b} = \lambda\,\vec{a}$, for some scalar λ
(b) $\hat{a} = \pm\hat{b}$
(c) The respective components of a and b are proportional
(d) Both the vectors a and b have same direction but different magnitudes

5. The direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$ are

(a) $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ (b) $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{4}{\sqrt{14}}$
(c) $\frac{11}{\sqrt{14}}, \frac{13}{\sqrt{14}}, \frac{21}{\sqrt{14}}$ (d) None of these

VERY SHORT ANSWER Type Questions

6 If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors, then find the value of $x + y + z$. **[Delhi 2013]**

7 Write a vector in the direction of the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ that has magnitude 9 units. **[Delhi 2014C]**

8 Find a vector in the direction of $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 6 units. **[NCERT Exemplar]**

9 Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$. **[Delhi 2012]**

10 Write a unit vector in the direction of the sum of the vectors $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$. **[Delhi 2014C]**

11 Find the unit vector in the direction of the sum of the vectors $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + 3\hat{k}$. **[NCERT Exemplar]**

12 If $\vec{a} = 4\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$, then find a unit vector parallel to the vector $\vec{a} + \vec{b}$. **[All India 2016]**

13 Find the value of p for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel. **[All India 2014]**

14 Write the value of cosine of the angle which the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ makes with Y-axis. **[Delhi 2014C]**

15 Find the angle between X-axis and the vector $\hat{i} + \hat{j} + \hat{k}$. **[All India 2014C]**

16 Write the direction cosines of vector $-2\hat{i} + \hat{j} - 5\hat{k}$. **[Delhi 2011]**

SHORT ANSWER Type I Questions

17 If A, B, P, Q and R be the five points in a plane, then show that the sum of the vectors $\overrightarrow{AP}, \overrightarrow{AQ}, \overrightarrow{AR}, \overrightarrow{PB}, \overrightarrow{QB}$ and $\overrightarrow{RB}$ is $3\,\overrightarrow{AB}$.

18 If O be the centre of a regular hexagon $PQRSTU$, then prove that

$$\overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR} + \overrightarrow{OS} + \overrightarrow{OT} + \overrightarrow{OU} = \vec{0}.$$

19 If $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$, then find the unit vector in the direction of $2\vec{a} - \vec{b}$. **[NCERT Exemplar]**

SHORT ANSWER Type II Questions

20 Find a vector of magnitude 5 units and parallel to the resultant of $\vec{a} = 2\hat{i} + 3\hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$. **[NCERT]**

21 Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY and OZ.

22 A vector $\vec{r}$ is inclined at equal angles to the three axes. If the magnitude of $\vec{r}$ is $2\sqrt{3}$ units, then find the value of $\vec{r}$.

23 Show that the vectors $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right angled triangle.

24 If a vector $\vec{r}$ has magnitude 14 and direction ratios 2, 3 and – 6. Then, find the direction cosines and components of $\vec{r}$, given that $\vec{r}$ makes an acute angle with X-axis. **[NCERT Exemplar]**

25 If a unit vector $\vec{a}$ makes angle $\frac{\pi}{4}$ with $\hat{i}$, $\frac{\pi}{3}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find the components of $\vec{a}$ and the angle θ.

26 If a unit vector $\hat{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find the angle θ and hence the components of $\vec{a}$. **[NCERT]**

27 Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$ and find a vector of magnitude 6 units which is parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.

HINTS & SOLUTIONS

1. (b) Let $\vec{a} = 6\hat{i} + 2\hat{j} + 3\hat{k}$, then $|\vec{a}| = \sqrt{6^2 + 2^2 + 3^2}$
$= \sqrt{36 + 4 + 9} = \sqrt{49} = 7$

2. (c) Let $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

Unit vector in the direction of vector $\vec{a}$ is given by $\frac{\vec{a}}{|\vec{a}|}$.

$$= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

$\therefore$ Vector in the direction of $\vec{a}$ with magnitude 12

$$= 12 \cdot \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3} = 4(\hat{i} - 2\hat{j} + 2\hat{k})$$

3. (c) **Hint** By triangle law of vector addition,
$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ or $\overrightarrow{AB} + \overrightarrow{BC} = -\overrightarrow{CA}$

4. (d) By definition collinear vectors may have same or opposite directions.

5. (a) Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, then $|\vec{a}| = \sqrt{1 + 4 + 9} = \sqrt{14}$

Now, $l = \frac{1}{|\vec{a}|} = \frac{1}{\sqrt{14}}$, $m = \frac{2}{|\vec{a}|} = \frac{2}{\sqrt{14}}$ and $n = \frac{3}{|\vec{a}|} = \frac{3}{\sqrt{14}}$

6. We have, $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are equal vectors.

$\therefore \quad x\hat{i} + 2\hat{j} - z\hat{k} = 3\hat{i} - y\hat{j} + \hat{k}$

On comparing the corresponding elements, we get

$\therefore \quad x = 3,\ y = -2$ and $z = -1$

Hence, $x + y + z = 3 - 2 - 1 = 0$.

7. Similar as Example 7. [**Ans.** $3\hat{i} - 6\hat{j} + 6\hat{k}$]

8. Similar as Example 7. [**Ans.** $4\hat{i} - 2\hat{j} + 4\hat{k}$]

9. We have, $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$

and $\quad \vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

Now, sum of the vectors is

$$\vec{a}+\vec{b}+\vec{c} = (\hat{i}-2\hat{j}+\hat{k})+(-2\hat{i}+4\hat{j}+5\hat{k})+(\hat{i}-6\hat{j}-7\hat{k})$$
$$= (1-2+1)\hat{i}+(-2+4-6)\hat{j}+(1+5-7)\hat{k}$$
$$= -4\hat{j}-\hat{k}$$

10. Given, $\vec{a} = 2\hat{i}+2\hat{j}-5\hat{k}$ and $\vec{b} = 2\hat{i}+\hat{j}-7\hat{k}$

Unit vector along the direction of $\vec{a}+\vec{b} = \dfrac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}$

Now, $\vec{a}+\vec{b} = 2\hat{i}+2\hat{j}-5\hat{k}+2\hat{i}+\hat{j}-7\hat{k}$
$$= 4\hat{i}+3\hat{j}-12\hat{k}$$

and $|\vec{a}+\vec{b}| = \sqrt{(4)^2+(3)^2+(-12)^2}$
$$= \sqrt{16+9+144} = \sqrt{169} = 13$$

So, the required unit vector is

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} = \frac{4\hat{i}+3\hat{j}-12\hat{k}}{13} = \frac{4}{13}\hat{i}+\frac{3}{13}\hat{j}-\frac{12}{13}\hat{k}$$

11. Similar as Question 10. $\left[\textbf{Ans. } \dfrac{1}{\sqrt{26}}\hat{i}+\dfrac{5}{\sqrt{26}}\hat{k}\right]$

12. Similar as Question 10. $\left[\textbf{Ans. } \dfrac{6\hat{i}-3\hat{j}+2\hat{k}}{7}\right]$

13. Given, $3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}-2p\hat{j}+3\hat{k}$ are two parallel vectors, so their direction ratios will be proportional.

$$\therefore \quad \frac{3}{1} = \frac{2}{-2p} = \frac{9}{3}$$
$$\Rightarrow \quad \frac{2}{-2p} = \frac{3}{1} \Rightarrow -6p = 2$$
$$\Rightarrow \quad p = \frac{2}{-6} \Rightarrow p = -\frac{1}{3}$$

14. Given, $\vec{a} = \hat{i}+\hat{j}+\hat{k}$

Now, unit vector in the direction of $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{(1)^2+(1)^2+(1)^2}} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$$

Hence, cosine of angle which the given vector makes with Y-axis, is $\dfrac{1}{\sqrt{3}}$.

15. Similar as Question 14. $\left[\textbf{Ans. } \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)\right]$

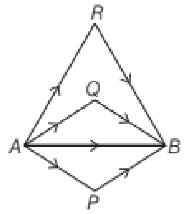
16. Similar as Example 11. $\left[\textbf{Ans. } \dfrac{-2}{\sqrt{30}}, \dfrac{1}{\sqrt{30}}, \dfrac{-5}{\sqrt{30}}\right]$

17. Applying triangle law of addition of vectors in triangles APB, AQB and ARB, we get

$$\overrightarrow{AP}+\overrightarrow{PB} = \overrightarrow{AB}, \ \overrightarrow{AQ}+\overrightarrow{QB} = \overrightarrow{AB}$$

and $\overrightarrow{AR}+\overrightarrow{RB} = \overrightarrow{AB}$

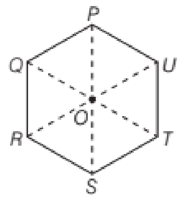
On adding all these, we get



$$\overrightarrow{AP}+\overrightarrow{PB}+\overrightarrow{AQ}+\overrightarrow{QB}+\overrightarrow{AR}+\overrightarrow{RB} = 3\overrightarrow{AB}$$

18. We know that the centre of a regular hexagon bisects all the diagonals passing through it.

$$\therefore \quad \overrightarrow{OP} = -\overrightarrow{OS}, \ \overrightarrow{OQ} = -\overrightarrow{OT} \text{ and } \overrightarrow{OU} = -\overrightarrow{OR}$$
$$\Rightarrow \overrightarrow{OP}+\overrightarrow{OS} = \vec{0}, \ \overrightarrow{OQ}+\overrightarrow{OT} = \vec{0} \text{ and } \overrightarrow{OU}+\overrightarrow{OR} = \vec{0}$$



On adding all of them, we get

$$\overrightarrow{OP}+\overrightarrow{OS}+\overrightarrow{OQ}+\overrightarrow{OT}+\overrightarrow{OU}+\overrightarrow{OR} = \vec{0}$$
$$\Rightarrow \overrightarrow{OP}+\overrightarrow{OQ}+\overrightarrow{OR}+\overrightarrow{OS}+\overrightarrow{OT}+\overrightarrow{OU} = \vec{0}$$

Hence proved.

19. Solve as Question 10. $\left[\textbf{Ans. } \dfrac{\hat{j}+6\hat{k}}{\sqrt{37}}\right]$

20. **Hint** Required vector $= \dfrac{5(\vec{a}+\vec{b})}{|\vec{a}+\vec{b}|}$. $\left[\textbf{Ans. } \dfrac{15}{\sqrt{10}}\hat{i}+\dfrac{5}{\sqrt{10}}\hat{j}\right]$

21. The direction ratios of $\hat{i}+\hat{j}+\hat{k}$ are (1, 1, 1).

So, its direction cosines are $\dfrac{1}{\sqrt{1+1+1}}, \dfrac{1}{\sqrt{1+1+1}}, \dfrac{1}{\sqrt{1+1+1}}$

i.e. $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$

Suppose vector $\hat{i}+\hat{j}+\hat{k}$ makes angles α, β, γ with the coordinates axes.

Then, $\cos\alpha = \cos\beta = \cos\gamma = \dfrac{1}{\sqrt{3}}$

Hence, the vector $\hat{i}+\hat{j}+\hat{k}$ is equally inclined to the axes OX, OY and OZ, respectively.

22. We have, $|\vec{r}| = 2\sqrt{3}$

Since, $\vec{r}$ is equally inclined to the three axes. So, direction cosines of $\vec{r}$ will be same, i.e. $l = m = n$

We know that, $l^2+m^2+n^2=1$

$\Rightarrow \quad l^2+l^2+l^2=1$

$\Rightarrow \quad l^2=\frac{1}{3}$

$\Rightarrow \quad l=\pm\left(\frac{1}{\sqrt{3}}\right)$

So, $\quad \hat{r}=\pm\left(\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}\right)$

Now, $\quad \vec{r}=\hat{r}\cdot|\vec{r}| \qquad \left[\because \hat{r}=\frac{\vec{r}}{|\vec{r}|}\right]$

$=\pm\left[\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}\right]2\sqrt{3} \quad [\because |\vec{r}|=2\sqrt{3}]$

$=\pm(2\hat{i}+2\hat{j}+2\hat{k})=\pm 2(\hat{i}+\hat{j}+\hat{k})$

23. Given, $\vec{a}=3\hat{i}-2\hat{j}+\hat{k}, \vec{b}=\hat{i}-3\hat{j}+5\hat{k}$

and $\quad \vec{c}=2\hat{i}+\hat{j}-4\hat{k}$

Now, $|\vec{a}|=\sqrt{(3)^2+(-2)^2+(1)^2}=\sqrt{9+4+1}=\sqrt{14}$,

$|\vec{b}|=\sqrt{(1)^2+(-3)^2+(5)^2}=\sqrt{1+9+25}=\sqrt{35}$

and $\quad |\vec{c}|=\sqrt{(2)^2+(1)^2+(-4)^2}=\sqrt{4+1+16}=\sqrt{21}$

$\because \quad |\vec{a}|^2+|\vec{c}|^2=14+21=35$

$\therefore \quad |\vec{a}|^2+|\vec{c}|^2=|\vec{b}|^2$

Hence, the vectors form a right angled triangle.

24. Given, $|\vec{r}|=14$ and if $\vec{r}=a\hat{i}+b\hat{j}+c\hat{k}$, then $a=2\lambda, b=3\lambda$ and $c=-6\lambda$ for some $\lambda\neq 0$.

$\therefore$ Direction cosines l, m and n are $l=\frac{a}{|\vec{r}|}=\frac{2\lambda}{14}=\frac{\lambda}{7}$,

$m=\frac{b}{|\vec{r}|}=\frac{3\lambda}{14}$ and $n=\frac{c}{|\vec{r}|}=\frac{-6\lambda}{14}=\frac{-3\lambda}{7}$

Also, we know that $l^2+m^2+n^2=1$

$\therefore \quad \frac{\lambda^2}{49}+\frac{9\lambda^2}{196}+\frac{9\lambda^2}{49}=1$

$\Rightarrow \quad \frac{4\lambda^2+9\lambda^2+36\lambda^2}{196}=1$

$\Rightarrow \quad 49\lambda^2=196 \Rightarrow \lambda^2=\frac{196}{49}$

$\lambda^2=4 \Rightarrow \lambda=\pm 2$

So, the direction cosines l, m and n are $\frac{2}{7},\frac{3}{7}$ and $\frac{-6}{7}$.

[$\because \vec{r}$ makes an acute angle with X-axis, so we will take positive value of λ]

$\because \quad \vec{r}=\hat{r}.|\vec{r}|$

$\therefore \quad \vec{r}=(l\hat{i}+m\hat{j}+n\hat{k})\cdot|\vec{r}|=\left(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right)\cdot 14$

$=4\hat{i}+6\hat{j}-12\hat{k}$

Thus, the components of $\vec{r}$ are $4\hat{i}, 6\hat{j}$ and $-12\hat{k}$.

25. We know that if a vector $\vec{a}$ makes angles α, β and γ with $\hat{i},\hat{j}$ and $\hat{k}$ respectively, then

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

It is given that, $\alpha=\frac{\pi}{4}, \beta=\frac{\pi}{3}$ and $\gamma=\theta$ an acute angle.

$\therefore \quad \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$\Rightarrow \quad \cos^2\frac{\pi}{4}+\cos^2\frac{\pi}{3}+\cos^2\theta=1$

$\Rightarrow \quad \frac{1}{2}+\frac{1}{4}+\cos^2\theta=1 \Rightarrow \cos^2\theta=\frac{1}{4}$

$\Rightarrow \quad \cos\theta=\frac{1}{2} \qquad [\because \theta \text{ is an acute angle, } \cos\theta>0]$

$\Rightarrow \quad \theta=\frac{\pi}{3} \Rightarrow \gamma=\frac{\pi}{3}$

Now, $\quad \vec{a}=|\vec{a}|\{(\cos\alpha)\hat{i}+(\cos\beta)\hat{j}+(\cos\gamma)\hat{k}\}$

$\Rightarrow \quad \vec{a}=\left(\cos\frac{\pi}{4}\right)\hat{i}+\left(\cos\frac{\pi}{3}\right)\hat{j}+\left(\cos\frac{\pi}{3}\right)\hat{k}$

$\left[\because |\vec{a}|=1, \alpha=\frac{\pi}{4}, \beta=\frac{\pi}{3} \text{ and } \gamma=\frac{\pi}{3}\right]$

$\Rightarrow \quad \vec{a}=\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{2}\hat{j}+\frac{1}{2}\hat{k}$

Thus, the components of $\vec{a}$ are $\left(\frac{1}{\sqrt{2}}\hat{i},\frac{1}{2}\hat{j},\frac{1}{2}\hat{k}\right)$.

