Chapter

2

Logarithms, Indices and Surds, Partial Fractions

Logarithms

Introduction

"The Logarithm of a given number to a given base is the index of the power to which the base must be raised in order to equal the given number."

If $a > 0$ and $\neq 1$, then logarithm of a positive number N is defined as the index x of that power of 'a' which equals N *i.e.,* $\log_a N = x$ iff $a^x = N \Rightarrow a^{\log_a N} = N, a > 0, a \neq 1$ and $N > 0$

It is also known as fundamental logarithmic identity.

Its domain is $(0, \infty)$ and range is R. a is called the base of the logarithmic function.

When base is 'e' then the logarithmic function is called natural or Napierian logarithmic function and when base is 10, then it is called common logarithmic function.

Characteristic and mantissa

(1) The integral part of a logarithm is called the characteristic and the fractional part is called mantissa.

$$\log_{10} N = \underset{\text{Characterstics}}{\underset{\downarrow}{\text{integer}}} + \underset{\text{Mantissa}}{\underset{\downarrow}{\text{fraction}\,(+ve)}}$$

(2) The mantissa part of log of a number is always kept positive.

(3) If the characteristics of $\log_{10} N$ be n, then the number of digits in N is $(n+1)$.

(4) If the characteristics of $\log_{10} N$ be $(-n)$ then there exists $(n-1)$ number of zeros after decimal part of N.

Properties of logarithms

Let m and n be arbitrary positive numbers such that $a > 0,\ a \neq 1,\ b > 0,\ b \neq 1$ then

(1) $\log_a a = 1,\ \log_a 1 = 0$

(2) $\log_a b . \log_b a = 1 \Rightarrow \log_a b = \dfrac{1}{\log_b a}$

(3) $\log_c a = \log_b a .\ \log_c b$ or $\log_c a = \dfrac{\log_b a}{\log_b c}$

(4) $\log_a (mn) = \log_a m + \log_a n$

(5) $\log_a \left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

(6) $\log_a m^n = n \log_a m$ (7) $a^{\log_a m} = m$

(8) $\log_a \left(\dfrac{1}{n}\right) = -\log_a n$ (9) $\log_{a^\beta} n = \dfrac{1}{\beta} \log_a n$

(10) $\log_{a^\beta} n^\alpha = \dfrac{\alpha}{\beta} \log_a n$, $(\beta \neq 0)$

(11) $a^{\log_c b} = b^{\log_c a}$, $(a, b, c > 0$ and $c \neq 1)$

Logarithmic inequalities

(1) If $a > 1, p > 1 \Rightarrow \log_a p > 0$

(2) If $0 < a < 1, p > 1 \Rightarrow \log_a p < 0$

(3) If $a > 1, 0 < p < 1 \Rightarrow \log_a p < 0$

(4) If $p > a > 1 \Rightarrow \log_a p > 1$

(5) If $a > p > 1 \Rightarrow 0 < \log_a p < 1$

(6) If $0 < a < p < 1 \Rightarrow 0 < \log_a p < 1$

(7) If $0 < p < a < 1 \Rightarrow \log_a p > 1$

(8) If $\log_m a > b \Rightarrow \begin{cases} a > m^b, & \text{if } m > 1 \\ a < m^b, & \text{if } 0 < m < 1 \end{cases}$

(9) $\log_m a < b \Rightarrow \begin{cases} a < m^b, & \text{if } m > 1 \\ a > m^b, & \text{if } 0 < m < 1 \end{cases}$

(10) $\log_p a > \log_p b \Rightarrow a \geq b$ if base p is positive and >1 or $a \leq b$ if base p is positive and < 1 *i.e.,* $0 < p < 1$.

In other words, if base is greater than 1 then inequality remains same and if base is positive but less than 1 then the sign of inequality is reversed.

Indices and Surds

Definition of indices

If a is any non zero real or imaginary number and m is the positive integer, then $a^m = a.a.a.a.......... a$ (m times). Here a is called the base and m is the index, power or exponent.

Laws of indices

(1) $a^0 = 1$, $(a \neq 0)$

(2) $a^{-m} = \frac{1}{a^m}$, $(a \neq 0)$

(3) $a^{m+n} = a^m.a^n$, where m and n are rational numbers

(4) $a^{m-n} = \frac{a^m}{a^n}$, where m and n are rational numbers, $a \neq 0$

(5) $(a^m)^n = a^{mn}$

(6) $a^{p/q} = \sqrt[q]{a^p}$

(7) If $x = y$, then $a^x = a^y$, but the converse may not be true.

For example : $(1)^6 = (1)^8$, but $6 \neq 8$

(i) If $a \neq \pm 1$, or 0, then $x = y$

(ii) If $a = 1$, then x, y may be any real number

(iii) If $a = -1$, then x, y may be both even or both odd

(iv) If $a = 0$, then x, y may be any non-zero real number

But if we have to solve the equations like $[f(x)]^{\phi(x)} = [f(x)]^{\Psi(x)}$ then we have to solve :

(a) $f(x) = 1$ (b) $f(x) = -1$

(c) $f(x) = 0$ (d) $\phi(x) = \Psi(x)$

Verification should be done in (b) and (c) cases

(8) $a^m.b^m = (ab)^m$ is not always true

In real domain, $\sqrt{a}\sqrt{b} = \sqrt{(ab)}$, only when $a \geq 0, b \geq 0$

In complex domain, $\sqrt{a}.\sqrt{b} = \sqrt{(ab)}$, if at least one of a and b is positive.

(9) If $a^x = b^x$ then consider the following cases :

(i) If $a \neq \pm b$, then $x = 0$

(ii) If $a = b \neq 0$, then x may have any real value

(iii) If $a = -b$, then x is even.

If we have to solve the equation of the form $[f(x)]^{\phi(x)} = [g(x)]^{\phi(x)}$ *i.e.,* same index, different bases, then we have to solve (a) $f(x) = g(x)$, (b) $f(x) = -g(x)$, (c) $\phi(x) = 0$

Verification should be done in (b) and (c) cases.

Definition of surds

Any root of a number which can not be exactly found is called a surd.

Let a be a rational number and n is a positive integer. If the n^{th} root of x *i.e.,* $x^{1/n}$ is irrational, then it is called surd of order n.

Order of a surd is indicated by the number denoting the root.

For example, $\sqrt{7}, \sqrt[3]{9}, (11)^{3/5}, \sqrt[n]{3}$ are surds of second, third, fifth and n^{th} order respectively.

A second order surd is often called a quadratic surd, a surd of third order is called a cubic surd.

Types of surds

(1) **Simple surds :** A surd consisting of a single term. For example $2\sqrt{3}, 6\sqrt{5}, \sqrt{5}$ etc.

(2) **Pure and mixed surds :** A surd consisting of wholly of an irrational number is called pure surd.

A surd consisting of the product of a rational number and an irrational number is called a mixed surd.

(3) **Compound surds :** An expression consisting of the sum or difference of two or more surds.

(4) **Similar surds :** If the surds are different multiples of the same surd, they are called similar surds.

(5) **Binomial surds :** A compound surd consisting of two surds is called a binomial surd.

(6) **Binomial quadratic surds :** Binomial surds consisting of pure (or simple) surds of order two *i.e.,* the surds of the form $a\sqrt{b} \pm c\sqrt{d}$ or $a \pm b\sqrt{c}$ are called binomial quadratic surds.

Two binomial quadratic surds which differ only in the sign which connects their terms are said to be conjugate or complementary to each other. The product of a binomial quadratic surd and its conjugate is always rational.

For example: The conjugate of the surd $2\sqrt{7} + 5\sqrt{3}$ is the surd $2\sqrt{7} - 5\sqrt{3}$.

Properties of quadratic surds

(1) The square root of a rational number cannot be expressed as the sum or difference of a rational number and a quadratic surd.

(2) If two quadratic surds cannot be reduced to others, which have not the same irrational part, their product is irrational.

(3) One quadratic surd cannot be equal to the sum or difference of two others, not having the same irrational part.

(4) If $a + \sqrt{b} = c + \sqrt{d}$, where a and c are rational, and $\sqrt{b}, \sqrt{d}$ are irrational, then $a = c$ and $b = d$.

Rationalisation factors

If two surds be such that their product is rational, then each one of them is called rationalising factor of the other.

Thus each of $2\sqrt{3}$ and $\sqrt{3}$ is a rationalising factor of each other. Similarly $\sqrt{3}+\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$ are rationalising factors of each other, as $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})=1$, which is rational.

To find the factor which will rationalize any given binomial surd :

Case I : Suppose the given surd is $\sqrt[p]{a}-\sqrt[q]{b}$

Let $a^{1/P}=x, b^{1/q}=y$ and let n be the L.C.M. of p and q. Then x^n and y^n are both rational.

Now x^n-y^n is divisible by $x-y$ for all values of n, and $x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+.....+y^{n-1})$.

Thus the rationalizing factor is $x^{n-1}+x^{n-2}y+x^{n-3}y^2+.....+y^{n-1}$ and the rational product is x^n-y^n.

Case II : Let the given surd be $\sqrt[p]{a}+\sqrt[q]{b}$.

Let x, y, n have the same meaning as in Case I.

(1) If n is even, then x^n-y^n is divisible by $x+y$ and $x^n-y^n=(x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^3-.....-y^{n-1})$

Thus the rationalizing factor is $x^{n-1}-x^{n-2}y+x^{n-3}y^2-...-y^{n-1}$ and the rational product is x^n-y^n.

(2) If n is odd, x^n+y^n is divisible by $x+y$ and $x^n+y^n=(x+y)(x^{n-1}-x^{n-2}y+....+y^{n-1})$

Thus the rationalizing factor is $x^{n-1}-x^{n-2}y+..... -xy^{n-2}+y^{n-1}$ and the rational product is x^n+y^n.

Square roots of *a* +√b and *a* + √b + √c + √d where √b , √c , √d are Surds

Let $\sqrt{(a+\sqrt{b})}=\sqrt{x}+\sqrt{y}$, where $x, y>0$ are rational numbers.

Then squaring both sides we have, $a+\sqrt{b}=x+y+2\sqrt{x}\sqrt{y}$

$\Rightarrow a=x+y,\ \sqrt{b}=2\sqrt{xy} \Rightarrow b=4xy$

So, $(x-y)^2=(x+y)^2-4xy=a^2-b$

After solving we can find x and y.

Similarly square root of $a-\sqrt{b}$ can be found by taking $\sqrt{(a-\sqrt{b})}=\sqrt{x}-\sqrt{y},\ x>y$

To find square root of *a* + √*b* + √c + √*d* :

Let $\sqrt{(a+\sqrt{b}+\sqrt{c}+\sqrt{d})}=\sqrt{x}+\sqrt{y}+\sqrt{z}, (x, y, z>0)$ and take $\sqrt{(a+\sqrt{b}-\sqrt{c}-\sqrt{d})}=\sqrt{x}+\sqrt{y}-\sqrt{z}$. Then by squaring and equating, we get equations in *x, y, z*. On solving these equations, we can find the required square roots.

(1) If a^2-b is not a perfect square, the square root of $a+\sqrt{b}$ is complicated *i.e.*, we can't find the value of $\sqrt{(a+\sqrt{b})}$ in the form of a compound surd.

(2) If $\sqrt{(a+\sqrt{b})}=\sqrt{x}+\sqrt{y}, x>y$ then $\sqrt{(a-\sqrt{b})}=\sqrt{x}-\sqrt{y}$

(3) $\sqrt{a+\sqrt{(b)}}=\sqrt{\left(\frac{a+\sqrt{a^2-b}}{2}\right)}+\sqrt{\left(\frac{a-\sqrt{a^2-b}}{2}\right)}$

(4) $\sqrt{a-\sqrt{(b)}}=\sqrt{\left(\frac{a+\sqrt{a^2-b}}{2}\right)}-\sqrt{\left(\frac{a-\sqrt{a^2-b}}{2}\right)}$

(5) If a is a rational number and $\sqrt{b}, \sqrt{c}, \sqrt{d}$ are surds then

(i) $\sqrt{a+\sqrt{b}+\sqrt{c}+\sqrt{d}}=\sqrt{\frac{bd}{4c}}+\sqrt{\frac{bc}{4d}}+\sqrt{\frac{cd}{4b}}$

(ii) $\sqrt{a-\sqrt{b}-\sqrt{c}+\sqrt{d}}=\sqrt{\frac{bd}{4c}}+\sqrt{\frac{cd}{4b}}+\sqrt{\frac{bc}{4d}}$

(iii) $\sqrt{a-\sqrt{b}-\sqrt{c}+\sqrt{d}}=\sqrt{\frac{bc}{4d}}-\sqrt{\frac{bd}{4c}}-\sqrt{\frac{cd}{4b}}$

Cube root of a binomial quadratic surd

If $(a+\sqrt{b})^{1/3}=x+\sqrt{y}$ then $(a-\sqrt{b})^{2/3}=x-\sqrt{y}$, where a is a rational number and b is a surd.

Procedure of finding $(a+\sqrt{b})^{1/3}$ is illustrated with the help of an example :

Taking $(37-30\sqrt{3})^{1/3}=x+\sqrt{y}$ we get on cubing both sides, $37-30\sqrt{3}=x^3+3xy-(3x^2+y)\sqrt{y}$

$\therefore x^3+3xy=37$

$(3x^2+y)\sqrt{y}=30\sqrt{3}=15\sqrt{12}$

As $\sqrt{3}$ can not be reduced, let us assume $y=3$ we get $3x^2+y=3x^2+3=30$. $\therefore x=3$,

which doesn't satisfy $x^3+3xy=37$.

Again taking $y=12$, we get $3x^2+12=15$, $\therefore x=1$

$x=1, y=12$ satisfy $x^3+3xy=37$

$\therefore \sqrt[3]{37-30\sqrt{3}}=1-\sqrt{12}=1-2\sqrt{3}$

Equations involving surds

While solving equations involving surds, usually we have to square, on squaring the domain of the equation extends and we may get some extraneous solutions, and so we must verify the solutions and neglect those which do not satisfy the equation.

Note that from $ax=bx$, to conclude $a=b$ is not correct. The correct procedure is $x(a-b)=0$ *i.e.* $x=0$ or $a=b$. Here, necessity of verification is required.

Partial Fractions

Definition

An expression of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial in *x*, is called a rational fraction.

(1) **Proper rational functions :** Functions of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials and $g(x) \neq 0$,are called rational functions of *x*.

If degree of $f(x)$ is less than degree of $g(x)$,then $\frac{f(x)}{g(x)}$ is called a proper rational function.

(2) **Improper rational functions :** If degree of $f(x)$ is greater than or equal to degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called an improper rational function.

(3) **Partial fractions :** Any proper rational function can be broken up into a group of different rational fractions, each having a simple factor of the denominator of the original rational function. Each such fraction is called a partial fraction.

If by some process, we can break a given rational function $\frac{f(x)}{g(x)}$ into different fractions, whose denominators are the factors of $g(x)$,then the process of obtaining them is called the resolution or decomposition of $\frac{f(x)}{g(x)}$ into its partial fractions.

Different cases of partial fractions

(1) **When the denominator consists of non-repeated linear factors :** To each linear factor $(x-a)$ occurring once in the denominator of a proper fraction, there corresponds a single partial fraction of the form $\frac{A}{x-a}$, where *A* is a constant to be determined.

If $g(x) = (x-a_1)(x-a_2)(x-a_3)\text{.......}(x-a_n)$, then we assume that, $\frac{f(x)}{g(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \text{......} + \frac{A_n}{x-a_n}$

where A_1, A_2, A_3 , ... A_n are constants, can be determined by equating the numerator of L.H.S. to the numerator of R.H.S. (after L.C.M.) and substituting $x = a_1, a_2, \text{......} a_n$.

(2) **When the denominator consists of linear factors, some repeated :** To each linear factor (*x* – *a*) occurring *r* times in the denominator of a proper rational function, there corresponds a sum of *r* partial fractions.

Let $g(x) = (x-a)^k (x-a_1)(x-a_2)\text{.......}(x-a_r)$. Then we assume that

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \text{......}$$
$$+ \frac{A_k}{(x-a)^k} + \frac{B_1}{(x-a_1)} + \text{......} + \frac{B_r}{(x-a_r)}$$

Where $A_1, A_2, \text{......}, A_k$ are constants. To determine the value of constants adopt the procedure as above.

(3) **When the denominator consists of non-repeated quadratic factors :** To each irreducible non repeated quadratic factor ax^2+bx+c, there corresponds a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$, where *A* and *B* are constants to be determined.

***Example* :**

$$\frac{4x^2+2x+3}{(x^2+4x+9)(x-2)(x+3)} = \frac{Ax+B}{x^2+4x+9} + \frac{C}{x-2} + \frac{D}{x+3}$$

(1) $\frac{px+q}{x^2(x-a)} = \frac{-q}{ax^2} - \frac{pa+q}{a^2x} + \frac{pa+q}{a^2(x-a)}$

(2) $\frac{px+q}{x(x-a)^2} = \frac{q}{a^2x} - \frac{q}{a^2(x-a)} + \frac{pa+q}{a(x-a)^2}$

(3) $\frac{px+q}{x(x^2+a^2)} = \frac{q}{a^2x} + \frac{pa^2-qx}{a^2(x^2+a^2)}$

(4) **When the denominator consists of repeated quadratic factors :** To each irreducible quadratic factor ax^2+bx+c occurring *r* times in the denominator of a proper rational fraction there corresponds a sum of *r* partial fractions of the form.

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \text{........} + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

where, *A*'*s* and *B*'*s* are constants to be determined.

