

Class 12th Relations & Functions

Q.1) Show that the number of equivalence relation in the set $\{1,2,3\}$ containing $(1, 2)$ and $(2, 1)$ is two.

Sol.1) $A = \{1, 2, 3\}$

The maximum possible relation (i.e. universal relation) is

$$R = \{(1,1), (2,2), (3,3), (1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$$

The smallest equivalence relation R_1 containing $(1, 2)$ and $(2, 1)$ is

$$R_1 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$$

we are left with four pairs (from universal relation) i.e. $(2,3), (3,2), (1,3)$ and $(3,1)$

If we add $(2,3)$ to R_1 , then for symmetric by we must add $(3,2)$ and now for transitivity we are forced to add $(1,3)$ and $(3,1)$

Thus the only relation bigger than R_1 is universal relation i.e R

$\therefore$ The no. of equivalence relations containing $(1,2)$ and $(2,1)$ is two. ans.

Q.2) If $R = \{(x, y) : x^2 + y^2 \leq 4 ; x, y \in \mathbb{Z}\}$ is a relation on $\mathbb{Z}$. Write the domain of R .

Sol.2) $R = \{(0,1), (0,-1), (0,2), (0,-2), (1,1), (1,-1), (-1,0), (-1,1), (-1,-1), (2,0), (-2,0)\}$

$\therefore$ Domain of $R = \{0, 1, -1, 2, -2\}$ ans.

(i.e the first domain of each ordered pairs)

Q.3) Let $R = \{(x, y) : |x^2 - y^2| < 1\}$ be a relation on set $A = \{1,2,3,4,5\}$. Write R as a set of ordered pairs.

Sol.3) $A = \{1,2,3,4,5\}$

for $|x^2 - y^2| < 1$: x should be equal to y

$\therefore R = \{(1,1), (2,2), (3,3), (4,4), (5,5)\}$ ans.

Q.4) R is a relation in $\mathbb{Z}$ defined as $(a, b) \in R \Leftrightarrow a^2 + b^2 = 25$. Find the range.

Sol.4) We have, $a^2 + b^2 = 25$ and $a, b \in \mathbb{Z}$

$\therefore R = \{(0,5), (0,-5), (3,4), (3,-4), (-3,4), (-3,-4), (4,3), (4,-3), (-4,3), (-4,-3), (5,0), (-5,0)\}$

$\therefore$ Range = $\{-5, 5, 4, -4, 3, -3, 0\}$

(i.e. second elements of each order pairs) ans.

Topic : Binary Operations

Q.5) $* : R \times R \rightarrow R$

(1) $a * b = a + b$

(2) $a * b = a - b$

(3) $a * b = ab$

Find identity element and inverse in both cases.

Sol.5) (1) $a * b = a + b$

Identity element

$$\begin{array}{l|l} a * e = a & e * a = a \\ \Rightarrow a + e = a & e + a = a \\ \Rightarrow e = 0 \in R & e = 0 \in R \end{array}$$

$\therefore 0$ is the identity element

Inverse

$$a * b = e$$

$$\Rightarrow a + b = 0$$

$$\Rightarrow b = -a \in R \quad \{a \in R \dots -a \text{ also } \in R\}$$

$\therefore -a$ is the inverse of a i.e $a^{-1} = -a$

$$(2) a * b = a - b$$

Identity element

$$a * e = a \quad e * a = a$$

$$a - e = a \quad e - a = a$$

$$-e = 0 \quad e = 2a$$

$$e = 0 \in R \quad \left| \quad \begin{array}{l} \text{but 'e' cannot be a variable as when a changes e also change but should be same} \\ \text{for all } a \in R \end{array} \right.$$

$\therefore$ Identity element does not exist

hence inverse does not exist

$$(3) a * b = ab$$

Identity element

$$a * e = a \quad e * a = a$$

$$ae = a \quad ea = a$$

$$e = 1 \in R \quad e = 1 \in R$$

$\therefore$ 1 is the identity element

Inverse

$$a * b = e$$

$$ab = 1$$

$$b = \frac{1}{a} \in R; a \neq 0$$

$\therefore$ all elements of R are invertible except '0' and $a^{-1} = \frac{1}{a}; a \neq 0$ ans.

Q.6) Let $*$ be a binary operation on R (real no's)

$$*: R \times R \rightarrow R$$

$$a * b = a + b + ab$$

(.) Check whether $*$ is binary operation or not

(.) Check the commutativity and Associativity

(.) Find identity element and inverse.

