

DEFINITE INTEGRALS

Q.1)	$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log \left(\frac{2 - \sin x}{2 + \sin x} \right) dx$
Sol.1)	$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log \left(\frac{2 - \sin x}{2 + \sin x} \right) dx$ Here $f(x) = \log \left[\frac{2 - \sin x}{2 + \sin x} \right]$ $f(-x) = \log \left[\frac{2 - \sin(-x)}{2 + \sin(-x)} \right]$ $= \log \left(\frac{2 + \sin x}{2 - \sin x} \right)$ $= -\log \left(\frac{2 - \sin x}{2 + \sin x} \right) \quad \dots\dots \left[\log \left(\frac{a}{b} \right) = -\log \left(\frac{b}{a} \right) \right]$ $f(-x) = -f(x)$ $\therefore f(x) \rightarrow \text{odd function}$ $\therefore I = 0 \quad \text{Ans}$
Q.2)	$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, dx$
Sol.2)	$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, dx$ $f(x) = \sin^2 x$ $f(-x) = \sin^2(-x) = (-\sin x)^2 = \sin^2 x = f(x)$ $\therefore f(x) \rightarrow \text{an even function}$ $\therefore I = 2 \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \quad \dots\dots \{ \because \int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx \}$ $I = \frac{2}{2} \int_0^{\frac{\pi}{2}} 1 - \cos(2x) \, dx$ $I = \left[x - \frac{\sin(2x)}{2} \right]_0^{\frac{\pi}{2}}$ $I = \left[\frac{\pi}{2} - \frac{\sin(\pi)}{2} \right] - [0]$ $I = \frac{\pi}{2} \quad \text{Ans} \quad \{ \because \sin(\pi) = 0 \}$
Q.3)	$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^4 x \, dx$
Sol.3)	$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^4 x \, dx$ Clearly $f(x)$ is an even function $\therefore I = 2 \int_0^{\frac{\pi}{4}} \sin^4 x \, dx$ $= 2 \int_0^{\frac{\pi}{4}} (\sin^2 x)^2 \, dx$

	$= 2 \int_0^{\frac{\pi}{4}} \left(\frac{1 - \cos(2x)}{2} \right)^2 dx$ $= \frac{2}{4} \int_0^{\frac{\pi}{4}} 1 + \cos^2(2x) - 2 \cos(2x) dx$ $= \frac{1}{2} \int_0^{\frac{\pi}{4}} 1 + \frac{1 + \cos(4x)}{2} - 2 \cos(2x) dx$ $= \frac{1}{4} \int_0^{\frac{\pi}{4}} 3 + \cos(4x) - 4 \cos(2x) dx$ $= \frac{1}{4} \left[3x + \frac{\sin(4x)}{4} - 2 \sin(2x) \right]_0^{\frac{\pi}{4}}$ $= \frac{1}{4} \left[\left(\frac{3\pi}{4} + \frac{\sin \pi}{4} - 2 \sin \left(\frac{\pi}{2} \right) \right) - (0 + 0 - 0) \right]$ $= \frac{1}{4} \left[\frac{3\pi}{4} + 0 - 2 \right] \quad \dots \dots \dots \{ \because \sin(\pi) = \sin(2\pi) = 0 \}$ $= \frac{3\pi}{16} - \frac{1}{2} \quad \text{Ans} \dots$
Q.4)	$\int_{-a}^a \sqrt{\frac{a-x}{a+x}} dx$
Sol.4)	$I = \int_{-a}^a \sqrt{\frac{a-x}{a+x}} dx$ rationalize $I = \int_{-a}^a \frac{a-x}{\sqrt{a^2-x^2}} dx$ $I = a \int_{-a}^a \frac{1}{\sqrt{a^2-x^2}} dx - \int_{-a}^a \frac{x}{\sqrt{a^2-x^2}} dx$ Let $f(x) = \frac{1}{\sqrt{a^2-x^2}} \quad ; \quad g(x) = \frac{x}{\sqrt{a^2-x^2}}$ $f(-x) = \frac{1}{\sqrt{a^2-(-x)^2}} \quad ; \quad g(-x) = \frac{-x}{\sqrt{a^2-(-x)^2}}$ $f(-x) = \frac{1}{\sqrt{a^2-x^2}} \quad ; \quad g(-x) = \frac{-x}{\sqrt{a^2-x^2}}$ $f(-x) = f(x) \quad ; \quad g(-x) = -g(x)$ $f(x) \rightarrow \text{even function} \quad g(x) \rightarrow \text{odd function}$ $\therefore I = 2a \int_0^a \frac{1}{\sqrt{a^2-x^2}} dx - 0 \quad \dots \dots \dots \left\{ \because \int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx : f(x) \text{ even} \\ 0 : f(x) \text{ odd} \end{cases} \right\}$ $I = 2a \left(\sin^{-1} \frac{x}{a} \right)_0^a$ $I = 2a (\sin^{-1}(1) - \sin^{-1}(0))$ $I = 2a \left(\frac{\pi}{2} \right)$ $I = a\pi \quad \text{ans.}$
Q.5)	$I = \int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx$
Sol.5)	$I = \int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx$