Partial fractions of improper rational functions

If degree of $f(x)$ is greater than or equal to degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called an improper rational function and every rational function can be transformed to a proper rational function by dividing the numerator by the denominator.

We divide the numerator by denominator until a remainder is obtained which is of lower degree than the denominator.

i.e., $\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}$, where degree of $R(x) <$ degree of $g(x)$.

For example, $\frac{x^3}{x^2-5x+6}$ is an improper rational function and can be expressed as $(x+5) + \frac{19x-30}{x^2-5x+6}$ which is the sum of a polynomial $(x+5)$ and a proper rational function $\frac{19x-30}{x^2-5x+6}$.

General method of finding out the constants

(1) Express the given fraction into its partial fractions in accordance with the rules written above.

(2) Then multiply both sides by the denominator of the given fraction and you will get an identity which will hold for all values of *x*.

(3) Equate the coefficients of like powers of *x* in the resulting identity and solve the equations so obtained simultaneously to find the various constant is short method. Sometimes, we substitute particular values of the variable *x* in the identity obtained after

clearing of fractions to find some or all the constants. For non-repeated linear factors, the values of x used as those for which the denominator of the corresponding partial fractions become zero.

Tips & Tricks

- The logarithm of a number is unique *i.e.*, no number can have two different log to a given base.

$\log_e a = \log_e 10 . \log_{10} a$ or $\log_{10} a = \dfrac{\log_e a}{\log_e 10} = 0.434 \log_e a$

- If a is not rational, $\sqrt[n]{a}$ is not a surd.
- Remainder of polynomial $f(x)$, when divided by $(x-a)$ is $f(a)$. *e.g.*, remainder of x^2+3x-7, when divided by $x-2$ is $(2)^2+3(2)-7=3$.
- $\dfrac{px+q}{(x-a)(x-b)} = \dfrac{pa+q}{(x-a)(a-b)} + \dfrac{pb+q}{(b-a)(x-b)}$
- If the given fraction is improper, then before finding partial fractions, the given fraction must be expressed as sum of a polynomial and a proper fraction by division.
- Some times a suitable substitution transforms the given function to a rational fraction which can be integrated by breaking it into partial fractions.

Ordinary Thinking

Objective Questions

Logarithms

1. For $y = \log_a x$ to be defined 'a' must be **[IIT 1990]**

(a) Any positive real number
(b) Any number
(c) $\geq e$
(d) Any positive real number $\neq 1$

2. Logarithm of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$ is

(a) 3.6 (b) 5
(c) 5.6 (d) None of these

3. The number $\log_2 7$ is **[IIT 1990; Pb CET 2002]**

(a) An integer (b) A rational number
(c) An irrational number (d) A prime number

4. If $\log_7 2 = m$, then $\log_{49} 28$ is equal to **[Roorkee 1999]**

(a) $2(1+2m)$ (b) $\dfrac{1+2m}{2}$
(c) $\dfrac{2}{1+2m}$ (d) $1+m$

5. If $\log_e\left(\dfrac{a+b}{2}\right) = \dfrac{1}{2}(\log_e a + \log_e b)$, then relation between a and b will be **[UPSEAT 2000]**

(a) $a=b$ (b) $a=\dfrac{b}{2}$
(c) $2a=b$ (d) $a=\dfrac{b}{3}$

6. Which is the correct order for a given number α in increasing order **[Roorkee 2000]**

(a) $\log_2 \alpha, \log_3 \alpha, \log_e \alpha, \log_{10} \alpha$
(b) $\log_{10} \alpha, \log_3 \alpha, \log_e \alpha, \log_2 \alpha$
(c) $\log_{10} \alpha, \log_e \alpha, \log_2 \alpha, \log_3 \alpha$
(d) $\log_3 \alpha, \log_e \alpha, \log_2 \alpha, \log_{10} \alpha$

7. $\log ab - \log|b| =$

(a) $\log a$ (b) $\log|a|$
(c) $-\log a$ (d) None of these

8. The value of $\sqrt{(\log_{0.5}^2 4)}$ is

(a) –2 (b) $\sqrt{(-4)}$
(c) 2 (d) None of these

9. The value of $\log_3 4 \log_4 5 \log_5 6 \log_6 7 \log_7 8 \log_8 9$ is **[IIIT Allahabad 2000]**

(a) 1 (b) 2
(c) 3 (d) 4

10. $\log_7 \log_7 \sqrt{7(\sqrt{7\sqrt{7}})} =$

(a) $3\log_2 7$ (b) $1-3\log_3 7$
(c) $1-3\log_7 2$ (d) None of these

11. The value of $81^{(1/\log_5 3)} + 27^{\log_9 36} + 3^{4/\log_7 9}$ is equal to

(a) 49 (b) 625
(c) 216 (d) 890

12. $7\log\left(\dfrac{16}{15}\right) + 5\log\left(\dfrac{25}{24}\right) + 3\log\left(\dfrac{81}{80}\right)$ is equal to **[EAMCET 1990]**

(a) 0 (b) 1
(c) $\log 2$ (d) $\log 3$

13. If $\log_4 5 = a$ and $\log_5 6 = b$, then $\log_3 2$ is equal to

(a) $\dfrac{1}{2a+1}$ (b) $\dfrac{1}{2b+1}$
(c) $2ab+1$ (d) $\dfrac{1}{2ab-1}$

14. If $\log_k x . \log_5 k = \log_x 5, k \neq 1, k > 0$, then x is equal to

(a) k (b) $\dfrac{1}{5}$
(c) 5 (d) None of these

15. If $\log_5 a . \log_a x = 2$, then x is equal to

(a) 125 (b) a^2
(c) 25 (d) None of these

16. If $a^2 + 4b^2 = 12ab$, then $\log(a+2b)$ is

(a) $\dfrac{1}{2}[\log a + \log b - \log 2]$ (b) $\log\dfrac{a}{2} + \log\dfrac{b}{2} + \log 2$

(c) $\frac{1}{2}[\log a + \log b + 4\log 2]$ (d) $\frac{1}{2}[\log a - \log b + 4\log 2]$

17. If $A = \log_2 \log_2 \log_4 256 + 2\log_{\sqrt{2}} 2$, then *A is* equal to **[WB JEE 1992]**

(a) 2 (b) 3
(c) 5 (d) 7

18. If $\log_{10} x = y$, then $\log_{1000} x^2$ is equal to

(a) y^2 (b) $2y$
(c) $\frac{3y}{2}$ (d) $\frac{2y}{3}$

19. If $x = \log_a(bc), y = \log_b(ca), z = \log_c(ab)$, then which of the following is equal to 1

(a) $x + y + z$
(b) $(1+x)^{-1} + (1+y)^{-1} + (1+z)^{-1}$
(c) xyz
(d) None of these

20. If $a = \log_{24} 12, b = \log_{36} 24$ and $c = \log_{48} 36$, then 1+*abc* is equal to **[SCRA 2000]**

(a) $2ab$ (b) $2ac$
(c) $2bc$ (d) 0

21. If $a^x = b, b^y = c, c^z = a$, then value of *xyz* is

(a) 0 (b) 1
(c) 2 (d) 3

22. If $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.47712$, the number of digits in $3^{12} \times 2^8$ is

(a) 7 (b) 8
(c) 9 (d) 10

23. $\sum_{n=1}^{n} \frac{1}{\log_{2^n}(a)} =$

(a) $\frac{n(n+1)}{2}\log_a 2$ (b) $\frac{n(n+1)}{2}\log_2 a$
(c) $\frac{(n+1)^2 n^2}{4}\log_2 a$ (d) None of these

24. The solution of the equation $\log_7 \log_5 (\sqrt{x^2+5+x}) = 0$ **[UPSEAT 2000]**

(a) $x = 2$ (b) $x = 3$
(c) $x = 4$ (d) $x = -2$

25. $\log_4 18$ is

(a) A rational number (b) An irrational number
(c) A prime number (d) None of these

26. The value of $(0.05)^{\log_{\sqrt{20}}(0.1+0.01+0.001+......)}$ is

(a) 81 (b) $\frac{1}{81}$
(c) 20 (d) $\frac{1}{20}$

27. If *a, b, c* are distinct positive numbers, each different from 1, such that $[\log_b a \log_c a - \log_a a] + [\log_a b \log_c b - \log_b b] + [\log_a c \log_b c - \log_c c] = 0$, then *abc* =

(a) 1 (b) 2
(c) 3 (d) None of these

28. If $\log_{12} 27 = a$, then $\log_6 16 =$ **[EAMCET 1990]**

(a) $2.\frac{3-a}{3+a}$ (b) $3.\frac{3-a}{3+a}$
(c) $4.\frac{3-a}{3+a}$ (d) None of these

29. If $\frac{\log x}{b-c} = \frac{\log y}{c-a} = \frac{\log z}{a-b}$, then which of the following is true **[Karnataka CET 2004]**

(a) $xyz = 1$ (b) $x^a y^b z^c = 1$
(c) $x^{b+c} y^{c+a} z^{a+b} = 1$ (d) $xyz = x^a y^b z^c$

30. The number of real values of the parameter *k* for which $(\log_{16} x)^2 - \log_{16} x + \log_{16} k = 0$ with real coefficients will have exactly one solution is

(a) 2 (b) 1
(c) 4 (d) None of these

31. If $x^{\frac{3}{4}(\log_3 x)^2 + \log_3 x - \frac{5}{4}} = \sqrt{3}$ then *x* has

(a) One positive integral value
(b) One irrational value
(c) Two positive rational values
(d) None of these

32. If $x = \log_5(1000)$ and $y = \log_7(2058)$ then

(a) $x > y$ (b) $x < y$
(c) $x = y$ (d) None of these

33. If $\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} > x$, then *x* be

(a) 2 (b) 3
(c) 3.5 (d) π

34. If $\log_{1/\sqrt{2}} \sin x > 0, x \in [0, 4\pi]$, then the number of values of *x* which are integral multiples of $\frac{\pi}{4}$, is

(a) 4 (b) 12
(c) 3 (d) None of these

35. The set of real values of *x* satisfying $\log_{1/2}(x^2 - 6x + 12) \ge -2$ is

(a) $(-\infty, 2]$ (b) $[2, 4]$
(c) $[4, +\infty)$ (d) None of these

36. The set of real values of *x* for which $2^{\log_{\sqrt{2}}(x-1)} > x + 5$ is

(a) $(-\infty, -1) \cup (4, +\infty)$ (b) $(4, +\infty)$
(c) $(-1, 4)$ (d) None of these

37. If $\log_{0.04}(x-1) \ge \log_{0.2}(x-1)$ then *x* belongs to the interval

(a) $(1, 2]$ (b) $(-\infty, 2]$
(c) $[2, +\infty)$ (d) None of these

38. The set of real values of *x* for which $\log_{0.2} \frac{x+2}{x} \le 1$ is

(a) $\left(-\infty, -\frac{5}{2}\right] \cup (0, +\infty)$ (b) $\left[\frac{5}{2}, +\infty\right)$
(c) $(-\infty, -2) \cup (0, +\infty)$ (d) None of these

39. If $x = \log_b a,\ y = \log_c b,\ z = \log_a c$, then xyz is **[UPSEAT 2003]**
(a) 0 (b) 1
(c) 3 (d) None of these

40. The value of $\log_2 . \log_3 .... \log_{100} 100^{99^{98^{\cdot^{\cdot^{2^1}}}}}$ is **[AMU 2005]**
(a) 0 (b) 1
(c) 2 (d) 100!

Indices and Surds

1. For $x \neq 0, \left(\frac{x^l}{x^m}\right)^{(l^2+lm+m^2)} \left(\frac{x^m}{x^n}\right)^{(m^2+nm+n^2)} \left(\frac{x^n}{x^l}\right)^{(n^2+nl+l^2)} =$
(a) 1 (b) x
(c) Does not exist (d) None of these

2. If $2^x = 4^y = 8^z$ and $xyz = 288$, then $\frac{1}{2x} + \frac{1}{4y} + \frac{1}{8z} =$
(a) 11/48 (b) 11/24
(c) 11/8 (d) 11/96

3. $\frac{2.3^{n+1} + 7.3^{n-1}}{3^{n+2} - 2(1/3)^{1-n}} =$
(a) 1 (b) 3
(c) –1 (d) 0

4. If $\left(\frac{2}{3}\right)^{x+2} = \left(\frac{3}{2}\right)^{2-2x}$, then $x =$ **[UPSEAT 1999]**
(a) 1 (b) 3
(c) 4 (d) 0

5. The greatest number among $\sqrt[3]{9}, \sqrt[4]{11}, \sqrt[6]{17}$ is
(a) $\sqrt[3]{9}$ (b) $\sqrt[4]{11}$
(c) $\sqrt[6]{17}$ (d) Can not be determined

6. The value of $\frac{15}{\sqrt{10} + \sqrt{20} + \sqrt{40} - \sqrt{5} - \sqrt{80}}$ is
(a) $\sqrt{5}(5 + \sqrt{2})$ (b) $\sqrt{5}(2 + \sqrt{2})$
(c) $\sqrt{5}(1 + \sqrt{2})$ (d) $\sqrt{5}(3 + \sqrt{2})$

7. The rationalising factor of $a^{1/3} + a^{-1/3}$ is
(a) $a^{1/3} - a^{-1/3}$ (b) $a^{2/3} + a^{-2/3}$
(c) $a^{2/3} - a^{-2/3}$ (d) $a^{2/3} + a^{-2/3} - 1$

8. $\sqrt{(3 + \sqrt{5})}$ is equal to
(a) $\sqrt{5} + 1$ (b) $\sqrt{3} + \sqrt{2}$
(c) $(\sqrt{5} + 1)/\sqrt{2}$ (d) $\frac{1}{2}(\sqrt{5} + 1)$

9. $\sqrt[4]{(17 + 12\sqrt{2})} =$
(a) $\sqrt{2} + 1$ (b) $2^{1/4}(\sqrt{2} + 1)$
(c) $2\sqrt{2} + 1$ (d) None of these

10. The equation $\sqrt{(x+1)} - \sqrt{(x-1)} = \sqrt{(4x-1)}$, $x \in R$ has
(a) One solution (b) Two solution
(c) Four solution (d) No solution

11. $a^{m \log_a n} =$
(a) a^{mn} (b) m^n
(c) n^m (d) None of these

12. If $(a^m)^n = a^{m^n}$, then the value of 'm' in terms of 'n' is
(a) n (b) $n^{1/m}$
(c) $n^{1/(n-1)}$ (d) None of these

13. $(x^5)^{1/3}(16x^3)^{2/3}\left(\frac{1}{4}x^{4/9}\right)^{-3/2} =$
(a) $(x/4)^3$ (b) $(4x)^3$
(c) $8x^3$ (d) None of these

14. If $a^{1/x} = b^{1/y} = c^{1/z}$ and $b^2 = ac$ then $x + z =$
(a) y (b) $2y$
(c) $2xyz$ (d) None of these

15. If $a^x = bc, b^y = ca, c^z = ab$, then $xyz =$
(a) 0 (b) 1
(c) $x + y + z$ (d) $x + y + z + 2$

16. If $a^x = (x+y+z)^y, a^y = (x+y+z)^z, a^z = (x+y+z)^x$, then
(a) $x = y = z = a/3$ (b) $x + y + z = a/3$
(c) $x + y + z = 0$ (d) None of these

17. If $a^{x-1} = bc, b^{y-1} = ca, c^{z-1} = ab$, then $\sum(1/x) =$
(a) 1 (b) 0
(c) abc (d) None of these

18. If $\frac{(2^{n+1})^m (2^{2n}) 2^n}{(2^{m+1})^n 2^{2m}} = 1$, then $m =$
(a) 0 (b) 1
(c) n (d) $2n$

19. If $x^y = y^x$, then $(x/y)^{(x/y)} = x^{(x/y)-k}$, where $k =$
(a) 0 (b) 1
(c) –1 (d) None of these

20. If $x^{x\sqrt[3]{x}} = (x.\sqrt[3]{x})^x$, then $x =$
(a) 1 (b) –1
(c) 0 (d) 64/27

21. If $a^x = b^y = (ab)^{xy}$, then $x + y =$
(a) 0 (b) 1
(c) xy (d) None of these

22. If $x = 2^{1/3} - 2^{-1/3}$, then $2x^3 + 6x =$
(a) 1 (b) 2
(c) 3 (d) None of these

23. Solution of the equation $(x)^{x\sqrt{x}} = (x\sqrt{x})^x$ are
(a) 9/4 (b) 1
(c) –1 (d) 0

24. If $5^{x-1} + 5.(0.2)^{x-2} = 26$, then x may have the value
(a) 25 (b) 1
(c) 3 (d) None of these

25. Let $\frac{7}{2^{1/2} + 2^{1/4} + 1} = A + B.2^{1/4} + C.2^{1/2} + D.2^{3/4}$, then
(a) $A = 1$ (b) $B = 3$
(c) $C = 2$ (d) $D = 1$

26. Solution of the equation $4.9^{x-1} = 3\sqrt{(2^{2x+1})}$ has the solution
(a) 3 (b) 2

(c) 3/2 (d) 2/3

27. Solution of the equation $9^x - 2^{x+\frac{1}{2}} = 2^{x+\frac{3}{2}} - 3^{2x-1}$

(a) $\log_9(9/\sqrt{8})$ (b) $\log_{(9/2)}(9/\sqrt{8})$

(c) $\log_e(9/\sqrt{8})$ (d) None of these

28. $\frac{[4+\sqrt{(15)}]^{3/2}+[4-\sqrt{(15)}]^{3/2}}{[6+\sqrt{(35)}]^{3/2}-[6-\sqrt{(35)}]^{3/2}} =$

