Determinants

Class 12^{th}

Solving System of Linear Equations (Matrix Method)

Q.1)	An amount of Rs 5000 is put in to three investments at the rate of interest of 6%,7% and 8 % per annum. The total annual income is Rs 358. If the combined income from the first two investments is Rs 70 more than the income from the third, find the amount of each investment by matrix method.
Sol.1)	Let $Rs.x$, $Rs.y$ and $Rs.z$ be the investments from given conditions: $x + y + z = 5000$ ....(1) $\frac{6}{100}\times x+\frac{7}{100}\times y+\frac{8}{100}\times z=358$ (or) $6x + 7y + 8z = 35800$ ......(2) and $\frac{6x}{100}+\frac{7y}{100}=\frac{8z}{100}+70$ {combined income from first two is 70 more than 3rd} (or) $6x + 7y - 8z = 7000$ ......(3) ∴ the equations are $x + y + z = 5000$ $6x + 7y + 8z = 35800$ $6x + 7y - 8z = 7000$ → Now solve by yourself using $x = A^{-1}B$ $x = Rs\,1000$; $y = Rs\,2200$; $z = Rs\,1800$ ans.
Q.2)	Two institutions decided to award their employees for the three values of resourcefulness, competence and determination in the form of prizes at the rate Rs x , Rs y , Rs ,z respectively per person. The first institution decided to award respectively 4 , 3 and 2employees with a total prize money of Rs.37000 and the second institution decided to award respectively 5 , 3 and 4 employees with a total prize money of Rs.47000. If all the three prizes per person together amount to Rs.12000 then by matrix method. Find the value of x , y and z.
Sol.2)	Here Rs ., $Rs.y$ and $Rs.z$ are the award money for resourcefulness , competence and determination respectively from above data/condition , the equation are $4x + 3y + 2z = 37000$ $5x + 3y + 4z = 47000$ $x + y + z = 12000$ (Do yourself using $X = A^{-1}B$) $Rs.4000$, $Rs.5000$, $Rs.3000$ ans.
Q.3)	Two school's P and Q decided to award prizes for (1) academic (2) sports (3) all-rounder achievements. School P awarded Rs 12000 to 3, 1, 1 students while Q awarded Rs 7,000 to 1 , 0 , 2 students in the above categories. All the three prizes amount to Rs 6000. Find matrix representation of the above situation form equations and solve them by matrix method to find value of each prize. Do you agree that prizes should be given for honestly and good character also? Give reasons.
Sol.3)	Let $Rs.x, Rs.y$, and Rs z are the awarded money for academic, sports and all rounder achievement respectively the matrix form is $\begin{bmatrix}3&1&1\\1&0&2\\1&1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}12000\\7000\\6000\end{bmatrix}$ (or) A X = B ⇒ $X = A^{-1}B$ Equations are $3x + y + z = 12000$ $x + 0y + 2z = 7000$

	and $x + y + z = 600$ $Rs. 3000, Rs\ 1000\ and\ Rs\ 2000$ ans.

Properties of Determinates & Adjoint

Q.4) Show that A = $\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$ satisfies the equation $f(x) = x^2 - 6x + 17 = 0$. Hence find A^{-1}.

Sol.4) We have A = $\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$

$A^2 = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -5 & -18 \\ 18 & 7 \end{bmatrix}$

given $f(x) = x^2 - 6x + 17$

$\Rightarrow f(A) = A^2 - 6A + 17I$

$= \begin{bmatrix} -5 & -18 \\ 18 & 7 \end{bmatrix} - \begin{bmatrix} 12 & -18 \\ 18 & 24 \end{bmatrix} + \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}$

$A^2 - 6A + 17I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Clearly A satisfies the equation $x^2 - 6x + 17 = 0$

Now we have, $A^2 - 6A + 17I = 0$

Pre-multiply by A^{-1}

$\Rightarrow A^{-1}A^2 - 6A^{-1}A + 17A^{-1}I = A^{-1}0$

$\Rightarrow A^{-1}A . A - 6\,I + 17A^{-1} = 0$

$\Rightarrow IA - 6\,I + 17A^{-1} = 0$

$\Rightarrow A - 6\,I + 17A^{-1} = 0$

$\Rightarrow 17A^{-1} = 6\,I - A$

$\Rightarrow 17A^{-1} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$ ans.

