

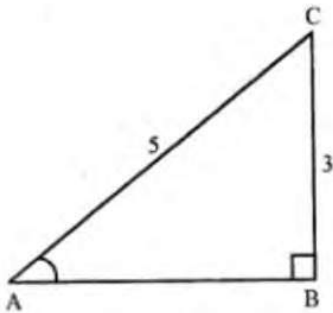
Chapter 18

Trigonometric identities

Exercise 18.1

1. If A is an acute angle and $\sin A = \frac{3}{5}$, find all other trigonometric ratios of angle A (using trigonometric identities).

Solution:



Given,

$\sin A = \frac{3}{5}$ and A is an acute angle

So, in $\triangle ABC$ we have $\angle B = 90^\circ$

And,

$AC = 5$ and $BC = 3$

By Pythagoras theorem,

$$AB = \sqrt{(AC)^2 - (BC)^2}$$

$$= \sqrt{(5^2 - 3^2)}$$

$$= \sqrt{25 - 9}$$

$$= \sqrt{16}$$

$$= 4$$

Now,

$$\cos A = \frac{AB}{AC} = \frac{4}{5}$$

$$\tan A = \frac{BC}{AB} = \frac{3}{4}$$

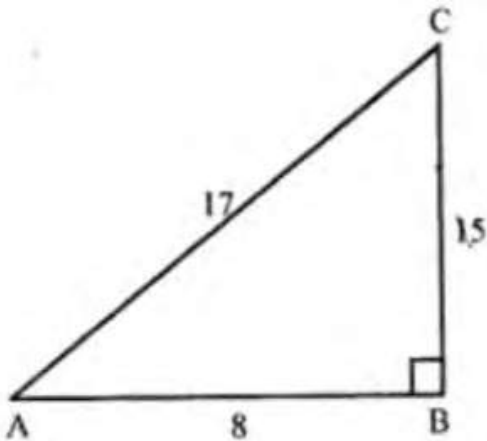
$$\cot A = \frac{1}{\tan \theta} = \frac{4}{3}$$

$$\sec A = \frac{1}{\cos \theta} = \frac{5}{4}$$

$$\operatorname{cosec} A = \frac{1}{\sin \theta} = \frac{5}{3}$$

2. If A is an acute angle and $\sec A = \frac{17}{8}$, find all other trigonometric ratios of angle A (using trigonometric identities).

Solution:



Given,

$\sec A = \frac{17}{8}$ and A is an acute angle

So, in $\triangle ABC$ we have $\angle B = 90^\circ$

And,

$$AC = 17 \text{ and } AB = 8$$

By Pythagoras theorem,

$$BC = \sqrt{AC^2 - AB^2}$$

$$= \sqrt{17^2 - 8^2}$$

$$= \sqrt{289 - 64}$$

$$= \sqrt{225}$$

$$= 15$$

Now,

$$\sin A = \frac{BC}{AC} = \frac{15}{17}$$

$$\cos A = \frac{1}{\sec A} = \frac{8}{17}$$

$$\tan A = \frac{BC}{AB} = \frac{15}{8}$$

$$\cot A = \frac{1}{\tan A} = \frac{8}{15}$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \frac{17}{15}$$

3. Express the ratios $\cos A$, $\tan A$ and $\sec A$ in terms of $\sin A$.

Solution:

We know that,

$$\sin^2 A + \cos^2 A = 1$$

So,

$$\cos A = \sqrt{1 - \sin^2 A}$$

$$\tan A = \frac{\sin A}{\cos A}$$

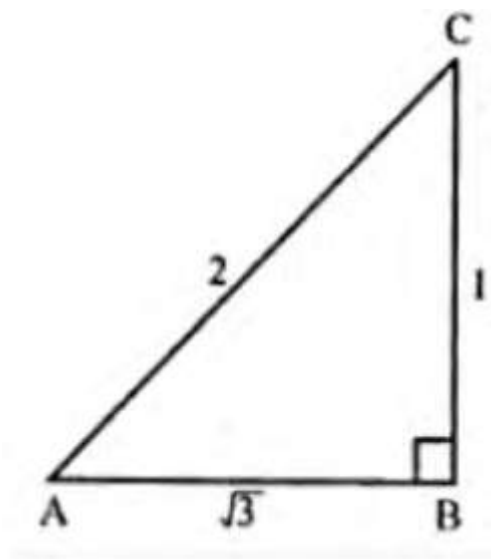
$$= \frac{\sin A}{\sqrt{1-\sin^2 A}}$$

$$\sec A = \frac{1}{\cos A}$$

$$= \frac{1}{(\sqrt{1-\sin^2 A})}$$

4. If $\tan A = \frac{1}{\sqrt{3}}$, find all other trigonometric ratios of angle A.

Solution:



Given, $\tan A = \frac{1}{\sqrt{3}}$

In right $\triangle ABC$,

$$\tan A = \frac{BC}{AB} = \frac{1}{\sqrt{3}}$$

So,

$$BC = 1 \text{ and } AB = \sqrt{3}$$

By Pythagoras theorem,

$$\begin{aligned}AC &= \sqrt{AB^2 + BC^2} \\&= \left[\sqrt{(\sqrt{3})^2 + (1)^2} \right] \\&= (\sqrt{3} + 1) = \sqrt{4} = 2\end{aligned}$$

Hence,

$$\sin A = \frac{BC}{AC} = \frac{1}{2}$$

$$\cos A = \frac{1}{\sec A} = \frac{\sqrt{3}}{2}$$

$$\tan A = \frac{BC}{AB} = \sqrt{3}$$

$$\cot A = \frac{1}{\tan A} = \frac{2}{\sqrt{3}}$$

$$\operatorname{Cosec} A = \frac{1}{\sin A} = \frac{2}{1} = 2$$

5. If $12 \operatorname{cosec} \theta = 13$, find the value of $\frac{(2 \sin \theta - 3 \cos \theta)}{(4 \sin \theta - 9 \cos \theta)}$

Solution:

Given,

$$12 \operatorname{cosec} \theta = 13$$

$$\Rightarrow \operatorname{cosec} \theta = \frac{13}{12}$$

in right $\triangle ABC$,

$$\angle A = \theta$$

$$\text{So, } \operatorname{cosec} \theta = \frac{AC}{BC} = \frac{13}{12}$$

$$AC = 13 \text{ and } BC = 12$$

By Pythagoras theorem,

$$AB = \sqrt{(AC^2 - BC^2)}$$

$$= \sqrt{((13)^2 - (12)^2)}$$

$$= \sqrt{169 - 144}$$

$$= \sqrt{25}$$

$$= 5$$

Now,

$$\sin\theta = \frac{BC}{AC} = \frac{12}{13}$$

$$\cos\theta = \frac{AB}{AC} = \frac{5}{13}$$

Hence,

$$\begin{aligned} & \frac{2\sin\theta - 3\cos\theta}{4\sin\theta - 9\cos\theta} \\ &= \frac{2 \times \frac{12}{13} - 3 \times \frac{5}{13}}{4 \times \frac{12}{13} - 9 \times \frac{5}{13}} \\ &= \frac{\frac{24}{13} - \frac{15}{13}}{\frac{48}{13} - \frac{45}{13}} = \frac{\frac{9}{13}}{\frac{3}{13}} \\ &= \frac{9}{13} \times \frac{13}{3} = 3 \end{aligned}$$

6. Without using trigonometric tables, evaluate the following :

$$(i) \cos^2 26^\circ + \cos^2 64^\circ \sin^2 26^\circ + \frac{(\tan 36^\circ)}{(\cot 54^\circ)}$$

$$(ii) \frac{(\sec 17^\circ)}{(\operatorname{cosec} 73^\circ)} + \frac{(\tan 68^\circ)}{(\cot 22^\circ)} + \cos^2 44^\circ + \cos^2 46^\circ$$

Solution:

Given,

$$(i) \cos^2 26^\circ + \cos^2 64^\circ \sin^2 26^\circ + \frac{(\tan 36^\circ)}{(\cot 54^\circ)}$$

$$= \cos^2 26^\circ + \cos (90^\circ - 16^\circ) \sin^2 26^\circ + \left[\frac{\tan 36^\circ}{\cos(90^\circ - 54^\circ)} \right]$$

$$= [\cos^2 26^\circ + \sin^2 26^\circ] + \frac{(\tan 36^\circ)}{(\tan 36^\circ)}$$

$$= 1 + 1$$

$$= 2$$

$$(ii) \frac{(\sec 17^\circ)}{(\operatorname{cosec} 73^\circ)} + \frac{(\tan 68^\circ)}{(\cot 22^\circ)} + \cos^2 44^\circ + \cos^2 46^\circ$$

$$= \left[\frac{\sec 17^\circ}{\operatorname{cosec}(90^\circ - 73^\circ)} \right] + \left[\frac{\tan 90^\circ - 22^\circ}{\cot 22^\circ} \right] + \cos^2(90^\circ - 44^\circ) + \cos^2 46^\circ$$

$$= \left[\frac{\sec 17^\circ}{\sec 17^\circ} \right] + \left[\frac{\cot 22^\circ}{\cot 22^\circ} \right] + [\sin^2 46^\circ + \cos^2 46^\circ]$$

$$= 1 + 1 + 1$$

$$= 3$$

$$7. \text{ (i) } \left(\frac{\sin 65^\circ}{\cos 25^\circ} \right) + \left(\frac{\cos 32^\circ}{\sin 58^\circ} \right) - \sin 28^\circ \sec 62^\circ + \operatorname{cosec}^2 30^\circ$$

$$\text{(ii)} \left[\frac{\sin 29^\circ}{\operatorname{cosec} 61^\circ} \right] + 2 \cot 8^\circ \cot 17^\circ \cot 45^\circ \cot 73^\circ \cot 82^\circ - 3(\sin^2 38^\circ + \sin^2 52^\circ)$$

Solution:

$$\text{(i) } \left(\frac{\sin 65^\circ}{\cos 25^\circ} \right) + \left(\frac{\cos 32^\circ}{\sin 58^\circ} \right) - \sin 28^\circ \sec 62^\circ + \operatorname{cosec}^2 30^\circ$$

$$= \left(\frac{\sin 65^\circ}{\cos (90^\circ - 65^\circ)} \right) + \left(\frac{\cos 32^\circ}{\sin (90^\circ - 32^\circ)} \right) - \sin 28^\circ \sec (90^\circ - 28^\circ) + 2^2$$

$$= \left(\frac{\sin 65^\circ}{\sin 65^\circ} \right) + \left(\frac{\cos 32^\circ}{\cos 32^\circ} \right) - [\sin 28^\circ \times \operatorname{cosec} 28^\circ] + 4$$

$$= 1 + 1 - 1 + 4$$

$$= 5$$

$$\text{(ii) } \left[\frac{\sin 29^\circ}{\operatorname{cosec} 61^\circ} \right] + 2 \cot 8^\circ \cot 17^\circ \cot 45^\circ \cot 73^\circ \cot 82^\circ - 3(\sin^2 38^\circ + \sin^2 52^\circ)$$

$$= \left[\frac{\sin 29^\circ}{\operatorname{cosec} (90^\circ - 29^\circ)} \right] + [2 \cot 8^\circ \cot 17^\circ \cot 45^\circ \cot (90^\circ - 17^\circ) \cot (90^\circ - 8^\circ)] - 3(\sin^2 38^\circ + \sin^2 (90^\circ - 38^\circ))$$

$$= \left[\frac{\sin 29^\circ}{\sin 29^\circ} \right] + [2 \cot 8^\circ \cot 17^\circ \cot 45^\circ \tan 17^\circ \tan 8^\circ] - 3(\sin^2 38^\circ + \cos^2 38^\circ)$$

$$= 1 + 2[(\cot 8^\circ \tan 8^\circ)(\cot 17^\circ \tan 17^\circ) \cot 45^\circ] - 3(1)$$

$$= 1 + 2[1 \times 1 \times 1] - 3$$

$$= 1 + 2 - 3$$

$$= 0$$

$$8. (i) \frac{(\sin 35^\circ \cos 55^\circ + \cos 35^\circ \sin 55^\circ)}{(\operatorname{Cosec}^2 10^\circ - \tan^3 80^\circ)}$$

$$(ii) \sin^2 34^\circ + \sin^2 35^\circ + 2 \tan 18^\circ \tan 72^\circ - \cot^2 30^\circ$$

Solution:

$$(i) \frac{(\sin 35^\circ \cos 55^\circ + \cos 35^\circ \sin 55^\circ)}{(\operatorname{Cosec}^2 10^\circ - \tan^3 80^\circ)}$$

$$= \frac{\sin 35^\circ \cos(90^\circ - 35^\circ) + \cos 35^\circ \sin(90^\circ - 35^\circ)}{\operatorname{Cosec}^2 10^\circ - \tan^2(90^\circ - 10^\circ)}$$

$$= \frac{\sin 35^\circ \sin 35^\circ + \cos 35^\circ + \cos 35^\circ}{\operatorname{Cosec}^2 10^\circ - \cot^2 10^\circ}$$

$$= \frac{1}{1}$$

$$= 1$$

$$(ii) \sin^2 34^\circ + \sin^2 35^\circ + 2 \tan 18^\circ \tan 72^\circ - \cot^2 30^\circ$$

$$= [\sin^2 34^\circ + \sin^2(90^\circ - 34^\circ)] + 2 \tan 18^\circ \tan(90^\circ - 18^\circ) - \cot^2 30^\circ$$

$$= [1 + 2 \times 1 - (\sqrt{3})^2]$$

$$= 1 + 2 - 3$$

$$= 0$$

$$9. (i) \left(\frac{\tan 25^\circ}{\operatorname{Cosec} 65^\circ} \right)^2 + \left(\frac{\cot 25^\circ}{\sec 65^\circ} \right)^2 + 2 \tan 18^\circ \tan 45^\circ \tan 75^\circ$$

$$(ii) (\cos^2 25^\circ + \cos^2 65^\circ) + \operatorname{cosec} \theta \sec (90^\circ - \theta) - \cot \theta \tan (90^\circ - \theta)$$

