

DEFINITE INTEGRALS

	Properties of Definite Integrals :-
P-I	$\int_a^b f(x)dx = \int_a^b f(t)dt$ e.g $\int_0^{\frac{\pi}{2}} \sin t dt = \int_{\frac{\pi}{2}}^{\pi} \sin x dx$
P-II	$\int_a^b f(x)dx = \int_{-b}^{-a} f(x)dx$
P-III	$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ Where $a < c < b$ e.g $\int_0^a f(x)dx = \int_0^{2a} f(x)dx + \int_{2a}^a f(x)dx$
P-IV	$\int_0^a f(x)dx = \int_0^a f(a-x)dx$ Proof: Taking RHS $\int_0^a f(a-x)dx$ Put $a-x = t$ when $x = 0 \Rightarrow t = a$ $-dx = dt \Rightarrow dx = -dt$ when $x = a \Rightarrow t = 0$ $\therefore \text{RHS} = -\int_a^0 f(t)dt$ $= \int_0^a f(t)dt$(by P-II) $= \int_0^a f(x)dx$(by P-I) $= \text{LHS}$ $\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$ proved
P-V	$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ Proof: Do yourself by put $a+b-x = t$
P-VI	$\int_b^{2a} f(x)dx = \begin{cases} 2 \int_0^a f(x)dx & ; f(2a-x) = f(x) \\ 0 & ; \text{if } f(2a-x) = -f(x) \end{cases}$ Mainly $\int_0^{2a} f(x)dx = 2 \int_0^a f(x)dx$
P-VII	Even - function property $\int_{-a}^a f(x)dx = \begin{cases} 2 \int_0^a f(x)dx & ; \text{if } f(x) \rightarrow \text{even} \\ 0 & ; \text{if } f(x) \rightarrow \text{odd} \end{cases}$ If $f(-x) = f(x)$ then $f(x)$ is an even function If $f(-x) = -f(x)$ then $f(x)$ is an odd function
Q.1)	Evaluate I = $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$.
Sol.1)	$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$(1)

	$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(\frac{\pi}{2}-x)}}{\sqrt{\sin(\frac{\pi}{2}-x)} + \sqrt{\cos(\frac{\pi}{2}-x)}} dx \dots\dots\dots \int_0^a f(x)dx = \int_0^a f(a-x)dx$ $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \dots\dots\dots(2)$ Adding (1) & (2) $2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ $= \int_0^{\frac{\pi}{2}} 1 \cdot dx$ $= (x)_0^{\frac{\pi}{2}}$ $2I = \frac{\pi}{2}$ $I = \frac{\pi}{4} \text{ ans.}$
Q.2)	Evaluate $I = \int_0^{\frac{\pi}{2}} \sin(2x) \log(\tan x) dx$
Sol.2)	$I = \int_0^{\frac{\pi}{2}} \sin(2x) \log(\tan x) dx \dots\dots\dots(1)$ $I = \int_0^{\frac{\pi}{2}} \sin\left[2\left(\frac{\pi}{2} - x\right)\right] \cdot \log\left[\tan\left(\frac{\pi}{2} - x\right)\right] dx \dots\dots\dots\left[\int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$ $I = \int_0^{\frac{\pi}{2}} \sin(\pi - 2x) \cdot \log(\cot x) dx$ $I = \int_0^{\frac{\pi}{2}} \sin(2x) \cdot \log(\cot x) dx \dots\dots\dots(2) \quad [\because \sin(\pi - 2x) = \sin(2x)]$ Adding eq. (1) & (2) $2I = \int_0^{\frac{\pi}{2}} \sin(2x) [\log(\tan x) + \log(\cot x)] dx$ $2I = \int_0^{\frac{\pi}{2}} \sin(2x) \log(\tan x \cdot \cot x) dx$ $2I = \int_0^{\frac{\pi}{2}} \sin(2x) \cdot \log(1) dx$ $2I = \int_0^{\frac{\pi}{2}} 0 dx \dots\dots\dots\{\because \log 1 = 0\}$ $\therefore I = 0 \text{ ans.....}$
Q.3)	$I = \int_0^{\frac{\pi}{4}} \log(1 + \tan x) dx$
Sol.3)	$I = \int_0^{\frac{\pi}{4}} \log(1 + \tan x) dx \dots\dots\dots(1)$ $I = \int_0^{\frac{\pi}{4}} \log\left[1 + \tan\left(\frac{\pi}{4} - x\right)\right] dx \dots\dots\dots\left[\int_0^a f(x)dx = \int_0^a f(a-x) dx\right]$ $I = \int_0^{\frac{\pi}{4}} \log\left[1 + \frac{1 - \tan x}{1 + \tan x}\right] dx \dots\dots\dots\{\tan(A - B) \text{ formula}\}$ $I = \int_0^{\frac{\pi}{4}} \log\left[\frac{1 + \tan x + 1 - \tan x}{1 + \tan x}\right] dx$

