DEFINITE INTEGRALS

Q.1)	(a) $I \;=\; \int_0^2 \frac{1}{4+x-x^2}\,dx$
Sol.1)	(a) $I \;= \int_0^2 \frac{1}{4+x-x^2}\,dx$ $\quad Perfect\ square$ $I \;=\; -\int_0^2 \frac{1}{x^2-x-4}$ $I \;=\; -\int_0^2 \frac{1}{\left(x-\frac{1}{2}\right)^2-\frac{1}{4}-4}\,dx$ $I \;=\; -\int_0^2 \frac{1}{\left(x-\frac{1}{2}\right)^2-\left(\frac{\sqrt{17}}{2}\right)^2}\,dx$ $I \;=\; \int_0^2 \frac{1}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx$ $I \;=\; \frac{1}{2\times\frac{\sqrt{17}}{2}} \times \left(\log\left\|\frac{\frac{\sqrt{17}}{2}+x-\frac{1}{2}}{\frac{\sqrt{17}}{2}-x+\frac{1}{2}}\right\|\right)_0^2$ $I \;=\; \frac{1}{\sqrt{17}}\left(log\left\|\frac{\sqrt{17}+2x-1}{\sqrt{17}-2x+1}\right\|\right)_0^2$ $I \;=\; \frac{1}{\sqrt{17}}\left[log\left\|\frac{\sqrt{17}+3}{\sqrt{17}-3}\right\| - log\left\|\frac{\sqrt{17}-1}{\sqrt{17}+1}\right\|\right]$ $I \;=\; \frac{1}{\sqrt{17}} log\left\|\frac{\sqrt{17}+3}{\sqrt{17}-3}\times\frac{\sqrt{17}+1}{\sqrt{17}-1}\right\|$ $\quad\quad \ldots\ldots\{log\ A - \ log\ B \;=\; log\ (A/B)\}$ $I \;=\; \frac{1}{\sqrt{17}} log\left\|\frac{17+\sqrt{17}+3\sqrt{17}+3}{17-\sqrt{17}-3\sqrt{17}+3}\right\|$ $I \;=\; \frac{1}{\sqrt{17}} log\left\|\frac{4\sqrt{17}+20}{20-4\sqrt{17}}\right\|$ $I \;=\; \frac{1}{\sqrt{17}} log\left\|\frac{\sqrt{17}+5}{5-\sqrt{17}}\right\|$ $I \;=\; \frac{1}{\sqrt{17}} log\left\|\frac{(\sqrt{17}+5)(5+\sqrt{17})}{25-17}\right\|$ $\quad\quad \ldots\ldots.(rationalize\ )$ $I \;=\; \frac{1}{\sqrt{17}} log\left(\frac{17+25+2\sqrt{17}}{8}\right)$ $I \;=\; \frac{1}{\sqrt{17}} log\left(\frac{42+2\sqrt{17}}{8}\right)$ $\therefore\ I \;=\; \frac{1}{\sqrt{17}} log\left(\frac{21+\sqrt{17}}{4}\right) \quad Ans\ \ldots.$
Q.2)	$I \;=\; \int_1^2 \frac{5(x^2-x-1)}{x^2+3x+2}\,dx$
Sol.2)	$I \;=\; 5\int_1^2 \frac{(x^2-x-1)}{x^2+3x+2}\,dx$ Here , degree of numerator = degree of denominator Express $q = \frac{R}{D}$ $I \;=\; 5\int_1^2 \left(1-\frac{4x+3}{x^2+3x+2}\right)dx$

	$= 5\int_1^2 1\,.dx - 5\int_1^2 \frac{4x+3}{x^2+3x+3}dx$ $= 5(x)_1^2 - 5\int_1^2 \frac{4x+3}{(x+1)(x+2)}dx$ $I = 5 - 5\int_1^2 \frac{4x+3}{(x+1)(x+2)}dx$ Let $\frac{4x+3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$ $\Rightarrow 4x + 3 = A(x + 2) + B(x + 1)$ Comp. the coff. of x and constant $4 = A + B$ $3 = 2A + B$ Solving then equation, we get $A = -1$ & $B = 5$ $\therefore I = 5 - 5\int_1^2 \frac{-1}{x+2} + \frac{5}{x+2}dx$ $I = 5 - 5\,[-log\|x+1\| + 5\,log\|x+2\|]_1^2$ $= 5 - 5[(-log\,3 + 5\,log\,4) — (log\,2 + 5\,log\,3)]$ $= 5 - 5[-log\,3 + 10\,log\,2 + log\,2 - 5\,log\,3]$ $= 5 - 5[11\,log\,2 - 6\,log\,3]$ $I = 5 - 55\,log\,2 + 30\,log\,3$ *Ans …*
Q.3)	If $\int_a^b x^3\,dx = 0$ and if $\int_a^b x^2\,dx = \frac{2}{3}$. Find value of a & b.