Q.5) If A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ show that $A^3 - 6A^2 + 5A + 11\,I = 0$ and hence find A^{-1}.

Sol.5) A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$A^2 = A . A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$

$A^3 = A^2 . A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$

Now $A^3 - 6A^2 + 5A + 11\,I$

$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$ (proved)

(ii) we have $A^3 - 6A^2 + 5A + 11\,I = 0$

Pre-multiply by A^{-1}

$\Rightarrow A^{-1}A^3 - 6A^{-1}A^2 + 5AA^{-1} + 11A^{-1}I = A^{-1}0$

$\Rightarrow A^{-1}A . A^2 - 6A^{-1}A\,A + 5\,I + 11A^{-1}I = 0$

$\Rightarrow I\,A^2 - 6\,I\,A + 5\,I + 11A^{-1} = 0$

$\Rightarrow A^2 - 6\,A + 5\,I + 11A^{-1} = 0$

$\Rightarrow 11A^{-1} = 6A - A^2 - 5\,I$

$$= 6\begin{bmatrix}1 & 1 & 1\\1 & 2 & -3\\2 & -1 & 3\end{bmatrix} - \begin{bmatrix}4 & 2 & 1\\-3 & 8 & -14\\7 & -3 & 14\end{bmatrix} - \begin{bmatrix}5 & 0 & 0\\0 & 5 & 0\\0 & 0 & 5\end{bmatrix}$$

$$\Rightarrow\ 11A^{-1} = \begin{bmatrix}-3 & 4 & 5\\9 & -1 & -4\\5 & -3 & -1\end{bmatrix}$$

$$\Rightarrow\ A^{-1} = \frac{1}{11}\begin{bmatrix}-3 & 4 & 5\\9 & -1 & -4\\5 & -3 & -1\end{bmatrix}$$ ans.

Q.6) If $B\begin{bmatrix}1 & -2\\1 & 4\end{bmatrix} = \begin{bmatrix}6 & 0\\0 & 6\end{bmatrix}$. Find matrix B using inverse concept.

Sol.6) Let A = $\begin{bmatrix}1 & -2\\1 & 4\end{bmatrix}$ and C = $\begin{bmatrix}6 & 0\\0 & 6\end{bmatrix}$

then we have, ABA = C

post multiply by A^{-1}

$\Rightarrow$ $BAA^{-1} = CA^{-1}$

$\Rightarrow$ $B I = CA^{-1}$

$\Rightarrow$ $B = CA^{-1}$

$|A| = 4 + 2 = 6$

$(AdjA) = \begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}$

$A^{-1} = \frac{1}{6}\begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}$

Now $B = CA^{-1}$

$B = \frac{1}{6}\begin{bmatrix}6 & 0\\0 & 6\end{bmatrix}\begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}$

$B = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}4 & 2\\1 & 1\end{bmatrix} = \begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}$ ans.

Q.7) Find matrix A if $\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix} A \begin{bmatrix}3 & 2\\5 & -3\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

Sol.7) Let $B = \begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}, C = \begin{bmatrix}3 & 2\\5 & -3\end{bmatrix}$ and $D = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

then we have B A C = D

pre-multiply by B^{-1} and post multiply by C^{-1}

$\Rightarrow$ $B B^{-1} A C C^{-1} = B^{-1} DC^{-1}$

$\Rightarrow$ $I A I = B^{-1} DC^{-1}$

$\Rightarrow$ $A = B^{-1} DC^{-1}$

$B^{-1} = \begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix}$

and $C^{-1} = \begin{bmatrix}3 & 2\\5 & 3\end{bmatrix}$ } (find yourself)

$A = \begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix}$

$A = \begin{bmatrix}3 & -1\\-3 & 2\end{bmatrix}\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$

$\therefore$ $A = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$ Ans...