(a) 1 (b) 7/13

(c) 13/7 (d) None of these

29. If $x = \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}, y = \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$, then $3x^2 + 4xy - 3y^2 =$

(a) $\frac{1}{3}[56\sqrt{10} - 12]$ (b) $\frac{1}{3}[56\sqrt{10} + 12]$

(c) $\frac{1}{3}[56 + 12\sqrt{10}]$ (d) None of these

30. $\frac{12}{3+\sqrt{5}-2\sqrt{2}} =$

(a) $1+\sqrt{5}+\sqrt{(10)}+\sqrt{2}$ (b) $1+\sqrt{5}-\sqrt{(10)}+\sqrt{2}$

(c) $1+\sqrt{5}+\sqrt{10}-\sqrt{2}$ (d) $1-\sqrt{5}-\sqrt{2}+\sqrt{(10)}$

31. $\frac{\sqrt{(5/2)}+\sqrt{(7-3\sqrt{5})}}{\sqrt{(7/2)}+\sqrt{(16-5\sqrt{7})}} =$

(a) Rational (b) Surd

(c) Multiple of $\sqrt{7}$ (d) None of these

32. $\frac{\sqrt{2}}{\sqrt{(2+\sqrt{3})}-\sqrt{(2-\sqrt{3})}} =$

(a) 0 (b) 1

(c) $\sqrt{2}$ (d) $1/\sqrt{2}$

33. $\frac{4}{1+\sqrt{2}-\sqrt{3}} =$

(a) $2+\sqrt{2}+\sqrt{6}$ (b) $1+\sqrt{2}+\sqrt{3}$

(c) $3+\sqrt{2}+\sqrt{3}$ (d) None of these

34. $\frac{3\sqrt{2}}{\sqrt{6}+\sqrt{3}} - \frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} =$

(a) $5\sqrt{2}$ (b) $3\sqrt{2}$

(c) $2\sqrt{3}$ (d) 0

35. The rationalising factor of $2\sqrt{3}-\sqrt{7}$ is

(a) $\sqrt{3}+\sqrt{7}$ (b) $2\sqrt{3}+\sqrt{7}$

(c) $\sqrt{3}+2\sqrt{7}$ (d) None of these

36. The value of $\sqrt{[12-\sqrt{(68+48\sqrt{2})}]} =$

(a) $2+\sqrt{2}$ (b) $2-\sqrt{2}$

(c) $\sqrt{2}-1$ (d) None of these

37. The square root of $\sqrt{(50)}+\sqrt{(48)}$ is

(a) $2^{1/4}(3+\sqrt{2})$ (b) $2^{1/4}(\sqrt{3}+2)$

(c) $2^{1/4}(2+\sqrt{2})$ (d) $2^{1/4}(\sqrt{3}+\sqrt{2})$

38. $\sqrt{(3+\sqrt{5})}-\sqrt{(2+\sqrt{3})} =$

(a) $\sqrt{(5/2)}+\sqrt{(3/2)}$ (b) $\sqrt{(5/2)}-\sqrt{(3/2)}$

(c) $\sqrt{(5/2)}-\sqrt{(1/2)}$ (d) $\sqrt{(3/2)}-\sqrt{(1/2)}$

39. The value of $\sqrt{[12\sqrt{5}+2\sqrt{(55)}]}$ is

(a) $5^{1/2}[\sqrt{(11)}+1]$ (b) $5^{1/2}[\sqrt{(11)}-1]$

(c) $5^{1/4}[\sqrt{(11)}+1]$ (d) $5^{1/4}[\sqrt{(11)}-1]$

40. The cube root of $9\sqrt{3}+11\sqrt{2}$ is

(a) $2\sqrt{3}+\sqrt{2}$ (b) $\sqrt{3}+2\sqrt{2}$

(c) $3\sqrt{3}+\sqrt{2}$ (d) $\sqrt{3}+\sqrt{2}$

41. If $x+\sqrt{(x^2+1)} = a$, then x =

(a) $\frac{1}{2}(a+1/a)$ (b) $\frac{1}{2}(a-1/a)$

(c) $(a+a^{-1})$ (d) None of these

42. If $x=\sqrt{7}+\sqrt{3}$ and $xy=4$, then $x^4+y^4 =$

(a) 400 (b) 368

(c) 352 (d) 200

43. If $x = 3-\sqrt{5}$, then $\frac{\sqrt{x}}{\sqrt{2}+\sqrt{(3x-2)}} =$

(a) 5 (b) $\sqrt{5}$

(c) 1/5 (d) $1/\sqrt{5}$

44. If $a=\sqrt{(21)}-\sqrt{(20)}$ and $b=\sqrt{(18)}-\sqrt{(17)}$, then

(a) $a=b$ (b) $a+b=0$

(c) $a>b$ (d) $a<b$

45. Solution of the equation $\sqrt{(x+10)}+\sqrt{(x-2)}=6$ are

(a) 0 (b) 6

(c) 4 (d) None of these

46. $\sqrt{[6+2\sqrt{3}+2\sqrt{2}+2\sqrt{6}]}-1/\sqrt{(5+2\sqrt{6})} =$

(a) 1 (b) –1

(c) 0 (d) None of these

Partial fractions

1. If $\frac{2x+3}{(x+1)(x-3)} = \frac{a}{x+1}+\frac{b}{(x-3)}$, then $a+b$ **[MNR 1993]**

(a) 1 (b) 2

(c) 9/4 (d) –1/4

2. If $\frac{3x+a}{x^2-3x+2} = \frac{A}{(x-2)} - \frac{10}{x-1}$, then

(a) $a=7$ (b) $a=-7$

(c) $A=-13$ (d) $A=13$

3. If $\frac{3x+4}{(x+1)^2(x-1)} = \frac{A}{(x-1)}+\frac{B}{(x+1)}+\frac{C}{(x+1)^2}$, then $A =$ **[EAMCET 1994]**

(a) –1/2 (b) 15/4

(c) 7/4 (d) –1/4

4. The partial fractions of $\frac{3x-1}{(1-x+x^2)(2+x)}$ are **[MNR 1995]**

(a) $\frac{x}{(x^2-x+1)}+\frac{1}{x+2}$ (b) $\frac{1}{x^2-x+1}+\frac{x}{x+2}$

(c) $\frac{x}{x^2-x+1}-\frac{1}{x+2}$ (d) $\frac{-1}{x^2-x+1}+\frac{x}{x+2}$

5. If $\frac{(x+1)^2}{x^3+x}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$, then $\sin^{-1}\left(\frac{A}{C}\right)=$ **[EAMCET 1997, 98]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

6. If $\frac{x}{(x-1)(x^2+1)^2}=\frac{1}{4}\left[\frac{1}{(x-1)}-\frac{x+1}{x^2+1}\right]+y$ then $y=$

(a) $\frac{(1-x)}{2(x^2+1)^2}$ (b) $\frac{(1-x)}{3(x^2+1)}$

(c) $\frac{1+x}{2(x^2-1)^2}$ (d) None of these

7. The coefficient of x^n in the expression $\frac{5x+6}{(2+x)(1-x)}$ when expanded in ascending order is

(a) $\frac{-2}{3}\frac{(-1)^n}{2^n}+\frac{11}{3}$ (b) $\frac{2}{3}+\frac{(-1)^n}{2^n}-\frac{11}{3}$

(c) $-\frac{2}{3}+\frac{(-1)^n}{3}-\frac{11}{2^n}$ (d) None of these

8. The remainder obtained when the polynomial $1+x+x^3+x^9+x^{27}+x^{81}+x^{243}$ is divided by $x-1$ is **[EAMCET 1991]**

(a) 3 (b) 5

(c) 7 (d) 11

9. If

$$\frac{1}{x(x+1)(x+2)....(x+n)}=\frac{A_0}{x}+\frac{A_1}{x+1}+\frac{A_2}{x+2}+....+\frac{A_n}{x+n}$$

then $A_r=$

(a) $\frac{r!(-1)^r}{(n-r)!}$ (b) $\frac{(-1)^r}{r!(n-r)!}$

(c) $\frac{1}{r!(n-r)!}$ (d) None of these

10. $\frac{x+1}{(x-1)(x-2)(x-3)}=$ **[IIT 1996]**

(a) $\frac{1}{x-1}+\frac{3}{x-2}+\frac{1}{x-3}$ (b) $-\frac{3}{x-1}+\frac{1}{x-2}+\frac{2}{x-3}$

(c) $\frac{1}{x-1}-\frac{3}{x-2}+\frac{2}{x-3}$ (d) None of these

11. If $\frac{ax^2+bx+c}{(x-1)(x+2)(2x+3)}=\frac{3}{x-1}+\frac{2}{x+2}-\frac{5}{2x+3}$, then

(a) $a=5$ (b) $b=-18$

(c) $c=22$ (d) None of these

12. If $\frac{(e^x+2)}{(e^x-1)(2e^x-3)}=-\frac{3}{e^x-1}+\frac{B}{2e^x-3}$, then $B=$

(a) 1 (b) 3

(c) 5 (d) 7

13. If $\frac{3x+4}{x^2-3x+2}=\frac{A}{x-2}-\frac{B}{x-1}$, then $(A,B)=$ **[EAMCET 1996]**

(a) (7, 10) (b) (10, 7)

(c) (10, – 7) (d) (– 10, 7)

14. If the remainders of the polynomial $f(x)$ when divided by $x+1, x-2, x+2$ are 6, 3, 15 then the remainder of $f(x)$ when divided by $(x+1)(x+2)(x-2)$ is

(a) $2x^2-3x+1$ (b) $3x^2-2x+1$

(c) $2x^2-x-3$ (d) $3x^2+2x+1$

15. If $\frac{1-\cos x}{\cos x(1+\cos x)}=\frac{\sin\alpha}{\cos x}-\frac{2}{1+\cos x}$, then $\alpha=$

(a) $\pi/8$ (b) $\pi/4$

(c) $\pi/2$ (d) π

16. If $\frac{x^2}{(x^2+a^2)(x^2+b^2)}=k\left(\frac{a^2}{x^2+a^2}-\frac{b^2}{x^2+b^2}\right)$ then $k=$

(a) a^2-b^2 (b) $\frac{1}{a+b}$

(c) $\frac{1}{a-b}$ (d) $\frac{1}{a^2-b^2}$

17. **[MNR 1993]** If $\frac{9}{(x-1)(x+2)^2}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$ then $A-B-C=$

(a) 3 (b) – 1

(c) 5 (d) None of these

18. If $\frac{ax+b}{(3x+4)^2}=\frac{1}{3x+4}-\frac{3}{(3x+4)^2}$ then

(a) $a=2$ (b) $b=1$

(c) $a=3$ (d) $b=4$

19. $\frac{x^2+13x+15}{(2x+3)(x+3)^2}=$

(a) $\frac{1}{x+3}-\frac{1}{2x+3}+\frac{5}{(x+3)^2}$

(b) $\frac{1}{2x+3}-\frac{1}{x+3}+\frac{5}{(x+3)^2}$

(c) $\frac{1}{2x+3}+\frac{1}{x+3}-\frac{5}{(x+3)^2}$

(d) $\frac{1}{2x+3}-\frac{1}{x+3}-\frac{5}{(x+3)^2}$

20. The partial fractions of $\frac{3x^3-8x^2+10}{(x-1)^4}$ is

(a) $\frac{3}{(x-1)}+\frac{1}{(x-1)^2}+\frac{7}{(x-1)^3}+\frac{5}{(x-1)^4}$

(b) $\frac{3}{(x-1)}+\frac{1}{(x-1)^2}-\frac{7}{(x-1)^3}+\frac{5}{(x-1)^4}$

(c) $\frac{3}{(x-1)}+\frac{1}{(x-1)^2}-\frac{7}{(x-1)^3}+\frac{5}{(x-1)^4}$

(d) None of these

21. If $\frac{(x-1)^2}{x^3+x}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$, then

(a) $A=1, B=0, C=2$ (b) $A=1, B=0, C=-2$

(c) $A=-1, B=0, C=-2$ (d) None of these

22. If $\frac{2x}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$, then

(a) $A=B=C$ (b) $A=B\neq C$

(c) $A\neq B=C$ (d) $A\neq B\neq C$

23. $\frac{x^2+1}{(2x-1)(x^2-1)}=$ **[MNR 1994]**

(a) $\frac{-5}{3(2x-1)}+\frac{3}{(x+1)}+\frac{1}{(x-1)}$ (b) $\frac{-5}{3(2x-1)}+\frac{1}{3(x+1)}+\frac{1}{(x-1)}$

(c) $\frac{1}{2x-1}+\frac{5}{(x+1)}-\frac{3}{(x-1)}$ (d) None of these

24. If $\frac{ax-1}{(1-x+x^2)(2+x)}=\frac{x}{1-x+x^2}-\frac{1}{2+x}$, then $a=$.....

(a) 2 (b) 3
(c) 4 (d) 5

25. $\frac{1}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{(x^2+1)}$, then $(A,B,C)=$.... **[IIT 1995]**

(a) (1, – 1, 0) (b) $(-1,0,-1)$
(c) $(0,1,1)$ (d) None of these

26. $\frac{2x}{x^4+x^2+1}=$

(a) $\frac{x+1}{x^2-x+1}+\frac{x-1}{x^2+x-1}$ (b) $\frac{x-1}{x^2-x+1}-\frac{x+1}{x^2+x-1}$

(c) $\frac{x}{x^2-x+1}+\frac{x+1}{x^2+x-1}$ (d) $\frac{1}{x^2-x+1}-\frac{1}{x^2+x+1}$

27. $\frac{3x^2+5}{(x^2+1)^2}=\frac{a}{x^2+1}+\frac{b}{(x^2+1)^2}$, then $(a,b)=$

(a) (2, 3) (b) (3, 2)
(c) (– 2,3) (d) (–3, 2)

28. $\frac{(x-a)(x-b)}{(x-c)(x-d)}=\frac{A}{x-c}-\frac{B}{(x-d)}+C$, then C =

(a) 5 (b) 4
(c) 3 (d) 1

29. The partial fractions of $\frac{x^2-5}{x^2-3x+2}$ are

(a) $1+\frac{1}{(x-1)}-\frac{1}{(x-2)^2}$ (b) $\frac{1}{(x-1)}+\frac{1}{(x-2)^2}$

(c) $\frac{1}{(x-1)}-\frac{1}{(x-2)^2}$ (d) $1+\frac{4}{(x-1)}-\frac{1}{(x-2)}$

30. The partial fraction of $\frac{6x^4+5x^3+x^2+5x+2}{1+5x+6x^2}=$

(a) $x^2+\frac{1}{1+2x}+\frac{1}{1+3x}$ (b) $x^2-\frac{1}{1+2x}+\frac{1}{1+3x}$

(c) $x^2+\frac{1}{1+2x}-\frac{1}{1-3x}$ (d) None of these

31. If $\frac{\sin^2 x+1}{2\sin^2 x-5\sin x+3}=\frac{A}{(2\sin x-3)}+\frac{B}{(\sin x-1)}+C$, then

(a) $A=\frac{13}{2}$ (b) $B=2$

(c) $C=1$ (d) $A+B+C=5$

32. The coefficient of x^4 in the expansion of the expression $\frac{3x}{(x-2)(x+1)}$ is

(a) –15/16 (b) 15/16
(c) –16/15 (d) 16/15

33. The coefficient of x^5 in the expansion of $\frac{x^2+1}{(x^2+4)(x-2)}$ is

(a) 1/256 (b) 1/562
(c) 1/265 (d) –1/256

Critical Thinking

Objective Questions

1. If $x=\log_3 5$, $y=\log_{17} 25$, which one of the following is correct **[WB JEE 1993]**

(a) $x<y$ (b) $x=y$
(c) $x>y$ (d) None of these

2. If $\log_{0.3}(x-1)<\log_{0.09}(x-1)$, then x lies in the interval

(a) $(2,\infty)$ (b) (– 2, –1)
(c) (1, 2) (d) None of these

3. If $\log x:\log y:\log z=(y-z):(z-x):(x-y)$ then **[UPSEAT 2001]**

(a) $x^y.y^z.z^x=1$ (b) $x^x y^y z^z=1$
(c) $\sqrt[x]{x}\sqrt[y]{y}\sqrt[z]{z}=1$ (d) None of these

4. The number of solution of $\log_2(x+5)=6-x$ is

(a) 2 (b) 0
(c) 3 (d) None of these

5. The number $\log_{20} 3$ lies in

(a) $(1/4,1/3)$ (b) $(1/3,1/2)$
(c) $(1/2,3/4)$ (d) $(3/4,4/5)$

6. If $\frac{1}{2}\le\log_{0.1} x\le 2$ then

(a) The maximum value of x is $1/\sqrt{10}$
(b) x lies between 1/100 and $1/\sqrt{10}$
(c) x does not lie between 1/100 and $1/\sqrt{10}$
(d) The minimum value of x is 1/100

7. The equation $4^{(x^2+2)}-9.2^{(x^2+2)}+8=0$ has the solution

(a) $x=1$ (b) $x=-1$
(c) $x=\sqrt{2}$ (d) $x=-\sqrt{2}$

8. $\sqrt{[10-\sqrt{(24)}-\sqrt{(40)}+\sqrt{(60)}]}=$

(a) $\sqrt{5}+\sqrt{3}+\sqrt{2}$ (b) $\sqrt{5}+\sqrt{3}-\sqrt{2}$
(c) $\sqrt{5}-\sqrt{3}+\sqrt{2}$ (d) $\sqrt{2}+\sqrt{3}-\sqrt{5}$