	$I = \int_{-\pi}^{\pi} \frac{2x + 2x \sin x}{1 + \cos^2 x} dx$ $I = \int_{-\pi}^{\pi} \frac{2x}{1 + \cos^2 x} dx + 2 \int_{-\pi}^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$ $f(x) = \frac{2x}{1 + \cos^2 x} \quad ; \quad g(x) = \frac{x \sin x}{1 + \cos^2 x}$ $f(-x) = \frac{-2x}{1 + \cos^2 x} \quad ; \quad g(-x) = \frac{(-x) \sin(-x)}{1 + \cos^2 x}$ $f(-x) = -f(x) \quad ; \quad g(-x) = \frac{x \sin x}{1 + \cos^2 x} = g(x)$ $f(x) \rightarrow \text{odd function} \quad ; \quad g(x) \rightarrow \text{odd function}$ $\therefore I = 0 + 4 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$  Removal of x (already done) Do yourself $I = \pi^2 \quad \text{ans.}$
Q.6)	$I = \int_0^{2\pi} \cos^5 x \, dx$
Sol.6)	$I = 2 \int_0^{\pi} \cos^5 x \, dx \quad \dots \dots \left\{ \int_0^{2a} f(x) \, dx = 2 \int_0^a f(x) \, dx \quad f(2a - x) = f(x) \right\}$ $I = 2 \int_0^{\pi} \cos^5 x \, dx \quad \dots \dots (1)$ $I = 2 \int_0^{\pi} \cos^5(\pi - x) \, dx \quad \dots \dots \left[\int_0^a f(x) \, dx = \int_0^a f(a - x) \, dx \right]$ $I = -2 \int_0^{\pi} \cos^5 x \, dx \quad \dots \dots (2) \quad \dots \dots [\because \cos(\pi - x) = -\cos x]$ $(1) + (2)$ $2I = 0$ $I = 0 \quad \text{Ans} \dots \dots$
Q.7)	$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x^2 \tan x + \sin x - x^2) \, dx$
Sol.7)	$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 + x^2 \tan x + \sin x \, dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \, dx$ $f(x) = x^3 + x^2 \tan x + \sin x \quad g(x) = x^2$ $f(-x) = -x^3 - x^2 \tan x - \sin x \quad g(-x) = (-x)^2 = x^2$ $f(-x) = -(x^3 + x^2 \tan x + \sin x) \quad g(-x) = g(x)$ $f(-x) = -f(x) \quad g(x) \rightarrow \text{even function}$ $\therefore f(x) \rightarrow \text{odd function}$ $\therefore I = 0 - 2 \int_0^{\frac{\pi}{2}} x^2 \, dx \quad \dots \dots \{\text{property-VII}\}$