Q.8) If $A = \begin{bmatrix}3 & 2\\7 & 5\end{bmatrix}$ and $B = \begin{bmatrix}6 & 7\\8 & 9\end{bmatrix}$ verify that $(AB)^{-1} = B^{-1} A^{-1}$.

Sol.8) $A = \begin{bmatrix}3 & 2\\7 & 5\end{bmatrix}$ & $B = \begin{bmatrix}6 & 7\\8 & 9\end{bmatrix}$

$|A| = 15 - 14 = 1$ $|B| = 54 - 56 = -2$

$\text{Adj } A = \begin{bmatrix}5 & -2\\-7 & 3\end{bmatrix}$ $\text{Adj } B = \begin{bmatrix}9 & -7\\-8 & 6\end{bmatrix}$

$A^{-1} = \frac{1}{1}\begin{bmatrix}5 & -2\\-7 & 3\end{bmatrix}$ $B^{-1} = -\frac{1}{2}\begin{bmatrix}9 & -7\\-8 & 6\end{bmatrix}$

Now $AB = \begin{bmatrix}3 & 2\\7 & 5\end{bmatrix}\begin{bmatrix}6 & 7\\8 & 9\end{bmatrix} = \begin{bmatrix}34 & 39\\82 & 94\end{bmatrix}$

$|AB| = 34 \times 94 - 82 \times 39 = -2$

$$\text{Adj}(AB) = \begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$$

$$(AB)^{-1} = -\frac{1}{2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$$

Taking RHS = $B^{-1} A^{-1}$

$$= -\frac{1}{2}\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 94 & -39 \\ -84 & 34 \end{bmatrix} = (AB)^{-1}$$

Hence $(AB)^{-1} = B^{-1} A^{-1}$ verified ans.

Q.9) If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ show that $A' A^{-1} = \begin{bmatrix} \cos(2x) & -\sin(2x) \\ \sin(2x) & \cos(2x) \end{bmatrix}$

Sol. $A' = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$|A| = 1 + \tan^2 x$

$A^{-1} = \frac{1}{|A|} . \text{Adj } A = \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

Taking LHS $A' A^{-1}$

$$= \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \frac{1}{1+\tan^2 x}\begin{bmatrix} 1-\tan^2 x & -2\tan x \\ 2\tan x & 1-\tan^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-\tan^2 x}{1+\tan^2 x} & \frac{-2\tan x}{1+\tan^2 x} \\ \frac{2\tan x}{1+\tan^2 x} & \frac{1-\tan^2 x}{1+\tan^2 x} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2x) & -\sin(2x) \\ \sin(2x) & \cos(2x) \end{bmatrix} \quad = \text{RHS} \quad \text{ans.}$$

Q.10) (a) Find area of ΔABC whose vertices are A(3, 8), B(–4, 2), C(5, –1).

(b) Find equation of line joining A(3, 5) & B(4, 2) using determinants.

(c) Find value of λ so that points (1, –5), (–4, 7) and (λ, 7) are collinear.

Sol.10) (a) A(3, 8) , B(–4, 2) , C(5, –1)

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|3(2+1) - 8(-4-5) + 1(4-10)|$$

$$= \frac{1}{2}|9 + 72 - 6| = \frac{75}{2} \text{ square units}$$

(b) equation of Ab is given by

$$\begin{vmatrix} x & y & 1 \\ 3 & 5 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow x(5-2) - y(3-4) + 1(6-20) = 0$

$\Rightarrow 3x + y - 14 = 0$ ans.

(c) since (1, –5), (–4, 5) and (λ, 7) are collinear

Area of Δ = 0

$$\therefore \frac{1}{2}\begin{vmatrix} 1 & -5 & 1 \\ -4 & 5 & 1 \\ \lambda & 7 & 1 \end{vmatrix} = 0$$

$\Rightarrow 1(5-7) + 5(-4-\lambda) + 1(-28-5\lambda) = 0$

$\Rightarrow -2 - 20 - 5\lambda - 28 - 5\lambda = 0$

$\Rightarrow -10\lambda = -50$

$\lambda = -5$ ans.