9. $\sum\frac{1}{1+x^{a-b}+x^{a-c}}=$

(a) 1 (b) –1
(c) 0 (d) None of these

10. $\frac{1}{\sqrt{(11-2\sqrt{30})}}-\frac{3}{\sqrt{(7-2\sqrt{10})}}-\frac{4}{\sqrt{(8+4\sqrt{3})}}=$

(a) 0 (b) –1
(c) 1 (d) None of these

11. The square root of 134 + $\sqrt{(6292)}$ is

(a) $21+\sqrt{(13)}$ (b) $11+\sqrt{(13)}$
(c) $13+\sqrt{(11)}$ (d) $13+\sqrt{(21)}$

12. If $x = 2 + \sqrt{3}, xy = 1$, then $\frac{x}{\sqrt{2} + \sqrt{x}} + \frac{y}{\sqrt{2} - \sqrt{y}} =$

(a) $\sqrt{2}$ (b) $\sqrt{3}$

(c) 1 (d) None of these

13. The partial fractions of $\frac{x^2}{(x-1)^3(x-2)}$ are **[IIT 1992]**

(a) $\frac{-1}{(x-1)^3} + \frac{3}{(x-1)^2} - \frac{4}{(x-1)} + \frac{4}{(x-2)}$

(b) $\frac{-1}{(x-1)^3} - \frac{3}{(x-1)^2} + \frac{4}{(x-1)} + \frac{4}{(x-2)}$

(c) $\frac{-1}{(x-1)^3} + \frac{-3}{(x-1)^2} + \frac{-4}{(x-1)} + \frac{4}{(x-2)}$

(d) None of these

14. If $\frac{x^3 - 6x^2 + 10x - 2}{x^2 - 5x + 6} = f(x) + \frac{A}{(x-2)} + \frac{B}{(x-3)}$, then $f(x) =$

(a) $x - 1$ (b) $x + 1$

(c) x (d) None of these

15. The partial fractions of $\frac{x^4 + 24x^2 + 28}{(x^2+1)^3}$ are

[EAMCET 1986]

(a) $\frac{1}{(x^2+1)} + \frac{22}{(x^2+1)^2} + \frac{5}{(x^2+1)^3}$

(b) $\frac{1}{(x^2+1)} + \frac{22}{(x^2+1)^2} - \frac{5}{(x^2+1)^2}$

(c) $\frac{1}{(x^2+1)} - \frac{22}{(x^2+1)^2} - \frac{5}{(x^2+1)^3}$

(d) None of these

16. Which of the following is not true **[UPSEAT 2000]**

(a) $\log(1+x) < x$ for $x > 0$ (b) $\frac{x}{1+x} < \log(1+x)$ for $x > 0$

(c) $e^x > 1 + x$ for $x > 0$ (d) $e^x < 1 - x$ for $x > 0$

Answers

Logarithms

1	d	2	a	3	c	4	b	5	a
6	b	7	b	8	c	9	b	10	c
11	d	12	c	13	d	14	b,c	15	c
16	c	17	c	18	d	19	b	20	c
21	b	22	c	23	a	24	c	25	b
26	a	27	a	28	c	29	a,b,c,d	30	b
31	a,b,c	32	a	33	a	34	a	35	b
36	b	37	c	38	a	39	b	40	b

Indices and Surds

1	a	2	d	3	a	4	c	5	a
6	c	7	d	8	c	9	a	10	d
11	c	12	c	13	d	14	b	15	d
16	a	17	a	18	d	19	b	20	a,d
21	b	22	c	23	a,b	24	b,c	25	a,c,d
26	c	27	b	28	b	29	b	30	c
31	a	32	b	33	a	34	d	35	b
36	b	37	d	38	b	39	c	40	d
41	b	42	b	43	d	44	d	45	b
46	a								

Partial fractions

1	b	2	a,d	3	c	4	c	5	a
6	a	7	a	8	c	9	b	10	c
11	a,c	12	d	13	b	14	a	15	c
16	d	17	c	18	b,c	19	a	20	c
21	b	22	d	23	b	24	b	25	a
26	d	27	b	28	d	29	d	30	a
31	a,d	32	b	33	d				

Critical Thinking Questions

1	c	2	a	3	b	4	d	5	b
6	a,b,d	7	a,b	8	b	9	a	10	a
11	b	12	a	13	c	14	a	15	a
16	d								

Answers and Solutions

Logarithms

1. (d) It is obvious.

2. (a) Let x be the required logarithm, then by definition $(2\sqrt{2})^x = 32\sqrt[5]{4} \Rightarrow (2.2^{1/2})^x = 2^5.2^{2/5}$;

$\therefore\ 2^{\frac{3x}{2}} = 2^{5+\frac{2}{5}}$ Equating the indices, $\frac{3}{2}x = \frac{27}{5}$

$\therefore x = \frac{18}{5} = 3.6$.

3. (c) Suppose, if possible, $\log_2 7$ is rational, say p/q where p and q are integers, prime to each other.

Then, $\frac{p}{q} = \log_2 7 \quad \Rightarrow 7 = 2^{p/q} \quad \Rightarrow 2^p = 7^q$,

which is false since L.H.S is even and R.H.S is odd. Obviously $\log_2 7$ is not an integer and hence not a prime number.

4. (b) $\log_{49} 28 = \frac{\log 28}{\log 49} = \frac{\log 7 + \log 4}{2\log 7}$

$= \frac{\log 7}{2\log 7} + \frac{\log 4}{2\log 7} = \frac{1}{2} + \frac{1}{2}\log_7 4$

$= \frac{1}{2} + \frac{1}{2}.2\log_7 2 = \frac{1}{2} + \log_7 2 = \frac{1}{2} + m = \frac{1+2m}{2}$

5. (a) $\log_e\left(\frac{a+b}{2}\right) = \frac{1}{2}(\log_e a + \log_e b)$

$= \frac{1}{2}\log_e(ab) = \log_e\sqrt{ab}$

$\Rightarrow \frac{a+b}{2} = \sqrt{ab} \Rightarrow a+b = 2\sqrt{ab}$

$\Rightarrow \left(\sqrt{a}-\sqrt{b}\right)^2 = 0 \Rightarrow \sqrt{a}-\sqrt{b} = 0 \Rightarrow a = b$.

6. (b) Since 10, 3, e, 2 are in decreasing order. Obviously, $\log_{10}\alpha, \log_3\alpha, \log_e\alpha, \log_2\alpha$ are in increasing order.

7. (b) $\log ab - \log|b| = \log\left(\frac{ab}{|b|}\right) = \log|a|$.

8. (c) $\sqrt{\log^2_{0.5} 4} = \sqrt{\{\log_{0.5}(0.5)^{-2}\}^2} = \sqrt{(-2)^2} = 2$.

9. (b) $\log_3 4.\log_4 5.\log_5 6.\log_6 7.\log_7 8.\log_8 9$

$= \frac{\log 4}{\log 3}.\frac{\log 5}{\log 4}.\frac{\log 6}{\log 5}.\frac{\log 7}{\log 6}.\frac{\log 8}{\log 7}.\frac{\log 9}{\log 8} = \frac{\log 9}{\log 3}$

$= \log_3 9 = \log_3 3^2 = 2$.

10. (c) $\log_7\log_7\sqrt{7\sqrt{7\sqrt{7}}} = \log_7\log_7 7^{7/8} = \log_7(7/8)$

$= \log_7 7 - \log_7 8 = 1 - \log_7 2^3 = 1 - 3\log_7 2$.

11. (d) $81^{(1/\log_5 3)} + 27^{\log_9 36} + 3^{4/\log_7 9}$

$= 3^{4\log_3 5} + 3^{3.\frac{1}{2}\log_3 36} + 3^{4\log_9 7}$

$= 3^{\log_3 5^4} + 3^{\log_3 36^{3/2}} + 3^{\log_3 7^{4/2}}$

$= 5^4 + 36^{3/2} + 7^2 = 890$.

12. (c) Given expression= $\log\left(\frac{16^7}{15^7}.\frac{25^5}{24^5}.\frac{81^3}{80^3}\right) = \log 2$.

13. (d) $ab = \log_4 5.\log_5 6 = \log_4 6 = \frac{1}{2}\log_2 6$

$ab = \frac{1}{2}(1+\log_2 3) \Rightarrow 2ab - 1 = \log_2 3$

$\therefore \log_3 2 = \frac{1}{2ab-1}$.

14. (b,c) $\log_k x.\log_5 k = \log_x 5 \Rightarrow \log_5 x = \log_x 5$

$\Rightarrow \log_x 5 = \frac{1}{\log_x 5} \Rightarrow (\log_x 5)^2 = 1 \Rightarrow \log_x 5 = \pm 1$

$\Rightarrow x^{\pm 1} = 5 \Rightarrow x = 5, \frac{1}{5}$.

15. (c) $\log_5 a.\log_a x = 2 \Rightarrow$

$\log_5 x = 2 \Rightarrow x = 5^2 = 25$.

16. (c) $a^2 + 4b^2 = 12ab$

$\Rightarrow a^2 + 4b^2 + 4ab = 16ab \Rightarrow (a+2b)^2 = 16ab$

$\Rightarrow 2\log(a+2b) = \log 16 + \log a + \log b$

$\therefore \log(a+2b) = \frac{1}{2}[\log a + \log b + 4\log 2]$

17. (c) $A = \log_2 \log_2 \log_4 256 + 2\log_{2^{1/2}} 2$

$= \log_2 \log_2 \log_4 4^4 + 2\times\frac{1}{(1/2)}\log_2 2$

$= \log_2 \log_2 4 + 4 = \log_2 \log_2 2^2 + 4$

$= \log_2 2 + 4 = 1 + 4 = 5$.

18. (d) $\log_{1000} x^2 = \log_{10^3} x^2 = 2\log_{10^3} x = \frac{2}{3}\log_{10} x = \frac{2}{3}y$.

19. (b) $x = \log_a bc \Rightarrow 1 + x = \log_a a + \log_a bc = \log_a abc$

$\therefore (1+x)^{-1} = \log_{abc} a$

$\therefore$

$(1+x)^{-1} + (1+y)^{-1} + (1+z)^{-1} = \log_{abc} a + \log_{abc} b + \log_{abc} c$

$= \log_{abc} abc = 1$.

20. (c) $a = \log_{24} 12 = \frac{\log 12}{\log 24} = \frac{2\log 2 + \log 3}{3\log 2 + \log 3}$

$b = \log_{36} 24 = \frac{3\log 2 + \log 3}{2(\log 2 + \log 3)}$

$c = \log_{48} 36 = \frac{2(\log 2 + \log 3)}{4\log 2 + \log 3}$

$\therefore abc = \frac{2\log 2 + \log 3}{4\log 2 + \log 3}$

$\Rightarrow$

$1 + abc = \frac{6\log 2 + 2\log 3}{4\log 2 + \log 3} = 2.\frac{3\log 2 + \log 3}{4\log 2 + \log 3} = 2bc$.

21. (b) $a^x = b \Rightarrow x\log a = \log b$

$\Rightarrow x = \frac{\log b}{\log a} = \log_a b$

Similarly $y = \log_b c, z = \log_c a$

$\therefore xyz = \log_a b.\log_b c.\log_c a = 1$.

22. (c) $y = 3^{12}\times 2^8 \Rightarrow \log_{10} y = 12\log_{10} 3 + 8\log_{10} 2$

$= 12\times 0.47712 + 8\times 0.30103$

$= 5.72544 + 2.40824 = 8.13368$

$\therefore$ Number of digits in y = 8 + 1 = 9.

23. (a) $\sum_{n=1}^{n}\frac{1}{\log_{2^n}(a)} = \sum_{n=1}^{n}\log_a 2^n = \sum_{n=1}^{n} n\log_a 2 = \log_a 2.\sum_{n=1}^{n} n$

$= \log_a 2.\frac{n(n+1)}{2} = \frac{n(n+1)}{2}\log_a 2$.

24. (c) $\log_7 \log_5(\sqrt{x^2+5+x}) = 0 = \log_7 1$

$\Rightarrow \log_5(x^2+5+x)^{1/2} = 1 = \log_5 5$

$\Rightarrow (x^2+5+x)^{1/2} = 5$

$\Rightarrow (x^2+x+5) = 25 \Rightarrow x^2 + x - 20 = 0$

$\Rightarrow (x-4)(x+5) = 0 \Rightarrow x = 4, -5 \Rightarrow x = 4$.

25. (b) $\log_4 18 = \frac{1}{2}\log_2(3^2.2) = \frac{1}{2}(2\log_2 3 + \log_2 2)$

$= \log_2 3 + \frac{1}{2}$, which is irrational.

26. (a) $(0.05)^{\log_{\sqrt{20}}(0.1+0.01+......)} = \left(\frac{1}{20}\right)^{2\log_{20}\left(\frac{0.1}{1-0.1}\right)}$

$= 20^{-2\log_{20}(1/9)} = 20^{2\log_{20} 9} = 20^{\log_{20} 9^2} = 9^2 = 81$.

27. (a) $[\log_b a.\log_c a - \log_a a] + [\log_a b.\log_c b - \log_b b]$

$+[\log_a c\log_b c - \log_c c] = 0$

$\Rightarrow [(\ln a)^3 + (\ln b)^3 + (\ln c)^3]\frac{1}{\ln a.\ln b.\ln c} - 3 = 0$

$\Rightarrow$

$\frac{1}{\ln a.\ln b.\ln c}[(\ln a)^3 + (\ln b)^3 + (\ln c)^3 - 3\ln a.\ln b.\ln c] = 0$

$\Rightarrow (\ln a)^3 + (\ln b)^3 + (\ln c)^3 - 3\ln a.\ln b.\ln c = 0$

$\Rightarrow \ln a + \ln b + \ln c = 0$

$\Rightarrow \ln(abc)$ = ln 1, $[a^3 + b^3 + c^3 - 3abc = 0$

$\Rightarrow a + b + c = 0]$, $\therefore abc = 1$.

28. (c) $a = \frac{\log 27}{\log 12} = \frac{3\log 3}{\log 3 + 2\log 2} \Rightarrow \log 3 = \frac{2a\log 2}{3-a}$

$\log_6 16 = \frac{\log 16}{\log 6} = \frac{4\log 2}{\log 2 + \log 3}$

$= \frac{4\log 2}{\log 2 + \frac{2a\log 2}{3-a}} = \frac{4(3-a)}{3-a+2a} = 4.\frac{3-a}{3+a}$.

29. (a,b,c,d) $\frac{\log x}{b-c} = \frac{\log y}{c-a} = \frac{\log z}{a-b} = k\,(\text{say})$

$\Rightarrow \log x = k(b-c), \log y = k(c-a), \log z = k(a-b)$

$\Rightarrow x = e^{k(b-c)}, y = e^{k(c-a)}, z = e^{k(a-b)}$

$\therefore xyz = e^{k(b-c)+k(c-a)+k(a-b)} = e^0 = 1$

$x^a y^b z^c = e^{k(b-c)a+k(c-a)b+k(a-b)c} = e^0 = 1 = xyz$

$x^{b+c}y^{c+a}z^{a+b} = e^{k(b^2-c^2)+k(c^2-a^2)+k(a^2-b^2)} = e^0 = 1$.

30. (b) Let $\log_{16} x = y \Rightarrow y^2 - y + \log_{16} k = 0$

This quadratic equation will have exactly one solution if its discriminant vanishes.

$\therefore (-1)^2 - 4.1.\log_{16} k = 0 \Rightarrow 1 = \log_{16} k^4$

$\Rightarrow k^4 = 16 \Rightarrow k^2 = 4 \Rightarrow k = \pm 2$.

But $\log_{16} k$ is not defined $k < 0$, $\therefore k = 2$.

$\therefore$ Number of real values of $k = 1$.

31. (a,b,c) $x^{\frac{3}{4}(\log_3 x)^2 + \log_3 x - \frac{5}{4}} = \sqrt{3} = 3^{\frac{1}{2}}$.

There is a possibility of a solution $x = 3$

For this value, LHS = $3^{\frac{3}{4}.1^2 + 1 - \left(\frac{5}{4}\right)} = 3^{\frac{2}{4}} = 3^{\frac{1}{2}} = \text{RHS}$.

$\therefore x = 3$ is a solution, which is a $+ve$ integer.

Next, $\left[\frac{3}{4}(\log_3 x)^2 + \log_3 x - \frac{5}{4}\right]\log_3 x = \frac{1}{2}$

$\Rightarrow [3(\log_3 x)^2 + 4\log_3 x - 5]\log_3 x - 2 = 0$

$\Rightarrow 3t^3 + 4t^2 - 5t - 2 = 0$, $[\,t = \log_3 x\,]$

$\Rightarrow 3t^3 - 3t^2 + 7t^2 - 7t + 2t - 2 = 0$

$\Rightarrow (3t^2 + 7t + 2)(t-1) = 0 \Rightarrow (3t+1)(t+2)(t-1) = 0$

$\Rightarrow t = 1, -2, -\frac{1}{3} \Rightarrow \log_3 x = 1, -2, -\frac{1}{3}$

$\Rightarrow x = 3^1, 3^{-2}, 3^{-1/3}$; $\therefore x = 3, \frac{1}{9}, \frac{1}{\sqrt[3]{3}}$

Thus, there is one $+ve$ integral value, one irrational value, two positive rational values.

