Class 12^{th}
Relations & Functions

Q.1) $*:P(x) \times P(x) \to P(x)$ defined by $A * B = (A - B) \cup (B - A)$ for all $A, B \in P(x)$
Show that φ in the identity element and all the elements of $P(x)$ are invertible with $A^{-1} = A$.

Sol.1) We have,
$A * B = (A - B) \cup (B - A)$
(1) To show φ is the identity elements , we have to show
$A * \varphi = A$ and $\varphi * A = A$

consider, $A * \varphi$
$= (A - \varphi) \cup (\varphi - A)$
$= A \cup \varphi$
$= A$

consider, $\varphi * A$
$= (\varphi - A) \cup (A - \not\subset)$
$= \varphi \cup A$
$= A$

clearly $\not\subset$ is the identity element
(2) $A * B = E$
$\Rightarrow (A - B) \cup (B - A) = \varphi$
this is possible only when $B = A$
since, $(A - A) \cup (A - A) = \varphi \cup \varphi = \varphi = E$
$\therefore$ all element of $P(x)$ are invertible with $A = A$ i.e $B = A$ ans.

Q.2) Consider the binary operation * $:R \times R \to R$ and $o: R \times R \to R$ defined by $a * b = |a - b|$ and $aob = a$
(.) Show that * is commutative but not Associative
(.) Show that o is associative but not commutative
(.) Show that $a * (b\, 0\, c) = (a * b)\, 0\, (a * c)$
(.) Does 0 distributes over * ?

Sol.2) $a * b = |a - b|$ and $a\, o\, b = a$
(.) consider $a * b = |a - b|$
commutative $a * b = |a - b|$
$b * a = |b - a|$
$= |a - b|$
$= a * b$
$\therefore$ * is commutative on R
Associative $(a * b) * c = |a - b| * c$
$= ||a - b| - c|$
$a * (b * c) = a * |b - c|$
$= |a - |b - c||$
$\neq (a * b) * c$
$e.g$ $(1 * 2) * 3 = |1 - 2| * 3$
$= 1 * 3$
$= |1 - 3| = 2$
$1 * (2 * 3) = 1 * |2 - 3|$
$= 1 * 1$
$= |1 - 1|$
$= 0$
clearly * is not Associate on R
(.) Consider $a\, o\, b = a$
Commutative: $a\, o\, b = a$
$b\, o\, a = b$
$a\, o\, b \Rightarrow b\, o\, a$

$e.g.$ $1\,o\,2 = 1$
$2\,o\,1 = 2$

clearly o is not commutative on R

Associative :

$$(a\,o\,b)\,o\,c = a\,o\,c = a$$
$$a\,o\,(b\,o\,c) = a\,o\,b = a$$

clearly $(a\,o\,b)\,o\,c = a\,o\,(b\,o\,c)$

$\therefore$ o is Associative on R

(.)To prove $a * (b\,o\,c) = (a * b)\,o\,(a * c)$

LHS $a * (b\,o\,c)$
$= a * b$
$= |a - b|$

RHS $(a * b)\ o\,(a * c)$
$= |a - b|o|a - c|$
$= |a - b|$

clearly LHS = RHS

(.) o distributes over when,

$a\,o\,(b * c) = (a\,o\,b) * (a\,o\,c)$

LHS $a\,o\,(b * c)$
$= a\,o\,a\,o\,|b - c|$
$= a$

RHS $(a\,o\,b)\,(a\,o\,c)$
$= a\ \ a$
$= |a - a|$
$= 0$

clearly LHS ≠ RHS

$\therefore$ o does not distributes over ans.

Q.3) Let * be a binary operation on set z (integers) defined by $a * b = 2a + b - 3$. Find
(i) (3 * 4) * 2 (ii) (2 * 3) * 4

Sol.3) We have a * b = 2a + b – 3

(i) (3 * 4) * 2
= (6 + 4 – 3) * 2
= 7 * 2
= 14 + 2 – 3
= 13 ans.

(ii) (2 * 3) * 4
= (4 + 3 – 3) * 4
= 4 * 4
= 8 + 4 – 3
= 9 ans.

Q.4) Let * be a binary operation on set A where A = {1,2,3,4}
(i) write the total number of binary operations
(ii) If a * b = HCF of a & b construct the operation table.

Sol.4) A = {1,2,3,4}

(i) we know that no. of binary operation = n^{n^2}
here $x = 4$

$\therefore$ no. of binary operations = $4^{4^2} = 4^{16}$ ans.

