DEFINITE INTEGRALS

Q.1)	$\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$
Sol.1)	$I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$ ......(1) $I = \int_0^1 \tan^{-1}\left(\frac{2(1-x)-1}{1+(1-x)-(1-x)^2}\right) dx$ ....(P-IV) $I = \int_0^1 tan^{-1}\left(\frac{2-2x-1}{1+1-x-1-x^2+2x}\right) dx$ $I = \int_0^1 \tan^{-1}\left(\frac{-2x+1}{1+x-x^2}\right) dx$ $I = -\int_0^1 tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$ ......(2) ......$\{\because \tan^{-1}(-x) = -\tan^{-1}x\}$ $(1) + (2)$ $2I = 0$ $I = 0$ *ans .* *A*lternate: $I = \int_0^1 tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$ $I = \int_0^1 tan^{-1}\left(\frac{(x)+(x-1)}{1-x(x-1)}\right) dx$ ....$\{adjustment\}$ $I = \int_0^1 tan^{-1}x\, dx + \int_0^1 tan^{-1}(x-1)\, dx$ ... ...$\left[tan^{-1}\left(\frac{x+y}{1-xy}\right) = tan^{-1}x + tan^{-1}y\right]$ $I = \int_0^1 tan^{-1}x\, dx + \int_0^1 tan^{-1}[(1-x)-1]dx$ ... .$(P-IV)$ $I = \int_0^1 tan^{-1}x\, dx + \int_0^1 tan^{-1}(-x)\, dx$ $I = \int_0^1 tan^{-1}x\, dx - \int_0^1 tan^{-1}x dx$ ......$[tan^{-1}](-x) = -tan^{-1}x$ $I = 0$ *ans.*
Q.2)	$\int_0^1 \cot^{-1}(1-x+x^2)\, dx$
Sol.2)	$I = \int_0^1 cot^{-1}(1-x+x^2)\, dx$ $I = \int_0^1 tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx$ ... ...$\left[tan^{-1}\left(\frac{1}{x}\right) = cot^{-1}x\right]$ $I =$ $\int_0^1 tan^{-1}\left[\frac{x+(1-x)}{1-x(1-x)}\right] dx$ ... ...$[adjustment]$ $I = \int_0^1 tan^{-1}(x)\, dx + \int_0^1 tan^{-1}(1-x)\, dx$ ...$\left[tan^{-1}\left(\frac{x+y}{1-xy}\right) = tan^{-1}x + tan^{-1}y\right]$ $I = \int_0^1 tan^{-1}(x)\, dx + \int_0^1 tan^{-1}[1-(1-x)]\, dx$ ... ...$(P-IV)$ $I = \int_0^1 tan^{-1}x\, dx + \int_0^1 tan^{-1}(x) dx$ $I = 2\int_0^1 tan^{-1}x\, dx$ $I = 2\int_0^1 tan^{-1}x\, .1\, dx$

	$I = 2\left[(tan^{-1}x - x)_0^1 - \int_0^1 \frac{1}{1+x^2} .x\,dx\right]$ $I = 2\left[\left(\frac{\pi}{4} - 0\right) - \int_0^1 \frac{x}{1+x^2} dx\right]$ $Put\ \ 1 + x^2 = t \qquad when\ x = 0\ ;$ $\quad x\,dx = \frac{dt}{2} \qquad when\ x = 1\ ;\ t = 2$ $\therefore\ I = 2\left[\frac{\pi}{4} - \frac{1}{2}\int_1^2 \frac{dt}{t}\right]$ $= \frac{\pi}{2} - \int_1^2 \frac{dt}{t}$ $= \frac{\pi}{2} - [\log t]_1^2$ $= \frac{\pi}{2} - [\log 2 - \log 1]_1^2$ $I = \frac{\pi}{2} - \log 2$ ans . $\qquad [\because\ \log(1) = 0]$
Q.3)	$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ ....$[Removal\ of\ x]$
Sol.3)	$I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ ....(1) $I = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)} dx$ .....(P-IV) $I = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}$ ....(2) $\quad \begin{cases}\cos(\pi - x) = -\cos x \\ \sin(\pi - x) = \sin x\end{cases}$ $(1) + (2)$ $2\,I = \int_0^{\pi} \frac{x \sin x + \pi \sin x - x \sin x}{1+\cos^2 x} dx$ $2\,I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$ Put $cos\,x = t$ when $x = 0$; $t - 1$ $\therefore\ \sin x\,dx = -dt$ when $x = \pi$; $t = -1$ $\therefore\ 2\,I = -\pi \int_1^{-1} \frac{dt}{1+t^2}$ $2\,I = -\pi[\tan^{-1} t]_1^{-1}$ $2\,I = -\pi[\tan^{-1}(-1) - \tan^{-1}(1)]$ $2\,I = -\pi\left[-\frac{\pi}{4} - \frac{\pi}{4}\right]$ $2\,I = -\pi\left(-\frac{\pi}{2}\right)$ $\therefore\ I = \frac{\pi^2}{4}$ ans.
Q.4)	$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$
Sol.4)	$I = \int_0^{\pi} \frac{x\ tan\ x}{sec\ x + tan\ x} dx$ ....$\{change\ in\ \sin x\ \&\ \cos x\}$ $I = \int_0^{\pi} \frac{x\ sin\ x}{1 + sin\ x} dx$ .... (1)

