Determinants
Class 12th

Solve for x

Q.1) Solve $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

Sol.1) $c_1 \to c_1 + c_2 + c_3$

$$\Rightarrow \begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} = 0$$

taking $(3a - x)$ common from

$$\Rightarrow (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix} = 0$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} = 0$$

taking 2x common from $R_2 \& R_3$ both

$$\Rightarrow (3a-x)(2x)(2x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

expanding along R_1

$$\Rightarrow (3a-x)(4x^2)[1(1) - 0 + 0] = 0$$

$$\Rightarrow (3a-x)(4x^2) = 0$$

$$\Rightarrow x = 3a, x = 0$$ ans.

Q.2) Solve $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

Sol.2) We have $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Rightarrow \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix} = 0$$

$c_2 \to c_2 - 2c_1$ and $c_3 \to c_3 - 3c_1$

$$\Rightarrow \begin{vmatrix} x-2 & 1 & 2 \\ -2 & -2 & -6 \\ -6 & -12 & -42 \end{vmatrix} = 0$$

taking (–2) and (–6) common from R_2 & R_3 respectively

$$\Rightarrow 12\begin{vmatrix} x-2 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 7 \end{vmatrix} = 0$$

expanding along R_1

$$\Rightarrow 12[(x-2)(7-6) - 1(7-3) + 2(2-1)] = 0$$

$$\Rightarrow 12[x - 2 - 4 + 2] = 0$$

$\Rightarrow 12(x-4) = 0$

$\Rightarrow x = 4$ ans.

Proving Questions

Q.3)

Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Sol.3)

let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

$R_2 \rightarrow R_2 - R_1 \text{and} R_3 \rightarrow R_3 - R_1$

$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$

taking (b-a) and (c-a) common from R_2 & R_3 respectively

$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$

expanding along R_1

$= (b-a)(c-a)[1(c+a-b-a) - a(0) + a^2(0)]$

$= (b-a)(c-a)(c-b)$

$= (a-b)(b-c)(c-a)$ = RHS Proved

Q.4)

Show that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

Sol.4)

let $\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

$c_1 \rightarrow ac_1; c_2 \rightarrow bc_2 \text{and} c_3 \rightarrow ac_3$

$= \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix}$

taking abc common from C_3

$= \frac{abc}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$

$c_2 \leftrightarrow c_3$

$= -\begin{vmatrix} a^2 & b^2 & c^2 \\ 1 & 1 & 1 \\ a^3 & b^3 & c^3 \end{vmatrix}$

$c_1 \leftrightarrow c_2$

$= \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

$c_2 \rightarrow c_2 - c_1 \text{and} c_3 \rightarrow c_3 - c_1$

$= \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2-a^2 & c^2-a^2 \\ a^3 & (b-a)(b^2+ab+a^2) & (c-b)(c^2+ac+a^2) \end{vmatrix}$

