

12. Higher Order Derivatives

Exercise 12.1

26. Question

If $y = \tan^{-1} x$, show that $(1 + x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = 0$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d(\tan^{-1} x)}{dx} = \frac{1}{1 + x^2}$$

$$(iii) \frac{d}{dx} x^n = nx^{n-1}$$

$$(iv) \text{ chain rule } \frac{df}{dx} = \frac{d(\text{wou})}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Given: -

$$Y = \tan^{-1} x$$

Differentiating w.r.t x

$$\frac{dy}{dx} = \frac{d(\tan^{-1} x)}{dx}$$

Using formula(ii)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1 + x^2}$$

$$\Rightarrow (1 + x^2) \frac{dy}{dx} = 1$$

Again Differentiating w.r.t x

Using formula(iii)

$$(1 + x^2) \frac{dy}{dx} + 2x \frac{dy}{dx} = 0$$

Hence proved.

27. Question

If $y = \log(x + \sqrt{x^2 + 1})$, show that $(1 + x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 2$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d(\log x)}{dx} = \frac{1}{x}$$

$$(iii) \frac{d}{dx} x^n = nx^{n-1}$$

$$(iv) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Given: -

$$y = \left[\log(x + \sqrt{1 + x^2}) \right]^2$$

Differentiating w.r.t x

$$\frac{dy}{dx} = \frac{d[\log(x + \sqrt{1 + x^2})]^2}{dx}$$

Using formula(ii)

$$\Rightarrow \frac{dy}{dx} = 2 \log(x + \sqrt{1 + x^2}) \cdot \frac{1}{(x + \sqrt{1 + x^2})} \cdot \left(1 + \frac{2x}{2\sqrt{1 + x^2}} \right)$$

Using formula(i)

$$\Rightarrow y_1 = \frac{2 \log(x + \sqrt{1 + x^2})}{x + \sqrt{1 + x^2}} \cdot \frac{x + \sqrt{1 + x^2}}{\sqrt{1 + x^2}}$$

$$\Rightarrow y_1 = \frac{2 \log(x + \sqrt{1 + x^2})}{\sqrt{1 + x^2}}$$

Squaring both sides

$$(y_1)^2 = \frac{4}{1 + x^2} [\log(x + \sqrt{1 + x^2})]^2$$

Differentiating w.r.t x

$$\Rightarrow (1 + x^2)y_2y_1 + 2x(y_1)^2 = 4y_1$$

Using formula(iii)

$$\Rightarrow (1 + x^2)y_2 + xy_1 = 2$$

Hence proved

28. Question

If $y = (\tan^{-1} x)^2$, then prove that $(1 + x^2)y_2 + 2x(1 + x^2)y_1 = 2$

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d(\tan^{-1} x)}{dx} = \frac{1}{1 + x^2}$$

$$(iii) \frac{d}{dx} x^n = nx^{n-1}$$

Given: -

$$Y = (\tan^{-1} x)^2$$

Then

$$\frac{dy}{dx} = \frac{d(\tan^{-1} x)^2}{dx}$$

Using formula (ii)&(i)

$$y_1 = 2 \tan^{-1} x \frac{dy}{dx} (\tan^{-1} x)$$

$$\Rightarrow y_1 = 2 \tan^{-1} x \cdot \frac{1}{1+x^2}$$

Again differentiating with respect to x on both the sides, we obtain

$$(1+x^2)y_2 + 2xy_1 = 2 \left(\frac{1}{1+x^2} \right) \text{ using formula (i) \& (iii)}$$

$$\Rightarrow (1+x^2)^2 y_2 + 2x(1+x^2)y_1 = 2$$

Hence proved.

29. Question

If $y = \cot x$ show that $\frac{d^2 y}{dx^2} + 2y \frac{dy}{dx} = 0$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2 y}{dx^2} = y_2$$

$$(ii) \frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$$

$$(iii) \frac{d}{dx} x^n = nx^{n-1}$$

$$(iv) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Given: -

$$Y = \cot x$$

Differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{d(\cot x)}{dx}$$

Using formula (ii)

$$\Rightarrow \frac{dy}{dx} = -\operatorname{cosec}^2 x$$

Differentiating w.r.t x

$$\frac{d^2 y}{dx^2} = -[2 \operatorname{cosec} x (-\operatorname{cosec} x \cot x)]$$

Using formula (iii)

$$\Rightarrow \frac{d^2 y}{dx^2} = 2 \operatorname{cosec}^2 x \cot x$$

$$\Rightarrow \frac{d^2 y}{dx^2} = -2 \frac{dy}{dx} \cdot y$$

$$\Rightarrow \frac{d^2y}{dx^2} + 2y \frac{dy}{dx} = 0$$

Hence proved.

30. Question

Find $\frac{d^2y}{dx^2}$, where $y = \log \left(\frac{x^2}{e^2} \right)$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d(e^{ax})}{dx} = ae^{ax}$$

$$(iii) \frac{d}{dx} x^n = nx^{n-1}$$

Given: -

$$y = \log \left(\frac{x^2}{e^2} \right)$$

Differentiating w.r.t x

$$\frac{dy}{dx} = \frac{1}{\frac{x^2}{e^2}} \cdot \frac{1}{e^2} 2x = \frac{2}{x}$$

Again Differentiating w.r.t x

$$\frac{d^2y}{dx^2} = 2 \left(-\frac{1}{x^2} \right) = -\frac{2}{x^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2}{x^2}$$

31. Question

If $y = e^x(\sin x + \cos x)$ prove that $\frac{d^2y}{dx^2} - 1 \frac{dy}{dx} + 2y = 0$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d(e^{ax})}{dx} = ae^{ax}$$

$$(iii) \frac{d}{dx} x^n = nx^{n-1}$$

$$(iv) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Given: -

$$y = ae^{2x} + be^{-2x}$$

Differentiating w.r.t x

$$\frac{dy}{dx} = 2ae^{2x} + be^{(-x)}(-1)$$

$$\Rightarrow \frac{dy}{dx} = 2ae^{2x} - be^{-x}$$

Differentiating w.r.t x

$$\frac{d^2y}{dx^2} = 2ae^{2x}(2) - be^{-x}(-1)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x}$$

Adding and subtracting be^{-x} on RHS

$$\frac{d^2y}{dx^2} = 4ae^{2x} + 2be^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2(ae^{2x} + be^{-x}) + 2ae^{2x} - be^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2y + \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

32. Question

If $y = e^x (\sin x + \cos x)$ Prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d(e^{ax})}{dx} = ae^{ax}$$

$$(iii) \frac{d}{dx} x^n = nx^{n-1}$$

$$(iv) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Given: -

$$y = e^x(\sin x + \cos x)$$

differentiating w.r.t x

$$\frac{dy}{dx} = e^x(\cos x - \sin x) + (\sin x + \cos x)e^x$$

$$\Rightarrow \frac{dy}{dx} = y + e^x(\cos x - \sin x)$$

Differentiating w.r.t x

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-\sin x - \cos x) + (\cos x - \sin x)e^x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} - y + (\cos x - \sin x)e^x$$

Adding and subtracting y on RHS

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} - y + (\cos x - \sin x)e^x + y - y$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0$$

Hence proved

33. Question

If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$(iii) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Given: - $y = \cos^{-1} x$

Then,

$$\frac{dy}{dx} = \frac{d(\cos^{-1} x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d[-(\sqrt{1-x^2})]^{-1}}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1 \cdot (1-x^2)^{-\frac{3}{2}} d(1-x^2)}{2} \cdot \frac{d(1-x^2)}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2\sqrt{(1-x^2)^3}} \cdot (-2x)$$

$$\frac{d^2y}{dx^2} = \frac{-x}{\sqrt{(1-x^2)^3}} \dots \dots (i)$$

$$y = \cos^{-1} x$$

$$\Rightarrow x = \cos y$$

Putting $x = \cos y$ in equation(i), we obtain

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{(1 - \cos^2 y)^3}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\cos y}{\sqrt{(\sin^2 y)^3}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\cos y}{\sin^3 y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\cos y}{\sin y} \cdot \frac{1}{\sin^2 y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\cot y \cdot \operatorname{cosec}^2 y$$

34. Question

If $y = e^{a \cos^{-1} x}$, prove that $(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d(\log x)}{dx} = \frac{1}{x}$$

$$(ii) \frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1 - x^2}}$$

$$(iii) \frac{d}{dx} x^n = nx^{n-1}$$

$$(iv) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

$$(v) \text{ logarithms differentiation } \frac{dy}{dx} = y \left[\frac{v(x)}{u(x)} \cdot u'(x) + v'(x) \cdot \log[u(x)] \right]$$

Given: -

$$y = e^{a \cos^{-1} x}$$

Taking logarithm on both sides we obtain

$$\frac{1}{y} \frac{dy}{dx} = a \frac{-1}{\sqrt{1 - x^2}}$$

$$\frac{dy}{dx} = \frac{-ay}{\sqrt{1 - x^2}}$$

By squaring both sides, we obtain

$$\left(\frac{dy}{dx} \right)^2 = \frac{a^2 y^2}{1 - x^2}$$

$$\Rightarrow (1 - x^2) \cdot \left(\frac{dy}{dx} \right)^2 = a^2 y^2$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Again differentiating both sides with respect to x, we obtain

$$\left(\frac{dy}{dx}\right)^2 \cdot \frac{d(1-x^2)}{dx} + (1-x^2) \cdot \frac{d\left[\left(\frac{dy}{dx}\right)^2\right]}{dx} = a^2 \frac{d(y^2)}{dx}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 [(-2x) + (1-x^2)] \cdot 2 \cdot \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} = a^2 \cdot 2y \cdot \frac{dy}{dx}$$

$$\Rightarrow -x \frac{dy}{dx} + (1-x^2) \frac{d^2y}{dx^2} = a^2y$$

$$\Rightarrow (1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2y = 0$$

Hence proved

35. Question

If $y = 500 e^{7x} + 600 e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d(e^{ax})}{dx} = ae^{ax}$$

$$(iii) \frac{d}{dx} x^n = nx^{n-1}$$

$$(iv) \text{ chain rule } \frac{df}{dx} = \frac{d(\text{wou})}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Given: -

$$y = 500e^{7x} + 600e^{-7x}$$

$$\frac{dy}{dx} = 500 \cdot \frac{d(e^{7x})}{dx} + 600 \cdot \frac{d(e^{-7x})}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 500e^{7x} \cdot \frac{d(7x)}{dx} + 600 \cdot e^{7x} \cdot \frac{d(-7x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 3500e^{7x} - 4200e^{-7x}$$

$$\Rightarrow \frac{dy}{dx} = 49(500e^{7x} + 600e^{-7x})$$

$$\Rightarrow \frac{dy}{dx} = 49y$$

Hence proved.

36. Question

If $x = 2 \cos t - \cos 2t$, $y = 2 \sin t - \sin 2t$, find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{2}$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} \cos x = -\sin x$$

$$(iii) \frac{d}{dx} \sin x = \cos x$$

$$(iv) \frac{d}{dx} x^n = nx^{n-1}$$

$$(v) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

$$(vi) \text{ parameteric forms } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Given: -

$$x = 2 \cos t - \cos 2t$$

$$y = 2 \sin t - \sin 2t$$

differentiating w.r.t t

$$\frac{dy}{dt} = 2(-\sin t) - 2(-\sin 2t)$$

$$\Rightarrow \frac{dy}{dt} = 2 \cos t - 2 \cos 2t$$

Dividing both

$$\frac{dy}{dx} = \frac{2(\cos t - \cos 2t)}{2(\sin 2t - \sin t)}$$

Differentiating w.r.t t

$$\Rightarrow \frac{d \frac{dy}{dx}}{dt} = \frac{(\sin 2t - \sin t)(-\sin t + 2 \sin 2t) - (\cos t - \cos 2t)(2 \cos 2t - \cos t)}{(\sin 2t - \sin t)^2}$$

Dividing

$$\frac{d^2y}{dx^2} = \frac{(\sin 2t - \sin t)(2 \sin t - \sin t) - (\cos t - \cos 2t)(2 \cos 2t - \cos t)}{2(\sin 2t - \sin t)^3}$$

Putting $t = \frac{\pi}{2}$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1 + 2}{-2} = -\frac{3}{2}$$

37. Question

If $x = 4z^2 + 5$, $y = 6z^2 + 7z + 3$, find $\frac{d^2y}{dx^2}$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} x^n = nx^{n-1}$$

$$(iii) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

$$(iv) \text{ parameteric forms } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Given: -

$$x = 4z^2 + 5, y = 6z^2 + 7z + 3$$

Differentiating both w.r.t z

$$\frac{dx}{dz} = 8z + 0$$

$$\Rightarrow \frac{dx}{dz} = \frac{12z + 7}{8z}$$

$$\text{and} \Rightarrow \frac{dy}{dz} = 12z + 7$$

differentiating w.r.t z

$$\frac{d\left(\frac{dy}{dz}\right)}{dz} = 0 + \frac{7}{8} \left(\frac{-1}{z^2}\right)$$

Dividing

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-7}{8z^2 \times 8z} = \frac{-7}{64z^3}$$

38. Question

If $y = \log(1 + \cos x)$, prove that $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} \cdot \frac{dy}{dx} = 0$

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} \cos x = -\sin x$$

$$(iii) \frac{d}{dx} \sin x = \cos x$$

$$(iv) \frac{d}{dx} x^n = nx^{n-1}$$

$$(v) \text{ chain rule } \frac{df}{dx} = \frac{d(\text{wou})}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Given: -

$$Y = \log(1 + \cos x)$$

Differentiating w.r.t x

$$\frac{dy}{dx} = \frac{1}{1 + \cos x} \cdot (-\sin x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin x}{1 + \cos x}$$

Differentiating w.r.t.x

$$\frac{d^2 y}{dx^2} = - \left[\frac{(1 + \cos x)\cos x - \sin x(-\sin x)}{(1 + \cos x)^2} \right]$$

$$\Rightarrow \frac{d^2 y}{dx^2} = - \left[\frac{(\cos x) + \cos^2 x + \sin^2 x}{(1 + \cos x)^2} \right]$$

$$\Rightarrow \frac{d^2 y}{dx^2} = - \left[\frac{1 + \cos x}{(1 + \cos x)^2} \right]$$

$$\Rightarrow \frac{d^2 y}{dx^2} = - \frac{1}{1 + \cos x}$$

Differentiating w.r.t x

$$\frac{d^3 y}{dx^3} = - \left(\frac{1}{(1 + \cos x)^2} \times -\sin x \right)$$

$$\Rightarrow \frac{d^3 y}{dx^3} = - \left(\frac{-\sin x}{1 + \cos x} \right) \times \left(\frac{-1}{1 + \cos x} \right)$$

$$\Rightarrow \frac{d^3 y}{dx^3} = - \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2}$$

$$\Rightarrow \frac{d^3 y}{dx^3} + \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2} = 0$$

39. Question

If $y = \sin(\log x)$, prove that $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2 y}{dx^2} = y_2$$

$$(ii) \frac{d(\log x)}{dx} = \frac{1}{x}$$

$$(iii) \frac{d}{dx} \cos x = -\sin x$$

$$(iv) \frac{d}{dx} \sin x = -\cos x$$

$$(v) \frac{d}{dx} x^n = nx^{n-1}$$

$$(vi) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Given: -

$$y = \sin(\log x)$$

$$\frac{dy}{dx} = \cos(\log x) \frac{1}{x}$$

$$\Rightarrow x \frac{dy}{dx} = \cos(\log x)$$

$$\Rightarrow x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = -\sin(\log x) \frac{1}{x}$$

$$\Rightarrow x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = -y$$

$$\Rightarrow x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$$

Hence proved.

40. Question

$$\text{If } y = 3e^{2x} + 2e^{3x}, \text{ prove that } \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0.$$

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2 y}{dx^2} = y_2$$

$$(ii) \frac{d(e^{ax})}{dx} = ae^{ax}$$

$$(iii) \frac{d}{dx} x^n = nx^{n-1}$$

Given: -

$$y = 3e^{2x} + 2e^{3x}$$

$$\Rightarrow \frac{dy}{dx} = 6e^{2x} + 6e^{3x}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = 12e^{2x} + 18e^{3x}$$

Hence

$$\begin{aligned} \Rightarrow \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6y &= 6(2e^{2x} + 3e^{3x}) - 30(e^{2x} + e^{3x}) + 6(3e^{2x} + 2e^{3x}) \\ &= 0 \end{aligned}$$

41. Question

If $y = (\cot^{-1} x)^2$, prove that $y_2(x^2 + 1)^2 + 2x(x^2 + 1)y_1 = 2$.

Answer

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} \cot^{-1} x = \frac{-1}{1 + x^2}$$

$$(iii) \frac{d}{dx} x^n = nx^{n-1}$$

$$(iv) \text{ chain rule } \frac{df}{dx} = \frac{d(\text{wou})}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Given: -

$$y = (\cot^{-1} x)^2$$

differentiating w.r.t x

$$\frac{dy}{dx} = y_1 = 2 \cot^{-1} x \cdot \left[\frac{-1}{1 + x^2} \right]$$

$$\Rightarrow y_1 = \frac{-2 \cot^{-1} x}{1 + x^2}$$

Differentiating w.r.t x

$$\Rightarrow (1 + x^2)y_2 + 2xy_1 = 2 \left(\frac{1}{1 + x^2} \right)$$

$$\Rightarrow (1 + x^2)^2 y_2 + 2x(1 + x^2)y_1 = 2$$

Hence proved

42. Question

If $y = \operatorname{cosec}^{-1} x, x > 1$, then show that $x(x^2 - 1) \frac{d^2y}{dx^2} + (2x^2 - 1) \frac{dy}{dx} = 0$

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d(\operatorname{cosec}^{-1} x)}{dx} = \frac{-1}{|x|\sqrt{x^2 - 1}}$$

$$(iii) \frac{d}{dx} x^n = nx^{n-1}$$

$$(iv) \text{ chain rule } \frac{df}{dx} = \frac{d(\text{wou})}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Given: -

$$Y = \operatorname{cosec}^{-1} x$$

We know that

$$\frac{d(\operatorname{cosec}^{-1}x)}{dx} = \frac{-1}{|x|\sqrt{x^2-1}}$$

Let $y = \operatorname{cosec}^{-1}x$

$$\frac{dy}{dx} = \frac{-1}{|x|\sqrt{x^2-1}}$$

Since $x > 1, |x| = x$

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$$

Differentiating the above function with respect to x

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{x \frac{2x}{2\sqrt{x^2-1}} + \sqrt{x^2-1}}{x^2(x^2-1)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{x^2}{\sqrt{x^2-1}} + \sqrt{x^2-1}}{x^2(x^2-1)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{x^2 + x^2 - 1}{x^2(x^2-1)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2x^2 - 1}{x^2(x^2-1)^{\frac{3}{2}}}$$

Thus

$$x(x^2-1) \frac{d^2y}{dx^2} = \frac{2x^2-1}{x\sqrt{x^2-1}} \dots \dots (2)$$

Similarly

$$\Rightarrow [2x^2-1] \frac{dy}{dx} = \frac{-2x^2+1}{x\sqrt{x^2-1}}$$

$$\Rightarrow x(x^2-1) \frac{d^2y}{dx^2} + [2x^2-1] \frac{dy}{dx} = \frac{2x^2-1}{x\sqrt{x^2-1}} + \frac{-2x^2+1}{x\sqrt{x^2-1}} = 0$$

Hence proved.