	$I = \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan x}\right) dx \quad \dots(2)$ Eq. (1) + (2) $2I = \int_0^{\frac{\pi}{4}} \log\left(1 + \tan x \cdot \frac{2}{1+\tan x}\right) dx$ $2I = \int_0^{\frac{\pi}{4}} \log(2) dx$ $2I = \log 2 (x)_0^{\frac{\pi}{4}}$ $2I = \log 2 \left[\frac{\pi}{4} - 0\right]$ $\therefore I = \frac{\pi}{8} \log 2 \quad \text{ans.}$
Q.4)	$I = \int_0^{\frac{\pi}{2}} 2 \log(\cos x) - \log(\sin(2x)) dx$
Sol.4)	$I = \int_0^{\frac{\pi}{2}} 2 \log(\cos x) - \log(\sin(2x)) dx$ $I = \int_0^{\frac{\pi}{2}} \log(\cos^2 x) - \log(\sin(2x)) dx$ $I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\cos^2 x}{\sin(2x)}\right) dx$ $I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\cos^2 x}{\sin(2x)}\right) dx$ $I = \int_0^{\frac{\pi}{2}} \log\left[\frac{\cot x}{2}\right] dx \quad \dots \dots (1)$ $I = \int_0^{\frac{\pi}{2}} \log\left[\frac{\cot\left(\frac{\pi-x}{2}\right)}{2}\right] dx \quad \dots [\int_0^a f(x) dx = \int_0^a f(a-x) dx]$ $I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\tan x}{2}\right) dx \quad \dots \dots (2)$ Eq. (1) + (2) $2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\cot x}{2} \cdot \frac{\tan x}{2}\right) dx$ $2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{1}{4}\right) dx \quad \dots \dots \{\tan x \cdot \cot x = 1\}$ $2I = \int_0^{\frac{\pi}{2}} \log(1) - \log(4) dx$ $2I = \int_0^{\frac{\pi}{2}} -\log 4 dx \quad \dots \dots \{\log 1 = 0\}$ $2I = -\log 4 (x)_0^{\frac{\pi}{2}}$ $2I = -\log 4 \left[\frac{\pi}{2}\right]$ $I = -\frac{\pi}{4} \log 4 \quad \text{ans.}$ (or) $I = \frac{-\pi}{4} \log(2)^2$ $I = \frac{-\pi}{2} \log 2 \quad \text{ans.}$