Solution:

$$(i) \left(\frac{\tan 25^\circ}{\operatorname{Cosec} 65^\circ} \right)^2 + \left(\frac{\cot 25^\circ}{\sec 65^\circ} \right)^2 + 2 \tan 18^\circ \tan 45^\circ \tan 75^\circ$$

$$= \left(\frac{\tan 25^\circ}{\operatorname{Cosec} (90^\circ - 25^\circ)} \right)^2 + \left(\frac{\cot 25^\circ}{\sec (90^\circ - 25^\circ)} \right)^2 + 2 \tan 18^\circ \tan (90^\circ - 18^\circ) \tan 45^\circ.$$

$$= \left(\frac{\tan 25^\circ}{\sec 25^\circ} \right)^2 + \left(\frac{\cot 25^\circ}{\operatorname{Cosec} 25^\circ} \right)^2 + 2 \tan 18^\circ \cot 18^\circ \tan 45^\circ.$$

$$= \left(\frac{\sin 25^\circ \times \cos 25^\circ}{\cos 25^\circ \times 1} \right) + \left[\frac{\cos 25^\circ \times \sin 25^\circ}{\sin 25^\circ \times 1} \right] + 2 \times 1 \times 1$$

$$= \sin^2 25^\circ + \cos^2 25^\circ + 2$$

$$= 1 + 2$$

$$= 3$$

$$(ii) (\cos^2 25^\circ + \cos^2 65^\circ) + \operatorname{cosec} \theta \sec (90^\circ - \theta) - \cot \theta \tan (90^\circ - \theta)$$

$$= \cos^2 25^\circ + \cos^2 (90^\circ - 25^\circ) + \operatorname{cosec} \theta \sec (90^\circ - \theta) - \cot \theta \cdot \cot \theta$$

$$= (\cos^2 25^\circ + \sin^2 25^\circ) + (\operatorname{cosec}^2 \theta + \cot^2 \theta)$$

$$= 1 + 1$$

$$= 2$$

$$10. (i) 2(\sec^2 35^\circ - \cot^2 55^\circ) - \frac{\cos 28^\circ \operatorname{cosec} 62^\circ}{\tan 18^\circ \tan 36^\circ \tan 30^\circ \tan 54^\circ \tan 72^\circ}$$

$$(ii) \frac{\operatorname{cosec}^2 (90-\theta) - \tan^2 \theta}{2(\cos^2 48^\circ + \cos^2 42^\circ)} - \frac{2 \tan^2 30^\circ \sec^2 52^\circ \sin^2 38^\circ}{\operatorname{cosec}^2 70^\circ - \tan^2 20^\circ}$$

Solution:

$$\begin{aligned} (i) & 2(\sec^2 35^\circ - \cot^2 55^\circ) - \frac{\cos 28^\circ \operatorname{cosec} 62^\circ}{\tan 18^\circ \tan 36^\circ \tan 30^\circ \tan 54^\circ \tan 72^\circ} \\ &= 2 \left(\sec^2 35^\circ - \cot^2 (90^\circ - 35^\circ) \right) - \left[\frac{\cos 28^\circ \operatorname{cosec} (90^\circ - 28^\circ)}{\tan 18^\circ \tan (90^\circ - 18^\circ) \tan 36^\circ \tan (90^\circ - 36^\circ) \tan 30^\circ} \right] \\ &= 2[\sec^2 35^\circ - \tan^2 35^\circ] - \frac{\cos 28^\circ \sec 28^\circ}{\tan 18^\circ \cot 18^\circ \tan 36^\circ \cot 36^\circ \tan 30^\circ} \end{aligned}$$

$$\left\{ \begin{array}{l} \because \sec^2 \theta - \tan^2 \theta = 1 \\ \tan \theta \cot \theta = 1 \\ \cos \theta \sec \theta = 1 \end{array} \right\}$$

$$= 2(1) - \frac{1}{1 \times 1 \times \frac{1}{\sqrt{3}}}$$

$$= 2 - \frac{\sqrt{3}}{1}$$

$$= 2 - \sqrt{3}$$

$$(ii) \frac{\operatorname{cosec}^2 (90-\theta) - \tan^2 \theta}{2(\cos^2 48^\circ + \cos^2 42^\circ)} - \frac{2 \tan^2 30^\circ \sec^2 52^\circ \sin^2 38^\circ}{\operatorname{cosec}^2 70^\circ - \tan^2 20^\circ}$$

$$= \frac{\sec^2 \theta - \tan^2 \theta}{2 (\cos^2 48^\circ + \cos^2 (90^\circ - 48^\circ))} - \frac{2 [\tan^2 30^\circ \sec^2 52^\circ \sin^2 (90^\circ - 52^\circ)]}{\operatorname{cosec}^2 70^\circ - \tan^2 (90^\circ - 70^\circ)}$$

$$\left\{ \begin{array}{l} \because \sin^2 \theta + \cos^2 \theta = 1 \\ \sec^2 \theta - \tan^2 \theta = 1 \\ \operatorname{cosec}^2 \theta - \cot^2 \theta = 1 \end{array} \right\}$$

$$= \frac{1}{2 [\cos^2 48^\circ + \sin^2 48^\circ]} - \frac{2 \left[\left(\frac{1}{\sqrt{3}} \right)^2 \sec^2 52^\circ \sec^2 52^\circ \right]}{\operatorname{cosec}^2 70^\circ - \cot^2 70^\circ}$$

$$= \frac{1}{2 \times 1} - \frac{2 \left[\frac{1}{3} \times 1 \right]}{1}$$

$$= \frac{1}{2} - \frac{2}{3}$$

$$= \frac{3-4}{6}$$

$$= \frac{-1}{6}$$

11. Prove that following :

(i) $\cos \theta \sin(90^\circ - \theta) + \sin \theta \cos(90^\circ - \theta) = 1$

(ii) $\frac{\tan \theta}{\tan(90^\circ - \theta)} + \frac{\sin(90^\circ - \theta)}{\cos \theta} = \sec^2 \theta$

(iii) $\frac{(\cos(90^\circ - \theta) \cos \theta)}{\tan \theta + \cos^2(90^\circ - \theta)} = 1$

(iv) $\sin(90^\circ - \theta) \cos(90^\circ - \theta) = \frac{\tan \theta}{(1 + \tan^2 \theta)}$

Solution:

$$(i) \cos\theta \sin(90^\circ - \theta) + \sin\theta \cos(90^\circ - \theta) = 1$$

$$\text{L.H.S.} = \cos\theta \sin(90^\circ - \theta) + \sin\theta \cos(90^\circ - \theta)$$

$$= \cos\theta \times \cos\theta + \sin\theta \times \sin\theta$$

$$= \cos^2\theta + \sin^2\theta$$

$$= 1$$

$$= \text{R.H.S.}$$

$$(ii) \text{L.H.S.} = \frac{\tan\theta}{\tan(90^\circ - \theta)} + \frac{\sin(90^\circ - \theta)}{\cos\theta}$$

$$= \frac{\tan\theta}{\cot\theta} + \frac{\cos\theta}{\cos\theta}$$

$$= \frac{\tan\theta}{\left(\frac{1}{\tan\theta}\right)} + 1$$

$$= \tan^2\theta + 1$$

$$= \sec^2\theta = \text{R.H.S.}$$

$$(iii) \text{L.H.S.} = \frac{(\cos(90^\circ - \theta)\cos\theta)}{\tan\theta + \cos^2(90^\circ - \theta)}$$

$$= \frac{(\sin\theta\cos\theta)}{\tan\theta + \sin^2\theta}$$

$$= \frac{(\sin\theta\cos\theta)}{\left(\frac{\sin\theta}{\cos\theta}\right)} + \sin^2\theta$$

$$= \cos^2\theta + \sin^2\theta$$

$$=1$$

$$= \text{R.H.S.}$$

$$\text{(iv)} \sin(90^\circ - \theta) \cos(90^\circ - \theta) = \frac{\tan \theta}{(1 + \tan^2 \theta)}$$

$$\text{L.H.S.} = \sin(90^\circ - \theta) \cos(90^\circ - \theta)$$

$$= \cos \theta \sin \theta$$

$$\left\{ \begin{array}{l} \because \sin(90^\circ - \theta) = \cos \theta \\ \sin^2 \theta + \cos^2 \theta = 1 \end{array} \right\}$$

$$\text{R.H.S.} = \frac{\tan \theta}{1 + \tan^2 \theta} = \frac{\left(\frac{\sin \theta}{\cos \theta} \right)}{1 + \frac{\sin^2 \theta}{\cos^2 \theta}}$$

$$= \frac{\frac{\sin \theta}{\cos \theta}}{\frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta}}$$

$$= \frac{\frac{\sin \theta}{\cos \theta}}{\frac{1}{\cos^2 \theta}} = \frac{\sin \theta}{\cos \theta} \times \cos^2 \theta = \sin \theta \cos \theta$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

Prove that following (12 to 30) identities, where the angles involved are acute angles for which the trigonometric ratios as defined :

12. (i) $(\sec A + \tan A) (1 - \sin A) = \cos A$

(ii) $(1 + \tan^2 A)(1 - \sin A)(1 + \sin A) = 1$.

Solution:

Given .

(i) L.H.S. = $(\sec A + \tan A) (1 - \sin A)$

$$= \left(\frac{1}{\cos A} + \frac{\sin A}{\cos A} \right) (1 - \sin A)$$

$$= \frac{(1 + \sin A) \times (1 - \sin A)}{\cos A} = \frac{1 - \sin^2 A}{\cos A}$$

$$= \frac{\cos^2 A}{\cos A} = \cos A$$

= R.H.S.

$$(1 - \sin^2 A = \cos^2 A)$$

$$(ii) \text{ L.H.S.} = (1 + \tan^2 A)(1 - \sin A)(1 + \sin A)$$

$$= \left(1 + \frac{\sin^2 A}{\cos^2 A}\right) (1 - \sin^2 A)$$

$$= \frac{\cos^2 A + \sin^2 A}{\cos^2 A} \times \cos^2 A$$

$$= \frac{1}{\cos^2 A} \times \cos^2 A$$

$$= 1$$

$$= \text{R.H.S.}$$

$$\left\{ \begin{array}{l} \because 1 - \sin^2 A = \cos^2 A \\ \sin^2 A + \cos^2 A = 1 \end{array} \right\}$$

$$**13. (i) \tan A + \cot A = \sec A \operatorname{cosec} A**$$

$$**(ii) (1 - \cos A)(1 + \sec A) = \tan A \sin A**$$

Solution:

$$**(i) \text{ L.H.S.} = \tan A + \cot A**$$

$$= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$$

$$= \frac{(\sin^2 A + \cos^2 A)}{(\sin A \cos A)}$$

$$= \frac{1}{(\sin A \operatorname{cosec} A)}$$

$$= \sec A \operatorname{cosec} A$$

$$= \text{R.H.S.}$$

$$(ii) \text{ L.H.S.} = (1 - \cos A) (1 + \sec A)$$

$$= (1 - \cos A) \left(1 + \frac{1}{\cos A}\right)$$

$$= (1 - \cos A) \frac{(\cos A + 1)}{\cos A}$$

$$= \frac{(1 - \cos A)(1 + \cos A)}{\cos A} = \frac{1 - \cos^2 A}{\cos A} = \frac{\sin^2 A}{\cos A}$$

$$= \frac{\sin^2 A}{\cos A} = \sin A \times \frac{\sin A}{\cos A}$$

$$\{1 - \cos^2 A = \sin^2 A\}$$

$$= \tan A \sin A = \text{R.H.S.}$$

$$14. (i) \frac{1}{(1 + \cos A)} + \frac{1}{(1 - \cos A)} = 2 \operatorname{cosec}^2 A$$

$$(ii) \frac{1}{(\sec A + \tan A)} + \frac{1}{(\sec A - \tan A)} = 2 \sec A$$

Solution:

$$(i) \frac{1}{(1 + \cos A)} + \frac{1}{(1 - \cos A)} = 2 \operatorname{cosec}^2 A$$

$$= \frac{1 - \cos A + 1 + \cos A}{(1 + \cos A)(1 - \cos A)}$$

$$= \frac{2}{1 - \cos^2 A}$$

$$= \frac{2}{\sin^2 A}$$

$$(\because 1 - \cos^2 A = \sin^2 A)$$

$$= 2 \operatorname{cosec}^2 A$$

$$= \text{R.H.S.}$$

$$(ii) \frac{1}{(\sec A + \tan A)} + \frac{1}{(\sec A - \tan A)} = 2\sec A$$

L.H.S. =

$$\frac{1}{(\sec A + \tan A)} + \frac{1}{(\sec A - \tan A)}$$

$$= \frac{\sec A - \tan A + \sec A + \tan A}{(\sec A + \tan A)(\sec A - \tan A)}$$