43. Question

If $x = \cos t + \log \tan \frac{t}{2}$, $y = \sin t$, then find the value of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} \cos x = -\sin x$$

$$(iii) \frac{d}{dx} \sin x = \cos x$$

$$(iv) \frac{d}{dx} \log x = \frac{1}{x}$$

$$(v) \frac{d}{dx} \tan x = \sec^2 x$$

$$(vi) \frac{d}{dx} x^n = nx^{n-1}$$

$$(v) \text{ chain rule } \frac{df}{dx} = \frac{d(\text{wou})}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

$$(vi) \text{ parameteric forms } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Given: -

$$x = \cos t + \log \tan \frac{t}{2}, y = \sin t$$

Differentiating with respect to t, we have

$$\frac{dx}{dt} = -\sin t + \frac{1}{\tan \frac{t}{2}} \times \sec^2 \left(\frac{t}{2} \right) \times \frac{1}{2}$$

$$\Rightarrow \frac{dx}{dt} = -\sin t + \frac{1}{\frac{\sin \left(\frac{t}{2} \right)}{\cos \left(\frac{t}{2} \right)}} \times \frac{1}{\cos^2 \frac{t}{2}} \times \frac{1}{2}$$

$$\Rightarrow \frac{dx}{dt} = -\sin t + \frac{1}{2 \sin \left(\frac{t}{2} \right) \cos \left(\frac{t}{2} \right)}$$

$$\Rightarrow \frac{dx}{dt} = -\sin t + \frac{1}{\sin t}$$

$$\Rightarrow \frac{dx}{dt} = \frac{1 - \sin^2 t}{\sin t}$$

$$\Rightarrow \frac{dx}{dt} = \frac{\cos^2 t}{\sin t}$$

$$\Rightarrow \frac{dx}{dt} = \cos t \cdot \cot t$$

Now find the value of $\frac{dy}{dt}$

$$\frac{dy}{dt} = \cos t$$

Now

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \cos t \times \frac{1}{\cos t \cdot \cot t}$$

$$\Rightarrow \frac{dy}{dx} = \tan t$$

We have

$$\frac{dy}{dt} = \cos t$$

Differentiating with w.r.t t

$$\frac{d^2y}{dt^2} = -\sin t$$

$$\text{At } t = \frac{\pi}{4}$$

$$\left(\frac{d^2y}{dt^2}\right)_{t=\frac{\pi}{4}} = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\tan t)}{\cos t \cdot \cot t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^2 t}{\cos t \cdot \cot t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^2 t}{\cos^2 t} \cdot \sin t$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^4 t \times \sin t$$

$$\text{Now putting } t = \frac{\pi}{4}$$

$$\left(\frac{d^2y}{dx^2}\right)_{t=\frac{\pi}{4}} = \sec^4 \frac{\pi}{4} \cdot \sin\left(\frac{\pi}{4}\right) = 2$$

44. Question

If $x = a \sin t$ and $y = a \left(\cos t + \log \tan \frac{t}{2} \right)$, find $\frac{d^2y}{dx^2}$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} \cos x = -\sin x$$

$$(iii) \frac{d}{dx} \sin x = \cos x$$

$$(iv) \frac{d}{dx} x^n = nx^{n-1}$$

$$(v) \text{ chain rule } \frac{df}{dx} = \frac{d(\text{wou})}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

(vi) parametric forms $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Given: -

$$x = at \sin t \text{ and } y = a \left(\cos t + \log \tan \left(\frac{t}{2} \right) \right)$$

$$\frac{dx}{dt} = a \cos t$$

$$\Rightarrow \frac{d^2y}{dt^2} = -a \sin t$$

$$\Rightarrow \frac{dy}{dt} = -a \sin t + \frac{a}{\tan \left(\frac{t}{2} \right)} \times \sec^2 \frac{t}{2} \times \frac{1}{2}$$

$$\Rightarrow \frac{dy}{dt} = -a \sin t + \frac{a}{2 \sin \left(\frac{t}{2} \right) \cos \left(\frac{t}{2} \right)}$$

$$\Rightarrow \frac{dy}{dt} = -a \sin t + a \operatorname{cosec} t$$

$$\Rightarrow \frac{d^2y}{dt^2} = -a \cos t - a \operatorname{cosec} t \cot t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt} \frac{d^2y}{dt^2} - \frac{dy}{dt} \frac{d^2x}{dt^2}}{\left(\frac{dx}{dt} \right)^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{a \cos t (-a \cos t - a \operatorname{cosec} t \cot t) - (-a \sin t + a \operatorname{cosec} t)(-a \sin t)}{(a \cos t)^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-a^2 (\cos^2 t + \sin^2 t) - a^2 \cot^2 t + a^2}{a^3 \cos^3 t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{a \sin^2 t \cos t}$$

45. Question

If $x = a (\cos t + t \sin t)$ and $y = a (\sin t - t \cos t)$, then find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} \cos x = -\sin x$$

$$(iii) \frac{d}{dx} \sin x = \cos x$$

$$(iv) \frac{d}{dx} x^n = nx^{n-1}$$

$$(v) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

$$(vi) \text{ parameteric forms } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Given: -

$$x = a(\cos t + t \sin t) \text{ and } y = a(\sin t - t \cos t)$$

$$\frac{dy}{dt} = a \cos t - a \cos t + a t \sin t = a t \sin t$$

$$\Rightarrow \frac{d^2 y}{dt^2} = a t \cos t + a \sin t$$

$$\Rightarrow \frac{dx}{dt} = -a \sin t + a \cos t + a \sin t = a \cos t$$

$$\Rightarrow \frac{d^2 x}{dt^2} = -a \sin t + a \cos t$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{\frac{dx}{dt} \frac{d^2 y}{dt^2} - \frac{dy}{dt} \frac{d^2 x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{a \cos t (a t \cos t + a \sin t) - (-a \sin t + a \cos t)(a t \sin t)}{(a \cos t)^3}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{1}{(a \cos t)^3}$$

$$\text{Putting } t = \frac{\pi}{4}$$

$$\left(\frac{d^2 y}{dx^2}\right)_{t=\frac{\pi}{4}} = \frac{1}{a \cos^3 \frac{\pi}{4} \cdot a \frac{\pi}{4}} = \frac{8\sqrt{2}}{\pi a}$$

46. Question

$$\text{If } x = a\left(\cos t + \log \tan \frac{t}{2}\right), y = a \sin t, \text{ evaluate } \frac{d^2 y}{dx^2} \text{ at } t = \frac{\pi}{3}$$

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2 y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} \cos x = -\sin x$$

$$(iii) \frac{d}{dx} \sin x = \cos x$$

$$(iv) \frac{d}{dx} x^n = n x^{n-1}$$

$$(v) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

(vi) parameteric forms $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Given: -

$$x = a(\cos t + \log \tan \frac{t}{2}), y = \sin t$$

Differentiating with respect to t, we have

$$\Rightarrow \frac{dx}{dt} = -a \sin t + a \frac{1}{\tan \frac{t}{2}} \times \sec^2 \left(\frac{t}{2} \right) \times \frac{1}{2}$$

$$\Rightarrow \frac{dx}{dt} = -a \sin t + a \frac{1}{\frac{\sin \left(\frac{t}{2} \right)}{\cos \left(\frac{t}{2} \right)}} \times \frac{1}{\cos^2 \frac{t}{2}} \times \frac{1}{2}$$

$$\Rightarrow \frac{dx}{dt} = -a \sin t + a \frac{1}{2 \sin \left(\frac{t}{2} \right) \cos \left(\frac{t}{2} \right)}$$

$$\Rightarrow \frac{dx}{dt} = -a \sin t + a \frac{1}{\sin t} = -a \sin t + a \operatorname{cosec} t$$

Now find the value of $\frac{dy}{dt}$

$$\frac{dy}{dt} = a \cos t$$

$$\Rightarrow \frac{d^2 y}{dt^2} = -a \sin t$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{\frac{dx}{dt} \frac{d^2 y}{dt^2} - \frac{dy}{dt} \frac{d^2 x}{dt^2}}{\left(\frac{dx}{dt} \right)^3}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{-a \sin t (-a \sin t + a \operatorname{cosec} t) - (-a \cos t - a \operatorname{cosec} t \cot t)(-a \cos t)}{(a \operatorname{cosec} t - a \sin t)^3}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{a^2 (\cos^2 t + \sin^2 t) + a^2 \cot^2 t - a^2}{(a \operatorname{cosec} t - a \sin t)^3}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{\sin t}{a \cos^4 t}$$

$$\left(\frac{d^2 y}{dx^2} \right)_{t=\frac{\pi}{3}} = \frac{\sin \frac{\pi}{3}}{a \cos^4 \frac{\pi}{3}} = \frac{8\sqrt{3}}{a}$$

47. Question

If $x = a (\cos 2t + 2t \sin 2t)$ and $y = a (\sin 2t - 2t \cos 2t)$, then find $\frac{d^2 y}{dx^2}$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} \cos x = -\sin x$$

$$(iii) \frac{d}{dx} \sin x = \cos x$$

$$(iv) \frac{d}{dx} x^n = nx^{n-1}$$

$$(v) \text{ chain rule } \frac{df}{dx} = \frac{d(f \circ g)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

$$(vi) \text{ parametric forms } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Given: -

$$x = a (\cos 2t + 2t \sin 2t)$$

$$\Rightarrow \frac{dx}{dt} = -2a \sin 2t + 2a \sin 2t + 4at \cos 2t = 4at \cos 2t$$

$$\text{and } y = a (\sin 2t - 2t \cos 2t)$$

$$\Rightarrow \frac{dy}{dt} = 2a \cos 2t - 2a \cos 2t + 4at \sin 2t = 4at \sin 2t$$

$$\Rightarrow \frac{dy}{dx} = \frac{dt}{dx} \times \frac{dy}{dt} = \frac{\sin 2t}{\cos 2t} = \tan 2t$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d(\tan 2t)}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 2t \frac{d(2t)}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \sec^2 2t \frac{d(t)}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^3 2t}{2a}$$

48. Question

If $x = 3 \cot t - 2 \cos^3 t$, $y = 3 \sin t - 2 \sin^3 t$, find $\frac{d^2y}{dx^2}$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} \cos x = -\sin x$$

$$(iii) \frac{d}{dx} \sin x = \cos x$$

$$(iv) \frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$$

$$(v) \frac{d}{dx} x^n = nx^{n-1}$$

$$(vi) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

$$(vii) \text{ parameteric forms } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

given: -

$$x = 3 \cot t - 2 \cos^3 t, y = 3 \sin t - 2 \sin^3 t$$

differentiating both w.r.t t

$$\frac{dx}{dt} = -3 \sin t - 6 \cos^2 t (-\sin t)$$

$$\frac{dx}{dt} = -3 \sin t + 6 \cos^2 t \sin t$$

$$\text{And } y = 3 \sin t - 2 \sin^3 t$$

differentiating both w.r.t t

$$\frac{dy}{dt} = 3 \cos t - 6 \sin^2 t \cos t$$

Now,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos t - 2 \sin^2 t \cos t}{-\sin t + 2 \cos^2 t \sin t}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos t [1 - 2 \sin^2 t]}{\sin t [2 \cos^2 t - 1]}$$

$$\Rightarrow \frac{dy}{dx} = \cot t$$

differentiating both w.r.t x

$$\frac{d^2 y}{dx^2} = \frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$$

49. Question

If $x = a \sin t - b \cos t$, $y = a \cos t + b \sin t$, prove that $\frac{d^2 y}{dx^2} = -\frac{x^2 + y^2}{y^3}$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2 y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} \cos x = -\sin x$$

$$(iii) \frac{d}{dx} \sin x = \cos x$$

$$(iv) \frac{d}{dx} x^n = nx^{n-1}$$

$$(v) \text{ chain rule } \frac{df}{dx} = \frac{d(f \circ g)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

$$(vi) \text{ parameteric forms } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Given: -

$$x = a \sin t - b \cos t, y = a \cos t + b \sin t$$

differentiating both w.r.t t

$$\frac{dx}{dt} = a \cos t + b \sin t, \frac{dy}{dt} = -a \sin t + b \cos t$$

$$\Rightarrow \frac{dx}{dt} = y, \Rightarrow \frac{dy}{dt} = x$$

Dividing both

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\frac{x}{y}$$

Differentiating w.r.t t

$$\Rightarrow \frac{d\left(\frac{dy}{dx}\right)}{dt} = -\frac{y\left(\frac{dx}{dt}\right) - x\left(\frac{dy}{dt}\right)}{y^2}$$

Putting the value

$$\Rightarrow \frac{d\left(\frac{dy}{dx}\right)}{dt} = -\frac{\{y^2 + x^2\}}{y^2}$$

Dividing them

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{\{y^2 + x^2\}}{y^2 \cdot y} = -\frac{\{x^2 + y^2\}}{y^3}$$

Hence proved.

50. Question

Find A and B so that $y = A \sin 3x + B \cos 3x$ satisfies the equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 10 \cos 3x$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} \cos x = -\sin x$$

$$(iii) \frac{d}{dx} \sin x = -\cos x$$

$$(iv) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Given: -

$$y = A \sin 3x + B \cos 3x$$

differentiating w.r.t x

$$\frac{dy}{dx} = 3A \cos 3x + 3B(-\sin 3x)$$

Again differentiating w.r.t x

$$\frac{d^2y}{dx^2} = 3A(-\sin 3x) \cdot 3 - 3B(\cos 3x) \cdot 3$$

$$\Rightarrow \frac{d^2y}{dx^2} = -9(A \sin 3x + B \cos 3x) = -9y$$

Now adding

$$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y = -9y + 4(3A \cos 3x - 3B \sin 3x) + 3y$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y = 12(A \cos 3x - B \sin 3x) - 6(A \sin 3x + B \cos 3x)$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y = (12A - 6B) \cos 3x - (12B + 6A) \sin 3x$$

But given,

$$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y = 10 \cos 3x$$

$$\Rightarrow 12A - 6B = 10$$

$$\Rightarrow -(12B + 6A) = 0$$

$$\Rightarrow 6A = -12B$$

$$\Rightarrow A = -2B$$

Putting A

$$\Rightarrow 12(-2B) - 6B = 10$$

$$\Rightarrow -24B - 6B = 10$$

$$\Rightarrow B = -\frac{1}{3}$$

$$A = -2 \times -\frac{1}{3} = \frac{2}{3}$$

$$\text{And } A = \frac{2}{3}, B = -\frac{1}{3}$$

51. Question

If $y = A e^{-kt} \cos(pt + c)$, prove that $\frac{d^2y}{dt^2} + 2k \frac{dy}{dt} + n^2 y = 0$, where $n^2 = p^2 + k^2$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} e^{ax} = ae^{ax}$$

$$(iii) \frac{d}{dx} \cos x = -\sin x$$

$$(iv) \frac{d}{dx} \sin x = \cos x$$

$$(v) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

$$(vi) \text{ parameteric forms } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Given: -

$$y = A e^{-kt} \cos(pt + c)$$

Differentiating w.r.t t

$$\frac{dy}{dt} = A \left(e^{-kt} (-\sin(pt + c) \cdot p) + (\cos(pt + c)) (-k e^{-kt}) \right)$$

$$\Rightarrow \frac{dy}{dt} = -A p e^{-kt} \sin(pt + c) - k A e^{-kt} \cos(pt + c)$$

$$\Rightarrow \frac{dy}{dt} = -A p e^{-kt} \sin(pt + c) - k y$$

Differentiating w.r.t t

$$\frac{d^2y}{dt^2} = A p k e^{-kt} \sin(pt + c) - p^2 y - k \frac{dy}{dt} + k y$$

$$\Rightarrow \frac{d^2y}{dt^2} = A p k e^{-kt} \sin(pt + c) - p^2 y - k \frac{dy}{dt} + k y$$

$$\Rightarrow \frac{d^2y}{dt^2} = -(p^2 + k^2) y - k \frac{dy}{dt}$$

$$\Rightarrow \frac{d^2y}{dt^2} + k \frac{dy}{dt} + (p^2 + k^2) y = 0$$

Hence proved

52. Question

If $y = x^n \{a \cos(\log x) + b \sin(\log x)\}$, prove that $x^2 \frac{d^2y}{dx^2} + (1 - 2n) \frac{dy}{dx} + (1 + n^2) y = 0$

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} \cos x = -\sin x$$

$$(iii) \frac{d}{dx} \sin x = \cos x$$

$$(iv) \frac{d(\log x)}{dx} = \frac{1}{x}$$

$$(v) \frac{d}{dx} x^n = nx^{n-1}$$

$$(vi) \text{ chain rule } \frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Given: -

$$y = x^n(a \cos(\log x) + b \sin(\log x))$$

$$\Rightarrow y = ax^n \cos(\log x) + bx^n \sin(\log x)$$

$$\frac{dy}{dx} = anx^{n-1} \cos(\log x) - ax^{n-1} \sin(\log x) + bx^{n-1} \sin(\log x) + bx^{n-1} \cos(\log x)$$

$$\Rightarrow \frac{dy}{dx} = x^{n-1} \cos(\log x)(na + b) + x^{n-1} \sin(\log x)(bn - a)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} (x^{n-1} \cos(\log x)(na + b) + x^{n-1} \sin(\log x)(bn - a))$$

$$\Rightarrow \frac{d^2y}{dx^2} = (na + b)[(n-1)x^{n-2} \cos(\log x) - x^{n-2} \sin(\log x)] + (bn - a)[(n-1)x^{n-2} \sin(\log x) + x^{n-2} \cos(\log x)]$$

$$x^2 \frac{d^2y}{dx^2} + (1 - 2n) \frac{dy}{dx} + (1 + n^2)y$$

$$= x^n (na + b)[(n-1) \cos(\log x) - \sin(\log x)] + (bn - a)x^n [(n-1) \sin(\log x) + \cos(\log x)] + (1 - 2n)x^{n-1} \cos(\log x)(na + b) + (1 - 2n)x^{n-1} \sin(\log x)(bn - a) + a(1 + n^2)x^n \cos(\log x) + b(1 + n^2)x^n \sin(\log x)$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} + (1 - 2n) \frac{dy}{dx} + (1 + n^2)y = 0$$

53. Question

If $y = a \{x + \sqrt{x^2 + 1}\}^n + b \{x - \sqrt{x^2 + 1}\}^{-n}$, prove that $(x^2 - 1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - n^2 y = 0$.