	taking $(b- a)$and $(c- b)$common from c_2& c_3 $= (b-a)(c-a)\begin{vmatrix}1 & 0 & 0\\ a^2 & b+a & c+a\\ a^3 & b^2+ab+q^2 & c^2+ac+a^2\end{vmatrix}$ expanding along R_1 $= (b-a)(c-a)[(b+a)(c^2+ac+a^2)-(c+a)(b^2+ab+a^2)]$ $= (b-a)(c-a)[bc^2+abc+a^2b+ac^2+a^2c+a^3-b^2c-abc-a^2c-ab^2-a^2b-a^3]$ $= (b-a)(c-a)(bc^2+ac^2-b^2c-ab^2)$ $= (b-a)(c-a)[bc(c-b)+a(c^2-b^2)]$ $= (b-a)(c-a)(c-b)[bc+a(c+b)]$ $= (b-a)(c-a)(c-b)(bc+ac+ab)$ $= (a-b)(b-c)(c-a)(ab+bc+ac)$= RHS
Q.5)	Show that $\begin{vmatrix}a+b+2c & a & b\\ c & b+c+2a & b\\ c & a & c+a+2b\end{vmatrix} = 2(a+b+c)^3$
Sol.5)	$c_1 \to c_1 + c_2 + c_3$ $= \begin{vmatrix}2(a+b+c) & a & b\\ 2(a+b+c) & b+c+2a & b\\ 2(a+b+c) & a & c+a+2b\end{vmatrix}$ taking 2(a + b + c) common from C_1 $= 2(a+b+c)\begin{vmatrix}1 & a & b\\ 1 & b+c+2a & b\\ 1 & a & c+a+2b\end{vmatrix}$ $R_2 \to R_2 - R_1 \text{and} R_3 \to R_3 - R_1$ $= 2(a+b+c)\begin{vmatrix}1 & a & b\\ 0 & b+c+2a & b\\ 0 & a & c+a+2b\end{vmatrix}$ expanding along R_1 $= 2(a+b+c)[(a+b+c)(a+b+c)] = 2(a+b+c)^3$ ans.
Q.6)	Show $\begin{vmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ \beta+\gamma & \gamma+\alpha & \alpha+\beta\end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$
Sol.6)	$R_3 \to R_3 + R_1$ $= \begin{vmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma\end{vmatrix}$ taking$(\alpha+\beta+\gamma)$common from R_3 $= (\alpha+\beta+\gamma)\begin{vmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ 1 & 1 & 1\end{vmatrix}$ $c_2 \to c_2 - c_1 \text{and} c_3 \to c_3 - c_1$ $= (\alpha+\beta+\gamma)\begin{vmatrix}\alpha & \beta-\alpha & \gamma-\alpha\\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2\\ 1 & 0 & 0\end{vmatrix}$ taking$(\beta-\alpha)$and$(\gamma-\alpha)$common from C_2 & C_3 respectively

	$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \alpha & 1 & 1 \\ \alpha^2 & \beta-\alpha & \gamma-\alpha \\ 1 & 0 & 0 \end{vmatrix}$ expanding $= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)[\alpha(0)+1(\gamma+\alpha)+1(-\beta-\alpha)]$ $= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma+\alpha-\beta-\alpha)$ $= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)$ $= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$= RHS
Q.7)	Show $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3$
Sol.7)	$R_1 \to R_1 + R_2 + R_3$ $= \begin{vmatrix} 2(a+b+c) & 0 & a+b+c \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$ taking (a + b + c) common from R_1 $= (a+b+c)\begin{vmatrix} 2 & 0 & 1 \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$ $c_1 \to c_1 - 2c_3$ $= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ c+a-2b & b-c & b \\ a+b-2c & c-a & c \end{vmatrix}$ expanding along R_1 $= (a+b+c)[1(c+a-2b)(c-a)-(a+b-2c)(b-c)]$ $= (a+b+c)[c^2-ac+ac-a^2-2bc+2ab-ab+ac-b^2+bc+2bc-2c^2]$ $= (a+b+c)(-a^2-b^2-c^2+ab+bc+ca)$ $= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ $= -[a^3+b^3+c^3-3abc]$ $= 3abc - a^3 - b^3 - c^3$ = RHS ans.
Q.8)	Show $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (x^3-1)^2$
Sol.8)	$c_1 \to c_1 + c_2 + c_3$ $= \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix}$ taking $(1+x+x^2)$ common from C_1 $= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}$ $R \to R_2 - R_1 \text{and} R_3 \to R_3 - R_1$ $= (1+x+x^2)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x-x^2 \\ 1 & x^2-x & 1-x^2 \end{vmatrix}$