	$I = \int_0^{\pi} \frac{(\pi-x)sin(\pi-x)}{1+sin(\pi-x)}\, dx$ ....(P-IV) $I = \int_0^{\pi} \frac{(\pi-x)\, sin\, x}{1+sin\, x}\, dx$ ....(2) $(1) + (2)$ $2\, I = \int_0^{\pi} \frac{x \sin x + \pi \sin x - x \sin x}{1+\sin x}\, dx$ $2\, I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x}\, dx$ $Type: rationalize$ $2\, I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} \times \frac{(1-\sin x)}{(1-\sin x)}\, dx$ $2\, I = \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{\cos^2 x}\, dx$ $2\, I = \pi \int_0^{\pi} \frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}\, dx$ $2\, I = \pi \int_0^{\pi} \tan x \sec x - \tan^2 x\, dx$ $2\, I = \pi \int_0^{\pi} \tan x \sec x - (\sec^2 x - 1)\, dx$ $2\, I = \pi[\sec x - \tan x + x]_0^{\pi}$ $2\, I = \pi[(\sec x - \tan x + \pi) - (\sec 0 - \tan 0 + 0)]$ $2\, I = \pi[(-1 - 0 + \pi) - (1 - 0)]$ $2\, I = \pi[\pi - 2]$ $\therefore\ I = \frac{\pi}{2}(\pi - 2)$ ans.
Q.5)	$\int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$
Sol.5)	$I = \int_0^{\frac{\pi}{2}} \frac{x \sin x\, .\, \cos x}{\sin^4 x + \cos^4 x} dx$ ....(1) $I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right) sin\left(\frac{\pi}{2} - x\right).\ cos\left(\frac{\pi}{2} - x\right)}{sin^4\left(\frac{\pi}{2} - x\right) + cos^4\left(\frac{\pi}{2} - x\right)} dx$ ....(P-IV) $I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)\cos x\, .\, \sin x}{\cos^4 x + \sin^4 x} dx$ ....(2) $(1) + (2)$ $2\, I = \int_0^{\frac{\pi}{2}} \frac{x\, sin\, x\, cos\, x + \left(\frac{\pi}{2} - x\right) sin\, x\, cos\, x}{sin^4 x + cos^4 x} dx$ $2\, I = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x\, .\, \cos x}{\sin^4 x + \cos^4 x} dx$ Divide N & D by $\cos^4 x$ $2\, I = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\tan x \sec^2 x}{\tan^4 x + 1} dx$ Put $\tan^2 x = t$ when $x = 0$; $t = 0$ $2 \tan x \sec^2 x\ dx = dt$ when $x = \frac{\pi}{2}$; $t = \infty$