26. Solve as Question 25. $\left[\textbf{Ans. } \theta=\frac{\pi}{3};\left(\frac{1}{2},\frac{1}{\sqrt{2}},\frac{1}{2}\right)\right]$

27. Given, $\vec{a}=\hat{i}+\hat{j}+\hat{k}$,

$\vec{b}=4\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$

Now, $2\vec{a}-\vec{b}+3\vec{c}$

$=2(\hat{i}+\hat{j}+\hat{k})-(4\hat{i}-2\hat{j}+3\hat{k})+3(\hat{i}-2\hat{j}+\hat{k})$

$=2\hat{i}+2\hat{j}+2\hat{k}-4\hat{i}+2\hat{j}-3\hat{k}+3\hat{i}-6\hat{j}+3\hat{k}$

$=\hat{i}-2\hat{j}+2\hat{k}$

Now, we find a unit vector in the direction of vector $2\vec{a}-\vec{b}+3\vec{c}$, which is equal to

$\frac{2\vec{a}-\vec{b}+3\vec{c}}{|2\vec{a}-\vec{b}+3\vec{c}|}=\frac{\hat{i}-2\hat{j}+2\hat{k}}{\sqrt{(1)^2+(-2)^2+(2)^2}}$

$=\frac{\hat{i}-2\hat{j}+2\hat{k}}{\sqrt{1+4+4}}=\frac{\hat{i}-2\hat{j}+2\hat{k}}{\sqrt{9}}$

$=\frac{\hat{i}-2\hat{j}+2\hat{k}}{3}=\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$

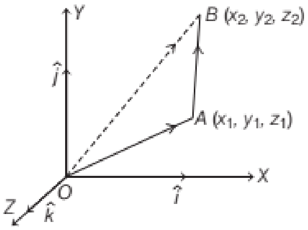
So, vector of magnitude 6 units parallel to the vector

$2\vec{a}-\vec{b}+3\vec{c}=6\left(\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}\right)=2\hat{i}-4\hat{j}+4\hat{k}$

|TOPIC 2|

Vector Joining Two Points

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points on the plane. Then, position vectors of A and B with respect to the origin O are $\overrightarrow{OA} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\overrightarrow{OB} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, respectively.



In ΔOAB, by applying triangle law of addition, we get

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

$$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

and $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

EXAMPLE |1| Find the vector joining the points $P(2, 3, 0)$ and $Q(-1, -2, -4)$ directed from Q to P.

Sol. Given points are $P(2, 3, 0)$ and $Q(-1, -2, -4)$. Then, the position vector of P, $\overrightarrow{OP} = 2\hat{i} + 3\hat{j} + 0\hat{k}$ and position vector of Q, $\overrightarrow{OQ} = -\hat{i} - 2\hat{j} - 4\hat{k}$.

Here, the vector is directed from Q to P. So, Q is the initial point and P is the terminal point.

$\therefore$ Required vector, $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$

$$= (2\hat{i} + 3\hat{j} + 0\hat{k}) - (-\hat{i} - 2\hat{j} - 4\hat{k})$$

$$= (2+1)\hat{i} + (3+2)\hat{j} + (0+4)\hat{k}$$

$$= 3\hat{i} + 5\hat{j} + 4\hat{k}$$

EXAMPLE |2| Find the scalar and vector components of the vector with initial point $(3, 4)$ and terminal point $(-5, 7)$.

Sol. Let the initial point be $A(3, 4)$ and the terminal point be $B(-5, 7)$, then the component form of $\overrightarrow{AB}$ is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -5\hat{i} + 7\hat{j} - (3\hat{i} + 4\hat{j})$$

$$= (-5-3)\hat{i} + (7-4)\hat{j} = -8\hat{i} + 3\hat{j}$$

Hence, the scalar components of $\overrightarrow{AB}$ are -8 and 3 and vector components of AB are $-8\hat{i}$ and $3\hat{j}$.

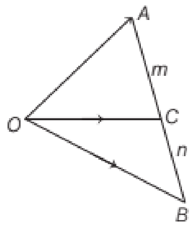
Section Formulae

Let A and B be two points represented by position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively, with respect to the origin.

Then, the line segment joining the points A and B may be divided by a third point C (say) in two ways which are given below.

INTERNAL DIVISION

Let a point C divide the line joining A, B internally in the ratio of $m:n$. Then, the position vector of point C is given by



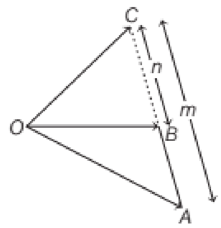
$$\overrightarrow{OC} = \frac{m\,\overrightarrow{OB} + n\,\overrightarrow{OA}}{m+n}$$

where, m and n are positive scalars.

EXTERNAL DIVISION

Let a point C divide the line joining AB externally in the ratio of $m:n$.

Then, the position vector of point C is given by



$$\overrightarrow{OC} = \frac{m\,\overrightarrow{OB} - n\,\overrightarrow{OA}}{m-n}$$

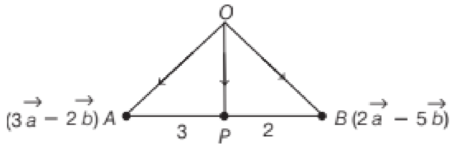
where, m and n are positive scalars.

Note If C is a mid-point, then the position vector of point C is given by $\overrightarrow{OC} = \dfrac{\overrightarrow{OB} + \overrightarrow{OA}}{2}$.

EXAMPLE |3| Find the position vectors of the points which divide the line joining the two points $3\vec{a} - 2\vec{b}$ and $2\vec{a} - 5\vec{b}$ internally and externally in the ratio 3 : 2.

Sol. Let A and B be the given points whose position vectors are $3\vec{a} - 2\vec{b}$ and $2\vec{a} - 5\vec{b}$ respectively, with respect to the origin O, i.e. $\overrightarrow{OA} = 3\vec{a} - 2\vec{b}$ and $\overrightarrow{OB} = 2\vec{a} - 5\vec{b}$.

Let P and Q be the points, which divides the line joining A and B internally and externally respectively, in the ratio 3 : 2.



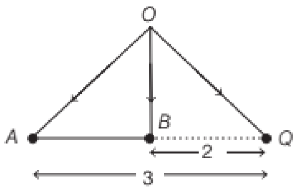
Then, by using section formula of internal division, we get

$$\overrightarrow{OP} = \frac{3\overrightarrow{OB} + 2\overrightarrow{OA}}{3+2} = \frac{3(2\vec{a} - 5\vec{b}) + 2(3\vec{a} - 2\vec{b})}{5}$$

$$= \frac{6\vec{a} - 15\vec{b} + 6\vec{a} - 4\vec{b}}{5} = \frac{12\vec{a} - 19\vec{b}}{5}$$

$$= \frac{12}{5}\vec{a} - \frac{19}{5}\vec{b}$$

Now, by using section formula of external division, we get



$$\overrightarrow{OQ} = \frac{3\,\overrightarrow{OB} - 2\,\overrightarrow{OA}}{3-2} = \frac{3(2\vec{a} - 5\vec{b}) - 2(3\vec{a} - 2\vec{b})}{1}$$

$$= \frac{6\vec{a} - 15\vec{b} - 6\vec{a} + 4\vec{b}}{1} = -11\vec{b}$$

TOPIC PRACTICE 2

OBJECTIVE TYPE QUESTIONS

1. The vector having initial and terminal points as (2, 5, 0) and (−3, 7, 4), respectively is **[NCERT Exemplar]**

(a) $-\hat{i} + 12\hat{j} + 4\hat{k}$ (b) $5\hat{i} + 2\hat{j} - 4\hat{k}$
(c) $-5\hat{i} + 2\hat{j} + 4\hat{k}$ (d) $\hat{i} + \hat{j} + \hat{k}$

2. $2\hat{i} + 4\hat{k}$, $5\hat{i} + 3\sqrt{3}\hat{j} + 4\hat{k}$, $-2\sqrt{3}\,\hat{j} + \hat{k}$ and $2\hat{i} + \hat{k}$ are the position vectors of points A, B, C and D respectively, then $\overrightarrow{CD}$ is equal to

(a) $\frac{2}{3}\overrightarrow{AB}$ (b) $\frac{1}{3}\overrightarrow{AB}$
(c) $5\overrightarrow{AB}$ (d) None of these

3. The position vector of the point which divides the join of points with position vectors $\vec{a} + \vec{b}$ and $2\vec{a} - \vec{b}$ in the ratio 1 : 2 is **[NCERT Exemplar]**

(a) $\frac{3\vec{a} + 2\vec{b}}{3}$ (b) $\vec{a}$
(c) $\frac{5\vec{a} - \vec{b}}{3}$ (d) $\frac{4\vec{a} + \vec{b}}{3}$

4. The ratio in which $\hat{i} + 2\hat{j} + 3\hat{k}$ divides the join of $-2\hat{i} + 3\hat{j} + 5\hat{k}$ and $7\hat{i} - \hat{k}$ is

(a) 1 : 2 (b) 1 : 4 (c) 2 : 3 (d) 3 : 4

VERY SHORT ANSWER Type Questions

5 If P (1, 5, 4) and Q (4, 1, − 2), then find the direction ratios of $\overrightarrow{PQ}$.

6 Find the direction cosines of the vector joining the points $A(1, 2, -3)$ and $B(-1, -2, 1)$ directed from B to A. **[All India 2016C]**

7 Find the scalar components of $\overrightarrow{AB}$ with initial point A (2, 1) and terminal point B (− 5, 7). **[All India 2012]**

8 If $\vec{a}$ and $\vec{b}$ denote the position vectors of points A and B respectively and C is a point on AB such that $AC = 2CB$, then write the position vector of C. **[Delhi 2016C]**

9 A and B are two points with position vectors $2\vec{a} - 3\vec{b}$ and $6\vec{b} - \vec{a}$, respectively.
Write the position vector of a point P which divides the line segment AB internally in the ratio 1 : 2. **[All India 2013]**

10 L and M are two points with position vectors $2\vec{a} - \vec{b}$ and $\vec{a} + 2\vec{b}$, respectively.
Write the position vector of a point N which divides the line segment LM in the ratio 2 : 1 externally. **[All India 2013]**

11 Find the position vector of a point which divides the join of points with position vectors $\vec{a}-2\vec{b}$ and $2\vec{a}+\vec{b}$ externally in the ratio 2 : 1.

[Delhi 2016]

SHORT ANSWER Type I Questions

12 If $\overrightarrow{PO}+\overrightarrow{OQ}=\overrightarrow{QO}+\overrightarrow{OR}$, then show that the points P, Q and R are collinear.

13 Find the scalar and vector component of $\overrightarrow{QP}$ with initial point $Q(2, 3, 5)$ and terminal point $P(7,1, 5)$. **[NCERT Exemplar]**

SHORT ANSWER Type II Questions

14 If $\vec{a}$ and $\vec{b}$ are the position vectors of $\vec{A}$ and $\vec{B}$ respectively, then find the position vector of a point $\vec{C}$ in $\overrightarrow{BA}$ produced such that $\overrightarrow{BC}=1.5\,\overrightarrow{BA}$. Also, it is shown by graphically.

15 Find the position vector of a point C which divides the line segment joining A and B, whose position vectors are $2\vec{a}+\vec{b}$ and $\vec{a}-3\vec{b}$, externally in the ratio 1 : 2. Also, show that A is the mid-point of the line segment BC.

16 Points L, M and N divide the side BC, CA and AB of ΔABC in the ratio 1 : 4, 3 : 2 and 3 : 7, respectively. Prove that $\overrightarrow{AL}+\overrightarrow{BM}+\overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$, where K divides AB in the ratio 1 : 3.

17 If the position vectors of the points A, B, C and D are $2\hat{i}+4\hat{k}$, $5\hat{i}+3\sqrt{3}\hat{j}+4\hat{k}$, $-2\sqrt{3}\hat{j}+\hat{k}$ and $2\hat{i}+\hat{k}$ respectively, then prove that $\overrightarrow{CD}$ is parallel to $\overrightarrow{AB}$ and $\overrightarrow{CD}=\frac{2}{3}\overrightarrow{AB}$.

18 Show that the points $A(6,-7,0)$, $B(16,-19,-4)$, $C(0,3,-6)$ and $D(2,-5,10)$ are such that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ intersect at the point $P(1,-1,2)$.

19 Using vectors show that the points $A(-2,3,5)$, $B(7,0,-1)$, $C(-3,-2,-5)$ and $D(3,4,7)$ are such that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ intersect at the point $P(1, 2, 3)$.

HINTS & SOLUTIONS

1. (c) Required vector

$$=(-3-2)\,\hat{i}+(7-5)\,\hat{j}+(4-0)\hat{k}=-5\hat{i}+2\hat{j}+4\hat{k}$$

2. (a) We know that,

$$\overrightarrow{AB}=\text{Position vector of } B-\text{Position vector of } A$$
$$=(5\hat{i}+3\sqrt{3}\,\hat{j}+4\hat{k})-(2\hat{i}+4\hat{k})$$
$$=3\hat{i}+3\sqrt{3}\,\hat{j}=3(\hat{i}+\sqrt{3}\,\hat{j}) \quad \text{...(i)}$$

and $\overrightarrow{CD}$ = Position vector of D − Position vectors of C

$$=(2\hat{i}+\hat{k})-(-2\sqrt{3}\,\hat{j}+\hat{k})$$
$$=2\hat{i}+2\sqrt{3}\,\hat{j}=2(\hat{i}+\sqrt{3})\,\hat{j} \quad \text{...(ii)}$$

So, from Eqs. (i) and (ii), we get

$$\overrightarrow{CD}=\frac{2}{3}\overrightarrow{AB}$$

3. (d) Applying section formula the position vector of the required point is

$$\frac{2(\vec{a}+\vec{b})+1(2\vec{a}-\vec{b})}{2+1}=\frac{4\vec{a}+\vec{b}}{3}$$

4. (a) Let the required ratio be $\lambda:1$, then applying section formula, we get

$$\hat{i}+2\hat{j}+3\hat{k}=\frac{(-2\hat{i}+3\hat{j}+5\hat{k})+\lambda\,(7\hat{i}-\hat{k})}{\lambda+1}$$

$$\Rightarrow \quad \hat{i}+2\hat{j}+3\hat{k}=\left(\frac{7\lambda-2}{\lambda+1}\right)\hat{i}+\frac{3}{\lambda+1}\,\hat{j}+\frac{(5-\lambda)}{\lambda+1}\,\hat{k}$$

On equating the coefficient of $\hat{j}$, we get

$$2=\frac{3}{\lambda+1}\Rightarrow \lambda+1=\frac{3}{2}\Rightarrow \lambda=\frac{3}{2}-1=\frac{1}{2}$$

5. Let O be the origin of reference.

Then, $\overrightarrow{OP}=\hat{i}+5\hat{j}+4\hat{k}$ and $\overrightarrow{OQ}=4\hat{i}+\hat{j}-2\hat{k}$

Now, $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$

$$=4\hat{i}+\hat{j}-2\hat{k}-\hat{i}-5\hat{j}-4\hat{k}$$
$$=3\hat{i}-4\hat{j}-6\hat{k}$$

So, the direction ratios of $\overrightarrow{PQ}$ are 3, − 4, − 6.

6. **Hint** $\overrightarrow{BA}=2\hat{i}+4\hat{j}-4\hat{k}$ ∴ DR's of $\overrightarrow{BA}$ are 2, 4, −4.

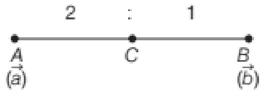
∴ DC's of $\overrightarrow{BA}$ are

$$\frac{2}{\sqrt{2^2+4^2+(-4)^2}},\ \frac{4}{\sqrt{2^2+4^2+(-4)^2}},\ \frac{-4}{\sqrt{2^2+4^2+(-4)^2}}$$

Ans. $\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}$

7. Similar as Example 2. [**Ans.** –7, 6]

8. **Hint**



$$\left[\textbf{Ans. } \text{Position vector of } C = \frac{2\vec{b}+\vec{a}}{3}\right]$$

9. Similar as Example 3. [**Ans.** $\vec{a}$]

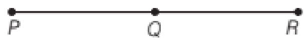
10. Similar as Example 3. [**Ans.** $5\vec{b}$]

11. Similar as Example 3. [**Ans.** $3\vec{a}+4\vec{b}$]

12. We have, $\overrightarrow{PO}+\overrightarrow{OQ}=\overrightarrow{QO}+\overrightarrow{OR}$

$\Rightarrow \quad \overrightarrow{OQ}-\overrightarrow{OP}=\overrightarrow{OR}-\overrightarrow{OQ}$

$\Rightarrow \quad \overrightarrow{PQ}=\overrightarrow{QR}$



As Q is a common point for both vectors.
Hence, the points P, Q and R are collinear.

