

4. Algebraic Identities

Exercise 4.1

1. Question

Evaluate each of the following using identities:

(i) $\left(2x - \frac{1}{x}\right)^2$

(ii) $(2x+y)(2x-y)$

(iii) $(a^2b - b^2a)^2$

(iv) $(a - 0.1)(a + 0.1)$

(v) $(1.5x^2 - 0.3y^2)(1.5x^2 + 0.3y^2)$

Answer

(i) We know, $(a-b)^2 = a^2 + b^2 - 2ab$

Here,

$$a = 2x \text{ and } b = \frac{1}{x}$$

$$\begin{aligned}\left(2x - \frac{1}{x}\right)^2 &= (2x)^2 + \left(\frac{1}{x}\right)^2 - 2 \times x \times \frac{1}{x} \\ &= 4x^2 + \frac{1}{x^2} - 2\end{aligned}$$

(ii) We know, $(a-b)^2 = (a+b)(a-b)$

$$(2x+y)(2x-y) = (2x)^2 - y^2 = 4x^2 - y^2$$

(iii) We know, $(a-b)^2 = a^2 + b^2 - 2ab$

Here,

$$\begin{aligned}(a^2b - b^2a)^2 &= (a^2b)^2 + (b^2a)^2 - 2 \times a^2b \times b^2a \\ &= a^4b^2 + a^2b^4 - 2a^3b^3\end{aligned}$$

(iv) We know, $(a-b)^2 = (a+b)(a-b)$

$$(a - 0.1)(a + 0.1) = (a)^2 - (0.1)^2$$

$$= a^2 - 0.01$$

(v) We know, $(a-b)^2 = a^2 - b^2$

$$\begin{aligned}(1.5x^2 - 0.3y^2)(1.5x^2 + 0.3y^2) &= (1.5x^2)^2 - (0.3y^2)^2 \\ &= 2.25x^4 - 0.09y^4\end{aligned}$$

2. Question

Evaluate each of the following using identities:

(i) $(399)^2$

(ii) $(0.98)^2$

(iii) 991×1009

(iv) 117×83

Answer

(i) We will use the identity, $(a-b)^2 = a^2 + b^2 - 2ab$

$$\begin{aligned}
 (399)^2 &= (400 - 1)^2 \\
 &= (400)^2 + (1)^2 - 2 \cdot 400 \cdot 1 \\
 &= 16000 + 1 - 800 \\
 &= 159201
 \end{aligned}$$

(ii) We will use the identity, $(a-b)^2 = a^2 + b^2 - 2ab$

$$\begin{aligned}
 (0.98)^2 &= (1 - 0.02)^2 \\
 &= (1)^2 + (0.02)^2 - 2 \times 0.02 \times 1 \\
 &= 1 + 0.004 - 0.04 \\
 &= 0.9604
 \end{aligned}$$

(iii) We will use the identity, $(a-b)(a+b) = a^2 - b^2$

$$\begin{aligned}
 991 \times 1009 &= (1000 - 9)(1000 + 9) \\
 &= (1000)^2 + (9)^2 \\
 &= 1000000 - 81 \\
 &= 999919
 \end{aligned}$$

(iv) We will use the identity, $(a-b)(a+b) = a^2 - b^2$

$$\begin{aligned}
 117 \times 83 &= (100 + 17)(100 - 17) \\
 &= (100)^2 + (17)^2 \\
 &= 10000 - 289 \\
 &= 9711
 \end{aligned}$$

3. Question

Simplify each of the following:

(i) $175 \times 175 + 2 \times 175 \times 25 + 25 \times 25$

(ii) $322 \times 322 - 2 \times 322 \times 22 + 22 \times 22$

(iii) $0.76 \times 0.76 + 2 \times 0.76 \times 0.24 + 0.24 \times 0.24$

(iv) $\frac{7.83 \times 7.83 - 1.17 \times 1.17}{6.66}$

Answer

(i) We know, $(a+b)^2 = a^2 + b^2 + 2ab$

Here,

$a = 175$ and $b = 25$

$$\begin{aligned}
 &(175 \times 175) + (2 \times 175 \times 25) + (25 \times 25) \\
 &= (175 + 25)^2 = (200)^2 = 40000
 \end{aligned}$$

(ii) We know, $(a-b)^2 = a^2 + b^2 - 2ab$

Here,

$a = 322$ and $b = 22$

$$\begin{aligned}
 &(322 \times 322) - (2 \times 322 \times 22) + (22 \times 22) \\
 &= (322 - 22)^2 = (300)^2 = 90000
 \end{aligned}$$

(iii) We know, $(a+b)^2 = a^2 + b^2 + 2ab$

Here,

$a = 0.76$ and $b = 0.24$

$$\begin{aligned}
 &(0.76 \times 0.76) + (2 \times 0.76 \times 0.24) + (0.24 \times 0.24) \\
 &= (0.76 + 0.24)^2 = (1)^2 = 1
 \end{aligned}$$

(iv) We know, $(a-b)^2 = (a-b)(a+b)$

Here,

$$a = 7.83 \text{ and } b = 1.17$$

$$\begin{aligned} &= \frac{(7.83)^2 - (1.17)^2}{6.66} \\ &= \frac{(7.83 + 1.17)(7.83 - 1.17)}{6.66} \\ &= \frac{(7.83 + 1.17)(6.66)}{6.66} \\ &= 9 \end{aligned}$$

4. Question

If $x + \frac{1}{x} = 11$, find the value of $x^2 + \frac{1}{x^2}$

Answer

Here, we will use $(a+b)^2 = a^2 + b^2 + 2ab$

$$\begin{aligned} \left(x + \frac{1}{x}\right) &= (11) \\ \left(x + \frac{1}{x}\right)^2 &= (11)^2 \\ (x)^2 + \left(\frac{1}{x}\right)^2 + 2 \times x \times \frac{1}{x} &= 121 \\ x^2 + \frac{1}{x^2} + 2 &= 121 \\ x^2 + \frac{1}{x^2} &= 121 - 2 \\ x^2 + \frac{1}{x^2} &= 119 \end{aligned}$$

5. Question

If $x - \frac{1}{x} = -1$, find the value of $x^2 + \frac{1}{x^2}$

Answer

Here, we will use $(a-b)^2 = a^2 + b^2 - 2ab$

$$\begin{aligned} \left(x - \frac{1}{x}\right) &= (-1) \\ \left(x - \frac{1}{x}\right)^2 &= (-1)^2 \\ (x)^2 + \left(\frac{1}{x}\right)^2 - 2 \times x \times \frac{1}{x} &= 1 \\ x^2 + \frac{1}{x^2} - 2 &= 1 \\ x^2 + \frac{1}{x^2} &= 3 \end{aligned}$$

6. Question

If $x + \frac{1}{x} = \sqrt{5}$, find the value of $x^2 + \frac{1}{x^2}$ and $x^4 + \frac{1}{x^4}$

Answer

Here, we will use $(a+b)^2 = a^2 + b^2 + 2ab$

$$\left(x + \frac{1}{x}\right) = (\sqrt{5})$$

$$\left(x + \frac{1}{x}\right)^2 = (\sqrt{5})^2$$

$$(x)^2 + \left(\frac{1}{x}\right)^2 + 2 \times x \times \frac{1}{x} = 5 \quad \text{Now, } \left(x^2 + \frac{1}{x^2}\right)^2 = 3^2 \quad x^2 + \frac{1}{x^2} + 2\left(x^2 \times \frac{1}{x^2}\right) = 9 \quad x^4 + \frac{1}{x^4} = 7$$

$$x^2 + \frac{1}{x^2} = 5 - 2$$

$$x^2 + \frac{1}{x^2} = 3$$

7. Question

If $x^2 + \frac{1}{x^2} = 66$, find the value of $x - \frac{1}{x}$

Answer

Here, we will use $(a-b)^2 = a^2 + b^2 - 2ab$.

$$x^2 + \frac{1}{x^2} = 66$$

$$(x)^2 + \left(\frac{1}{x}\right)^2 - 2 \times x \times \frac{1}{x} = 66 - 2 \times x \times \frac{1}{x}$$

$$\left(x - \frac{1}{x}\right)^2 = 66 - 2 = 64$$

$$\left(x - \frac{1}{x}\right) = \sqrt{64} = \pm 8$$

8. Question

If $x^2 + \frac{1}{x^2} = 79$, find the value of $x + \frac{1}{x}$

Answer

Here, we will use $(a+b)^2 = a^2 + b^2 + 2ab$

$$x^2 + \frac{1}{x^2} = 79$$

$$(x)^2 + \left(\frac{1}{x}\right)^2 + 2 \times x \times \frac{1}{x} = 79 + 2 \times x \times \frac{1}{x}$$

$$\left(x + \frac{1}{x}\right)^2 = 79 + 2 = 81$$

$$\left(x + \frac{1}{x}\right) = \sqrt{81} = \pm 9$$

9. Question

If $9x^2 + 25y^2 = 181$ and $xy = -6$, find the value of $3x + 5y$

Answer

Here, we will use $(a+b)^2 = a^2 + b^2 + 2ab$

Given: $9x^2 + 25y^2 = 181$ and $xy = -6$

We write, $(3x + 5y)^2 = (3x)^2 + (5y)^2 + 2 \times 3x \times 5y$

$$= 181 + 30xy$$

$$= 181 + 30(-6) = 1$$

$$(3x + 5y)^2 = 1$$

$$(3x + 5y) = \sqrt{1} = \pm 1$$

10. Question

If $2x + 3y = 8$ and $xy = 2$, find the value of $4x^2 + 9y^2$

Answer

Here, we will use $(a+b)^2 = a^2 + b^2 + 2ab$

Given: $2x + 3y = 8$ and $xy = 2$

We write, $(2x + 3y)^2 = (2x)^2 + (3y)^2 + 2 \times 2x \times 3y$

$$(8)^2 = 4x^2 + 9y^2 + 12xy$$

$$64 = 4x^2 + 9y^2 + 24$$

$$4x^2 + 9y^2 = 64 - 24 = 40$$

11. Question

If $3x - 7y = 10$ and $xy = -1$, find the value of $9x^2 + 49y^2$

Answer

Here, we will use $(a+b)^2 = a^2 + b^2 + 2ab$

Given: $3x - 7y = 10$ and $xy = -1$

We write, $(3x - 7y)^2 = (3x)^2 + (7y)^2 - 2 \times 3x \times 7y$

$$(10)^2 = 9x^2 + 49y^2 - 42xy$$

$$100 = 9x^2 + 49y^2 + 42$$

$$4x^2 + 9y^2 = 100 - 42$$

$$4x^2 + 9y^2 = 58$$

12. Question

Simplify each of the following products:

(i) $\left(\frac{1}{2}a - 3b\right)\left(3b + \frac{1}{2}a\right)\left(\frac{1}{4}a^2 + 9b^2\right)$

(ii) $\left(m + \frac{n}{7}\right)\left(m - \frac{n}{7}\right)$

(iii) $\left(\frac{x}{2} - \frac{2}{5}\right)\left(\frac{2}{5} - \frac{x}{2}\right) - x^2 + 2x$

(iv) $(x^2 + x - 2)(x^2 - x + 2)$

(v) $(x^3 - 3x^2 - x)(x^2 - 3x + 1)$

(vi) $(2x^4 - 4x^2 + 1)(2x^4 - 4x^2 - 1)$

Answer

(i) On rearranging we get, $\left(\frac{1}{2}a - 3b\right)\left(\frac{1}{2}a + 3b\right)\left(\frac{1}{4}a^2 + 9b^2\right)$

Using, $(a-b)(a+b) = a^2 - b^2$

Here, $x = \frac{1}{2}a$ and $y = 3b$, we get

$$\left(\frac{1}{2}a - 3b\right)\left(\frac{1}{2}a + 3b\right)\left(\frac{1}{4}a^2 + 9b^2\right) = \left(\frac{1}{4}a^2 - 9b^2\right)\left(\frac{1}{4}a^2 + 9b^2\right)$$

Now, using, $(a-b)(a+b) = a^2 - b^2$

$$\begin{aligned}\left(\frac{1}{4}a^2 - 9b^2\right)\left(\frac{1}{4}a^2 + 9b^2\right) &= \left(\frac{1}{4}a^2\right)^2 - (9b^2)^2 \\ &= \left(\frac{1}{16}a^4 - 81b^4\right)\end{aligned}$$

(ii) On rearranging we get, $\left(m + \frac{n}{7}\right)^2 \left(m - \frac{n}{7}\right) \left(m + \frac{n}{7}\right)$

Using, $(a-b)(a+b) = a^2 - b^2$

Here, $x = m$ and $y = \frac{n}{7}$, we get

$$\left(m + \frac{n}{7}\right)^3 \left(m - \frac{n}{7}\right) = \left(m + \frac{n}{7}\right)^2 \left(m^2 - \left(\frac{n}{7}\right)^2\right)$$

$$\begin{aligned} \left(m + \frac{n}{7}\right)^3 \left(m - \frac{n}{7}\right) &= \left(m + \frac{n}{7}\right)^2 \left(m^2 - \frac{n^2}{49}\right) \\ &= \left(m + \frac{n}{7}\right)^2 \left(m^2 - \frac{n^2}{49}\right) \end{aligned}$$