	$\tan x \sec^2 x \ dx = \frac{dt}{2}$ $\therefore \ 2I = \frac{\pi}{4}\int_0^{\infty} \frac{dt}{t^2+1}$ $2I = \frac{\pi}{4}(\tan^{-1}t)_0^{\infty}$ $2I = \frac{\pi}{4}[\tan^{-1}(\infty) - \tan^{-1}(0)]$ $2I = \frac{\pi}{4}\left[\frac{\pi}{2} - 0\right]$ $\Rightarrow 2I = \frac{\pi^2}{8}$ $\Rightarrow \quad I = \frac{\pi^2}{16}$ ans.
Q.6)	$I = \int_0^{\pi} \frac{x \ dx}{a^2\cos^2 x + b^2\sin^2 x}$
Sol.6)	$I = \int_0^{\pi} \frac{x \ dx}{a^2cos^2x + b^2sin^2x}$ ....(1) $I = \int_0^{\pi} \frac{(\pi - x)\ dx}{a^2cos^2(\pi-x) + b^2sin^2(\pi-x)}$ ....(P-IV) $I = \int_0^{\pi} \frac{x \ dx}{a^2\cos^2 x + b^2\sin^2 x}$ ....(2) $(1) + (2)$ $2I = \int_0^{\pi} \frac{x+\pi-x}{a^2\cos^2x + b^2\sin^2x}dx$ $2I = \pi\int_0^{\pi} \frac{1}{a^2\cos^2x + b^2\sin^2x}dx$ Type: Divide by cos^2x Divide N & D by cos^2x $\therefore \ 2I = \pi\int_0^{\pi} \frac{\sec^2x}{a^2 + b^2\tan^2x}dx$ $2I = 2\pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 + b^2 \ \tan^2x}$ ......$\left[\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx\right]$ ....(P-VI) Put $tan\, x = t \qquad x = 0 \ ; \ t = 0$ $\sec^2 x \ dx = dT \qquad x = \frac{\pi}{2} \ ; \ t = \infty$ $\therefore \ 2I = 2\pi\int_0^{\infty} \frac{dt}{a^2+b^2t^2}$ $I = \frac{\pi}{b^2}\int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$ $I = \frac{\pi}{b^2} \times \frac{b}{a}\left[tan^{-1}\left(\frac{bt}{a}\right)\right]_0^{\infty}$ $I = \frac{\pi}{ab}[\tan^{-1}(\infty) - \tan^{-1}(0)]$ $I = \frac{\pi}{ab}\left[\frac{\pi}{2} - 0\right]$ $I = \frac{\pi^2}{2ab}$ ans.
Q.7)	$I = \int_0^{\pi} \frac{x}{1-cos\,\alpha\ sin\,x}dx$

Sol.7)	$I = \int_0^{\pi} \frac{x}{1-\cos\alpha \sin x} dx$ ....(1) $I = \int_0^{\pi} \frac{\pi - x}{1-\cos\alpha \sin(\pi - x)} dx$ ....(P-IV) $I = \int_0^{\pi} \frac{\pi - x}{1-\cos\alpha \sin x} dx$ ....(2) $(1) + (2)$ $2I = \int_0^{\pi} \frac{x + \pi - x}{1-\cos\alpha \sin x} dx$ $2I = \pi \int_0^{\pi} \frac{1}{1-\cos\alpha \sin x} dx$ (Type: single $\sin x$, $\cos x$) $2I = \pi \int_0^{\pi} \frac{1}{1-\cos\alpha . \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}} dx$ $2I = \pi \int_0^{\pi} \frac{1+\frac{\tan^2 x}{2}}{1+\tan^2\frac{x}{2} - 2\cos\alpha .\tan\frac{x}{2}} dx$ $2I = \pi \int_0^{\pi} \frac{1+\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2} - 2\cos\alpha .\ \tan\frac{x}{2} + 1} dx$ Put $\tan\frac{x}{2} = t$ when $x = 0$; $t = 0$ $\sec^2\frac{x}{2} . dx = 2\,dt$ $x = \pi$; $t = \infty$ $2I = \frac{\pi}{2} \int_0^{\infty} \frac{dt}{t^2 - \frac{2t}{\cos\alpha} + 1}$ $2I = \frac{\pi}{2} \int_0^{\infty} \frac{dt}{t^2 - 2\cos\alpha\, t + 1}$ (perfect square) $2I = \frac{\pi}{2} \int_0^{\infty} \frac{1}{(t-\cos\alpha)^2 - \cos^2\alpha + 1} dt$ $2I = \frac{\pi}{2} \int_0^{\infty} \frac{dt}{(t-\cos\alpha)^2 - \sin^2\alpha} dt$ .....$[1 - \cos^2\alpha = \sin^2\alpha]$ $2I = \frac{\pi}{2} \times \frac{1}{\sin\alpha} \left[\tan^{-1}\left(\frac{t-\cos\alpha}{\sin\alpha}\right)\right]_0^{\infty}$ $2I = \frac{\pi}{2\sin\alpha} \left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{-\cos\alpha}{\sin\alpha}\right)\right]$ $I = \frac{\pi}{4} \sin\alpha \left[\frac{\pi}{2} - \tan^{-1}(-\cot\alpha)\right]$ $I = \frac{\pi}{4} \sin\alpha \left[\frac{\pi}{2} + \tan^{-1}(\cot\alpha)\right]$ ....$[\tan^{-1}(-x) = -\tan^{-1}x]$ $I = \frac{\pi}{4} \sin\alpha \left[\frac{\pi}{2} + \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \alpha\right)\right)\right]$ $I = \frac{\pi}{4\sin\alpha} \left[\frac{\pi}{2} + \frac{\pi}{2} - \alpha\right]$ $I = \frac{\pi}{4\sin\alpha} [\pi - \alpha]$ ans.
Q.8)	$I = \int_0^{\infty} \frac{\log x}{1+x^2} dx$
Sol.8)	$I = \int_0^{\infty} \frac{\log x}{1+x^2} dx$