$$= \frac{2 \sec A}{\sec^2 A - \tan^2 A}$$

$$= \frac{2 \sec A}{1}$$

$$(\because \sec^2 A - \tan^2 A = 1)$$

$$= 2 \sec A = \text{R.H.S.}$$

$$15. (i) \frac{\sin A}{(1 + \cos A)} = \frac{(1 - \cos A)}{\sin A}$$

$$(ii) \frac{(1 - \tan^2 A)}{(\cot^2 A - 1)} = \tan^2 A$$

$$(iii) \frac{\sin A}{(1 + \cos A)} = \operatorname{cosec} A - \cot A$$

Solution:

$$(i) \text{ L.H.S. } = \frac{\sin A}{(1 + \cos A)}$$

On multiplying and dividing by (1 - cos A), we have

$$= \frac{\sin A (1 - \cos A)}{(1 - \cos^2 A)}$$

$$= \frac{\sin A (1 - \cos A)}{\sin^2 A}$$

$$(\because 1 - \cos^2 A = \sin^2 A)$$

$$= \frac{(1 - \cos A)}{\sin A} = \text{R.H.S.}$$

$$(ii) \frac{1 - \tan^2 A}{\cot^2 A - 1} = \frac{1 - \frac{\sin^2 A}{\cos^2 A}}{\frac{\cos^2 A}{\sin^2 A} - 1}$$

$$= \frac{\frac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\frac{\cos^2 A - \sin^2 A}{\sin^2 A}}$$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A} \times \frac{\sin^2 A}{\cos^2 A - \sin^2 A}$$

$$= \frac{\sin^2 A}{\cos^2 A}$$

$$= \tan^2 A$$

$$= \text{R.H.S.}$$

$$(iii) \text{R.H.S.} = \operatorname{cosec} A - \cot A$$

$$= \frac{1}{\sin A} - \frac{\cos A}{\sin A} = \frac{1 - \cos A}{\sin A}$$

$$= \frac{(1 - \cos A)(1 + \cos A)}{\sin A(1 + \cos A)}$$

{ Multiplying and dividing by $1 + \cos A$ }

$$= \frac{1 - \cos^2 A}{\sin A(1 + \cos A)} = \frac{\sin^2 A}{\sin A(1 + \cos A)}$$

$$\{\because 1 - \cos^2 A = \sin^2 A\}$$

$$= \frac{\sin A}{1 + \cos A} = \text{L.H.S.}$$

$$16. \text{ (i) } \frac{(\sec A - 1)}{(\sec A + 1)} = \frac{(1 - \cos A)}{(1 + \cos A)}$$

$$\text{(ii) } \frac{\tan^2 \theta}{(\sec \theta - 1)^2} = \frac{(1 + \cos \theta)}{(1 - \cos \theta)}$$

$$\text{(iii) } (1 + \tan A)^2 + (1 - \tan A)^2 = 2 \sec^2 A$$

$$\text{(iv) } \sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \cdot \operatorname{cosec}^2 A$$

Solution:

$$\text{(i) } \frac{(\sec A - 1)}{(\sec A + 1)} = \frac{(1 - \cos A)}{(1 + \cos A)}$$

$$= \frac{\frac{1 - \cos A}{\cos A}}{\frac{1 + \cos A}{\cos A}} = \frac{1 - \cos A}{\cos A} \times \frac{\cos A}{1 + \cos A}$$

$$= \frac{1 - \cos A}{1 + \cos A} = \text{R.H.S.}$$

$$(ii) \frac{\tan^2 \theta}{(\sec \theta - 1)^2} = \frac{(1 + \cos \theta)}{(1 - \cos \theta)}$$

$$= \frac{\frac{\sin^2 \theta}{\cos^2 \theta}}{\frac{1}{\cos^2 \theta} + 1 - \frac{2}{\cos \theta}}$$

$$= \frac{\sin^2 \theta}{\cos^2 \theta} \times \frac{\cos^2 \theta}{1 + \cos^2 \theta - 2 \cos \theta}$$

$$= \frac{\sin^2 \theta}{\cos^2 \theta} \times \frac{\cos^2 \theta}{1 + \cos^2 \theta - 2 \cos \theta}$$

$$= \frac{\sin^2 \theta}{(1 - \cos \theta)^2} = \frac{(1 - \cos^2 \theta)}{(1 - \cos \theta)^2}$$

$$= \frac{(1 + \cos \theta)(1 - \cos \theta)}{(1 - \cos \theta)^2}$$

$$= \frac{1 + \cos \theta}{1 - \cos \theta}$$

R.H.S. Proved

$$\begin{aligned}
& \text{(iii)} \quad (1 + \tan A)^2 + (1 - \tan A)^2 = 2\sec^2 A \\
& = 1 + 2\tan A + \tan^2 A + 1 - 2\tan A + \tan^2 A \\
& = 2 + 2\tan^2 A \\
& = 2(1 + \tan^2 A) \quad [\text{As } 1 + \tan^2 A = \sec^2 A] \\
& = 2\sec^2 A \\
& = \text{R.H.S.}
\end{aligned}$$

$$\begin{aligned}
& \text{(iv)} \quad \sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \cdot \operatorname{cosec}^2 A \\
& \text{L.H.S.} = \sec^2 A + \operatorname{cosec}^2 A \\
& = \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} \\
& = \frac{(\sin^2 A + \cos^2 A)}{(\sin^2 A \cos^2 A)} \\
& = \frac{1}{(\sin^2 A \cos^2 A)} \\
& = \sec^2 A \operatorname{cosec}^2 A \\
& = \text{R.H.S.}
\end{aligned}$$

$$17. (i) \frac{(1+\sin A)}{\cos A} + \frac{\cos A}{(1+\sin A)} = 2\sec A$$

$$(ii) \frac{\tan A}{(\sec A - 1)} + \frac{\tan A}{(\sec A + 1)} = 2\operatorname{cosec} A$$

Solution:

$$\text{L.H.S.} = \frac{(1+\sin A)}{\cos A} + \frac{\cos A}{(1+\sin A)}$$

$$= \frac{(1+\sin A)(1+\sin A) + \cos^2 A}{\cos A (1+\sin A)}$$

$$= \frac{1 + \sin A + \sin A + \sin^2 A + \cos^2 A}{\cos A (1+\sin A)}$$

$$= \frac{1 + 2\sin A + 1}{\cos A (1+\sin A)}$$

$$= \frac{2 + 2\sin A}{\cos A (1+\sin A)}$$

$$= \frac{2(1+\sin A)}{\cos A (1+\sin A)}$$

$$= \frac{2}{\cos A}$$

$$= 2\sec A$$

$$= \text{R.H.S.}$$

$$(ii) \frac{\tan A}{(\sec A - 1)} + \frac{\tan A}{(\sec A + 1)} = 2 \operatorname{cosec} A$$

$$\text{L.H.S.} = \frac{\tan A}{(\sec A - 1)} + \frac{\tan A}{(\sec A + 1)}$$

$$= \tan A \left(\frac{1}{\sec A - 1} + \frac{1}{\sec A + 1} \right)$$

$$= \tan A \left(\frac{\sec A + 1 + \sec A - 1}{(\sec A - 1)(\sec A + 1)} \right)$$

$$= \frac{\tan A \times 2 \sec A}{\sec^2 A - 1}$$

$$= \frac{2 \sec A \tan A}{\tan^2 A}$$

$$= \frac{2 \sec A}{\tan A}$$

$$= \frac{2 \times 1 \times \cos A}{\cos A \times \sin A}$$

$$= \frac{2}{\sin A}$$

$$= 2 \operatorname{cosec} A$$

$$= \text{R.H.S.}$$

$$18. (i) \frac{\operatorname{cosec} A}{(\operatorname{cosec} A - 1)} + \frac{\operatorname{cosec} A}{(\operatorname{cosec} A + 1)} = 2 \sec^2 A$$

$$(ii) \cot A - \tan A = \frac{(2 \cos^2 A - 1)}{(\sin A - \cos A)}$$

$$(iii) \frac{(\cot A - 1)}{(2 - \sec^2 A)} = \frac{\cot A}{(1 + \tan A)}$$

Solution:

$$(i) \frac{\operatorname{cosec} A}{(\operatorname{cosec} A - 1)} + \frac{\operatorname{cosec} A}{(\operatorname{cosec} A + 1)} = 2\sec^2 A$$

Solution:

$$\text{L.H.S.} = (i) \frac{\operatorname{cosec} A}{(\operatorname{cosec} A - 1)} + \frac{\operatorname{Cosec} A}{(\operatorname{cosec} A + 1)} = 2\sec^2 A$$

$$= \operatorname{cosec} A \left[\frac{1}{(\operatorname{cosec} A - 1)} + \frac{1}{(\operatorname{cosec} A + 1)} \right]$$

$$= \operatorname{cosec} A \left[\frac{\operatorname{cosec} A + 1 + \operatorname{cosec} A - 1}{(\operatorname{cosec} A - 1)(\operatorname{cosec} A + 1)} \right]$$

$$= \frac{\operatorname{cosec} A \times 2\operatorname{cosec} A}{\operatorname{Cosec}^2 A - 1} = \frac{2\operatorname{Cosec}^2 A}{\operatorname{Cot}^2 A}$$

$$= 2\sec^2 A$$

$$= \text{R.H.S.}$$

$$(ii) \cot A - \tan A = \frac{(2\cos^2 A - 1)}{(\sin A - \cos A)}$$

$$= \frac{\cos A}{\sin A} - \frac{\sin A}{\cos A} = \frac{\cos^2 A - \sin^2 A}{\sin A \cos A}$$

$$= \frac{\cos^2 A - (1 - \cos^2 A)}{\sin A \cos A}$$

$$= \frac{\cos^2 A - 1 + \cos^2 A}{\sin A \cos A}$$

$$= \frac{2\cos^2 A - 1}{\sin A \cos A} = \text{R.H.S.}$$

$$(iii) \frac{(\cot A - 1)}{(2 - \sec^2 A)} = \frac{\cot A}{(1 + \tan A)}$$

$$= \frac{\frac{\cos A - \sin A}{\sin A}}{\frac{2\cos^2 A - 1}{\cos^2 A}}$$

$$= \frac{\cos A - \sin A}{\sin A} \times \frac{\cos^2 A}{2\cos^2 A - 1}$$

$$= \frac{\cos^2 A (\cos A - \sin A)}{\sin A (2\cos^2 A - 1)}$$

$$= \frac{\cos^2 A (\cos A - \sin A)}{\sin A [2\cos^2 A - (\sin^2 A + \cos^2 A)]}$$

$$= \frac{\cos^2 A (\cos A - \sin A)}{\sin A [2\cos^2 A - \sin^2 A - \cos^2 A]}$$

$$= \frac{\cos^2 A (\cos A - \sin A)}{\sin A (\cos^2 A - \sin^2 A)}$$

$$= \frac{\cos^2 A (\cos A - \sin A)}{\sin A (\cos A + \sin A)(\cos A - \sin A)}$$

$$= \frac{\cos^2 A}{\sin A (\cos A + \sin A)}$$

$$\text{R.H.S.} = \frac{\cot A}{1+\tan A} = \frac{\frac{\cos A}{\sin A}}{1+\frac{\sin A}{\cos A}}$$

$$= \frac{\frac{\cos A}{\sin A}}{\frac{\cos A + \sin A}{\cos A}}$$

$$= \frac{\cos A}{\sin A} \times \frac{\cos A}{\cos A + \sin A}$$

$$= \frac{\cos^2 A}{\sin A (\cos A + \sin A)}$$

$$\therefore L.H.S. = R.H.S.$$

$$19. \text{ (i) } \tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$$

$$\text{(ii) } \frac{\cos \theta}{(1-\tan \theta)} - \frac{\sin^2 \theta}{(\cos \theta - \sin \theta)} = \cos \theta + \sin \theta$$

Solution:

$$\text{(i) L.H.S.} = \tan^2 \theta - \sin^2 \theta$$

$$= \frac{\sin^2 \theta}{\cos^2 \theta} - \sin^2 \theta$$

$$= \frac{\sin^2 \theta - \sin^2 \theta \cos^2 \theta}{\cos^2 \theta}$$

$$= \frac{\sin^2 \theta (1 - \cos^2 \theta)}{\cos^2 \theta}$$

$$\begin{aligned}
&= \sin^2 \theta \times \frac{\sin^2 \theta}{\cos^2 \theta} \\
&= \sin^2 \theta \times \tan^2 \theta \\
&= \tan^2 \theta \sin^2 \theta \\
&= \text{R.H.S.}
\end{aligned}$$

$$(ii) \frac{\cos \theta}{(1-\tan \theta)} - \frac{\sin^2 \theta}{(\cos \theta - \sin \theta)} = \cos \theta + \sin \theta$$

$$\text{L.H.S.} \frac{\cos \theta}{(1-\tan \theta)} - \frac{\sin^2 \theta}{(\cos \theta - \sin \theta)}$$

$$= \frac{\cos \theta}{1 - \frac{\sin \theta}{\cos \theta}} - \frac{\sin^2 \theta}{(\cos \theta - \sin \theta)}$$

$$= \frac{\cos \theta}{\frac{\cos \theta - \sin \theta}{\cos \theta}} - \frac{\sin^2 \theta}{(\cos \theta - \sin \theta)}$$

$$= \frac{\cos^2 \theta}{\cos \theta - \sin \theta} - \frac{\sin^2 \theta}{(\cos \theta - \sin \theta)}$$

$$= \frac{\cos^2 \theta - \sin^2 \theta}{\cos \theta - \sin \theta}$$

$$= \frac{(\cos \theta + \sin \theta)(\cos \theta - \sin \theta)}{\cos \theta - \sin \theta}$$

$$= \cos \theta + \sin \theta$$

$$= \text{R.H.S.}$$

$$20. (i) \operatorname{cosec}^4 \theta - \operatorname{cosec}^2 \theta = \cot^4 \theta + \cot^2 \theta$$

$$(ii) 2\sec^2 \theta - \sec^4 \theta - 2\operatorname{cosec}^2 \theta + \operatorname{cosec}^4 \theta = \cot^4 \theta - \tan^4 \theta$$