	$= (1+x+x^2)\begin{vmatrix}1 & x & x^2\\ 0 & 1-x & x(1-x)\\ 0 & -x(1-x) & (1+x)(1-x)\end{vmatrix}$ taking (1 – x) common from R_2 & R_3 both $= (1+x+x^2)(1-x)^2\begin{vmatrix}1 & x & x^2\\ 0 & 1 & x\\ 0 & -x & 1+x\end{vmatrix}$ expanding $= (1+x+x^2)(1-x)^2[1+x+x^2]$ $= (1-x)^2(1+x+x^2)^2$ $= [(1-x)(1+x+x^2)]^2$ $= (1-x^3)^2$ .....$\{a^3-b^3=(a-b)(a^2+ab+b^2)\}$ $= (x^3-1)^2$ = RHS .....$\{(a-b)^2=(b-a)^2\}$
Q.9)	Show $\begin{vmatrix}a^2 & 2ab & b^2\\ b^2 & a^2 & 2ab\\ 2ab & b^2 & b^2\end{vmatrix} = (a^3+b^3)^2$
Sol.9)	$c_1 \to c_1+c_2+c_3$ $= \begin{vmatrix}a^2+2\mathrm{ab}+b^2 & 2ab & b^2\\ a^2+2\mathrm{ab}+b^2 & a^2 & 2ab\\ a^2+2\mathrm{ab}+b^2 & b^2 & a^2\end{vmatrix}$ $= (a+b)^2\begin{vmatrix}1 & 2\mathrm{ab} & b^2\\ 1 & a^2 & 2\mathrm{ab}\\ 1 & b^2 & a^2\end{vmatrix}$ $R_2 \to R_2 - R_1 \mathrm{and} R_3 \to R_3 - R_1$ $= (a+b)^2\begin{vmatrix}1 & 2ab & b^2\\ 0 & a^2-2ab & 2ab-b^2\\ 0 & b^2-2ab & a^2-b^2\end{vmatrix}$ expanding along R_1 $= (a+b)^2[(a^2-2\mathrm{ab})(a^2-b^2)-(b^2-2\mathrm{ab})(2\mathrm{ab}-b^2)]$ $= (a+b)^2[a^4-a^2b^2-2\mathrm{a}^3b+2\mathrm{ab}^3-2\mathrm{ab}^3+b^4+2\mathrm{a}^2b^2-2\mathrm{ab}^3]$ $= (a+b)^2[a^4+b^4+a^2b^2-2\mathrm{a}^3b+2\mathrm{a}^2b^2-2\mathrm{ab}^3]$ $= (a+b)^2(a^2=ab+b^2)^2$ .....$\{(a+b+c)^2=a^2+b^2+c^2+2\mathrm{ab}+2\mathrm{bc}+2\mathrm{ca}\}$ $= [(a+b)(a^2-ab+b^2)]^2$ $= (a^3+b^3)^2$ ans.
Q.10)	Show $\begin{vmatrix}a^2+1abac\\ abb^2+1bc\\ cacbc^2+1\end{vmatrix} = 1+a^2+b^2+c^2$
Sol.10)	$R_1 \to aR_1, R_2 \to bR_2 \mathrm{and} R_3 \to cR_3$ $= \frac{1}{abc}\begin{vmatrix}a^3+aa^2ba^2c\\ ab^2b^3+bb^2c\\ c^2ac^2bc^3+c\end{vmatrix}$ taking a , b , c common from c_1, c_2 & c_3 respectively $= \frac{abc}{abc}\begin{vmatrix}a^2+1a^2a^2\\ b^2b^2+1b^2\\ c^2c^2c^2+1\end{vmatrix}$ $R_1 \to R_1+R_2+R_3$

$$= \begin{vmatrix} 1 + a^2 + b^2 + c^2 1 + a^2 + b^2 + c^2 1 + a^2 + b^2 + c^2 \\ b^2 b^2 + 1 b^2 \\ c^2 c^2 c^2 + 1 \end{vmatrix}$$

$$= (1 + a^2 + b^2 + c^2) \begin{vmatrix} 111 \\ b^2 b^2 + 1 b^2 \\ c^2 c^2 c^2 + 1 \end{vmatrix}$$

$c_2 \to c_2 - c_1 \text{and} c_3 \to c_3 - c_1$

$$= (1 + a^2 + b^2 + c^2) \begin{vmatrix} 100 \\ b^2 10 \\ c^2 01 \end{vmatrix}$$

expanding

$= (1 + a^2 + b^2 + c^2)[1] = 1 + a^2 + b^2 + c^2$= RHS