Answer

Formula: -

$$(i) \frac{dy}{dx} = y_1 \text{ and } \frac{d^2y}{dx^2} = y_2$$

$$(ii) \frac{d}{dx} x^n = nx^{n-1}$$

(iii) chain rule $\frac{df}{dx} = \frac{d(wou)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$

Given: -

$$y = a \{x + \sqrt{x^2 + 1}\}^n + b \{x - \sqrt{x^2 + 1}\}^{-n}$$

$$\frac{dy}{dx} = na\{x + x^2 + 1\}^{n-1} \left[1 + x(x^2 + 1)^{-\frac{1}{2}}\right] - nb\{x - \sqrt{x^2 + 1}\}^{-n-1} \left[1 - x(x^2 + 1)^{-\frac{1}{2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{na\{x + x^2 + 1\}^n}{\sqrt{x^2 + 1}} + \frac{nb\{x + x^2 + 1\}^{-n}}{\sqrt{x^2 + 1}}$$

$$\Rightarrow \frac{xdy}{dx} = \frac{nx}{\sqrt{x^2 + 1}}y$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{nx}{\sqrt{x^2 + 1}} \frac{dy}{dx} + y \left[\frac{\sqrt{x^2 + 1} - x^2(x^2 + 1)^{-\frac{1}{2}}}{x^2 + 1} \right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{n^2x^2}{x^2 + 1} + y \left[\frac{1}{(x^2 + 1)\sqrt{x^2 + 1}} \right]$$

$$\Rightarrow (x^2 - 1) \frac{d^2y}{dx^2} = \frac{n^2x^4(\sqrt{x^2 + 1}) + x^2y}{x^2 + 1\sqrt{x^2 + 1}} - \frac{n^2x^2(\sqrt{x^2 + 1}) + y}{x^2 + 1(\sqrt{x^2 + 1})}$$

Now

$$\begin{aligned} \Rightarrow (x^2 - 1) \frac{d^2y}{dx^2} + \frac{xdy}{dx} - ny &= \frac{n^2x^4(\sqrt{x^2 + 1}) + x^2y}{(x^2 + 1)\sqrt{x^2 + 1}} - \frac{n^2x^2(\sqrt{x^2 + 1}) + y}{(x^2 + 1)(\sqrt{x^2 + 1})} - ny = 0 \end{aligned}$$

1 A. Question

Find the second order derivatives of each of the following functions:

$$x^3 + \tan x$$

Answer

Basic idea:

✓Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v , i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

✓Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, $y = x^3 + \tan x$

We have to find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find $\frac{dy}{dx}$ and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (x^3 + \tan x) = \frac{d}{dx} (x^3) + \frac{d}{dx} (\tan x)$$

$$[\because \frac{d}{dx} (\tan x) = \sec^2 x \text{ \& } \frac{d}{dx} (x^n) = nx^{n-1}]$$

$$= 3x^2 + \sec^2 x$$

$$\therefore \frac{dy}{dx} = 3x^2 + \sec^2 x$$

Differentiating again with respect to x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (3x^2 + \sec^2 x) = \frac{d}{dx} (3x^2) + \frac{d}{dx} (\sec^2 x)$$

$$\frac{d^2y}{dx^2} = 6x + 2 \sec x \sec x \tan x$$

$$[\text{differentiated } \sec^2 x \text{ using chain rule, let } t = \sec x \text{ and } z = t^2 \therefore \frac{dz}{dx} = \frac{dz}{dt} \times \frac{dt}{dx}]$$

$$\frac{d^2y}{dx^2} = 6x + 2 \sec^2 x \tan x$$

1 B. Question

Find the second order derivatives of each of the following functions:

$$\sin (\log x)$$

Answer

✓Basic Idea: Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

✓Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

$$\text{Given, } y = \sin (\log x)$$

We have to find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\sin (\log x))$$

differentiating $\sin(\log x)$ using the chain rule,

let, $t = \log x$ and $y = \sin t$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \text{ [using chain rule]}$$

$$\frac{dy}{dx} = \cos t \times \frac{1}{x}$$

$$\frac{dy}{dx} = \cos(\log x) \times \frac{1}{x} \left[\because \frac{d}{dx}(\log x) = \frac{1}{x} \text{ \& } \frac{d}{dx}(\sin x) = \cos x \right]$$

Differentiating again with respect to x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\cos(\log x) \times \frac{1}{x} \right)$$

$$\frac{d^2 y}{dx^2} = \cos(\log x) \times \frac{-1}{x^2} + \frac{1}{x} \times \frac{1}{x} (-\sin(\log x))$$

[using product rule of differentiation]

$$= \frac{-1}{x^2} \cos(\log x) - \frac{1}{x^2} \sin(\log x)$$

$$\frac{d^2 y}{dx^2} = \frac{-1}{x^2} \cos(\log x) - \frac{1}{x^2} \sin(\log x)$$

1 C. Question

Find the second order derivatives of each of the following functions:

$\log(\sin x)$

Answer

✓Basic Idea: Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v , i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, $y = \log(\sin x)$

We have to find $\frac{d^2 y}{dx^2}$

$$\text{As } \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\log(\sin x))$$

differentiating $\sin(\log x)$ using the chain rule,

let, $t = \sin x$ and $y = \log t$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \text{ [using chain rule]}$$

$$\frac{dy}{dx} = \cos x \times \frac{1}{t}$$

$$[\because \frac{d}{dx} \log x = \frac{1}{x} \& \frac{d}{dx} (\sin x) = \cos x]$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

Differentiating again with respect to x:

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (\cot x)$$

$$\frac{d^2 y}{dx^2} = -\operatorname{cosec}^2 x \quad [\because \frac{d}{dx} \cot x = -\operatorname{cosec}^2 x]$$

$$\frac{d^2 y}{dx^2} = -\operatorname{cosec}^2 x$$

1 D. Question

Find the second order derivatives of each of the following functions:

$$e^x \sin 5x$$

Answer

✓Basic Idea: Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

$$\text{Given, } y = e^x \sin 5x$$

We have to find $\frac{d^2 y}{dx^2}$

$$\text{As, } \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (e^x \sin 5x)$$

$$\text{Let } u = e^x \text{ and } v = \sin 5x$$

$$\text{As, } y = uv$$

$\therefore$ Using product rule of differentiation:

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = e^x \frac{d}{dx}(\sin 5x) + \sin 5x \frac{d}{dx} e^x$$

$$\frac{dy}{dx} = 5e^x \cos 5x + e^x \sin 5x$$

$$[\because \frac{d}{dx}(\sin ax) = a \cos ax, \text{ where } a \text{ is any constant \& } \frac{d}{dx} e^x = e^x]$$

Again differentiating w.r.t x:

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (5e^x \cos 5x + e^x \sin 5x)$$

$$= \frac{d}{dx} (5e^x \cos 5x) + \frac{d}{dx} (e^x \sin 5x)$$

Again using the product rule :

$$\frac{d^2 y}{dx^2} = e^x \frac{d}{dx}(\sin 5x) + \sin 5x \frac{d}{dx} e^x + 5e^x \frac{d}{dx}(\cos 5x) + \cos 5x \frac{d}{dx} (5e^x)$$

$$\frac{d^2 y}{dx^2} = 5e^x \cos 5x - 25e^x \sin 5x + e^x \sin 5x + 5e^x \cos 5x [\because \frac{d}{dx}(\cos ax) = -a \sin ax, a \text{ is any constant}]$$

$$\frac{d^2 y}{dx^2} = 10e^x \cos 5x - 24e^x \sin 5x$$

1 E. Question

Find the second order derivatives of each of the following functions:

$$e^{6x} \cos 3x$$

Answer

✓Basic Idea: Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

$$\text{Given, } y = e^{6x} \cos 3x$$

We have to find $\frac{d^2 y}{dx^2}$

$$\text{As, } \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So lets first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (e^{6x} \cos 3x)$$

$$\text{Let } u = e^{6x} \text{ and } v = \cos 3x$$

$$\text{As, } y = uv$$

∴ Using product rule of differentiation:

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = e^{6x} \frac{d}{dx}(\cos 3x) + \cos 3x \frac{d}{dx} e^{6x}$$

$$\frac{dy}{dx} = -3e^{6x} \sin 3x + 6e^{6x} \cos 3x \quad \because \frac{d}{dx}(\cos ax) = -a \sin ax, a \text{ is any constant} \& \frac{d}{dx} e^{ax} = ae^x$$

Again differentiating w.r.t x:

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (-3e^{6x} \sin 3x + 6e^{6x} \cos 3x)$$

$$= \frac{d}{dx} (-3e^{6x} \sin 3x) + \frac{d}{dx} (6e^{6x} \cos 3x)$$

Again using the product rule :

$$\frac{d^2 y}{dx^2} = -3e^{6x} \frac{d}{dx}(\sin 3x) - 3 \sin 3x \frac{d}{dx} e^{6x} + 6e^{6x} \frac{d}{dx}(\cos 3x) + \cos 3x \frac{d}{dx} (6e^{6x})$$

$$\frac{d^2 y}{dx^2} = -9e^{6x} \cos 3x - 18e^{6x} \sin 3x - 18e^{6x} \sin 3x + 36e^{6x} \cos 3x$$

$$\frac{d^2 y}{dx^2} = 27e^{6x} \cos 3x - 36e^{6x} \sin 3x$$

1 F. Question

Find the second order derivatives of each of the following functions:

$$x^3 \log x$$

Answer

✓Basic Idea: Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

$$\text{Given, } y = x^3 \log x$$

We have to find $\frac{d^2 y}{dx^2}$

$$\text{As } \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (x^3 \log x)$$

$$\text{Let } u = x^3 \text{ and } v = \log x$$

As, $y = uv$

∴ Using product rule of differentiation:

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = x^3 \frac{d}{dx}(\log x) + \log x \frac{d}{dx} x^3$$

$$\frac{dy}{dx} = 3x^2 \log x + \frac{x^3}{x}$$

$$[\because \frac{d}{dx}(\log x) = \frac{1}{x} \text{ and } \frac{d}{dx}(x^n) = nx^{n-1}]$$

Again differentiating w.r.t x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (3x^2 \log x + x^2)$$

$$= \frac{d}{dx} (3x^2 \log x) + \frac{d}{dx} (x^2)$$

Again using the product rule :

$$\frac{d^2 y}{dx^2} = 3 \log x \frac{d}{dx} x^2 + 3x^2 \frac{d}{dx} \log x + \frac{d}{dx} x^2$$

$$[\because \frac{d}{dx}(\log x) = \frac{1}{x} \text{ and } \frac{d}{dx}(x^n) = nx^{n-1}]$$

$$\frac{d^2 y}{dx^2} = 6x \log x + \frac{3x^2}{x} + 2x$$

$$\frac{d^2 y}{dx^2} = 6x \log x + 5x$$

1 G. Question

Find the second order derivatives of each of the following functions:

$$\tan^{-1} x$$

Answer

Basic idea:

√Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

√The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v , i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

√Product rule of differentiation- $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

√Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

$$\text{Given, } y = \tan^{-1} x$$

We have to find $\frac{d^2 y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\tan^{-1} x) \left[\because \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2} \right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2} \left[\because \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2} \right]$$

Differentiating again with respect to x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{1+x^2} \right)$$

Differentiating $\frac{1}{1+x^2}$ using chain rule,

let $t = 1 + x^2$ and $z = 1/t$

$$\therefore \frac{dz}{dx} = \frac{dz}{dt} \times \frac{dt}{dx} \text{ [from chain rule of differentiation]}$$

$$\therefore \frac{dz}{dx} = \frac{-1}{t^2} \times 2x = -\frac{2x}{1+x^2} \left[\because \frac{d}{dx} (x^n) = nx^{n-1} \right]$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{2x}{(1+x^2)^2}$$

1 H. Question

Find the second order derivatives of each of the following functions:

$x \cos x$

Answer

✓Basic Idea: Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v , i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, $y = x \cos x$

We have to find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (x \cos x)$$

Let $u = x$ and $v = \cos x$

As, $y = uv$

$\therefore$ Using product rule of differentiation:

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}x$$

$$\frac{dy}{dx} = -x \sin x + \cos x$$

$$[\because \frac{d}{dx}(\cos x) = -\sin x \text{ and } \frac{d}{dx}(x^n) = nx^{n-1}]$$

Again differentiating w.r.t x:

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (-x \sin x + \cos x)$$

$$= \frac{d}{dx} (-x \sin x) + \frac{d}{dx} \cos x$$

Again using the product rule :

$$\frac{d^2y}{dx^2} = -x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} (-x) + \frac{d}{dx} \cos x$$

$$[\because \frac{d}{dx}(\sin x) = \cos x \text{ and } \frac{d}{dx}(x^n) = nx^{n-1}]$$

$$\frac{d^2y}{dx^2} = -x \cos x - \sin x - \sin x$$

$$\frac{d^2y}{dx^2} = -x \cos x - 2 \sin x$$

1 I. Question

Find the second order derivatives of each of the following functions:

$\log (\log x)$

Answer

✓Basic Idea: Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v, i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, $y = \log (\log x)$

We have to find $\frac{d^2y}{dx^2}$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So lets first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\log \log x)$$

Let $y = \log t$ and $t = \log x$

Using chain rule of differentiation:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{t} \times \frac{1}{x} = \frac{1}{x \log x} \left[\because \frac{d}{dx}(\log x) = \frac{1}{x} \right]$$

Again differentiating w.r.t x :

$$\text{As, } \frac{dy}{dx} = u \times v$$

$$\text{Where } u = \frac{1}{x} \text{ and } v = \frac{1}{\log x}$$

$\therefore$ using product rule of differentiation:

$$\frac{d^2y}{dx^2} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{x} \frac{d}{dx} \left(\frac{1}{\log x} \right) + \frac{1}{\log x} \frac{d}{dx} \left(\frac{1}{x} \right) \left[\text{use chain rule to find } \frac{d}{dx} \left(\frac{1}{\log x} \right) \right]$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2 (\log x)^2} - \frac{1}{x^2 \log x} \left[\because \frac{d}{dx}(\log x) = \frac{1}{x} \text{ and } \frac{d}{dx}(x^n) = nx^{n-1} \right]$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{1}{x^2 (\log x)^2} - \frac{1}{x^2 \log x}$$

2. Question

If $y = e^{-x} \cos x$, show that : $\frac{d^2y}{dx^2} = 2e^{-x} \sin x$.

Answer

Basic idea:

✓Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v , i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

✓Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given,

$$y = e^{-x} \cos x$$

TO prove :

$$\frac{d^2y}{dx^2} = 2e^{-x} \sin x.$$

Clearly from the expression to be proved we can easily observe that we need to just find the second derivative of given function.