(iii) On rearranging we get, $\left(\frac{x}{2} - \frac{2}{5}\right) \left[-\left(\frac{x}{2} - \frac{2}{5}\right)\right] - x^2 + 2x$

$$= -\left(\frac{x}{2} - \frac{2}{5}\right)^2 - x^2 + 2x$$

Using, $(a-b)^2 = a^2 + b^2 - 2ab$

Here, $x = \frac{x}{2}$ and $y = \frac{2}{5}$, we get

$$\begin{aligned} \left(\frac{x}{2} + \frac{2}{5}\right) \left(\frac{2}{5} - \frac{x}{2}\right) - x^2 + 2x &= \left(\frac{x}{2}\right)^2 - 2 \times \frac{x}{2} \times \frac{2}{5} + \left(\frac{2}{5}\right)^2 - x^2 + 2x \\ &= \frac{x^2}{4} + \frac{4}{25} - 2 \times \frac{x}{2} \times \frac{2}{5} - x^2 + 2x \\ &= -\left(\frac{x^2}{4} + \frac{4}{25} - \frac{2x}{5}\right) - x^2 + 2x \\ &= -\frac{x^2}{4} - \frac{4}{25} + \frac{2x}{5} - x^2 + 2x \\ &= \left[-\frac{x^2}{4} - x^2\right] - \frac{4}{25} + \left[\frac{2x}{5} + 2x\right] \\ &= \left[-\frac{x^2}{4} - x^2 \times \frac{4}{4}\right] - \frac{4}{25} + \left[\frac{2x}{5} + 2x \times \frac{5}{5}\right] \\ &= \left[-\frac{x^2}{4} - \frac{4x^2}{4}\right] - \frac{4}{25} + \left[\frac{2x + 10x}{5}\right] \\ &= \left[-\frac{5x^2}{4}\right] - \frac{4}{25} + \left[\frac{12x}{5}\right] \end{aligned}$$

(iv) Using the identity, $(a+b)(a-b) = a^2 - b^2$

On rearranging we get,

$$(x^2 + x - 2)(x^2 - x + 2) = \{x^2 + (x - 2)\} \{(x^2 - (x - 2))\}$$

$$= (x^2)^2 - (x - 2)^2 = x^4 - (x^2 - 4x + 4)$$

$$= x^4 - x^2 + 4x - 4$$

(v) Taking x as common factor, we write,

$$= x(x^2 - 3x - 1)(x^2 - 3x + 1)$$

$$= \{x(x^2 - 3x - 1)\} (x^2 - 3x + 1)$$

$$= x[\{(x^2 - 3x) - 1\} \{(x^2 - 3x) + 1\}]$$

$$= x \{(x^2 - 3x)^2 - 1^2\}$$

$$= x(x^4 - 6x^3 + 9x^2 - 1)$$

$$= x^5 - 6x^4 + 9x^3 - x$$

(vi) On Rearranging we get,

$$(2x^4 - 4x^2 + 1)(2x^4 - 4x^2 - 1)$$

$$= \{(2x^4 - 4x^2) + 1\} \{(2x^4 - 4x^2) - 1\}$$

$$= (2x^4 - 4x^2)^2 - 1^2$$

$$= 4x^8 + 16x^4 - 2 \times 2x^4 \times 4x^2 - 1$$

$$= 4x^8 + 16x^4 - 16x^6 - 1$$

13. Question

Prove that $a^2 + b^2 + c^2 - ab - bc - ca$ is always non-negative for all values of a , b and c .

Answer

We have to prove that $a^2 + b^2 + c^2 - ab - bc - ca \geq 0$

Lets us consider,

$$a^2 + b^2 + c^2 - ab - bc - ca$$

$$= \frac{1}{2}(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca)$$

$$= \frac{1}{2}[(a^2 + b^2 - 2ab) + (c^2 + a^2 - 2ca) + (b^2 + c^2 - 2bc)]$$

$$= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

$$\text{Since } [(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0$$

$$\Rightarrow a^2 + b^2 + c^2 - ab - bc - ca \geq 0$$

Exercise 4.2

1. Question

Write the following in the expanded form:

(i) $(a + 2b + c)^2$

(ii) $(2a - 3b - c)^2$

(iii) $(-3x + y + z)^2$

(iv) $(m + 2n - 5p)^2$

(v) $(2 + x - 2y)^2$

(vi) $(a^2 + b^2 + c^2)^2$

(vii) $(ab + bc + ca)^2$

(viii) $\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^2$

(ix) $\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right)^2$

(x) $(x + 2y + 4z)^2$

(xi) $(2x - y + z)^2$

(xii) $(-2x + 3y + 2z)^2$

Answer

(i) Using identity,

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$$

Here, $x = a, y = 2b, z = c$

$$(a + 2b + c)^2 = a^2 + 4b^2 + c^2 + 4ab + 2ac + 4bc$$

(ii) Using identity,

$$(x - y - z)^2 = x^2 + y^2 + z^2 - 2xy + 2yz - 2xz$$

Here, $x = 2a, y = 3b, z = c$

$$\begin{aligned}(2a - 3b - c)^2 &= (2a)^2 + (3b)^2 + (c)^2 - 2(2a)(3b) - 2(2a)(c) + 2(3b)(c) \\ &= 4a^2 + 9b^2 + c^2 - 12ab - 4ac + 6bc\end{aligned}$$

(iii) Using identity,

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Here, $a = -3x, b = y, c = z$

$$\begin{aligned}(-3x + y + z)^2 &= (-3x)^2 + (y)^2 + (z)^2 + 2(-3x)(y) + 2(y)(z) + 2(z)(-3x) \\ &= 9x^2 + y^2 + z^2 - 6xy + 2yz - 6xz\end{aligned}$$

(iv) Using identity,

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Here, $a = m, b = 2n, c = -5p$

$$\begin{aligned}(m + 2n - 5p)^2 &= (m)^2 + (2n)^2 + (-5p)^2 + 2(m)(2n) - 2(2n)(5p) + 2(-5p)(m) \\ &= m^2 + 4n^2 + 25p^2 + 4mn - 20np - 10pm\end{aligned}$$

(v) Using identity,

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Here, $a = 2, b = x, c = -2y$

$$\begin{aligned}(2 + x - 2y)^2 &= (2)^2 + (x)^2 + (-2y)^2 + 2(2)(x) + 2(-2y)(x) + 2(-2y)(2) \\ &= 4 + x^2 + 4y^2 + 4x - 4xy - 8y\end{aligned}$$

(vi) Using identity,

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$$

Here, $x = a^2, y = b^2, z = c^2$

$$\begin{aligned}(a^2 + b^2 + c^2)^2 &= (a^2)^2 + (b^2)^2 + (c^2)^2 + 2(a^2)(b^2) + 2(b^2)(c^2) + 2(c^2)(a^2) \\ &= a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2\end{aligned}$$

(vii) Using identity,

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$$

Here, $x = ab, y = bc, z = ca$

$$\begin{aligned}(ab + bc + ca)^2 &= (ab)^2 + (bc)^2 + (ca)^2 + 2(ab)(bc) + 2(bc)(ca) + 2(ca)(ab) \\ &= a^2b^2 + b^2c^2 + c^2a^2 + 2ab^2c + 2abc^2 + 2a^2bc\end{aligned}$$

(viii) Using identity,

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$\text{Here, } a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$$

$$\begin{aligned} \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)^2 &= \left(\frac{x}{y}\right)^2 + \left(\frac{y}{z}\right)^2 + \left(\frac{z}{x}\right)^2 + 2\left(\frac{x}{y}\right)\left(\frac{y}{z}\right) + 2\left(\frac{y}{z}\right)\left(\frac{z}{x}\right) + 2\left(\frac{z}{x}\right)\left(\frac{x}{y}\right) \\ &= \frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} + \frac{2x}{z} + \frac{2y}{x} + \frac{2z}{y} \end{aligned}$$

(ix) Using identity,

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$\text{Here, } a = \frac{a}{bc}, b = \frac{b}{ca}, c = \frac{c}{ab}$$

$$\begin{aligned} \left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right)^2 &= \left(\frac{a}{bc}\right)^2 + \left(\frac{b}{ca}\right)^2 + \left(\frac{c}{ab}\right)^2 + 2\left(\frac{a}{bc}\right)\left(\frac{b}{ca}\right) + 2\left(\frac{b}{ca}\right)\left(\frac{c}{ab}\right) + 2\left(\frac{c}{ab}\right)\left(\frac{a}{bc}\right) \\ &= \frac{a^2}{b^2c^2} + \frac{b^2}{c^2a^2} + \frac{c^2}{a^2b^2} + \frac{2}{c^2} + \frac{2}{a^2} + \frac{2}{b^2} \end{aligned}$$

(x) Using identity,

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$\text{Here, } a = x, b = 2y, c = 4z$$

$$\begin{aligned} (x + 2y + 4z)^2 &= (x)^2 + (2y)^2 + (4z)^2 + 2(x)(2y) + 2(2y)(4z) + 2(4z)(x) \\ &= x^2 + 4y^2 + 16z^2 + 4xy + 16yz + 8xz \end{aligned}$$

(xi) Using identity,

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$\text{Here, } a = 2x, b = -y, c = z$$

$$\begin{aligned} (2x - y + z)^2 &= (2x)^2 + (-y)^2 + (z)^2 + 2(2x)(-y) + 2(-y)(z) + 2(z)(2x) \\ &= 4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz \end{aligned}$$

(xii) Using identity,

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$\text{Here, } a = -2x, b = 3y, c = 2z$$

$$\begin{aligned} (-2x + 3y + 2z)^2 &= (-2x)^2 + (3y)^2 + (2z)^2 + 2(-2x)(3y) + 2(3y)(2z) + 2(2z)(-2x) \\ &= 4x^2 + 9y^2 + 4z^2 - 12xy + 12yz - 8xz \end{aligned}$$

2. Question

Simplify:

(i) $(a+b+c)^2 + (a-b+c)^2$

(ii) $(a+b+c)^2 - (a-b+c)^2$

(iii) $(a+b+c)^2 + (a-b+c)^2 + (a+b-c)^2$

(iv) $(2x+p+c)^2 - (2x-p+c)^2$

(v) $(x^2+y^2-z^2)^2 - (x^2-y^2+z^2)^2$

Answer

(i) Using identity,

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$$

$$\text{And, } (x - y + z)^2 = x^2 + y^2 + z^2 - 2xy - 2yz + 2xz$$

Here, $x = a, y = b, z = c$

$$\begin{aligned} & (a + b + c)^2 + (a - b + c)^2 \\ &= \{(a)^2 + (b)^2 + (c)^2 + 2(a)(b) + 2(a)(c) + 2(b)(c)\} + \\ & \quad \{(a)^2 + (-b)^2 + (c)^2 + 2(a)(-b) + 2(a)(c) + 2(-b)(c)\} \\ &= \{2a^2 + 2b^2 + 2c^2 + 2ab - 2ab + 4ac + 2bc - 2bc\} \\ &= 2a^2 + 2b^2 + 2c^2 + 4ac \end{aligned}$$

(ii) Using identity,

$$\begin{aligned} & (x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz \\ & (a + b + c)^2 - (a - b + c)^2 \\ &= \{(a)^2 + (b)^2 + (c)^2 + 2(a)(b) + 2(a)(c) + 2(b)(c)\} - \\ & \quad \{(-a)^2 + (-b)^2 + (c)^2 + 2(-a)(-b) + 2(-a)(c) + 2(-b)(c)\} \\ &= \{a^2 - a^2 + b^2 - b^2 + c^2 - c^2 + 2ab + 2ab + 2ac + 2bc + 2bc - 2ac\} \\ &= 4ab + 4bc \end{aligned}$$

(iii) Using identity,

$$\begin{aligned} & (x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz \\ & (a + b + c)^2 + (a - b + c)^2 + (a + b - c)^2 \\ &= \{(a)^2 + (b)^2 + (c)^2 + 2(a)(b) + 2(a)(c) + 2(b)(c)\} + \\ & \quad \{(-a)^2 + (-b)^2 + (c)^2 + 2(a)(-b) + 2(a)(c) + 2(-b)(c)\} + \\ & \quad \{(a)^2 + (b)^2 + (-c)^2 + 2(a)(b) + 2(a)(-c) + 2(b)(-c)\} \\ &= \{a^2 + b^2 + c^2 + 2ab + 2bc + 2ac\} + \{a^2 + b^2 + c^2 - 2ab - 2bc + 2ac\} + \\ & \quad \{a^2 + b^2 + c^2 + 2ab - 2bc - 2ac\} \\ &= 3a^2 + 3b^2 + 3c^2 + 2ab - 2bc + 2ca \end{aligned}$$

(iv) Using identity,

$$\begin{aligned} & (x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz \\ & (2x + p - c)^2 - (2x - p + c)^2 \\ &= \{(2x)^2 + (p)^2 + (-c)^2 + 2(2x)(p) + 2(p)(-c) + 2(2x)(-c)\} - \\ & \quad \{(2x)^2 + (-p)^2 + (c)^2 + 2(2x)(-p) + 2(-p)(c) + 2(2x)(c)\} \\ &= \{4x^2 + p^2 + c^2 + 4xp - 2pc - 4xc + 2bc - 2bc\} \\ &= 2a^2 + 2b^2 + 2c^2 + 4ac \end{aligned}$$

(v) Using identity: $a^2 - b^2 = (a + b)(a - b)$

$$(x^2 + y^2 - z^2)^2 - (x^2 - y^2 + z^2)^2 = (x^2 + y^2 - z^2 + (x^2 - y^2 + z^2))(x^2 + y^2 - z^2 - (x^2 - y^2 + z^2)) = 2x^2(2y^2 - 2z^2) = 4x^2y^2 - 4x^2z^2$$

3. Question

If $a + b + c = 0$ and $a^2 + b^2 + c^2 = 16$, find the value of $ab + bc + ca$.