	$= -2 \left(\frac{x^3}{3} \right)_0^{\frac{\pi}{2}}$ $= -2 \left(\frac{\pi^3}{24} \right)$ $= \frac{-\pi^3}{12} \quad \text{ans.}$
Q.8)	Evaluation using limit as sum $I = \int_2^4 3x - 2 \, dx$
Sol.8)	Here $a = 2, b = 4$ and $nh = 4 - 2 = 2$ $I = \lim_{h \rightarrow 0} h[f(2) + f(2+h) + f(2+2h) + \dots + f(2+(n-1)h)]$ $= \lim_{h \rightarrow 0} h[(6-2) + (6+3h-2) + (6+6h-2) + \dots + (6+3(n-1)h-2)]$ $= \lim_{h \rightarrow 0} h[4 + (4+3h) + (4+6h) + \dots + (4+3(n-1)h)]$ $= \lim_{h \rightarrow 0} h[(4+4+4 \dots n \text{ term}) + (3h+6h+\dots+3(n-1)h)]$ $= \lim_{h \rightarrow 0} h[4n + 3h(1+2+\dots+(n-1))]$ $= \lim_{h \rightarrow 0} h \left[4n + 3h \cdot \frac{n(n-1)}{2} \right]$ $= \lim_{h \rightarrow 0} \left[4nh + 3 \frac{(nh)(nh-h)}{2} \right]$ Put $nh = 2$ $\therefore I = \lim_{h \rightarrow 0} \left[8 + \frac{3(2)(2-h)}{2} \right]$ $= 8 + \frac{3(2)(2)}{2}$ $I = 14$ Ans Check: $I = \int_2^4 3x - 2 \, dx = \left[\frac{3x^2}{2} - 2x \right]_2^4 = [(24-8) - (6-4)]$ $= 16 - 2 = 14$
Q.9)	$I = \int_{-1}^2 3x^2 + 2x - 5 \, dx$
Sol.9)	Here $a = -1, b = 2$ and $nh = 3$ $I = \lim_{h \rightarrow 0} h[f(-1) + f(-1+h) + f(-1+2h) + \dots + f(-1+(n-1)h)]$ $= \lim_{h \rightarrow 0} h[(3-2-5) + [3(1+h^2-2h)-2+2h-5] + [3(1+4h^2-4h)-2+4h-5] + \dots + [3(1+(n-1)^2h^2-2(n-1)h)-2+2(n-1)h-5]]$ $= \lim_{h \rightarrow 0} h[(-4) + (3n^2-4h-h) + (12h^2-8h-4) + \dots + (3(n-1)^2h^2-4(n-1)h-4)]$ $= \lim_{h \rightarrow 0} h[(-4-4-4 \dots n \text{ terms}) + (3h^2+12h^2+\dots+3(n-1)^2h^2) + (-4h-8 \dots -4(n-1)h)]$ $= \lim_{h \rightarrow 0} h[-4n + 3h^2(1^2+2^2+\dots+(n-1)^2) - 4h(1+2+3+\dots+(n-1))]$

	$= \lim_{h \rightarrow 0} h \left[-4n + 3h^2 \cdot \frac{n(n-1)(2n-1)}{6} - 4h \cdot \frac{n(n-1)}{2} \right]$ $= \lim_{h \rightarrow 0} \left[-4nh + \frac{(nh)(nh-h)(2nh-h)}{2} - 2(nh)(nh-h) \right]$ Put $nh = 3$ $\therefore I = \lim_{h \rightarrow 0} \left[-12 + \frac{(3)(3-h)(6-h)}{2} - 2(3)(3-h) \right]$ $= \left[-12 + \frac{(3)(3)(6)}{2} - 2(3)(3) \right]$ $= -12 + 27 - 18$ $= -30 + 27$ $I = -3 \quad \text{Ans}$ Check: $I = \int_{-1}^2 3x^2 + 2x - 5 \, dx$ $= (x^3 + x^2 - 5x)_{-1}^2 = (8 + 4 - 10) - (-1 + 1 + 5)$ $= 2 - 5 = -3 \quad \text{Ans}$
Q.10)	$I = \int_2^5 e^{2x+1} \, dx$
Sol.10)	Here $a = 2$, $b = 5$ & $nh = 3$ $I = \lim_{h \rightarrow 0} h [f(2) + f(2+h) + f(2+2h) + \dots (f(2+(n-1)h))]]$ $= \lim_{h \rightarrow 0} h [e^{4+1} + e^{4+2h+1} + e^{4+4h+1} + \dots e^{4+2(n-1)h+1}]$ $= \lim_{h \rightarrow 0} h [e^5 + e^{2h+5} + e^{4h+5} + \dots e^{2(n-1)h+5}]$ $= e^5 \lim_{h \rightarrow 0} h [1 + e^{2h} + e^{4h} + \dots e^{2(n-1)h}]$ G.P. $a = 1$, $r = e^{2h}$ $= e^5 \lim_{h \rightarrow 0} h \left[1 \cdot \left(\frac{(e^{2h})^n - 1}{e^{2h} - 1} \right) \right]$ $= e^5 \lim_{h \rightarrow 0} \left[\frac{e^{2nh} - 1}{e^{2h} - 1} \right]$ $= e^5 \lim_{h \rightarrow 0} h \left[\frac{e^{2nh} - 1}{\frac{e^{2h} - 1}{2h} \times 2h} \right] \quad \dots \dots \{adjustment\}$ $= \frac{e^5 [e^6 - 1]}{\lim_{h \rightarrow 0} \left(\frac{e^{2h} - 1}{2h} \right) \times 2} \quad \dots \dots \{ \because nh = 3 \}$ $I = \frac{e^5(e^6 - 1)}{2} \quad \text{Ans} \quad \dots \dots \left\{ \because \lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right) = 1 \right\}$ Check: $I = \int_2^5 e^{2x+1} \, dx$ $= \left(\frac{e^{2x+1}}{2} \right)_2^5 = \frac{1}{2} [e^{11} - e^5] = \frac{e^5(e^6 - 1)}{2} \quad \text{ans,}$