Solution:

$$\begin{aligned} (ii) \text{ L.H.S.} &= \operatorname{cosec}^4 \theta - \operatorname{cosec}^2 \theta \\ &= \operatorname{cosec}^2 \theta (\operatorname{cosec}^2 \theta - 1) \\ &= \operatorname{cosec}^2 \theta \cot^2 \theta [\operatorname{cosec}^2 \theta - 1 = \cot^2 \theta] \\ &= (\cot^2 \theta + 1) \cot^2 \theta \\ &= \cot^4 \theta + \cot^2 \theta \\ &= \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} (ii) \text{ L.H.S.} &= \\ &2\sec^2 \theta - \sec^4 \theta - 2\operatorname{cosec}^2 \theta + \operatorname{cosec}^4 \theta = \cot^4 \theta - \tan^4 \theta \\ &= 2(\tan^2 \theta + 1) - (\tan^2 \theta + 1)^2 - 2(1 + \cot^2 \theta) + (1 + \cot^2 \theta)^2 \end{aligned}$$

$$\left\{ \begin{array}{l} \because \sec^2 \theta = \tan^2 \theta + 1 \\ \operatorname{Cosec}^2 \theta = 1 + \cot^2 \theta \end{array} \right\}$$

$$\begin{aligned} &= 2\tan^2 \theta + 2 - (\tan^4 \theta + 2\tan^2 \theta + 1) - 2 - 2\cot^2 \theta + (1 + 2\cot^2 \theta + \cot^4 \theta) \\ &= 2\tan^2 \theta + 2 - \tan^4 \theta - 2\tan^2 \theta - 1 - 2 - 2\cot^2 \theta + 1 + 2\cot^2 \theta + \cot^4 \theta \\ &= \cot^4 \theta - \tan^4 \theta = \text{R.H.S.} \end{aligned}$$

21.

$$(i) \frac{1+\cos\theta-\sin^2\theta}{\sin\theta (1+\cos\theta)} = \cot\theta$$

$$(ii) \frac{(\tan^3\theta-1)}{(\tan\theta-1)} = \sec^2\theta + \tan\theta$$

Solution:

$$(i) \text{ L.H.S.} = \frac{1+\cos\theta-\sin^2\theta}{\sin\theta (1+\cos\theta)}$$

$$= \frac{\cos\theta + \cos^2\theta}{\sin\theta (1+\cos\theta)}$$

$$\{\because 1 - \sin^2\theta = \cos^2\theta\}$$

$$= \frac{\cos\theta(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

$$= \text{R.H.S.}$$

$$(ii) \frac{(\tan^3\theta-1)}{(\tan\theta-1)} = \sec^2\theta + \tan\theta$$

$$\text{L.H.S.} = \frac{(\tan^3\theta-1)}{\tan\theta-1} (\tan^2\theta + \tan\theta + 1)$$

$$\{\because a^3 - b^3 = (a - b)(a^2 + ab + b^2)\}$$

$$= \tan^2\theta + \tan\theta + 1$$

$$= \tan^2\theta + 1 + \tan\theta$$

$$= \sec^2\theta + \tan\theta$$

$$= \text{R.H.S.} \quad \{\because \sec^2\theta = \tan^2\theta + 1\}$$

$$22. (i) \frac{(1+\operatorname{cosec} A)}{\operatorname{cosec} A} = \frac{\cos^2 A}{(1-\sin A)}$$

$$(ii) \sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A}$$

Solution:

$$(i) \frac{(1+\operatorname{cosec} A)}{\operatorname{cosec} A} = \frac{\cos^2 A}{(1-\sin A)}$$

$$\text{L.H.S.} = \frac{(1+\operatorname{cosec} A)}{\operatorname{cosec} A}$$

$$= \frac{1 + \frac{1}{\sin A}}{\frac{1}{\sin A}}$$

$$= \frac{\sin A + 1}{\sin A} \times \frac{\sin A}{1}$$

$$= \sin A + 1$$

$$\text{R.H.S.} = \frac{\cos^2 A}{(1-\sin A)} = \frac{1-\sin^2 A}{1-\sin A}$$

$$= \frac{(1+\sin A)(1-\sin A)}{1-\sin A} = 1 + \sin A = \sin A + 1$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

$$(ii) \sqrt{\frac{1-\cos A}{1+\cos A}} = \operatorname{cosec} A - \cot A$$

$$\text{L.H.S.} = \sqrt{\frac{1-\cos A}{1+\cos A}}$$

Rationalising the denominators

$$= \sqrt{\frac{(1-\cos A)(1-\cos A)}{(1+\cos A)(1+\cos A)}}$$

$$= \sqrt{\frac{(1-\cos A)^2}{1-\cos^2 A}}$$

$$= \sqrt{\frac{(1-\cos A)^2}{\sin^2 A}}$$

$$= \frac{1-\cos A}{\sin A}$$

$$= \frac{1}{\sin A} - \frac{\cos A}{\sin A}$$

$$= \operatorname{cosec} A - \cot A = R.H.S.$$

$$23. \text{ (i) } \sqrt{\frac{1+\sin A}{1-\sin A}} = \tan A + \sec A$$

$$\text{(ii) } \sqrt{\frac{1-\cos A}{1+\cos A}} = \operatorname{cosec} A - \cot A$$

Solution:

$$\text{(i) } \sqrt{\frac{1+\sin A}{1-\sin A}} = \tan A + \sec A$$

$$\text{L.H.S.} = \sqrt{\frac{1+\sin A}{1-\sin A}}$$

Rationalising the denominators

$$= \frac{\sqrt{(1+\sin A)(1-\sin A)}}{\sqrt{(1-\sin A)(1+\sin A)}}$$

$$= \sqrt{\frac{(1+\sin A)^2}{1-\sin^2 A}} = \sqrt{\frac{(1+\sin A)^2}{\cos^2 A}}$$

$$= \frac{1+\sin A}{\cos A}$$

$$= \frac{1}{\cos A} + \frac{\sin A}{\cos A}$$

$$= \sec A + \tan A = \tan A + \sec A = R.H.S.$$

$$(ii) \sqrt{\frac{1-\cos A}{1+\cos A}} = \operatorname{cosec} A - \cot A$$

$$\text{L.H.S.} = \sqrt{\frac{1-\cos A}{1+\cos A}}$$

Rationalising the denominators

$$= \frac{\sqrt{(1-\cos A)(1-\cos A)}}{\sqrt{(1+\cos A)(1-\cos A)}}$$

$$= \sqrt{\frac{(1-\cos A)^2}{1-\cos^2 A}}$$

$$= \sqrt{\frac{(1-\cos A)^2}{\sin^2 A}}$$

$$\begin{aligned}
&= \frac{1 - \cos A}{\sin A} \\
&= \frac{1}{\sin A} - \frac{\cos A}{\sin A} \\
&= \operatorname{cosec} A - \cot A = R.H.S.
\end{aligned}$$

$$24. (i) \sqrt{\frac{\sec A - 1}{\sec A + 1}} + \sqrt{\frac{\sec A + 1}{\sec A - 1}} = 2 \operatorname{cosec} A$$

$$(ii) \frac{\cos A \cot A}{(1 - \sin A)} = 1 + \operatorname{cosec} A$$

Solution:

$$(i) \sqrt{\frac{\sec A - 1}{\sec A + 1}} + \sqrt{\frac{\sec A + 1}{\sec A - 1}} = 2 \operatorname{cosec} A$$

$$\text{L.H.S.} = \sqrt{\frac{\sec A - 1}{\sec A + 1}} + \sqrt{\frac{\sec A + 1}{\sec A - 1}}$$

$$= \frac{\sqrt{\sec A - 1}}{\sqrt{\sec A + 1}} + \frac{\sqrt{\sec A + 1}}{\sqrt{\sec A - 1}}$$

$$= \frac{\sec A - 1 + \sec A + 1}{\sqrt{(\sec A + 1)(\sec A - 1)}} = \frac{2 \sec A}{\sqrt{\sec^2 A - 1}}$$

$$\{\because \sec^2 A - 1 = \tan^2 A\}$$

$$= \frac{2 \sec A}{\sqrt{\tan^2 A}} = \frac{2 \sec A}{\tan A}$$

$$= \frac{2 \times \cos A}{\cos A \times \sin A} = \frac{2}{\sin A}$$

$$= 2 \operatorname{cosec} A$$

$$= R.H.S.$$

$$(ii) \frac{\cos A \cot A}{(1 - \sin A)} = 1 + \operatorname{cosec} A$$

$$\text{L.H.S.} = \frac{\cos A \cot A}{(1 - \sin A)} = \frac{\cos A \cos A}{\sin A (1 - \sin A)}$$

$$\left\{ \cos A = \frac{\cos A}{\sin A} \right\}$$

$$= \frac{\cos^2 A}{\sin A (1 - \sin A)} = \frac{1 - \sin^2 A}{\sin A (1 - \sin A)}$$

$$\{ \because \cos^2 A = 1 - \sin^2 A \}$$

$$= \frac{(1 + \sin A)(1 - \sin A)}{\sin A (1 - \sin A)} = \frac{1 + \sin A}{\sin A}$$

$$= \frac{1}{\sin A} + \frac{\sin A}{\sin A}$$

$$= \operatorname{cosec} A + 1$$

$$= 1 + \operatorname{cosec} A \quad \text{R.H.S.}$$

$$25. (i) \frac{(1 + \tan A)}{\sin A} + \frac{(1 + \cot A)}{\cos A} = 2(\sec A + \operatorname{cosec} A)$$

$$(ii) \sec^4 A - \tan^4 A = 1 + 2 \tan^2 A$$

Solution:

$$(i) \frac{(1 + \tan A)}{\sin A} + \frac{(1 + \cot A)}{\cos A} = 2(\sec A + \operatorname{cosec} A)$$

$$\text{L.H.S.} = \frac{(1 + \tan A)}{\sin A} + \frac{(1 + \cot A)}{\cos A}$$

$$= \frac{1 + \left(\frac{\sin A}{\cos A} \right)}{\sin A} + \frac{1 + \left(\frac{\cos A}{\sin A} \right)}{\cos A}$$

$$= \frac{\cos A + \sin A}{\sin A \times \cos A} + \frac{\sin A + \cos A}{\cos A \times \sin A}$$

$$= 2 \left[\frac{\cos A + \sin A}{\cos A \sin A} \right]$$

$$= 2 \left[\frac{\cos A}{\cos A \sin A} + \frac{\sin A}{\sin A \cos A} \right]$$

$$= 2 \left[\frac{1}{\sin A} + \frac{1}{\cos A} \right]$$

$$= 2 [\operatorname{cosec} A + \sec A]$$

$$= 2 (\sec A + \operatorname{cosec} A)$$

$$= \text{R.H.S.}$$

$$(ii) \sec^4 A - \tan^4 A = 1 + 2\tan^2 A$$

$$\text{L.H.S.} = \sec^4 A - \tan^4 A$$

$$= (\sec^2 A - \tan^2 A)(\sec^2 A + \tan^2 A)$$

$$= (1 + \tan^4 A - \tan^4 A)(1 + \tan^4 A + \tan^4 A)$$

$$[As \sec^2 A = \tan^2 A + 1]$$

$$= 1(1 + 2\tan^2 A)$$

$$= 1 + 2\tan^2 A = \text{R.H.S.}$$

$$\mathbf{26. (i) \operatorname{cosec}^6 A - \cot^6 A = 3\cot^2 A \operatorname{cosec}^2 A + 1}$$

$$(ii) \sec^6 A - \tan^6 A = 1 + 3\tan^2 A + 3\tan^4 A$$

Solution:

$$(i) \operatorname{cosec}^6 A - \cot^6 A = 3\cot^2 A \operatorname{cosec}^2 A + 1$$

$$\text{L.H.S.} = \operatorname{cosec}^6 A - \cot^6 A$$

$$= (\operatorname{cosec}^2 A)^3 - (\cot^2 A)^3$$

$$= (\operatorname{cosec}^2 \theta - \cot^2 A)^3 + 3 \operatorname{cosec}^2 A \cot^2 A (\operatorname{cosec}^2 \theta - \cot^2 A)$$

$$= (1)^3 + 3 \operatorname{cosec}^2 A \cot^2 A \times 1$$

$$= \mathbf{1 + 3\cot^2 A \operatorname{cosec}^2 A + 1}$$

$$= \text{R.H.S.}$$

$$(ii) \sec^6 A - \tan^6 A = 1 + 3\tan^2 A + 3\tan^4 A$$

Solution:

$$= (\sec^2 A)^3 - (\tan^2 A)^3$$

$$= (\sec^2 \theta - \tan^2 A)^3 + 3 \sec^2 A \tan^2 A (\sec^2 \theta - \tan^2 A)$$

$$= (1)^3 + 3 \sec^2 A \tan^2 A \times 1$$

$$= 1 + 3 \sec^2 A \tan^2 A$$

$$= 1 + 3[(1 + \tan^2 A) \tan^2 A]$$

$$= 1 + 3[\tan^2 A + \tan^4 A]$$

$$= 1 + 3\tan^2 A + 3 \tan^4 A$$

$$= \text{R.H.S.}$$

$$27. (i) \frac{\cot\theta - \operatorname{cosec}\theta - 1}{\cot\theta - \operatorname{cosec}\theta + 1} = \frac{1 + \cos\theta}{\sin\theta}$$

$$(ii) \frac{\sin\theta}{\cot\theta + \operatorname{cosec}\theta} = 2 + \frac{\sin\theta}{\cot\theta + \operatorname{cosec}\theta}$$