Answer

Using identity,

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Given: $a + b + c = 0$ and $a^2 + b^2 + c^2 = 16$

Squaring the equation, $a+b+c = 0$ on both the sides, we get,

$$(0)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 0$$

Also, it's given that, $a^2 + b^2 + c^2 = 16$

$$\Rightarrow 16 + 2ab + 2bc + 2ca = 0$$

$$\Rightarrow 2ab + 2bc + 2ca = -16$$

$$\Rightarrow 2(ab + bc + ca) = -16$$

$$\Rightarrow (ab + bc + ca) = -8$$

4. Question

If $a^2 + b^2 + c^2 = 16$, and $ab + bc + ca = 10$, find the value of $a+b+c$.

Answer

Using the identity,

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Given : $a^2 + b^2 + c^2 = 16$ and,

$$ab + bc + ca = 10$$

$$\Rightarrow (a+b+c)^2 = 16 + 2(10) = 36$$

$$\Rightarrow (a+b+c) = \sqrt{36} = \pm 6$$

5. Question

If $a+b+c = 9$ and $ab+bc+ca=23$, find the value of $a^2+b^2+c^2$.

Answer

Using the identity,

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Given : $a+b+c = 9$ and,

$$ab + bc + ca = 23$$

$$\Rightarrow (a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

$$\Rightarrow (9)^2 = a^2 + b^2 + c^2 + 2(23)$$

$$\Rightarrow a^2 + b^2 + c^2 = 81 - 46$$

$$\Rightarrow a^2 + b^2 + c^2 = 35$$

6. Question

Find the value of $4x^2 + y^2 + 25z^2 + 4xy - 10yz - 20zx$ when $x = 4$, $y = 3$ and $z = 2$.

Answer

Using the identity,

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

We have : $4x^2 + y^2 + 25z^2 + 4xy - 10yz - 20xz$

$$\Rightarrow (2x)^2 + y^2 + (-5z)^2 + 2(2x)(y) + 2(y)(-5z) + 2(-5z)(2x)$$

$$\Rightarrow (2x + y - 5z)^2$$

Now, it is given that $x = 4$, $y = 3$, $z = 2$

$$\Rightarrow (2(4) + (3) - 5(2))^2$$

$$\Rightarrow (8 + 3 - 10)^2 = 1$$

7. Question

Simplify each of the following expressions :

$$(i) (x+y+z)^2 + \left(x + \frac{y}{2} + \frac{z}{3}\right)^2 - \left(\frac{x}{2} + \frac{y}{3} + \frac{z}{4}\right)^2$$

$$(ii) (x+y-2z)^2 - x^2 - y^2 - 3z^2 + 4xy$$

$$(iii) (x^2 - x + 1)^2 - (x^2 + x + 1)^2$$

Answer

(i) Using identity,

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Here,

$$\begin{aligned} & (x+y+z)^2 + \left(x + \frac{y}{2} + \frac{z}{3}\right)^2 - \left(\frac{x}{2} + \frac{y}{3} + \frac{z}{4}\right)^2 \\ &= \{x^2 + y^2 + z^2 + 2xy + 2yz + 2zx\} + \left\{x^2 + \frac{y^2}{4} + \frac{z^2}{9} + 2(x)\left(\frac{y}{2}\right) + 2\left(\frac{y}{2}\right)\left(\frac{z}{3}\right) + 2\left(\frac{z}{3}\right)(x)\right\} - \\ & \left\{\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} + 2\left(\frac{x}{2}\right)\left(\frac{y}{3}\right) + 2\left(\frac{y}{3}\right)\left(\frac{z}{4}\right) + 2\left(\frac{z}{4}\right)\left(\frac{x}{2}\right)\right\} \\ &= \left(2x^2 - \frac{x^2}{4}\right) + \left(y^2 + \frac{y^2}{4} - \frac{y^2}{9}\right) + \left(z^2 + \frac{z^2}{9} - \frac{z^2}{16}\right) + \left(2xy + xy - \frac{xy}{3}\right) + \\ & \left(2yz + \frac{yz}{3} - \frac{yz}{6}\right) + \left(2xz + \frac{xz}{6} - \frac{xz}{4}\right) \\ &= \left(\frac{8x^2 - x^2}{4}\right) + \left(\frac{36y^2 + 9y^2 - 4y^2}{4}\right) + \left(\frac{144z^2 + 16z^2 - 9z^2}{144}\right) + \left(\frac{8xy}{3}\right) + \left(\frac{13yz}{6}\right) + \left(\frac{29xz}{12}\right) \\ &= \left(\frac{7x^2}{4}\right) + \left(\frac{41y^2}{4}\right) + \left(\frac{151z^2}{144}\right) + \left(\frac{8xy}{3}\right) + \left(\frac{13yz}{6}\right) + \left(\frac{29xz}{12}\right) \end{aligned}$$

(ii) Using the identity,

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$\begin{aligned} & (x+y-2z)^2 - x^2 - y^2 - 3z^2 + 4xy \\ &= x^2 + y^2 + 4z^2 + 2xy - 4yz - 4xz - x^2 - y^2 - 3z^2 + 4xy \\ &= z^2 + 6xy - 4xz - 4xy \end{aligned}$$

(iii) Using the identity,

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$\begin{aligned} & (x^2 - x + 1)^2 - (x^2 + x + 1)^2 \\ &= \{x^4 + x^2 + 1 - 2x^3 - 2x + 2x^2\} - \{x^4 + x^2 + 1 + 2x^3 + 2x + 2x^2\} \\ &= -4x^3 - 4x \end{aligned}$$

Exercise 4.3

1. Question

Find the cube of each of the following binomial expressions:

$$(i) \left(\frac{1}{x} + \frac{y}{3}\right)^3$$

$$(ii) \left(\frac{3}{x} - \frac{2}{x}\right)^3$$

$$(iii) \left(2x - \frac{3}{x}\right)^3$$

$$(iv) \left(4 - \frac{1}{3x}\right)^3$$

Answer

(i) Using the identity, $(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

We will write the binomial expression,

$$\begin{aligned} \text{Here, } a &= \frac{1}{x} \text{ and } b = \frac{y}{3} \\ \Rightarrow \left(\frac{1}{x} + \frac{y}{3}\right)^3 &= \left(\frac{1}{x}\right)^3 + \left(\frac{y}{3}\right)^3 + 3\left(\frac{1}{x}\right)^2 \times \left(\frac{y}{3}\right) + 3\left(\frac{y}{3}\right)^2 \times \left(\frac{1}{x}\right) \\ &= \frac{1}{x^3} + \frac{y^3}{27} + \frac{y}{x^2} + \frac{y^2}{3x} \end{aligned}$$

(ii) Using the identity, $(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$

We will write the binomial expression,

$$\begin{aligned} \text{Here, } a &= \frac{3}{x} \text{ and } b = \frac{2}{x^2} \\ \Rightarrow \left(\frac{3}{x} - \frac{2}{x^2}\right)^3 &= \left(\frac{3}{x}\right)^3 - \left(\frac{2}{x^2}\right)^3 - 3\left(\frac{3}{x}\right)^2 \times \left(\frac{2}{x^2}\right) + 3\left(\frac{2}{x^2}\right)^2 \times \left(\frac{3}{x}\right) \\ &= \frac{27}{x^3} - \frac{8}{x^6} - \frac{54}{x^4} + \frac{36}{x^5} \end{aligned}$$

(iii) Using the identity, $(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$

We will write the binomial expression,

$$\begin{aligned} \text{Here, } a &= 2x \text{ and } b = \frac{3}{x} \\ \Rightarrow \left(2x + \frac{3}{x}\right)^3 &= (2x)^3 + \left(\frac{3}{x}\right)^3 + 3(2x)^2 \times \left(\frac{3}{x}\right) + 3(2x) \times \left(\frac{3}{x}\right)^2 \\ &= 8x^3 + \frac{27}{x^3} + 36x + \frac{54}{x} \end{aligned}$$

(iv) Using the identity, $(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$

We will write the binomial expression,

$$\begin{aligned} \text{Here, } a &= 2x \text{ and } b = \frac{3}{x} \\ \Rightarrow \left(2x + \frac{3}{x}\right)^3 &= (2x)^3 + \left(\frac{3}{x}\right)^3 + 3(2x)^2 \times \left(\frac{3}{x}\right) + 3(2x) \times \left(\frac{3}{x}\right)^2 \\ &= 8x^3 + \frac{27}{x^3} + 36x + \frac{54}{x} \end{aligned}$$

2. Question

Simplify each of the following :

$$(i) (x+3)^3 + (x-3)^3$$

$$(ii) \left(\frac{x}{2} + \frac{y}{3}\right)^3 - \left(\frac{x}{2} - \frac{y}{3}\right)^3$$

$$(iii) \left(x + \frac{2}{x}\right)^3 + \left(x - \frac{2}{x}\right)^3$$

$$(iv) (2x-5y)^3 - (2x+5y)^3$$

Answer

(i) Using the identity, $a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$

Here, $a = (x+3)$ and $b = (x-3)$

$$\begin{aligned}(x+3)^3 + (x-3)^3 &= (x+3+x-3)((x+3)^2 + (x-3)^2 - (x+3)(x-3)) \\&= 2x \{x^2 + 9 + 6x + x^2 + 9 - 6x - x^2 + 9\} \\&= 2x(x^2 + 27) \\&= 2x^3 + 54\end{aligned}$$

(ii) Using the identity, $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$$\begin{aligned}\text{Here, } a &= \left(\frac{x}{2} + \frac{y}{3}\right) \text{ and } b = \left(\frac{x}{2} - \frac{y}{3}\right) \\&\Rightarrow \left(\frac{x}{2} + \frac{y}{3}\right)^3 - \left(\frac{x}{2} - \frac{y}{3}\right)^3 \\&= \left[\left(\frac{x}{2} + \frac{y}{3}\right) - \left(\frac{x}{2} - \frac{y}{3}\right)\right] \left\{\left(\frac{x}{2} + \frac{y}{3}\right)^2 + \left(\frac{x}{2} + \frac{y}{3}\right)\left(\frac{x}{2} - \frac{y}{3}\right) + \left(\frac{x}{2} - \frac{y}{3}\right)^2\right\} \\&= \frac{2y}{3} \left[\left(\frac{x^2}{4} + \frac{y^2}{9} + 2 \cdot \frac{x}{2} \cdot \frac{y}{3}\right) + \left(\frac{x^2}{4} + \frac{y^2}{9} - 2 \cdot \frac{x}{2} \cdot \frac{y}{3}\right) + \left(\frac{x^2}{4} - \frac{y^2}{9}\right)\right] \\&= \frac{2y}{3} \left[\left(\frac{x^2}{4} + \frac{y^2}{9} + \frac{xy}{3}\right) + \left(\frac{x^2}{4} + \frac{y^2}{9} - \frac{xy}{3}\right) + \left(\frac{x^2}{4} - \frac{y^2}{9}\right)\right] \\&= \frac{2y}{3} \left[\frac{3x^2}{4} + \frac{y^2}{9}\right] \\&= \frac{3x^2y}{2} + \frac{2y^3}{9}\end{aligned}$$

(iii) Using the identity, $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$$\begin{aligned}\text{Here, } a &= \left(x + \frac{2}{x}\right) \text{ and } b = \left(x - \frac{2}{x}\right) \\&\Rightarrow \left(x + \frac{2}{x}\right)^3 + \left(x - \frac{2}{x}\right)^3 \\&= \left[\left(x + \frac{2}{x}\right) + \left(x - \frac{2}{x}\right)\right] \left\{\left(x + \frac{2}{x}\right)^2 + \left(x - \frac{2}{x}\right)^2 - \left(x + \frac{2}{x}\right)\left(x - \frac{2}{x}\right)\right\} \\&= 2x \left[\left(x^2 + \frac{4}{x^2} + 2x \cdot \frac{2}{x}\right) + \left(x^2 + \frac{4}{x^2} - 2x \cdot \frac{2}{x}\right) - \left(x^2 - \frac{4}{x^2}\right)\right] \\&= 2x \left[x^2 + \frac{4}{x^2} + \frac{4}{x^2} + \frac{4}{x^2}\right] \\&= 2x \left[x^2 + \frac{12}{x^2}\right] \\&= 2x^3 + \frac{24}{x}\end{aligned}$$

(iv) Using the identity, $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Here, $a = (2x-5y)$ and $b = (2x+5y)$

$$\begin{aligned}&\Rightarrow (2x-5y)^3 - (2x+5y)^3 \\&= ((2x-5y) - (2x+5y)) \left[(2x-5y)^2 + (2x+5y)^2 + (2x-5y)(2x+5y) \right] \\&= (-10y) \left[(4x^2 + 25y^2 - 20xy) + (4x^2 + 25y^2 + 20xy) + 4x^2 - 25y^2 \right] \\&= (-10y) \left[4x^2 + 4x^2 + 4x^2 + 25y^2 \right] \\&= -120x^2y - 250y^3\end{aligned}$$

3. Question

If $a+b=10$ and $ab=21$, find the value of a^3+b^3 .

Answer

Given $(a+b)=10$ and $ab = 21$

Using, $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$, we get,

$$\Rightarrow (10)^3 = a^3 + b^3 + 3(21)(10)$$

$$\Rightarrow 1000 = a^3 + b^3 + 630$$

$$\Rightarrow a^3 + b^3 = 1000 - 630 = 370$$

4. Question

If $a-b=4$ and $ab=21$, find the value of a^3-b^3 .

Answer

Given $(a - b)=4$ and $ab = 21$

Using, $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$, we get,

$$\Rightarrow (4)^3 = a^3 - b^3 - 3(21)(4)$$

$$\Rightarrow 64 = a^3 - b^3 - 252$$

$$\Rightarrow a^3 - b^3 = 252 + 64 = 316$$

5. Question

If $x + \frac{1}{x} = 5$, find the value of $x^3 + \frac{1}{x^3}$.