32. (a) $x = \log_5 1000 = 3\log_5 10 = 3 + 3\log_5 2 = 3 + \log_5 8$

$y = \log_7 2058 = \log_7(7^3.6) = 3 + \log_7 6$

As $\log_5 8 > \log_5 5$ *i.e.,* $\log_5 8 > 1$. $\therefore x > 4$

And $\log_7 6 < \log_7 7$ *i.e.,* $\log_7 6 < 1$

$\therefore y < 4$; $\therefore x > y$.

33. (a) $\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} > x$

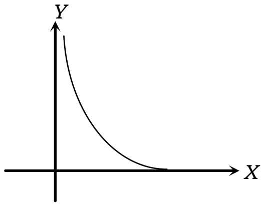
$\Rightarrow \log_\pi 3 + \log_\pi 4 > x \Rightarrow \log_\pi 12 > x$

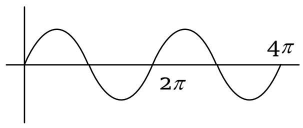
$\pi^2 < 12 < \pi^3$

$\therefore 12 > \pi^2$; $\therefore \log_\pi 12 > \log_\pi \pi^2$

i.e., $\log_\pi 12 > 2$; $\therefore x$ will be 2.

34. (a) $0 < \frac{1}{\sqrt{2}} < 1$





$\log_{1/\sqrt{2}} \sin x > 0$, $x \in [0, 4\pi] \Rightarrow 0 < \sin x < 1$

$\therefore$ Integral multiple of $\frac{\pi}{4}$ will be $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$

Number of required values = 4.

35. (b) $\log_{1/2}(x^2 - 6x + 12) \ge -2$ .....(i)

For log to be defined, $x^2 - 6x + 12 > 0$

$\Rightarrow (x-3)^2 + 3 > 0$, which is true $\forall x \in R$.

From (i), $x^2 - 6x + 12 \le \left(\frac{1}{2}\right)^{-2}$

$\Rightarrow x^2 - 6x + 12 \le 4 \Rightarrow x^2 - 6x + 8 \le 0$

$\Rightarrow (x-2)(x-4) \le 0 \Rightarrow 2 \le x \le 4$; $\therefore x \in [2, 4]$.

36. (b) $2^{\log_{\sqrt{2}}(x-1)} > x + 5 \Rightarrow (\sqrt{2})^{2\log_{\sqrt{2}}(x-1)} > x + 5$

$\Rightarrow (x-1)^2 > x + 5 \Rightarrow x^2 - 3x - 4 > 0$

$\Rightarrow (x-4)(x+1) > 0 \Rightarrow x > 4$ or $x < -1$

But for $\log_{\sqrt{2}}(x-1)$ to be defined, $x - 1 > 0$

i.e., $x > 1$ $\therefore x > 4 \Rightarrow x \in (4, \infty)$.

37. (c) $\log_{0.04}(x-1) \ge \log_{0.2}(x-1)$ .....(i)

For log to be defined $x - 1 > 0 \Rightarrow x > 1$

From (i), $\log_{(0.2)^2}(x-1) \ge \log_{0.2}(x-1)$

$\Rightarrow \frac{1}{2}\log_{0.2}(x-1) \ge \log_{0.1}(x-1) \Rightarrow \sqrt{x-1} \le (x-1)$

$\Rightarrow \sqrt{x-1}(1 - \sqrt{x-1}) \le 0 \Rightarrow 1 - \sqrt{x-1} \le 0$

$\Rightarrow \sqrt{x-1} \ge 1 \Rightarrow x \ge 2$, $\therefore x \in [2, \infty)$.

38. (a) $\log_{0.2}\frac{x+2}{x} \le 1$ .....(i)

For log to be defined, $\frac{x+2}{x} > 0 \Rightarrow x > 0$ or $x < -2$

Now from (i), $\log_{0.2}\frac{x+2}{x} \le \log_{0.2} 0.2$

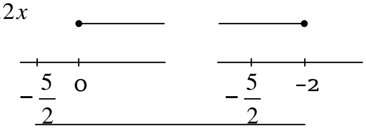
$\Rightarrow \frac{x+2}{x} \ge 0.2$ .....(ii)

Case (i) $x > 0$

From (ii), $x + 2 \ge 0.2x$

$\Rightarrow 0.8x \ge -2$

$\Rightarrow x \ge -\frac{5}{2}$.



Case (ii) $x < -2$

From (ii), $x + 2 \le 0.2x \Rightarrow 0.8x \le -2 \Rightarrow x \le -\frac{5}{2}$

$\Rightarrow x \in (0, \infty) \cup \left(-\infty, -\frac{5}{2}\right]$; $\therefore x \in \left(-\infty, -\frac{5}{2}\right] \cup (0, \infty)$.

39. (b) We have $xyz = \log_b a \times \log_c b \times \log_a c$

$= \frac{\log_e a}{\log_e b} \times \frac{\log_e b}{\log_e c} \times \frac{\log_e c}{\log_e a} = 1$.

40. (b) $\log_2 . \log_3 ..... \log_{99} \log_{100} 100^{99^{98^{\cdot^{\cdot^{\cdot^{2^1}}}}}}$

$= \log_2 \log_3 .... \log_{99} 99^{98^{\cdot^{\cdot^{\cdot^{2^1}}}}}$ $[\log_{100} 100 = 1]$

$$= \log_2 \log_3 .... \log_{98} {98^{97^{\cdot^{\cdot^{\cdot^{2^1}}}}}}$$

$$= \log_2 \log_3 .... \log_{97} {97^{96^{\cdot^{\cdot^{\cdot^{2^1}}}}}} \quad = \log_2 \log_3 3^{2^1}$$

$$= \log_2 \ 2' \log_3 3 = \log_2 2 = 1 .$$

I. Indices and Surds

1. (a) $\left(\frac{x^l}{x^m}\right)^{l^2+lm+m^2} \left(\frac{x^m}{x^n}\right)^{m^2+nm+n^2} \left(\frac{x^n}{x^l}\right)^{n^2+nl+l^2}$

$= (x^{l-m})^{(l^2+lm+m^2)} (x^{m-n})^{m^2+nm+n^2} \ (x^{n-l})^{n^2+nl+l^2}$

$= x^{l^3-m^3} . x^{m^3-n^3} . x^{n^3-l^3} = x^{l^3-m^3+m^3-n^3+n^3-l^3} = x^0 = 1$

2. (d) $2^x = 2^{2y} = 2^{3z}$ *i.e.*, $x = 2y = 3z = k$ (say).

Then $xyz = \frac{k^3}{6} = 288$, So $k = 12$

$\therefore x = 12, y = 6, z = 4$.

Therefore, $\frac{1}{2x} + \frac{1}{4y} + \frac{1}{8z} = \frac{11}{96}$

3. (a) $\frac{2.3^{n+1} + 7.3^{n-1}}{3^{n+2} - 2\left(\frac{1}{3}\right)^{1-n}} = \frac{2.3^{n-1}.3^2 + 7.3^{n-1}}{3^{n-1}.3^3 - 2.3^{n-1}} = \frac{3^{n-1}[18+7]}{3^{n-1}[27-2]} = 1$

4. (c) $\left(\frac{2}{3}\right)^{x+2} = \left(\frac{3}{2}\right)^{2-2x} \Rightarrow \left(\frac{2}{3}\right)^{x+2} = \left(\frac{2}{3}\right)^{2x-2}$

Clearly $x + 2 = 2x - 2 \ \Rightarrow \ x = 4$

5. (a) $\sqrt[3]{9}, \sqrt[4]{11}, \sqrt[6]{17}$

$\because$ L.C.M. of 3, 4, 6 is 12

$\therefore \sqrt[3]{9} = 9^{1/3} = (9^4)^{1/12} = (6561)^{1/12}$,

$\sqrt[4]{11} = (11)^{1/4} (11^3)^{1/12} = (1331)^{1/12}$,

$\sqrt[6]{17} = (17)^{1/6} = (17^2)^{1/2} = (289)^{1/12}$

Hence, $\sqrt[3]{9}$ is the greatest number.

6. (c) Given fraction

$= \frac{15}{\sqrt{10} + \sqrt{20} + \sqrt{40} - \sqrt{5} - \sqrt{80}}$

$= \frac{15}{\sqrt{10} + 2\sqrt{5} + 2\sqrt{10} - \sqrt{5} - 4\sqrt{5}}$

$= \frac{15}{3\sqrt{10} - 3\sqrt{5}} = \frac{5}{\sqrt{10} - \sqrt{5}} . \frac{\sqrt{10} + \sqrt{5}}{\sqrt{10} + \sqrt{5}}$

$= \sqrt{10} + \sqrt{5} = \sqrt{5}(\sqrt{2} + 1)$

7. (d) Let $x = a^{1/3}, y = a^{-1/3}$ then $a = x^3, a^{-1} = y^3$

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

So, rationlising factor is $(x^2 - xy + y^2)$. Put the value of x and y. Thus the required rationlising factor is $a^{2/3} + a^{-2/3} - 1$.

8. (c) Let $\sqrt{3 + \sqrt{5}} = \sqrt{x} + \sqrt{y}$

$3 + \sqrt{5} = x + y + 2\sqrt{xy}$. Obviously $x + y = 3$

and $4xy = 5$. So $(x - y)^2 = 9 - 5 = 4$ or $(x - y) = 2$

After solving $x = \frac{5}{2}, y = \frac{1}{2}$.

Hence, $\sqrt{3 + \sqrt{5}} = \sqrt{\frac{5}{2}} + \sqrt{\frac{1}{2}} = \frac{\sqrt{5} + 1}{\sqrt{2}}$.

9. (a) $\sqrt{(17 + 12\sqrt{2})} = \sqrt{[3^2 + (2\sqrt{2})^2 + 2.3.2\sqrt{2}]} = 3 + 2\sqrt{2}$

$\therefore \sqrt[4]{(17 + 12\sqrt{2})} = \sqrt{(3 + 2\sqrt{2})} = \sqrt{2} + 1$.

10. (d) Given $\sqrt{(x+1)} - \sqrt{(x-1)} = \sqrt{(4x-1)}$ .....(i)

Squaring both sides, we get,

$-2\sqrt{(x^2 - 1)} = 2x - 1$

Squaring again, we get, $x = \frac{5}{4}$, which does not satisfy eq. (i). Hence, there is no solution of the given equation.

11. (c) $a^{m \log_a n} = a^{\log_a n^m} = n^m$.

12. (c) $(a^m)^n = a^{m^n} \Rightarrow a^{mn} = a^{m^n} \Rightarrow mn = m^n$

$\Rightarrow n = m^{n-1} \Rightarrow m = n^{\frac{1}{n-1}}$.

13. (d) $(x^5)^{1/3}(16x^3)^{2/3}\left(\frac{1}{4}x^{4/9}\right)^{-3/2} \quad x^{\frac{5}{3}+3.\frac{2}{3}-\frac{4}{9}.\frac{3}{2}} 2^{\frac{2}{3}.4+3} = 2^{\frac{17}{3}} x^3$

14. (b) $a^{1/x} = b^{1/y} = c^{1/z} = k\,(\text{say}) \Rightarrow a = k^x, b = k^y, c = k^z$

$b^2 = ac \Rightarrow (k^y)^2 = k^x.k^z \ \Rightarrow \ k^{2y} = k^{x+z} \Rightarrow x + z = 2y$.

15. (d) $a^x.b^y.c^z = bc.ca.ab = a^2b^2c^2$

$\Rightarrow a^{x-2}b^{y-2}c^{z-2} = 1 = a^0b^0c^0$

$\therefore \ x = y = z = 2$

$\therefore xyz = 2^3 = 8 = x + y + z + 2$.

16. (a) $a^x.a^y.a^z = (x + y + z)^{y+z+x}$

$\Rightarrow a^{x+y+z} = (x + y + z)^{x+y+z} \Rightarrow \ x + y + z = a$

Now, $a^x = (x + y + z)^y = a^y \Rightarrow \ x = y$, similarly $y = z$

$\therefore x = y = z = \frac{a}{3}$.

17. (a) $a^{x-1} = bc \Rightarrow a^x = abc$

$\therefore a^x = b^y = c^z = abc = k\,(\text{say})$

$\Rightarrow a = k^{1/x} \Rightarrow \frac{1}{x} = \log_k a$;

$\sum \frac{1}{x} = \log_k a + \log_k b + \log_k c = \log_k abc = \log_{abc} abc = 1$.

18. (d) $2^{m(n+1)+2n+n} = 2^{(m+1)n+2m}$

$\Rightarrow mn + m + 3n = mn + 2m + n \Rightarrow m = 2n$.

19. (b) $x^y = y^x \Rightarrow (x^y)^{1/x} = y$

Now, $\left(\frac{x}{y}\right)^{x/y} = \left(\frac{x}{x^{y/x}}\right)^{x/y} = \left(x^{1-\frac{y}{x}}\right)^{x/y}$

$= x^{(x/y)-1} = x^{(x/y)-k} \Rightarrow k = 1$.

20. (a, d) $x^{x.x^{1/3}}=(x.x^{1/3})^x \Rightarrow x^{x^{1+\frac{1}{3}}}=\left(x^{1+\frac{1}{3}}\right)^x$

$\Rightarrow x^{x^{4/3}}=\left(x^{4/3}\right)^x = x^{x^{4/3}}=x^{\frac{4}{3}x} \Rightarrow x^{4/3}=\frac{4}{3}x$

$\Rightarrow x^{\frac{4}{3}-1}=\frac{4}{3} \Rightarrow x^{1/3}=\frac{4}{3}$; $\therefore x=\left(\frac{4}{3}\right)^3=\frac{64}{27}$

Also $x=1$ is an obvious solution.

21. (b) $a^x=b^y=(ab)^{xy}$

$\Rightarrow x\ln a=y\ln b=xy\ln(ab)=k\,(\text{say})$

$\ln a=\frac{k}{x},\ \ln b=\frac{k}{y}$

$\ln(ab)=\frac{k}{xy} \Rightarrow \ln a+\ln b=\frac{k}{xy} \Rightarrow \frac{k}{x}+\frac{k}{y}=\frac{k}{xy}$

$\Rightarrow \frac{1}{x}+\frac{1}{y}=\frac{1}{xy} \Rightarrow \frac{x+y}{xy}=\frac{1}{xy}$; $\therefore$ $x+y=1$.

22. (c) $x=2^{1/3}-2^{-1/3}$

$\Rightarrow x^3=2-2^{-1}-3.2^{1/3}.2^{-1/3}(2^{1/3}-2^{-1/3})$

$\Rightarrow x^3=2-\frac{1}{2}-3x \Rightarrow x^3+3x=\frac{3}{2}$

$\therefore\ 2x^3+6x=3$.

23. (a,b)

$x^{x\sqrt{x}}=(x\sqrt{x})^x \Rightarrow x^{x^{3/2}}=(x^{3/2})^x \Rightarrow x^{x^{3/2}}=x^{(3/2)x}$

$\Rightarrow x^{3/2}=\frac{3}{2}x \Rightarrow x^{1/2}=\frac{3}{2} \Rightarrow x=\frac{9}{4}$

Also $x=1$ is an obvious solution.

24. (b,c) $5^{x-1}+5(0.2)^{x-2}=26 \Rightarrow 5^{x-1}+5.\left(\frac{1}{5}\right)^{x-2}=26$

$\Rightarrow 5^{x-1}+5^{3-x}=26 \Rightarrow 5^{x-1}+25.5^{-(x-1)}-26=0$

$\Rightarrow 5^{2(x-1)}-26.5^{(x-1)}+25=0$

$\Rightarrow 5^{2(x-1)}-5^{x-1}-25.5^{x-1}+25=0$

$\Rightarrow 5^{x-1}(5^{x-1}-1)-25(5^{x-1}-1)=0$

$\Rightarrow (5^{x-1}-25)(5^{x-1}-1)=0 \Rightarrow$

$(5^{x-1}-5^2)(5^{x-1}-5^0)=0$

$\Rightarrow 5^{x-1}=5^2$ or $5^{x-1}=5^0 \Rightarrow x=3,1$.

25. (a, c, d)

$\frac{7}{2^{1/2}+2^{1/4}+1}=\frac{7.(2^{1/4}-1)}{(2^{1/4}-1)[(2^{1/4})^2+2^{1/4}.1+1^2]}$

$=\frac{7.(2^{1/4}-1)}{2^{3/4}-1}=A+B.2^{1/4}+C.2^{1/2}+D.2^{3/4}$

$\Rightarrow 7.2^{1/4}-7=(A-D)2^{3/4}+(2B-A)+(2C-B).2^{1/4}+(2D-C)2^{1/2}$

$\Rightarrow$

$(2B-A+7)+(A-D)2^{3/4}+(2C-B-7)2^{1/4}+(2D-C)2^{1/2}=0$

$\Rightarrow 2B-A+7=A-D=2C-B-7=2D-C=0$

$\Rightarrow A=D=1, B=-3, C=2$.

26. (c) $4.9^{x-1}=3.\sqrt{(2^{2x+1})} \Rightarrow 3^{2x-2-1}=2^{\frac{2x+1}{2}-2}$

$\Rightarrow 3^{2x-3}=2^{\frac{2x-3}{2}} \Rightarrow 2^{\frac{2x-3}{2}}=\left(3^{\frac{2x-3}{2}}\right)^2$

$\Rightarrow 2x-3=0$, $\therefore x=\frac{3}{2}$.