Sol.6) We have,

$$a * b = a + b + ab$$

since $*$ carries each pair (a, b) in $R \times R$ to a unique element $a + b + ab$ in R

$\therefore$ $*$ is a binary operation on R

Alternate : since $(a, b) \in R \times R$ and addition and multiplication of real no.s is also a real no.

$$a + b + ab \in R$$

$\therefore$ $*$ is a binary operation on R

Commutative :

let $a, b \in R$

$$a * b = a + b + ab$$

$$b * a = b + a + ba$$

$$= a + b + ab \quad \dots \{ \because \text{addition and multiplication are itself commutative} \}$$

$$= a * b$$

$\therefore b * a = a * b$ for all $a, b \in R$

$\therefore$ $*$ is commutative on R

Associative:

let $a, b, c \in R$

$$\begin{aligned}(a * b) * c &= (a + b + ab) * c \\&= a + b + ab + c(a + b + ab)c \\&= a + b + ab + c + ac + bc + abc \\&= a + b + c + ab + bc + ac + abc\end{aligned}$$

$$\begin{aligned}\text{Now } a * (b * c) &= a * (b + c + bc) \\&= a + b + c + bc + a(b + c + bc) \\&= a + b + c + bc + ab + ac + abc \\&= a + b + c + ab + bc + ac + abc\end{aligned}$$

clearly $(a * b) * c = a * (b * c)$ for all $a, b, c \in R$

$\therefore *$ is Associative on R .

Identity element :

let e be the identity element in R

$$\begin{array}{l|l} a * e = a \text{ and } e * a = a \text{ for all } a \in R & \\ \Rightarrow a + e + ae = a & e + a + ea = a \\ \Rightarrow e(1 + a) = 0 & e(1 + a) = 0 \\ \Rightarrow e = 0 \in R & e = 0 \in R \end{array}$$

$\therefore 0$ is the identity element

Inverse :

$$a * b = e$$

$$a + b + ab = 0$$

$$b(1 + a) = -a$$

$$b = \frac{-a}{1+a} \in R \quad \{\text{except } a = -1\}$$

$\therefore -1$ is not the invertible element

(.) all elements of R are invertible except -1

(.) and $a^{-1} = \frac{-a}{1+a}; a \neq -1$

(.) e.g. inverse of $2 = \frac{-2}{1+2} = \frac{-2}{3}$ ans.

Q.7) Let $*$ be a binary operation on Z (integers) $a * b = a + ab$
Check the commutative, Associativity, identify element and inverse (if it exists).

Sol.7) We have

$$a * b = a + ab \text{ where } a, b \in Z$$

Commutative :

let $a, b \in Z$, then

$$a * b = a + ab$$

$$b * a = b + ba$$

$$= b + ab$$

$$b * a \neq a * b$$

$$\text{e.g. } (1 * 2) = 1 + (1)(2) = 1 + 2 = 3$$

$$(2 * 1) = 2 + 2(1) = 2 + 2 = 4$$

clearly $1 * 2 \neq 2 * 1$

$\therefore *$ is not commutative on Z

Associative :

let $a, b, c \in Z$ then

$$(a * b) * c = (a + ab) * c$$

$$\begin{aligned}
&= a + ab + (a + ab)c \\
&= a + ab + ac + abc \\
a * (b * c) &= a * (b + bc) \\
&= a + a(b + bc) \\
&= a + ab + abc \\
&\neq (a * b) * c \\
\text{e.g. } (1 * 2) * 3 &= (1 + 2) * 3 \\
&= 3 * 3 \\
&= 3 + 3(3) = 12 \\
1 * (2 * 3) &= 1 * (2 + 6) = 1 * 8 \\
&= 1 + 1(8) = 9
\end{aligned}$$

Clearly * is not Associative on Z

Identity element:

let e be the identity element $\forall z$, then

$$\begin{array}{l|l}
a * e = a & e * a = a \\
a + ae = a & e + ea = a \\
ae = 0 & e(1 + a) = a \\
e = 0 \in \mathbb{Z} & e = \frac{a}{1+a}
\end{array}$$

as a changes e also changes, but e must be constant for all value of a

$\therefore$ identity element does not exist and hence inverse not possible. ans.

Q.8) Let * be a binary operation on N given by $a * b = \text{LCM of } a \text{ \& } b$

- (1) Find $5 * 7, 20 * 16$
- (2) Is * commutative ?
- (3) If * Associative ?
- (4) Find the identity element in N.
- (5) which elements of N are invertible ?

Sol.8) We have

$$a * b = \text{LCM of } a \text{ \& } b ; a, b \in N$$

$$\begin{aligned}
(1) \quad 5 * 7 &= \text{LCM of 5 and 7} = 35 \\
20 * 16 &= \text{LCM of 20 and 16} = 80
\end{aligned}$$

(2) Commutative :

$$\text{let } a, b \in N$$

$$a * b = \text{LCM of } a \text{ and } b$$

$$b * a = \text{LCM of } b \text{ and } a$$

$$= \text{LCM of } a \text{ and } b$$

$$= a * b$$

$$\therefore b * a = a * b \text{ for all } a, b \in N$$

$$\therefore * \text{ is commutative on } N$$

(3) Associative :

$$\text{let } a, b, c \in N$$

$$\begin{aligned}
(a * b) * c &= (\text{LCM of } a \text{ and } b) * c \\
&= \text{LCM of } [(\text{LCM of } a \text{ and } b) \text{ and } c] \\
&= \text{LCM of } a, b \text{ and } c
\end{aligned}$$

$$\begin{aligned}
a * (b * c) &= a * (\text{LCM of } b \text{ and } c) \\
&= \text{LCM of } [a \text{ and } (\text{LCM of } b \text{ and } c)] \\
&= \text{LCM of } a, b \text{ and } c
\end{aligned}$$