Given, $y = e^{-x} \cos x$

We have to find $\frac{d^2y}{dx^2}$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (e^{-x} \cos x)$$

Let $u = e^{-x}$ and $v = \cos x$

As, $y = u \cdot v$

$\therefore$ using product rule of differentiation:

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = e^{-x} \frac{d}{dx} (\cos x) + \cos x \frac{dy}{dx} e^{-x}$$

$$\frac{dy}{dx} = -e^{-x} \sin x - e^{-x} \cos x$$

$$[\because \frac{d}{dx} (\cos x) = -\sin x \text{ \& \& } \frac{d}{dx} e^{-x} = -e^{-x}]$$

Again differentiating w.r.t x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (-e^{-x} \sin x - e^{-x} \cos x)$$

$$= \frac{d}{dx} (-e^{-x} \sin x) - \frac{d}{dx} (e^{-x} \cos x)$$

Again using the product rule :

$$\frac{d^2y}{dx^2} = -e^{-x} \frac{d}{dx} (\sin x) - \sin x \frac{d}{dx} e^{-x} - e^{-x} \frac{d}{dx} (\cos x) - \cos x \frac{d}{dx} (e^{-x})$$

$$\frac{d^2y}{dx^2} = -e^{-x} \cos x + e^{-x} \sin x + e^{-x} \sin x + e^{-x} \cos x$$

$$[\because \frac{d}{dx} (\cos x) = -\sin x, \frac{d}{dx} e^{-x} = -e^{-x}]$$

$$\frac{d^2y}{dx^2} = 2e^{-x} \sin x \dots \text{proved}$$

3. Question

If $y = x + \tan x$, show that: $\cos^2 x \frac{d^2y}{dx^2} - 2y - 2x = 0$

Answer

Basic idea:

✓Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v , i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

✓Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, $y = x + \tan x$ equation 1

As we have to prove: $\cos^2 x \frac{d^2y}{dx^2} - 2y + 2x = 0$

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x + \tan x) = \frac{d}{dx}(x) + \frac{d}{dx}(\tan x) \left[\because \frac{d}{dx}(\tan x) = \sec^2 x \text{ \& } \frac{d}{dx}(x^n) = nx^{n-1} \right]$$

$$= 1 + \sec^2 x$$

$$\therefore \frac{dy}{dx} = 1 + \sec^2 x$$

Differentiating again with respect to x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (1 + \sec^2 x) = \frac{d}{dx}(1) + \frac{d}{dx}(\sec^2 x)$$

$$\frac{d^2y}{dx^2} = 0 + 2 \sec x \sec x \tan x$$

[differentiated $\sec^2 x$ using chain rule, let $t = \sec x$ and $z = t^2 \therefore \frac{dz}{dx} = \frac{dz}{dt} \times \frac{dt}{dx}$]

$$\frac{d^2y}{dx^2} = 2 \sec^2 x \tan x \text{equation 2}$$

As we got an expression for the second order, as we need $\cos^2 x$ term with $\frac{d^2y}{dx^2}$

Multiply both sides of equation 1 with $\cos^2 x$:

$\therefore$ we have,

$$\cos^2 x \frac{d^2y}{dx^2} = 2 \cos^2 x \sec^2 x \tan x \left[\because \cos x \times \sec x = 1 \right]$$

$$\cos^2 x \frac{d^2y}{dx^2} = 2 \tan x$$

From equation 1:

$$\tan x = y - x \therefore \cos^2 x \frac{d^2 y}{dx^2} = 2(y - x)$$

$$\therefore \cos^2 x \frac{d^2 y}{dx^2} - 2y + 2x = 0 \dots \text{proved}$$

4. Question

$$\text{If } y = x^3 \log x, \text{ prove that } \frac{d^4 y}{dx^4} = \frac{6}{x}.$$

Answer

Basic idea:

✓Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v , i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

✓Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

$$\text{As we have to prove : } \frac{d^4 y}{dx^4} = \frac{6}{x}$$

We notice a third order derivative in the expression to be proved so first take the step to find the third order derivative.

$$\text{Given, } y = x^3 \log x$$

$$\text{Let's find - } \frac{d^4 y}{dx^4}$$

$$\text{As } \frac{d^4 y}{dx^4} = \frac{d}{dx} \left(\frac{d^3 y}{dx^3} \right) = \frac{d}{dx} \frac{d}{dx} \left(\frac{d^2 y}{dx^2} \right) = \frac{d}{dx} \left(\frac{d}{dx} \left(\frac{d}{dx} \left(\frac{dy}{dx} \right) \right) \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (x^3 \log x)$$

differentiating using product rule:

$$\frac{dy}{dx} = x^3 \frac{d}{dx} \log x + \log x \frac{d}{dx} x^3$$

$$\frac{dy}{dx} = \frac{x^3}{x} + 3x^2 \log x$$

$$\left[\frac{d}{dx} (x^n) = nx^{n-1} \text{ \& } \frac{d}{dx} (\log x) = \frac{1}{x} \right]$$

$$\frac{dy}{dx} = x^2(1 + 3 \log x)$$

Again differentiating using product rule:

$$\frac{d^2y}{dx^2} = x^2 \frac{d}{dx}(1 + 3\log x) + (1 + 3\log x) \frac{d}{dx}x^2$$

$$\frac{d^2y}{dx^2} = x^2 \times \frac{3}{x} + (1 + 3\log x) \times 2x$$

$$\left[\frac{d}{dx}(x^n) = nx^{n-1} \text{ \& } \frac{d}{dx}(\log x) = \frac{1}{x} \right]$$

$$\frac{d^2y}{dx^2} = x(5 + 6\log x)$$

Again differentiating using product rule:

$$\frac{d^3y}{dx^3} = x \frac{d}{dx}(5 + 6\log x) + (5 + 6\log x) \frac{d}{dx}x$$

$$\frac{d^3y}{dx^3} = x \times \frac{6}{x} + (5 + 6\log x)$$

$$\left[\frac{d}{dx}(x^n) = nx^{n-1} \text{ \& } \frac{d}{dx}(\log x) = \frac{1}{x} \right]$$

$$\frac{d^3y}{dx^3} = 11 + 6\log x$$

Again differentiating w.r.t x :

$$\frac{d^4y}{dx^4} = \frac{6}{x} \dots \dots \text{proved}$$

5. Question

If $y = \log(\sin x)$, prove that: $\frac{d^3y}{dx^2} = 2 \cos x \operatorname{cosec}^3 x$.

Answer

Basic idea:

✓Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v , i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

✓Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

As we have to prove: $\frac{d^3y}{dx^2} = 2 \cos x \operatorname{cosec}^3 x$

We notice a third order derivative in the expression to be proved so first take the step to find the third order derivative.

Given, $y = \log(\sin x)$

Let's find $-\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{d}{dx} \left(\frac{dy}{dx} \right) \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\log(\sin x))$$

differentiating $\sin(\log x)$ using the chain rule,

let, $t = \sin x$ and $y = \log t$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \text{ [using chain rule]}$$

$$\frac{dy}{dx} = \cos x \times \frac{1}{t}$$

$$[\because \frac{d}{dx} \log x = \frac{1}{x} \text{ \& } \frac{d}{dx} (\sin x) = \cos x]$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$$

Differentiating again with respect to x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (\cot x)$$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$$

$$[\because \frac{d}{dx} \cot x = -\operatorname{cosec}^2 x]$$

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$$

Differentiating again with respect to x :

$$\frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = \frac{d}{dx} (-\operatorname{cosec}^2 x)$$

using the chain rule and $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$

$$\frac{d^3y}{dx^3} = -2\operatorname{cosec} x (-\operatorname{cosec} x \cot x)$$

$$= 2\operatorname{cosec}^2 x \cot x = 2 \operatorname{cosec}^2 x \frac{\cos x}{\sin x} [\because \cot x = \cos x / \sin x]$$

$$\therefore \frac{d^3y}{dx^3} = 2\operatorname{cosec}^2 x \cos x \dots \dots \text{proved}$$

6. Question

If $y = 2 \sin x + 3 \cos x$, show that: $\frac{d^2y}{dx^2} + y = 0$

Answer

Basic idea:

$\sqrt{\text{Second order derivative is nothing but derivative of derivative i.e. } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)}$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v , i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

✓Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, $y = 2\sin x + 3\cos x$ equation 1

As we have to prove : $\frac{d^2y}{dx^2} + y = 0$.

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (2\sin x + 3\cos x) = 2 \frac{d}{dx} (\sin x) + 3 \frac{d}{dx} (\cos x)$$

$$[\because \frac{d}{dx} (\sin x) = \cos x \text{ \& } \frac{d}{dx} (\cos x) = -\sin x]$$

$$= 2 \cos x - 3 \sin x$$

$$\therefore \frac{dy}{dx} = 2 \cos x - 3 \sin x$$

Differentiating again with respect to x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (2 \cos x - 3 \sin x) = \frac{2d}{dx} \cos x - 3 \frac{d}{dx} \sin x$$

$$\frac{d^2y}{dx^2} = -2 \sin x - 3 \cos x$$

From equation 1 we have :

$$y = 2 \sin x + 3 \cos x$$

$$\therefore \frac{d^2y}{dx^2} = -(2 \sin x + 3 \cos x) = -y$$

$$\therefore \frac{d^2y}{dx^2} + y = 0 \text{proved}$$

7. Question

If $y = \frac{\log x}{x}$, show that $\frac{d^2y}{dx^2} = \frac{2 \log x - 3}{x^3}$.

Answer

Basic idea:

✓Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v , i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

✓Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given, $y = \frac{\log x}{x}$ equation 1

As we have to prove : $\frac{d^2 y}{dx^2} = \frac{2 \log x - 3}{x^3}$..

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2 y}{dx^2}$

As $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

So, let's first find dy/dx and differentiate it again.

As y is the product of two functions u and v

Let $u = \log x$ and $v = 1/x$

Using product rule of differentiation:

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\frac{d}{dx} \left(\frac{\log x}{x} \right) = \log x \frac{d}{dx} \frac{1}{x} + \frac{1}{x} \frac{d}{dx} \log x$$

$$[\because \frac{d}{dx}(\log x) = \frac{1}{x} \text{ \& \; } \frac{d}{dx}(x^n) = nx^{n-1}]$$

$$\frac{dy}{dx} = -\frac{1}{x^2} \log x + \frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x^2} (1 - \log x)$$

Again using the product rule to find $\frac{d^2 y}{dx^2}$:

$$\frac{d^2 y}{dx^2} = (1 - \log x) \frac{d}{dx} \frac{1}{x^2} + \frac{1}{x^2} \frac{d}{dx} (1 - \log x)$$

$$[\because \frac{d}{dx}(\log x) = \frac{1}{x} \text{ \& \; } \frac{d}{dx}(x^n) = nx^{n-1}]$$

$$= -2 \left(\frac{1 - \log x}{x^3} \right) - \frac{1}{x^3}$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{2 \log x - 3}{x^3} \text{ proved}$$

8. Question

If $x = a \sec \theta$, $y = b \tan \theta$, prove that $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.

Answer

Idea of parametric form of differentiation:

If $y = f(\theta)$ and $x = g(\theta)$ i.e. y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

We can write : $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

$$x = a \sec \theta \dots\dots\text{equation 1}$$

$$y = b \tan \theta \dots\dots\text{equation 2}$$

to prove : $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a \sec \theta = a \sec \theta \tan \theta \dots\dots\text{equation 3}$$

$$\text{Similarly, } \frac{dy}{d\theta} = b \sec^2 \theta \dots\dots\text{equation 4}$$

$$[\because \frac{d}{dx} \sec x = \sec x \tan x, \frac{d}{dx} \tan x = \sec^2 x]$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b \sec^2 \theta}{a \sec \theta \tan \theta} = \frac{b}{a} \operatorname{cosec} \theta$$

Differentiating again w.r.t x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{b}{a} \operatorname{cosec} \theta \right)$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a} \operatorname{cosec} \theta \cot \theta \frac{d\theta}{dx} \dots\dots\text{equation 5 [using chain rule]}$$

From equation 3:

$$\frac{dx}{d\theta} = a \sec \theta \tan \theta$$

$$\therefore \frac{d\theta}{dx} = \frac{1}{a \sec \theta \tan \theta}$$

Putting the value in equation 5 :

$$\frac{d^2y}{dx^2} = -\frac{b}{a} \operatorname{cosec} \theta \cot \theta \frac{1}{a \sec \theta \tan \theta}$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a^2 \tan^3 \theta}$$

From equation 1:

$$y = b \tan \theta$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{-b}{\frac{a^2 y^3}{b^3}} = -\frac{b^4}{a^2 y^3} \dots \text{proved.}$$

9. Question

If $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$ prove that

$$\frac{d^2 x}{d\theta^2} = a(\cos \theta - \theta \sin \theta), \frac{d^2 y}{d\theta^2} = a(\sin \theta + \theta \cos \theta) \text{ and } \frac{d^2 y}{dx^2} = \frac{\sec^3 \theta}{a\theta}.$$

Answer

Basic idea:

✓ Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

✓ The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v , i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓ Product rule of differentiation- $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

✓ Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

The idea of parametric form of differentiation:

If $y = f(\theta)$ and $x = g(\theta)$, i.e. y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

$$\text{We can write : } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Given,

$$x = a(\cos \theta + \theta \sin \theta) \dots \text{equation 1}$$

$$y = a(\sin \theta - \theta \cos \theta) \dots \text{equation 2}$$

to prove :

$$\text{i) } \frac{d^2 x}{d\theta^2} = a(\cos \theta - \theta \sin \theta)$$

$$\text{ii) } \frac{d^2 y}{d\theta^2} = a(\sin \theta + \theta \cos \theta)$$

$$\text{iii) } \frac{d^2 y}{dx^2} = \frac{\sec^3 \theta}{a\theta}.$$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2 y}{dx^2}$

$$\text{As } \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a(\cos \theta + \theta \sin \theta)$$

$$= a(-\sin \theta + \theta \cos \theta + \sin \theta)$$

[differentiated using product rule for $\theta \sin \theta$]

$$= a\theta \cos \theta \text{ ..eqn 4}$$

Again differentiating w.r.t θ using product rule:-

$$\frac{d^2x}{d\theta^2} = a(-\theta \sin \theta + \cos \theta)$$

$$\therefore \frac{d^2x}{d\theta^2} = a(\cos \theta - \theta \sin \theta) \text{ proved (i)}$$

Similarly,

$$\frac{dy}{d\theta} = \frac{d}{d\theta} a(\sin \theta - \theta \cos \theta) = a \frac{d}{d\theta} \sin \theta - a \frac{d}{d\theta} (\theta \cos \theta)$$

$$= a \cos \theta + a\theta \sin \theta - a \cos \theta$$

$$\therefore \frac{dy}{d\theta} = a\theta \sin \theta \text{equation 5}$$

Again differentiating w.r.t θ using product rule:-

$$\frac{d^2x}{d\theta^2} = a(\theta \cos \theta + \sin \theta)$$

$$\therefore \frac{d^2x}{d\theta^2} = a(\sin \theta + \theta \cos \theta) \text{ proved (ii)}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Using equation 4 and 5 :

$$\frac{dy}{dx} = \frac{a\theta \sin \theta}{a\theta \cos \theta} = \tan \theta$$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$\therefore$ again differentiating w.r.t x :-

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \tan \theta$$

$$= \sec^2 \theta \frac{d\theta}{dx} \text{ [using chain rule]}$$

$$\therefore \frac{dx}{d\theta} = a\theta \cos \theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{a\theta \cos \theta}$$

Putting a value in the above equation-

We have :

$$\frac{d^2y}{dx^2} = \sec^2 \theta \times \frac{1}{a\theta \cos \theta}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^3 \theta}{a\theta} \text{ proved (iii)}$$

10. Question

If $y = e^x \cos x$, prove that $\frac{d^2y}{dx^2} = 2e^x \cos \left(x + \frac{\pi}{2} \right)$

Answer

Basic idea:

✓Second order derivative is nothing but derivative of derivative i.e. $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

✓The idea of chain rule of differentiation: If f is any real-valued function which is the composition of two functions u and v , i.e. $f = v(u(x))$. For the sake of simplicity just assume $t = u(x)$

Then $f = v(t)$. By chain rule, we can write the derivative of f w.r.t to x as:

$$\frac{df}{dx} = \frac{dv}{dt} \times \frac{dt}{dx}$$

✓Product rule of differentiation- $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

✓Apart from these remember the derivatives of some important functions like exponential, logarithmic, trigonometric etc..

Let's solve now:

Given,

$$y = e^x \cos x$$

TO prove :

$$\frac{d^2y}{dx^2} = 2e^x \cos \left(x + \frac{\pi}{2} \right)$$

Clearly from the expression to be proved we can easily observe that we need to just find the second derivative of given function.

Given, $y = e^x \cos x$

We have to find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So let's first find dy/dx and differentiate it again.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (e^x \cos x)$$

Let $u = e^x$ and $v = \cos x$

As, $y = u \cdot v$

$\therefore$ Using product rule of differentiation:

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = e^x \frac{d}{dx} (\cos x) + \cos x \frac{d}{dx} e^x$$

$$\frac{dy}{dx} = -e^x \sin x + e^x \cos x \left[\because \frac{d}{dx} (\cos x) = -\sin x \text{ \& } \frac{d}{dx} e^x = e^x \right]$$

Again differentiating w.r.t x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (-e^x \sin x + e^x \cos x)$$

$$= \frac{d}{dx} (-e^x \sin x) + \frac{d}{dx} (e^x \cos x)$$

Again using the product rule :

$$\frac{d^2y}{dx^2} = -e^x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}e^x + e^x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(e^x)$$

$$\frac{d^2y}{dx^2} = -e^x \cos x - e^x \sin x - e^x \sin x + e^x \cos x$$

$$[\because \frac{d}{dx}(\cos x) = -\sin x, \frac{d}{dx}e^{-x} = -e^{-x}]$$

$$\frac{d^2y}{dx^2} = -2e^x \sin x [\because -\sin x = \cos(x + \pi/2)]$$

$$\frac{d^2y}{dx^2} = -2e^x \cos(x + \frac{\pi}{2}) \dots \text{proved}$$

11. Question

If $x = a \cos \theta$, $y = b \sin \theta$, show that $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.

Answer

Idea of parametric form of differentiation:

If $y = f(\theta)$ and $x = g(\theta)$ i.e. y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

We can write : $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

$$x = a \cos \theta \dots \dots \text{equation 1}$$

$$y = b \sin \theta \dots \dots \text{equation 2}$$

$$\text{to prove : } \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a \cos \theta = -a \sin \theta \dots \dots \text{equation 3}$$

$$\text{Similarly, } \frac{dy}{d\theta} = b \cos \theta \dots \dots \text{equation 4}$$

$$[\because \frac{d}{dx} \cos x = -\sin x \tan x, \frac{d}{dx} \sin x = \cos x]$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{b \cos \theta}{a \sin \theta} = -\frac{b}{a} \cot \theta$$

Differentiating again w.r.t x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(-\frac{b}{a} \cot \theta \right)$$

$$\frac{d^2y}{dx^2} = \frac{b}{a} \operatorname{cosec}^2 \theta \frac{d\theta}{dx} \dots \text{equation 5}$$

[using chain rule and $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$]

From equation 3:

$$\frac{dx}{d\theta} = -a \sin \theta$$

$$\therefore \frac{d\theta}{dx} = \frac{-1}{a \sin \theta}$$

Putting the value in equation 5 :

$$\frac{d^2y}{dx^2} = -\frac{b}{a} \operatorname{cosec}^2 \theta \frac{1}{a \sin \theta}$$

$$\frac{d^2y}{dx^2} = \frac{-b}{a^2 \sin^3 \theta}$$

From equation 1:

$$y = b \sin \theta$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-b}{\frac{a^2 y^2}{b^2}} = -\frac{b^4}{a^2 y^3} \dots \text{proved.}$$

12. Question

If $x = a(1 - \cos^3 \theta)$, $y = a \sin^3 \theta$, Prove that $\frac{d^2y}{dx^2} = \frac{32}{27a}$ at $\theta = \frac{\pi}{6}$.