27. (b) $9^x-2^{x+(1/2)}=2^{x+(3/2)}-3^{2x-1}$

$\Rightarrow 3^{2x}+\frac{1}{3}.3^{2x}=2.2^{x+\frac{1}{2}}+2^{x+\frac{1}{2}-2}$

$\Rightarrow 4.3^{2x-1}=3.2^{x+\frac{1}{2}} \Rightarrow 3^{2x-2}=2^{x+\frac{1}{2}-2}$

$\Rightarrow 3^{2x-2}=2^{x-\frac{3}{2}} \Rightarrow \left(\frac{9}{2}\right)^{x-1}=2^{-1/2}$

$\Rightarrow (x-1)\log_{9/2}9/2=-\frac{1}{2}\log_{9/2}2$

$\Rightarrow x-1=-\frac{1}{2}\log_{9/2}2$

$\Rightarrow x=1-\log_{9/2}\sqrt{2}=\log_{9/2}9/2-\log_{9/2}\sqrt{2}$

$\Rightarrow x=\log_{9/2}(9/2\sqrt{2})$; $\therefore x=\log_{9/2}(9/\sqrt{8})$.

28. (b) Let $4+\sqrt{15}=x$, then $4-\sqrt{15}=\frac{1}{x}$

$6+\sqrt{35}=y$, then $6-\sqrt{35}=\frac{1}{y}$

$\therefore$ Given expression

$=\frac{x^{3/2}+\frac{1}{x^{3/2}}}{y^{3/2}-\frac{1}{y^{3/2}}}=\frac{x^3+1}{y^3-1}.\left(\frac{y}{x}\right)^{3/2}$

$=\frac{(4+\sqrt{15})^3+1}{(6+\sqrt{35})^3-1}.\left(\frac{6+\sqrt{35}}{4+\sqrt{15}}\right)^{3/2}$

$=\frac{(4+\sqrt{15}+1)\{(4+\sqrt{15})^2-(4+\sqrt{15})+1\}}{(6+\sqrt{35}-1)\{(6+\sqrt{35})^2+(6+\sqrt{35})+1\}}\times\left(\frac{6+\sqrt{35}}{4+\sqrt{15}}\right)^{3/2}$

$=\frac{5+\sqrt{15}}{5+\sqrt{35}}.\frac{\{31+8\sqrt{15}-4-\sqrt{15}+1\}}{\{71+12\sqrt{35}+6+\sqrt{35}+1\}}.\left(\frac{6+\sqrt{35}}{4+\sqrt{15}}\right)^{3/2}$

$=\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{7}}\times\frac{28+7\sqrt{15}}{78+13\sqrt{35}}\left(\frac{6+\sqrt{35}}{4+\sqrt{15}}\right)^{3/2}$

$=\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{7}}.\frac{7}{13}.\sqrt{\frac{6+\sqrt{35}}{4+\sqrt{15}}}$

$=\frac{7}{13}.\frac{\sqrt{3}+\sqrt{5}}{\sqrt{5}+\sqrt{7}}.\sqrt{\frac{(\sqrt{5}+\sqrt{7})^2}{2}.\frac{2}{(\sqrt{3}+\sqrt{5})^2}}$

$=\frac{7}{13}.\frac{\sqrt{3}+\sqrt{5}}{\sqrt{5}+\sqrt{7}}.\frac{\sqrt{5}+\sqrt{7}}{\sqrt{3}+\sqrt{5}}=\frac{7}{13}$.

29. (b) $y=\frac{1}{x} \Rightarrow xy=1$

$\therefore\ 3x^2+4xy-3y^2=3(x-y)(x+y+4)$

$=3.\left(\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}-\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}\right)\left(\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}+\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}\right)+4$

$=\frac{3[(\sqrt{5}+\sqrt{2})^2-(\sqrt{5}-\sqrt{2})^2]}{(5-2)(5-2)}[(\sqrt{5}+\sqrt{2})^2+(\sqrt{5}-\sqrt{2})^2]+4$

$=\frac{1}{3}.4\sqrt{10}.2(5+2)+4=\frac{56}{3}\sqrt{10}+4=\frac{1}{3}(56\sqrt{10}+12)$

30. (c) $\frac{12}{3+\sqrt{5}-2\sqrt{2}}=\frac{12[(3-2\sqrt{2})-\sqrt{5}]}{[(3-2\sqrt{2})+\sqrt{5}][(3-2\sqrt{2})-\sqrt{5}]}$

$=\frac{12(3-2\sqrt{2}-\sqrt{5})}{(3-2\sqrt{2})^2-5}=\frac{12(3-2\sqrt{2}-\sqrt{5})}{17-12\sqrt{2}-5}$

$=\frac{(3-2\sqrt{2}-\sqrt{5})}{1-\sqrt{2}}=\frac{(\sqrt{5}+2\sqrt{2}-3)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$

$=\frac{\sqrt{10}+4-3\sqrt{2}+\sqrt{5}+2\sqrt{2}-3}{2-1}=1+\sqrt{5}+\sqrt{10}-\sqrt{2}$

31. (a) $\frac{\sqrt{5/2}+\sqrt{7-3\sqrt{5}}}{\sqrt{7/2}+\sqrt{16-5\sqrt{7}}}=\frac{\sqrt{5}+\sqrt{14-6\sqrt{5}}}{\sqrt{7}+\sqrt{32-10\sqrt{7}}}$

$=\frac{\sqrt{5}+(3-\sqrt{5})}{\sqrt{7}+(5-\sqrt{7})}=\frac{3}{5}$, which is rational.

32. (b) $\frac{\sqrt{2}}{\sqrt{2+\sqrt{3}}-\sqrt{2-\sqrt{3}}}=\frac{\sqrt{2}\left(\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}\right)}{(2+\sqrt{3})-(2-\sqrt{3})}$

$=\frac{\sqrt{4+2\sqrt{3}}+\sqrt{4-2\sqrt{3}}}{2\sqrt{3}}=\frac{(\sqrt{3}+1)+(\sqrt{3}-1)}{2\sqrt{3}}=1$.

33. (a) $\frac{4}{1+\sqrt{2}-\sqrt{3}}=\frac{4(1+\sqrt{2}+\sqrt{3})}{(1+\sqrt{2})^2-3}$

$=\frac{4(1+\sqrt{2}+\sqrt{3})}{3+2\sqrt{2}-3}+\frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{3-2}$

$=\sqrt{2}(1+\sqrt{2}+\sqrt{3})=2+\sqrt{2}+\sqrt{6}$.

34. (d) $\frac{3\sqrt{2}}{\sqrt{6}+\sqrt{3}}-\frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$

$=\frac{3\sqrt{2}(\sqrt{6}-\sqrt{3})}{6-3}-\frac{4\sqrt{3}(\sqrt{6}-\sqrt{2})}{6-2}+\frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{3-2}$

$=\sqrt{2}(\sqrt{6}-\sqrt{3})-\sqrt{3}(\sqrt{6}-\sqrt{2})+\sqrt{6}(\sqrt{3}-\sqrt{2})=0$.

35. (b) $(2\sqrt{3}-\sqrt{7})(2\sqrt{3}+\sqrt{7})=12-7=5$ (a rational)

$\therefore$ Rationalising factor = $2\sqrt{3}+\sqrt{7}$

36. (b) $\sqrt{12-\sqrt{68+48\sqrt{2}}}=\sqrt{12-\sqrt{6^2+(4\sqrt{2})^2+2.6.4\sqrt{2}}}$

$=\sqrt{12-\sqrt{(6+4\sqrt{2})^2}}=\sqrt{12-6-4\sqrt{2}}=\sqrt{6-4\sqrt{2}}$

$=\sqrt{2^2+(\sqrt{2})^2-2.2\sqrt{2}}=2-\sqrt{2}$.

37. (d) $\sqrt{50}+\sqrt{48}=5\sqrt{2}+4\sqrt{3}=\sqrt{2}[5+2.\sqrt{2}.\sqrt{3}]$

$=\sqrt{2}(\sqrt{3}+\sqrt{2})^2$;

$\therefore\ \sqrt{\sqrt{50}+\sqrt{48}}=2^{1/4}(\sqrt{3}+\sqrt{2})$.

38. (b) $\sqrt{(3+\sqrt{5})}-\sqrt{2+\sqrt{3}}=\sqrt{\frac{6+2\sqrt{5}}{2}}-\sqrt{\frac{4+2\sqrt{3}}{2}}$

$=\frac{1}{\sqrt{2}}[(1+\sqrt{5})-(1+\sqrt{3})]=\frac{1}{\sqrt{2}}(\sqrt{5}-\sqrt{3})=\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}$

39. (c) $\sqrt{12\sqrt{5}+2\sqrt{55}}=\sqrt{\sqrt{5}(12+2\sqrt{11})}$

$=5^{1/4}\sqrt{11+1+2\sqrt{11}}=5^{1/4}(\sqrt{11}+1)$

40. (d) Let $x=(9\sqrt{3}+11\sqrt{2})^{1/3}$

$\Rightarrow x^3=9\sqrt{3}+11\sqrt{2}$

$=6\sqrt{3}+3\sqrt{3}+9\sqrt{2}+2\sqrt{2}$

$=3\sqrt{3}+2\sqrt{2}+6\sqrt{3}+9\sqrt{2}$

$=3\sqrt{3}+2\sqrt{2}+3(2\sqrt{3}+3\sqrt{2})$

$=3\sqrt{3}+2\sqrt{2}+3\sqrt{2}.\sqrt{3}(\sqrt{2}+\sqrt{3})$

$=(\sqrt{3})^3+(\sqrt{2})^3+3.\sqrt{2}.\sqrt{2}(\sqrt{3}+\sqrt{2})=(\sqrt{3}+\sqrt{2})^3$

So, $x^3=(\sqrt{3}+\sqrt{2})^3$

$x=\sqrt{3}+\sqrt{2}$.

41. (b) $x+\sqrt{x^2+1}=a\Rightarrow\sqrt{x^2+1}=a-x$

$\Rightarrow x^2+1=(a-x)^2=x^2-2ax+a^2$

$\Rightarrow x=\frac{1-a^2}{-2a}=\frac{a^2-1}{2a}=\frac{1}{2}\left(a-\frac{1}{a}\right)$.

42. (b) $x=\sqrt{7}+\sqrt{3},\ xy=4$

$\Rightarrow y=\frac{4}{x}=\frac{4}{\sqrt{7}+\sqrt{3}}=\frac{4(\sqrt{7}-\sqrt{3})}{7-3}=\sqrt{7}-\sqrt{3}$

$x^4+y^4=(x^2+y^2)^2-2x^2y^2$

$=[(x+y)^2-2xy]^2-2(xy)^2=[(2\sqrt{7})^2-8]^2-2.4^2=368$

43. (d) $x=3-\sqrt{5}$

$\sqrt{x}=\sqrt{3-\sqrt{5}}=\frac{1}{\sqrt{2}}.\sqrt{6-2\sqrt{5}}=\frac{1}{\sqrt{2}}(\sqrt{5}-1)$

$3x-2=9-3\sqrt{5}-2=7-3\sqrt{5}=\frac{14-6\sqrt{5}}{2}$

$=\frac{(3-\sqrt{5})^2}{2}$; $\therefore\ \sqrt{3x-2}=\frac{3-\sqrt{5}}{\sqrt{2}}$

$\sqrt{2}+\sqrt{3x-2}=\frac{5-\sqrt{5}}{\sqrt{2}}=\sqrt{5}\left(\frac{\sqrt{5}-1}{\sqrt{2}}\right)$

$\Rightarrow\sqrt{2}+\sqrt{3x-2}=\sqrt{5}.\sqrt{x}$;

$\therefore\ \frac{\sqrt{x}}{\sqrt{2}+\sqrt{3x-2}}=\frac{1}{\sqrt{5}}$.

44. (d) $a-b=\sqrt{21}-\sqrt{20}-\sqrt{18}+\sqrt{17}$

$=(\sqrt{21}-\sqrt{18})-(\sqrt{20}-\sqrt{17})$

$=\frac{(\sqrt{21}-\sqrt{18})(\sqrt{21}+\sqrt{18})}{\sqrt{21}+\sqrt{18}}-\frac{20-17}{\sqrt{20}+\sqrt{17}}$

$=3\left[\frac{1}{\sqrt{21}+\sqrt{18}}-\frac{1}{\sqrt{20}+\sqrt{17}}\right]$

$=\frac{3[\sqrt{20}+\sqrt{17}-\sqrt{21}-\sqrt{18}]}{(\sqrt{21}+\sqrt{18})(\sqrt{20}+\sqrt{17})}$

$=\frac{3[(\sqrt{20}-\sqrt{21})+(\sqrt{17}-\sqrt{18})]}{(\sqrt{21}+\sqrt{18})(\sqrt{20}+\sqrt{17})}$

$=\frac{-3[(\sqrt{21}-\sqrt{20})+(\sqrt{18}-\sqrt{17})]}{(\sqrt{21}+\sqrt{18})(\sqrt{20}+\sqrt{17})}<0$, $\therefore\ a<b$.

45. (b) $\sqrt{x+10}+\sqrt{x-2}=6\Rightarrow\sqrt{x+10}=6-\sqrt{x-2}$

$\Rightarrow\ x+10=36+x-2-12\sqrt{x-2}$

$\Rightarrow\ 2=\sqrt{x-2}\ \Rightarrow\ 4=x-2\ \Rightarrow\ x=-6$

This value satisfies the given equation. $\therefore x=6$.

46. (a) $\dfrac{\sqrt{6+2\sqrt{3}+2\sqrt{2}+2\sqrt{6}}-1}{\sqrt{5+2\sqrt{6}}}$

$=\dfrac{(1+\sqrt{2}+\sqrt{3})-1}{(\sqrt{3}+\sqrt{2})}=\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=1$.

II. Partial fractions

1. (b) $2x+3=a(x-3)+b(x+1)$

Put $x=-1$;

$2(-1)+3=a(-1-3)\Rightarrow 1=-4a\Rightarrow a=\dfrac{-1}{4}$

Now put $x=3$; $2(3)+3=b(3+1)\Rightarrow\ 9=4b$

$\Rightarrow\ b=\dfrac{9}{4}$

Therefore, $a+b=\dfrac{-1}{4}+\dfrac{9}{4}=2$.

2. (a, d) $\dfrac{3x+a}{x^2-3x+2}=\dfrac{A}{(x-2)}-\dfrac{10}{(x-1)}$

$\Rightarrow\ (3x+a)=A(x-1)-10(x-2)$

$\Rightarrow\ 3=A-10$, $a=-A+20$

(On equating coefficients of x and constant term)

$\Rightarrow A=13,\ a=7$.

3. (c) We have, $\dfrac{3x+4}{(x+1)^2(x-1)}=\dfrac{A}{(x-1)}+\dfrac{B}{(x+1)}+\dfrac{C}{(x+1)^2}$

$\Rightarrow\ 3x+4=A(x+1)^2+B(x+1)(x-1)+C(x-1)$

Putting $x=1$, we get $7=A(2)^2\ \Rightarrow\ A=\dfrac{7}{4}$.

4. (c) $\dfrac{3x-1}{(1-x+x^2)(2+x)}=\dfrac{Ax+B}{x^2-x+1}+\dfrac{C}{x+2}$

$\Rightarrow\ (3x-1)=(Ax+B)(x+2)+C(x^2-x+1)$

Comparing the coefficient of like terms, we get $A+C=0$, $2A+B-C=3$, $2B+C=-1$

$\Rightarrow\ A=1$, $B=0$, $C=-1$

$\therefore\ \dfrac{3x-1}{(1-x+x^2)(2+x)}=\dfrac{x}{x^2-x+1}-\dfrac{1}{x+2}$.

5. (a) $\dfrac{(x+1)^2}{x^3+x}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}$

$\Rightarrow\ (x+1)^2=A(x^2+1)+(Bx+C)x$

$\Rightarrow\ A+B=1$, $C=2$, $A=1\ \Rightarrow\ B=0$

Therefore, $\sin^{-1}\left(\dfrac{A}{C}\right)=\sin^{-1}\left(\dfrac{1}{2}\right)=30^o=\dfrac{\pi}{6}$.

6. (a) $\dfrac{x}{(x-1)(x^2+1)^2}=\dfrac{1}{4}\left[\dfrac{1}{(x-1)}-\dfrac{x+1}{x^2+1}\right]+y$

$\Rightarrow\dfrac{x}{(x-1)(x^2+1)^2}=\dfrac{1}{4}\left[\dfrac{1}{(x-1)}-\dfrac{x+1}{x^2+1}\right]+\dfrac{Ax+B}{(x^2+1)^2}$

$\Rightarrow$

$4x=(x^2+1)^2-(x+1)(x-1)(x^2+1)+4(Ax+B)(x-1)$

$\Rightarrow\ 4A+2=0$, $4B-4A=4\ \Rightarrow\ A=\dfrac{-1}{2}$, $B=\dfrac{1}{2}$

$\therefore\ y=\dfrac{Ax+B}{(x^2+1)^2}=\dfrac{1}{2}\dfrac{(1-x)}{(x^2+1)^2}$

7. (a) $\dfrac{5x+6}{(2+x)(1-x)}=\dfrac{\frac{-4}{3}}{2+x}+\dfrac{\frac{11}{3}}{1-x}$

Rewriting the denominators for expressions,

$=\dfrac{\frac{-4}{3}}{2\left(1+\frac{x}{2}\right)}+\dfrac{\frac{11}{3}}{1-x}=\dfrac{-2}{3}\left(1+\dfrac{x}{2}\right)^{-1}+\dfrac{11}{3}(1-x)^{-1}$

$=\dfrac{-2}{3}\left[1-\dfrac{x}{2}+\dfrac{x^2}{4}-\dfrac{x^3}{8}+......+(-1)^n\dfrac{x^n}{2^n}+......\right]$

$+\dfrac{11}{3}[1+x+x^2+.......+x^n+.....]$

The coefficient of x^n in the given expression is

$\dfrac{-2}{3}(-1)^n\dfrac{1}{2^n}+\dfrac{11}{3}$.