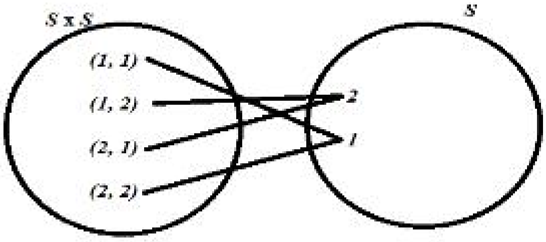
(ii) a * b = HCF of a & b
operation table :

	b				
	*	1	2	3	4
a	1	1	1	1	1
	2	1	2	1	2
	3	1	1	3	1
	4	1	2	1	4

Q.5) Show that the number of binary operations on {1 , 2} having 1 as identity element and having 2 as inverse of 2 is exactly one

Sol.5) (.) We know that a binary operation on set S is a function from $S \times S$ to S.
(.) so a binary operation on set s : {1 , 2} is a function from {(1,1), (1,2), (2,1), (2, 2)} to {1,2}
(.) let * be the required binary operation
(.) If 1 is the identity element and 2 is the inverse of 2 , then
1 * 1 = 1
1 * 2 = 2 $a * e = a \text{ and } e * a = a$
2 * 1 = 2 here $e = 1, a = 1 \& 2$

and 2 * 2 = 1 a * b = e
here a = 2 ; b = 2 & e = 1
(2 is the inverse of 2 given)



Clearly * can be defined in a unique way
∴ Hence no. of required binary operations is 1 ans.

Q.6) Define a binary operation * on the set {0,1,2,3,4,5} as
a * b {a + b if a + b < 6}
{a + b – 6 if a + b ≥ 6}
show that zero is the identity for thus operation and each element a ≠ 0 of the set is invertible with 6 – a being the inverse of a.

Sol.6) Identity element :
Consider a * b = a + b

a * e = a	e * a = a
a + e = a	e + a = a
e = 0 ∈ A	e = 0 ∈ A

∴ 0 is the identity element
Consider , $a * b = a + b - 6$

$a * e = a$	$e * a = a$
$a + e - 6 = a$	$e + a - 6 = a$

$e = 6 \notin A \qquad e = 6 \notin A$

$\therefore$ 0 is the identity element ans.

Inverse

Consider a * b = a + b

$a * b = e$

$a + b = 0$

$b = -a \notin A$

Consider,

$a * b = a + b - 6$

$a * b = e$

$a + b - 6 = 0$

$b = 6 - a \in A \; ; (a \neq 0)$

$\therefore$ $6 - a$ is the inverse of a. ans.

Q.7) Show that zero is the identity element for addition on R (real no's) and 1 is the identity element for multiplication on R but there is no identity element for subtraction on R and division on R – {0}.

Sol.7) (i) * : $R \times R \to R$

a * b = a + b

$a + e = a$	$e * a = a$
$e = 0 \in R$	$e = 0 \in R$

$\therefore$ 0 is the identity element for addition on R

(ii) * : $R \times R \to R$

a * b = ab

$a * e = a$	$e * a = a$
$ae = a$	$ea = a$
$e = 1 \in R$	$e = 1 \in R$

$\therefore$ 1 is the identity element for multiplication on R

(iii) * :$R \times R \to R$

$a * b = a - b$

$a * e = a$	$e * a = a$
$a - e = a$	$e - a = a$
$-e = 0$	$e = 2a$
$e = 0 \in R$	but e can not be in terms of a or variable

$\therefore$ identity element does not exist

(iv) * :$R - \{0\} \times R - \{0\} \to R - \{0\}$

a * b =$\frac{a}{b}$

$a * e = a$	$e * a = a$
$\frac{a}{e} = a$	$\frac{e}{a} = a$
$e = 1 \in R - \{0\}$	$e = a^2$
	but e cannot be a variable

$\therefore$ identity element does not exist. ans.

Topic : Functions

Q.8) Let$f: R \to \left\{\frac{-4}{3}\right\} \to R$defined as $f(x) = \frac{4x}{3x+4}$.

Show that f is invertible and find its inverse.

Sol.8) We have

$$f: R - \left\{\frac{-4}{3}\right\} \to R$$

and $f(x) = \frac{4x}{3x+4}$ .... (1)

ONE-ONE :-

let $x_1, x_2 \in R - \left\{\frac{-4}{3}\right\}$(domain)

and $f(x_1) = f(x_2)$

$\Rightarrow \frac{4x_1}{3x_1+4} = \frac{4x_2}{3x_2+4}$

$\Rightarrow 12x_1x_2 + 16x_1 = 12x_1x_2 + 16x_2$

$\Rightarrow 16x_1 = 16x_2$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one function

ON-TO :-

let $y = f(x)$

$\Rightarrow y = \frac{4x}{3x+4}$

$\Rightarrow 3xy + 4y = 4x$

$\Rightarrow x(3y-4) - 4y$

$\Rightarrow x = \frac{-4y}{3y-4}$

for each$y$R (co-domain), there exists an element x in domain such that

$f(x) = f\left(\frac{-4y}{3y-4}\right)$

$f(x) = \frac{4\left(\frac{-4y}{3y-4}\right)}{3\left(\frac{-4y}{3y-4}\right)+4}$ ....{from eq. (1)}

$= \frac{\frac{-16y}{3y-4}}{\frac{-12y+12y-16}{3y-4}}$

$= \frac{-16y}{-16} = y$

$\therefore f(x) = y$

$\therefore$ f is on-to function

$\therefore$ f is bijective function

$\therefore$ f is invertible function

and $f^{-1} = \frac{-4y}{3y-4}$

and $f^{-1}(x) = \frac{-4x}{3x-4}$ ans.