13. Similar as Example 2. [**Ans.** $(5,-2,0);(5\hat{i},-2\hat{j},0\hat{k})$]

14. Given, $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OB}=\vec{b}$

Now, $\overrightarrow{BA}=\overrightarrow{OA}-\overrightarrow{OB}=\vec{a}-\vec{b}$

and $1.5\,\overrightarrow{BA}=1.5(\vec{a}-\vec{b})$

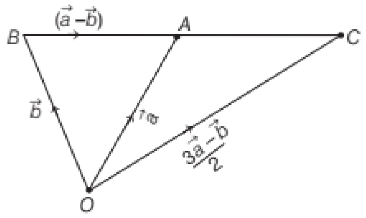
Since, $\overrightarrow{BC}=1.5\,\overrightarrow{BA}=1.5(\vec{a}-\vec{b})$

$\therefore \quad \overrightarrow{OC}-\overrightarrow{OB}=1.5\vec{a}-1.5\vec{b}$

$\overrightarrow{OC}=1.5\vec{a}-1.5\vec{b}+\vec{b}$ $\quad [\because \overrightarrow{OB}=\vec{b}]$

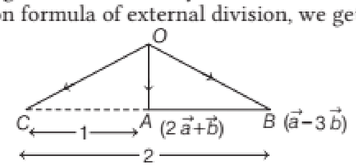
$=1.5\vec{a}-0.5\vec{b}=\dfrac{3\vec{a}-\vec{b}}{2}$

Graphically, explanation of the above solution is shown below



15. Given, $\overrightarrow{OA}=2\vec{a}+\vec{b}$ and $\overrightarrow{OB}=\vec{a}-3\vec{b}$ $\quad$ ...(i)

Also, it is given that C is the point which divides the line joining A and B externally in the ratio 1:2. Then, by using section formula of external division, we get



$$\overrightarrow{OC}=\frac{2\overrightarrow{OA}-\overrightarrow{OB}}{2-1}$$

$\Rightarrow \quad \overrightarrow{OC}=\dfrac{2(2\vec{a}+\vec{b})-1(\vec{a}-3\vec{b})}{1}$ $\quad$ [from Eq. (i)]

$=4\vec{a}+2\vec{b}-\vec{a}+3\vec{b}=3\vec{a}+5\vec{b}$ $\quad$ ...(ii)

Now, we have to show that A is the mid-point of BC

i.e. to show $\overrightarrow{OA}=\dfrac{\overrightarrow{OB}+\overrightarrow{OC}}{2}$

Consider, $\dfrac{\overrightarrow{OB}+\overrightarrow{OC}}{2}=\dfrac{\vec{a}-3\vec{b}+3\vec{a}+5\vec{b}}{2}$

[from Eqs. (i) and (ii)]

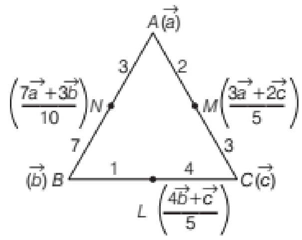
$=\dfrac{4\vec{a}+2\vec{b}}{2}=2\vec{a}+\vec{b}=\overrightarrow{OA}$ $\quad$ [from Eq. (i)]

Thus, $\dfrac{\overrightarrow{OB}+\overrightarrow{OC}}{2}=\overrightarrow{OA}$

Hence, A is mid-point of line segment BC.

16. Let $\vec{a},\vec{b}$ and $\vec{c}$ be the position vectors of the vertices A, B and C of ΔABC, then the position vectors of L, M and N are $\dfrac{4\vec{b}+\vec{c}}{5}$, $\dfrac{3\vec{a}+2\vec{c}}{5}$ and $\dfrac{7\vec{a}+3\vec{b}}{10}$, respectively.

Also, K divides AB in the ratio 1 : 3. Therefore, the position vector of K is $\dfrac{\vec{b}+3\vec{a}}{4}$.



Now, $\overrightarrow{AL}+\overrightarrow{BM}+\overrightarrow{CN}$

$=\dfrac{4\vec{b}+\vec{c}}{5}-\vec{a}+\dfrac{3\vec{a}+2\vec{c}}{5}-\vec{b}+\dfrac{7\vec{a}+3\vec{b}}{10}-\vec{c}$

$=\dfrac{3\vec{a}+\vec{b}-4\vec{c}}{10}$

$=\dfrac{4}{10}\overrightarrow{CK}$

Hence, $\overrightarrow{AL}+\overrightarrow{BM}+\overrightarrow{CN}$ is parallel to $\overrightarrow{CK}$.

17. We have,

$\overrightarrow{AB}=$ Position vector of $B-$ Position vector of A

$=(5\hat{i}+3\sqrt{3}\hat{j}+4\hat{k})-(2\hat{i}+4\hat{k})$

$=3\hat{i}+3\sqrt{3}\hat{j}+0\hat{k}$

$=3(\hat{i}+\sqrt{3}\hat{j}+0\hat{k})$

and $\overrightarrow{CD}$ = Position vector of D – Position vector of C

$$= (2\hat{i} + \hat{k}) - (-2\sqrt{3}\hat{j} + \hat{k})$$
$$= 2\hat{i} + 2\sqrt{3}\hat{j} + 0\hat{k} = 2(\hat{i} + \sqrt{3}\hat{j} + 0\hat{k})$$
$$= \frac{2}{3}(3\hat{i} + 3\sqrt{3}\hat{j} + 0\hat{k}) = \frac{2}{3}\overrightarrow{AB}$$

Hence, $\overrightarrow{CD}$ is parallel to $\overrightarrow{AB}$ and $\overrightarrow{CD} = \frac{2}{3}\overrightarrow{AB}$.

18. We have, $\overrightarrow{AP}$ = Position vector of P – Position vector of A

$$= (\hat{i} - \hat{j} + 2\hat{k}) - (6\hat{i} - 7\hat{j} + 0\hat{k})$$
$$= -5\hat{i} + 6\hat{j} + 2\hat{k}$$

and $\overrightarrow{PB}$ = Position vector of B – Position vector of P

$$= (16\hat{i} - 19\hat{j} - 4\hat{k}) - (\hat{i} - \hat{j} + 2\hat{k})$$
$$= 15\hat{i} - 18\hat{j} - 6\hat{k}$$
$$= -3(-5\hat{i} + 6\hat{j} + 2\hat{k})$$

Clearly, $\overrightarrow{PB} = -3\overrightarrow{AP}$

So, the vectors $\overrightarrow{AP}$ and $\overrightarrow{PB}$ are collinear.

But P is a common point to $\overrightarrow{AP}$ and $\overrightarrow{PB}$.

Hence, P, A and B are collinear points.

Now, $\overrightarrow{CP} = (\hat{i} - \hat{j} + 2\hat{k}) - (0\hat{i} + 3\hat{j} - 6\hat{k}) = \hat{i} - 4\hat{j} + 8\hat{k}$

and $\overrightarrow{PD} = (2\hat{i} - 5\hat{j} + 10\hat{k}) - (\hat{i} - \hat{j} + 2\hat{k}) = \hat{i} - 4\hat{j} + 8\hat{k}$

Clearly, $\overrightarrow{CP} = \overrightarrow{PD}$.

But, P is a common point to $\overrightarrow{CP}$ and $\overrightarrow{PD}$.

Hence, C, P and D are collinear points.

Thus, A, B, C, D and P are points such that A, P, B and C, P, D are two sets of collinear points.

Hence, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ intersect at the point P.

19. Solve as Question 18.

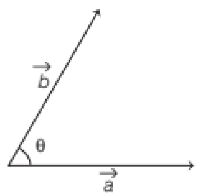
|TOPIC 3|

Scalar Product of Two Vectors

Multiplication or product of two vectors is defined in two ways, namely **scalar (or dot) product** where the result is a scalar and **vector (or cross) product** where the result is a vector.

SCALAR (OR DOT) PRODUCT OF TWO VECTORS

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors inclined at an angle θ. Then, the scalar product or dot product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \cdot \vec{b}$ and defined as



$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta, 0 \le \theta \le \pi$$

or $$\vec{a} \cdot \vec{b} = ab\cos\theta, 0 \le \theta \le \pi$$

where $a = |\vec{a}|$ and $b = |\vec{b}|$.

Note If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined and in this case we define, $\vec{a} \cdot \vec{b} = 0$.

IMPORTANT RESULTS ON SCALAR PRODUCT OF TWO VECTORS

(i) The scalar product of two vectors (i.e. $\vec{a} \cdot \vec{b}$) is a real number.

(ii) Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors, then $\vec{a} \cdot \vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other. i.e. $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$.

(iii) If $\theta = 0$, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$. Also, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$.

(iv) If $\theta = \pi$, then $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$.

Also, $\vec{a} \cdot (-\vec{a}) = -|\vec{a}|^2$.

(v) For mutually perpendicular unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$, $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$.

(vi) The angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \text{ or } \theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right).$$

EXAMPLE |1| Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitude 2 and 1 respectively, such that $\vec{a}\cdot\vec{b} = \sqrt{3}$.

Sol. Given, $|\vec{a}| = 2, |\vec{b}| = 1$ and $\vec{a}\cdot\vec{b} = \sqrt{3}$.

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Then, $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \cos\theta = \dfrac{\sqrt{3}}{2\times 1}$ [given]

$\Rightarrow \cos\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \cos\theta = \cos\dfrac{\pi}{6}$ $\left[\because \cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}\right]$

$\Rightarrow \theta = \dfrac{\pi}{6}$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{6}$.

Properties of Scalar Product of Two Vectors

(i) Scalar product of two vectors is **commutative**, i.e. for vectors $\vec{a}$ and $\vec{b}$, we have $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

(ii) Scalar product of vectors is **distributive over addition**, i.e. for vectors $\vec{a}, \vec{b}$ and $\vec{c}$, we have

$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$.

(iii) If m is any scalar and $\vec{a}$ and $\vec{b}$ be two non-zero vectors. Then, $(m\vec{a})\cdot(\vec{b}) = m(\vec{a}\cdot\vec{b}) = (\vec{a})\cdot(m\vec{b})$

(iv) **Scalar product of two vectors in component form.**

Suppose two vectors $\vec{a}$ and $\vec{b}$ are given in component form, say

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then, their scalar product is given by

$$\vec{a}\cdot\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$
$$= a_1b_1 + a_2b_2 + a_3b_3$$

EXAMPLE |2| Find $|\vec{a}|$ and $|\vec{b}|$, if $|\vec{a}| = 2|\vec{b}|$ and $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 12$. **[Delhi 2020]**

Sol. Given $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 12$ and $|\vec{a}| = 2|\vec{b}|$

$\Rightarrow \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} - \vec{b}\cdot\vec{b} = 12$

$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 12$

$\Rightarrow (2|\vec{b}|)^2 - |\vec{b}|^2 = 12$ [given, $|\vec{a}| = 2|\vec{b}|$]

$\Rightarrow 4|\vec{b}|^2 - |\vec{b}|^2 = 12 \Rightarrow 3|\vec{b}|^2 = 12$

$\Rightarrow |\vec{b}|^2 = 4 \Rightarrow |\vec{b}| = 2$

$\therefore |\vec{a}| = 2|\vec{b}| = 2(2) = 4$

EXAMPLE |3| Find $(\vec{a}+2\vec{b})\cdot(3\vec{a}-\vec{b})$, if $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$ and $\vec{b} = 3\hat{i}+2\hat{j}-\hat{k}$.

Sol. We have, $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$ and $\vec{b} = 3\hat{i}+2\hat{j}-\hat{k}$

Clearly, $\vec{a}+2\vec{b} = (\hat{i}+\hat{j}+2\hat{k}) + 2(3\hat{i}+2\hat{j}-\hat{k})$
$= \hat{i}+\hat{j}+2\hat{k}+6\hat{i}+4\hat{j}-2\hat{k}$
$= 7\hat{i}+5\hat{j}+0\hat{k}$

and $3\vec{a}-\vec{b} = 3(\hat{i}+\hat{j}+2\hat{k}) - (3\hat{i}+2\hat{j}-\hat{k})$
$= (3\hat{i}+3\hat{j}+6\hat{k}) - (3\hat{i}+2\hat{j}-\hat{k})$
$= 0\hat{i}+\hat{j}+7\hat{k}$

Now, $(\vec{a}+2\vec{b})\cdot(3\vec{a}-\vec{b})$
$= (7\hat{i}+5\hat{j}+0\hat{k})\cdot(0\hat{i}+\hat{j}+7\hat{k})$
$= 7(0)+5(1)+0(7) = 0+5+0 = 5$

EXAMPLE |4| Find angle θ between the vectors $\vec{a} = \hat{i}+\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}+\hat{k}$. **[NCERT]**

Sol. Given, $\vec{a} = \hat{i}+\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}+\hat{k}$

Clearly, $|\vec{a}| = |\hat{i}+\hat{j}-\hat{k}|$
$= \sqrt{1^2+1^2+(-1)^2} = \sqrt{1+1+1} = \sqrt{3}$

and $|\vec{b}| = |\hat{i}-\hat{j}+\hat{k}|$
$= \sqrt{(1)^2+(-1)^2+(1)^2} = \sqrt{1+1+1} = \sqrt{3}$

Also, $\vec{a}\cdot\vec{b} = (\hat{i}+\hat{j}-\hat{k})\cdot(\hat{i}-\hat{j}+\hat{k})$
$= (1)(1)+(1)(-1)+(-1)(1)$
$= 1-1-1 = -1$

Now, let the angle between two vectors $\vec{a}$ and $\vec{b}$ be θ.

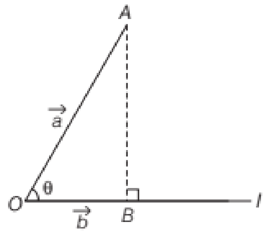
Then, $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{-1}{\sqrt{3}\sqrt{3}}$

$\Rightarrow \cos\theta = -\dfrac{1}{3} \Rightarrow \theta = \cos^{-1}\left(\dfrac{-1}{3}\right)$

PROJECTION OF A VECTOR ON THE LINE

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively and let θ be the angle made by $\vec{a}$ with directed line l in the anti-clockwise direction.

Then, the projection of $\overrightarrow{OA}$ on the line l is $\overrightarrow{OB}$, which is given by $|\overrightarrow{OA}|\cos\theta$ and the direction of $\vec{b}$, called **projection vector**, being the same (or opposite) to that of the line l, depending upon whether $\cos\theta$ is positive or negative.



Note $\vec{b} = |\overrightarrow{OA}|\cos\theta \cdot \hat{b} = |\vec{a}|\cos\theta \cdot \hat{b}$

Some Results on Projection of a Vector

(i) If $\hat{p}$ is the unit vector along a line l, then the projection of a vector $\vec{a}$ on the line l is given by $\vec{a}\cdot\hat{p}$.

(ii) Projection of vector $\vec{a}$ on $\vec{b}$ is given by $\vec{a}\cdot\hat{b}$ or $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ and projection of vector $\vec{b}$ on $\vec{a}$ is given by $\vec{b}\cdot\hat{a}$ or $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

(iii) If $\theta = 0$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{AB}$ itself and if $\theta = \pi$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{BA}$.

(iv) If $\theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{3\pi}{2}$, then the projection vector of $\overrightarrow{AB}$ will be zero vector.

(v) Let α, β and γ be the direction angles of vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then its direction cosines may be given as

$$\cos\alpha = \frac{\vec{a}\cdot\hat{i}}{|\vec{a}|\,|\hat{i}|} = \frac{a_1}{|\vec{a}|},\quad \cos\beta = \frac{\vec{a}\cdot\hat{j}}{|\vec{a}|\,|\hat{j}|} = \frac{a_2}{|\vec{a}|}$$

and $\cos\gamma = \dfrac{\vec{a}\cdot\hat{k}}{|\vec{a}|\,|\hat{k}|} = \dfrac{a_3}{|\vec{a}|}$.

Here, $|\vec{a}|\cos\alpha$, $|\vec{a}|\cos\beta$ and $|\vec{a}|\cos\gamma$ are respectively, the projection of $\vec{a}$ along OX, OY and OZ, i.e. the scalar components a_1, a_2 and a_3 of the vector $\vec{a}$ are precisely the projections of $\vec{a}$ along X, Y and Z-axes, respectively.

Note if $\hat{a}$ is a unit vector, then it may be expressed as
$$\hat{a} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}.$$

EXAMPLE |5| If $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$.

Sol. We have, $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

Clearly, $\vec{a}\cdot\vec{b} = (7\hat{i} + \hat{j} - 4\hat{k})\cdot(2\hat{i} + 6\hat{j} + 3\hat{k})$

$= 7(2) + 1(6) - 4(3) = 14 + 6 - 12 = 8$

and $|\vec{b}| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{4 + 36 + 9} = \sqrt{49} = 7$

Now, projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \dfrac{8}{7}$

TOPIC PRACTICE 3

OBJECTIVE TYPE QUESTIONS

1 The angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitude $\sqrt{3}$ and 4, respectively and $\vec{a}\cdot\vec{b} = 2\sqrt{3}$ is **[NCERT Exemplar]**

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{2}$ (d) $\dfrac{5\pi}{2}$

2. The angle between the vectors $\hat{i} - \hat{j}$ and $\hat{j} - \hat{k}$ is **[NCERT Exemplar]**

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{2\pi}{3}$ (c) $-\dfrac{\pi}{3}$ (d) $\dfrac{5\pi}{6}$

3. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\cdot\vec{b} \ge 0$ only when **[NCERT]**

(a) $0 < \theta < \dfrac{\pi}{2}$ (b) $0 \le \theta \le \dfrac{\pi}{2}$

(c) $0 < \theta < \pi$ (d) $0 \le \theta \le \pi$

4. The value of λ for which the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ are orthogonal. os **[NCERT Exemplar]**

(a) 0 (b) 1 (c) $\dfrac{3}{2}$ (d) $\dfrac{-5}{2}$

5. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a}\cdot\vec{b} = -\dfrac{1}{2}$. Then, $|\vec{a} + \vec{b}|$ is

(a) 2 (b) 0

(c) 1 (d) None of these

VERY SHORT ANSWER Type Questions

6 Write the angle between vectors $\vec{a}$ and $\vec{b}$ with magnitude $\sqrt{3}$ and 2 respectively, having $\vec{a}\cdot\vec{b}=\sqrt{6}$. **[All India 2011]**

7 If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. **[Delhi 2014]**

8 If $\hat{a}$ is a unit vector and $(\vec{x}-\hat{a})\cdot(\vec{x}+\hat{a})=8$, then find $|\vec{x}|$. **[NCERT Exemplar]**

9 If $\vec{a}$ and $\vec{b}$ are two vectors, such that $|\vec{a}|=2$, $|\vec{b}|=1$ and $\vec{a}\cdot\vec{b}=1$, then find $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$.