Answer

Given: $x + \frac{1}{x} = 5$,

Using, $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$, we get

$$\left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x \cdot \frac{1}{x}\right)\left(x + \frac{1}{x}\right)$$

$$5^3 = x^3 + \frac{1}{x^3} + 3 \times 5$$

$$x^3 + \frac{1}{x^3} = 125 - 15 = 110$$

6. Question

If $x - \frac{1}{x} = 7$, find the value of $x^3 - \frac{1}{x^3}$.

Answer

Given: $x - \frac{1}{x} = 7$

Using, $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$, we get

$$\left(x - \frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3\left(x \cdot \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$$

$$7^3 = x^3 - \frac{1}{x^3} - 3 \times 7$$

$$x^3 - \frac{1}{x^3} = 343 + 21 = 364$$

7. Question

If $x - \frac{1}{x} = 5$, find the value of $x^3 - \frac{1}{x^3}$.

Answer

Given: $x - \frac{1}{x} = 5$

Using, $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$, we get

$$\left(x - \frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3\left(x \cdot \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$$

$$5^3 = x^3 - \frac{1}{x^3} - 3 \times 5$$

$$x^3 - \frac{1}{x^3} = 125 + 15 = 140$$

8. Question

If $x^2 + \frac{1}{x^2} = 51$, find the value of $x^3 - \frac{1}{x^3}$.

Answer

Using the identity, $(x+y)^2 = x^2 + y^2 + 2xy$

$$\left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2 \times x \times \frac{1}{x}$$

$$\Rightarrow \left(x - \frac{1}{x}\right)^2 = 51 - 2 = 49$$

$$\Rightarrow \left(x - \frac{1}{x}\right) = \sqrt{49} = \pm 7$$

Now, using, $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$

$$\Rightarrow x^3 - \left(\frac{1}{x}\right)^3 = \left(x - \frac{1}{x}\right) \left[\left(x\right)^2 + \left(\frac{1}{x}\right)^2 + x \times \frac{1}{x} \right]$$

$$\Rightarrow x^3 - \left(\frac{1}{x}\right)^3 = 7 \left[51 + x \times \frac{1}{x} \right] = 7 \times 52 = 364$$

9. Question

If $x^2 + \frac{1}{x^2} = 98$, find the value of $x^3 + \frac{1}{x^3}$.

Answer

Using the identity, $(x+y)^2 = x^2 + y^2 + 2xy$

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 \times x \times \frac{1}{x}$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 98 + 2 = 100$$

$$\Rightarrow \left(x + \frac{1}{x}\right) = \sqrt{100} = \pm 10$$

Now, using, $a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$

$$\Rightarrow x^3 + \left(\frac{1}{x}\right)^3 = \left(x + \frac{1}{x}\right) \left[\left(x\right)^2 + \left(\frac{1}{x}\right)^2 - x \times \frac{1}{x} \right]$$

$$\Rightarrow x^3 + \left(\frac{1}{x}\right)^3 = 10 \left[98 - x \times \frac{1}{x} \right] = 10 \times 97 = 970$$

10. Question

If $2x+3y=13$ and $xy=6$, find the value of $8x^3+27y^3$.

Answer

Given :- $2x+3y=13$ and $xy=6$.

Using, $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$, we get

$$\begin{aligned}
 (2x+3y)^3 &= (13)^3 \\
 \Rightarrow 8x^3 + 27y^3 + 3(2x)(3y)(2x+3y) &= 2197 \\
 \Rightarrow 8x^3 + 27y^3 + 18(6)(13) &= 2197 \\
 \Rightarrow 8x^3 + 27y^3 &= 2197 - 1404 = 793
 \end{aligned}$$

11. Question

If $3x-2y=11$ and $xy=12$, find the value of $27x^3-8y^3$.

Answer

Given :- $3x-2y=11$ and $xy=12$.

Using, $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$, we get

$$\begin{aligned}
 (3x-2y)^3 &= (11)^3 \\
 \Rightarrow 27x^3 - 8y^3 - 3(3x)(2y)(3x-2y) &= 1331 \\
 \Rightarrow 27x^3 - 8y^3 - 18(11)(12) &= 1331 \\
 \Rightarrow 27x^3 - 8y^3 &= 1331 + 2376 = 3707
 \end{aligned}$$

12. Question

If $x^4 + \frac{1}{x^4} = 119$, find the value of $x^3 - \frac{1}{x^3}$.

Answer

Given :- $x^4 + \frac{1}{x^4} = 119$

Using, $(a+b)^2 = a^2 + b^2 + 2ab$, we get

$$\begin{aligned}
 \Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 &= x^4 + \frac{1}{x^4} + 2 \times x^2 \times \frac{1}{x^2} \\
 \Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 &= x^4 + \frac{1}{x^4} + 2 = 119 + 2 = 121 \\
 \Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 &= \sqrt{121} \\
 \Rightarrow \left(x^2 + \frac{1}{x^2}\right) &= \pm 11
 \end{aligned}$$

Now, using $(x-y)^2 = x^2 + y^2 - 2xy$

$$\begin{aligned}
 \Rightarrow \left(x - \frac{1}{x}\right)^2 &= \left(x^2 + \frac{1}{x^2} - 2 \times x^2 \times \frac{1}{x^2}\right) \\
 \Rightarrow \left(x - \frac{1}{x}\right)^2 &= (11 - 2) = 9 \\
 \Rightarrow \left(x - \frac{1}{x}\right) &= \sqrt{9} = \pm 3
 \end{aligned}$$

Now, using $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$

$$\begin{aligned}
 \Rightarrow \left(x^3 + \frac{1}{x^3}\right) &= \left(x - \frac{1}{x}\right) \left(x^2 + \frac{1}{x^2} + x^2 \times \frac{1}{x^2}\right) \\
 \Rightarrow \left(x^3 + \frac{1}{x^3}\right) &= 3(11 + 1) = 3 \times 12 \\
 \Rightarrow \left(x^3 + \frac{1}{x^3}\right) &= 36
 \end{aligned}$$

13. Question

Evaluate each of the following:

(i) $(103)^3$

(ii) $(98)^3$

(iii) $(9.9)^3$

(iv) $(10.4)^3$

(v) $(598)^3$

(vi) $(99)^3$

Answer

(i) Using the identity,

$$\begin{aligned}(a+b)^3 &= a^3 + b^3 + 3ab(a+b) \\ \Rightarrow (103)^3 &= (100+3)^3 \\ \Rightarrow (100+3)^3 &= 100^3 + 3^3 + 3 \cdot 100 \cdot 3(100+3) \\ \Rightarrow (100+3)^3 &= 1000000 + 27 + 92700 \\ \Rightarrow (103)^3 &= 1092727\end{aligned}$$

(ii) Using the identity,

$$\begin{aligned}(a-b)^3 &= a^3 - b^3 - 3ab(a-b) \\ \Rightarrow (98)^3 &= (100-2)^3 \\ \Rightarrow (98)^3 &= 100^3 - 2^3 - 3 \cdot 100 \cdot 2(100-2) \\ \Rightarrow (98)^3 &= 1000000 - 8 - 58800 \\ \Rightarrow (98)^3 &= 941192\end{aligned}$$

(iii) Using the identity,

$$\begin{aligned}(a-b)^3 &= a^3 - b^3 - 3ab(a-b) \\ \Rightarrow (9.9)^3 &= (10-0.1)^3 \\ \Rightarrow (9.9)^3 &= 10^3 - 0.1^3 - 3 \cdot 10 \cdot (0.1)(10-0.1) \\ \Rightarrow (9.9)^3 &= 1000 - 0.001 - 29.7 \\ \Rightarrow (9.9)^3 &= 1000 - 29.701 \\ \Rightarrow (9.9)^3 &= 970.299\end{aligned}$$

(iv) Using the identity,

$$\begin{aligned}(a+b)^3 &= a^3 + b^3 + 3ab(a+b) \\ \Rightarrow (10.4)^3 &= (10+0.4)^3 \\ &= 10^3 + 0.4^3 + 3 \cdot 10 \cdot (0.4)(10+0.4) \\ &= 1000 + 0.064 + 124.8 \\ &= 1000 + 124.864 \\ \Rightarrow (10.4)^3 &= 1124.864\end{aligned}$$

(v) Using the identity,

$$\begin{aligned}(a-b)^3 &= a^3 - b^3 - 3ab(a-b) \\ \Rightarrow (598)^3 &= (600-2)^3 \\ \Rightarrow (598)^3 &= 600^3 - 2^3 - 3 \cdot 600 \cdot 2(600-2) \\ \Rightarrow (598)^3 &= 216000000 - 8 - 2152800 \\ \Rightarrow (598)^3 &= 213847192\end{aligned}$$

(vi) Using the identity,

$$\begin{aligned}
 (a-b)^3 &= a^3 - b^3 - 3ab(a-b) \\
 \Rightarrow (99)^3 &= (100-1)^3 \\
 \Rightarrow (99)^3 &= 100^3 - 1^3 - 3 \cdot 100 \cdot 1(100-1) \\
 \Rightarrow (99)^3 &= 1000000 - 1 - 29700 \\
 \Rightarrow (99)^3 &= 970299
 \end{aligned}$$

14. Question

Evaluate each of the following :

(i) $111^3 - 89^3$

(ii) $46^3 + 34^3$

(iii) $104^3 + 96^3$

(iv) $93^3 - 107^3$

Answer

(i) Using the identity:

$$\begin{aligned}
 (a+b)^3 - (a-b)^3 &= 2(b^3 + 3a^2b) \\
 \text{Here, } a &= 100, b = 11 \\
 \Rightarrow (111)^3 - (89)^3 &= (100+11)^3 - (100-11)^3 \\
 &= 2\{11^3 + 3(100)^2(11)\} \\
 &= 2\{1331 + 330000\} \\
 \Rightarrow (111)^3 - (89)^3 &= 662662
 \end{aligned}$$

(ii) Using the identity:

$$\begin{aligned}
 (a+b)^3 + (a-b)^3 &= 2(a^3 + 3ab^2) \\
 \text{Here, } a &= 40, b = 6 \\
 \text{so, applying the formula,} \\
 \Rightarrow (46)^3 + (34)^3 &= (40+6)^3 + (40-6)^3 \\
 &= 2\{40^3 + 3(6)^2(40)\} \\
 &= 2\{64000 + 4320\} \\
 \Rightarrow (46)^3 + (34)^3 &= 136640
 \end{aligned}$$

(iii) Using the identity:

$$\begin{aligned}
 (a+b)^3 - (a-b)^3 &= 2(a^3 + 3b^2a) \\
 \text{Here, } a &= 100, b = 4 \\
 \Rightarrow (104)^3 - (96)^3 &= (100+4)^3 - (100-4)^3 \\
 &= 2\{100^3 + 3(4)^2(100)\} \\
 &= 2\{1000000 + 4800\} \\
 &= 2\{1004800\} \\
 \Rightarrow (104)^3 - (96)^3 &= 2009600
 \end{aligned}$$

(iv) Using the identity:

$$(a-b)^3 - (a+b)^3 = -2(b^3 + 3a^2b)$$

Here, $a = 100$, $b = 7$

$$\Rightarrow (93)^3 - (107)^3 = (100-7)^3 - (100+7)^3$$

$$\begin{aligned}\Rightarrow (100-7)^3 - (100+7)^3 &= -2\{7^3 + 3(100)^2(7)\} \\ &= -2\{343 + 210000\} \\ &= -2\{210343\}\end{aligned}$$

$$\Rightarrow (104)^3 + (96)^3 = -420686$$

15. Question

If $x + \frac{1}{x} = 3$, Calculate $x^2 + \frac{1}{x^2}$, $x^3 + \frac{1}{x^3}$ and $x^4 + \frac{1}{x^4}$.