Q.9) Consider$f: R_+ \to [4, \infty]$given by $f(x) = x^2 + 4$. Show that f is bijective. Also find the inverse.

Sol.9) We have

$$f: R_+ \to [4, \infty]$$

and $f(x) = x^2 + 4$

One-One :

let$x_1, x_2 \in R_+$

and $f(x_1) = f(x_2)$

$\Rightarrow x_1^2 + 4 = x_2^2 + 4$

$\Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = \pm x_2$

but $x_1 \neq x_2$ .....$\{\because x_1, x_2 \in R_+\}$

$\Rightarrow\ x_1 = x_2$

$\therefore$ f is one-one function

ON-TO :

let$y = f(x)$

$\Rightarrow\ y = x^2 + 4$

$\Rightarrow\ x^2 = y - 4$

$\Rightarrow\ x = \sqrt{y-4}$

for each $y \in [4, \infty]$, there exists an element x in R_+ such that

$f(x) = f(\sqrt{y-4})$

$=(\sqrt{y-4})^2 + 4$

$=y - 4 + 4$

$f(x) = y$

$\therefore$ f is on-to function

$\therefore$ f is bijective

$\therefore$ f is invertible

and $f^{-1} = \sqrt{y-4}$

and $f^{-1}(x) = \sqrt{x-4}$ ans.

Q.10) Let f : N → S, where S is the range of$f$. $f(x) = 4x^2 + 12x + 15$. Show f is invertible and find its inverse.

Sol.10) We have,

f : N → S

$f(x) = 4x^2 + 12x + 15$

One-One :-

let x_1 , x_2 ∈ N (domain)

and $f(x_1) = f(x_2)$

$\Rightarrow 4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15$

$\Rightarrow 4x_1^2 - 4x_2^2 + 12x_1 - 12x^2 = 0$

$\Rightarrow 4(x_1^2 - x_2^2) + 12(x_1 - x_2) = 0$

$\Rightarrow 4(x_1 + x_2)(x_1 - x_2) + 12(x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)[4x_1 + 4x_2 + 12] = 0$

$\Rightarrow x_1 - x_2 = 0$ and$4x_1 + 4x_2 + 12 = 0$

$\Rightarrow x_1 = x_2$ but $4x_1 + 4x_2 + 12 \neq 0$ ......$\{\because x_1, x_2 \in N\}$

$\therefore$ f is one-one function

On-To

let $y = f(x)$

$\Rightarrow y = 4x^2 + 12x + 15$

$\Rightarrow 4x^2 + 12x + (15 - y) = 0$ {quadratic equation}

here $a = 4$, $b = 12$ and $c = 15 - y$

by quadratic formula,

$x = \frac{-12 \pm \sqrt{144 - 4(4)(15-y)}}{8}$

$x = \frac{-12 \pm \sqrt{144 - 240 + 16y}}{8}$

$x = \frac{-12 \pm \sqrt{16y - 96}}{8}$

$x = \frac{-12 \pm 4\sqrt{y-6}}{8}$

$x = \frac{-12 \pm \sqrt{144 - 4(4)(15-y)}}{8}$

$x = \frac{-3 \pm \sqrt{y-6}}{2}$

$x = \frac{-3+\sqrt{y-6}}{2}$ but $x \neq \frac{-3-\sqrt{y-6}}{2}$ .....{$\because$ x ∈ N}

for each $y \in S$ (co-domain) , there exists

on element x in N (domain) such that

$$f(x) = f\left(\frac{-3+\sqrt{y-6}}{2}\right)$$

$$= \quad 4\left[\frac{-3+\sqrt{y-6}}{2}\right]^2 + 12\left[\frac{-3+\sqrt{y-6}}{2}\right] + 15$$

$$= \quad 4\left(\frac{9+y-6-6\sqrt{y-6}}{4}\right) + 6\left(-3+\sqrt{y-6}\right) + 15$$

$$= \quad 3 + y - 6\sqrt{y-6} - 18 + 6\sqrt{6-y} + 15$$

$f(x) = y$

$\therefore$ f is on-to function

$\therefore$ f is bijective

$\therefore$ f is invertible

and $f^{-1} = \frac{-3+\sqrt{y-6}}{2}$

and $f^{-1}(x) = \frac{-3\sqrt{x-6}}{2}$ ans.