Solution:

$$(i) \frac{\cot\theta - \operatorname{cosec}\theta - 1}{\cot\theta - \operatorname{cosec}\theta + 1} = \frac{1 + \cos\theta}{\sin\theta}$$

$$\text{L.H.S.} = \frac{\cot\theta - \operatorname{cosec}\theta - 1}{\cot\theta - \operatorname{cosec}\theta + 1}$$

$$= \frac{\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} - 1}{\frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta} + 1}$$

$$= \frac{\cos\theta + 1 - \sin\theta}{\sin\theta} \times \frac{\sin\theta}{\cos\theta - 1 + \sin\theta}$$

$$= \frac{\cos\theta + 1 - \sin\theta}{\cos\theta - 1 + \sin\theta}$$

$$= \frac{\cos\theta + (1 - \sin\theta)}{\cos\theta - (1 - \sin\theta)}$$

$$= \frac{[\cos\theta + (1 - \sin\theta)][\cos\theta + (1 - \sin\theta)]}{[\cos\theta - (1 - \sin\theta)][\cos\theta + (1 - \sin\theta)]}$$

$$= \frac{(\cos\theta + 1 - \sin\theta)^2}{\cos^2\theta - (1 + \sin^2\theta - 2\sin\theta)}$$

$$= \frac{\cos^2 \theta + \sin^2 \theta + 1 + 2\cos \theta - 2\sin \theta - 2\sin \theta \cos \theta}{\cos^2 \theta - 1 - \sin^2 \theta + 2\sin \theta}$$

$$= \frac{1 + 1 + 2\cos \theta - 2\sin \theta - 2\sin \theta \cos \theta}{1 - \sin^2 \theta - 1 - \sin^2 \theta + 2\sin \theta}$$

$$= \frac{2 + 2\cos \theta - 2\sin \theta - 2\sin \theta \cos \theta}{1 - \sin^2 \theta - 1 - \sin^2 \theta + 2\sin \theta}$$

$$= \frac{2 + 2\cos \theta - 2\sin \theta - 2\sin \theta \cos \theta}{2\sin \theta - 2\sin^2 \theta}$$

$$= \frac{2(1 + \cos \theta) - 2\sin \theta(1 + \cos \theta)}{2\sin \theta(1 - \sin \theta)}$$

$$= \frac{1 + \cos \theta}{\sin \theta}$$

$$= \text{R.H.S.}$$

$$(ii) \quad \frac{\sin \theta}{\cot \theta + \operatorname{cosec} \theta} = 2 + \frac{\sin \theta}{\cot \theta + \operatorname{cosec} \theta}$$

$$\text{L.H.S.} = \frac{\frac{\sin \theta}{\cos \theta}}{\frac{1}{\sin \theta}} + \sin \theta = \frac{\sin \theta}{\frac{\cos \theta + 1}{\sin \theta}}$$

$$= \frac{\sin^2 \theta}{1 + \cos \theta}$$

$$\begin{aligned}
 &= \frac{1-\cos^2\theta}{1+\cos\theta} = \frac{(1+\cos\theta)(1-\cos\theta)}{1+\cos\theta} \\
 &= 1 - \cos\theta
 \end{aligned}$$

$$\text{R.H.S.} = 2 + \frac{\sin\theta}{\cot\theta + \operatorname{cosec}\theta}$$

$$= 2 + \frac{\sin\theta}{\frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}}$$

$$= 2 + \frac{\sin^2\theta}{\cos\theta - 1}$$

$$= \frac{2\cos\theta - 2 + (1 - \cos^2\theta)}{\cos\theta - 1}$$

$$= \frac{2(\cos\theta - 1) + (1 + \cos\theta)(1 - \cos\theta)}{\cos\theta - 1}$$

$$= \frac{(\cos\theta - 1)(2 - 1 - \cos\theta)}{(\cos\theta - 1)}$$

$$= 1 - \cos\theta$$

$$\text{L.H.S.} = \text{R.H.S.}$$

$$28. (i) (\sin\theta + \cos\theta)(\sec\theta + \csc\theta) = 2 + \sec\theta\csc\theta$$

$$(ii) (\csc A - \sin A)(\sec A - \cos A)\sec^2 A = \tan A$$

Solution:

$$(i) (\sin\theta + \cos\theta)(\sec\theta + \csc\theta) = 2 + \sec\theta\csc\theta$$

$$\text{L.H.S.} = (\sin\theta + \cos\theta)(\sec\theta + \csc\theta)$$

$$= (\sin\theta + \cos\theta) \left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta} \right)$$

$$= \frac{(\sin\theta + \cos\theta)(\sin\theta + \cos\theta)}{(\sin\theta\cos\theta)}$$

$$= \frac{\sin^2\theta + \sin\theta\cos\theta + \cos\theta\sin\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1 + 2\sin\theta\cos\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta} + \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$$

$$= \csc\theta \sec\theta + 2$$

$$= 2 + \sec\theta\csc\theta.$$

$$= \text{R.H.S.}$$

$$(ii) (\csc A - \sin A)(\sec A - \cos A)\sec^2 A = \tan A$$

$$\text{L.H.S.} = (\csc A - \sin A)(\sec A - \cos A)\sec^2 A$$

$$= \left(\frac{1}{\sin A} - \sin A \right) \left(\frac{1}{\cos A} - \cos A \right) \frac{1}{\cos^2 A}$$

$$= \left(\frac{1 - \sin^2 A}{\sin A} \right) \left(\frac{1 - \cos^2 A}{\cos A} \right) \frac{1}{\cos^2 A}$$

$$= \frac{\cos^2 A}{\sin A} \cdot \frac{\sin^2 A}{\cos A} \cdot \frac{1}{\cos^2 A} = \frac{\sin A}{\cos A}$$

$$= \tan A$$

$$= \text{R.H.S.}$$

$$29. (i) \frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A} = 2$$

$$(ii) \frac{\tan^2 A}{1 + \tan^2 A} + \frac{\cot^2 A}{1 + \cot^2 A} = 1$$

Solution:

$$(i) \frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A} = 2$$

$$\text{L.H.S.} = \frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A}$$

$$= \frac{(\sin A + \cos A)(\sin^2 A - \sin A \cos A + \cos^2 A)}{(\sin A + \cos A)} + \frac{(\sin A - \cos A)(\sin^2 A + \sin A \cos A + \cos^2 A)}{(\sin A - \cos A)}$$

$$= (1 - \sin A \cos A) + (1 + \sin A \cos A)$$

$$[\because \sin^2 A + \cos^2 A = 1]$$

$$= 1 - \sin A \cos A + 1 + \sin A \cos A = 2 = \text{R.H.S.}$$

$$(ii) \frac{\tan^2 A}{1+\tan^2 A} + \frac{\cot^2 A}{1+\cot^2 A} = 1$$

Solution:

$$\text{L.H.S.} = \frac{\tan^2 A}{1+\tan^2 A} + \frac{\frac{1}{\tan^2 A}}{1+\frac{1}{\tan^2 A}}$$

$$= \frac{\tan^2 A}{1+\tan^2 A} + \frac{\frac{1}{\tan^2 A}}{\frac{\tan^2 A + 1}{\tan^2 A}}$$

$$= \frac{\tan^2 A}{1+\tan^2 A} + \frac{\tan^2 A}{\tan^2 A (\tan^2 A + 1)}$$

$$= \frac{\tan^2 A}{1+\tan^2 A} + \frac{1}{1+\tan^2 A}$$

$$= \frac{1+\tan^2 A}{1+\tan^2 A}$$

$$= 1$$

$$= \text{R.H.S.}$$

$$30. (i) \frac{1}{(\sec A + \tan A)} - \frac{1}{\cos A} = \frac{1}{\cos A} - \frac{1}{(\sec A - \tan A)}$$

$$(ii) (\sin A + \sec A)^2 + (\cos A + \operatorname{cosec} A)^2 = (1 + \sec A \operatorname{cosec} A)^2$$

$$(iii) \frac{(\tan A + \sin A)}{(\tan A - \sin A)} = \frac{(\sec A + 1)}{(\sec A - 1)}$$

Solution:

$$(i) \frac{1}{(\sec A + \tan A)} - \frac{1}{\cos A} = \frac{1}{\cos A} - \frac{1}{(\sec A - \tan A)}$$

$$\text{L.H.S.} = \frac{1}{(\sec A + \tan A)} - \frac{1}{\cos A}$$

$$= \frac{1}{\left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)} - \frac{1}{\cos A}$$

$$= \frac{\cos A}{(1 + \sin A)} - \frac{1}{\cos A}$$

$$= \frac{\cos^2 A - 1 - \sin A}{\cos A(1 + \sin A)} = \frac{-\sin^2 A - \sin A}{\cos A(1 + \sin A)}$$

$$= \frac{-\sin A (1 + \sin A)}{\cos A(1 + \sin A)} = -\tan A$$

$$\text{R.H.S.} = \frac{1}{\cos A} - \frac{1}{(\sec A - \tan A)}$$

$$= \frac{1}{\cos A} - \frac{1}{\frac{1}{\cos A} - \frac{\sin A}{\cos A}}$$

$$= \frac{1}{\cos A} - \frac{\cos A}{1 - \sin A}$$

$$= \frac{1 - \sin A - \cos^2 A}{\cos A(1 - \sin A)}$$

$$= \frac{\sin^2 A - \sin A}{\cos A(1 - \sin A)}$$

$$\begin{aligned}
&= \frac{-\sin A + \sin^2 A}{\cos A(1 - \sin A)} \\
&= \frac{-\sin A(1 - \sin A)}{\cos A(1 - \sin A)} \\
&= \frac{-\sin A}{\cos A} = -\tan A
\end{aligned}$$

$\therefore \text{L.H.S.} = \text{R.H.S.}$

$$(ii) (\sin A + \sec A)^2 + (\cos A + \operatorname{cosec} A)^2 = (1 + \sec A \operatorname{cosec} A)^2$$

$$\begin{aligned}
&\textbf{Solution: } (\sin A + \sec A)^2 + (\cos A + \operatorname{cosec} A)^2 \\
&= \sin^2 A + \sec^2 A + 2\sin A \sec A + \cos^2 A + \operatorname{cosec}^2 A + 2\sec A \operatorname{cosec} A \\
&= (\sin^2 A + \cos^2 A) + (\sec^2 A + \operatorname{cosec}^2 A) + 2\sin A \times \frac{1}{\cos A} + 2 \times \\
&\quad \cos A \times \frac{1}{\sin A} \\
&= 1 + \left[\frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} \right] + \frac{2\sin^2 A + 2\cos^2 A}{\sin A \cos A} \\
&= 1 + \left[\frac{\sin^2 A + \cos^2 A}{\cos^2 A \cdot \sin^2 A} \right] + \frac{2[\sin^2 A + 2\cos^2 A]}{\sin A \cos A} \\
&= 1 + \frac{1}{\cos^2 A \cdot \sin^2 A} + \frac{2}{\sin A \cdot \cos A} \quad [\because \sin^2 \theta + \cos^2 \theta = 1] \\
&= \left(1 + \frac{1}{\cos A \cdot \sin A} \right)^2 \quad [\because (a + b)^2 = a^2 + (b)^2 + 2ab] \\
&= (1 + \operatorname{cosec} A \sec A)^2 \\
&= \text{R.H.S.}
\end{aligned}$$

$$(iii) \frac{\tan A + \sin A}{\tan A - \sin A} = \frac{\sec A + 1}{\sec A - 1}$$

$$\begin{aligned} \text{L.H.S.} &= \frac{\tan A + \sin A}{\tan A - \sin A} = \frac{\frac{\sin A}{\cos A} + \sin A}{\frac{\sin A}{\cos A} - \sin A} \\ &= \frac{\frac{\sin A + \sin A \cos A}{\cos A}}{\frac{\sin A - \sin A \cos A}{\cos A}} \\ &= \frac{\sin A (1 + \cos A)}{\sin A (1 - \cos A)} = \frac{1 + \cos A}{1 - \cos A} \end{aligned}$$

On dividing each term by $\cos A$, we have

$$= \frac{\frac{1}{\cos A} + 1}{\frac{1}{\cos A} - 1} = \frac{\sec A + 1}{\sec A - 1} = \text{R.H.S.}$$

31. If $\sin \theta + \cos \theta = \sqrt{2} \sin (90^\circ - \theta)$, show that $\cot \theta = \sqrt{2} + 1$

Solution:

Given, $\sin \theta + \cos \theta = \sqrt{2} \sin (90^\circ - \theta)$

$$\sin \theta + \cos \theta = \sqrt{2} \cos \theta$$

On dividing by $\sin \theta$, we have

$$1 + \cot \theta = \sqrt{2} \cot \theta$$

$$1 = \sqrt{2} \cot \theta - \cot \theta$$

$$(\sqrt{2} - 1) \cot \theta = 1$$

$$\cot \theta = \frac{1}{(\sqrt{2} - 1)}$$

(Rationalising the denominators)

$$= \frac{1 \times (\sqrt{2} + 1)}{(\sqrt{2} - 1)(\sqrt{2} + 1)}$$

$$= \frac{(\sqrt{2} + 1)}{(\sqrt{2}^2) - (1^2)} = \frac{\sqrt{2} + 1}{2 - 1} = \frac{\sqrt{2} + 1}{1}$$

$$= \sqrt{2} + 1 = \text{R.H.S.}$$

$$\text{Hence, } \cot \theta = \sqrt{2} + 1$$

32. If $7\sin^2 \theta + 3\cos^2 \theta = 4$, $\theta^\circ \leq \theta \leq 90^\circ$, then find the value of θ .

Solution:

Given,

$$7\sin^2 \theta + 3\cos^2 \theta = 4, \theta^\circ \leq \theta \leq 90^\circ$$

$$3 \sin^2 \theta + 3\cos^2 \theta + 4 \sin^2 \theta = 4$$

$$3(\sin^2 \theta + \cos^2 \theta) + 4\sin^2 \theta = 4$$

$$3(1) + 4\sin^2 \theta = 4$$

$$4\sin^2 \theta = 4 - 3$$

$$\sin^2 \theta = \frac{1}{4}$$

Taking square-root on both sides, we get

$$\sin \theta = \frac{1}{2}$$

$$\text{Thus, } \theta = 30^\circ$$

33. If $\sec\theta + \tan\theta = m$ and $\sec\theta - \tan\theta = n$, prove that $mn = 1$.

Solution:

Given,

$$\sec\theta + \tan\theta = m$$

$$\sec\theta - \tan\theta = n$$

Now,

$$mn = (\sec\theta + \tan\theta)(\sec\theta - \tan\theta)$$

$$= \sec^2\theta - \tan^2\theta = 1$$

Thus, $mn = 1$

34. If $x = a\sec\theta + b\tan\theta$ and $y = a\tan\theta + b\sec\theta$, prove that $x^2 - y^2 = a^2 - b^2$

Solution:

Given,

$$x = a\sec\theta + b\tan\theta$$

$$y = a\tan\theta + b\sec\theta$$

Now,

$$x^2 - y^2 = (a\sec\theta + b\tan\theta)^2 - (a\tan\theta + b\sec\theta)^2$$

$$= (a^2\sec^2\theta + b^2\tan^2\theta + 2ab\sec\theta\tan\theta) - (a^2\tan^2\theta + b^2\sec^2\theta + 2ab\sec\theta\tan\theta)$$

$$= a^2\sec^2\theta + b^2\tan^2\theta + 2ab\sec\theta\tan\theta - a^2\tan^2\theta - b^2\sec^2\theta - 2ab\sec\theta\tan\theta$$

$$\begin{aligned}
&= a^2(\sec^2\theta - \tan^2\theta) - b^2(\sec^2\theta - \tan^2\theta) \\
&= a^2 \times 1 - b^2 \times 1 \quad \{ \sec^2\theta - \tan^2\theta = 1 \} \\
&= a^2 - b^2 \\
&= \text{Hence proved.}
\end{aligned}$$

35. If $x = h + a\cos\theta$ and $y = k + a\sin\theta$, prove that $(x - h)^2 + (y - k)^2 = a^2$

Solution:

Given,

$$x = h + a\cos\theta$$

$$y = k + a\sin\theta$$

Now,

$$x - h = a\cos\theta$$

$$y - k = a\sin\theta$$

On squaring and adding we get

$$(x - h)^2 + (y - k)^2 = a^2\cos^2\theta + a^2\sin^2\theta$$

$$= a^2(\sin^2\theta + \cos^2\theta)$$

$$= a^2(1)$$

$$[\text{Since, } \sin^2\theta + \cos^2\theta = 1]$$

= Hence proved.

Chapter Test

1. If θ is an acute angle $\operatorname{cosec} \theta = \sqrt{5}$, find the value of $\cot \theta - \cos \theta$.

(ii) If θ is an acute angle and $\tan \theta = \frac{8}{15}$, find the value of $\sec \theta + \operatorname{cosec} \theta$.

Solution:

Given, θ is an acute angle and $\operatorname{cosec} \theta = \sqrt{5}$

So,

$$\sin \theta = \frac{1}{\sqrt{5}}$$

$$\text{And, } \cos \theta = \sqrt{1 - \sin^2 \theta}$$

$$\cos \theta = \sqrt{1 - \left(\frac{1}{\sqrt{5}}\right)^2}$$

$$= \sqrt{\left(1 - \left(\frac{1}{5}\right)\right)}$$

$$= \sqrt{\frac{4}{5}}$$

$$\cos \theta = \frac{2}{\sqrt{5}}$$

Now,

$$\cot \theta - \cos \theta = \left(\frac{\cos \theta}{\sin \theta}\right) - \cos \theta$$

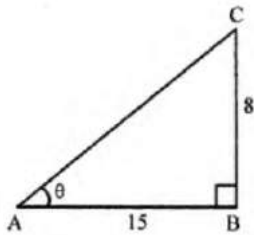
$$= \sqrt{1 - \left(\frac{1}{\sqrt{5}}\right)^2}$$

$$= \sqrt{1 - \frac{1}{5}}$$

$$= \sqrt{\frac{5-1}{5}}$$

$$= \sqrt{\frac{4}{5}}$$

$$= \frac{2}{\sqrt{5}}$$



(ii) Given, θ is an acute angle and $\tan \theta = \frac{8}{15}$

In Fig. we have

$$\tan \theta = \frac{BC}{AB} = \frac{8}{15}$$

So, $BC = 8$ and $AB = 15$

By Pythagoras theorem, we have

$$AC = \sqrt{AB^2 + BC^2}$$

$$= \sqrt{(15^2 + 8^2)}$$

$$= \sqrt{225 + 64}$$

$$= \sqrt{289}$$

$$\Rightarrow AC = 17$$

Now,

$$\sec \theta = \frac{AC}{AB} = \frac{17}{15}$$

$$\operatorname{cosec} \theta = \frac{AC}{BC} = \frac{17}{8}$$

So,

$$\sec \theta + \operatorname{Cosec} \theta = \frac{17}{15} + \frac{17}{8}$$

$$= \frac{(136+255)}{120}$$

$$= \frac{391}{120}$$

$$= 3\frac{31}{120}$$

2. Evaluate the following :

$$(i) 2 \times \left(\frac{\cos^2 20^\circ + \cos^2 70^\circ}{\sin^2 25^\circ + \sin^2 65^\circ} \right) - \tan 45^\circ + \tan 13^\circ \tan 23^\circ \tan 30^\circ \tan 67^\circ \tan 77^\circ$$

$$(ii) \frac{\sin^2 22^\circ + \sin^2 68^\circ}{\cos^2 22^\circ + \cos^2 68^\circ} + \sin^2 63^\circ + \cos^2 63^\circ \sin 27^\circ$$

Solution;

$$(i) 2 \times \left(\frac{\cos^2 20^\circ + \cos^2 70^\circ}{\sin^2 25^\circ + \sin^2 65^\circ} \right) - \tan 45^\circ + \tan 13^\circ \tan 23^\circ \tan 30^\circ \tan 67^\circ \tan 77^\circ$$

$$= 2 \left[\frac{\cos^2 (90^\circ - 70^\circ) + \cos^2 70^\circ}{\sin^2 25^\circ + \sin^2 (90^\circ - 25^\circ)} \right] - \tan 45^\circ + \tan 13^\circ \tan 77^\circ \tan 23^\circ \tan 67^\circ \tan 30^\circ$$

$$= 2 \left[\frac{\sin^2 70^\circ + \cos^2 70^\circ}{\sin^2 25^\circ + \cos^2 25^\circ} \right] - 1 + \tan 13^\circ \tan (90^\circ - 13^\circ) \tan 23^\circ \tan (90^\circ - 23^\circ) \times \frac{1}{\sqrt{3}}$$

$$= 2 \left(\frac{1}{1} \right) - 1 + \tan 13^\circ \cot 13^\circ \tan 23^\circ \cot 23^\circ \times \frac{1}{\sqrt{3}}$$

$$= 2 - 1 + 1 \times 1 \times \frac{1}{\sqrt{3}}$$

$$= 2 - 1 + \frac{1}{\sqrt{3}}$$

$$= 1 + \frac{1}{\sqrt{3}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}}$$

$$= \frac{(\sqrt{3}+1)\sqrt{3}}{\sqrt{3} \times \sqrt{3}}$$

$$= \frac{3+\sqrt{3}}{3}$$

$$(ii) \frac{\sec 29^\circ}{\operatorname{cosec} 61^\circ} + 2 \cot 8^\circ \cot 17^\circ \cot 45^\circ \cot 73^\circ \cot 82^\circ - 3(\sin^2 38^\circ + \sin^2 52^\circ)$$

$$= \frac{\sec 29^\circ}{\operatorname{cosec}(90^\circ - 29^\circ)} + 2 \cot 8^\circ \times \cot(90^\circ - 8^\circ) \times \cot 17^\circ \times \cot(90^\circ - 17^\circ) \cot 45^\circ - 3[\sin^2 38^\circ + \sin^2(90^\circ - 38^\circ)]$$

$$= \frac{\sec 29^\circ}{\sec 29^\circ} + 2 \cot 8^\circ \tan 8^\circ \times \cot 17^\circ \tan 17^\circ \times 1 - 3(\sin^2 38^\circ + \cos^2 38^\circ)$$

$$= 1 + 2 \times 1 \times 1 \times 1 - 3 \times 1 = 1 + 2 - 3 = 0$$

$$\begin{aligned}
& \text{(iii)} \quad \frac{\sin^2 22^\circ + \sin^2 68^\circ}{\cos^2 22^\circ + \cos^2 68^\circ} + \sin^2 63^\circ + \cos^2 63^\circ \sin 27^\circ \\
&= \frac{\sin^2 22^\circ + \sin^2 (90^\circ - 22^\circ)}{\cos^2 22^\circ + \cos^2 (90^\circ - 22^\circ)} + \sin^2 63^\circ + \cos^2 63^\circ \sin(90^\circ - 63^\circ) \\
&= \frac{\sin^2 22^\circ + \cos^2 22^\circ}{\cos^2 22^\circ + \sin^2 22^\circ} + \sin^2 63^\circ + \cos^2 63^\circ \times \cos^2 63^\circ \\
&= \frac{1}{1} + (\sin^2 63^\circ + \cos^2 63^\circ) \\
&= 1 + 1 \\
&= 2
\end{aligned}$$

3. If $\frac{4}{3} (\sec^2 59^\circ + \cot^2 31^\circ) - \frac{2}{3} \sin 90^\circ + 3 \tan^2 56^\circ \tan^2 34^\circ = \frac{x}{2}$, then find the value of x.

Solution :

Given

$$\begin{aligned}
& \frac{4}{3} (\sec^2 59^\circ + \cot^2 31^\circ) - \frac{2}{3} \sin 90^\circ + 3 \tan^2 56^\circ \tan^2 34^\circ = \frac{x}{2} \\
& \Rightarrow \frac{4}{3} [\sec^2 59^\circ - \tan^2 59^\circ] - \frac{2}{3} \times 1 + 3 \tan^2 56^\circ \cot^2 56^\circ = \frac{x}{2} \\
&= \frac{4}{3} \times 1 - \frac{2}{3} + 3 \times 1 = \frac{x}{2} \\
&= \frac{4}{3} - \frac{2}{3} + 3 = \frac{x}{2} \\
&= \frac{4-2+9}{3} = \frac{x}{2} \\
& \frac{11}{3} = \frac{x}{2} \\
& \Rightarrow x = \frac{11 \times 2}{3} = 11
\end{aligned}$$

$$\therefore x = 11$$

$$4. (i) \frac{\cos A}{(1-\sin A)} + \frac{\cos A}{(1+\sin A)} = 2 \sec A$$

$$(ii) \frac{\cos A}{(\operatorname{Cosec} A + 1)} + \frac{\cos A}{(\operatorname{Cosec} A - 1)} = 2 \tan A$$

Solution:

$$(i) \frac{\cos A}{(1-\sin A)} + \frac{\cos A}{(1+\sin A)} = 2 \sec A$$

$$\text{L.H.S.} = \frac{\cos A}{(1-\sin A)} + \frac{\cos A}{(1+\sin A)}$$

$$= \cos A \left[\frac{1}{(1-\sin A)} + \frac{1}{(1+\sin A)} \right]$$

$$= \cos A \left[\frac{1-\sin A + 1+\sin A}{(1-\sin A)(1+\sin A)} \right]$$

$$= \cos A \left[\frac{2}{2-\sin^2 A} \right] = \frac{2 \cos A}{\cos^2 A}$$

$$= \frac{2}{\cos A} = 2 \sec A$$

$$= \text{R.H.S.}$$

$$(ii) \frac{\cos A}{\operatorname{cosec} A + 1} + \frac{\cos A}{\operatorname{cosec} A - 1} = 2 \tan A$$

$$\text{L.H.S.} = \frac{\cos A}{\operatorname{cosec} A + 1} + \frac{\cos A}{\operatorname{cosec} A - 1}$$

$$= \cos A \left[\frac{1}{\operatorname{cosec} A + 1} + \frac{1}{\operatorname{cosec} A - 1} \right]$$

$$= \cos A \left[\frac{\operatorname{cosec} A - 1 + \operatorname{cosec} A + 1}{(\operatorname{cosec} A + 1)(\operatorname{cosec} A - 1)} \right]$$

$$= \frac{\cos A [2 \operatorname{cosec} A]}{\operatorname{cosec}^2 A - 1} = \frac{2 \cos A}{\sin A (\cot^2 A)}$$

$$= \frac{2 \cot A}{\cot^2 A} = \frac{2}{\cot A}$$

$$= 2 \tan A$$

$$= \text{R.H.S.}$$

$$5. (i) \frac{(\cos \theta - \sin \theta)(1 + \tan \theta)}{2 \cos^2 \theta - 1} = \sec \theta$$

$$(ii) (\operatorname{cosec} \theta - \sin \theta)(\sec \theta - \cos \theta)(\tan \theta + \cot \theta) = 1$$