8. (c) Putting $x=1$, remainder = 7

9. (b)

$1=A_0(x+1)(x+2)....(x+n)+A_1x(x+2)(x+3)...(x+n)$

$+...+A_rx(x+1)(x+2)....(x+r-1)(x+r+1)(x+r+2)$

$.......(x+n)$

Putting $x=-r$,

$1=A_r(-r)(-r+1)(-r+2),.....(-1).1.2....(-r+n)$

$\Rightarrow\ \ 1=A_r.(-1)^r r!.(n-r)!$; $\therefore\ A_r=\dfrac{(-1)^r}{r!(n-r)!}$.

10. (c) $\dfrac{x+1}{(x-1)(x-2)(x-3)}=\dfrac{A}{x-1}+\dfrac{B}{x-2}+\dfrac{C}{x-3}$

$\Rightarrow$

$x+1=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$

Putting $x=1, A=1$; $x=2$ gives $B=-3$,

For $x=3, C=2$

$\therefore$ Given expression = $\frac{1}{x-1}-\frac{3}{x-2}+\frac{2}{x-3}$.

11. (a,c) $ax^2+bx+c=3(x+2)(2x+3)+2(x-1)(2x+3)$
$-5(x-1)(x+2)$

$\Rightarrow$ $a=6+4-5=5$, $b=21+2-5=18$,

$c=18-6+10=22$.

12. (d) $e^x+2=-3(2e^x-3)+B(e^x-1)$

$\Rightarrow$ $1=-6+B$, $2=9-B$ $\Rightarrow$ $B=7$.

13. (b) $3x+4=A(x-1)-B(x-2)$

$\Rightarrow$ $3=A-B,\ \ 4=-A+2B$

$\Rightarrow$ $A=10,\ \ B=7$

$\therefore$ $(A,B)=(10,7)$.

14. (a) $\frac{f(x)}{x+1}=\phi_1(x)+\frac{6}{x+1},\frac{f(x)}{x-2}=\phi_2(x)+\frac{3}{x-2}$

and $\frac{f(x)}{x+2}=\phi_3(x)+\frac{15}{x+2}$

$\frac{f(x)}{(x+1)(x+2)(x-2)}=\phi(x)+\frac{Q(x)}{(x+1)(x+2)(x-2)}$

We have to find $Q(x)$, which will be a second degree polynomial. When $Q(x)$ is divided by $(x+1)$, we should get the same remainder as being obtained by dividing $f(x)$ by $(x+1)$ *i.e.*, 6. Similarly when $Q(x)$ is divided by $(x-2)$, remainder should be 3 and when $f(x)$ is divided by $x+2$, the remainder should be 15.

$\therefore$ $Q(-1)=6$

$Q(2)=3$, $Q(-2)=15$

Let $Q(x)=\alpha x^2+\beta x+\gamma$, $\therefore$ $\alpha-\beta+\gamma=6$ .....(i)

$4\alpha+2\beta+\gamma=3$ .....(ii); $4\alpha-2\beta+\gamma=15$ .....(iii)

$\Rightarrow \alpha=2,\beta=-3,\gamma=1$; $\therefore Q(x)=2x^2-3x+1$.

15. (c) $1-\cos x=\sin\alpha(1+\cos x)-2\cos x$

$\Rightarrow 1=\sin\alpha,\ -1=-2+\sin\alpha \Rightarrow \alpha=\frac{\pi}{2}$.

16. (d) $x^2=k\,[a^2(x^2+b^2)-b^2(x^2+a^2)]$

$\Rightarrow x^2=k\,[(a^2-b^2)x^2] \Rightarrow 1=k(a^2-b^2)$

$\therefore k=\frac{1}{a^2-b^2}$.

17. (c) 9 = $A(x+2)^2+B(x-1)(x+2)+C(x-1)$

For $x=1,\ 9=9A \Rightarrow A=1$

For $x=-2, 9=-3C \Rightarrow C=-3$

Equating coefficient of $x^2, 0=A+B \Rightarrow B=-A=-1$

$\therefore$ $A-B-C=1-(-1)-(-3)=1+1+3=5$.

18. (b,c) $ax+b=(3x+4)-3$ $\Rightarrow$ $a=3, b=4-3=1$.

19. (a) $\frac{x^2+13x+15}{(2x+3)(x+3)^2}=\frac{A}{2x+3}+\frac{B}{x+3}+\frac{C}{(x+3)^2}$

$\Rightarrow$

$x^2+13x+15=A(x+3)^2+B(2x+3)(x+3)+C(2x+3)$

For $x=-3, C=5$ and for $x=-\frac{3}{2}; A=-1$

Equating coefficient of x^2

$1=A+2B \Rightarrow B=\frac{1-A}{2}=1$

$\therefore$ Expression= $\frac{1}{x+3}-\frac{1}{2x+3}+\frac{5}{(x+3)^2}$.

20. (c)

$\frac{3x^3-8x^2+10}{(x-1)^4}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{(x-1)^4}$

$\Rightarrow$

$3x^3-8x^2+10=A(x-1)^3+B(x-1)^2+C(x-1)+D$

Equating coefficients of different powers of x, $3=A$

$-8=-3A+B \Rightarrow B=1$

0 = $3A-2B+C \Rightarrow C=-7$

$10=-A+B-C+D \Rightarrow D=5$

$\therefore$ Given expression

= $\frac{3}{x-1}+\frac{1}{(x-1)^2}-\frac{7}{(x-1)^3}+\frac{5}{(x-1)^4}$.

21. (b) $A(x^2+1)+x(Bx+C)=(x-1)^2$

For $x=i, -B+Ci=-2i$ $\Rightarrow$ $B=0, C=-2$

Equating coefficient of x^2,

$A+B=1 \Rightarrow A=1-B=1-0=1$;

$\therefore$ $A=1, B=0, C=-2$.

22. (d) $2x=A(x^2+x+1)+(Bx+C)(x-1)$

For $x=1$, $2=3A$ $\Rightarrow$ $A=\frac{2}{3}$

For $x=\omega, 2\omega=A(1+\omega+\omega^2)+B\omega^2+(C-B)\omega-C$

$\Rightarrow$ $2\omega=A.0+B\omega^2+(C-B)\omega-C$

$\omega=\frac{-1+\sqrt{3}i}{2},\ \omega^2=\frac{-1-\sqrt{3}i}{2}$

$\therefore -1+\sqrt{3}i=B\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)+(C-B)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)-C$

$\Rightarrow$ $-1+\sqrt{3}i=\left(-\frac{B}{2}-\frac{C}{2}+\frac{B}{2}-C\right)+\frac{i\sqrt{3}}{2}(C-2B)$

$\Rightarrow$ $-1=-\frac{3}{2}C, \sqrt{3}=\frac{\sqrt{3}}{2}(C-2B)$

$C=\frac{2}{3}, B=\frac{C-2}{2}=-\frac{2}{3}$

$\therefore$ $A=C\neq B$ $\Rightarrow$ $A\neq B\neq C$.

23. (b) $\frac{x^2+1}{(2x-1)(x^2-1)}=\frac{A}{(2x-1)}+\frac{B}{x+1}+\frac{C}{x-1}$

$\Rightarrow x^2+1=A(x^2-1)+B(2x-1)(x-1)+C(x+1)(2x-1)$

For $x=1,\ \ 2=2C\Rightarrow C=1$

For $x=-1$, $2=6B\Rightarrow B=\frac{1}{3}$

For $x=\frac{1}{2}$, $\frac{5}{4}=-\frac{3}{4}A\Rightarrow A=-\frac{5}{3}$

$\therefore$ Given expression

$$= -\frac{5}{3}\frac{1}{(2x-1)}+\frac{1}{3}\frac{1}{x+1}+\frac{1}{x-1}$$

24. (b) $ax-1=x(2+x)-(1-x+x^2)=3x-1$

$\therefore a=3$.

25. (a) $A(x^2+1)+(Bx+C)x=1$

For $x=0, A=1$ and for $x=i$, $-B+Ci=1$

$\Rightarrow B=-1, C=0 \ \Rightarrow\ (A,B,C)=(1,-1,0)$.

26. (d) $\frac{2x}{x^4+x^2+1}=\frac{2x}{(x^2+1)^2-x^2}=\frac{2x}{(x^2-x+1)(x^2+x+1)}$

$$=\frac{1}{x^2-x+1}-\frac{1}{x^2+x+1}.$$

27. (b) $3x^2+5=a(x^2+1)+b$

$\Rightarrow\ a=3, a+b=5\Rightarrow b=2$; $\therefore (a,b)=(3,2)$.

28. (d) $A(x-d)-B(x-c)+C(x-c)(x-d)=(x-a)\,(x-b)$

Equating coefficient of $x^2, C=1$.

29. (d) $\frac{x^2-5}{x^2-3x+2}=\frac{x^2-5}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}+C$

$\Rightarrow\ x^2-5=A(x-2)+B(x-1)+C(x-1)(x-2)$

$\Rightarrow\ C=1, A+B-3C=0, -2A-B+2C=-5$

$\therefore\ A=4, B=-1, C=1$

$\therefore$ Given expression = $1+\frac{4}{x-1}-\frac{1}{x-2}$

30. (a) $\frac{6x^4+5x^3+x^2+5x+2}{6x^2+5x+1}=\frac{x^2(6x^2+5x+1)+(5x+2)}{(6x^2+5x+1)}$

$$= x^2+\frac{5x+2}{(2x+1)(3x+1)}=x^2+\frac{(2x+1)+(3x+1)}{(2x+1)(3x+1)}$$

$$= x^2+\frac{1}{2x+1}+\frac{1}{3x+1}.$$

31. (a,d) $\sin^2 x+1=A(\sin x-1)+B(2\sin x-3)$
$+C(\sin x-1)(2\sin x-3)$

$\Rightarrow\ 1=2C\Rightarrow C=\frac{1}{2}$

$0=A+2B-5C, 1=-A-3B+3C$.

$\therefore A=\frac{13}{2}$, $B=-2$, $C=\frac{1}{2}$, $A+B+C=5$.

32. (b) $\frac{3x}{(x-2)(x+1)}=-\frac{3x}{2}(1+x)^{-1}\left(1-\frac{x}{2}\right)^{-1}=-\frac{3}{2}x$

$$(1-x+x^2-x^3+x^4-..)\left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+...\right)$$

Coefficient of $x^4=-\frac{3}{2}\left[-1.1+1.\frac{1}{2}-1.\frac{1}{4}+1.\frac{1}{8}\right]$

$$=-\frac{3}{2}\left[-1+\frac{1}{2}-\frac{1}{4}+\frac{1}{8}\right]=\frac{15}{16}.$$

33. (d) $\frac{x^2+1}{(x^2+4)(x-2)}=\frac{Ax+B}{x^2+4}+\frac{C}{x-2}$

$\Rightarrow\ x^2+1=(Ax+B)(x-2)+C(x^2+4)\Rightarrow\ 1=A+C$

$-2A+B=0$, $1=-2B+4C$

$\therefore\ A=\frac{3}{8}, B=\frac{3}{4}, C=\frac{5}{8}$

$$\therefore\ \frac{x^2+1}{(x^2+4)\,(x-2)}=\frac{\frac{3}{8}x+\frac{3}{4}}{x^2+4}+\frac{\frac{5}{8}}{x-2}$$

$$=\frac{1}{4}\left(\frac{3}{8}x+\frac{3}{4}\right)\left(1+\frac{x^2}{4}\right)^{-1}+\frac{5}{8}\frac{1}{(-2)}\left(1-\frac{x}{2}\right)^{-1}$$

$$=\frac{1}{4}\left(\frac{3}{8}x+\frac{3}{4}\right)\left(1-\frac{x^2}{4}+\left(\frac{x^2}{4}\right)^2-\left(\frac{x^2}{4}\right)^3+....\right)$$

$$-\frac{5}{16}\left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+.....\right)$$

Coefficient of x^5 = $\frac{3}{32}.\frac{1}{4^2}+\frac{3}{16}\times 0-\frac{5}{16}\left(\frac{1}{2}\right)^5$

$$=\frac{3}{2^9}-\frac{5}{2^9}=-\frac{1}{2^8}=-\frac{1}{256}.$$

III. Critical Thinking Questions

1. (c) $y=\log_{17}25=2\log_{17}5$; $\therefore\ \frac{1}{y}=\frac{1}{2}\log_5 17$

$\frac{1}{x}=\log_5 3=\frac{1}{2}\log_5 9$. Clearly, $\frac{1}{y}>\frac{1}{x}$; $\therefore x>y$

2. (a) $\log_{0.3}(x-1)<\log_{(0.3)^2}(x-1)=\frac{1}{2}\log_{0.3}(x-1)$

$\therefore\ \frac{1}{2}\log_{0.3}(x-1)<0$

or $\log_{0.3}(x-1)<0=\log 1$ or $(x-1)>1$ or $x>2$

As base is less than 1, therefore the inequality is reversed, now $x>2\Rightarrow x$ lies in $(2,\infty)$.

3. (b) $\log x:\log y:\log z=y-z:z-x:x-y$

$\Rightarrow\ \frac{\log x}{y-z}=\frac{\log y}{z-x}=\frac{\log z}{x-y}=k\,(\text{say})$

$\Rightarrow\ \log x=k(y-z), \log y=k(z-x), \log z=k(x-y)$

$\therefore \log x+\log y+\log z=0\ \Rightarrow\ \log(xyz)=0$

$\Rightarrow xyz=1$.

$x\log x+y\log y+z\log z$

$=x.k.(y-z)+y.k.(z-x)+z.k(x-y)=0$

$\Rightarrow \log(x^x.y^y.z^z)=\log 1$

$\therefore x^x y^y z^z=1$.

4. (d) $\log_2(x+5)=6-x\ \Rightarrow\ x+5=2^{6-x}\ \Rightarrow\ x+5=64.2^{-x}$

Let $y=x+5$, $y=64.2^{-x}$ will intersect at one point.

Number of solutions = 1.

5. (b) $20^{1/3} < 3 < 20^{1/2} \Rightarrow \frac{1}{3} < \log_{20} 3 < \frac{1}{2}$

$\therefore \log_{20} 3 \in \left(\frac{1}{3}, \frac{1}{2}\right)$.

6. (a,b,d) $\frac{1}{2} \le \log_{0.1} x \le 2$

$\frac{1}{2} \le \log_{0.1} x \Rightarrow \log_{0.1}(0.1)^{1/2} \le \log_{0.1} x$

$\Rightarrow (0.1)^{1/2} \ge x \Rightarrow x \le \frac{1}{\sqrt{10}}$

$\log_{0.1} x \le 2 \Rightarrow \log_{0.1} x \le \log_{0.1}(0.1)^2$

$x \ge (0.1)^2 \Rightarrow x \ge \frac{1}{100}$, $\therefore \frac{1}{100} \le x \le \frac{1}{\sqrt{10}}$.

Hence, $x_{\max} = \frac{1}{\sqrt{10}}, x_{\min.} = \frac{1}{100}$.

7. (a, b) $4^{(x^2+2)} - 9.2^{(x^2+2)} + 8 = 0$

$\Rightarrow \left(2^{(x^2+2)}\right)^2 - 9.2^{(x^2+2)} + 8 = 0$

Put $2^{(x^2+2)^2} = y$. Then $y^2 - 9y + 8 = 0$, which gives $y = 8, y = 1$.

when $y = 8 \Rightarrow 2^{x^2+2} = 8 \Rightarrow 2^{x^2+2} = 2^3 \Rightarrow x^2 + 2 = 3$

$\Rightarrow x^2 = 1 \Rightarrow x = 1, -1$.

when $y = 1 \Rightarrow 2^{x^2+2} = 1 \Rightarrow 2^{x^2+2} = 2^o$

$\Rightarrow x^2 + 2 = 0 \Rightarrow x^2 = -2$, which is not possible.

8. (b) Let $10 - \sqrt{24} - \sqrt{40} + \sqrt{60} = (\sqrt{a} - \sqrt{b} + \sqrt{c})^2$

$= a + b + c - 2\sqrt{ab} - 2\sqrt{bc} + 2\sqrt{ca}$

$a, b, c > 0$. Then $a + b + c = 10$,

$ab = 6$, $bc = 10$, $ca = 15$

$a^2b^2c^2 = 900 \Rightarrow abc = 30$ $(\ne \pm 30)$.

So, $a = 3$, $b = 2$, $c = 5$

Therefore,

$\sqrt{(10 - \sqrt{24} - \sqrt{40} + \sqrt{60})} = \pm(\sqrt{3} + \sqrt{5} - \sqrt{2})$

9. (a) $\sum \frac{1}{1 + x^{a-b} + x^{a-c}} = \sum \frac{x^{b+c}}{x^{b+c} + x^{c+a} + x^{a+b}}$

$= \frac{1}{x^{b+c} + x^{c+a} + x^{a+b}} \sum x^{b+c}$

$= \frac{1}{x^{b+c} + x^{c+a} + x^{a+b}}(x^{b+c} + x^{c+a} + x^{a+b}) = 1$.