Sol.3)	Consider $\int_a^b x^3\,dx = 0$ $\Rightarrow \left(\frac{x^4}{4}\right)_a^b = 0$ $\Rightarrow \frac{1}{4}[b^4 - a^4] = 0$ $\Rightarrow a^4 = b^4$ $\Rightarrow a = -b$ Consider $\int_a^b x^2\,dx = \frac{2}{3}$ $\Rightarrow \left(\frac{x^3}{3}\right)_a^b = \frac{2}{3}$ $\Rightarrow \frac{1}{3}(b^3 - a^3) = \frac{2}{3}$ $\Rightarrow b^3 - a^3 = 2$ $\Rightarrow (-a)^3 - a^3 = 2$ $\Rightarrow -a^3 - a^3 = 2$ $\Rightarrow -2a^3 = 2$ $\Rightarrow a^3 = -1$

	$\Rightarrow\ a = -1$ Since $b = -a$ $\therefore\ b = 1$ $\therefore\ a = -1\ \&\ b = 1$ Ans….
Q.4)	$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cos(2x).\log(sin\ x)\,dx$
Sol.4)	$I = \left[\left(\log(sin\ x).\frac{sin(2x)}{2}\right)_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{sin\ x}\ .cos\ x\ .\frac{sin(2x)}{2}dx\right]$ $= \left[\left(\log 1\ .\frac{sin\ \pi}{2}\right) - \left(\log\left(\frac{1}{\sqrt{2}}\right).\frac{sin\frac{\pi}{2}}{2}\right)\right] - \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{sin\ x}\ .cos\ x\ .2sin\ x\ cos\ x\ dx$ $= \left[0 + log(\sqrt{2})\frac{1}{2}\right] - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cos^2x\ dx$ $\ldots\ldots..\{log\ (a/b) = -log\ (b/a)\}$ $= \frac{1}{2}log\ 2\frac{1}{2} - \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 + cos(2x)\ dx$ $= \frac{1}{2}log\ 2\ \frac{1}{2} - \frac{1}{2}\left[x + \frac{sin(2x)}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ $= \frac{1}{2}log\ 2\ \frac{1}{2} - \frac{1}{2}\left[\left(\frac{\pi}{2} + 0\right) - \left(\frac{\pi}{4} + \frac{1}{2}\right)\right]$ $= \frac{1}{2}log\ 2\ \frac{1}{2} - \frac{\pi}{4} + \frac{\pi}{8} + \frac{1}{4}$ $I = \frac{1}{4}\ log\ 2 - \frac{\pi}{8} + \frac{1}{4}$ ans.