Solution:

$$(i) \frac{(\cos \theta - \sin \theta)(1 + \tan \theta)}{2 \cos^2 \theta - 1} = \sec \theta$$

$$\text{L.H.S.} = \frac{(\cos \theta - \sin \theta)(1 + \tan \theta)}{2 \cos^2 \theta - 1}$$

$$= \frac{(\cos \theta - \sin \theta)(\cos \theta + \sin \theta)}{\cos \theta (2 \cos^2 \theta - 1)}$$

$$= \frac{\cos^2 \theta - \sin^2 \theta}{\cos \theta (2\cos^2 \theta - 1)} = \frac{\cos^2 \theta - (1 - \cos^2 \theta)}{\cos \theta (2\cos^2 \theta - 1)}$$

$$= \frac{\cos^2 \theta - 1 + \cos^2 \theta}{\cos \theta (2\cos^2 \theta - 1)} = \frac{2\cos^2 \theta - 1}{\cos \theta (2\cos^2 \theta - 1)}$$

$$= \frac{1}{\cos \theta}$$

$$= \sec \theta$$

$$= \text{R.H.S.}$$

$$(ii) (\operatorname{cosec} \theta - \sin \theta)(\sec \theta - \cos \theta)(\tan \theta + \cot \theta) = 1.$$

$$\text{L.H.S.} = (\operatorname{cosec} \theta - \sin \theta)(\sec \theta - \cos \theta)(\tan \theta + \cot \theta)$$

$$= \left(\frac{1}{\sin \theta} - \sin \theta \right) \left(\frac{1}{\cos \theta} - \cos \theta \right) \left(\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \right)$$

$$= \left(\frac{1 - \sin^2 \theta}{\sin \theta} \right) \left(\frac{1 - \cos^2 \theta}{\cos \theta} \right) \left(\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} \right)$$

$$= \frac{(\cos^2 \theta \times \sin^2 \theta)}{\sin \theta \cos \theta} \times \frac{1}{\sin \theta \cos \theta}$$

$$= \frac{\sin^2 \theta \cos^2 \theta}{\sin \theta \cos \theta}$$

$$= 1$$

$$= \text{R.H.S.}$$

$$6. (i) \sin^2 \theta + \cos^4 \theta = \cos^2 \theta + \sin^4 \theta$$

$$(ii) \frac{\cot \theta}{\operatorname{cosec} \theta + 1} + \frac{\operatorname{cosec} \theta + 1}{\cot \theta} = 2 \sec \theta$$

Solution:

Given,

$$(i) \sin^2 \theta + \cos^4 \theta = \cos^2 \theta + \sin^4 \theta$$

$$\text{L.H.S.} = \sin^2 \theta + \cos^4 \theta$$

$$= (1 - \cos^2 \theta) + \cos^4 \theta$$

$$= \cos^4 \theta - \cos^2 \theta + 1$$

$$= \cos^2 \theta (\cos^2 \theta - 1) + 1$$

$$= \cos^2 \theta (-\sin^2 \theta) + 1$$

$$= 1 - \sin^2 \theta \cos^2 \theta$$

Now,

$$\text{R.H.S.} = \cos^2 \theta + \sin^4 \theta$$

$$= (1 - \sin^2 \theta) + \sin^4 \theta$$

$$= \sin^4 \theta - \sin^2 \theta + 1$$

$$= \sin^2 \theta (\sin^2 \theta - 1) + 1$$

$$= \sin^2 \theta (-\cos^2 \theta) + 1$$

$$= 1 - \sin^2 \theta \cos^2 \theta$$

Hence, L.H.S. = R.H.S.

$$= \frac{\cot\theta}{\operatorname{cosec}\theta+1} + \frac{\operatorname{cosec}\theta+1}{\cot\theta} = 2\sec\theta$$

$$= \text{L.H.S.} = \frac{\cot\theta}{\operatorname{cosec}\theta+1} + \frac{\operatorname{cosec}\theta+1}{\cot\theta}$$

$$= \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}+1} + \frac{\frac{1}{\sin\theta}+1}{\frac{\cos\theta}{\sin\theta}}$$

$$7. \text{ (i) } \sec^4 A(1 - \sin^4 A) - 2\tan^2 A = 1$$

$$\text{ (ii) } \frac{1}{\sin A + \cos A + 1} + \frac{1}{\sin A + \cos A - 1} = \sec A + \operatorname{cosec} A$$

Solution:

$$\text{ (i) } \sec^4 A(1 - \sin^4 A) - 2\tan^2 A = 1$$

$$\text{L.H.S.} = \sec^4 A(1 - \sin^4 A) - 2\tan^2 A$$

$$= \frac{1}{\cos^4 A} (1 + \sin^2 A)(1 - \sin^2 A) - 2\tan^2 A$$

$$[\because a^2 - b^2 = (a + b)(a - b)]$$

$$= \frac{(1 + \sin^2 A)\cos^2 A}{\cos^4 A} - 2 \frac{\sin^2 A}{\cos^2 A}$$

$$= \frac{(1 + \sin^2 A)}{\cos^2 A} - \frac{2\sin^2 A}{\cos^2 A}$$

$$= \frac{1+\sin^2 A-2\sin^2 A}{\cos^2 A}$$

$$= \frac{1-\sin^2 A}{\cos^2 A}$$

$$= \frac{\cos^2 A}{\cos^2 A}$$

$$= 1$$

$$(\because 1 - \sin^2 A = \cos^2 A)$$

$$= \text{R.H.S.}$$

$$(ii) \frac{1}{\sin A + \cos A + 1} + \frac{1}{\sin A + \cos A - 1} = \sec A + \operatorname{cosec} A$$

$$\text{L.H.S.} = \frac{1}{\sin A + \cos A + 1} + \frac{1}{\sin A + \cos A - 1}$$

$$= \frac{\sin A + \cos A - 1 + \sin A + \cos A + 1}{(\sin A + \cos A + 1)(\sin A + \cos A - 1)}$$

$$= \frac{2(\sin A + \cos A)}{(\sin A + \cos A)^2 - (1)^2}$$

$$= \frac{2(\sin A + \cos A)}{\sin^2 A + \cos^2 A + 2\sin A \cos A - 1}$$

$$= \frac{2(\sin A + \cos A)}{2\sin A \cos A}$$

$$= \frac{\sin A + \cos A}{\sin A \cos A} = \frac{\sin A}{\sin A \cos A} + \frac{\cos A}{\sin A \cos A}$$

$$= \frac{1}{\cos A} + \frac{1}{\sin A}$$

$$= \sec A + \operatorname{cosec} A = \text{R.H.S.}$$

$$8. (i) \frac{\sin^3\theta + \cos^3\theta}{\sin\theta\cos\theta} + \sin\theta\cos\theta = 1$$

$$(ii) (\sec A - \tan A)^2(1 + \sin A) = 1 - \sin A$$

Solution:

$$\begin{aligned} \text{L.H.S.} &= \frac{\sin^3\theta + \cos^3\theta}{\sin\theta\cos\theta} + \sin\theta\cos\theta \\ &= \frac{(\sin\theta + \cos\theta)(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta)}{(\sin\theta + \cos\theta)} + \sin\theta\cos\theta \\ &= \sin^2\theta + \cos^2\theta - \sin\theta\cos\theta + \sin\theta\cos\theta \\ &= 1 \\ &= \text{R.H.S.} \end{aligned}$$

$$(ii) (\sec A - \tan A)^2(1 + \sin A) = 1 - \sin A$$

$$\begin{aligned} \text{L.H.S.} &= \left(\frac{1}{\cos A} - \frac{\sin A}{\cos A} \right)^2 (1 + \sin A) \\ &= \left(\frac{1 - \sin A}{\cos A} \right)^2 (1 + \sin A) \\ &= \frac{(1 - \sin A)^2(1 + \sin A)}{1 - \sin^2 A} \\ &= \frac{(1 - \sin A)^2(1 + \sin A)}{(1 - \sin A)(1 + \sin A)} \\ &= 1 - \sin A \\ &= \text{R.H.S.} \end{aligned}$$

$$9. (i) \frac{\cos A}{1-\tan A} - \frac{\sin^2 A}{\cos A - \sin A} = \sin A + \cos A$$

$$(ii) (\sec A - \operatorname{cosec} A)(1 + \tan A + \cot A) = \tan A \sec A - \cot A \operatorname{cosec} A$$

$$(iii) \frac{\tan^2 \theta}{\tan^2 \theta - 1} - \frac{\operatorname{cosec}^2 \theta}{\sec^2 \theta - \operatorname{cosec}^2 \theta} = \frac{1}{\sin^2 \theta - \cos^2 \theta}$$

Solution:

$$(i) \frac{\cos A}{1-\tan A} - \frac{\sin^2 A}{\cos A - \sin A} = \sin A + \cos A$$

$$\begin{aligned} \text{L.H.S.} &= \frac{\cos A}{1-\tan A} - \frac{\sin^2 A}{\cos A - \sin A} \\ &= \frac{\cos A}{1-\frac{\sin A}{\cos A}} - \frac{\sin^2 A}{\cos A - \sin A} \\ &= \frac{\cos A \times \cos A}{\cos A - \sin A} - \frac{\sin^2 A}{\cos A - \sin A} \\ &= \frac{\cos^2 A}{\cos A - \sin A} - \frac{\sin^2 A}{\cos A - \sin A} \\ &= \frac{\cos^2 A - \sin^2 A}{\cos A - \sin A} \\ &= \frac{(\cos A + \sin A)(\cos A - \sin A)}{(\cos A - \sin A)} = \cos A + \sin A \\ &= \text{R.H.S.} \end{aligned}$$

$$(ii) (\sec A - \operatorname{cosec} A) (1 + \tan A + \cot A) = \tan A \sec A - \cot A \operatorname{cosec} A$$

$$\text{L.H.S.} = (\sec A - \operatorname{cosec} A) (1 + \tan A + \cot A)$$

$$= \left(\frac{1}{\cos A} - \frac{1}{\sin A} \right) \left(1 + \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} \right)$$

$$= \frac{\sin A - \cos A}{\sin A \cos A} \times \frac{\sin A \cos A + \sin^2 A + \cos^2 A}{\sin A \cos A}$$

$$= \frac{(\sin A - \cos A)(\sin A \cos A + 1)}{\sin^2 A \cos^2 A}$$

$$\text{R.H.S.} = \tan A \sec A - \cot A \operatorname{cosec} A$$

$$= \frac{\sin A}{\cos A} \cdot \frac{1}{\cos A} - \frac{\cos A}{\sin A} \cdot \frac{1}{\sin A}$$

$$= \frac{\sin A}{\cos^2 A} - \frac{\cos A}{\sin^2 A}$$

$$= \frac{\sin^3 A - \cos^3 A}{\sin^2 A \cos^2 A}$$

$$= \frac{(\sin A - \cos A)(\sin^2 A + \cos^2 A + \sin A \cos A)}{\sin^2 A \cos^2 A}$$

$$= \frac{(\sin A - \cos A)(\sin A \cos A + 1)}{\sin^2 A \cos^2 A}$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

$$(iii) \frac{\tan^2 \theta}{\tan^2 \theta - 1} + \frac{\operatorname{cosec}^2 \theta}{\sec^2 \theta - \operatorname{cosec}^2 \theta} = \frac{1}{\sin^2 \theta - \cos^2 \theta}$$

$$\text{L.H.S.} = \frac{\tan^2 \theta}{\tan^2 \theta - 1} + \frac{\operatorname{cosec}^2 \theta}{\sec^2 \theta - \operatorname{cosec}^2 \theta}$$

$$= \frac{\frac{\sin^2 \theta}{\cos^2 \theta}}{\frac{\sin^2 \theta}{\cos^2 \theta} - 1} + \frac{\frac{1}{\sin^2 \theta}}{\frac{1}{\cos^2 \theta} - \frac{1}{\sin^2 \theta}}$$

$$= \frac{\sin^2 \theta}{\cos^2 \theta} \times \frac{\cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} + \frac{\frac{1}{\sin^2 \theta}}{\frac{\sin^2 \theta - \cos^2 \theta}{\sin^2 \theta \cos^2 \theta}}$$

$$= \frac{\sin^2 \theta}{\sin^2 \theta - \cos^2 \theta} + \frac{\sin^2 \theta \cos^2 \theta}{\sin^2 \theta (\sin^2 \theta - \cos^2 \theta)}$$

$$= \frac{\sin^2 \theta}{\sin^2 \theta - \cos^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta - \cos^2 \theta}$$

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta}$$

$$= \frac{1}{\sin^2 \theta - \cos^2 \theta}$$

$$= \text{R.H.S.}$$

$$10. \frac{\sin A + \cos A}{\sin A - \cos A} + \frac{\sin A - \cos A}{\sin A + \cos A} = \frac{2}{\sin^2 A - \cos^2 A} = \frac{2}{1 - 2\cos^2 A} = \frac{2\sec^2 A}{\tan^2 A - 1}$$