	Put $x = \tan\theta$ when $x = 0$; $\theta = 0$ $dx = sec^2\theta\, d\theta$ when $x = \infty$; $\theta = \frac{\pi}{2}$ $\therefore\ I = \int_0^{\frac{\pi}{2}} \frac{\log(tan\,\theta)}{1+tan^2\theta} . sec^2\theta\, d\theta$ $I = \int_0^{\frac{\pi}{2}} \frac{log\,(tan\,\theta)}{sec^2\theta} . sec^2\theta\, d\theta$ $I = \int_0^{\frac{\pi}{2}} log\,(tan\,\theta)\, d\theta$ ....(1) Proceed Yourself 0 ans.
Q.9)	$I = \int_0^1 \frac{log\,(1+x)}{1+x^2} dx$
Q.9)	$I = \int_0^1 \frac{log\,(1+x)}{1+x^2} dx$ Put $x = \tan\theta$ when $x = 0$; $\theta = 0$ $dx = sec^2\theta\, d\theta$ when $x = 1$; $\theta = \frac{\pi}{4}$ $\therefore\ I = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{1+tan^2\theta} . sec^2\theta\, d\theta$ $I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\, d\theta$ ....(1) Proceed yourself $I = \frac{\pi}{8} \log 2$ ans.
Q.10)	$I = \int_0^{\frac{\pi}{2}} \log(sin\,x)\, dx$
Sol.10)	$I = \int_0^{\frac{\pi}{2}} \log(sin\,x)\, dx$ ....(1) I = $\int_0^{\frac{\pi}{2}} \log\left(sin\left(\frac{\pi}{2} - x\right)\right) dx$ ....(P-IV) I = $\int_0^{\frac{\pi}{2}} \log(cos\,x)\, dx$ ....(2) (1) + (2) $2I = \int_0^{\frac{\pi}{2}} \log(sin\,x\ .\ cos\,x)\, dx$ $2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{sin\,(2x)}{2}\right) dx$ $2I = \int_0^{\frac{\pi}{2}} \log(sin\,(2x)) - \log 2\, dx$ $2I = \int_0^{\frac{\pi}{2}} \log(sin\,(2x))\, dx - \int_0^{\frac{\pi}{2}} \log 2 . dx$ $2I = \int_0^{\frac{\pi}{2}} \log(sin\,(2x))\, dx - \log 2\, (x)_0$

	$2\,I \;=\; \int_0^{\frac{\pi}{2}} \log\,(sin\;(2x))\,dx - \frac{\pi}{2}\log 2$ $2\,I \;= I_1 - \frac{\pi}{2}\,log\;2$ ....(3) Where $I_1 \;=\; \int_0^{\frac{\pi}{2}} log\;(sin\;(2x))\,dx$ Put $2x \;=\; t$ when $x \;=\; 0\;;\; t \;=\; 0$ $dx \;=\; \frac{dt}{2}$ $x \;=\; \frac{\pi}{2}\;;\; t \;=\; \pi$ $\therefore\; I_1 \;=\; \frac{1}{2}\int_0^{\pi} \log\,(sin\;t)\,dt$ $I_1 \;=\; \frac{1}{2} \times 2\int_0^{\frac{\pi}{2}} \log\,(sin\;t)\,dt$ ......(P-VI) $I_1 \;=\; \int_0^{\frac{\pi}{2}} log\;(\sin t\;)\,dt$ $I_1 \;=\; \int_0^{\frac{\pi}{2}} \log\,(sin\;x)\,dx$ ....(P-I) $I_1 = I$ $\therefore\;\; eq.(3)$ becomes $2I \;=\; I - \frac{\pi}{2}\log 2$ $\therefore\;\; I \;=\; -\frac{\pi}{2}\log 2 \quad ans.$