10 Write the value of λ, so that the vectors $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ are perpendicular to each other. **[Delhi 2013C]**

11 Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$. **[CBSE 2018]**

12 Find the angle between the vectors $\hat{i}-\hat{j}$ and $\hat{j}-\hat{k}$. **[All India 2015]**

13 Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $2\hat{i}-3\hat{j}+6\hat{k}$.

14 Write the projection of vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$. **[All India 2011]**

SHORT ANSWER Type I Questions

15 If $\vec{a}$ and $\vec{b}$ are two unit vectors, then find the angle between $\vec{a}$ and $\vec{b}$, given that $\sqrt{3}\vec{a}-\vec{b}$ is a unit vector.

16 Find $|\vec{a}-\vec{b}|$, if two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=2$, $|\vec{b}|=3$ and $\vec{a}\cdot\vec{b}=4$.

17 Prove that for any two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}\cdot\vec{b}|\le|\vec{a}||\vec{b}|$.

18 If $\vec{a}=5\hat{i}-\hat{j}-3\hat{k}$ and $\vec{b}=\hat{i}+3\hat{j}-5\hat{k}$, then show that the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are perpendicular.

19 If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

20 If $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+\hat{j}-2\hat{k}$ and $\vec{c}=\hat{i}+3\hat{j}-\hat{k}$, then find λ such that $\vec{a}$ is perpendicular to $\lambda\vec{b}+\vec{c}$. **[NCERT Exemplar]**

SHORT ANSWER Type II Questions

21 If the dot products of a vector with vectors $3\hat{i}-5\hat{k}$, $2\hat{i}+7\hat{j}$ and $\hat{i}+\hat{j}+\hat{k}$ are respectively –1, 6 and 5, then find the vector.

22 Vectors $\vec{a},\vec{b}$ and $\vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=3$, $|\vec{b}|=5$ and $|\vec{c}|=7$. Then, find the angle between $\vec{a}$ and $\vec{b}$. **[Delhi 2014]**

23 If $\vec{a},\vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

24 The scalar product of the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b}+\vec{c}$. **[All India 2014]**

25 If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ, then prove that $\sin\frac{\theta}{2}=\frac{1}{2}|\hat{a}-\hat{b}|$.

26 Find the values of x for which the angle between the vectors $\vec{a}=2x^2\hat{i}+4x\hat{j}+\hat{k}$ and $\vec{b}=7\hat{i}-2\hat{j}+x\hat{k}$ is obtuse.

27 Find the values of c for which the vectors $\vec{a}=(c\log_2 x)\hat{i}-6\hat{j}+3\hat{k}$ and $\vec{b}=(\log_2 x)\hat{i}-2\hat{j}+(2c\log_2 x)\hat{k}$ makes an obtuse angle for any $x\in(0,\infty)$.

28 Find the values of a for which the vectors $\vec{r}=(a^2-4)\hat{i}+2\hat{j}-(a^2-9)\hat{k}$ makes an acute angles with the coordinate axes.

29 Prove that for any two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$.

30 Find the value of p for which the vectors $\vec{a} = 3\hat{i}+2\hat{j}+9\hat{k}$ and $\vec{b} = \hat{i}+p\hat{j}+3\hat{k}$ are

(i) perpendicular. (ii) parallel.

31 If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of equal magnitude, then find the angle between $\vec{a}$ and $(\vec{a}+\vec{b}+\vec{c})$. **[Foreign 2011]**

32 If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of the same magnitude, then prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with the vectors $\vec{a}, \vec{b}$ and $\vec{c}$. Also, find the angle which $\vec{a}+\vec{b}+\vec{c}$ makes with $\vec{a}$ or $\vec{b}$ or $\vec{c}$. **[Delhi 2017, 2013C, 2011]**

33 Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors of magnitudes 3, 4 and 5, respectively. If each one is perpendicular to the sum of the other two vectors, then prove that $|\vec{a}+\vec{b}+\vec{c}| = 5\sqrt{2}$. **[NCERT; Delhi 2013]**

34 If with reference to a right handed system of mutually perpendicular unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$, we have $\vec{\alpha} = 3\hat{i}-\hat{j}$ and $\vec{\beta} = 2\hat{i}+\hat{j}-3\hat{k}$. Express $\vec{\beta}$ in the form of $\vec{\beta} = \vec{\beta}_1+\vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$. **[NCERT; Delhi 2012]**

35 If $\vec{a} = 2\hat{i}-\hat{j}-2\hat{k}$ and $\vec{b} = 7\hat{i}+2\hat{j}-3\hat{k}$, then express $\vec{b}$ in the form of $\vec{b} = \vec{b}_1+\vec{b}_2$, where $\vec{b}_1$ is parallel to $\vec{a}$ and $\vec{b}_2$ is prependicular to $\vec{a}$. **[All India 2017]**

HINTS & SOLUTIONS

1. (b) Here, $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 4$ and $\vec{a}\cdot\vec{b} = 2\sqrt{3}$

We know that,

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \quad 2\sqrt{3} = \sqrt{3}\cdot 4\cdot\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{2\sqrt{3}}{4\sqrt{3}} = \frac{1}{2}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

2. (b) **Hint** Apply the formula $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

3. (b) It is given that $\vec{a}\cdot\vec{b} \geq 0$

and we know that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\therefore \quad \cos\theta \geq 0 \qquad [|\vec{a}| \text{ and } |\vec{b}| \text{ are positive}]$$

$$\Rightarrow \quad 0 \leq \theta \leq \frac{\pi}{2}$$

4. (d) Let the vectors $\vec{a}$ and $\vec{b}$ are orthogonal, *i.e.*, perpendicular, then $\vec{a}\cdot\vec{b} = 0$

$$\Rightarrow \quad (2\hat{i}+\lambda\hat{j}+\hat{k})\cdot(\hat{i}+2\hat{j}+3\hat{k}) = 0$$

$$\Rightarrow \quad 2+2\lambda+3 = 0$$

$$\Rightarrow \quad \lambda = \frac{-5}{2}, \text{ which is the required value of } \lambda.$$

5. (c) Here, we have $|\vec{a}| = |\vec{b}| = 1$

Now, consider $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})^2$

$$= (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

$$= \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b} \qquad [\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}]$$

$$= 1^2+1^2+2\left(-\frac{1}{2}\right)$$

$$= 1+1-1 = 1$$

$$\Rightarrow \quad |\vec{a}+\vec{b}| = 1$$

6. Similar as Example 1. [**Ans.** $\pi/4$]

7. Given, $|\vec{a}| = 1$, $|\vec{b}| = 1$ and $|\vec{a}+\vec{b}| = 1$

Now, consider $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$$= \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2$$

$$[\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} \text{ and } \vec{x}\cdot\vec{x} = |\vec{x}|^2]$$

$$\Rightarrow \quad 1 = 1+2\vec{a}\cdot\vec{b}+1$$

$$[\because |\vec{a}+\vec{b}| = |\vec{a}| = |\vec{b}| = 1, \text{ given}]$$

$$\Rightarrow \quad 2\vec{a}\cdot\vec{b} = -1$$

$$\Rightarrow \quad |\vec{a}||\vec{b}|\cos\theta = -\frac{1}{2} \qquad [\because \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta]$$

$$\Rightarrow \quad \cos\theta = -\frac{1}{2} \qquad [\because |\vec{a}| = |\vec{b}| = 1]$$

$$\Rightarrow \quad \cos\theta = \cos\frac{2\pi}{3} \Rightarrow \theta = \frac{2\pi}{3}$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{2\pi}{3}$.

8. Given, $\hat{a}$ is a unit vector.

$\therefore \quad |\hat{a}| = 1$

Now, consider $(\vec{x} - \hat{a})\cdot(\vec{x} + \hat{a}) = \vec{x}\cdot\vec{x} + \vec{x}\cdot\hat{a} - \hat{a}\cdot\vec{x} - \hat{a}\cdot\hat{a}$

$= |\vec{x}|^2 + \vec{x}\cdot\hat{a} - \vec{x}\cdot\hat{a} - |\hat{a}|^2 \qquad [\because \vec{x}\cdot\hat{a} = \hat{a}\cdot\vec{x}]$

$\Rightarrow \quad (\vec{x} - \hat{a})\cdot(\vec{x} + \hat{a}) = |\vec{x}|^2 - |\hat{a}|^2$

$\Rightarrow \quad 8 = |\vec{x}|^2 - (1)^2 \qquad [\text{given}, (\vec{x} - \hat{a})\cdot(\vec{x} + \hat{a}) = 8]$

$\Rightarrow \quad 8 + 1 = |\vec{x}|^2 \quad \Rightarrow \quad 9 = |\vec{x}|^2$

$\therefore \quad |\vec{x}| = 3 \qquad$ [magnitude is always positive]

9. Given, $|\vec{a}| = 2, |\vec{b}| = 1$ and $\vec{a}\cdot\vec{b} = 1$

Now, $(3\vec{a} - 5\vec{b})\cdot(2\vec{a} + 7\vec{b})$

$= 6|\vec{a}|^2 + 21\vec{a}\cdot\vec{b} - 10\vec{b}\cdot\vec{a} - 35|\vec{b}|^2$

$= 6|\vec{a}|^2 + 11\vec{a}\cdot\vec{b} - 35|\vec{b}|^2 \qquad [\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}]$

$= 6(2)^2 + 11(1) - 35(1)^2$

$= 6 \times 4 + 11 - 35$

$= 24 + 11 - 35 = 0$

10. Given vectors are $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$.

Since, the vectors are perpendicular.

$\therefore \quad \vec{a}\cdot\vec{b} = 0$

$\Rightarrow \quad (2\hat{i} + \lambda\hat{j} + \hat{k})\cdot(\hat{i} - 2\hat{j} + 3\hat{k}) = 0$

$\Rightarrow \quad 2 - 2\lambda + 3 = 0$

$\therefore \quad \lambda = 5/2$

11. Given, two vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{a}| = |\vec{b}|$, $\vec{a}\cdot\vec{b} = \frac{9}{2}$ and angle between them is 60º.

We know that

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta,$$

where θ is angle between $\vec{a}$ and $\vec{b}$.

$\therefore \quad \frac{9}{2} = |\vec{a}|\cdot|\vec{a}|\cos 60^\circ$

$\Rightarrow \quad \frac{1}{2}\cdot|\vec{a}|^2 = \frac{9}{2} \qquad \left[\because \cos 60^\circ = \frac{1}{2}\right]$

$\Rightarrow \quad |\vec{a}|^2 = 9$

$\Rightarrow \quad |\vec{a}| = 3 \quad$ [$\because$ magnitude cannot be negative]

Thus, $\quad |\vec{a}| = |\vec{a}| = 3$

12. Similar as Example 4. [**Ans.** $2\pi/3$]

13. Similar as Example 5. [**Ans.** 5]

14. Similar as Example 5. [**Ans.** 0]

15. Solve as Question 7. $\left[\textbf{Ans. } \theta = \frac{\pi}{6}\right]$

16. Given, $|\vec{a}| = 2, |\vec{b}| = 3$ and $\vec{a}\cdot\vec{b} = 4$

Now, consider $|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b})^2$

$= (\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b})$

$= \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$

$= |\vec{a}|^2 - \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{b} + |\vec{b}|^2$

$= |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 \qquad [\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}]$

$\therefore |\vec{a} - \vec{b}|^2 = (2)^2 - 2(4) + (3)^2$

$= 4 - 8 + 9 = 5$

$\therefore \quad |\vec{a} - \vec{b}| = \sqrt{5} \qquad$ [taking square root on both sides]

17. When, either $\vec{a} = 0$ or $\vec{b} = 0$, then $|\vec{a}\cdot\vec{b}| = 0 = |\vec{a}||\vec{b}|$.

Thus, inequality holds trivially when either $\vec{a} = 0$ or $\vec{b} = 0$.

When $|\vec{a}| \neq 0 \neq |\vec{b}|$, then $\frac{|\vec{a}\cdot\vec{b}|}{|\vec{a}||\vec{b}|} = |\cos\theta|$

But $\quad |\cos\theta| \le 1 \Rightarrow |\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$

18. We have, $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$

Now, $\vec{a} + \vec{b} = 5\hat{i} - \hat{j} - 3\hat{k} + \hat{i} + 3\hat{j} - 5\hat{k} = 6\hat{i} + 2\hat{j} - 8\hat{k}$

and $\vec{a} - \vec{b} = 5\hat{i} - \hat{j} - 3\hat{k} - \hat{i} - 3\hat{j} + 5\hat{k} = 4\hat{i} - 4\hat{j} + 2\hat{k}$

Now, $(\vec{a} + \vec{b})\cdot(\vec{a} - \vec{b}) = (6\hat{i} + 2\hat{j} - 8\hat{k})\cdot(4\hat{i} - 4\hat{j} + 2\hat{k})$

$= 24 - 8 - 16 = 0$

Hence, $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular vectors.

19. We have, $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$

and $\quad \vec{c} = 3\hat{i} + \hat{j}$

Now, $\vec{a} + \lambda\vec{b} = (2\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + 2\hat{j} + \hat{k})$

$= 2\hat{i} + 2\hat{j} + 3\hat{k} - \lambda\hat{i} + 2\lambda\hat{j} + \lambda\hat{k}$

$= (2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}$

Since, $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$.

$\therefore \quad (\vec{a} + \lambda\vec{b})\cdot\vec{c} = 0$

$\Rightarrow \quad [(2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}]\cdot(3\hat{i} + \hat{j}) = 0$

$\Rightarrow \quad (2 - \lambda)(3) + (2 + 2\lambda)(1) + (3 + \lambda)(0) = 0$

$\Rightarrow \quad 6 - 3\lambda + 2 + 2\lambda = 0$

$\Rightarrow \quad -\lambda + 8 = 0$

$\therefore \quad \lambda = 8$

20. Solve as Question 19.

[**Ans.** $\lambda = -2$]

21. Let $\vec{a}=3\hat{i}-5\hat{k}$, $\vec{b}=2\hat{i}+7\hat{j}$ and $\vec{c}=\hat{i}+\hat{j}+\hat{k}$ be three given vectors. Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ be a vector such that its dot products with $\vec{a},\vec{b}$ and $\vec{c}$ are -1, 6 and 5 respectively, we have

$$\vec{r}\cdot\vec{a}=(x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-5\hat{k})\Rightarrow -1=3x-5z \quad ...(i)$$

$$\vec{r}\cdot\vec{b}=(x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}+7\hat{j})\Rightarrow 6=2x+7y \quad ...(ii)$$

and $\vec{r}\cdot\vec{c}=(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})\Rightarrow 5=x+y+z$ ...(iii)

On solving Eqs. (i), (ii) and (iii), we get

$x=3$, $y=0$ and $z=2$

Hence, the required vector is $\vec{r}=3\hat{i}+2\hat{k}$.

22. Given, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$$\Rightarrow \vec{a}+\vec{b}=-\vec{c}$$

$$\Rightarrow (\vec{a}+\vec{b})^2=(-\vec{c})^2 \quad \text{[squaring on both sides]}$$

$$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=(-\vec{c})\cdot(-\vec{c})$$

$$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=\vec{c}\cdot\vec{c}$$

$$\Rightarrow |\vec{a}|^2+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{b}+|\vec{b}|^2=|\vec{c}|^2 \quad [\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}]$$

$$\Rightarrow |\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2=|\vec{c}|^2$$

$$\Rightarrow |\vec{a}|^2+2|\vec{a}||\vec{b}|\cos\theta+|\vec{b}|^2=|\vec{c}|^2 \quad ...(i)$$

$[\because \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta]$

On putting the values of $|\vec{a}|=3$, $|\vec{b}|=5$ and $|\vec{c}|=7$ in Eq. (i), we get

$$(3)^2+2\times3\times5\times\cos\theta+(5)^2=(7)^2$$

$$\Rightarrow 9+30\cos\theta+25=49$$

$$\Rightarrow 30\cos\theta=49-34$$

$$\Rightarrow \cos\theta=\frac{15}{30}=\frac{1}{2}$$

$$\Rightarrow \cos\theta=\cos\frac{\pi}{3}$$

$$\therefore \theta=\frac{\pi}{3}$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$.