clearly $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$

$\therefore *$ is Associative on N

(4) Identity element

let e be an identity element $\in N$

$$a * e = a$$

$$\Rightarrow \text{LCM of } a \text{ and } e = a \quad e * a = a$$

$$\Rightarrow \text{LCM of } a \text{ \& } 1 = a \quad \text{LCM of } e \text{ and } a = a$$

$$\Rightarrow e = 1 \in N \quad \text{LCM of } 1 \text{ and } a = a$$

$$\Rightarrow e = 1 \in N$$

$\therefore 1$ is the identity element for all $a \in N$

Inverse

$$a * b = e$$

$$\Rightarrow \text{LCM of } a \text{ and } b = 1$$

this is possible only when $a = 1$ & $b \neq 1$

$\therefore 1$ is the only invertible element and 1 is its inverse ans.

- Q.9) Let R be a set of real no.s and $A = R \times R$ is a binary operation on A given by $(a, b) * (c, d) = (ac, bd)$ for all $(a, b), (c, d) \in A$
- (1) Show that $*$ is Commutative
 - (2) Show that $*$ is Associative
 - (3) Find the identity element
 - (4) Find invertible elements and their inverse.

Sol.9) We have,

$$(a, b) * (c, d) = (ac, bd)$$

Commutative :

let $(a, b) \& (c, d) \in A$, then

$$(a, b) * (c, d) = (ac, bd)$$

$$(c, d) * (a, b) = (ca, db)$$

$$= (ac, bd)$$

$$= (a, b) * (c, d)$$

$\therefore *$ is commutative on A

Associative :

let $(a, b), (c, d) \& (e, f) \in A$

$$[(a, b) * (c, d)] * (e, f)$$

$$= (ac, bd) * (e, f)$$

$$= (ace, bdf)$$

$$(a, b) * [(c, d) * (e, f)]$$

$$= (a, b) * (ce, df)$$

$$= (ace, bdf)$$

$$\text{clearly } ((a, b) * (c, d)) * (e, f) = (a, b) * ((c, d) * (e, f))$$

$\therefore *$ is Associative on A

Identity element

let (x, y) be the identity element

$$(a, b) * (x, y) = (a, b)$$

$$\Rightarrow (ax, by) = (a, b)$$

$$\Rightarrow ax = a \& by = b$$

$$\Rightarrow x = 1 \text{ and } y = 1$$

$$(x, y) * (a, b) = (a, b)$$

$$\Rightarrow (xa, yb) = (a, b)$$

$$\Rightarrow xa = a \& yb = b$$

$$\Rightarrow x = 1 \& y = 1$$

$\therefore (1, 1)$ is the identity element

Inverse

$$(a, b) * (c, d) = (x, y)$$

$$\Rightarrow (ac, bd) = (1, 1)$$

$$\Rightarrow ac = 1 \text{ and } bd = 1$$

$$\Rightarrow c = \frac{1}{a} \text{ and } d = \frac{1}{b}$$

$$\therefore (c, d) = \left(\frac{1}{a}, \frac{1}{b}\right) \in R \text{ except } (0, b) = (0, 0)$$

(.) all elements of A are invertible except $(0, 0)$

(.) inverse of $(0, b)$ is $\left(\frac{1}{a}, \frac{1}{b}\right)$; $(a, b) \neq (0, 0)$ ans.

Q.10) X is a non-empty set and $*$ is a binary operation $*$: $P(X) \rightarrow P(X)$ given by $A \times B = A \cap B$

(.) Show $*$ is Commutative

(.) Show $*$ is Associative

(.) Find the Identity element

(.) Find the Invertible elements in $P(X)$ and their inverse.

Sol.10) We have, $A \times B = A \cap B$

Commutative :

$$\text{let } A, B \in P(X)$$

$$A * B = A \cap B$$

$$B * A = B \cap A$$

$$= A \cap B \quad \dots\{\because \cap \text{ is its of commutative}\}$$

clearly $A * B = B * A$ for all $A, B \in P(X)$

$\therefore *$ is commutative on $P(X)$

Associative :

$$\text{let } A, B, C \in P(X)$$

$$(A * B) * C = (A \cap B) * C$$

$$= (A \cap B) \cap C$$

$$A * (B * C) = A * (B \cap C)$$

$$= A \cap (B \cap C) \quad \{\cap \text{ is itself Associative}\}$$

clearly $(A * B) * C = A * (B * C)$ for all $A, B, C \in P(X)$

$\therefore *$ is Associative on $P(X)$

Identity element

let E is an identity element then

$$A * E = A \quad E * A = A$$

$$\Rightarrow A \cap E = A \quad \Rightarrow E \cap A = A$$

$$\Rightarrow E = X \in (P(X)) \quad \Rightarrow E = X \in P(X)$$

{reason : X is the largest subset in $P(X)$ }

$\therefore X$ is the identity element

Inverse :

$$A * B = E$$

$$\Rightarrow A \cap B = X$$

this is possible only when $A = B = X$

since $X \cap X = X$

$\therefore X$ is only the invertible element in $P(X)$ and X is its inverse

ans.