Answer

Idea of parametric form of differentiation:

If $y = f(\theta)$ and $x = g(\theta)$ i.e. y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

$$\text{We can write : } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Given,

$$x = a(1 - \cos^3 \theta) \dots \text{equation 1}$$

$$y = a \sin^3 \theta, \dots \text{equation 2}$$

$$\text{to prove : } \frac{d^2y}{dx^2} = \frac{32}{27a} \text{ at } \theta = \frac{\pi}{6}$$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a(1 - \cos^3 \theta) = 3a \cos^2 \theta \sin \theta \dots \text{equation 3 [using chain rule]}$$

Similarly,

$$\frac{dy}{d\theta} = \frac{d}{d\theta} a \sin^3 \theta = 3a \sin^2 \theta \cos \theta \dots \text{equation 4}$$

$$[\because \frac{d}{dx} \cos x = -\sin x \text{ \& } \frac{d}{dx} \sin x = \cos x]$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a \sin^2 \theta \cos \theta}{3a \cos^2 \theta \sin \theta} = \tan \theta$$

Differentiating again w.r.t x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (\tan \theta)$$

$$\frac{d^2y}{dx^2} = \sec^2 \theta \frac{d\theta}{dx} \dots \text{equation 5}$$

$$[\text{using chain rule and } \frac{d}{dx} \tan x = \sec^2 x]$$

From equation 3:

$$\frac{dx}{d\theta} = 3a \cos^2 \theta \sin \theta$$

$$\therefore \frac{d\theta}{dx} = \frac{1}{3a \cos^2 \theta \sin \theta}$$

Putting the value in equation 5 :

$$\frac{d^2y}{dx^2} = \sec^2 \theta \frac{1}{3a \cos^2 \theta \sin \theta}$$

$$\frac{d^2y}{dx^2} = \frac{1}{3a \cos^4 \theta \sin \theta}$$

Put $\theta = \pi/6$

$$\left(\frac{d^2y}{dx^2} \right) \text{ at } \left(x = \frac{\pi}{6} \right) = \frac{1}{3a \cos^4 \frac{\pi}{6} \sin \frac{\pi}{6}} = \frac{1}{3a \left(\frac{\sqrt{3}}{2} \right)^4 \frac{1}{2}}$$

$$\therefore \left(\frac{d^2y}{dx^2} \right) \text{ at } \left(x = \frac{\pi}{6} \right) = \frac{32}{27a} \dots \text{proved}$$

13. Question

If $x = a(\theta + \sin \theta)$, $y = a(1 + \cos \theta)$, prove that $\frac{d^2y}{dx^2} = -\frac{a}{y^2}$.

Answer

Idea of parametric form of differentiation:

If $y = f(\theta)$ and $x = g(\theta)$ i.e. y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

$$\text{We can write : } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Given,

$$x = a(\theta + \sin \theta) \dots \text{equation 1}$$

$$y = a(1 + \cos \theta) \dots \text{equation 2}$$

to prove : $\frac{d^2y}{dx^2} = -\frac{a}{y^2}$

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a(\theta + \sin \theta) = a(1 + \cos \theta) = y \quad [\because \text{from equation 2}] \dots \text{equation 3}$$

Similarly,

$$\frac{dy}{d\theta} = \frac{d}{d\theta} a(1 + \cos \theta) = -a \sin \theta \dots \text{equation 4}$$

$$[\because \frac{d}{dx} \cos x = -\sin x, \frac{d}{dx} \sin x = \cos x]$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a \sin \theta}{a(1 + \cos \theta)} = \frac{-\sin \theta}{(1 + \cos \theta)} = \frac{-\sin \theta}{y} \quad [\because \text{from equation 2}] \dots \text{equation 5}$$

Differentiating again w.r.t x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = -a \frac{d}{dx} \left(\frac{\sin \theta}{y} \right)$$

Using product rule and chain rule of differentiation together:

$$\frac{d^2y}{dx^2} = -a \left(\frac{\sin \theta}{-y^2} \frac{dy}{dx} + \frac{1}{y} \cos \theta \frac{d\theta}{dx} \right)$$

$$\frac{d^2y}{dx^2} = -a \left(\frac{\sin \theta (-\sin \theta)}{-y^2} + \frac{1}{y} \cos \theta \frac{1}{y} \right) [\text{using equation 3 and 5}]$$

$$\frac{d^2y}{dx^2} = -a \left(\frac{a \sin^2 \theta}{y^3} + \frac{1}{y^2} \cos \theta \right)$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2} \left(\frac{a \sin^2 \theta}{a(1 + \cos \theta)} + \cos \theta \right) [\text{from equation 1}]$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2} \left(\frac{1 - \cos^2 \theta}{(1 + \cos \theta)} + \cos \theta \right)$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2} \left(\frac{(1 - \cos \theta)(1 + \cos \theta)}{(1 + \cos \theta)} + \cos \theta \right)$$

$$\frac{d^2y}{dx^2} = -\frac{a}{y^2} (1 - \cos \theta + \cos \theta)$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{a}{y^2} \dots \text{proved}$$

14. Question

If $x = a(\theta - \sin \theta)$, $y = a(1 + \cos \theta)$ find $\frac{d^2y}{dx^2}$.

Answer

Idea of parametric form of differentiation:

If $y = f(\theta)$ and $x = g(\theta)$ i.e. y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

We can write : $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

$$x = a(\theta - \sin \theta) \dots \dots \text{equation 1}$$

$$y = a(1 + \cos \theta) \dots \dots \text{equation 2}$$

to find : $\frac{d^2y}{dx^2}$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a(\theta - \sin \theta) = a(1 - \cos \theta) \dots \dots \text{equation 3}$$

Similarly,

$$\frac{dy}{d\theta} = \frac{d}{d\theta} a(1 + \cos \theta) = -a \sin \theta \dots \dots \text{equation 4}$$

$$[\because \frac{d}{dx} \cos x = -\sin x, \frac{d}{dx} \sin x = \cos x]$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a \sin \theta}{a(1 - \cos \theta)} = \frac{-\sin \theta}{(1 - \cos \theta)} \dots \dots \text{equation 5}$$

Differentiating again w.r.t x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = - \frac{d}{dx} \left(\frac{\sin \theta}{1 - \cos \theta} \right)$$

Using product rule and chain rule of differentiation together:

$$\frac{d^2y}{dx^2} = \left\{ -\frac{1}{1 - \cos \theta} \frac{d}{d\theta} \sin \theta - \sin \theta \frac{d}{d\theta} \frac{1}{(1 - \cos \theta)} \right\} \frac{d\theta}{dx}$$

Apply chain rule to determine $\frac{d}{d\theta} \frac{1}{(1 - \cos \theta)}$

$$\frac{d^2y}{dx^2} = \left\{ \frac{-\cos \theta}{1 - \cos \theta} + \frac{\sin^2 \theta}{(1 - \cos \theta)^2} \right\} \frac{1}{a(1 - \cos \theta)} \text{ [using equation 3]}$$

$$\frac{d^2y}{dx^2} = \left\{ \frac{-\cos \theta(1 - \cos \theta) + \sin^2 \theta}{(1 - \cos \theta)^2} \right\} \frac{1}{a(1 - \cos \theta)}$$

$$\frac{d^2y}{dx^2} = \left\{ \frac{-\cos \theta + \cos^2 \theta + \sin^2 \theta}{(1 - \cos \theta)^2} \right\} \frac{1}{a(1 - \cos \theta)}$$

$$\frac{d^2y}{dx^2} = \left\{ \frac{1 - \cos \theta}{(1 - \cos \theta)^2} \right\} \frac{1}{a(1 - \cos \theta)} \text{ [} \because \cos^2 \theta + \sin^2 \theta = 1]$$

$$\frac{d^2y}{dx^2} = \frac{1}{a(1 - \cos \theta)^2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{a(2 \sin^2 \frac{\theta}{2})^2} \text{ [} \because 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}]$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{4a} \operatorname{cosec}^4 \frac{\theta}{2}$$

15. Question

If $x = a(1 - \cos \theta)$, $y = a(\theta + \sin \theta)$, prove that $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$

Answer

Idea of parametric form of differentiation:

If $y = f(\theta)$ and $x = g(\theta)$ i.e. y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

We can write : $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

$$y = a(\theta + \sin \theta) \text{equation 1}$$

$$x = a(1 - \cos \theta) \text{equation 2}$$

to prove : $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$.

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dy}{d\theta} = \frac{d}{d\theta} a(\theta + \sin \theta) = a(1 + \cos \theta) \text{equation 3}$$

Similarly,

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a(1 - \cos \theta) = a \sin \theta \text{equation 4}$$

$$[\because \frac{d}{dx} \cos x = -\sin x, \frac{d}{dx} \sin x = \cos x]$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(1 + \cos \theta)}{a \sin \theta} = \frac{(1 + \cos \theta)}{\sin \theta} \text{equation 5}$$

Differentiating again w.r.t x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{(1 + \cos \theta)}{\sin \theta} \right) = \frac{d}{dx} (1 + \cos \theta) \operatorname{cosec} \theta$$

Using product rule and chain rule of differentiation together:

$$\frac{d^2y}{dx^2} = \{ \operatorname{cosec} \theta \frac{d}{d\theta} (1 + \cos \theta) + (1 + \cos \theta) \frac{d}{d\theta} \operatorname{cosec} \theta \} \frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = \{ \operatorname{cosec} \theta (-\sin \theta) + (1 + \cos \theta) (-\operatorname{cosec} \theta \cot \theta) \} \frac{1}{a \sin \theta} \text{ [using equation 4]}$$

$$\frac{d^2y}{dx^2} = \{ -1 - \operatorname{cosec} \theta \cot \theta - \cot^2 \theta \} \frac{1}{a \sin \theta}$$

$$\text{As we have to find } \frac{d^2y}{dx^2} = -\frac{1}{a} \text{ at } \theta = \frac{\pi}{2}$$

$\therefore$ put $\theta = \pi/2$ in above equation:

$$\frac{d^2y}{dx^2} = \{-1 - \operatorname{cosec} \frac{\pi}{2} \cot \frac{\pi}{2} - \cot^2 \frac{\pi}{2}\} \frac{1}{a \sin \frac{\pi}{2}} = \frac{\{-1 - 0 - 0\}1}{a}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{a} \dots \text{ans}$$

16. Question

If $x = a(1 + \cos \theta)$, $y = a(\theta + \sin \theta)$ Prove that $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$.

Answer

Idea of parametric form of differentiation:

If $y = f(\theta)$ and $x = g(\theta)$ i.e. y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

$$\text{We can write : } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Given,

$$y = a(\theta + \sin \theta) \dots \dots \text{equation 1}$$

$$x = a(1 + \cos \theta) \dots \dots \text{equation 2}$$

$$\text{to prove : } \frac{d^2y}{dx^2} = -\frac{1}{a} \text{ at } \theta = \frac{\pi}{2}.$$

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

$$\text{Let's find } \frac{d^2y}{dx^2}$$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dy}{d\theta} = \frac{d}{d\theta} a(\theta + \sin \theta) = a(1 + \cos \theta) \dots \dots \text{equation 3}$$

Similarly,

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a(1 + \cos \theta) = -a \sin \theta \dots \dots \text{equation 4}$$

$$[\because \frac{d}{dx} \cos x = -\sin x, \frac{d}{dx} \sin x = \cos x]$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(1 + \cos \theta)}{-a \sin \theta} = -\frac{(1 + \cos \theta)}{\sin \theta} \dots \dots \text{equation 5}$$

Differentiating again w.r.t x :

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(-\frac{(1 + \cos \theta)}{\sin \theta} \right) = -\frac{d}{dx} (1 + \cos \theta) \operatorname{cosec} \theta$$

Using product rule and chain rule of differentiation together:

$$\frac{d^2y}{dx^2} = -\left\{ \operatorname{cosec} \theta \frac{d}{d\theta} (1 + \cos \theta) + (1 + \cos \theta) \frac{d}{d\theta} \operatorname{cosec} \theta \right\} \frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = -\left\{ \operatorname{cosec} \theta (-\sin \theta) + (1 + \cos \theta) (-\operatorname{cosec} \theta \cot \theta) \right\} \frac{1}{(-a \sin \theta)}$$

[using equation 4]

$$\frac{d^2y}{dx^2} = \{-1 - \operatorname{cosec}\theta \cot\theta - \cot^2\theta\} \frac{1}{a \sin\theta}$$

As we have to find $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$

$\therefore$ put $\theta = \pi/2$ in above equation:

$$\frac{d^2y}{dx^2} = \{-1 - \operatorname{cosec}\frac{\pi}{2} \cot\frac{\pi}{2} - \cot^2\frac{\pi}{2}\} \frac{1}{a \sin\frac{\pi}{2}} = \frac{\{-1 - 0 - 0\}1}{a}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{a}$$

17. Question

If $x = \cos \theta$, $y = \sin^3 \theta$. Prove that $y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3 \sin^2 \theta (5 \cos^2 \theta - 1)$

Answer

The idea of parametric form of differentiation:

If $y = f(\theta)$ and $x = g(\theta)$, i.e. y is a function of θ and x is also some other function of θ .

Then $dy/d\theta = f'(\theta)$ and $dx/d\theta = g'(\theta)$

We can write : $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

Given,

$$y = \sin^3 \theta \dots\dots \text{equation 1}$$

$$x = \cos \theta \dots\dots \text{equation 2}$$

$$\text{To prove: } y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3 \sin^2 \theta (5 \cos^2 \theta - 1)$$

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx using parametric form and differentiate it again.

$$\frac{dx}{d\theta} = -\sin \theta \dots\dots\dots \text{equation 3}$$

Applying chain rule to differentiate $\sin^3 \theta$:

$$\frac{dy}{d\theta} = 3 \sin^2 \theta \cos \theta \dots\dots\dots \text{equation 4}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3 \sin^2 \theta \cos \theta}{-\sin \theta} = -3 \sin \theta \cos \theta \dots\dots\dots \text{equation 5}$$

Again differentiating w.r.t x :

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} (-3 \sin \theta \cos \theta)$$

Applying product rule and chain rule to differentiate:

$$\frac{d^2y}{dx^2} = -3\left\{\sin \theta \frac{d}{d\theta} \cos \theta + \cos \theta \frac{d}{d\theta} \sin \theta\right\} \frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = 3\{-\sin^2 \theta + \cos^2 \theta\} \frac{1}{\sin \theta}$$

[using equation 3 to put the value of $d\theta/dx$]

Multiplying y both sides to approach towards the expression we want to prove-

$$y \frac{d^2y}{dx^2} = 3\{-\sin^2 \theta + \cos^2 \theta\} \frac{y}{\sin \theta}$$

$$y \frac{d^2y}{dx^2} = 3\{-\sin^2 \theta + \cos^2 \theta\} \sin^2 \theta$$

[from equation 1, substituting for y]

Adding equation 5 after squaring it:

$$y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\{-\sin^2 \theta + \cos^2 \theta\} \sin^2 \theta + 9 \sin^2 \theta \cos^2 \theta$$

$$y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3 \sin^2 \theta \{-\sin^2 \theta + \cos^2 \theta + 3 \cos^2 \theta\}$$

$$y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3 \sin^2 \theta \{5 \cos^2 \theta - 1\} \dots \dots \text{proved}$$

18. Question

If $y = \sin(\sin x)$, prove that : $\frac{d^2y}{dx^2} + \tan x \cdot \frac{dy}{dx} + y \cos^2 x = 0$

Answer

Given,

$$y = \sin(\sin x) \dots \dots \text{equation 1}$$

$$\text{To prove: } \frac{d^2y}{dx^2} + \tan x \cdot \frac{dy}{dx} + y \cos^2 x = 0$$

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx

$$\frac{dy}{dx} = \frac{d}{dx} \sin(\sin x)$$

Using chain rule, we will differentiate the above expression

$$\text{Let } t = \sin x \Rightarrow \frac{dt}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}$$

$$\frac{dy}{dx} = \cos t \cos x = \cos(\sin x) \cos x \dots\dots \text{equation 2}$$

Again differentiating with respect to x applying product rule:

$$\frac{d^2y}{dx^2} = \cos x \frac{d}{dx} \cos(\sin x) + \cos(\sin x) \frac{d}{dx} \cos x$$

Using chain rule again in the next step-

$$\frac{d^2y}{dx^2} = -\cos x \cos x \sin(\sin x) - \sin x \cos(\sin x)$$

$$\frac{d^2y}{dx^2} = -y \cos^2 x - \tan x \cos x \cos(\sin x)$$

[using equation 1 : $y = \sin(\sin x)$]

And using equation 2, we have:

$$\frac{d^2y}{dx^2} = -y \cos^2 x - \tan x \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} + y \cos^2 x + \tan x \frac{dy}{dx} = 0 \dots\dots \text{proved}$$

19. Question

If $y = (\sin^{-1} x)^2$, prove that: $(1-x^2) y_2 - xy_1 - 2=0$

Answer

Note: y_2 represents second order derivative i.e. $\frac{d^2y}{dx^2}$ and $y_1 = \frac{dy}{dx}$

Given,

$$y = (\sin^{-1} x)^2 \dots\dots \text{equation 1}$$

to prove : $(1-x^2) y_2 - xy_1 - 2=0$

We notice a second-order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx} (\sin^{-1} x)^2$$

Using chain rule we will differentiate the above expression

$$\text{Let } t = \sin^{-1} x \Rightarrow \frac{dt}{dx} = \frac{1}{\sqrt{1-x^2}} \text{ [using formula for derivative of } \sin^{-1} x]$$

$$\text{And } y = t^2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}$$

$$\frac{dy}{dx} = 2t \frac{1}{\sqrt{1-x^2}} = 2 \sin^{-1} x \frac{1}{\sqrt{1-x^2}} \dots\dots \text{equation 2}$$