23. Given, $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$ and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ ...(i)

Now, consider $(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{0}\cdot\vec{0}=0$

$$\Rightarrow \vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})+\vec{b}\cdot(\vec{a}+\vec{b}+\vec{c})+\vec{c}\cdot(\vec{a}+\vec{b}+\vec{c})=0$$

$$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c}=0$$

$$\Rightarrow |\vec{a}|^2+\vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{a}+\vec{a}\cdot\vec{b}+|\vec{b}|^2+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{b}\cdot\vec{c}+|\vec{c}|^2=0$$

[dot product is commutative]

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}=0$$

$$\Rightarrow 1+1+1+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow 2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=-3$$

$$\Rightarrow (\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=-\frac{3}{2}$$

24. Given, $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$

and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$

Now, $\vec{b}+\vec{c}=2\hat{i}+4\hat{j}-5\hat{k}+\lambda\hat{i}+2\hat{j}+3\hat{k}$

$$=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$$

$$\therefore |\vec{b}+\vec{c}|=\sqrt{(2+\lambda)^2+(6)^2+(-2)^2}$$

$$=\sqrt{4+\lambda^2+4\lambda+36+4}$$

$$=\sqrt{\lambda^2+4\lambda+44}$$

The unit vector along $\vec{b}+\vec{c}$ is given by

$$\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}} \quad ...(i)$$

Given, scalar product of $(\hat{i}+\hat{j}+\hat{k})$ with above unit vector is 1,

i.e. $$(\hat{i}+\hat{j}+\hat{k})\cdot\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}=1$$

$$\Rightarrow (\hat{i}+\hat{j}+\hat{k})\cdot\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}=1$$

$$\Rightarrow \frac{1(2+\lambda)+1(6)+1(-2)}{\sqrt{\lambda^2+4\lambda+44}}=1$$

$$\Rightarrow \frac{2+\lambda+6-2}{\sqrt{\lambda^2+4\lambda+44}}=1$$

$$\Rightarrow \lambda+6=\sqrt{\lambda^2+4\lambda+44}$$

$$\Rightarrow (\lambda+6)^2=\lambda^2+4\lambda+44 \quad \text{[squaring on both sides]}$$

$$\Rightarrow \lambda^2+36+12\lambda=\lambda^2+4\lambda+44$$

$$\Rightarrow 8\lambda=8 \Rightarrow \lambda=1$$

Hence, the value of λ is 1.

On putting the value of λ in Eq. (i), we get

Unit vector along $\vec{b}+\vec{c}=\dfrac{(2+1)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(1)^2+4(1)+44}}$

$$=\frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{1+4+44}}$$

$$=\frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{49}}$$

$$=\frac{3}{7}\hat{i}+\frac{6}{7}\hat{j}-\frac{2}{7}\hat{k}$$

25. We know that, $|\vec{x}\cdot\vec{x}| = |\vec{x}|^2$

$$\Rightarrow \quad |\hat{a}-\hat{b}|^2 = (\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})$$

$$\Rightarrow \quad |\hat{a}-\hat{b}|^2 = \hat{a}\cdot\hat{a} - \hat{a}\cdot\hat{b} - \hat{b}\cdot\hat{a} + \hat{b}\cdot\hat{b}$$

$$\Rightarrow \quad |\hat{a}-\hat{b}|^2 = |\hat{a}|^2 - 2\hat{a}\cdot\hat{b} + |\hat{b}|^2 \qquad [\because \hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{a}]$$

$$\Rightarrow \quad |\hat{a}-\hat{b}|^2 = |\hat{a}|^2 - 2|\hat{a}||\hat{b}|\cos\theta + |\hat{b}|^2$$

$$[\because \hat{a}\cdot\hat{b} = |\hat{a}||\hat{b}|\cos\theta]$$

$$\Rightarrow \quad |\hat{a}-\hat{b}|^2 = 1 - 2\cdot1\cdot1\cos\theta + 1 \qquad [\text{given } |\hat{a}| = |\hat{b}| = 1]$$

$$\Rightarrow \quad |\hat{a}-\hat{b}|^2 = 2 - 2\cos\theta$$

$$\Rightarrow \quad |\hat{a}-\hat{b}|^2 = 2(1-\cos\theta) = 2\left[2\sin^2\left(\frac{\theta}{2}\right)\right] = 4\sin^2\frac{\theta}{2}$$

$$\Rightarrow \quad \sin^2\frac{\theta}{2} = \frac{1}{4}|\hat{a}-\hat{b}|^2 \Rightarrow \sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}-\hat{b}|$$

26. Let θ be the angle between vectors $\vec{a}$ and $\vec{b}$.

Then, $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

For the angle θ to be obtuse, we must have $\cos\theta < 0$

$$\Rightarrow \quad \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} < 0$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b} < 0 \qquad [\because |\vec{a}|, |\vec{b}| > 0]$$

$$\Rightarrow \quad (2x^2\hat{i} + 4x\hat{j} + \hat{k})\cdot(7\hat{i} - 2\hat{j} + x\hat{k}) < 0$$

$$\Rightarrow \quad 14x^2 - 8x + x < 0 \Rightarrow 7x(2x-1) < 0$$

$$\Rightarrow \quad x(2x-1) < 0 \Rightarrow 0 < x < \frac{1}{2}$$

Hence, the angle between the given vectors is obtuse, if $x \in (0, 1/2)$.

27. Solve as Question 26.
[Ans. $c \in (-4/3, 0)$**]**

28. For vector $\vec{r}$ to be inclined acute angles with the coordinate axes, we must have

$$\vec{r}\cdot\hat{i} > 0, \vec{r}\cdot\hat{j} > 0 \text{ and } \vec{r}\cdot\hat{k} > 0$$

Consider, $\quad \vec{r}\cdot\hat{i} > 0 \text{ and } \vec{r}\cdot\hat{k} > 0 \qquad [\because \vec{r}\cdot\vec{j} = 2 > 0]$

$$\Rightarrow \quad (a^2-4) > 0 \text{ and} -(a^2-9) > 0$$

$$[\because \vec{r}\cdot\hat{i} = a^2 - 4 \text{ and } \vec{r}\cdot\hat{k} = -(a^2-9)]$$

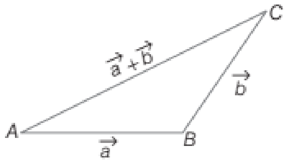
$$\Rightarrow \quad (a-2)(a+2) > 0 \text{ and} (a+3)(a-3) < 0$$

$$\Rightarrow \quad a < -2 \text{ or } a > 2 \text{ and } -3 < a < 3$$

$$\Rightarrow \quad a \in (-3, -2) \cup (2, 3).$$

29. When either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$.

Thus, the inequality holds trivially in this case.



When $|\vec{a}| \neq 0 \neq |\vec{b}|$.

Then, $|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})^2 = (\vec{a} + \vec{b})\cdot(\vec{a} + \vec{b})$

$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$$

$$\Rightarrow \quad |\vec{a} + \vec{b}|^2 \le |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}\cdot\vec{b}| \qquad [\because x \le |x|, \forall\, x \in R]$$

$$\Rightarrow \quad |\vec{a} + \vec{b}|^2 \le |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}| \qquad [\because |\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|]$$

$$\Rightarrow \quad |\vec{a} + \vec{b}|^2 \le (|\vec{a}| + |\vec{b}|)^2$$

$$\Rightarrow \quad |\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$$

30. (i) If vectors $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a}\cdot\vec{b} = 0$

$$\Rightarrow \quad (3\hat{i} + 2\hat{j} + 9\hat{k}).(\hat{i} + p\hat{j} + 3\hat{k}) = 0$$

$$\Rightarrow \quad 3 + 2p + 27 = 0$$

$$\Rightarrow \quad p = -15$$

(ii) We know that, the vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are parallel iff

$$\Rightarrow \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = l$$

So, the given vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$

and $\quad \vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ will be parallel iff

$$\frac{3}{1} = \frac{2}{p} = \frac{9}{3}$$

$$\Rightarrow \quad 3 = \frac{2}{p} \Rightarrow p = \frac{2}{3}$$

31. Given $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular vectors with equal magnitude.

$\therefore$ We have, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$

and $\qquad |\vec{a}| = |\vec{b}| = |\vec{c}| \qquad$ ...(i)

Let θ be the angle between $\vec{a}$ and $(\vec{a} + \vec{b} + \vec{c})$.

Then, $\cos\theta = \dfrac{\vec{a}\cdot(\vec{a} + \vec{b} + \vec{c})}{|\vec{a}||\vec{a} + \vec{b} + \vec{c}|} = \dfrac{\vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}}{|\vec{a}||\vec{a} + \vec{b} + \vec{c}|}$

$$= \frac{|\vec{a}|^2 + 0 + 0}{|\vec{a}||\vec{a} + \vec{b} + \vec{c}|} = \frac{|\vec{a}|}{|\vec{a} + \vec{b} + \vec{c}|} \qquad [\text{from Eq. (i)}]$$

Now, $|\vec{a} + \vec{b} + \vec{c}|^2 = (\vec{a} + \vec{b} + \vec{c})\cdot(\vec{a} + \vec{b} + \vec{c})$

$$= \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2\vec{a}\cdot\vec{b} + 2\vec{b}\cdot\vec{c} + 2\vec{c}\cdot\vec{a}$$

$$\therefore |\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 0 + 0 + 0$$

$$= |\vec{a}|^2 + |\vec{a}|^2 + |\vec{a}|^2 \qquad [\text{from Eq. (i)}]$$

$$= 3|\vec{a}|^2$$

$\therefore \quad \cos\theta = \dfrac{|\vec{a}|}{\sqrt{3}|\vec{a}|} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad \theta = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

32. Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ (say)

Since, $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular vectors.

Therefore, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$ ...(i)

Now, $|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$= \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{c}+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 0+0+0$ [using Eq. (i)]

$= 3\lambda^2$ $[\because |\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda]$

$\Rightarrow |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}\lambda$ ...(ii)

Suppose $\vec{a}+\vec{b}+\vec{c}$ makes angles θ_1, θ_2 and θ_3 with $\vec{a}, \vec{b}$ and $\vec{c}$, respectively.

Then, $\cos\theta_1 = \dfrac{\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} = \dfrac{\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|}$

$\Rightarrow \quad \cos\theta_1 = \dfrac{|\vec{a}|^2}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} = \dfrac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} = \dfrac{\lambda}{\sqrt{3}\lambda} = \dfrac{1}{\sqrt{3}}$

[using Eqs. (i) and (ii)]

$\Rightarrow \quad \theta_1 = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

Similarly, we have

$\theta_2 = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

and $\quad \theta_3 = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

$\therefore \quad \theta_1 = \theta_2 = \theta_3$

Hence, $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with $\vec{a}, \vec{b}$ and $\vec{c}$.

Now, $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = 1$

$\Rightarrow \quad 3\cos^2\theta_1 = 1$

$\Rightarrow \quad \cos^2\theta_1 = \dfrac{1}{3}$

$\Rightarrow \quad \cos\theta_1 = \pm\dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad \theta_1 = \cos^{-1}\left(\pm\dfrac{1}{\sqrt{3}}\right)$

$\Rightarrow \quad \theta_1 = \theta_2 = \theta_3 = \cos^{-1}\left(\pm\dfrac{1}{\sqrt{3}}\right)$

33. We have, $|\vec{a}| = 3, |\vec{b}| = 4$ and $|\vec{c}| = 5$

It is given that,

$\vec{a}\perp(\vec{b}+\vec{c}), \vec{b}\perp(\vec{c}+\vec{a})$ and $\vec{c}\perp(\vec{a}+\vec{b})$

$\Rightarrow \vec{a}\cdot(\vec{b}+\vec{c}) = 0, \vec{b}\cdot(\vec{c}+\vec{a}) = 0$ and $\vec{c}\cdot(\vec{a}+\vec{b}) = 0$

$\Rightarrow \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} = 0, \vec{b}\cdot\vec{c}+\vec{b}\cdot\vec{a} = 0$ and $\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b} = 0$

Adding all these, we get

$2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$

$\Rightarrow \quad \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} = 0$ ...(i)

Now, consider

$|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$

$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2 = 3^2+4^2+5^2+0$ [using Eq. (i)]

$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}$

34. Since, $\vec{\beta}_1$ is parallel to $\vec{\alpha}$.

Therefore, $\vec{\beta}_1 = \lambda\vec{\alpha}$ for some scalar λ. ...(i)

It is given that, $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$

$\Rightarrow \quad \vec{\beta}_2 = \vec{\beta} - \vec{\beta}_1 = \vec{\beta} - \lambda\vec{\alpha}$ ...(ii)

Also, it is given that, $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$, therefore

$\Rightarrow \quad \vec{\beta}_2\cdot\vec{\alpha} = 0$

$\Rightarrow \quad (\vec{\beta}-\lambda\vec{\alpha})\cdot\vec{\alpha} = 0$

$\Rightarrow \quad \vec{\beta}\cdot\vec{\alpha} - \lambda(\vec{\alpha}\cdot\vec{\alpha}) = 0$ [using Eq. (ii)]

$\Rightarrow \quad \lambda = \dfrac{\vec{\beta}\cdot\vec{\alpha}}{\vec{\alpha}\cdot\vec{\alpha}}$

Also, given $\vec{\alpha} = 3\hat{i} - \hat{j}$ and $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$

Now, $\vec{\beta}\cdot\vec{\alpha} = (2\hat{i}+\hat{j}-3\hat{k})\cdot(3\hat{i}-\hat{j}) = 6-1+0 = 5$

and $\quad \vec{\alpha}\cdot\vec{\alpha} = (3\hat{i}-\hat{j})\cdot(3\hat{i}-\hat{j}) = 9+1 = 10$

$\therefore \quad \lambda = \dfrac{\vec{\beta}\cdot\vec{\alpha}}{\vec{\alpha}\cdot\vec{\alpha}} = \dfrac{5}{10} = \dfrac{1}{2}$

From Eq. (i), we get

$\vec{\beta}_1 = \lambda\vec{\alpha} \Rightarrow \vec{\beta}_1 = \dfrac{1}{2}(3\hat{i}-\hat{j}) = \dfrac{3}{2}\hat{i} - \dfrac{1}{2}\hat{j}$

and $\quad \vec{\beta}_2 = \vec{\beta} - \vec{\beta}_1$

$\Rightarrow \quad \vec{\beta}_2 = (2\hat{i}+\hat{j}-3\hat{k}) - \left(\dfrac{3}{2}\hat{i} - \dfrac{1}{2}\hat{j}\right) = \dfrac{1}{2}\hat{i} + \dfrac{3}{2}\hat{j} - 3\hat{k}$

Hence, $\vec{\beta}_1 = \dfrac{3}{2}\hat{i} - \dfrac{1}{2}\hat{j}$ and $\vec{\beta}_2 = \dfrac{1}{2}\hat{i} + \dfrac{3}{2}\hat{j} - 3\hat{k}$

35. Solve as Question 34.

[**Ans.** $b_1 = 4\hat{i} - 2\hat{j} - 4\hat{k}, b_2 = 3\hat{i} + 4\hat{j} + \hat{k}$].

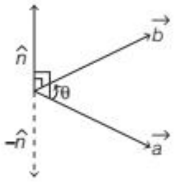
|TOPIC 4|

Vector (or Cross) Product of Two Vectors

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors inclined at an angle θ. Then, the vector (or cross) product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$, read as $\vec{a}$ cross $\vec{b}$ and defined as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$$

where, $\hat{n}$ is a unit vector perpendicular to both vectors $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system.



Also, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \, |\hat{n}|$

$$= ab \sin\theta \qquad [\because |\vec{a}| = a, |\vec{b}| = b \text{ and } |\hat{n}| = 1]$$

Note

(i) Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined as $\vec{0}$ has no direction and in this case $\vec{a} \times \vec{b} = \vec{0}$.

(ii) $\hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

(iii) The relation between dot and cross product is

$$(\vec{a} \cdot \vec{b})^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2.$$

EXAMPLE |1| If vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 3, |\vec{b}| = \dfrac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.

Sol. Given, $|\vec{a}| = 3, |\vec{b}| = \dfrac{2}{3}$.

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Since, $|\vec{a} \times \vec{b}| = 1$

$\therefore \quad |\vec{a}||\vec{b}| \sin\theta = 1$

$\Rightarrow \quad 3 \times \dfrac{2}{3} \sin\theta = 1$

$\Rightarrow \quad 2\sin\theta = 1$

$\Rightarrow \quad \sin\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}$

IMPORTANT RESULTS ON VECTOR PRODUCT OF TWO VECTORS

(i) The vector product of two vectors $(\vec{a} \times \vec{b})$ is a vector.

(ii) Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors. Then, $\vec{a} \times \vec{b} = \vec{0}$ if and only if $\vec{a}$ and $\vec{b}$ are parallel (or **collinear**) to each other, i.e. $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$.

(iii) $\vec{a} \times \vec{a} = \vec{0}$, as $\theta = 0°$ and $\vec{a} \times (-\vec{a}) = \vec{0}$, as $\theta = \pi$.

(iv) If $\theta = \dfrac{\pi}{2}$, then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|$.

(v) For mutually perpendicular unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$,

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

and
$$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j};$$
$$\hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i} \text{ and } \hat{i} \times \hat{k} = -\hat{j}$$

(vi) The angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ in terms of vector products is given by $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$.

(vii) Vectors of magnitude λ normal to the plane of $\vec{a}$ and $\vec{b}$ are given as $\dfrac{\pm \lambda (\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$.

Properties of Vector Product of Two Vectors

(i) Vector product is not **commutative**, for vectors $\vec{a}$ and $\vec{b}$, we have $(\vec{a} \times \vec{b}) = -(\vec{b} \times \vec{a})$

(ii) Vector product is distributive over addition, i.e. for vectors $\vec{a}, \vec{b}$ and $\vec{c}$, we have

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

(iii) If m is any scalar and $\vec{a}$ and $\vec{b}$ be the two non-zero vectors, then $(m\,\vec{a}) \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times (m\,\vec{b})$

EXAMPLE |2| For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, prove that $\vec{a}\times(\vec{b}-\vec{c})=(\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})$.