10. (a) $\frac{1}{\sqrt{11 - 2\sqrt{30}}} - \frac{3}{\sqrt{7 - 2\sqrt{10}}} - \frac{4}{2\sqrt{2 + \sqrt{3}}}$

$= \frac{\sqrt{11 + 2\sqrt{30}}}{\sqrt{1}} - \frac{3(7 + 2\sqrt{10})}{\sqrt{9}} - \frac{2\sqrt{2 - \sqrt{3}}}{\sqrt{1}}$

$= \sqrt{11 + 2\sqrt{30}} - \sqrt{7 + 2\sqrt{10}} - 2\sqrt{2 - \sqrt{3}}$

$= (\sqrt{6} + \sqrt{5}) - (\sqrt{5} + \sqrt{2}) - \sqrt{8 - 4\sqrt{3}}$

$= (\sqrt{6} - \sqrt{2}) - (\sqrt{6} - \sqrt{2}) = 0$.

11. (b) $134 + \sqrt{6292} = [11^2 + (\sqrt{13})^2] + 2.11.\sqrt{13} = (11 + \sqrt{13})^2$

$\therefore \sqrt{134 + \sqrt{6292}} = 11 + \sqrt{13}$.

12. (a) $y = \frac{1}{x} = \frac{1}{2 + \sqrt{3}} = \frac{2 - \sqrt{3}}{4 - 3} = 2 - \sqrt{3}$

$\therefore \frac{x}{\sqrt{2} + \sqrt{x}} + \frac{y}{\sqrt{2} - \sqrt{y}} = \frac{x(\sqrt{2} - \sqrt{x})}{2 - x} + \frac{y(\sqrt{2} + \sqrt{y})}{2 - y}$

$\Rightarrow$

$\frac{x(\sqrt{x} - \sqrt{2})}{x - 2} + \frac{y(\sqrt{y} + \sqrt{2})}{2 - y} = \frac{x(\sqrt{x} - \sqrt{2})}{\sqrt{3}} + \frac{y(\sqrt{y} + \sqrt{2})}{\sqrt{3}}$

$= \frac{1}{\sqrt{3}}[x\sqrt{x} + y\sqrt{y} + \sqrt{2}(y - x)]$

$= \frac{1}{\sqrt{3}}[(2 + \sqrt{3})^{3/2} + (2 - \sqrt{3})^{3/2} + \sqrt{2}(-2\sqrt{3})]$

$=$

$\frac{1}{\sqrt{3}}\left[\frac{1}{2^{3/2}}(4 + 2\sqrt{3})^{3/2} + \frac{1}{2^{3/2}}(4 - 2\sqrt{3})^{3/2} - 2\sqrt{6}\right]$

$= \frac{1}{\sqrt{3}}\left[\frac{1}{2\sqrt{2}}\{(\sqrt{3} + 1]^3 + (\sqrt{3} - 1)^3\} - 2\sqrt{6}\right]$

$= \frac{1}{\sqrt{3}}\left[\frac{1}{2\sqrt{2}}\{2.3\sqrt{3} + 6.\sqrt{3}\} - 2\sqrt{6}\right]$

$= \frac{1}{\sqrt{3}}(3\sqrt{6} - 2\sqrt{6}) = \sqrt{2}$.

13. (c) Put the repeated factor

$(x - 1) = y \Rightarrow x = y + 1$

$\therefore \frac{x^2}{(x - 1)^3(x - 2)} = \frac{(1 + y)^2}{y^3(y - 1)} = \frac{1 + 2y + y^2}{y^3(-1 + y)}$

Dividing the numerator, $(1 + 2y + y^2)$ by $(-1 + y)$ till y^3 appears as factor,

We get

$\frac{1 + 2y + y^2}{-1 + y} = (-1 - 3y - 4y^2) + \frac{4y^3}{-1 + y}$

Given expression $= \frac{-1}{y^3} - \frac{3}{y^2} - \frac{4}{y} + \frac{4}{-1 + y}$

$= \frac{-1}{(x - 1)^3} + \frac{-3}{(x - 1)^2} + \frac{-4}{(x - 1)} + \frac{4}{(x - 2)}$.

14. (a)

$$
\begin{array}{r|l}
x^2 - 5x + 6 & x^3 - 6x^2 + 10x - 2 \;(\; x - 1 \\
 & x^3 - 5x^2 + 6x \\
 & \underline{-\quad +\qquad -} \\
 & -x^2 + 4x - 2 \\
 & -x^2 + 5x - 6 \\
 & \underline{+\quad -\quad +} \\
 & -x + 4
\end{array}
$$

$\therefore f(x) = x - 1$.

15. (a) $\frac{x^4 + 24x^2 + 28}{(x^2 + 1)^3} = \frac{A_1x + B_1}{x^2 + 1} + \frac{A_2x + B_2}{(x^2 + 1)^2} + \frac{A_3x + B_3}{(x^2 + 1)^3}$

$\Rightarrow x^4 + 24x^2 + 28 = (A_1x + B_1)(x^2 + 1)^2$

$+ (A_2x + B_2)(x^2 + 1) + (A_3x + B_3)$

Putting $x = i, 5 = A_3i + B_3 \Rightarrow A_3 = 0, B_3 = 5$

Equating different powers of x,

$0 = A_1, B_1 = 1, 2A_1 + A_2 = 0 \Rightarrow A_2 = 0$

$2B_1 + B_2 = 24 \Rightarrow B_2 = 22$.

$\therefore$ P.fraction= $\frac{1}{x^2+1} + \frac{22}{(x^2+1)^2} + \frac{5}{(x^2+1)^3}$.

16. (d) $\log_e(1+x) - x = \log_e(1+x) - \log_e e^x = \log_e \frac{1+x}{e^x}$

$= \ln \frac{1+x}{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...} < 0, \text{as } 1+x < 1+x+\frac{x^2}{2!}+....+$

$\therefore \log_e(1+x) < x$, for $x > 0$.

$\frac{x}{1+x} - \log(1+x) = 1 - \frac{1}{1+x} - \log(1+x)$

$= 1 - \left[\frac{1}{1+x} + \log(1+x)\right] < 0$, for $x > 0$

$\therefore \frac{x}{1+x} < \log(1+x)$, $\therefore$ (b) is true

$e^x - (1+x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... - (1+x)$

$= \frac{x^2}{2!} + \frac{x^3}{3!} + ... > 0$, for $x > 0$

$\therefore e^x > 1 + x$, for $x > 0$; $\therefore$ (c) is true

$e^x - (1-x) = 1 + x + \frac{x^2}{2!} + ... - 1 + x$

$= 2x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... > 0$, for $x > 0$

$\therefore e^x > 1 - x$, for $x > 0$

Thus, $e^x < (1-x)$, for $x > 0$ is not true.

Logarithms, Indices and Surds, Partial Fractions

SET Self Evaluation Test - 2

1. If $\log_{10} 3 = 0.477$, the number of digits in 3^{40} is **[IIT 1992]**

(a) 18 (b) 19

(c) 20 (d) 21

2. $\sum_{r=1}^{89} \log_3(\tan r^o) =$

(a) 3 (b) 1

(c) 2 (d) 0

3. If $n = 1983!$, then the value of expression $\frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \frac{1}{\log_4 n} + ... + \frac{1}{\log_{1983} n}$ is equal to **[DCE 2005]**

(a) – 1 (b) 0

(c) 1 (d) 2

4. If $x_n > x_{n-1} > ... > x_2 > x_1 > 1$ then the value of $\log_{x_1} \log_{x_2} \log_{x_3} ..... \log_{x_n} x_n^{x_{n-1}^{\cdot^{\cdot^{x_1}}}}$ is equal to

(a) 0 (b) 1

(c) 2 (d) None of these

5. Solution set of inequality $\log_{10}(x^2 - 2x - 2) \le 0$ is

(a) $[-1, 1-\sqrt{3}]$

(b) $[1+\sqrt{3}, 3]$

(c) $[-1, 1-\sqrt{3}) \cup (1+\sqrt{3}, 3]$

(d) None of these

6. If $x = \sqrt[3]{(\sqrt{2}+1)} - \sqrt[3]{(\sqrt{2}-1)}$, then $x^3 + 3x =$

(a) 2 (b) 6

(c) $6x$ (d) None of these

7. $\sqrt[3]{(61 - 46\sqrt{5})} =$

(a) $1 - 2\sqrt{5}$ (b) $1 - \sqrt{5}$

(c) $2 - \sqrt{5}$ (d) None of these

8. If $a > 0$, then $\sqrt{a + \sqrt{a + \sqrt{a + ...\infty}}}$ is

(a) $\frac{1}{2}\sqrt{(4a-1)}$ (b) $\frac{1}{2}[1 + \sqrt{(4a+1)}]$

(c) $\frac{1}{2}[1 - \sqrt{(4a-1)}]$ (d) $\frac{1}{2}[1 \pm \sqrt{(4a+1)}]$

9. If $\frac{4 + 3\sqrt{3}}{\sqrt{(7 + 4\sqrt{3})}} = a + \sqrt{b}$, then $(a, b) =$

(a) (12,1) (b) (1, 12)

(c) (–1, 12) (d) (–12, 1)

10. If $3^x - 3^{x-1} = 6$, then x^x is equal to **[UPSEAT 2003]**

(a) 2 (b) 4

(c) 9 (d) None of these

11. $\sqrt{[x + 2\sqrt{(x-1)}]} + \sqrt{[x - 2\sqrt{(x-1)}]} =$

(a) 2, if $1 \le x \le 2$ (b) 2, if $x > 2$

(c) $2\sqrt{(x-1)}$, if $1 \le x \le 2$ (d) $2\sqrt{(x-1)}$, if $x > 2$

12. The remainder obtained when the polynomial $x^{64} + x^{27} + 1$ is divided by $(x+1)$ is **[EAMCET 1992]**

(a) 1 (b) – 1

(c) 2 (d) – 2

13. If $\frac{x^3}{(2x-1)(x+2)(x-3)} = p + \frac{q}{2x-1} + \frac{r}{x+2} + \frac{s}{x-3}$, then

(a) $p = 1$ (b) $p = 2$

(c) $p = \frac{1}{2}$ (d) $6q - 3r + 2s = 3$

14. If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$ then $x \ne 1$ lies in **[Kerala (Engg.) 2005]**

(a) (1, 2) (b) (0,1)

(c) (1, ∞) (d) (2, ∞)

(e) (0.09, 0.3)

Answers and Solutions

(SET - 2)

1. (c) Let $y = 3^{40}$

Taking log both the sides, $\log y = \log 3^{40}$

$\Rightarrow \log y = 40 \log 3 \Rightarrow \log y = 19.08$

$\therefore$ Number of digits in $y = 19 + 1 = 20$.

2. (d) $\sum_{r=1}^{39} \log_3(\tan r^o) = \log_3(\tan 45^o) = \log_3 1 = 0$.

3. (c) $\frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \frac{1}{\log_4 n} + ... + \frac{1}{\log_{1983} n}$

$= \log_n 2 + \log_n 3 + \log_n 4 + ... + \log_n 1983$

$= \log_n(2.3.4.... 1983) = \log_n(1983 !) = \log_n n = 1$.

4. (b) $\log_{x_1} \log_{x_2} \log_{x_3} .... \log_{x_n} x_n^{x_{n-1}^{\cdot^{\cdot^{x_1}}}}$

$= \log_{x_1} \log_{x_2} \log_{x_3} .... \log_{x_{n-1}} x_{n-1}^{x_{n-2}^{\cdot^{\cdot^{x_1}}}}$

$= \log_{x_1} \log_{x_2} \log_{x_3} ...... \log_{x_{n-2}} x_{n-2}^{x_{n-3}^{\cdot^{\cdot^{x_1}}}}$

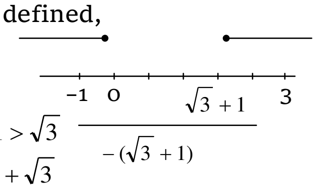
$= \log_{x_1} \log_{x_2} x_2^{x_1} = \log_{x_1} x_1 = 1$.

5. (c) $\log_{10}(x^2 - 2x - 2) \le 0$ .....(i)

For logarithm to be defined,

$x^2 - 2x - 2 > 0$

$\Rightarrow (x-1)^2 > 3$



$\Rightarrow x - 1 < -\sqrt{3}$ or $x - 1 > \sqrt{3}$

$\Rightarrow x < 1 - \sqrt{3}$ or $x > 1 + \sqrt{3}$

i.e., $x < -(\sqrt{3} - 1)$ or $x > (\sqrt{3} + 1)$

Now from (i), $x^2 - 2x - 2 \le 1$

$\Rightarrow x^2 - 2x - 3 \le 0$

$\Rightarrow (x-3)(x+1) \le 0 \Rightarrow -1 \le x \le 3$

$\therefore x \in [-1, -(\sqrt{3} - 1)[\cup] \sqrt{3} + 1, 3]$.

i.e., $x \in [-1, 1 - \sqrt{3}) \cup (1 + \sqrt{3}, 3)$.

6. (a) $x = (\sqrt{2} + 1)^{1/3} - (\sqrt{2} - 1)^{1/3}$

$x^3 = (\sqrt{2} + 1) - (\sqrt{2} - 1) - 3(\sqrt{2} + 1)^{1/3} (\sqrt{2} - 1)^{1/3}$

$\left[\sqrt[3]{(\sqrt{2} + 1)} - \sqrt[3]{\sqrt{2} - 1}\right]$

$x^3 = 2 - 3(2 - 1)^{1/3} x \Rightarrow x^3 + 3x = 2$.

7. (a) $\sqrt[3]{61 - 46\sqrt{5}} = a - \sqrt{b}$

$\Rightarrow 61 - 46\sqrt{5} = (a - \sqrt{b})^3 = a^3 + 3ab - (3a^2 + b)\sqrt{b}$

$\Rightarrow 61 = a^3 + 3ab, 46\sqrt{5} = (3a^2 + b)\sqrt{b}$

$\Rightarrow 61 = (a^2 + 3b)a$, $23\sqrt{20} = (3a^2 + b)\sqrt{b}$

So, $a = 1,\ b = 20$.

Therefore, $\sqrt[3]{61 - 46\sqrt{5}} = 1 - \sqrt{20} = 1 - 2\sqrt{5}$.

8. (b) $x = \sqrt{a + \sqrt{a + \sqrt{a + ...\infty}}}$

$\Rightarrow x = \sqrt{a + x} \Rightarrow x^2 - x - a = 0 \Rightarrow x = \frac{1 \pm \sqrt{1 + 4a}}{2}$

As $a > 0$, $x > 0$; $\therefore$ +*ve* sign should be considered.

$\therefore x = \frac{1}{2}(1 + \sqrt{4a + 1})$.

9. (c) $\frac{4 + 3\sqrt{3}}{\sqrt{7 + 4\sqrt{3}}} = a + \sqrt{b} \Rightarrow \frac{4 + 3\sqrt{3}}{2 + \sqrt{3}} = a + \sqrt{b}$

$\Rightarrow \frac{(4 + 3\sqrt{3})(2 - \sqrt{3})}{(2 + \sqrt{3})(2 - \sqrt{3})} = a + \sqrt{b} \Rightarrow -1 + 2\sqrt{3} = a + \sqrt{b}$

$\Rightarrow -1 + \sqrt{12} = a + \sqrt{b}$; $\therefore (a, b) = (-1, 12)$.

10. (b) We have $3^x - 3^{x-1} = 6 \Rightarrow 3^x - \frac{3^x}{3} = 6$

Let $3^x = t$, then given equation can be written as

$t - \frac{t}{3} = 6 \Rightarrow 3t - t = 18 \Rightarrow 2t = 18 \Rightarrow t = 9$

$\therefore 3^x = 3^2 \Rightarrow x = 2$. Hence, $x^x = 2^2 = 4$.

11. (a,d) $x - 1 \ge 0 \Rightarrow x \ge 1$

Next, $x \pm 2\sqrt{x - 1} \ge 0$

$\Rightarrow x^2 \ge 4(x-1) \Rightarrow x^2 - 4x + 4 \ge 0 \Rightarrow (x-2)^2 \ge 0$,

which is true $\forall\ x$, $\therefore x \ge 1$.

For $1 \le x \le 2$, $\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}$

$= \sqrt{1 + (x-1) + 2\sqrt{x-1}} + \sqrt{1 + (x-1) - 2\sqrt{x-1}}$

$= (1 + \sqrt{x-1}) + (1 - \sqrt{x-1}) = 2$

For $x > 2$, $\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}$

$= (1 + \sqrt{x-1}) + (\sqrt{x-1} - 1) = 2\sqrt{x-1}$.

12. (a) Remainder of $x^{64} + x^{27} + 1$, when divided by $x + 1$ is $(-1)^{64} + (-1)^{27} + 1 = 1 - 1 + 1 = 1$.

13. (c,d) $x^3 = p(2x-1)(x+2)(x-3) + q(x+2)(x-3)$

$+ r(2x-1)(x-3) + s(2x-1)(x+2)$

Equating coefficient of x^3 ; $1 = 2p \Rightarrow p = \frac{1}{2}$

Equating coefficient of x^0 *i.e.,* constant term,

$0 = 6p - 6q + 3r - 2s \Rightarrow 6q - 3r + 2s = 3$.

14. (c) $\log_{0.3}(x-1) < \log_{0.09}(x-1)$

$1 < \frac{\log_{0.09}(x-1)}{\log_{0.3}(x-1)} \Rightarrow 1 < \log_{0.3}(0.09)$

$\Rightarrow 1 < \log_{0.3}(0.3)^2 \Rightarrow 1 < 2$

Which of true therefore it is true for every positive value of 2. $\therefore x \in (1, \infty)$.

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