Q.5)	$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{sin\ x + cos\ x}{\sqrt{sin(2x)}}dx$
Sol.5)	$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{sin\ x+cos\ x}{\sqrt{1-1+sin(2x)}}dx$ $= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{sin\ x+cos\ x}{\sqrt{1-(1-sin\ 2x)}}dx$ $= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{sin\ x+cos\ x}{\sqrt{1-[sin^2x+cos^2x-2\ sin\ x\ cos\ x]}}dx$ $= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{sin\ x+cos\ x}{\sqrt{1-(sin\ x-cos\ x)^2}}dx$ $Put\ \ sin\ x - cos\ x = t$ $when\ x = \frac{\pi}{6}\ \ t = \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2}$ $(cos\ x + sin\ x)dx = dt$ $when\ x = \frac{\pi}{3}\ \ t = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$ $\therefore\ I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$ $= (sin^{-1}t)_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}}$ $= sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) - sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right)$

	$= sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) + sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$ ……. $\{\because\ sin-1(-x) = -sin-1\,x\}$ $I = 2\,sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$ *Ans …..*
Q.6)	$I = \int_0^{\frac{\pi}{4}} \frac{sin\,x+cos\,x}{9+16\,sin\,(2x)} dx$
Sol.6)	$I = \int_0^{\frac{\pi}{4}} \frac{sin\,x+cos\,x}{9+16\,[1-1+sin(2x)]} dx$ $I = \int_0^{\frac{\pi}{4}} \frac{sin\,x+cos\,x}{9+16\,[1-(1-sin\,2x)]} dx$ $I = \int_0^{\frac{\pi}{4}} \frac{sin\,x+cos\,x}{9+16\,[1-(sin^2 x+cos^2 x-2\,sin\,x\,cos\,x)]} dx$ $I = \int_0^{\frac{\pi}{4}} \frac{sin\,x+cos\,x}{9+16\,[1-(sin\,x-cos\,x)^2]} dx$ $Put\ sin\,x - cos\,x = t$ $when\ x = 0, t = 0-1 = -1$ $(cos\,x + sin\,x)dx = dt$ $when\ x = \frac{\pi}{4}, t = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$ $\therefore\ I = \int_{-1}^{0} \frac{dt}{9+16(1-t^2)}$ $= \int_{-1}^{0} \frac{1}{25-16t^2} dt$ $= \frac{1}{16}\int_{-1}^{0} \frac{1}{\left(\frac{5}{4}\right)^2 - t^2} dt$ $= \frac{1}{16} \times \frac{1}{2\times\frac{5}{4}} \left(log\left\|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right\|\right)_{-1}^{0}$ $= \frac{1}{40}\left(log\left\|\frac{5+4t}{5-4t}\right\|\right)_{-1}^{0}$ $= \frac{1}{40}\left[log\|1\| - log\left\|\frac{1}{9}\right\|\right]$ $= \frac{1}{40}[0 + log(9)]$ $I = \frac{1}{40} log\,9$ $I = \frac{1}{20} log\,3$ *Ans …..*
Q.7)	(a) $I = \int_1^2 e^{2x}\left(\frac{1}{x} - \frac{1}{2x^2}\right) dx$
Sol.7)	(a) $I = \int_1^2 e^{2x}.\frac{1}{x} dx - \frac{1}{2}\int_1^2 e^{2x}.\frac{1}{x^2} dx$ $= \left(\frac{1}{x}.\frac{e^{2x}}{2}\right)_1^2 + \frac{1}{2}\int_1^2 e^{2x}.\frac{1}{x^2} dx - \frac{1}{2}\int_1^2 e^{2x}.\frac{1}{x^2} dx$ $= \left(\frac{1}{2}.\frac{e^4}{2}\right) - \left(\frac{1}{1}.\frac{e^2}{2}\right)$ $= \frac{e^4}{4} - \frac{e^2}{2}$ $= \frac{e^2}{2}\left(\frac{e^2}{2} - 1\right)$ Ans…..