Sol. We have, LHS $=\vec{a}\times(\vec{b}-\vec{c})=\vec{a}\times[\vec{b}+(-\vec{c})]$

$$=(\vec{a}\times\vec{b})+\vec{a}\times(-\vec{c}) \quad \text{[by distributive law]}$$

$$=(\vec{a}\times\vec{b})+[-(\vec{a}\times\vec{c})] \quad [\because \vec{a}\times(m\vec{c})=m(\vec{a}\times\vec{c})]$$

$$=(\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})=\text{RHS}$$

Note We can use this result as standard result.

EXAMPLE |3| Show that $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ does not always imply that $\vec{b}=\vec{c}$.

Sol. We have, $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$

$$\Rightarrow \quad (\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})=\vec{0}$$

$$\Rightarrow \quad (\vec{a}\times\vec{b})+\{\vec{a}\times(-\vec{c})\}=\vec{0}$$

$$\Rightarrow \quad \vec{a}\times(\vec{b}+(-\vec{c}))=\vec{0}$$

$$\Rightarrow \quad \vec{a}\times(\vec{b}-\vec{c})=\vec{0}$$

$$\Rightarrow \quad \vec{a}=\vec{0} \text{ or } (\vec{b}-\vec{c})=\vec{0} \text{ or } \vec{a}\,||\,(\vec{b}-\vec{c})$$

$$\Rightarrow \quad \vec{a}=\vec{0} \text{ or } \vec{b}=\vec{c} \text{ or } \vec{a}\,||\,(\vec{b}-\vec{c})$$

Hence, $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ does not always imply that $\vec{b}=\vec{c}$.

Vector Product of Two Vectors in Component Form

Suppose, two vectors $\vec{a}$ and $\vec{b}$ are given in component form, say $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then their vector product is given by

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}.$$

EXAMPLE |4| If $\vec{a}=3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+4\hat{k}$, then find $\vec{a}\times\vec{b}$ and $|\vec{a}\times\vec{b}|$.

Sol. Given, $\vec{a}=3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+4\hat{k}$

$$\text{Now, } \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & 2\\ 2 & -2 & 4\end{vmatrix}$$

$$=\hat{i}(4+4)-\hat{j}(12-4)+\hat{k}(-6-2)$$

$$=8\hat{i}-8\hat{j}-8\hat{k}$$

$$\therefore \quad |\vec{a}\times\vec{b}|=\sqrt{8^2+(-8)^2+(-8)^2}=8\sqrt{1+1+1}=8\sqrt{3}$$

EXAMPLE |5| Find a vector of magnitude 3, which is perpendicular to both the vectors $3\hat{i}+\hat{j}-4\hat{k}$ and $6\hat{i}+5\hat{j}-2\hat{k}$.

Sol. Let two given vectors be $\vec{a}=3\hat{i}+\hat{j}-4\hat{k}$ and $\vec{b}=6\hat{i}+5\hat{j}-2\hat{k}$.

Using vector product of two vectors, we get

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & -4\\ 6 & 5 & -2\end{vmatrix}$$

$$=\hat{i}\,\{1(-2)-5(-4)\}-\hat{j}\,\{3(-2)-6(-4)\}+\hat{k}\,\{3(5)-6(1)\}$$

[expanding the determinant along R_1]

$$=\hat{i}(-2+20)-\hat{j}(-6+24)+\hat{k}(15-6)$$

$$=18\hat{i}-18\hat{j}+9\hat{k}$$

and $|\vec{a}\times\vec{b}|=\sqrt{18^2+(-18)^2+9^2}=\sqrt{729}=27$

Hence, required vector $=\dfrac{\lambda\,(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

$$=\frac{3(18\hat{i}-18\hat{j}+9\hat{k})}{27} \quad [\text{given } \lambda=3]$$

$$=\frac{18}{9}\hat{i}-\frac{18}{9}\hat{j}+\frac{9}{9}\hat{k}=2\hat{i}-2\hat{j}+\hat{k}$$

EXAMPLE |6| Find the unit vector perpendicular to the plane ABC, where the position vectors of A, B and C are $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+\hat{j}+2\hat{k}$ and $2\hat{i}+3\hat{k}$, respectively.

[All India 2014C]

Sol. We have, $\overrightarrow{OA}=2\hat{i}-\hat{j}+\hat{k}$, $\overrightarrow{OB}=\hat{i}+\hat{j}+2\hat{k}$

and $\overrightarrow{OC}=2\hat{i}+3\hat{k}$

Now, $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$$=(\hat{i}+\hat{j}+2\hat{k})-(2\hat{i}-\hat{j}+\hat{k})=-\hat{i}+2\hat{j}+\hat{k}$$

and $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(2\hat{i}+3\hat{k})-(2\hat{i}-\hat{j}+\hat{k})=\hat{j}+2\hat{k}$

$$\text{Now, } \overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 2 & 1\\ 0 & 1 & 2\end{vmatrix}$$

$$=\hat{i}(4-1)-\hat{j}(-2-0)+\hat{k}(-1-0)=3\hat{i}+2\hat{j}-\hat{k}$$

$\therefore$ Required unit vector perpendicular to the plane ABC

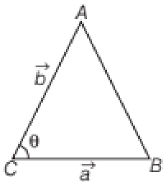
$$=\frac{\overrightarrow{AB}\times\overrightarrow{AC}}{|\overrightarrow{AB}\times\overrightarrow{AC}|}=\frac{3\hat{i}+2\hat{j}-\hat{k}}{\sqrt{3^2+2^2+(-1)^2}}$$

$$=\frac{3\hat{i}+2\hat{j}-\hat{k}}{\sqrt{9+4+1}}=\frac{3\hat{i}+2\hat{j}-\hat{k}}{\sqrt{14}}$$

Applications of Vector Product of Two Vectors

Vector product of two vectors can be used to find the area of some geometrical figures which are given below

1. **Area of a triangle** The area of triangle having adjacent sides $\vec{a}$ and $\vec{b}$ is given by

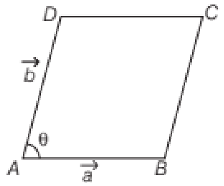


$$\text{Area} = \frac{1}{2}|\vec{a} \times \vec{b}|$$

2. **Area of a triangle** The area of ΔABC is

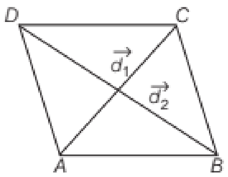
$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| \text{ or } \frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BA}| \text{ or } \frac{1}{2}|\overrightarrow{CB} \times \overrightarrow{CA}|.$$

3. **Area of a parallelogram** The area of a parallelogram having adjacent sides $\vec{a}$ and $\vec{b}$ is given by



$$\text{Area} = |\vec{a} \times \vec{b}|$$

4. **Area of a parallelogram** The area of a parallelogram having diagonals $\vec{d}_1$ and $\vec{d}_2$ is given by



$$\text{Area} = \frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$$

where, $\overrightarrow{AC} = \vec{d}_1$ and $\overrightarrow{BD} = \vec{d}_2$ are diagonals.

EXAMPLE |7| Find the area of the triangle whose vertices are $P(-1, 2, -1)$, $Q(3, -1, 2)$ and $R(2, 3, -1)$.

Sol. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of points P, Q and R, respectively. Then, $\vec{a} = -\hat{i} + 2\hat{j} - \hat{k}$, $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = 2\hat{i} + 3\hat{j} - \hat{k}$

Clearly, the area of $\Delta PQR = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$

Now, $\overrightarrow{PQ}$ = Position vector of Q − Position vector of P

$$= \vec{b} - \vec{a} = (3\hat{i} - \hat{j} + 2\hat{k}) - (-\hat{i} + 2\hat{j} - \hat{k})$$
$$= 4\hat{i} - 3\hat{j} + 3\hat{k}$$

$\overrightarrow{PR}$ = Position vector of R − Position vector of P

$$= \vec{c} - \vec{a} = (2\hat{i} + 3\hat{j} - \hat{k}) - (-\hat{i} + 2\hat{j} - \hat{k}) = 3\hat{i} + \hat{j}$$

$$\therefore \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -3 & 3 \\ 3 & 1 & 0 \end{vmatrix} = (0-3)\hat{i} - (0-9)\hat{j} + (4+9)\hat{k}$$
$$= -3\hat{i} + 9\hat{j} + 13\hat{k}$$

and $|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(-3)^2 + (9)^2 + (13)^2}$

$$= \sqrt{9 + 81 + 169} = \sqrt{259}$$

So, area of $\Delta PQR = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}\sqrt{259}$

EXAMPLE |8| If $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + \hat{k}$ and $\vec{c} = 2\hat{j} - \hat{k}$ are three vectors, then find the area of the parallelogram having diagonals $(\vec{a} + \vec{b})$ and $(\vec{b} + \vec{c})$.

[Delhi 2014C]

Sol. We have, $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + \hat{k}$ and $\vec{c} = 2\hat{j} - \hat{k}$

Now, $\vec{a} + \vec{b} = (2\hat{i} - 3\hat{j} + \hat{k}) + (-\hat{i} + \hat{k}) = \hat{i} - 3\hat{j} + 2\hat{k}$

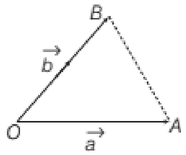
and $\vec{b} + \vec{c} = (-\hat{i} + \hat{k}) + (2\hat{j} - \hat{k}) = -\hat{i} + 2\hat{j}$

$$\therefore (\vec{a} + \vec{b}) \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ -1 & 2 & 0 \end{vmatrix}$$
$$= \hat{i}(0-4) - \hat{j}(0+2) + \hat{k}(2-3)$$
$$= -4\hat{i} - 2\hat{j} - \hat{k}$$

Hence, the area of parallelogram having diagonals $(\vec{a} + \vec{b})$ and $(\vec{b} + \vec{c})$

$$= \frac{|(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})|}{2} = \frac{|-4\hat{i} - 2\hat{j} - \hat{k}|}{2}$$
$$= \frac{\sqrt{16 + 4 + 1}}{2} = \frac{\sqrt{21}}{2} = \frac{1}{2}\sqrt{21} \text{ sq units}$$

EXAMPLE |9| A ΔOAB is determined by the vectors $\vec{a}$ and $\vec{b}$ as shown in the figure. Show that the triangle has the area, is given by $\Delta = \frac{1}{2}\sqrt{|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2}$.



Sol. We know that, area of triangle, $\Delta = \frac{1}{2}|\overrightarrow{OA}\times\overrightarrow{OB}|$

$$\Rightarrow \quad \Delta^2 = \frac{1}{4}|\vec{a}\times\vec{b}|^2 \quad \text{...(i)}$$

Now, $|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2\sin^2\theta+|\vec{a}|^2|\vec{b}|^2\cos^2\theta$

$$= |\vec{a}|^2|\vec{b}|^2(\sin^2\theta+\cos^2\theta)$$

$$= |\vec{a}|^2|\vec{b}|^2\times 1 = |\vec{a}|^2|\vec{b}|^2$$

$$\Rightarrow \quad |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2$$

$\therefore$ From Eq. (i), we get $\Delta^2 = \frac{1}{4}\left[|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2\right]$

$$\Rightarrow \quad \Delta = \frac{1}{2}\sqrt{|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2}$$

TOPIC PRACTICE 4

OBJECTIVE TYPE QUESTIONS

1 If θ is the angle between any two non-zero vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$ when θ is equal to **[NCERT]**

(a) zero (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

2 If $|\vec{a}|=10$, $|\vec{b}|=2$ and $\vec{a}\cdot\vec{b}=12$, then the value of $|\vec{a}\times\vec{b}|$ is **[NCERT Exemplar]**

(a) 5 (b) 10 (c) 14 (d) 16

3 The value of $\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{k}\times\hat{i})+\hat{k}\cdot(\hat{i}\times\hat{j})$ is **[NCERT]**

(a) zero (b) -1 (c) 1 (d) 3

4 The value of λ for which the vectors $3\hat{i}-6\hat{i}+\hat{k}$ and $2\hat{i}-4\hat{j}+\lambda\hat{k}$ are parallel, is **[NCERT Exemplar]**

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) $\frac{2}{5}$

5 The vectors from origin to the points A and B are $\vec{a}=2\hat{i}-3\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}+\hat{k}$ respectively, then the area of ΔOAB is equal to **[NCERT Exemplar]**

(a) 340 (b) $\sqrt{25}$
(c) $\sqrt{229}$ (d) $\frac{1}{2}\sqrt{229}$

VERY SHORT ANSWER Type Questions

6 If θ is the angle between two vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$, find $\sin\theta$. **[CBSE 2018]**

7 Find λ and μ, if $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k})=\vec{0}$. **[NCERT]**

8 Find λ and μ, if $(\hat{i}+3\hat{j}+9\hat{k})\times(3\hat{i}-\lambda\hat{j}+\mu\hat{k})=\vec{0}$. **[All India 2016]**

9 If $|\vec{a}|=2$, $|\vec{b}|=7$ and $\vec{a}\times\vec{b}=3\hat{i}+2\hat{j}+6\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$. **[All India 2019]**

10 Find the angle between two vectors $\vec{a}$ and $\vec{b}$ having the same length $\sqrt{2}$ and their vector product is $-\hat{i}-\hat{j}+\hat{k}$. **[All India 2016C]**

11 Evaluate $(2\vec{a}+3\vec{b})\times(5\vec{a}+7\vec{b})$.

SHORT ANSWER Type I Questions

12 Find a vector of magnitude 9, which is perpendicular to both the vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$.

13 If $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+5\hat{j}-2\hat{k}$, then find $|\vec{a}\times\vec{b}|$. **[All India 2019]**

14 Given $|\vec{a}|=10$, $|\vec{b}|=2$ and $\vec{a}\cdot\vec{b}=12$, find $|\vec{a}\times\vec{b}|$.

SHORT ANSWER Type II Questions

15 Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$. **[NCERT]**

16 If $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}$ and $\vec{c}=3\hat{i}-4\hat{j}-5\hat{k}$, then find a unit vector perpendicular to both of the vectors $(\vec{a}-\vec{b})$ and $(\vec{c}-\vec{b})$. **[All India 2015]**

17 If the three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

Then, show that $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$. **[NCERT Exemplar]**

18 For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, evaluate $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$.

19 If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, then show that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$.

20 Using vectors, find the area of the ΔABC, whose vertices are $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$. **[Delhi 2017; All India 2013]**

21 Using vectors, find the area of triangle with vertices $A(1, 1, 2)$, $B(2, 3, 5)$ and $C(1, 5, 6)$. **[All India 2011]**

22 Show that $(\vec{a} \times \vec{b})^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$

23 If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

24 If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then find $(\vec{r} \times \hat{i}) \cdot (\vec{r} \times \hat{j}) + xy$. **[Delhi 2015]**

25 If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, then find a vector $\vec{c}$ such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$. **[Delhi 2013]**

26 Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and satisfying $\vec{d} \cdot \vec{c} = 21$. **[Delhi 2016C]**

HINTS & SOLUTIONS

1. (b) We have, $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$

$\Rightarrow \quad |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\sin\theta$

$\cos\theta = \sin\theta \quad [|\vec{a}|$ and $|\vec{b}|$ are positive$]$

$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

2. (d) We know, $(\vec{a} \cdot \vec{b})^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2$

$\therefore \quad (12)^2 + |\vec{a} \times \vec{b}|^2 = 10^2 \cdot 2^2 = 400$

$\Rightarrow \quad |\vec{a} \times \vec{b}|^2 = 400 - 144 = 256 \Rightarrow |\vec{a} \times \vec{b}| = 16$

3. (d) We have,

$\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{k} \times \hat{i}) + \hat{k} \cdot (\hat{i} \times \hat{j}) = \hat{i} \cdot \hat{i} + \hat{j} \cdot \hat{j} + \hat{k} \cdot \hat{k}$

$[\because \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}, \hat{i} \times \hat{j} = \hat{k}]$

$= 1 + 1 + 1 = 3 \quad [\because \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1]$

4. (a) **Hint** $\vec{a} \parallel \vec{b}$ iff $\vec{a} \times \vec{b} = \vec{0}$

5. (d) Area of $\Delta OAB = \frac{1}{2}|\overrightarrow{OA} \times \overrightarrow{OB}|$

$= \frac{1}{2}|(2\hat{i} - 3\hat{j} + 2\hat{k}) \times (2\hat{i} + 3\hat{j} + \hat{k})|$

$= \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

$= \frac{1}{2}|[\hat{i}(-3-6) - \hat{j}(2-4) + \hat{k}(6+6)]|$

$= \frac{1}{2}|-9\hat{i} + 2\hat{j} + 12\hat{k}|$

$\therefore$ Area of $\Delta OAB = \frac{1}{2}\sqrt{(81 + 4 + 144)} = \frac{1}{2}\sqrt{229}$

6. Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

Then, $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$ ...(i)

$[\because |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta]$

Here, $|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$

$|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$

and $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 3 & -2 & 1 \end{vmatrix}$

$= \hat{i}(-2+6) - \hat{j}(1-9) + \hat{k}(-2+6)$

$= 4\hat{i} + 8\hat{j} + 4\hat{k} = 4(\hat{i} + 2\hat{j} + \hat{k})$

$\Rightarrow |\vec{a} \times \vec{b}| = 4\sqrt{1^2 + 2^2 + 1^2} = 4\sqrt{1+4+1} = 4\sqrt{6}$

Now, from Eq. (i), we get

$\sin\theta = \frac{4\sqrt{6}}{\sqrt{14} \cdot \sqrt{14}} = \frac{4\sqrt{6}}{14} = \frac{2\sqrt{6}}{7}$

7. We have, $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$

$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = \vec{0}$

$\Rightarrow (6\mu - 27\lambda)\hat{i} - (2\mu - 27)\hat{j} + (2\lambda - 6)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

On equating the coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$ both sides, we get

$6\mu - 27\lambda = 0,\ 2\mu - 27 = 0$ and $2\lambda - 6 = 0$

$\Rightarrow \quad \lambda = 3$ and $\mu = \frac{27}{2}$

8. Solve as Question 7. [**Ans.** $\lambda = -9, \mu = 27$]

9. Let θ be the angle between $\vec{a}$ and $\vec{b}$.

We have, $\vec{a}\times\vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$

$\therefore \quad |\vec{a}\times\vec{b}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$

$\Rightarrow |\vec{a}||\vec{b}|\sin\theta = 7 \qquad [\because |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta]$

$\Rightarrow \quad \sin\theta = \dfrac{7}{|\vec{a}||\vec{b}|} = \dfrac{7}{2\times 7} = \dfrac{1}{2}$

$\Rightarrow \quad \sin\theta = \sin\left(\dfrac{\pi}{6}\right) \Rightarrow \theta = \dfrac{\pi}{6}$

Hence, the required angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{6}$.