Q.5)	$I = \int_0^1 \log \left(\frac{1}{x} - 1 \right) dx$
Sol.5)	$I = \int_0^1 \log \left(\frac{1-x}{x} \right) dx \quad \dots\dots(1)$ $= \int_0^1 \log \left[\frac{1-(1-x)}{1-x} \right] dx \quad \dots\dots \left[\int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$ $= \int_0^1 \log \left[\frac{x}{1-x} \right] dx \quad \dots\dots(2)$ $(1) + (2)$ $2I = \int_0^1 \log \left(\frac{1-x}{x} \cdot \frac{x}{1-x} \right) dx$ $= \int_0^1 \log (1) dx$ $2I = 0 \quad \dots\dots \{ \because \log 1 = 0 \}$ $I = 0 \quad \text{ans.}$
Q.6)	$I = \int_0^5 \frac{\sqrt[3]{x}}{\sqrt[3]{x} + \sqrt[3]{5-x}} dx$
Sol.6)	$I = \int_0^5 \frac{\sqrt[3]{x}}{\sqrt[3]{x} + \sqrt[3]{5-x}} dx \quad \dots\dots(1)$ $I = \int_0^5 \frac{\sqrt[3]{5-x}}{\sqrt[3]{5-x} + \sqrt[3]{5-(5-x)}} dx \quad \dots\dots \left[\int_0^a f(x) dx = \int_0^a f(a-x) \right]$ $I = \int_0^5 \frac{\sqrt[3]{5-x}}{\sqrt[3]{5-x} + \sqrt[3]{x}} dx \quad \dots\dots(2)$ $(1) + (2)$ $2I = \int_0^5 \frac{\sqrt[3]{x} + \sqrt[3]{5-x}}{\sqrt[3]{x} + \sqrt[3]{5-x}} dx$ $= \int_0^5 1. dx$ $= (x)_0^5$ $2I = 5$ $I = \frac{5}{2} \quad \text{ans.}$
Q.7)	Show that $\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx$
Sol.7)	$R.H.S \int_0^a f(x) dx + \int_0^a f(2a-x) dx$ Put $2a-x = t$ when $x = 0$; $t = 2a$ $-dx = dt$ $dx = -dt$ when $x = a$; $t = a$ $\therefore R.H.S = \int_0^a f(x) dx - \int_{2a}^a f(t) dt$ $= \int_0^a f(x) dx + \int_a^{2a} f(t) dt \quad \dots\dots \left[\int_0^b f(x) dx = - \int_b^a f(x) \right]$ $= \int_0^a f(x) dx + \int_a^{2a} f(x) dx \quad \dots\dots \left[\int_a^b f(x) dx = \int_a^b f(t) dt \right]$

	$= \int_0^{2a} f(x)dx \quad \dots\dots \left[\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx \right]$ $= LHS \quad \text{Proved}$
Q.8)	Show that $I = \int_0^1 x(1-x)^n dx$
Sol.8)	$I = \int_0^1 (1-x)[1-(1-x)]^n dx \quad \dots\dots \left[\int_0^a f(x)dx = \int_0^a f(a-x)dx \right]$ $I = \int_0^1 (1-x)(x)^n dx$ $I = \int_0^1 x^n - x^{n+1} dx$ $I = \left[\frac{x^n}{n+1} - \frac{x^{n+2}}{n+2} \right]_0^1$ $I = \left[\frac{1}{n+1} - \frac{1}{n+2} \right] - [0 - 0]$ $I = \frac{n+2-n-1}{(n+1)(n+2)}$ $I = \frac{1}{(n+1)(n+2)} \quad \text{ans.}$
Q.9)	$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$
Sol.9)	$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx \quad \dots\dots (1)$ $I = \int_0^{\frac{\pi}{2}} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right) + \cos\left(\frac{\pi}{2}-x\right)} dx \quad \dots\dots (P-IV)$ $I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos x + \sin x} dx \quad \dots\dots (2)$ $(1) + (2)$ $2I = \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x} dx$ (Type: - single $\sin x$ & $\cos x$) $2I = \int_0^{\frac{\pi}{2}} \frac{1}{\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}} dx$ $2I = \int_0^{\frac{\pi}{2}} \frac{1 + \tan^2\left(\frac{x}{2}\right)}{2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx$ $2I = \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{2 \tan \frac{x}{2} + 1 - \tan^2\left(\frac{x}{2}\right)} dx$ Put $\tan\left(\frac{x}{2}\right) = t$ when $x = 0$; $\tan(0) = t$ $\sec^2\left(\frac{x}{2}\right) \cdot \frac{1}{2} dx = dt \quad \quad \quad t = 0$ $\sec^2\left(\frac{x}{2}\right) dx = 2dt \quad \quad \text{when } x = \frac{\pi}{2} ; \tan\left(\frac{\pi}{4}\right) = t ,$ $t = 1$