Q.8)	$I = \int_0^{\frac{\pi}{2}} \frac{cos^2x}{cos^2x+4sin^2x} dx$
Sol.8)	Divide N & D by $\cos^4 x$ $I = \int_0^{\frac{\pi}{2}} \frac{sec^2x}{sec^2x+4tan^2x\, sec^2x} dx$ $I = \int_0^{\frac{\pi}{2}} \frac{sec^2x}{sec^2x(1+4\, tan^2x)} dx$ $I = \int_0^{\frac{\pi}{2}} \frac{sec^2x}{(1+tan^2x)(1+4tan^2x)} dx$ put $tan\, x = t$; $sec^2 x\, dx = dt$; $when\ x = 0\ ;\ t = 0$; $when\ x = \frac{\pi}{2}\ ;\ t = \infty$ $I = \int_0^{\infty} \frac{dt}{(1+t^2)(1+4t^2)} dx$ Type: partial fraction type 4 Let $t^2 = y$ (temp.) $\therefore \frac{1}{(1+t^2)(1+4t^2)} = \frac{1}{(1+y)(1+4y)}$ let $\frac{1}{(1+y)(1+4y)} = \frac{A}{1+y} + \frac{B}{1+4y}$ $1 = A(1 + 4y) + B(1 + y)$ Comp. $0 = 4A + B$ $1 = A + B$ Solving these equations $A = -\frac{1}{3}$ & $B = \frac{4}{3}$ $\therefore I = \int_0^{\infty} \frac{-1}{3(1+t^2)} + \frac{4}{3} . \frac{1}{(1+4t^2)} dt$ $= \frac{-1}{3}\int_0^{\infty} \frac{1}{1+t^2} dt + \frac{4}{3}.\frac{1}{4}\int_0^{\infty} \frac{1}{\frac{1}{4}+t^2} dt$ $= \frac{-1}{3}\int_0^{\infty} \frac{1}{1+t^2} dt + \frac{1}{3}\int_0^{\infty} \frac{1}{\left(\frac{1}{2}\right)^2+t^2} dt$ $= \frac{-1}{3} tan^{-1}(t)_0^{\infty} + \frac{1}{3} \times 2\left(tan^{-1}(2t)\right)_0^{\infty}$ $= \frac{-1}{3}(tan^{-1}\infty - tan^{-1}0) + \frac{2}{3}[tan^{-1}(\infty) - tan^{-1}(0)]$ $= \frac{-1}{3}\left[\frac{\pi}{2}\right] + \frac{2}{3}\left[\frac{\pi}{2} - 0\right]$ $= \frac{-\pi}{6} + \frac{\pi}{3}$ $I = \frac{\pi}{6}$ Ans…..
Q.9)	$\int_0^1 \sqrt{\frac{1-x}{1+x}} dx$
Sol.9)	$\int_0^1 \sqrt{\frac{1-x}{1+x}} dx$

	rationalize $I = \int_0^1 \frac{1-x}{\sqrt{1-x^2}} dx$ Separate $I = \int_0^1 \frac{1}{\sqrt{1-x^2}} dx - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$ $Put\ 1 - x^2 = t \quad when\ x = 0\ ;\ t = 1$ $x\,dx = -\frac{dt}{2} \quad when\ x = 1\ ;\ t = 0$ $\therefore I = \int_0^1 \frac{1}{\sqrt{1-x^2}} dx + \frac{1}{2}\int_1^0 \frac{dt}{\sqrt{t}}$ $I = (sin^{-1}x)_0^1 + \frac{1}{2}\left(2\sqrt{t}\right)_1^0$ $I = \left(\frac{\pi}{2} - 0\right) + \frac{1}{2}(0-2)$ $I = \frac{\pi}{2} - 1$ ans.
Q.10)	$I = \int_0^{\frac{1}{\sqrt{2}}} \frac{sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}$
Sol.10)	$I = \int_0^{\frac{1}{\sqrt{2}}} \frac{sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}$ $I = \int_0^{\frac{1}{\sqrt{2}}} \frac{sin^{-1}x}{(1-x^2)\sqrt{1-x^2}} dx$ Put $sin^{-1}x = \text{t}$ when $x = 0\ ;\ t = 0$ $\frac{1}{\sqrt{1-x^2}} dx = dt$ when $x = \frac{1}{\sqrt{2}}\ ;\ t = \frac{\pi}{4}$ $I = \int_0^{\frac{\pi}{4}} \frac{t}{(1-x^2)} dt$ $= \int_0^{\frac{\pi}{4}} \frac{t}{(1-sin^2 t)} dt \quad \ldots\ldots\{\because\ x = sin\,t\}$ $= \int_0^{\frac{\pi}{4}} sec^2\,t\,.\,t\ dt$ $= (t\ tan\,)_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 1\,.\,tan\ t\ dt$ $= \left(\frac{\pi}{4}.\,tan\frac{\pi}{4} - 0\right) - (\log\lvert\sec t\mid)_0^{\frac{\pi}{4}}$ $= \frac{\pi}{4} - \left[log\left(\sqrt{2}\right) - log(1)\right] \quad \ldots\ldots\left\{\because\ \ sec\frac{\pi}{4} = \sqrt{2}sec\ \theta = 1\right\}$ $= \frac{\pi}{4} - \left[\frac{1}{2}\log 2\right]$ $I = \frac{\pi}{4} - \frac{1}{2}\log 2$ Ans...