10. Solve as Question 9.

Hint $|\vec{a}| = |\vec{b}| = \sqrt{2}$ and $|\vec{a}\times\vec{b}| = |-\hat{i} - \hat{j} + \hat{k}|$ [**Ans.** $\pi/3$]

11. We have, $(2\vec{a} + 3\vec{b})\times(5\vec{a} + 7\vec{b})$

$= 10(\vec{a}\times\vec{a}) + 14(\vec{a}\times\vec{b}) + 15(\vec{b}\times\vec{a}) + 21(\vec{b}\times\vec{b})$

$= 0 + 14(\vec{a}\times\vec{b}) - 15(\vec{a}\times\vec{b}) + 0$

$[\because \vec{a}\times\vec{a} = \vec{b}\times\vec{b} = \vec{0} \text{ and } \vec{a}\times\vec{b} = -\vec{b}\times\vec{a}]$

$= -(\vec{a}\times\vec{b})$

12. Let $\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$. Then,

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = (2-3)\hat{i} - (-8+6)\hat{j} + (4-2)\hat{k}$$

$= -\hat{i} + 2\hat{j} + 2\hat{k}$

and $|\vec{a}\times\vec{b}| = \sqrt{(-1)^2 + 2^2 + 2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

$\therefore$ Required vector $= 9\left\{\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right\} = \dfrac{9}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$

$= -3\hat{i} + 6\hat{j} + 6\hat{k}$

13. We have, $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$

Now, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$

$= \hat{i}(-2-15) - \hat{j}(-4-9) + \hat{k}(10-3)$

$= \hat{i}(-17) - \hat{j}(-13) + \hat{k}(7)$

$= -17\hat{i} + 13\hat{j} + 7\hat{k}$

$\therefore |\vec{a}\times\vec{b}| = \sqrt{(-17)^2 + (13)^2 + (7)^2}$

$= \sqrt{289 + 169 + 49} = \sqrt{507} = 13\sqrt{3}$

14. Given, $|\vec{a}| = 10$, $|\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = 12$

We know that, $(\vec{a}\cdot\vec{b})^2 + |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$

$\therefore \quad 12^2 + |\vec{a}\times\vec{b}|^2 = (10)^2\times(2)^2$

$\Rightarrow \quad |\vec{a}\times\vec{b}|^2 = 400 - 144 \Rightarrow |\vec{a}\times\vec{b}|^2 = 256$

$\Rightarrow \quad |\vec{a}\times\vec{b}| = 16$ [taking positive square root]

15. We have, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

Now, $\vec{a} + \vec{b} = (\hat{i} + \hat{j} + \hat{k}) + (\hat{i} + 2\hat{j} + 3\hat{k}) = 2\hat{i} + 3\hat{j} + 4\hat{k}$

and $\vec{a} - \vec{b} = (\hat{i} + \hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 0\hat{i} - \hat{j} - 2\hat{k}$

A vector $\vec{c}$ perpendicular to both $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is given by

$\vec{c} = (\vec{a} + \vec{b})\times(\vec{a} - \vec{b}) = (2\hat{i} + 3\hat{j} + 4\hat{k})\times(0\hat{i} - \hat{j} - 2\hat{k})$

$$\Rightarrow \quad \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = (-6+4)\hat{i} - (-4-0)\hat{j} + (-2-0)\hat{k}$$

$= -2\hat{i} + 4\hat{j} - 2\hat{k}$

and $|\vec{c}| = \sqrt{(-2)^2 + (4)^2 + (-2)^2} = \sqrt{4+16+4} = \sqrt{24} = 2\sqrt{6}$

$\therefore$ Required unit vector $= \dfrac{1}{|\vec{c}|}\vec{c} = \dfrac{1}{2\sqrt{6}}(-2\hat{i} + 4\hat{j} - 2\hat{k})$

$= -\dfrac{1}{\sqrt{6}}\hat{i} + \dfrac{2}{\sqrt{6}}\hat{j} - \dfrac{1}{\sqrt{6}}\hat{k}$

16. Solve as Question 15. $\left[\textbf{Ans. } \dfrac{-\hat{j}}{\sqrt{2}} + \dfrac{\hat{k}}{\sqrt{2}}\right]$

17. Clearly, $\vec{b} + \vec{c} = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$

$= (b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}$

$$\therefore \vec{a}\times(\vec{b} + \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

[by property of determinant]

$= (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$

Hence, $\vec{a}\times(\vec{b} + \vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$.

18. Clearly, $\vec{a}\times(\vec{b} + \vec{c}) + \vec{b}\times(\vec{c} + \vec{a}) + \vec{c}\times(\vec{a} + \vec{b})$

$= (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c}) + (\vec{b}\times\vec{a})$
$+ (\vec{c}\times\vec{a}) + (\vec{c}\times\vec{b})$

$= (\vec{a}\times\vec{b}) - (\vec{c}\times\vec{a}) + (\vec{b}\times\vec{c}) - (\vec{a}\times\vec{b})$
$+ (\vec{c}\times\vec{a}) - (\vec{b}\times\vec{c})$

$= 0 \qquad [\because \vec{b}\times\vec{a} = -(\vec{a}\times\vec{b}), \vec{a}\times\vec{c} = -(\vec{c}\times\vec{a})$
and $\vec{c}\times\vec{b} = -(\vec{b}\times\vec{c})]$

19. Given, $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ ...(i)

and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$\vec{a}\times\vec{b}-\vec{a}\times\vec{c}=\vec{c}\times\vec{d}-\vec{b}\times\vec{d}$$

$\Rightarrow\quad \vec{a}\times(\vec{b}-\vec{c})=(\vec{c}-\vec{b})\times\vec{d}$

$\Rightarrow\quad \vec{a}\times(\vec{b}-\vec{c})=-(\vec{b}-\vec{c})\times\vec{d}$

$\Rightarrow\quad \vec{a}\times(\vec{b}-\vec{c})=\vec{d}\times(\vec{b}-\vec{c})$ $[\because \vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})]$

$\Rightarrow \vec{a}\times(\vec{b}-\vec{c})-\vec{d}\times(\vec{b}-\vec{c})=\vec{0}$

$\Rightarrow\quad (\vec{a}-\vec{d})\times(\vec{b}-\vec{c})=\vec{0}$

Hence, $(\vec{a}-\vec{d})$ is parallel to $(\vec{b}-\vec{c})$.

20. Similar as Example 7. [**Ans.** $\frac{1}{2}\sqrt{274}$ sq units]

21. Similar as Example 7. [**Ans.** $2\sqrt{3}$ sq units]

22. We have, $(\vec{a}\times\vec{b})^2=|\vec{a}\times\vec{b}|^2$

$\Rightarrow\quad (\vec{a}\times\vec{b})^2=\{|\vec{a}||\vec{b}|\sin\theta\}^2$

$\Rightarrow\quad (\vec{a}\times\vec{b})^2=|\vec{a}|^2|\vec{b}|^2\sin^2\theta$

$\Rightarrow\quad (\vec{a}\times\vec{b})^2=\{|\vec{a}|^2|\vec{b}|^2\}(1-\cos^2\theta)$

$\Rightarrow\quad (\vec{a}\times\vec{b})^2=|\vec{a}|^2|\vec{b}|^2-|\vec{a}|^2|\vec{b}|^2\cos^2\theta$

$\Rightarrow\quad (\vec{a}\times\vec{b})^2=(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})-(\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{b})$

$[\because \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta]$

$$\Rightarrow\quad (\vec{a}\times\vec{b})^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b}\end{vmatrix}$$

23. To prove, $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$

We have, $\vec{a}+\vec{b}+\vec{c}=\vec{0}\Rightarrow\vec{a}\times(\vec{a}+\vec{b}+\vec{c})=\vec{a}\times\vec{0}$

[taking cross product with $\vec{a}$]

$\Rightarrow\quad (\vec{a}\times\vec{a})+(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})=\vec{0}$

[using distributive law]

$\Rightarrow\quad (\vec{a}\times\vec{b})-(\vec{c}\times\vec{a})=\vec{0}$

$[\because \vec{a}\times\vec{a}=0 \text{ and } \vec{a}\times\vec{c}=-\vec{c}\times\vec{a}]$

$\Rightarrow\quad \vec{a}\times\vec{b}=\vec{c}\times\vec{a}$ ...(i)

Again, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow\quad \vec{b}\times(\vec{a}+\vec{b}+\vec{c})=\vec{b}\times\vec{0}$

[taking cross product with $\vec{b}$]

$\Rightarrow\quad (\vec{b}\times\vec{a})+(\vec{b}\times\vec{b})+(\vec{b}\times\vec{c})=\vec{0}$

$\Rightarrow\quad -(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})=\vec{0}$

$[\because \vec{b}\times\vec{b}=0,\ \vec{b}\times\vec{a}=-\vec{a}\times\vec{b}]$

$\Rightarrow\quad \vec{a}\times\vec{b}=\vec{b}\times\vec{c}$ ...(ii)

From Eqs. (i) and (ii), we get

$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$ **Hence proved.**

24. We have, $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

Now, $\vec{r}\times\hat{i}=(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{i}$

$=x(\hat{i}\times\hat{i})+y(\hat{j}\times\hat{i})+z(\hat{k}\times\hat{i})$

$=x\cdot 0+y(-\hat{k})+z(\hat{j})=-y\hat{k}+z\hat{j}$

$[\because \hat{i}\times\hat{i}=\vec{0},\ \hat{j}\times\hat{i}=-\hat{k} \text{ and } \hat{k}\times\hat{i}=\hat{j}]$

and $\vec{r}\times\hat{j}=(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{j}$

$=x(\hat{i}\times\hat{j})+y(\hat{j}\times\hat{j})+z(\hat{k}\times\hat{j})$

$=x\hat{k}+y\cdot 0+z(-\hat{i})$

$=x\hat{k}-z\hat{i}$

$[\because \hat{j}\times\hat{j}=0,\ \hat{i}\times\hat{j}=\hat{k} \text{ and } \hat{k}\times\hat{j}=-\hat{i}]$

Now, $(\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j})=(-y\hat{k}+z\hat{j})\cdot(x\hat{k}-z\hat{i})$

$=-yx+0+0-0=-xy$

$[\because \hat{k}\cdot\hat{k}=1,\ \hat{k}\cdot\hat{i}=0 \text{ and } \hat{j}\cdot\hat{k}=0,\ \hat{j}\cdot\hat{i}=0]$

$\therefore\ (\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j})+xy=-xy+xy=0$

25. Given, $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{j}-\hat{k}$ and $\vec{c}=x\hat{i}+y\hat{j}+z\hat{k}$

$$\therefore\ \vec{a}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ x & y & z\end{vmatrix}=\hat{i}(z-y)-\hat{j}(z-x)+\hat{k}(y-x)$$

Now, $\vec{a}\times\vec{c}=\vec{b}$ [given]

$\Rightarrow \hat{i}(z-y)+\hat{j}(x-z)+\hat{k}(y-x)=0\hat{i}+1\hat{j}+(-1)\hat{k}$

$[\because \vec{b}=\hat{j}-\hat{k}]$

On comparing the coefficients from both sides, we get

$z-y=0,\ x-z=1$ and $y-x=-1$

$\Rightarrow\quad y=z$ and $x-y=1$ ...(i)

Also, given $\vec{a}\cdot\vec{c}=3\ \Rightarrow\ (\hat{i}+\hat{j}+\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})=3$

$\Rightarrow\quad x+y+z=3$

$\Rightarrow\quad x+2y=3$ $[\because z=y]$ ...(ii)

On solving Eqs. (i) and (ii), we get

$3y=2\ \Rightarrow\ y=\frac{2}{3}=z$ $[\because y=z]$

From Eq. (i), we get $x=1+y=1+\frac{2}{3}=\frac{5}{3}$

Hence, $\vec{c}=\frac{5}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$

26. Since, $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

Therefore, $\vec{d}=\lambda(\vec{a}\times\vec{b})$ ...(i)

$$=\lambda\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 5 & -1\\ 1 & -4 & 5\end{vmatrix}=\lambda[21\hat{i}-21\hat{j}-21\hat{k}]$$

Also, we have $\vec{d}\cdot\vec{c}=21$

$\therefore\quad 21\lambda[\hat{i}-\hat{j}-\hat{k}]\cdot(3\hat{i}+\hat{j}-\hat{k})=21$

$\Rightarrow\quad \lambda[3-1+1]=1\ \Rightarrow\ \lambda=\frac{1}{3}$

Thus, $\vec{d}=\frac{1}{3}[21\hat{i}-21\hat{j}-21\hat{k}]=7\hat{i}-7\hat{j}-7\hat{k}$

SUMMARY

- **Position Vector** The vector $\overrightarrow{OP}$ or $\vec{r}$ is called the position vector of the point $P(x, y, z)$ with respect to origin O and $|\overrightarrow{OP}| = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.
- **Triangle Law of Vector Addition** The vector sum of the three sides of a triangle taken in order is $\vec{0}$.
- **Parallelogram Law of Vector Addition** Suppose two vectors $\vec{a}$ and $\vec{b}$ are represented by the two adjacent sides of a parallelogram, then their sum $\vec{c}$ is represented by the diagonal of the parallelogram, which is coinitial with the given vectors.
- **Properties of vector Addition**
  (i) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$. [commutative]
  (ii) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ [associative]
  (iii) $\vec{a} + \vec{0} = \vec{a} = \vec{a} + \vec{0}$ [additive identity]
- **Vector Joining Two Points** If $A \equiv (x_1, y_1, z_1)$ and $B \equiv (x_2, y_2, z_2)$, then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$.
- **Section Formulae** Let the point C divides the line segment AB in the ratio $m:n$. Then,
  For internal division $\overrightarrow{OC} = \dfrac{m\,\overrightarrow{OB} + n\,\overrightarrow{OA}}{m+n}$
  For external division $\overrightarrow{OC} = \dfrac{m\,\overrightarrow{OB} - n\,\overrightarrow{OA}}{m-n}$

(vii) If $\theta = 0$, then $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$; if $\theta = \pi$, then $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$.

- **Projection of a Vector** Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ and projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.
- **Vector** (or Cross) **Product of Vectors** The cross product of $\vec{a}$ and $\vec{b}$, is given by $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $0 \le \theta \le \pi$.
- **Properties of Cross Product of Two Vectors**
  (i) $(\vec{a}\times\vec{b}) = -(\vec{b}\times\vec{a})$ [not commutative]
  (ii) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$ [distributive property]
  (iii) $\lambda(\vec{a}\times\vec{b}) = (\lambda\,\vec{a})\times\vec{b} = \vec{a}\times(\lambda\,\vec{b})$
- **Important Results on Cross Product of Two Vectors**
  (i) The unit vector $\hat{n}$ which is perpendicular to both the vectors $\vec{a}$ and $\vec{b}$, is given by $\hat{n} = \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.
  (ii) For vectors $\vec{a}$ and $\vec{b}$, if $\vec{a}\times\vec{b} = \vec{0}$, then either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}\,||\,\vec{b}$.
  (iii) The angle between two non-zero vectors is given by $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$ or $\theta = \sin^{-1}\left[\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\right]$.

- **Scalar** (or Dot) **Product of Two Vectors** The dot product of $\vec{a}$ and $\vec{b}$, is given by $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where $0 \le \theta \le \pi$.
  If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined.
- **Properties of Scalar Product of Two Vectors**
  (i) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ [i.e. scalar product is commutative]
  (ii) $(\vec{a}\cdot\vec{0}) = 0$
  (iii) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ [distributive property]
  (iv) If $\vec{a}$ and $\vec{b}$ are perpendicular to each other, then $\vec{a}\cdot\vec{b} = 0$, converse is also true.
- **Important Results on Scalar Product of Two Vectors**
  (i) The angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$ or $\theta = \cos^{-1}\left[\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}\right]$
  (ii) $\vec{a}\cdot\vec{a} = |\vec{a}|^2$ (iii) $\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = 1$
  (iv) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
  (v) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.
  (vi) $(\lambda\cdot\vec{a})\cdot\vec{b} = \lambda(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(\lambda\cdot\vec{b})$, where λ is any scalar.