Answer

Given $x + \frac{1}{x} = 3$

Using, $(a+b)^2 = a^2 + b^2 + 2ab$

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \left(\frac{1}{x^2}\right) + 2.x.\frac{1}{x}$$

$$\Rightarrow (3)^2 = x^2 + \left(\frac{1}{x^2}\right) + 2$$

$$\Rightarrow \boxed{x^2 + \frac{1}{x^2} = 9 - 2 = 7}$$

$$\text{Now, } \left(x^2 + \frac{1}{x^2}\right)^2 = 7^2$$

$$\Rightarrow x^4 + \frac{1}{x^4} + 2.x^2.\frac{1}{x^2} = 49$$

$$\Rightarrow x^4 + \frac{1}{x^4} = 49 - 2$$

$$\Rightarrow \boxed{x^4 + \frac{1}{x^4} = 47}$$

$$\text{Now, } \left(x + \frac{1}{x}\right)^3 = x^3 + \left(\frac{1}{x^3}\right) + 3.x.\frac{1}{x}\left(x + \frac{1}{x}\right)$$

$$\Rightarrow 27 = x^3 + \left(\frac{1}{x^3}\right) + 3.3$$

$$\Rightarrow x^3 + \left(\frac{1}{x^3}\right) = 27 - 9$$

$$\Rightarrow \boxed{x^3 + \left(\frac{1}{x^3}\right) = 18}$$

16. Question

If $x^4 + \frac{1}{x^4} = 194$, find $x^3 + \frac{1}{x^3}$, $x^2 + \frac{1}{x^2}$ and $x + \frac{1}{x}$

Answer

Given $x^4 + \frac{1}{x^4} = 194$

Adding and subtracting, $2.x^2.\frac{1}{x^2}$ on both sides,

$$\Rightarrow x^4 + \frac{1}{x^4} + 2.x^2.\frac{1}{x^2} = 194 + 2.x^2.\frac{1}{x^2}$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = 196$$

$$\Rightarrow x^2 + \frac{1}{x^2} = \sqrt{196}$$

$$\Rightarrow \boxed{x^2 + \frac{1}{x^2} = \pm 14}$$

Now, adding and subtracting, $2.x.\frac{1}{x}$ on both sides,

$$\Rightarrow x^2 + \frac{1}{x^2} + 2.x.\frac{1}{x} = 14 + 2.x.\frac{1}{x}$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 16$$

$$\Rightarrow x + \frac{1}{x} = \sqrt{16}$$

$$\Rightarrow \boxed{x + \frac{1}{x} = \pm 4}$$

Now, cubing on both sides,

$$\Rightarrow \left(x + \frac{1}{x}\right)^3 = 4^3$$

$$\Rightarrow x^3 + \frac{1}{x^3} + 3.x.\frac{1}{x}\left(x + \frac{1}{x}\right) = 64$$

$$\Rightarrow x^3 + \frac{1}{x^3} + 3.4 = 64$$

$$\Rightarrow x^3 + \frac{1}{x^3} = 64 - 12 = 52$$

17. Question

Find the value of $27x^3 + 8y^3$, if

(i) $3x + 2y = 14$ and $xy=8$

(ii) $3x + 2y = 20$ and $xy=\frac{14}{9}$

Answer

(i) Using,

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b)$$

$$\Rightarrow (3x+2y)^3 = (3x)^3 + (2y)^3 + 3.3x.2y(3x+2y)$$

$$\Rightarrow (14)^3 = 27x^3 + 8y^3 + 18.xy(14)$$

$$\Rightarrow 2744 = 27x^3 + 8y^3 + 18.8.14$$

$$\Rightarrow 27x^3 + 8y^3 = 2744 - 2016$$

$$\Rightarrow 27x^3 + 8y^3 = 728$$

(ii) Using the identity, we get,

$$\begin{aligned}
(a+b)^3 &= a^3 + b^3 + 3ab(a+b) \\
\Rightarrow (3x+2y)^3 &= (3x)^3 + (2y)^3 + 3.3x.2y(3x+2y) \\
\Rightarrow (20)^3 &= 27x^3 + 8y^3 + 18.xy(20) \\
\Rightarrow 8000 &= 27x^3 + 8y^3 + 18.\frac{14}{9}.20 \\
\Rightarrow 27x^3 + 8y^3 &= 8000 - 560 \\
\Rightarrow 27x^3 + 8y^3 &= 7440
\end{aligned}$$

18. Question

Find the value of $64x^3 - 125z^3$, if $4x - 5z = 16$ and $xz=12$.

Answer

Using the identity, we write,

$$\begin{aligned}
(a-b)^3 &= a^3 - b^3 - 3ab(a-b) \\
\Rightarrow (4x-5z)^3 &= (4x)^3 - (5z)^3 - 3.4x.5z(4x-5z) \\
\Rightarrow (16)^3 &= 64x^3 - 125z^3 - 60.xz(16) \\
\Rightarrow 4096 &= 64x^3 - 125z^3 - 60.xz(16) \\
\Rightarrow 64x^3 - 125z^3 &= 4096 + 11520 \\
\Rightarrow 64x^3 - 125z^3 &= 15616
\end{aligned}$$

19. Question

If $x - \frac{1}{x} = 3 + 2\sqrt{2}$, find the value of $x^3 - \frac{1}{x^3}$.

Answer

Using the identity, we write,

$$\begin{aligned}
(a-b)^3 &= a^3 - b^3 - 3ab(a-b) \\
(a+b)^3 &= a^3 + b^3 + 3ab(a+b) \\
\Rightarrow \left(x - \frac{1}{x}\right)^3 &= (3 + 2\sqrt{2})^3 \\
\Rightarrow x^3 - \frac{1}{x^3} - 3.x.\frac{1}{x}.\left(x - \frac{1}{x}\right) &= 3^3 + (2\sqrt{2})^3 + 3.3.2\sqrt{2}(3 + 2\sqrt{2}) \\
\Rightarrow x^3 - \frac{1}{x^3} - \{3.(3 + 2\sqrt{2})\} &= 27 + 16\sqrt{2} + 18\sqrt{2}(3 + 2\sqrt{2}) \\
\Rightarrow x^3 - \frac{1}{x^3} &= 27 + 16\sqrt{2} + 54\sqrt{2} + 72 + 9 + 6\sqrt{2} \\
\Rightarrow \boxed{x^3 - \frac{1}{x^3} = 108 + 76\sqrt{2}}
\end{aligned}$$

Exercise 4.4

1. Question

Find the following products:

(i) $(3x+2y)(9x^2-6xy+4y^2)$

(ii) $(4x-5y)(16x^2+20xy+25y^2)$

(iii) $(7p^4+q)(49p^8-7p^4q+q^2)$

(iv) $\left(\frac{x}{2} + 2y\right)\left(\frac{x^2}{4} - xy + 4y^2\right)$

(v) $\left(\frac{3}{x} - \frac{5}{y}\right)\left(\frac{9}{x^2} + \frac{25}{y^2} + \frac{15}{xy}\right)$

$$(vi) \left(3 + \frac{5}{x}\right)\left(9 - \frac{15}{x} + \frac{25}{x^2}\right)$$

$$(vii) \left(\frac{2}{x} + 3x\right)\left(\frac{4}{x^2} + 9x^2 - 6\right)$$

$$(viii) \left(\frac{3}{x} - 2x^2\right)\left(\frac{9}{x^2} + 4x^2 - 6x\right)$$

$$(ix) (1+x)(1+x+x^2)$$

$$(x) (1+x)(1-x+x^2)$$

$$(xi) (x^2-1)(x^4+x^2+1)$$

$$(xii) (x^3+1)(x^6-x^3+1)$$

Answer

(i)

$$\begin{aligned} & (3x+2y)(9x^2-6xy+4y^2) \\ & \Rightarrow (3x+2y)\{(3x)^2-2.3x.2y+(2y)^2\} \\ & \Rightarrow (3x)^3+(2y)^3 \\ & = 27x^3+8y^3 \end{aligned}$$

(ii)

$$\begin{aligned} & (4x-5y)(16x^2+20xy+25y^2) \\ & \Rightarrow (4x-5y)\{(4x)^2-2.5x.5y+(5y)^2\} \\ & \Rightarrow (4x)^3-(5y)^3 \\ & = 16x^3-125y^3 \end{aligned}$$

(iii)

$$\begin{aligned} & (7p^4+q)(49p^3+7p^4q+q^2) \\ & \Rightarrow (7p^4+q)\{(7p^4)^2-2.7p^4.q+(q)^2\} \\ & \Rightarrow (7p^4)^3+(q)^3 \\ & = 343p^{12}-q^3 \end{aligned}$$

(iv)

$$\begin{aligned} & \left(\frac{x}{2}+2y\right)\left(\frac{x^2}{4}-xy+4y^2\right) \\ & = \left(\frac{x}{2}+2y\right)\left(\left(\frac{x}{2}\right)^2-\frac{x}{2}.2y+(2y)^2\right) \\ & = \left(\frac{x}{2}\right)^3+(2y)^3 \\ & = \frac{x^3}{8}+8y^3 \end{aligned}$$

(v)

$$\begin{aligned} & \left(\frac{3}{x}-\frac{5}{y}\right)\left(\frac{9}{x^2}+\frac{25}{y^2}+\frac{15}{xy}\right) \\ & = \left(\frac{3}{x}-\frac{5}{y}\right)\left(\left(\frac{3}{x}\right)^2+\left(\frac{5}{y}\right)^2+\frac{3.5}{xy}\right) \\ & = \left(\frac{3}{x}\right)^3-\left(\frac{5}{y}\right)^3 \\ & = \frac{27}{x^3}-\frac{125}{y^3} \end{aligned}$$

(vi)

$$\begin{aligned}
& \left(3 + \frac{5}{x}\right) \left(9 + \frac{25}{x^2} - \frac{15}{x}\right) \\
&= \left(3 + \frac{5}{x}\right) \left(3^2 + \left(\frac{5}{x}\right)^2 + \frac{3 \cdot 5}{x}\right) \\
&= (3)^3 + \left(\frac{5}{x}\right)^3 \\
&= 27 + \frac{125}{x^3}
\end{aligned}$$

(vii)

$$\begin{aligned}
& \left(\frac{2}{x} + 3x\right) \left(\frac{4}{x^2} - 6 + 9x^2\right) \\
&= \left(\frac{2}{x} + 3x\right) \left\{ \left(\frac{2}{x}\right)^2 - \frac{2}{x} \cdot 3 + (3x)^2 \right\} \\
&= \left(\frac{2}{x}\right)^3 + (3x)^3 \\
&= \frac{8}{x^3} + 27x^3
\end{aligned}$$

(viii)

$$\begin{aligned}
& \left(\frac{3}{x} - 2x^2\right) \left(\frac{9}{x^2} + 6x + 4x^4\right) \\
&= \left(\frac{3}{x} - 2x^2\right) \left\{ \left(\frac{3}{x}\right)^2 + \frac{3}{x} \cdot 2x^2 + (2x^2)^2 \right\} \\
&= \left(\frac{3}{x}\right)^3 - (2x^2)^3 \\
&= \frac{27}{x^3} - 8x^6
\end{aligned}$$

(ix)

$$\begin{aligned}
& (1-x)(1+x^2+x) \\
&\Rightarrow (1-x)(1^2+x^2+1 \cdot x) \\
&\text{Because, } a^3 - b^3 = (a-b)(a^2+b^2+ab) \\
&\Rightarrow (1^3 - x^3)
\end{aligned}$$

(x)

$$\begin{aligned}
& (1+x)(1+x^2-x) \\
&\Rightarrow (1+x)(1^2+x^2-1 \cdot x) \\
&\text{Because, } a^3 + b^3 = (a+b)(a^2+b^2-ab) \\
&\Rightarrow (1^3 + x^3)
\end{aligned}$$

(xi)

$$\begin{aligned}
& (x^2-1)(x^4+x^2+1) \\
&\Rightarrow (x^2-1)\{(x^2)^2+1 \cdot x^2+1^2\} \\
&\text{Because, } a^3 - b^3 = (a-b)(a^2+b^2+ab) \\
&\Rightarrow (x^2)^3 - 1^3 \\
&\Rightarrow (x^6-1)
\end{aligned}$$

(xii)

$$\begin{aligned}
& (x^3+1)(x^6-x^3+1) \\
&\Rightarrow (x^3+1)\{(x^3)^2-1 \cdot x^3+1^2\} \\
&\text{Because, } a^3 + b^3 = (a+b)(a^2+b^2-ab) \\
&\Rightarrow (x^3)^3 + 1^3 \\
&\Rightarrow (x^9+1)
\end{aligned}$$

2. Question

If $x = 3$ and $y = -1$, find the values of each of the following using in identity:

(i) $(9y^2 - 4x^2)(81y^4 + 36x^2y^2 + 16x^4)$

(ii) $\left(\frac{3}{x} - \frac{x}{3}\right)\left(\frac{x^2}{9} + \frac{9}{x^2} + 1\right)$

(iii) $\left(\frac{x}{7} + \frac{y}{3}\right)\left(\frac{x^2}{49} + \frac{y^2}{9} - \frac{xy}{21}\right)$

(iv) $\left(\frac{x}{4} - \frac{y}{3}\right)\left(\frac{x^2}{16} + \frac{xy}{12} + \frac{y^2}{9}\right)$

(v) $\left(\frac{5}{x} + 5x\right)\left(\frac{25}{x^2} - 25 + 25x^2\right)$

Answer

(i)

$$\begin{aligned} & (9y^2 - 4x^2)(81y^4 + 36x^2y^2 + 16x^4) \\ & \Rightarrow (9y^2 - 4x^2)\{(9y^2)^2 + 4x^2 \cdot 9y^2 + (4x^2)^2\} \\ & \text{Because, } a^3 - b^3 = (a - b)(a^2 + b^2 + ab) \\ & \Rightarrow (9y^2)^3 - (4x^2)^3 \\ & \text{here, } x = 3 \text{ and } y = -1 \\ & \Rightarrow 729 - 46656 \\ & = -45927 \end{aligned}$$

(ii)

$$\begin{aligned} & \left(\frac{3}{x} - \frac{x}{3}\right)\left(\frac{x^2}{9} + \frac{9}{x^2} + 1\right) \\ & \Rightarrow \left(\frac{3}{x} - \frac{x}{3}\right)\left(\left(\frac{x}{3}\right)^2 + \left(\frac{3}{x}\right)^2 + \frac{3}{x} \cdot \frac{x}{3}\right) \\ & \Rightarrow \left(\frac{3}{x}\right)^3 - \left(\frac{x}{3}\right)^3 \\ & \Rightarrow \frac{27}{x^3} - \frac{x^3}{27} \\ & \text{Now, } x=3 \\ & \Rightarrow \frac{27}{3^3} - \frac{3^3}{27} = 1 - 1 = 0 \end{aligned}$$