Again differentiating with respect to x applying product rule:

$$\frac{d^2y}{dx^2} = 2 \sin^{-1} x \frac{d}{dx} \left(\frac{1}{\sqrt{1-x^2}} \right) + \frac{2}{\sqrt{1-x^2}} \frac{d}{dx} \sin^{-1} x$$

$$\frac{d^2y}{dx^2} = -\frac{2 \sin^{-1} x}{2(1-x^2)\sqrt{1-x^2}} (-2x) + \frac{2}{(1-x^2)} \left[\text{using } \frac{d}{dx} (x^n) = nx^{n-1} \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \right]$$

$$\frac{d^2y}{dx^2} = \frac{2x \sin^{-1} x}{(1-x^2)\sqrt{1-x^2}} + \frac{2}{(1-x^2)}$$

$$(1-x^2) \frac{d^2y}{dx^2} = 2 - \frac{2x \sin^{-1} x}{\sqrt{1-x^2}}$$

Using equation 2 :

$$(1-x^2) \frac{d^2y}{dx^2} = 2 - x \frac{dy}{dx}$$

$$\therefore (1-x^2) y_2 - x y_1 - 2 = 0 \dots\dots \text{proved}$$

20. Question

If $y = (\sin^{-1} x)^2$, prove that: $(1-x^2) y_2 - x y_1 - 2 = 0$

Answer

Note: y_2 represents second order derivative i.e. $\frac{d^2y}{dx^2}$ and $y_1 = \frac{dy}{dx}$

Given,

$$y = (\sin^{-1} x)^2 \dots\dots \text{equation 1}$$

$$\text{to prove : } (1-x^2) y_2 - x y_1 - 2 = 0$$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx} (\sin^{-1} x)^2$$

Using chain rule we will differentiate the above expression

$$\text{Let } t = \sin^{-1} x \Rightarrow \frac{dt}{dx} = \frac{1}{\sqrt{1-x^2}} \left[\text{using formula for derivative of } \sin^{-1} x \right]$$

$$\text{And } y = t^2$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}$$

$$\frac{dy}{dx} = 2t \frac{1}{\sqrt{1-x^2}} = 2 \sin^{-1} x \frac{1}{\sqrt{1-x^2}} \dots\dots \text{equation 2}$$

Again differentiating with respect to x applying product rule:

$$\frac{d^2y}{dx^2} = 2 \sin^{-1} x \frac{d}{dx} \left(\frac{1}{\sqrt{1-x^2}} \right) + \frac{2}{\sqrt{1-x^2}} \frac{d}{dx} \sin^{-1} x$$

$$\frac{d^2y}{dx^2} = -\frac{2 \sin^{-1} x}{2(1-x^2)\sqrt{1-x^2}} (-2x) + \frac{2}{(1-x^2)} \left[\text{using } \frac{d}{dx} (x^n) = nx^{n-1} \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \right]$$

$$\frac{d^2y}{dx^2} = \frac{2x \sin^{-1} x}{(1-x^2)\sqrt{1-x^2}} + \frac{2}{(1-x^2)}$$

$$(1-x^2) \frac{d^2y}{dx^2} = 2 + \frac{2x \sin^{-1} x}{\sqrt{1-x^2}}$$

Using equation 2 :

$$(1-x^2) \frac{d^2y}{dx^2} = 2 + x \frac{dy}{dx}$$

$$\therefore (1-x^2) y_2 - x y_1 - 2 = 0 \dots\dots \text{proved}$$

21. Question

If $y = e^{\tan^{-1}x}$, Prove that: $(1+x^2)y_2 + (2x-1)y_1 = 0$

Answer

Note: y_2 represents second order derivative i.e. $\frac{d^2y}{dx^2}$ and $y_1 = \frac{dy}{dx}$

Given,

$$y = e^{\tan^{-1}x} \dots\dots \text{equation 1}$$

to prove : $(1+x^2)y_2 + (2x-1)y_1 = 0$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx

$$\frac{dy}{dx} = \frac{d}{dx} e^{\tan^{-1}x}$$

Using chain rule we will differentiate the above expression

$$\text{Let } t = \tan^{-1}x \Rightarrow \frac{dt}{dx} = \frac{1}{1+x^2} \left[\frac{d}{dx} \tan^{-1}x = \frac{1}{1+x^2} \right]$$

$$\text{And } y = e^t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}$$

$$\frac{dy}{dx} = e^t \frac{1}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2} \dots\dots \text{equation 2}$$

Again differentiating with respect to x applying product rule:

$$\frac{d^2y}{dx^2} = e^{\tan^{-1}x} \frac{d}{dx} \left(\frac{1}{1+x^2} \right) + \frac{1}{1+x^2} \frac{d}{dx} e^{\tan^{-1}x}$$

Using chain rule we will differentiate the above expression-

$$\frac{d^2y}{dx^2} = \left(\frac{e^{\tan^{-1}x}}{(1+x^2)^2} \right) - \frac{2xe^{\tan^{-1}x}}{(1+x^2)^2} \left[\text{using equation 2 ; } \frac{d}{dx} (x^n) = nx^{n-1} \text{ \& } \frac{d}{dx} \tan^{-1}x = \frac{1}{1+x^2} \right]$$

$$(1+x^2) \frac{d^2y}{dx^2} = \frac{e^{\tan^{-1}x}}{1+x^2} - \frac{2xe^{\tan^{-1}x}}{1+x^2}$$

$$(1+x^2)\frac{d^2y}{dx^2} = \frac{e^{\tan^{-1}x}}{1+x^2} (1-2x)$$

Using equation 2 :

$$(1+x^2)\frac{d^2y}{dx^2} = \frac{dy}{dx} (1-2x)$$

$$\therefore (1+x^2)y_2 + (2x-1)y_1 = 0 \dots\dots\text{proved}$$

22. Question

If $y = 3 \cos (\log x) + 4 \sin (\log x)$, prove that: $x^2y_2 + xy_1 + y = 0$.

Answer

Note: y_2 represents second order derivative i.e. $\frac{d^2y}{dx^2}$ and $y_1 = dy/dx$

Given,

$$y = 3 \cos (\log x) + 4 \sin (\log x) \dots\dots\text{equation 1}$$

$$\text{to prove: } x^2y_2 + xy_1 + y = 0$$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx

$$\frac{dy}{dx} = \frac{d}{dx} (3 \cos(\log x) + 4 \sin(\log x))$$

$$\text{Let, } \log x = t$$

$$\therefore y = 3 \cos t + 4 \sin t \dots\dots\dots\text{equation 2}$$

$$\frac{dy}{dt} = -3 \sin t + 4 \cos t$$

$$\frac{dt}{dx} = \frac{1}{x} \dots\dots\dots\text{equation 3}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}$$

$$\frac{dy}{dx} = (-3 \sin t + 4 \cos t) \frac{1}{x} \dots\dots\dots\text{equation 4}$$

Again differentiating w.r.t x :

Using product rule of differentiation we have

$$\frac{d^2y}{dx^2} = (-3 \sin t + 4 \cos t) \frac{d}{dx} \frac{1}{x} + \frac{1}{x} \frac{d}{dx} (-3 \sin t + 4 \cos t)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2} (-3 \sin t + 4 \cos t) + \frac{1}{x} \frac{dt}{dx} (-3 \cos t - 4 \sin t)$$

Using equation 2,3 and 4 we can substitute above equation as:

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2} x \frac{dy}{dx} + \frac{1}{x} \frac{1}{x} (-y)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2}$$

Multiplying x^2 both sides:

$$x^2 \frac{d^2y}{dx^2} = -x \frac{dy}{dx} - y$$

$$\therefore x^2 y_2 + x y_1 + y = 0 \dots\dots\dots \text{proved}$$

23. Question

If $y = e^{2x}(ax + b)$, show that $y_2 - 4y_1 + 4y = 0$.

Answer

Note: y_2 represents second order derivative i.e. $\frac{d^2y}{dx^2}$ and $y_1 = \frac{dy}{dx}$

Given,

$$y = e^{2x}(ax + b) \dots\dots \text{equation 1}$$

to prove: $y_2 - 4y_1 + 4y = 0$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find $\frac{dy}{dx}$

$$\therefore y = e^{2x}(ax + b)$$

Using product rule to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = e^{2x} \frac{dy}{dx} (ax + b) + (ax + b) \frac{d}{dx} e^{2x}$$

$$\frac{dy}{dx} = ae^{2x} + 2(ax + b)e^{2x}$$

$$\frac{dy}{dx} = e^{2x}(a + 2ax + 2b) \dots\dots\dots \text{equation 2}$$

Again differentiating w.r.t x using product rule:

$$\frac{d^2y}{dx^2} = e^{2x} \frac{dy}{dx} (a + 2ax + 2b) + (a + 2ax + 2b) \frac{d}{dx} e^{2x}$$

$$\frac{d^2y}{dx^2} = 2ae^{2x} + 2(a + 2ax + 2b)e^{2x} \dots\dots\dots \text{equation 3}$$

In order to prove the expression try to get the required form:

Subtracting 4*equation 2 from equation 3:

$$\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} = 2ae^{2x} + 2(a + 2ax + 2b)e^{2x} - 4e^{2x}(a + 2ax + 2b)$$

$$\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} = 2ae^{2x} - 2e^{2x}(a + 2ax + 2b)$$

$$\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} = -4e^{2x}(ax + b)$$

Using equation 1:

$$\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} = -4y$$

$$\therefore y_2 - 4y_1 + 4y = 0 \dots\dots\dots \text{proved}$$

24. Question

If $x = \sin\left(\frac{1}{a} \log y\right)$, show that $(1-x^2)y_2 - xy_1 - a^2 y = 0$

Answer

Note: y_2 represents second order derivative i.e. $\frac{d^2y}{dx^2}$ and $y_1 = dy/dx$

Given,

$$x = \sin\left(\frac{1}{a} \log y\right)$$

$$(\log y) = a \sin^{-1} x$$

$$y = e^{a \sin^{-1} x} \dots\dots\dots \text{equation 1}$$

$$\text{to prove: } (1-x^2)y_2 - xy_1 - a^2 y = 0$$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As, } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find dy/dx

$$\therefore y = e^{a \sin^{-1} x}$$

$$\text{Let } t = a \sin^{-1} x \Rightarrow \frac{dt}{dx} = \frac{a}{\sqrt{1-x^2}} \left[\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \right]$$

$$\text{And } y = e^t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}$$

$$\frac{dy}{dx} = e^t \frac{a}{\sqrt{1-x^2}} = \frac{ae^{a \sin^{-1} x}}{\sqrt{1-x^2}} \dots\dots\dots \text{equation 2}$$

Again differentiating with respect to x applying product rule:

$$\frac{d^2y}{dx^2} = ae^{a \sin^{-1} x} \frac{d}{dx} \left(\frac{1}{\sqrt{1-x^2}} \right) + \frac{a}{\sqrt{1-x^2}} \frac{d}{dx} e^{a \sin^{-1} x}$$

Using chain rule and equation 2:

$$\frac{d^2y}{dx^2} = -\frac{ae^{a \sin^{-1} x}}{2(1-x^2)\sqrt{1-x^2}} (-2x) + \frac{a^2 e^{a \sin^{-1} x}}{(1-x^2)} \left[\text{using } \frac{d}{dx} (x^n) = nx^{n-1} \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \right]$$

$$\frac{d^2y}{dx^2} = \frac{xae^{a \sin^{-1} x}}{(1-x^2)\sqrt{1-x^2}} + \frac{a^2 e^{a \sin^{-1} x}}{(1-x^2)}$$

$$(1-x^2) \frac{d^2y}{dx^2} = a^2 e^{a \sin^{-1} x} + \frac{xae^{a \sin^{-1} x}}{\sqrt{1-x^2}}$$

Using equation 1 and equation 2 :

$$(1-x^2)\frac{d^2y}{dx^2} = a^2y + x\frac{dy}{dx}$$

$$\therefore (1-x^2)y_2 - xy_1 - a^2y = 0 \dots \text{proved}$$

25. Question

If $\log y = \tan^{-1} X$, show that : $(1+x^2)y_2 + (2x-1)y_1 = 0$.

Answer

Note: y_2 represents second order derivative i.e. $\frac{d^2y}{dx^2}$ and $y_1 = \frac{dy}{dx}$

Given,

$$\log y = \tan^{-1} X$$

$$\therefore y = e^{\tan^{-1} x} \dots \text{equation 1}$$

to prove : $(1+x^2)y_2 + (2x-1)y_1 = 0$

We notice a second order derivative in the expression to be proved so first take the step to find the second order derivative.

Let's find $\frac{d^2y}{dx^2}$

$$\text{As } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

So, let's first find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx} e^{\tan^{-1} x}$$

Using chain rule, we will differentiate the above expression

$$\text{Let } t = \tan^{-1} x \Rightarrow \frac{dt}{dx} = \frac{1}{1+x^2} \left[\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \right]$$

$$\text{And } y = e^t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}$$

$$\frac{dy}{dx} = e^t \frac{1}{1+x^2} = \frac{e^{\tan^{-1} x}}{1+x^2} \dots \text{equation 2}$$

Again differentiating with respect to x applying product rule:

$$\frac{d^2y}{dx^2} = e^{\tan^{-1} x} \frac{d}{dx} \left(\frac{1}{1+x^2} \right) + \frac{1}{1+x^2} \frac{d}{dx} e^{\tan^{-1} x}$$

Using chain rule we will differentiate the above expression-

$$\frac{d^2y}{dx^2} = \left(\frac{e^{\tan^{-1} x}}{(1+x^2)^2} \right) - \frac{2xe^{\tan^{-1} x}}{(1+x^2)^2}$$

$$[\text{using equation 2 ; } \frac{d}{dx} (x^n) = nx^{n-1} \text{ \& } \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}]$$

$$(1+x^2) \frac{d^2y}{dx^2} = \frac{e^{\tan^{-1} x}}{1+x^2} - \frac{2xe^{\tan^{-1} x}}{1+x^2}$$

$$(1+x^2) \frac{d^2y}{dx^2} = \frac{e^{\tan^{-1} x}}{1+x^2} (1-2x)$$

Using equation 2 :

$$(1+x^2)\frac{d^2y}{dx^2} = \frac{dy}{dx} (1-2x)$$

$\therefore (1+x^2)y_2 + (2x-1)y_1 = 0$ proved

MCQ

1. Question

Write the correct alternative in the following:

If $x = a \cos nt - b \sin nt$, then $\frac{d^2x}{dt^2}$ is

- A. n^2x
- B. $-n^2x$
- C. $-nx$
- D. nx

Answer

Given:

$$x = a \cos nt - b \sin nt$$

$$\frac{dx}{dt} = -an \sin nt - bn \cos nt$$

$$\frac{d^2x}{dt^2} = -an^2 \cos nt + bn^2 \sin nt$$

$$= -n^2 (a \cos nt - b \sin nt)$$

$$= -n^2 x$$

2. Question

Write the correct alternative in the following:

If $x = at^2$, $y = 2at$, then $\frac{d^2y}{dx^2} =$

- A. $-\frac{1}{t^2}$
- B. $\frac{1}{2at^3}$
- C. $-\frac{1}{t^3}$
- D. $-\frac{1}{2at^3}$

Answer

Given:

$$y = 2at, x = at^2$$

$$\frac{dx}{dt} = 2at; \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{-1}{t^2}}{2at}$$

$$= \frac{-1}{2at^3}$$

3. Question

Write the correct alternative in the following:

If $y = ax^{n+1} + bx^{-n}$, then $x^2 \frac{d^2y}{dx^2} =$

- A. $n(n-1)y$
- B. $n(n+1)y$
- C. ny
- D. n^2y

Answer

Given:

$$y = ax^{n+1} + bx^{-n}$$

$$\frac{dy}{dx} = (n+1)ax^n + (-n)bx^{-n-1}$$

$$\frac{d^2y}{dx^2} = n(n+1)ax^{n-1} + (-n)(-n-1)bx^{-n-2}$$

$$x^2 \frac{d^2y}{dx^2} = x^2 \{n(n+1)ax^{n-1} + (-n)(-n-1)bx^{-n-2}\}$$

$$= n(n+1)a x^{n-1+2} + n(n+1)bx^{-n-2+2}$$

$$= n(n+1)[a x^{n+1} + bx^{-n}]$$

$$= n(n+1)y$$

4. Question

Write the correct alternative in the following:

$$\frac{d^{20}}{dx^{20}} (2 \cos x \cos 3x) =$$

- A. $2^{20}(\cos 2x - 2^{20} \cos 4x)$

B. $2^{20}(\cos 2x + 2^{20} \cos 4x)$

C. $2^{20}(\sin 2x - 2^{20} \sin 4x)$

D. $2^{20}(\sin 2x - 2^{20} \sin 4x)$

Answer

Given:

Let $y = 2 \cos x \cos 3x$

$$2 \cos A \cos B = \cos\left(\frac{A+B}{2}\right) + \cos\left(\frac{A-B}{2}\right)$$

So $y = \cos 2x + \cos 4x$

$$\frac{dy}{dx} = -2 \sin 2x - 4 \sin 4x$$

$$= (-2)^1 (\sin 2x + 2^1 \sin 4x)$$

$$\frac{d^2y}{dx^2} = -4 \cos 2x - 16 \cos 4x$$

$$= (-2)^2 (\cos 2x + 2^2 \cos 4x)$$

$$\frac{d^3y}{dx^3} = 8 \sin 2x + 64 \sin 4x$$

$$= (-2)^3 (\cos 2x + 2^3 \cos 4x)$$

$$\frac{d^4y}{dx^4} = 16 \cos 2x + 256 \cos 4x$$

$$= (-2)^4 (\cos 2x + 2^4 \cos 4x)$$

For every odd degree; differential $= (-2)^n (\cos 2x + 2^n \cos 4x)$; $n = \{1, 3, 5, \dots\}$

For every even degree; differential $= (-2)^n (\cos 2x + 2^n \cos 4x)$; $n = \{0, 2, 4, \dots\}$

So, $\frac{d^{20}y}{dx^{20}} = (-2)^{20}(\cos 2x + 2^{20} \cos 4x)$

$$= (-2)^{20} (\cos 2x + 2^{20} \cos 4x);$$

5. Question

Write the correct alternative in the following:

If $x = t^2$, $y = t^3$, then $\frac{d^2y}{dx^2} =$

A. $\frac{3}{2}$

B. $\frac{3}{4t}$

C. $\frac{3}{2t}$

D. $\frac{3t}{2}$

Answer

Given:

$$x = t^2; y = t^3$$

$$\frac{dy}{dt} = 3t^2; \frac{dx}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t}{2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}}{2t}$$

$$= \frac{3}{4}$$

6. Question

Write the correct alternative in the following:

If $y = a + bx^2$, a, b arbitrary constants, then

A. $\frac{d^2y}{dx^2} = 2xy$

B. $x \frac{d^2y}{dx^2} = y_1$

C. $x \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0$

D. $x \frac{d^2y}{dx^2} = 2xy$

Answer

Given:

$$y = a + bx^2$$

$$\frac{dy}{dx} = 2bx$$

$$\frac{d^2y}{dx^2} = 2b \neq 2xy$$

$$x \frac{d^2y}{dx^2} = 2bx$$

$$= \frac{dy}{dx}$$

$$x \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 2bx - 2bx + y$$

$$= y$$

7. Question

Write the correct alternative in the following:

If $f(x) = (\cos x + i \sin x) (\cos 2x + i \sin 2x) (\cos 3x + i \sin 3x) \dots (\cos nx + i \sin nx)$ and $f(1) = 1$, then $f''(1)$ is equal to

A. $\frac{n(n+1)}{2}$

B. $\left\{ \frac{n(n+1)}{2} \right\}^2$

C. $-\left\{ \frac{n(n+1)}{2} \right\}^2$

D. none of these

Answer

Given:

$$f(x) = (\cos x + i \sin x) (\cos 2x + i \sin 2x) (\cos 3x + i \sin 3x) \dots (\cos nx + i \sin nx)$$

Since $e^{ix} = \cos x + i \sin x$

So, $f(x) = e^{ix} \times e^{i2x} \times e^{i3x} \times e^{i4x} \times \dots \times e^{inx}$

$$f(x) = e^{ix(1+2+3+4+\dots+n)}$$

$$= e^{ix \frac{n(n+1)}{2}}$$

$$f(1) = e^{i \frac{n(n+1)}{2}}$$

$$f'(x) = ix \frac{n(n+1)}{2} e^{ix \frac{n(n+1)}{2}}$$

$$f''(x) = i^2 x^2 \left(\frac{n(n+1)}{2} \right)^2 e^{ix \frac{n(n+1)}{2}}$$

$$f''(x) = -x^2 \left(\frac{n(n+1)}{2} \right)^2 e^{ix \frac{n(n+1)}{2}}$$

$$f''(1) = -1^2 \left(\frac{n(n+1)}{2} \right)^2 \times 1$$

$$= -\left(\frac{n(n+1)}{2} \right)^2$$

8. Question

Write the correct alternative in the following:

If $y = a \sin mx + b \cos mx$, then $\frac{d^2y}{dx^2}$ is equal to

A. $-m^2y$

B. m^2y

C. $-my$

D. my

Answer

Given:

$$y = a \sin mx + b \cos mx$$

$$\frac{dy}{dx} = ma \cos mx - mb \sin mx$$

$$\frac{d^2y}{dx^2} = -m^2 a \sin mx - m^2 b \cos mx$$

$$= -m^2 [a \sin mx + b \cos mx]$$

$$= -m^2 y$$

9. Question

Write the correct alternative in the following:

$$\text{If } f(x) = \frac{\sin^{-1} x}{\sqrt{(1-x^2)}} \text{ then } (1-x^2) f'(x) - xf(x) =$$

- A. 1
- B. -1
- C. 0
- D. none of these

Answer

Given:

$$y = f(x) = \frac{\sin^{-1} x}{\sqrt{(1-x^2)}}$$

$$\frac{dy}{dx} = \frac{1}{(\sqrt{(1-x^2)})^2} \left\{ \frac{1}{\sqrt{(1-x^2)}} \sqrt{(1-x^2)} - \sin^{-1} x \frac{(-2x)}{2\sqrt{(1-x^2)}} \right\}$$

$$= \frac{1}{(\sqrt{(1-x^2)})^2} \left\{ 1 + \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}} \right\}$$

$$= \frac{1 + xy}{(1-x^2)}$$

$$f'(x) = \frac{1 + xf(x)}{(1-x^2)}$$

$$(1-x^2)f'(x) = 1 + xf(x)$$

$$(1-x^2)f'(x) - xf(x) = 1$$

10. Question

Write the correct alternative in the following:

$$\text{If } y = \tan^{-1} \left\{ \frac{\log_e (e/x^2)}{\log_e (ex^2)} \right\} + \tan^{-1} \left(\frac{3 + 2 \log_e x}{1 - 6 \log_e x} \right), \text{ then } \frac{d^2y}{dx^2} =$$

- A. 2
- B. 1
- C. 0
- D. -1

Answer

Given:

$$y = \tan^{-1} \left\{ \frac{\log_e \left(\frac{e}{x^2} \right)}{\log_e (ex^2)} \right\} + \tan^{-1} \left\{ \frac{3 + 2 \log_e x}{1 - 6 \log_e x} \right\}$$

$$y = \tan^{-1} \left\{ \frac{\log_e e - \log_e x^2}{\log_e e + \log_e x^2} \right\} + \tan^{-1} \left\{ \frac{3 \log_e e + 2 \log_e x}{1 - 3 \log_e e \times 2 \log_e x} \right\}$$

$$y = \tan^{-1} \left\{ \frac{1 - \log_e x^2}{1 + \log_e x^2} \right\} + \tan^{-1}(3 \log_e e) + \tan^{-1}(2 \log_e x)$$

$$y = \tan^{-1} \left\{ \frac{\log_e e - 2 \log_e x}{1 + \log_e e \times 2 \log_e x} \right\} + \tan^{-1}(3 \log_e e) + \tan^{-1}(2 \log_e x)$$

$$y = \tan^{-1}(\log_e e) - \tan^{-1}(2 \log_e x) + \tan^{-1}(3 \log_e e) + \tan^{-1}(2 \log_e x)$$

$$y = \tan^{-1}(1) + \tan^{-1}(3)$$

$$y = \tan^{-1} \left(\frac{1+3}{1-3} \right) = \tan^{-1}(-2)$$

$$\frac{dy}{dx} = 0$$

11. Question

Write the correct alternative in the following:

Let $f(x)$ be a polynomial. Then, the second order derivative of $f(e^x)$ is

- A. $f''(e^x) e^{2x} + f'(e^x) e^x$
- B. $f''(e^x) e^x + f'(e^x)$
- C. $f''(e^x) e^{2x} + f''(e^x) e^x$
- D. $f''(e^x)$

Answer

Given:

$$\frac{d}{dx} \left[\frac{d}{dx} f(e^x) \right] = ?$$

$$\text{Since, } \frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

$$\text{So, } \frac{d}{dx} f(e^x) = f'(e^x)e^x$$

$$\text{Also, } \frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$$

$$\text{So, } \frac{d}{dx} f'(e^x)e^x = f''(x)e^x e^x + e^x f'(x)$$

$$= f''(x)e^{2x} + e^x f'(x)$$

12. Question

Write the correct alternative in the following:

If $y = a \cos(\log_e x) + b \sin(\log_e x)$, then $x^2 y_2 + xy_1 =$

- A. 0

- B. y
 C. $-y$
 D. none of these

Answer

Given:

$$y = a \cos(\log_e x) + b \sin(\log_e x)$$

$$\frac{dy}{dx} = -a \sin(\log_e x) \frac{1}{x} + b \cos(\log_e x) \frac{1}{x}$$

$$xy_1 = -a \sin(\log_e x) + b \cos(\log_e x)$$

$$\frac{d^2y}{dx^2} = -a \cos(\log_e x) \frac{1}{x^2} + \frac{1}{x^2} a \sin(\log_e x) - b \sin(\log_e x) \frac{1}{x^2} + b \cos(\log_e x) \frac{1}{x^2}$$

$$x^2 y_2 = -a \cos(\log_e x) + a \sin(\log_e x) - b \sin(\log_e x) - b \cos(\log_e x)$$

$$x^2 y_2 + xy_1 = -a \cos(\log_e x) + a \sin(\log_e x) - b \sin(\log_e x) - b \cos(\log_e x) - a \sin(\log_e x) + b \cos(\log_e x)$$

$$= -a \sin(\log_e x) - b \cos(\log_e x)$$

$$= -y$$

13. Question

Write the correct alternative in the following:

If $x = 2at$, $y = at^2$, where a is a constant, then $\frac{d^2y}{dx^2}$ at $x = \frac{1}{2}$ is

- A. $1/2a$
 B. 1
 C. $2a$
 D. none of these

Answer

Given:

$$x = 2at, y = at^2$$

$$\frac{dx}{dt} = 2a; \frac{dy}{dt} = 2at$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{1}{2a}$$

14. Question

Write the correct alternative in the following:

If $x = f(t)$ and $y = g(t)$, then $\frac{d^2y}{dx^2}$ is equal to

A. $\frac{f'g'' - g'f''}{(f')^3}$

B. $\frac{f'g'' - g'f''}{(f')^2}$

C. $\frac{g''}{f''}$

D. $\frac{f''g' - g''f'}{(g')^3}$

Answer

Given:

$$x = f(t) \text{ and } y = g(t)$$

$$\frac{dx}{dt} = f'(t); \frac{dy}{dt} = g'(t)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{1}{f'(t)} \left\{ \frac{1}{(f'(t))^2} (g''(t)f'(t) - f''(t)g'(t)) \right\}$$

$$= \frac{(g''(t)f'(t) - f''(t)g'(t))}{(f'(t))^3}$$

15. Question

Write the correct alternative in the following:

If $y = \sin(m \sin^{-1} x)$, then $(1 - x^2) y_2 - xy_1$ is equal to

A. m^2y

B. my

C. $-m^2y$

D. none of these

Answer

Given:

$$y = \sin(m \sin^{-1} x)$$

$$\frac{dy}{dx} = m \cos(m \sin^{-1} x) \frac{1}{\sqrt{1 - x^2}}$$

$$x \frac{dy}{dx} = \cos(m \sin^{-1} x) \frac{mx}{\sqrt{1 - x^2}}$$

$$\begin{aligned}
& \frac{d^2y}{dx^2} \\
&= m \left\{ \frac{-m \sin(m \sin^{-1} x) \sqrt{1-x^2} \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{1-x^2}} (-2x) \cos(m \sin^{-1} x)}{(\sqrt{1-x^2})^2} \right\} \\
&= \frac{m}{(1-x^2)} \left\{ -m \sin(m \sin^{-1} x) + \frac{x}{\sqrt{1-x^2}} \cos(m \sin^{-1} x) \right\} \\
(1-x^2) y_2 &= m \left\{ -m \sin(m \sin^{-1} x) + \frac{x}{\sqrt{1-x^2}} \cos(m \sin^{-1} x) \right\} \\
&= -m^2 \sin(m \sin^{-1} x) + \frac{mx}{\sqrt{1-x^2}} \cos(m \sin^{-1} x) \\
(1-x^2) y_2 - xy_1 & \\
&= -m^2 \sin(m \sin^{-1} x) + \frac{mx}{\sqrt{1-x^2}} \cos(m \sin^{-1} x) - \cos(m \sin^{-1} x) \frac{mx}{\sqrt{1-x^2}} \\
&= -m^2 \sin(m \sin^{-1} x) \\
&= -m^2 y
\end{aligned}$$

16. Question

Write the correct alternative in the following:

If $y = (\sin^{-1} x)^2$, then $(1-x^2) y_2$ is equal to

- A. $xy_1 + 2$
- B. $xy_1 - 2$
- C. $-xy_1 + 2$
- D. none of these

Answer

Given:

$$y = (\sin^{-1} x)^2$$

$$\frac{dy}{dx} = 2 \sin^{-1} x \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = 2 \left\{ \left(\frac{1}{\sqrt{1-x^2}} \right)^2 + \sin^{-1} x \frac{\frac{2x}{2\sqrt{1-x^2}}}{(\sqrt{1-x^2})^2} \right\}$$

$$= 2 \left\{ \frac{1}{1-x^2} + \sin^{-1} x \frac{x}{(\sqrt{1-x^2})^{3/2}} \right\}$$

$$(1-x^2) y_2 = 2 \left\{ 1 + \sin^{-1} x \frac{x}{\sqrt{1-x^2}} \right\}$$

$$= 2 + x \left\{ 2 \sin^{-1} x \frac{1}{\sqrt{1-x^2}} \right\}$$

$$= 2 + xy_1$$

17. Question

Write the correct alternative in the following:

If $y = e^{\tan x}$, then $(\cos^2 x)y_2 =$

- A. $(1 - \sin 2x) y_1$
- B. $-(1 + \sin 2x) y_1$
- C. $(1 + \sin 2x) y_1$
- D. none of these

Answer

Given:

$$y = e^{\tan x}$$

$$\frac{dy}{dx} = e^{\tan x}(\sec x)^2$$

$$\frac{d^2y}{dx^2} = e^{\tan x}(\sec x)^2(\sec x)^2 + e^{\tan x} \times 2 \sec x \times \tan x \times \sec x$$

$$= e^{\tan x}(\sec x)^2[(\sec x)^2 + 2 \tan x]$$

$$(\cos^2 x)y_2 = e^{\tan x}[(\sec x)^2 + 2 \tan x]$$

$$= e^{\tan x} \left[\frac{1 + 2 \sin x \cos x}{(\cos x)^2} \right]$$

$$= e^{\tan x}(\sec x)^2[1 + 2 \sin x \cos x]$$

$$= e^{\tan x}(\sec x)^2[1 + \sin 2x]$$

$$= [1 + \sin 2x]y_1$$

18. Question

Write the correct alternative in the following:

If $y = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left(\frac{a - b}{a + b} \tan \frac{x}{2} \right)$, $a > b > 0$, then

- A. $y_1 = \frac{-1}{a + b \cos x}$
- B. $y_2 = \frac{b \sin x}{(a + b \cos x)^2}$
- C. $y_1 = \frac{1}{a - b \cos x}$
- D. $y_2 = \frac{-b \sin x}{(a - b \cos x)^2}$

Answer

Given:

$$y = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left(\frac{a - b}{a + b} \tan \frac{x}{2} \right)$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2}{\sqrt{(a^2 - b^2)}} \left(\frac{1}{1 + \left(\frac{a-b}{a+b} \tan \frac{x}{2}\right)^2} \right) \left(\frac{a-b}{a+b} \right) \left(\sec \frac{x}{2} \right)^2 \\
&= \frac{2}{\sqrt{(a^2 - b^2)}} \left(\frac{(a+b)^2}{(a+b)^2 + (a-b)^2 \left(\tan \frac{x}{2} \right)^2} \right) \left(\frac{a-b}{a+b} \right) \left(\sec \frac{x}{2} \right)^2 \\
&= \frac{2}{\sqrt{(a^2 - b^2)}} \left(\frac{(a+b)}{a^2(1 + (\tan x)^2) + b^2(1 + (\tan x)^2) + 2ab(1 - (\tan x)^2)} \right) (a - b) \left(\sec \frac{x}{2} \right)^2 \\
&= 2 \left(\frac{1}{a^2 \left(1 + \left(\tan \frac{x}{2} \right)^2 \right) + b^2 \left(1 + \left(\tan \frac{x}{2} \right)^2 \right) + 2ab \left(1 - \left(\tan \frac{x}{2} \right)^2 \right)} \right) \sqrt{(a^2 - b^2)} \left(\sec \frac{x}{2} \right)^2
\end{aligned}$$

Divide numerator and denominator by $\left(1 + \left(\tan \frac{x}{2} \right)^2 \right)$;

We get:

$$\begin{aligned}
&= 2 \left(\frac{1}{a^2 + b^2 + 2ab \left(\frac{1 - \left(\tan \frac{x}{2} \right)^2}{1 + \left(\tan \frac{x}{2} \right)^2} \right)} \right) \sqrt{(a^2 - b^2)} \left(\sec \frac{x}{2} \right)^2 \frac{1}{1 + \left(\tan \frac{x}{2} \right)^2} \\
&= 2 \left(\frac{1}{a^2 + b^2 + 2ab \cos x} \right) \sqrt{(a^2 - b^2)} \left(\sec \frac{x}{2} \right)^2 \frac{1}{\left(\sec \frac{x}{2} \right)^2} \\
&= 2 \left(\frac{1}{a^2 + b^2 + 2ab \cos x} \right) \sqrt{(a^2 - b^2)} \\
\frac{d^2y}{dx^2} &= 2\sqrt{(a^2 - b^2)} \left(\frac{1}{a^2 + b^2 + 2ab \cos x} \right)^2 \{-2ab \sin x\}
\end{aligned}$$

19. Question

Write the correct alternative in the following:

If $y = \frac{ax + b}{x^2 + c}$, then $(2xy_1 + y)y_3 =$

- A. $3(xy_2 + y_1)y_2$
- B. $3(xy_2 + y_2)y_2$
- C. $3(xy_2 + y_1)y_1$
- D. none of these

Answer

Given:

$$y = \frac{ax + b}{x^2 + c}$$

$$\frac{dy}{dx} = \frac{a(x^2 + c) - 2x(ax + b)}{(x^2 + c)^2}$$

$$= \frac{-ax^2 - 2bx + ac}{(x^2 + c)^2}$$

$$2xy_1 = \frac{-ax^3 - 2bx^2 + acx}{(x^2 + c)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(-2ax - 2b)(x^2 + c)^2 - 2(2x)(x^2 + c)(-ax^2 - 2bx + ac)}{(x^2 + c)^4}$$

20. Question

Write the correct alternative in the following:

If $y = \log_e \left(\frac{x}{a + bx} \right)^2$, then $x^3 y_2 =$

A. $(xy_1 - y)^2$

B. $(x + y)^2$

C. $\left(\frac{y - xy_1}{y_1} \right)^2$

D. none of these

Answer

Given:

$$y = \left(\log_e \left(\frac{x}{a + bx} \right) \right)^2$$

$$= 2 \log_e \left(\frac{x}{a + bx} \right)$$

$$\frac{dy}{dx} = 2 \left(\frac{1}{\frac{x}{a + bx}} \right) \left[\frac{a + bx - bx}{(a + bx)^2} \right]$$

$$= 2 \left(\frac{a + bx}{x} \right) \left[\frac{a}{(a + bx)^2} \right]$$

$$= \frac{2a}{x(a + bx)}$$

$$= \frac{2a}{(ax + bx^2)}$$

$$x \frac{dy}{dx} = \frac{2ax}{(ax + bx^2)}$$

$$\frac{d^2y}{dx^2} = 2a \left\{ \frac{-(a + 2bx)}{(ax + bx^2)^2} \right\}$$

$$= (-a - 2bx) \frac{dy}{dx}$$

$$x^3 \frac{d^2y}{dx^2} = -x^3(a + 2bx) \frac{dy}{dx}$$

21. Question

Write the correct alternative in the following:

If $x = f(t) \cos t - f'(t) \sin t$ and $y = f(t) \sin t + f'(t) \cos t$, then $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 =$

- A. $f(t) - f''(t)$
- B. $\{f(t) - f''(t)\}^2$
- C. $\{f(t) + f''(t)\}^2$
- D. none of these

Answer

Given:

$$x = f(t) \cos t - f'(t) \sin t$$

$$y = f(t) \sin t + f'(t) \cos t$$

$$\frac{dx}{dt} = f'(t) \cos t - f(t) \sin t - f''(t) \sin t - f'(t) \cos t$$

$$= -f(t) \sin t - f''(t) \sin t$$

$$= -\sin t [f(t) + f''(t)]$$

$$\left(\frac{dx}{dt}\right)^2 = \{-\sin t [f(t) + f''(t)]\}^2$$

$$= (\sin t)^2 \{f(t) + f''(t)\}^2$$

$$\frac{dy}{dt} = f'(t) \sin t + f(t) \cos t + f''(t) \cos t - f'(t) \sin t$$

$$= f(t) \cos t + f''(t) \cos t$$

$$= \cos t [f(t) + f''(t)]$$

$$\left(\frac{dy}{dt}\right)^2 = \{\cos t [f(t) + f''(t)]\}^2$$

$$= (\cos t)^2 \{f(t) + f''(t)\}^2$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (\sin t)^2 \{f(t) + f''(t)\}^2 + (\cos t)^2 \{f(t) + f''(t)\}^2$$

$$= \{f(t) + f''(t)\}^2$$

22. Question

Write the correct alternative in the following:

If $y^{1/n} + y^{-1/n} = 2x$, then $(x^2 - 1)y_2 + xy_1 =$

- A. $-n^2y$
- B. n^2y
- C. 0
- D. none of these

Answer

Given:

$$y^{1/n} + y^{-1/n} = 2x$$

$$\frac{1}{n} y^{\frac{1}{n}-1} \frac{dy}{dx} + \frac{-1}{n} y^{\frac{-1}{n}-1} \frac{dy}{dx} = 2$$

$$\frac{1}{n} \frac{dy}{dx} \left\{ y^{\frac{1}{n}-1} - y^{\frac{-1}{n}-1} \right\} = 2$$

23. Question

Write the correct alternative in the following:

$$\text{If } \frac{d}{dx} \left\{ x^n - a_1 x^{n-1} + a_2 x^{n-2} + \dots + (-1)^n a_n \right\} e^x = x^n e^x,$$

Then the value of a_r , $0 < r \leq n$, is equal to

A. $\frac{n!}{r!}$

B. $\frac{(n-r)!}{r!}$

C. $\frac{n!}{(n-r)!}$

D. none of these

Answer

Given:

$$\frac{d}{dx} \{x^n - a_1 x^{n-1} + a_2 x^{n-2} + \dots + (-1)^n a_n\} e^x = x^n e^x$$

$$\frac{d}{dx} \{a_0 (-1)^0 x^n + a_1 (-1)^1 x^{n-1} + a_2 (-1)^2 x^{n-2} + \dots + (-1)^n a_n\} e^x$$

$$\frac{d}{dx} (x-1)^n$$

$$(x-1)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} (-1)^k$$

So, at $k=r$;

$$a_r = \binom{n}{r}$$

$$\text{Also, } \binom{n}{r} = \binom{n}{n-r}$$

$$\text{So, } a_r = \binom{n}{n-r}$$

24. Question

Write the correct alternative in the following:

If $y = x^{n-1} \log x$, then $x^2 y_2 + (3 - 2n) x y_1$ is equal to

A. $-(n-1)^2 y$

B. $(n-1)^2 y$

C. $-n^2y$

D. n^2y

Answer

Given:

$$y = x^{n-1} \log x$$

$$\frac{dy}{dx} = (n-1)x^{n-2} \log x + \frac{1}{x} x^{n-1}$$

$$= (n-1)x^{n-2} \log x + x^{n-2}$$

$$= x^{n-2} [(n-1) \log x + 1]$$

$$xy_1 = x^{n-1} [(n-1) \log x + 1]$$

$$= (n-1)y + x^{n-1}$$

$$(3-2n)xy_1 = (3-2n)[(n-1)y + x^{n-1}]$$

$$= (3n-3-2n^2+2n)y + 3x^{n-1} - 2nx^{n-1} \quad (1)$$

$$\frac{d^2y}{dx^2} = (n-1)(n-2)x^{n-3} \log x + \frac{1}{x}(n-1)x^{n-2} + (n-2)x^{n-3}$$

$$= (n-1)(n-2)x^{n-3} \log x + (n-1)x^{n-3} + (n-2)x^{n-3}$$

$$= x^{n-3} [(n-1)(n-2) \log x + (n-1) + (n-2)]$$

$$x^2 y_2 = x^{n-1} [(n-1)(n-2) \log x + (2n-3)]$$

$$= (n^2-3n+2)y + 2nx^{n-1} - 3x^{n-1} \quad (2)$$

$$x^2 y_2 + (3-2n)xy_1$$

$$= (n^2-3n+2)y + 2nx^{n-1} - 3x^{n-1} + (3n-3-2n^2+2n)y + 3x^{n-1} - 2nx^{n-1}$$

$$= (-n^2+2n-1)y$$

$$= -(n-1)^2y$$

25. Question

Write the correct alternative in the following:

If $xy - \log_e y = 1$ satisfies the equation $x(yy_2 + y_1^2) - y_2 + \lambda yy_1 = 0$, then $\lambda =$

A. -3

B. 1

C. 3

D. none of these

Answer

Given:

$$xy - \log_e y = 1$$

$$xy = \log_e y + 1$$

Differentiate w.r.t. 'x' on both sides;

$$y + x \frac{dy}{dx} = \frac{1}{y} \frac{dy}{dx}$$

$$\frac{dy}{dx} \left(\frac{1}{y} - x \right) = y$$

$$\frac{dy}{dx} = \frac{y^2}{(1 - xy)}$$

$$\left(\frac{dy}{dx} \right)^2 = \left[\frac{y^2}{(1 - xy)} \right]^2$$

$$= \frac{y^4}{(1 - xy)^2}$$

$$\frac{d}{dx} \left[\frac{dy}{dx} \right] = \frac{d}{dx} \left[\frac{y^2}{(1 - xy)} \right]$$

$$= \frac{1}{(1 - xy)^2} \left\{ 2y \frac{dy}{dx} (1 - xy) - y^2 \left(-y + x \frac{dy}{dx} \right) \right\}$$

$$= \frac{1}{(1 - xy)^2} \left\{ 2y \frac{dy}{dx} (1 - xy) - y^2 \left(-y + x \frac{dy}{dx} \right) \right\}$$

$$= \frac{1}{(1 - xy)^2} \left\{ 2y \frac{dy}{dx} \frac{y^2}{\frac{dy}{dx}} + y^3 + xy^2 \frac{dy}{dx} \right\}$$

$$= \frac{1}{(1 - xy)^2} \left\{ 2y^3 + y^3 + xy^2 \frac{dy}{dx} \right\}$$

$$= \frac{1}{(1 - xy)^2} \left\{ 3y^3 + xy^2 \frac{dy}{dx} \right\}$$

$$= \frac{y^2}{(1 - xy)^2} \left\{ 3y + x \frac{dy}{dx} \right\}$$

$$y \frac{d^2y}{dy^2} = \frac{y^3}{(1 - xy)^2} \left\{ 3y + x \frac{dy}{dx} \right\}$$

$$y \frac{d^2y}{dy^2} + \left(\frac{dy}{dx} \right)^2 = \frac{y^3}{(1 - xy)^2} \left\{ 3y + x \frac{dy}{dx} \right\} + \frac{y^4}{(1 - xy)^2}$$

$$= \frac{y^3}{(1 - xy)^2} \left\{ 3y + x \frac{dy}{dx} + y \right\}$$

$$= \frac{y^3}{(1 - xy)^2} \left\{ 4y + x \frac{dy}{dx} \right\}$$

$$x \left[y \frac{d^2y}{dy^2} + \left(\frac{dy}{dx} \right)^2 \right] = \frac{y^3 x}{(1 - xy)^2} \left\{ 4y + x \frac{dy}{dx} \right\}$$

$$x \left[y \frac{d^2y}{dy^2} + \left(\frac{dy}{dx} \right)^2 \right] - \frac{d^2y}{dy^2} = \frac{y^3 x}{(1 - xy)^2} \left\{ 4y + x \frac{dy}{dx} \right\} - \frac{y^2}{(1 - xy)^2} \left\{ 3y + x \frac{dy}{dx} \right\}$$

$$= \frac{y^2}{(1 - xy)^2} \left\{ xy \left(4y + x \frac{dy}{dx} \right) - 3y - x \frac{dy}{dx} \right\}$$

$$= \frac{y^2}{(1 - xy)^2} \left\{ 4xy^2 + x^2 y \frac{dy}{dx} - 3y - x \frac{dy}{dx} \right\}$$

$$\begin{aligned}
&= \frac{y^2}{(1-xy)^2} \left\{ y(4xy-3) + x \frac{dy}{dx} (xy-1) \right\} \\
&= \frac{y^2}{(1-xy)^2} \left\{ y(xy+3xy-3) - x \frac{dy}{dx} (1-xy) \right\} \\
&= \frac{y^2}{(1-xy)^2} \left\{ y(xy-3(1-xy)) - x \frac{dy}{dx} \frac{y^2}{\frac{dy}{dx}} \right\} \\
&= \frac{y^2}{(1-xy)^2} \left\{ y \left(xy - 3 \frac{y^2}{\frac{dy}{dx}} \right) - xy^2 \right\} \\
&= \frac{y^2}{(1-xy)^2} \left\{ xy^2 - 3 \frac{y^3}{\frac{dy}{dx}} - xy^2 \right\} \\
&= - \frac{y^2}{(1-xy)^2} \left\{ 3 \frac{y^3}{\frac{dy}{dx}} \right\}
\end{aligned}$$

$$\text{Since } x \left[y \frac{d^2y}{dy^2} + \left(\frac{dy}{dx} \right)^2 \right] - \frac{d^2y}{dy^2} + \lambda y \frac{dy}{dx} = 0$$

$$\text{So, } x \left[y \frac{d^2y}{dy^2} + \left(\frac{dy}{dx} \right)^2 \right] - \frac{d^2y}{dy^2} = -\lambda y \frac{dy}{dx}$$

$$-\lambda y \frac{dy}{dx} = - \frac{y^2}{(1-xy)^2} \left\{ 3 \frac{y^3}{\frac{dy}{dx}} \right\}$$

$$-\lambda y \frac{y^2}{(1-xy)} = - \frac{y^2}{(1-xy)^2} \left\{ 3 \frac{y^3}{\frac{dy}{dx}} \right\}$$

$$\lambda y = \frac{1}{(1-xy)} \left\{ 3 \frac{y^3}{\frac{dy}{dx}} \right\}$$

$$\lambda = \frac{3y^2}{(1-xy) \frac{dy}{dx}}$$

$$\lambda = \frac{3 \frac{dy}{dx}}{\frac{dy}{dx}}$$

$$\lambda = 3$$

26. Question

Write the correct alternative in the following:

If $y^2 = ax^2 + bx + c$, then $y^3 \frac{d^2y}{dx^2}$ is

- A. a constant
- B. a function of x only
- C. a function of y only

D. a function of x and y

Answer

Given:

$$y^2 = ax^2 + bx + c$$

$$y = \sqrt{(ax^2 + bx + c)}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{(ax^2 + bx + c)}} \times (2ax + b)$$

$$\frac{d^2y}{dx^2} = \frac{1}{2} \left\{ \frac{(2a \times \sqrt{ax^2 + bx + c}) - \left((2ax + b) \times \frac{1}{2\sqrt{(ax^2 + bx + c)}} \times (2ax + b) \right)}{(\sqrt{(ax^2 + bx + c)})^2} \right\}$$

$$= \frac{1}{2} \left\{ \frac{\frac{4a(ax^2 + bx + c) - (2ax + b)^2}{2\sqrt{(ax^2 + bx + c)}}}{(\sqrt{(ax^2 + bx + c)})^2} \right\}$$

$$= \frac{1}{2} \left\{ \frac{4a^2x^2 + 4abx + 4ac - 4a^2x^2 - b^2 - 4abx}{(\sqrt{(ax^2 + bx + c)})^2 \times 2\sqrt{(ax^2 + bx + c)}} \right\}$$

$$= \frac{1}{4} \left\{ \frac{4ac - b^2}{(\sqrt{(ax^2 + bx + c)})^3} \right\}$$

$$y^3 \frac{d^2y}{dx^2} = \frac{1}{4} \left\{ \frac{4ac - b^2}{(\sqrt{(ax^2 + bx + c)})^3} \right\} \times (\sqrt{(ax^2 + bx + c)})^3$$

$$= \frac{4ac - b^2}{4}$$

Hence, y is a constant.

Very short answer

1. Question

If $y = ax^{n+1} + bx^{-n}$ and $x^2 \frac{d^2y}{dx^2} = \lambda y$, then write the value of λ .

Answer

Given:

$$y = ax^{n+1} + bx^{-n}$$

$$\frac{dy}{dx} = (n+1)ax^n + (-n)bx^{-n-1}$$

$$\frac{d^2y}{dx^2} = n(n+1)ax^{n-1} + (-n)(-n-1)bx^{-n-2}$$

$$x^2 \frac{d^2y}{dx^2} = x^2 \{n(n+1)ax^{n-1} + (-n)(-n-1)bx^{-n-2}\} = \lambda y$$

$$\lambda y = n(n+1)a x^{n-1+2} + n(n+1)bx^{-n-2+2}$$

$$\lambda y = n(n+1)[a x^{(n+1)} + bx^{(-n)}]$$

$$\lambda y = n(n+1)$$

$$\lambda = n(n+1)$$

2. Question

If $x = a \cos nt - b \sin nt$ and $\frac{d^2y}{dt^2} = \lambda x$, then find the value of λ .

Answer

Given:

$$y = a \cos nt - b \sin nt$$

$$\frac{dy}{dt} = -an \sin nt - bn \cos nt$$

$$\frac{d^2y}{dt^2} = -an^2 \cos nt + bn^2 \sin nt = \lambda y$$

$$\lambda y = -n^2 (a \cos nt - b \sin nt)$$

$$\lambda y = -n^2 y$$

$$\lambda = -n^2$$

3. Question

If $x = t^2$ and $y = t^3$, where a is a constant, then find $\frac{d^2y}{dx^2}$ at $x = \frac{1}{2}$.

Answer

Given:

$$x = t^2, y = t^3$$

$$\frac{dy}{dt} = 3t^2; \frac{dx}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t}{2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{3t}{2}}{2t}$$

$$= \frac{3}{4}$$

4. Question

If $x = 2at$, $y = at^2$, where a is a constant, then find $\frac{d^2y}{dx^2}$ at $x = \frac{1}{2}$.

Answer

Given:

$$x = 2at, y = at^2$$

$$\frac{dx}{dt} = 2a; \frac{dy}{dt} = 2at$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$=t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{1}{2a}$$

5. Question

If $x = f(t)$ and $y = g(t)$, then write the value of $\frac{d^2y}{dx^2}$.

Answer

Given:

$$x = f(t) \text{ and } y = g(t)$$

$$\frac{dx}{dt} = f'(t); \frac{dy}{dt} = g'(t)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{1}{f'(t)} \left\{ \frac{1}{f'(t)^2} (g''(t)f'(t) - f''(t)g'(t)) \right\}$$

$$= \frac{(g''(t)f'(t) - f''(t)g'(t))}{(f'(t))^3}$$

6. Question

If $y = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \dots$ to ∞ , then write $\frac{d^2y}{dx^2}$ in terms of y .

Answer

Given:

$$y = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \infty$$

$$\frac{dy}{dx} = 0 - 1 + \frac{2x}{2!} - \frac{3x^2}{3!} - \frac{4x^3}{4!} + \dots \infty$$

$$\frac{d^2y}{dx^2} = 0 - 0 + 1 - \frac{2x}{2!} + \frac{3x^2}{3!} - \frac{4x^3}{4!} + \dots \infty$$

$$= 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \infty$$

$$\frac{d^2y}{dx^2} = y$$

7. Question

If $y = x + e^x$, find $\frac{d^2x}{dy^2}$.

Answer

Given:

$$y = x + e^x$$

$$\frac{d^2x}{d^2y} = \frac{1}{\frac{d^2y}{dx^2}}$$

$$\frac{dy}{dx} = 1 + e^x$$

$$\frac{d^2y}{dx^2} = e^x$$

$$\frac{d^2x}{d^2y} = \frac{1}{e^x}$$

$$= e^{-x}$$

8. Question

If $y = |x - x^2|$, then find $\frac{d^2y}{dx^2}$.

Answer

Given:

$$y = |x - x^2|$$

$$y = \begin{cases} x - x^2; x \geq 0 \\ x^2 - x; x \leq 0 \end{cases}$$

$$\frac{dy}{dx} = \begin{cases} 1 - 2x; x \geq 0 \\ 2x - 1; x \leq 0 \end{cases}$$

$$\frac{d^2y}{dx^2} = \begin{cases} -2; x \geq 0 \\ 2; x \leq 0 \end{cases}$$

9. Question

If $y = |\log_e x|$, find $\frac{d^2y}{dx^2}$.

Answer

Given:

$$y = |\log_e x| \quad \forall x > 0$$

$$y = \log_e x$$

$$\frac{dy}{dx} = \frac{1}{x} = x^{-1}$$

$$\frac{d^2y}{dx^2} = (-1)x^{-2}$$