Solution:

$$\text{L.H.S.} = \frac{\sin A + \cos A}{\sin A - \cos A} + \frac{\sin A - \cos A}{\sin A + \cos A}$$

$$= \frac{(\sin A + \cos A)^2 + (\sin A - \cos A)^2}{\sin^2 A - \cos^2 A}$$

$$= \frac{2(\sin^2 A + \cos^2 A)}{\sin^2 A - \cos^2 A}$$

$$= \frac{2 \times 1}{\sin^2 A - \cos^2 A}$$

$$= \frac{2}{\sin^2 A - \cos^2 A}$$

$$= \text{R.H.S.}$$

Now, we have

$$= \frac{2}{\sin^2 A - \cos^2 A}$$

$$= \frac{2}{1 - \cos^2 A - \cos^2 A}$$

$$= \frac{2}{1 - 2\cos^2 A}$$

$$= \text{R. H.S.}$$

now, we have

$$= \frac{2}{\sin^2 A - \cos^2 A}$$

$$= \frac{2}{\sin^2 A - (1 - \sin^2 A)}$$

$$= \frac{2}{\sin^2 A - \cos^2 A}$$

$$= \frac{\frac{2}{\cos^2 A}}{\frac{\sin^2 A}{\cos^2 A} - \frac{\cos^2 A}{\cos^2 A}}$$

$$= \frac{2\sec^2 A}{\tan^2 A - 1}$$

$$= \text{R.H.S.}$$

Hence,

$$\frac{\sin A + \cos A}{\sin A - \cos A} + \frac{\sin A - \cos A}{\sin A + \cos A} = \frac{2}{\sin^2 A - \cos^2 A} = \frac{2}{1 - 2\cos^2 A} = \frac{2\sec^2 A}{\tan^2 A - 1}$$

11. $2[\sin^6\theta + \cos^6\theta] - 3(\sin^4\theta + \cos^4\theta) + 1 = 0$

Solution:

Given,

$$2[\sin^6\theta + \cos^6\theta] - 3(\sin^4\theta + \cos^4\theta) + 1 = 0$$

L.H.S. =

$$2[\sin^6\theta + \cos^6\theta] - 3(\sin^4\theta + \cos^4\theta) + 1$$

$$\text{L.H.S.} = 2[\sin^6\theta + \cos^6\theta] - 3(\sin^4\theta + \cos^4\theta) + 1$$

$$= 2[(\sin^2\theta)^3 + (\cos^2\theta)^3] - 3(\sin^4\theta + \cos^4\theta) + 1$$

$$= 2[(\sin^2\theta + \cos^2\theta)(\sin^4\theta + \cos^4\theta - \sin^2\theta\cos^2\theta)] - 3(\sin^4\theta + \cos^4\theta) + 1$$

$$= 2(\sin^4\theta + \cos^4\theta - \sin^2\theta\cos^2\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$$

$$= 2\sin^4\theta + 2\cos^4\theta - 2\sin^2\theta\cos^2\theta - 3\sin^4\theta - 3\cos^4\theta + 1$$

$$= 1 - \sin^4\theta - \cos^4\theta - 2\sin^2\theta\cos^2\theta$$

$$= 1 - [\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta]$$

$$= 1 - 1$$

$$= 0$$

$$= \text{R.H.S.}$$

12. If $\cot\theta + \cos\theta = m$, $\cot\theta - \cos\theta = n$, then prove that $(m^2 - n^2)^2 = 16$.

Solution:

Given,

$$\cot\theta + \cos\theta = m \quad \dots\dots\dots(i)$$

$$\cot\theta - \cos\theta = n \quad \dots\dots\dots(ii)$$

Adding (i) and (ii), we get

$$2\cot\theta = m + n$$

$$\Rightarrow \cot\theta = \frac{m+n}{2}$$

$$\therefore \tan\theta = \frac{2}{m+n} \quad \dots\dots(iii)$$

Subtracting (ii) from (i),

$$2\cos\theta = m - n \Rightarrow \cos\theta = \frac{m-n}{2}$$

$$\therefore \sec\theta = \frac{2}{m-n} \quad \dots\dots(iv)$$

Now, squaring and subtracting (iii) from (iv), we have

$$\sec^2\theta - \tan^2\theta = \left(\frac{2}{m-n}\right)^2 - \left(\frac{2}{m+n}\right)^2$$

$$1 = \frac{4}{(m-n)^2} - \frac{4}{(m+n)^2}$$

$$\Rightarrow 4 \left[\frac{1}{(m-n)^2} - \frac{1}{(m+n)^2} \right] = 1$$

$$= 4 \left[\frac{1}{(m-n)^2} - \frac{1}{(m+n)^2} \right] = 1$$

$$= \frac{4(4mn)}{(m^2-n^2)^2} = 1$$

$$\therefore (m^2 - n^2)^2 = 16 mn.$$

$$2\cot\theta = m + n$$

$$\Rightarrow \cot\theta = \frac{m + n}{2}$$

$$\therefore \tan\theta = \frac{2}{m+n} \quad \dots\dots\dots(\text{iii})$$

Subtracting (ii) from (i),

$$2\cos\theta = m - n$$

$$\Rightarrow \cos\theta = \frac{m-n}{2}$$

$$\therefore \sec\theta = \frac{2}{m-n} \quad \dots\dots\dots(\text{iv})$$

Now, squaring and subtracting (iii) from (iv), we have

$$\sec^2\theta - \tan^2\theta = \left(\frac{2}{m-n}\right)^2 - \left(\frac{2}{m+n}\right)^2$$

$$1 = \frac{4}{(m-n)^2} - \frac{4}{(m+n)^2}$$

$$\Rightarrow 4 \left[\frac{1}{(m-n)^2} - \frac{1}{(m+n)^2} \right] = 1$$

$$= 4 \left[\frac{(m+n)^2 - (m-n)^2}{(m+n)^2(m-n)^2} \right] = 1$$

$$= \frac{4(4mn)}{(m^2 - n^2)^2} = 1$$

$$= \frac{16mn}{(m^2 - n^2)^2} = 1$$

$$\therefore (m^2 - n^2)^2 = 16mn$$

13. If $\sec\theta + \tan\theta = p$, prove that $\sin\theta = \frac{(p^2-1)}{(p^2+1)}$

Solution:

Given,

$$\sec\theta + \tan\theta = p$$

$$\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = p$$

$$\frac{1+\sin\theta}{\cos\theta} = p$$

Squaring on both sides,

$$\frac{(1+\sin\theta)^2}{\cos^2\theta} = p^2$$

$$\Rightarrow \frac{(1+\sin\theta)^2}{1-\sin^2\theta} = p^2$$

$$\frac{(1+\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)} = p^2$$

$$\Rightarrow \frac{(1+\sin\theta)}{(1-\sin\theta)} = \frac{p^2}{1}$$

Applying componendo and dividendo

$$\frac{1+\sin\theta+1-\sin\theta}{1+\sin\theta-1+\sin\theta} = \frac{p^2+1}{p^2-1}$$

$$= \frac{2}{2\sin\theta} = \frac{p^2+1}{p^2-1}$$

$$\Rightarrow \frac{1}{\sin\theta}$$

$$= \frac{p^2+1}{p^2-1}$$

14. If $\tan A = n \tan B$ and $\sin A = m \sin B$, prove that $\cos^2 A = \frac{(m^2-1)}{(n^2-1)}$

Solution:

Given,

$$\tan A = n \tan B \text{ and } \sin A = m \sin B$$

$$n = \frac{\tan A}{\tan B}$$

$$m = \frac{\sin A}{\sin B}$$

$$\frac{1}{\sin B} = \frac{m}{\sin A} \Rightarrow \operatorname{cosec} B = \frac{m}{\sin A}$$

$$\frac{1}{\tan B} = \frac{m}{\tan A} \Rightarrow \cot B = \frac{n}{\tan A}$$

$$\text{Now, } \operatorname{cosec}^2 B - \cot^2 B = 1$$

$$\Rightarrow \frac{m^2}{\sin^2 A} - \frac{n^2}{\tan^2 A} = 1$$

$$= \frac{m^2}{\sin^2 A} - \frac{n^2 \cos^2 A}{\sin^2 A} = 1$$

$$= m^2 - n^2 \cos^2 A = \sin^2 A$$

$$= m^2 - n^2 \cos^2 A = 1 - \cos^2 A$$

$$= m^2 - 1 = \cos^2 A + n^2 \cos^2 A$$

$$= m^2 - 1 = (n^2 - 1) \cos^2 A$$

$$\Rightarrow \cos^2 A = \frac{m^2 - 1}{n^2 - 1}$$

15. If $\sec A = x + \frac{1}{4}x$, then prove that $\sec A + \tan A = 2x$ or $\frac{1}{2}x$

Solution:

Given, $\sec A = x + \frac{1}{4}x$

we know that

$$\tan A = \pm \sqrt{\sec^2 A - 1}$$

$$= \pm \sqrt{\left(x + \frac{1}{4}x\right)^2 - 1}$$

$$= \pm \sqrt{x^2 + \frac{1}{16x^2} + \frac{1}{2} - 1}$$

$$= \pm \sqrt{x^2 + \frac{1}{16x^2} - \frac{1}{2}}$$

$$= \pm \left(x - \frac{1}{4x}\right)$$

$$\therefore \sec A + \tan A = x + \frac{1}{4x} + x - \frac{1}{4x} = 2x$$

$$\text{Or } x + \frac{1}{4x} - x + \frac{1}{4x} = \frac{1}{2x}$$

$$\text{Or } x + \frac{1}{4x} - x + \frac{1}{4x} = \frac{1}{2x}$$

$$\tan A = \pm \sqrt{\sec^2 A - 1}$$

$$= \pm \sqrt{\left(x + \frac{1}{4x}\right)^2 - 1}$$

$$= \pm \sqrt{x^2 + \frac{1}{16x^2} + \frac{1}{2} - 1}$$

$$= \pm \sqrt{x^2 + \frac{1}{16x^2} - \frac{1}{2}}$$

$$= \pm \left(x - \frac{1}{4x}\right)$$

$$\therefore \sec A + \tan A = x + \frac{1}{4x} + x - \frac{1}{4x} = 2x$$

$$\text{Or } x + \frac{1}{4x} - x + \frac{1}{4x} = \frac{1}{2x}$$

16. When $0^\circ < \theta < 90^\circ$, Solve the following equations:

(i) $2\cos^2\theta + \sin\theta - 2 = 0$

(ii) $3\cos\theta = 2\sin^2\theta$

(iii) $\sec^2\theta - 2\tan\theta = 0$

(iv) $\tan^2\theta = 3(\sec\theta - 1)$

Solution:

Given, $0^\circ < \theta < 90^\circ$

(i) $2\cos^2\theta + \sin\theta - 2 = 0$

$$2(1 - \sin^2\theta) + \sin\theta - 2 = 0$$

$$2 - 2\sin^2\theta + \sin\theta - 2 = 0$$

$$-2\sin^2\theta + \sin\theta = 0$$

$$\sin\theta(1 - 2\sin\theta) = 0$$

So, either $\sin\theta = 0$ or $1 - 2\sin\theta = 0$

If $\sin\theta = 0$

$$\Rightarrow \theta = 0^\circ$$

And, if $1 - 2\sin\theta = 0$

$$\sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = 30^\circ$$

Thus, $\theta = 0^\circ$ or 30°

$$(ii) 3 \cos\theta = 2\sin^2\theta$$

$$3 \cos\theta = 2(1 - \cos^2\theta)$$

$$3 \cos\theta = 2 - 2\cos^2\theta$$

$$2\cos^2\theta + 3 \cos\theta - 2 = 0$$

$$2\cos^2\theta + 4 \cos\theta - \cos\theta - 2 = 0$$

$$2\cos\theta(\cos\theta + 2) - 1(\cos\theta + 2)$$

$$(2\cos\theta - 1)(\cos\theta + 2) = 0$$

So, either $2\cos\theta - 1 = 0$ or $\cos\theta + 2 = 0$

If $2\cos\theta - 1 = 0$

$$\cos\theta = \frac{1}{2}$$

$$\Rightarrow \theta = 60^\circ$$

And, for $\cos\theta + 2 = 0$

$\Rightarrow \cos\theta = -2$ which is not possible being out of range.

Thus, $\theta = 60^\circ$

$$(iii) \sec^2\theta - 2\tan\theta = 0$$

$$(1 + \tan^2\theta) - 2\tan\theta = 0$$

$$\tan^2\theta - 2\tan\theta + 1 = 0$$

$$(\tan\theta - 1)^2 = 0$$

$$\tan\theta - 1 = 0$$

$$\Rightarrow \tan\theta = 1$$

$$\text{Thus, } \theta = 45^\circ$$

$$\text{(iv) } \tan^2\theta = 3(\sec\theta - 1)$$

$$= (\sec^2\theta - 1) = 3\sec\theta - 3$$

$$\sec^2\theta - 1 - 3\sec\theta + 3 = 0$$

$$\sec^2\theta - 3\sec\theta + 2 = 0$$

$$\sec^2\theta - 2\sec\theta - \sec\theta + 2 = 0$$

$$\sec\theta(\sec\theta - 2) - 1(\sec\theta - 2) = 0$$

$$(\sec\theta - 1)(\sec\theta - 2) = 0$$

$$\text{So, either } \sec\theta - 1 = 0 \text{ or } \sec\theta - 2 = 0$$

$$\text{If } \sec\theta - 1 = 0$$

$$\sec\theta = 1$$

$$\Rightarrow \theta = 0^\circ$$

$$\text{And, if } \sec\theta - 2 = 0$$

$$\sec\theta = 2$$

$$\Rightarrow \theta = 60^\circ$$

$$\text{Thus, } \theta = 0^\circ \text{ or } 60^\circ$$