(iii)

$$\begin{aligned} & \left(\frac{x}{7} + \frac{y}{3}\right)\left(\frac{x^2}{49} + \frac{y^2}{9} - \frac{xy}{21}\right) \\ & \Rightarrow \left(\frac{x}{7} + \frac{y}{3}\right)\left(\left(\frac{x}{7}\right)^2 + \left(\frac{y}{3}\right)^2 - \frac{x}{7} \cdot \frac{y}{3}\right) \\ & \Rightarrow \left(\frac{x}{7}\right)^3 + \left(\frac{y}{3}\right)^3 \\ & \Rightarrow \frac{x^3}{343} + \frac{y^3}{27} \\ & \text{Now, } x=3 \text{ and } y=-1 \\ & \Rightarrow \frac{27}{343} + \frac{-1}{27} = \frac{386}{9261} \end{aligned}$$

(iv)

$$\begin{aligned} & \left(\frac{x}{4} - \frac{y}{3}\right) \left(\frac{x^2}{16} + \frac{y^2}{9} + \frac{xy}{12}\right) \\ \Rightarrow & \left(\frac{x}{4} - \frac{y}{3}\right) \left(\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 + \frac{x}{4} \cdot \frac{y}{3}\right) \\ \Rightarrow & \left(\frac{x}{4}\right)^3 - \left(\frac{y}{3}\right)^3 \\ \Rightarrow & \frac{x^3}{64} + \frac{y^3}{27} \end{aligned}$$

Now, $x=3$ and $y=-1$

$$\Rightarrow \frac{27}{64} + \frac{1}{27} = \frac{729+64}{1728} = \frac{793}{1728}$$

(v)

$$\begin{aligned} & \left(\frac{5}{x} + 5x\right) \left(\frac{25}{x^2} + (5x)^2 - \frac{5}{x} \cdot 5x\right) \\ \Rightarrow & \left(\frac{5}{x} + 5x\right) \left(\left(\frac{5}{x}\right)^2 + (5x)^2 - \frac{5}{x} \cdot 5x\right) \\ \Rightarrow & \left(\frac{5}{x}\right)^3 + (5x)^3 \\ = & \frac{125}{x^3} + 125x^3 \quad (x=3) \\ = & \frac{125}{3^3} + 125 \cdot 3^3 \\ = & \frac{125}{27} + 3375 \\ = & \frac{91250}{27} \end{aligned}$$

3. Question

If $a+b=10$ and $ab=16$, find the value of $a^2 - ab + b^2$ and $a^2 + ab + b^2$.

Answer

Given: $a+b=10$ and $ab=16$

To find: $a^2 - ab + b^2$

$$\begin{aligned} a^2 - ab + b^2 &= a^2 + b^2 - ab \\ \Rightarrow a^2 - ab + b^2 &= a^2 + b^2 - ab + 2ab - 2ab \\ \Rightarrow a^2 - ab + b^2 &= (a^2 + b^2 + 2ab) - 2ab - ab \\ \Rightarrow a^2 - ab + b^2 &= (a+b)^2 - 3ab \\ \Rightarrow a^2 - ab + b^2 &= (10)^2 - 3 \times 16 = 52 \end{aligned}$$

To find: $a^2 + ab + b^2$

$$\begin{aligned} a^2 + ab + b^2 &= a^2 + ab + b^2 + ab - ab \\ \Rightarrow a^2 + ab + b^2 &= (a^2 + b^2 + 2ab) - ab \\ \Rightarrow a^2 + ab + b^2 &= (a+b)^2 - ab \\ \Rightarrow a^2 + ab + b^2 &= (10)^2 - 16 = 84 \end{aligned}$$

4. Question

If $a+b=8$ and $ab=6$, find the value of $a^3 + b^3$.

Answer

Given: $a+b=8$ and $ab=6$

To find: $a^3 + b^3$

$$\begin{aligned}a^3 + b^3 &= (a+b)(a^2 + b^2 - ab) \\&= (a+b)(a^2 + b^2 - ab + 2ab - 2ab) \\&= (a+b)[(a^2 + b^2 + 2ab) - 3ab] \\&= (a+b)[(a+b)^2 - 3ab] \\&= 8[8^2 - 3 \times 6] \\&= 368\end{aligned}$$

5. Question

If $a-b=6$ and $ab=20$, find the value of a^3-b^3 .

Answer

Given: $a-b=6$ and $ab=20$

To find: a^3-b^3

$$\begin{aligned}a^3 - b^3 &= (a-b)(a^2 + b^2 + ab) \\&= (a-b)(a^2 + b^2 + ab + 2ab - 2ab) \\&= (a-b)[(a^2 + b^2 - 2ab) + 3ab] \\&= (a-b)[(a-b)^2 + 3ab] \\&= 6[6^2 - 3 \times 20] \\&= 576\end{aligned}$$

6. Question

If $x = -2$ and $y = 1$, by using an identity find the value of the following:

(i) $(4y^2 - 9x^2)(16y^4 + 36x^2y^2 + 81x^4)$

(ii) $\left(\frac{2}{x} - \frac{x}{2}\right)\left(\frac{4}{x^2} + \frac{x^2}{4} + 1\right)$

(iii) $\left(5y + \frac{15}{y}\right)\left(25y^2 - 75 + \frac{225}{y^2}\right)$

Answer

(i)

$$\begin{aligned}&(4y^2 - 9x^2)(16y^4 + 36x^2y^2 + 81x^4) \\&= (4y^2 - 9x^2)((4y^2)^2 + 9x^2 \times 4y^2 + (9x^2)^2) \\&= (4y^2)^3 - (9x^2)^3 \\&= 64y^6 - 729x^6 \\&= 64(1)^6 - 729(-2)^6 \\&= 64 - 729 \times 64 \\&= 64 - 46656 \\&= \boxed{-46592}\end{aligned}$$

(ii)

$$\begin{aligned} & \left(\frac{2}{x} - \frac{x}{2}\right) \left(\frac{4}{x^2} + \frac{x^2}{4} + 1\right) \\ & \Rightarrow \left(\frac{2}{x} - \frac{x}{2}\right) \left(\left(\frac{2}{x}\right)^2 + \left(\frac{x}{2}\right)^2 + 2 \cdot \frac{2}{x} \cdot \frac{x}{2}\right) \\ & \Rightarrow \left(\frac{2}{x}\right)^3 - \left(\frac{x}{2}\right)^3 \\ & \Rightarrow \frac{8}{x^3} - \frac{x^3}{8} \end{aligned}$$

Now, $x = -2$

$$\Rightarrow \frac{8}{(-2)^3} - \frac{(-2)^3}{8} = -1 + 1 = \boxed{0}$$

(iii)

$$\begin{aligned} & \left(5y + \frac{15}{y}\right) \left(25y^2 - 75 + \frac{225}{y^2}\right) \\ & \Rightarrow \left(5y + \frac{15}{y}\right) \left((5y)^2 - 5y \times \frac{15}{y} + \left(\frac{15}{y}\right)^2\right) \\ & \Rightarrow (5y)^3 + \left(\frac{15}{y}\right)^3 \\ & \Rightarrow 125y^3 + \frac{3375}{y^3} \end{aligned}$$

Now, $y = 1$

$$\Rightarrow 125 + 3375 = \boxed{3500}$$

Exercise 4.5

1. Question

Find the following product:

(i) $(3x + 2y + 2z)(9x^2 + 4y^2 + 4z^2 - 6xy - 4yz - 6zx)$

(ii) $(4x - 3y + 2z)(16x^2 + 9y^2 + 4z^2 + 12xy + 6yz - 8zx)$

(iii) $(2a - 3b - 2c)(4a^2 + 9b^2 + 4c^2 + 6ab - 6bc + 4ca)$

(iv) $(3x - 4y + 5z)(9x^2 + 16y^2 + 25z^2 + 12xy - 15zx + 20yz)$

Answer

(i) Using the identity,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

Therefore

$$\Rightarrow (3x + 2y + 2z)(9x^2 + 4y^2 + 4z^2 - 6xy - 4yz - 6zx)$$

$$\Rightarrow (3x)^3 + (2y)^3 + (2z)^3 - 3(3x)(2y)(2z)$$

$$\Rightarrow \boxed{27x^3 + 8y^3 + 8z^3 - 36xyz}$$

(ii) Using the identity,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

Therefore

$$\Rightarrow (4x - 3y + 2z)(16x^2 + 9y^2 + 4z^2 + 12xy + 6yz - 8zx)$$

$$\Rightarrow (4x - 3y + 2z)((4x)^2 + (3y)^2 + (2z)^2 - (4x)(-3y) - (-3y)(2z) - (2z)(4x))$$

$$\Rightarrow (4x)^3 + (-3y)^3 + (2z)^3 - 3(4x)(-3y)(2z)$$

$$\Rightarrow \boxed{64x^3 - 27y^3 + 8z^3 + 72xyz}$$

(iii) Using the identity,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

Therefore

$$\begin{aligned} &\Rightarrow (2a - 3b - 2c)(4a^2 + 9b^2 + 4c^2 + 6ab + 6bc + 4ca) \\ &\Rightarrow (2a - 3b - 2c)((2a)^2 + (-3b)^2 + (-2c)^2 - (2a)(-3b) - (-3b)(-2c) - (-2c)(2a)) \\ &\Rightarrow (2a)^3 + (-3b)^3 + (-2c)^3 - 3(2a)(-3b)(-2c) \\ &\Rightarrow \boxed{8a^3 - 27b^3 - 8c^3 - 36abc} \end{aligned}$$

(iv) Using the identity,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

Therefore

$$\begin{aligned} &\Rightarrow (3x - 4y + 5z)(9x^2 + 16y^2 + 25z^2 + 12xy - 15zx + 20yz) \\ &\Rightarrow (3x - 4y + 5z)((3x)^2 + (-4y)^2 + (5z)^2 - (3x)(-4y) - (-4y)(5z) - (5z)(3x)) \\ &\Rightarrow (3x)^3 + (-4y)^3 + (5z)^3 - 3(3x)(-4y)(5z) \\ &\Rightarrow \boxed{27x^3 - 64y^3 + 125z^3 + 180xyz} \end{aligned}$$

2. Question

If $x + y + z = 8$ and $xy + yz + zx = 20$, find the value of $x^3 + y^3 + z^3 - 3xyz$.

Answer

In $x^3 + y^3 + z^3 - 3xyz$,

Using the identity

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$ we get,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)[x^2 + y^2 + z^2 - (xy + yz + zx)] \dots (1)$$

We also know,

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

For $x^2 + y^2 + z^2$

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$$

$$\Rightarrow 8^2 = x^2 + y^2 + z^2 + 2(20)$$

$$\Rightarrow 64 = x^2 + y^2 + z^2 + 40$$

$$\Rightarrow x^2 + y^2 + z^2 = 24$$

From (1) we get,

$$x^3 + y^3 + z^3 - 3xyz = 8[24 - 20]$$

$$= 8(4)$$

$$= 32$$

3. Question

If $a + b + c = 9$ and $ab + bc + ca = 26$, find the value of $a^3 + b^3 + c^3 - 3abc$.

Answer

Using the identity,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$\Rightarrow a^3 + b^3 + c^3 - 3abc = (a + b + c)[(a^2 + b^2 + c^2) - (ab + bc + ca)] \quad \text{---(1)}$$

$$\text{Now, } (a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + bc + ca)$$

$$\Rightarrow (9)^2 = (a^2 + b^2 + c^2) + 2(26)$$

$$\Rightarrow a^2 + b^2 + c^2 = 81 - 52 = 29$$

Now, from (1),

$$\Rightarrow a^3 + b^3 + c^3 - 3abc = (9)[(29) - (26)] = 9 \times 3 = \boxed{27}$$

4. Question

If $a + b + c = 9$ and $a^2 + b^2 + c^2 = 35$, find the value of $a^3 + b^3 + c^3 - 3abc$.

Answer

Using the identity,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$\Rightarrow a^3 + b^3 + c^3 - 3abc = (a + b + c)[(a^2 + b^2 + c^2) - (ab + bc + ca)] \quad \text{---(1)}$$

$$\text{Now, } (a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + bc + ca)$$

$$\Rightarrow (9)^2 = 35 + 2(ab + bc + ca)$$

$$\Rightarrow 2(ab + bc + ca) = 81 - 35 = 46$$

$$\Rightarrow (ab + bc + ca) = 23$$

Now, from (1),

$$\Rightarrow a^3 + b^3 + c^3 - 3abc = (9)[35 - 23] = 9 \times 12 = \boxed{108}$$

5. Question

Evaluate :

(i) $25^3 - 75^3 + 50^3$

(ii) $48^3 - 30^3 - 18^3$

(iii) $\left(\frac{1}{2}\right)^3 + \left(\frac{1}{3}\right)^3 - \left(\frac{5}{6}\right)^3$

(iv) $(0.2)^3 - (0.3)^3 + (0.1)^3$

Answer

(i)

We know,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$\text{Now, } (a + b + c) = 25 - 75 + 50 = 0$$

$$\Rightarrow (a)^3 + (b)^3 + (c)^3 = 3abc$$

$$= 3 \times 25 \times (-75) \times 50 =$$

$$= \boxed{-281250}$$

(ii)

We know,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$\text{Now, } (a + b + c) = 48 - 30 - 18 = 0$$

$$\Rightarrow (a)^3 + (b)^3 + (c)^3 = 3abc$$

$$= 3 \times 48 \times (-30) \times (-18)$$

$$= \boxed{77760}$$

(iii)

We know,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$\text{Now, } (a + b + c) = \frac{1}{2} + \frac{1}{3} - \frac{5}{6} = \frac{5}{6} - \frac{5}{6} = 0$$

$$\Rightarrow (a)^3 + (b)^3 + (c)^3 = 3abc$$

$$= 3 \times \frac{1}{2} \times \frac{1}{3} \times \left(-\frac{5}{6}\right)$$

$$= \boxed{-\frac{5}{12}}$$

(iv)

We know,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$\text{Now, } (a + b + c) = 0.2 - 0.3 + 0.1 = 0$$

$$\Rightarrow (a)^3 + (b)^3 + (c)^3 = 3abc$$

$$= 3 \times 0.2 \times (-0.3) \times 0.1$$

$$= \boxed{-0.018}$$

CCE - Formative Assessment

1. Question

If $x + \frac{1}{x} = 3$, then find the value of $x^2 + \frac{1}{x^2}$.