(iv) $\vec{a}\times\vec{a} = \vec{0}$

(v) Two non-zero vectors $\vec{a}, \vec{b}$ are collinear iff $\vec{a}\times\vec{b} = \vec{0}$

(vi) If $\theta = \dfrac{\pi}{2}$, then $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|$.

(vii) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$ and $\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$

(viii) $\hat{j}\times\hat{i} = -\hat{k}$, $\hat{k}\times\hat{j} = -\hat{i}$ and $\hat{i}\times\hat{k} = -\hat{j}$

(ix) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

- **Applications of Vector Product of Two Vectors**
  (i) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given by $|\vec{a}\times\vec{b}|$.
  (ii) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle, then its area is given by $= \dfrac{1}{2}|\vec{a}\times\vec{b}|$.
  (iii) If $\vec{d_1}$ and $\vec{d_2}$ represent the diagonals of a parallelogram, then its area is given by $\dfrac{1}{2}|\vec{d_1}\times\vec{d_2}|$.

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1 If m_1, m_2, m_3 and m_4 are respectively the magnitudes of the vectors $\vec{a}_1 = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{a}_2 = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{a}_3 = \hat{i} + \hat{j} - \hat{k}$ and $\vec{a}_4 = -\hat{i} + 3\hat{j} + \hat{k}$, then the correct order of m_1, m_2, m_3 and m_4 is

(a) $m_3 < m_1 < m_4 < m_2$ (b) $m_3 < m_1 < m_2 < m_4$
(c) $m_3 < m_4 < m_1 < m_2$ (d) $m_3 < m_4 < m_2 < m_1$

2 $ABCD$ is a rhombus whose diagonals intersect at E. Then $\vec{EA} + \vec{EB} + \vec{EC} + \vec{ED}$ equals

(a) $\vec{0}$ (b) $\vec{AD}$ (c) $2\vec{BC}$ (d) $2\vec{AD}$

3 If $\vec{a}$ is a non-zero vector of magnitude a and λ is a non-zero scalar, then $\lambda\vec{a}$ is unit vector if **[NCERT]**

(a) $\lambda = 1$ (b) $\lambda = -1$
(c) $a = |\lambda|$ (d) $a = 1/|\lambda|$

4 If $|\vec{a}| = 4$ and $-3 \le \lambda \le 2$, then the range of $|\lambda\vec{a}|$ is **[NCERT Exemplar]**

(a) [0, 8] (b) [−12, 8]
(c) [0, 12] (d) [8, 12]

5 The position vector of the point which divides the join of points $2\vec{a} - 3\vec{b}$ and $\vec{a} + \vec{b}$ in the ratio 3 : 1, is **[NCERT Exemplar]**

(a) $\dfrac{3\vec{a} - 2\vec{b}}{2}$ (b) $\dfrac{7\vec{a} - 8\vec{b}}{4}$

(c) $\dfrac{3\vec{a}}{4}$ (d) $\dfrac{5\vec{a}}{4}$

6 If $\vec{a}\cdot\vec{b} = \frac{1}{2}|\vec{a}||\vec{b}|$, then the angle between $\vec{a}$ and $\vec{b}$ is

(a) 0° (b) 30° (c) 60° (d) 90°

7 The projection vector of $\vec{a}$ on $\vec{b}$ is **[NCERT Exemplar]**

(a) $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$ (b) $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ (c) $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ (d) $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\hat{b}$

8 For any vector $\vec{a}$, the value of $(\vec{a}\times\hat{i})^2 + (\vec{a}\times\hat{j})^2 + (\vec{a}\times\hat{k})^2$ is **[NCERT Exemplar]**

(a) $\vec{a}^2$ (b) $3\vec{a}^2$ (c) $4\vec{a}^2$ (d) $2\vec{a}^2$

9 The number of vectors of unit length perpendicular to the vectors $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$ is **[NCERT Exemplar]**

(a) one (b) two
(c) three (d) infinite

10 If $\vec{a}\cdot\vec{b} = 0$ and $\vec{a}\times\vec{b} = 0$, then

(a) $|\vec{a}| = 0$ (b) $|\vec{b}| = 0$

(c) Both (a) and (b) are true (d) Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$

VERY SHORT ANSWER Type Questions

11 Write the direction ratio's of the vector $3\vec{a} + 2\vec{b}$, where $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$. **[All India 2015C]**

12 Find $\vec{a}\cdot\vec{b}$, if $\vec{a} = -\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$.

13 For what value of λ, the vectors $\hat{i} + 2\lambda\hat{j} + \hat{k}$ and $2\hat{i} + \hat{j} - 3\hat{k}$ are perpendicular? **[All India 2011C]**

14 If $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and the angle between $\vec{a}$ and $\vec{b}$ is 60°, then find $\vec{a}\cdot\vec{b}$. **[Delhi 2011C]**

15 Find λ, when projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units. **[Delhi 2012]**

16 Write the value of $(\hat{k}\times\hat{j})\cdot\hat{i} + \hat{j}\cdot\hat{k}$. **[All India 2012]**

17 If $\vec{a}\cdot\vec{a} = 0$ and $\vec{a}\cdot\vec{b} = 0$, then what can be concluded about $\vec{b}$? **[All India 2011]**

18 P and Q are two points with position vectors $3\vec{a} - 2\vec{b}$ and $\vec{a} + \vec{b}$, respectively. Write the position vector of a point R which divides the line segment PQ in the ratio 2 : 1 externally. **[All India 2013]**

19 Find $|\vec{x}|$, if for a unit vector $\hat{a}$, $(\vec{x}-\hat{a})\cdot(\vec{x}+\hat{a})=15$. **[All India 2013]**

20 Find the vector product of the vectors $3\hat{i}-\hat{k}$ and $-\hat{i}-\hat{j}+5\hat{k}$.

21 Find the area of parallelogram determined by the vectors $\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$.

22 If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}\cdot\vec{b}|=|\vec{a}\times\vec{b}|$. Find the value of θ.

23 Write the angle between the vectors $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$. **[Delhi 2017C]**

24 If $(\vec{a}\times\vec{b})^2+(\vec{a}\cdot\vec{b})^2=225$ and $|\vec{a}|=5$, then write the value of $|\vec{b}|$. **[All India 2017C]**

25 The two vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represent the two sides AB and AC respectively of triangle ABC. Find the length of the median through A. **[Delhi 2016; Foreign 2015]**

26 If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}|=|\vec{a}|$, then prove that vector $2\vec{a}+\vec{b}$ is perpendicular to vector $\vec{b}$. **[Delhi 2013]**

27 Find λ, if $(2\hat{i}+6\hat{j}+14\hat{k})\times(\hat{i}-\lambda\hat{j}+7\hat{k})=\vec{0}$. **[All India 2010]**

28 Find the projection of the vector $\vec{a}=2\hat{i}+3\hat{j}+2\hat{k}$ on the vector $\vec{b}=2\hat{i}+2\hat{j}+\hat{k}$. **[All India 2015C]**

29 If $\vec{a}$ and $\vec{b}$ are unit vectors, then what is the angle between $\vec{a}$ and $\vec{b}$ so that $\sqrt{2}\vec{a}-\vec{b}$ is a unit vector? **[Delhi 2015C]**

30 Write the number of vectors of unit length perpendicular to both the vectors $\vec{a}=2\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=\hat{j}+\hat{k}$. **[All India 2016]**

31 For vector $\vec{a}$, if $|\vec{a}|=a$, then write the value of $(\vec{a}\times\hat{i})^2+(\vec{a}\times\hat{j})^2+(\vec{a}\times\hat{k})^2$. **[Delhi 2016C]**

SHORT ANSWER Type I Questions

32 The position vectors of points A, B and C are $\lambda\hat{i}+3\hat{j}$, $12\hat{i}+\mu\hat{j}$ and $11\hat{i}-3\hat{j}$ respectively. If C divides the line segment joining A and B in the ratio 3 : 1, find the value of λ and μ. **[Delhi 2017C]**

33 If the points with position vectors $10\hat{i}+3\hat{j}$, $12\hat{i}-5\hat{j}$ and $\lambda\hat{i}+11\hat{j}$ are collinear, find the value of λ. **[Delhi 2017C]**

34 If $\vec{a}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b}=3\hat{i}+\hat{j}-5\hat{k}$, then find a unit vector in the direction of $\vec{a}-\vec{b}$.

35 Find the projection of $\vec{b}+\vec{c}$ on $\vec{a}$, where $\vec{a}=2\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$.

36 Find $|\vec{a}-\vec{b}|$, if two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=2$, $|\vec{b}|=3$ and $\vec{a}\cdot\vec{b}=5$.

37 Find the unit vector perpendicular to $3\hat{i}+\hat{j}+2\hat{k}$ and $2\hat{i}-2\hat{j}+4\hat{k}$.

SHORT ANSWER Type II Questions

38 Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i}+2\hat{j}-\hat{k}$ and $-\hat{i}+\hat{j}+\hat{k}$ respectively, in the ratio 2 : 1

(i) internally. (ii) externally.

39 Show that the projection vector of $\vec{b}$ on $\vec{a}$, $a\neq 0$ is $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$.

40 If $\hat{a}$ and $\hat{b}$ are two unit vectors and θ is angle between them, then what is the value of $\left|\dfrac{\hat{a}-\hat{b}}{2}\right|$?

41 If $\vec{a}=\hat{i}-\hat{j}+7\hat{k}$ and $\vec{b}=5\hat{i}-\hat{j}+\lambda\hat{k}$, then find the value of λ, so that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular vectors. **[All India 2013]**

42 Find a unit vector perpendicular to both of the vectors $3\vec{a}+2\vec{b}$ and $3\vec{a}-2\vec{b}$, where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$. **[Delhi 2016C]**

43 Show that the points A, B, C with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively, are the vertices of a right-angled triangle. Hence, find the area of triangle. **[All India 2017]**

44 The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram. **[All India 2016]**

45 If $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular vectors of equal magnitudes, find the angles which the vector $2\vec{a}+\vec{b}+2\vec{c}$ makes with the vectors $\vec{a}, \vec{b}$ and $\vec{c}$. **[All India 2017C]**

46 If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of equal magnitude x, then show that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}+\vec{b}$ and $\vec{c}$. Also, find the angle. **[NCERT]**

47 Find the unit vector perpendicular to the plane ABC, where the position vectors of A, B and C are $2\hat{i}-\hat{j}+\hat{k}, \hat{i}+\hat{j}+2\hat{k}$ and $2\hat{i}+3\hat{k}$, respectively.

48 If $\vec{b}\times\vec{c}=\vec{c}\times\vec{a}\neq\vec{0}$, then prove that $\vec{a}+\vec{b}=\lambda\vec{c}$, where λ is a scalar.

49 If $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}, \vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{p}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{p}\cdot\vec{c}=18$. **[All India 2012]**

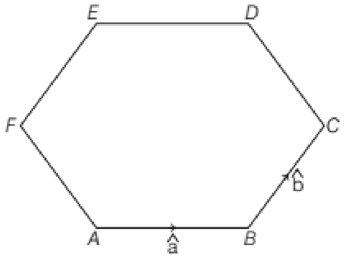
50 If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors such that $|\vec{a}|=5$, $|\vec{b}|=12$, $|\vec{c}|=13$ and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$. **[Delhi 2012]**

51 If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, $|\vec{a}|=\sqrt{37}$, $|\vec{b}|=3$ and $|\vec{c}|=4$, then what will be the angle between $\vec{b}$ and $\vec{c}$?

52 Prove that in any ΔABC, $\cos C=\dfrac{a^2+b^2-c^2}{2ab}$, where a, b and c are the magnitudes of the sides opposite to the vertices A, B and C, respectively. **[NCERT Exemplar]**

CASE BASED Questions

53. A boy see a design in a park which is shown below.



Now, he thought that if, $\hat{a}$ and $\hat{b}$ are the vectors determined by two adjacent sides of the given regular hexagon. **[CBSE Question Bank]**

On the basis of above information, answer the following questions.

(i) $\overrightarrow{AC}$ is equal to

(a) $\hat{a}-\hat{b}$ (b) $\hat{b}-\hat{a}$
(c) $\hat{a}+\hat{b}$ (d) $\hat{0}$

(ii) $\overrightarrow{AD}$ is equal to

(a) $2\hat{a}$ (b) $2\hat{b}$
(c) $2(\hat{a}+\hat{b})$ (d) $2(\hat{a}-\hat{b})$

(iii) $\overrightarrow{CD}$ is equal to

(a) $\hat{a}-\hat{b}$ (b) $2(\hat{a}-\hat{b})$
(c) $\hat{b}-\hat{a}$ (d) $2(\hat{b}-\hat{a})$

(iv) $\overrightarrow{EF}$ is equal to

(a) $\hat{a}$ (b) $\hat{b}$ (c) $-\hat{a}$ (d) $-\hat{b}$

(v) $\overrightarrow{FA}$ is equal to

(a) $\hat{a}-\hat{b}$ (b) $2(\hat{a}-\hat{b})$
(c) $\hat{b}-\hat{a}$ (d) $2(\hat{b}-\hat{a})$

54. Consider the points A, B, C with position Vctors $2\hat{i}-\hat{j}+\hat{k}, \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$. **[CBSE Question Bank]**

On the basis of above information, answer the following questions.

(i) $\overrightarrow{BC}$ is

(a) $-\hat{i}-2\hat{j}-6\hat{k}$ (b) $2\hat{i}-\hat{j}+\hat{k}$
(c) $-\hat{i}+3\hat{j}+5\hat{k}$ (d) None of these

(ii) $\overrightarrow{CA}$ is

(a) $-\hat{i}-2\hat{j}-6\hat{k}$ (b) $2\hat{i}-\hat{j}+\hat{k}$
(c) $-\hat{i}+3\hat{j}+5\hat{k}$ (d) None of these

(iii) $\overrightarrow{BC}\cdot\overrightarrow{CA}$ is

(a) -1 (b) 1 (c) 0 (d) 2

(iv) $\overrightarrow{BC}\times\overrightarrow{CA}$ is

(a) $8\hat{i}+11\hat{j}-5\hat{k}$ (b) $-8\hat{i}-11\hat{j}+5\hat{k}$
(c) $8\hat{i}-11\hat{j}-5\hat{k}$ (d) None of these

(v) Area of triangle ABC (in sq unit) is

(a) $\dfrac{\sqrt{210}}{2}$ (b) $\sqrt{210}$
(c) $2\sqrt{210}$ (d) None of these

ANSWERS

1. (a) **2.** (a) **3.** (d) **4.** (c) **5.** (d) **6.** (c) **7.** (a) **8.** (d)
9. (b) **10.** (d) **11.** 7, −5, 4 **12.** 3 **13.** $\lambda = \frac{1}{2}$ **14.** $\sqrt{3}$ **15.** $\lambda = 5$ **16.** −1
17. $\vec{b}$ can be any vector **18.** $-\vec{a} + 4\vec{b}$ **19.** 4 **20.** $-\hat{i} - 14\hat{j} - 3\hat{k}$ **21.** $8\sqrt{3}$ sq units **22.** $\frac{\pi}{4}$
23. π **24.** 3 **25.** $\frac{\sqrt{34}}{2}$ **27.** $\lambda = -3$ **28.** 4 units **29.** $\theta = \frac{\pi}{4}$ **30.** 2 **31.** $2a^2$
32. $\lambda = 8, \mu = -5$ **33.** $\lambda = 8$ **34.** $\frac{-2}{\sqrt{21}}\hat{i} + \frac{1}{\sqrt{21}}\hat{j} + \frac{4}{\sqrt{21}}\hat{k}$ **35.** 2 **36.** $\sqrt{3}$
37. $\frac{\hat{i}}{\sqrt{3}} - \frac{\hat{j}}{\sqrt{3}} - \frac{\hat{k}}{\sqrt{3}}$ **38.** (i) $\left(-\frac{1}{3}\right)\hat{i} + \left(\frac{4}{3}\right)\hat{j} + \left(\frac{1}{3}\right)\hat{k}$ (ii) $-3\hat{i} + 3\hat{k}$
40. $\sin\frac{\theta}{2}$ **41.** $\lambda = \pm 5$ **42.** $-12\hat{i} + 24\hat{j} - 12\hat{k}$ **43.** $\frac{\sqrt{210}}{2}$ sq units
44. $\frac{1}{\sqrt{6}}(2\hat{i} - \hat{j} - \hat{k}); \frac{1}{5}(3\hat{j} + 4\hat{k}); 2\sqrt{101}$ sq units **45.** $\cos^{-1}\frac{2}{3}, \cos^{-1}\frac{1}{3}, \cos^{-1}\frac{2}{3}$ **46.** $\cos^{-1}\left(\frac{x}{|\vec{a} + \vec{b} + \vec{c}|}\right)$
47. $\frac{3}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} - \frac{1}{\sqrt{14}}\hat{k}$ **49.** $\vec{p} = 64\hat{i} - 2\hat{j} - 28\hat{k}$ **50.** −169 **51.** $\theta = 60°$
53. (i) → (c), (ii) → (b), (iii) → (c), (iv) → (d), (v) → (a) **54.** (i) → (b), (ii) → (c), (iii) → (c), (iv) → (b), (v) → (a)