	$\therefore 2I = 2 \int_0^1 \frac{dt}{-t^2+2t+1}$ $I = - \int_0^1 \frac{1}{t^2-2t-1} dt$ $= - \int_0^1 \frac{1}{(t-1)^2-1-1} dt$ $= - \int_0^1 \frac{1}{(t-1)^2-(\sqrt{2})^2} dt$ $= \int_0^1 \frac{1}{(\sqrt{2})^2-(t-1)^2} dt$ $= \frac{1}{2\sqrt{2}} \left[\log \left \frac{\sqrt{2}+t-1}{\sqrt{2}-t+1} \right \right]_0^1$ $= \frac{1}{2\sqrt{2}} \left[\log \left \frac{\sqrt{2}-0}{\sqrt{2}+0} \right - \log \left \frac{\sqrt{2}-1}{\sqrt{2}+1} \right \right]$ $= \frac{1}{2\sqrt{2}} \left[\log(1) - \log \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \right) \right]$ $I = \frac{-1}{2\sqrt{2}} \log \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \right) \quad \text{ans.}$ (Or) $I = \frac{-1}{2\sqrt{2}} \log \left[\frac{(\sqrt{2}-1)(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} \right] \quad \dots \dots \{Rationalize\}$ $= \frac{-1}{2\sqrt{2}} \log \left[\frac{(\sqrt{2}-1)^2}{2-1} \right]$ $= - \frac{2}{2\sqrt{2}} \log(\sqrt{2}-1)$ $I = \frac{-1}{\sqrt{2}} \log(\sqrt{2}-1) \quad \text{ans.}$
Q.10)	$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin x \cdot \cos x} dx$
Sol.10)	$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1+\sin x \cdot \cos x} dx \quad \dots\dots(1)$ $I = \int_0^{\frac{\pi}{2}} \frac{\cos^2(\frac{\pi}{2}-x)}{1+\sin(\frac{\pi}{2}-x) \cdot \cos(\frac{\pi}{2}-x)} dx$ $I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\cos x \cdot \sin x} dx \quad \dots\dots(2)$ (1) + (2) $2I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x \cdot \cos x} dx \quad \dots\dots\{\sin^2 x + \cos^2 x = 1\}$ Type: Divide by $\cos^2 x$ Divide N & D by $\cos^2 x$ $2I = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 x + \tan x} dx$ $2I = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{1 + \tan^2 + \tan x} dx$

Put $\tan x = t$ when $x = 0$; $t = 0$

$\therefore \sec^2 x \cdot dx = dt$ when $x = \frac{\pi}{2}$; $t = \infty$

$$\therefore 2I = \int_0^{\infty} \frac{dt}{t^2+t+1}$$

Perfect square

$$2I = \int_0^{\infty} \frac{1}{\left(t+\frac{1}{2}\right)^2 - \frac{1}{4} + 1} dt$$

$$2I = \int_0^{\infty} \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$2I = \frac{2}{\sqrt{3}} \left[\tan^{-1} \left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) \right]_0^{\infty}$$

$$2I = \frac{2}{\sqrt{3}} \left[\tan^{-1} \left(\frac{2t+1}{\sqrt{3}} \right) \right]_0^{\infty}$$

$$2I = \frac{2}{\sqrt{3}} \left[\tan^{-1}(\infty) - \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) \right]$$

$$2I = \frac{2}{\sqrt{2}} \left[\frac{\pi}{2} - \frac{\pi}{6} \right]$$

$$2I = \frac{2}{\sqrt{2}} \left[\frac{\pi}{3} \right]$$

$$I = \frac{\pi}{3\sqrt{3}} \text{ ans.}$$