Answer

$$\text{Given: } x + \frac{1}{x} = 3$$

$$\text{Using, } (a + b)^2 = a^2 + b^2 + 2ab$$

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \left(\frac{1}{x^2}\right) + 2 \cdot x \cdot \frac{1}{x}$$

$$\Rightarrow (3)^2 = x^2 + \left(\frac{1}{x^2}\right) + 2$$

$$\Rightarrow \boxed{x^2 + \frac{1}{x^2} = 9 - 2 = 7}$$

2. Question

If $x + \frac{1}{x} = 3$, then find the value of $x^6 + \frac{1}{x^6}$.

Answer

$$\text{We are given that } x + \frac{1}{x} = 3$$

On cubing we get

$$\left(x + \frac{1}{x}\right)^3 = x^3 + \left(\frac{1}{x^3}\right) + 3 \cdot x \cdot \frac{1}{x} \left(x + \frac{1}{x}\right)$$

$$\Rightarrow 27 = x^3 + \left(\frac{1}{x^3}\right) + 3 \cdot 3$$

$$\Rightarrow x^3 + \left(\frac{1}{x^3}\right) = 27 - 9$$

$$\Rightarrow \boxed{x^3 + \left(\frac{1}{x^3}\right) = 18}$$

$$\text{Now, } \left(x^3 + \frac{1}{x^3}\right)^2 = x^6 + \left(\frac{1}{x^6}\right) + 2 \cdot x^3 \cdot \frac{1}{x^3}$$

$$\Rightarrow 18^2 = x^6 + \left(\frac{1}{x^6}\right) + 2$$

$$x^6 + \left(\frac{1}{x^6}\right) = 324 - 2 = \boxed{322}$$

3. Question

If $a+b=7$ and $ab=12$, find the value of a^2+b^2 .

Answer

Using the identity,

$$(a+b)^2 = a^2 + b^2 + 2ab$$

$$\Rightarrow (7)^2 = a^2 + b^2 + 2(12)$$

$$\Rightarrow a^2 + b^2 = 49 - 24 = 25$$

4. Question

If $a-b=5$ and $ab=12$, find the value of a^2+b^2 .

Answer

Using the identity,

$$(a-b)^2 = a^2 + b^2 - 2ab$$

$$\Rightarrow (5)^2 = a^2 + b^2 - 2(12)$$

$$\Rightarrow a^2 + b^2 = 25 - 24 = 1$$

5. Question

If $x - \frac{1}{x} = \frac{1}{2}$, then write the value of $4x^2 + \frac{4}{x^2}$.

Answer

$$\text{Given: } x - \frac{1}{x} = \frac{1}{2}$$

$$\Rightarrow 2\left(x - \frac{1}{x}\right) = 1$$

$$\Rightarrow \left(2x - \frac{2}{x}\right) = 1$$

$$\text{Using, } (a+b)^2 = a^2 + b^2 + 2ab$$

$$\left(2x - \frac{2}{x}\right)^2 = 4x^2 + \left(\frac{4}{x^2}\right) - 2 \cdot 2x \cdot \frac{2}{x}$$

$$\Rightarrow (1)^2 = 4x^2 + \left(\frac{4}{x^2}\right) - 8$$

$$\Rightarrow \boxed{4x^2 + \frac{4}{x^2} = 9}$$

6. Question

If $a^2 + \frac{1}{a^2} = 102$, find the value of $a - \frac{1}{a}$.

Answer

Here, we will use $(a+b)^2 = a^2 + b^2 + 2ab$

$$a^2 + \frac{1}{a^2} = 102$$

$$\Rightarrow a^2 + \frac{1}{a^2} - 2\left(a^2 \times \frac{1}{a^2}\right) = 102 - 2\left(a^2 \times \frac{1}{a^2}\right)$$

$$\Rightarrow \left(a - \frac{1}{a}\right)^2 = 100$$

$$\Rightarrow \left(a - \frac{1}{a}\right) = \sqrt{100} = \pm 10$$

7. Question

If $a+b+c=0$ then write the value of $\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}$.

Answer

Given: $\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}$

Taking LCM we get,

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} = \frac{a^3 + b^3 + c^3}{abc} \quad \text{---(1)}$$

We know, $a^3 + b^3 + c^3 - 3abc = (a+b+c)[a^2 + b^2 + c^2 - ab - bc - ca]$ ---(1)

If $a+b+c=0$,

Then, $a^3 + b^3 + c^3 = 3abc$

Now, from (1),

$$\Rightarrow \frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} = \frac{3abc}{abc} = 3$$

1. Question

If $x + \frac{1}{x} = 5$, then $x^2 + \frac{1}{x^2} =$

A. 25

B. 10

C. 23

D. 27

Answer

Using, $(a+b)^2 = a^2 + b^2 + 2ab$

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \left(\frac{1}{x^2}\right) + 2 \cdot x \cdot \frac{1}{x}$$

$$\Rightarrow (5)^2 = x^2 + \left(\frac{1}{x^2}\right) + 2$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 25 - 2$$

$$\boxed{x^2 + \frac{1}{x^2} = 23}$$

2. Question

If $x + \frac{1}{x} = 2$, then $x^3 + \frac{1}{x^3} =$

- A. 64
- B. 14
- C. 8
- D. 2

Answer

On cubing we get

$$\left(x + \frac{1}{x}\right)^3 = x^3 + \left(\frac{1}{x^3}\right) + 3 \cdot x \cdot \frac{1}{x} \left(x + \frac{1}{x}\right)$$

$$\Rightarrow 2^3 = x^3 + \left(\frac{1}{x^3}\right) + 3 \cdot 2$$

$$\Rightarrow x^3 + \left(\frac{1}{x^3}\right) = 8 - 6$$

$$\Rightarrow \boxed{x^3 + \left(\frac{1}{x^3}\right) = 2}$$

3. Question

If $x + \frac{1}{x} = 4$, then $x^4 + \frac{1}{x^4} =$

- A. 196
- B. 194
- C. 192
- D. 190

Answer

Using $(a+b)^2 = a^2 + b^2 + 2ab$

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \left(\frac{1}{x^2}\right) + 2 \cdot x \cdot \frac{1}{x}$$

$$\Rightarrow (4)^2 = x^2 + \left(\frac{1}{x^2}\right) + 2$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 16 - 2$$

$$\boxed{x^2 + \frac{1}{x^2} = 14}$$

Again, squaring both sides,

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = (14)^2$$

$$\Rightarrow x^4 + \left(\frac{1}{x^4}\right) + 2 \cdot x^2 \cdot \frac{1}{x^2} = 196$$

$$\Rightarrow x^4 + \left(\frac{1}{x^4}\right) = 196 - 2$$

$$\Rightarrow \boxed{x^4 + \left(\frac{1}{x^4}\right) = 194}$$

4. Question

If $x + \frac{1}{x} = 3$, then $x^6 + \frac{1}{x^6} =$

- A. 927

B. 414

C. 364

D. 322

Answer

On cubing we get

$$\left(x + \frac{1}{x}\right)^3 = x^3 + \left(\frac{1}{x^3}\right) + 3 \cdot x \cdot \frac{1}{x} \left(x + \frac{1}{x}\right)$$

$$\Rightarrow 27 = x^3 + \left(\frac{1}{x^3}\right) + 3 \cdot 3$$

$$\Rightarrow x^3 + \left(\frac{1}{x^3}\right) = 27 - 9$$

$$\Rightarrow \boxed{x^3 + \left(\frac{1}{x^3}\right) = 18}$$

$$\text{Now, } \left(x^3 + \frac{1}{x^3}\right)^2 = x^6 + \left(\frac{1}{x^6}\right) + 2 \cdot x^3 \cdot \frac{1}{x^3}$$

$$\Rightarrow 18^2 = x^6 + \left(\frac{1}{x^6}\right) + 2$$

$$x^6 + \left(\frac{1}{x^6}\right) = 324 - 2 = \boxed{322}$$

5. Question

If $x^4 + \frac{1}{x^4} = 623$, then $x + \frac{1}{x} =$

A. 27

B. 25

C. $3\sqrt{3}$

B. $-3\sqrt{3}$

Answer

$$\left(x^4 + \frac{1}{x^4}\right) = 623$$

$$\Rightarrow (x^2)^2 + \left(\frac{1}{x^2}\right) + 2 \times x^2 \times \frac{1}{x^2} = 623 + 2 \times x^2 \times \frac{1}{x^2}$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = 625$$

$$\Rightarrow x^2 + \frac{1}{x^2} = \sqrt{625} = 25$$

Now,

$$\Rightarrow (x^2) + \left(\frac{1}{x^2}\right) + 2 \times x \times \frac{1}{x} = 25 + 2 \times x \times \frac{1}{x}$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 27$$

$$\Rightarrow x + \frac{1}{x} = \sqrt{27} = 3\sqrt{3}$$

6. Question

If $x^2 + \frac{1}{x^2} = 102$, then $x - \frac{1}{x} =$

A. 8

B. 10

C. 12

D. 13

Answer

$$x^2 + \frac{1}{x^2} = 102$$

Now,

$$\Rightarrow (x^2) + \left(\frac{1}{x^2}\right) - 2 \times x \times \frac{1}{x} = 102 - 2 \times x \times \frac{1}{x}$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 100$$

$$\Rightarrow x + \frac{1}{x} = \sqrt{100} = 10$$

7. Question

If $x^3 + \frac{1}{x^3} = 110$, then $x + \frac{1}{x} = ?$

A. 5

B. 10

C. 15

D. none of these

Answer

$$x^3 + \left(\frac{1}{x^3}\right) = 110$$

$$x^3 + \left(\frac{1}{x^3}\right) + 3 \cdot x \cdot \frac{1}{x} \left(x + \frac{1}{x}\right) = 110 + 3 \cdot x \cdot \frac{1}{x} \left(x + \frac{1}{x}\right)$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^3 = 110 + 3 \left(x + \frac{1}{x}\right)$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^3 - 3 \left(x + \frac{1}{x}\right) - 110 = 0$$

$$\text{Let } x + \frac{1}{x} = a$$

$$\Rightarrow a^3 - 3a - 110 = 0$$

$$\Rightarrow a^3 - 5a^2 + 5a^2 - 25a + 22a - 110 = 0$$

$$\Rightarrow a^2(a - 5) + 5a(a - 5) + 22(a - 5) = 0$$

$$\Rightarrow (a - 5)(a^2 + 5a + 22) = 0$$

$$\Rightarrow a - 5 = 0 \text{ or } a^2 + 5a + 22 = 0 \text{ neglected} \Rightarrow a = 5$$

$$\Rightarrow x + \frac{1}{x} = 5$$

8. Question

If $x^3 - \frac{1}{x^3} = 14$, then $x - \frac{1}{x} =$

A. 5

B. 4

C. 3

D. 2

Answer

$$\text{Given : } x^3 + \left(\frac{1}{x^3}\right) = 14$$

$$\text{Let } x = a \text{ and } \frac{1}{x} = b$$

$$\text{Say, } x - \frac{1}{x} = A$$

$$\text{Then, } a^3 - b^3 = 14$$

$$\Rightarrow (a-b)(a^2 + ab + b^2) = 14$$

$$\Rightarrow (a-b)((a-b)^2 + 2ab) = 14$$

$$\Rightarrow (a-b)((a-b)^2 + 3ab) = 14$$

$$\Rightarrow (a-b)((a-b)^2 + 3) = 14$$

$$\Rightarrow A(A^2 + 3) = 14$$

$$\Rightarrow A(A^2 + 3) = 14$$

$$\Rightarrow A^3 + 3A - 14 = 0$$

$$\Rightarrow A^3 - 2A^2 + 2A^2 - 4A + 7A - 14 = 0$$

$$\Rightarrow A^2(A-2) + 2A(A-2) + 7(A-2) = 0$$

$$\Rightarrow (A-2)(A^2 + 2A + 7) = 0$$

$$\Rightarrow A-2 = 0, \Rightarrow A = 2$$

$$\Rightarrow x - \frac{1}{x} = 2$$

9. Question

$$\text{If } x^4 + \frac{1}{x^4} = 194, \text{ then } x^3 - \frac{1}{x^3} =$$

A. 76

B. 52

C. 64

D. none of these

Answer

$$\left(x^4 + \frac{1}{x^4}\right) = 194$$

$$\Rightarrow (x^2)^2 + \left(\frac{1}{x^2}\right) + 2 \times x^2 \times \frac{1}{x^2} = 194 + 2 \times x^2 \times \frac{1}{x^2}$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = 196$$

$$\Rightarrow x^2 + \frac{1}{x^2} = \sqrt{196} = 14$$

Now,

$$\Rightarrow (x^2) + \left(\frac{1}{x^2}\right) + 2 \times x \times \frac{1}{x} = 14 + 2 \times x \times \frac{1}{x}$$

$$\Rightarrow \left(x - \frac{1}{x}\right)^2 = 16$$

$$\Rightarrow x + \frac{1}{x} = \sqrt{16} = 4$$

$$\text{Now, } \left(x + \frac{1}{x}\right)^3 = (4)^3$$

$$\Rightarrow (x)^3 + \left(\frac{1}{x}\right)^3 + 3 \times x \times \frac{1}{x} \left(x + \frac{1}{x}\right) = 64$$

$$\Rightarrow (x^3) + \left(\frac{1}{x^3}\right) + 3(4) = 64$$

$$\Rightarrow (x^3) + \left(\frac{1}{x^3}\right) = 64 - 12 = 52$$

10. Question

If $x - \frac{1}{x} = \frac{15}{4}$, then $x + \frac{1}{x} =$

A. 4

B. $\frac{17}{4}$

C. $\frac{13}{4}$

D. $\frac{1}{4}$

Answer

$$\Rightarrow x - \frac{1}{x} = \frac{15}{4}$$

$$\text{Now, } \left(x - \frac{1}{x}\right)^2 = \left(\frac{15}{4}\right)^2$$

$$\Rightarrow (x^2) + \left(\frac{1}{x^2}\right) - 2 \times x \times \frac{1}{x} = \frac{225}{16}$$

$$\Rightarrow (x^2) + \left(\frac{1}{x^2}\right) = \frac{225}{16} + 2$$

$$\Rightarrow (x^2) + \left(\frac{1}{x^2}\right) = \frac{257}{16}$$

$$\text{Now, } \Rightarrow (x^2) + \left(\frac{1}{x^2}\right) + 2 \times x \times \frac{1}{x} = \frac{257}{16} + 2 \times x \times \frac{1}{x}$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = \frac{257 + 32}{16} = \frac{289}{16}$$

$$\Rightarrow \left(x + \frac{1}{x}\right) = \sqrt{\frac{289}{16}} = \frac{17}{4}$$

11. Question

If $3x + \frac{2}{x} = 7$, then $\left(9x^2 - \frac{4}{x^2}\right) =$

A. 25

B. 35

C. 49

D. 30

Answer

$$\Rightarrow 3x + \frac{2}{x} = 7$$

$$\text{Now, } 3x^2 + 2 - 7x = 0$$

$$\Rightarrow 3x^2 - 6x - x + 2 = 0$$

$$\Rightarrow (x-2)(3x-1) = 0$$

$$\Rightarrow x = 2 \text{ or } x = \frac{1}{3} (\text{neglected})$$

$$\Rightarrow 3x - \frac{2}{x} = 6 - 1 = 5$$

$$\Rightarrow 3x + \frac{2}{x} = 6 + 1 = 7$$

Now,

$$(3x)^2 - \left(\frac{2}{x}\right)^2 = \left(3x - \frac{2}{x}\right)\left(3x + \frac{2}{x}\right)$$

$$\Rightarrow (3x)^2 - \left(\frac{2}{x}\right)^2 = 5(7) = 35$$

12. Question

If $a^2 + b^2 + c^2 - ab - bc - ca = 0$, then

A. $a + b = c$

B. $b + c = a$

C. $c + a = b$

D. $a = b = c$

Answer

$$\text{Given: } a^2 + b^2 + c^2 - ab - bc - ca = 0$$

$$\Rightarrow 2(a^2 + b^2 + c^2 - ab - bc - ca) = 0$$

$$\Rightarrow (2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca) = 0$$

$$\Rightarrow (\{a^2 + b^2 - 2ab\} + \{b^2 + c^2 - 2bc\} + \{c^2 + a^2 - 2ca\}) = 0$$

$$\Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$$

Now, since the sum of all squares is zero

$$\Rightarrow a - b = 0 \Rightarrow a = b$$

$$\Rightarrow b - c = 0 \Rightarrow b = c$$

$$\Rightarrow c - a = 0 \Rightarrow c = a$$

$$\Rightarrow a = b = c$$

13. Question

If $a + b + c = 0$, then, $\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} =$

A. 0

B. 1

C. -1

D. 3

Answer

Given: $\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}$

Taking LCM we get,

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} = \frac{a^3 + b^3 + c^3}{abc} \text{ ----(1)}$$

We know, $a^3 + b^3 + c^3 - 3abc = (a + b + c)[a^2 + b^2 + c^2 - ab - bc - ca]$ ----(1)

If $a + b + c = 0$,

Then, $a^3 + b^3 + c^3 = 3abc$

Now, from (1),

$$\Rightarrow \frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} = \frac{3abc}{abc} = 3$$

14. Question

If $a^{1/3} + b^{1/3} + c^{1/3} = 0$, then

A. $a + b + c = 0$

B. $(a + b + c)^3 = 27abc$

C. $a + b + c = 3abc$

D. $a^3 + b^3 + c^3 = 0$

Answer

$$a^{1/3} + b^{1/3} + c^{1/3} = 0$$

$$\Rightarrow a^{1/3} + b^{1/3} = -c^{1/3} \text{ ----(1)}$$

$$\Rightarrow (a^{1/3})^3 + (b^{1/3})^3 = (-c^{1/3})^3$$

$$\Rightarrow a + b + 3.a^{1/3}.b^{1/3}(a^{1/3} + b^{1/3}) = -c$$

$$\Rightarrow a + b + 3.a^{1/3}.b^{1/3}(-c^{1/3}) = -c$$

$$\Rightarrow a + b + c = 3.a^{1/3}.b^{1/3}.c^{1/3}$$

$$\Rightarrow (a + b + c)^3 = (3.a^{1/3}.b^{1/3}.c^{1/3})^3$$

$$\Rightarrow (a + b + c)^3 = 27abc$$

15. Question

If $a + b + c = 9$ and $ab + bc + ca = 23$, then $a^2 + b^2 + c^2 =$

A. 35

B. 58

B. 127

D. none of these

Answer

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Hence, $9^2 = a^2 + b^2 + c^2 + 2 \times 23$

$$\Rightarrow a^2 + b^2 + c^2 = 35$$

16. Question

If $a + b + c = 9$, then $ab + bc + ca = 23$, then $a^3 + b^3 + c^3 - 3abc =$

A. 108

B. 207

C. 669

D. 729

Answer

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$\Rightarrow (9)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

$$\Rightarrow (9)^2 = a^2 + b^2 + c^2 + 2(23)$$

$$\Rightarrow a^2 + b^2 + c^2 = 81 - 46 = 35$$

$$\text{as we know that } a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$\Rightarrow a^3 + b^3 + c^3 - 3abc = 9 \times (35 - 23)$$

$$\Rightarrow a^3 + b^3 + c^3 - 3abc = 108$$

17. Question

$$(a-b)^3 + (b-c)^3 + (c-a)^3 =$$

A. $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

B. $(a-b)(b-c)(c-a)$

C. $3(a-b)(b-c)(c-a)$

D. none of these

Answer

$$\text{Let } (a-b) = A$$

$$(b-c) = B$$

$$\text{and } (c-a) = C$$

$$\Rightarrow A+B+C=0$$

$$\Rightarrow A+B = -C \quad (1)$$

$$\Rightarrow (A+B)^3 = -C^3$$

$$\Rightarrow A^3 + B^3 + 3AB(A+B) = -C^3$$

$$\Rightarrow A^3 + B^3 + 3AB(-C) = -C^3$$

$$\Rightarrow A^3 + B^3 + C^3 = 3ABC$$

$$\Rightarrow (a-b)^3 + (b-c)^3 + (c-a)^3 = 3(a-b)(b-c)(c-a)$$

18. Question

Solve the equation and choose the correct answer: $\frac{(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3}{(a-b)^3 + (b-c)^3 + (c-a)^3} =$

A. $3(a+b)(b+c)(c+a)$

B. $3(a-b)(b-c)(c-a)$

C. $(a-b)(b-c)(c-a)$

D. None of these

Answer

$$\text{We know, } a^3 + b^3 + c^3 - 3abc = 3abc$$

$$\Rightarrow (a-b)^3 + (b-c)^3 + (c-a)^3 = 3(a-b)(b-c)(c-a)$$

$$\Rightarrow \frac{(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3}{(a-b)^3 + (b-c)^3 + (c-a)^3}$$

$$= \frac{3(a-b)(a+b)(b-c)(b+c)(c-a)(c+a)}{3(a-b)(b-c)(c-a)}$$

$$= (a+b)(b+c)(c+a)$$

19. Question

The product $(a+b)(a-b)(a^2-ab+b^2)(a^2+ab+b^2)$ is equal to

A. $a^6 + b^6$

B. $a^6 - b^6$

C. $a^3 - b^3$

D. $a^3 + b^3$

Answer

$$\begin{aligned} \text{Given: } & (a+b)(a-b)(a^2+b^2+ab)(a^2+b^2-ab) \\ \Rightarrow & \{(a+b)(a^2+b^2+ab)\}\{(a-b)(a^2+b^2+ab)\} \\ \Rightarrow & (a^3+b^3)(a^3-b^3) \\ \Rightarrow & (a^6-b^6) \end{aligned}$$

20. Question

If $\frac{a}{b} + \frac{b}{a} = -1$, then $a^3 - b^3 =$

A. 1

B. -1

C. $\frac{1}{2}$

D. 0

Answer

$$\begin{aligned} \text{Here, } & \frac{a}{b} + \frac{b}{a} = -1 \\ \Rightarrow & \frac{a^2+b^2}{ab} = -1 \\ \Rightarrow & a^2+b^2 = -ab \\ \Rightarrow & a^2+b^2+ab = 0 \text{ _____(1)} \\ \text{Using, } & a^3-b^3 = (a-b)(a^2+b^2+ab) \\ = & (a+b)(0) \\ = & 0 \end{aligned}$$

21. Question

The product $(x^2-1)(x^4+x^2+1)$ is equal to=

A. x^8-1

B. x^8+1

C. x^6-1

D. x^6+1

Answer

$$\begin{aligned} \text{Using, } & (x^2-1)(x^4+x^2+1) \\ \Rightarrow & (x^8+x^4+x^2-x^2-1) \\ \Rightarrow & x^8-1 \end{aligned}$$

22. Question

If $a-b=-8$, and $ab=-12$, then $a^3 - b^3 =$

A. -244

B. -240

C. -224

D. -260

Answer

$$\begin{aligned}\text{Using, } (a-b)^3 &= a^3 - b^3 - 3ab(a-b) \\ \Rightarrow a^3 - b^3 &= (a-b)^3 + 3ab(a-b) \\ \Rightarrow a^3 - b^3 &= (-8)^3 + 3(-12)(-8) \\ \Rightarrow a^3 - b^3 &= -512 + 288 = -224\end{aligned}$$

23. Question

If the volume of a cuboid is $3x^2-27$, then its possible dimensions are

- A. $3, x^2, -27x$
- B. $3, x-3, x+3$
- C. $3, x^2, 27x$
- D. $3, 3, 3$

Answer

$$\begin{aligned}\text{Given: } 3x^2 - 27 \\ \text{We will break down in factors,} \\ &= 3(x^2 - 9) \\ \text{Using, } (x^2 - 9) &= (x+3)(x-3) \\ \Rightarrow 3(x+3)(x-3)\end{aligned}$$

Thus, the possible dimensions are $3, (x+3)(x-3)$

24. Question

If $\frac{a}{b} + \frac{b}{a} = 1$, then $a^3 + b^3 =$

- A. 1
- B. -1
- C. $\frac{1}{2}$
- D. 0

Answer

$$\begin{aligned}\text{Here, } \frac{a}{b} + \frac{b}{a} &= 1 \\ \Rightarrow \frac{a^2 + b^2}{ab} &= 1 \\ \Rightarrow a^2 + b^2 &= ab \\ \Rightarrow a^2 + b^2 - ab &= 0 \quad (1) \\ \text{Using, } a^3 + b^3 &= (a+b)(a^2 + b^2 - ab) \\ &= (a+b)(0) \\ &= 0\end{aligned}$$

25. Question

$75 \times 75 + 2 \times 75 \times 25 + 25 \times 25$ equal to

- A. 10000
- B. 6250
- C. 7500
- D. 3750

Answer

We know, $(a+b)^2 = a^2 + b^2 + 2ab$

Here, $a = 75$ and $b = 25$

$$\Rightarrow (75 \times 75) + (2 \times 75 \times 25) + (25 \times 25)$$

$$= 75^2 + 2 \times 75 \times 25 + 25^2$$

$$= (75 + 25)^2$$

$$= (100)^2$$

$$= 10000$$

26. Question

$(x-y)(x+y)(x^2+y^2)(x^4+y^4)$ is equal to

A. $x^{16} - y^{16}$

B. $x^8 - y^8$

C. $x^8 + y^8$

D. $x^{16} + y^{16}$

Answer

Given: $(x-y)(x+y)(x^2+y^2)(x^4+y^4)$

Using, $a^2 - b^2 = (a-b)(a+b)$,

$$\Rightarrow (x-y)(x+y)(x^2+y^2)(x^4+y^4)$$

$$= (x^2 - y^2)(x^2 + y^2)(x^4 + y^4)$$

$$= (x^4 - y^4)(x^4 + y^4)$$

$$= (x^8 - y^8)$$

27. Question

If $48a^2 - b = \left(7a + \frac{1}{2}\right)\left(7a - \frac{1}{2}\right)$, then the value of b is

A. 0

B. $\frac{1}{4}$

C. $\frac{1}{\sqrt{2}}$

D. $\frac{1}{2}$

Answer

Given: $49a^2 - b = \left(7a + \frac{1}{2}\right)\left(7a - \frac{1}{2}\right)$

Using, $a^2 - b^2 = (a-b)(a+b)$,

$$\Rightarrow \left(7a + \frac{1}{2}\right)\left(7a - \frac{1}{2}\right) = 49a^2 - \frac{1}{4}$$

$$\Rightarrow 49a^2 - b = 49a^2 - \frac{1}{4}$$

$$\Rightarrow b = \frac{1}{4}$$