
Exercise – 6.1

1. Which of the following expressions are polynomials in one variable and which are not?

State reasons for your answer:

(i) $3x^2 - 4x + 15$

(ii) $y^2 + 2\sqrt{3}$

(iii) $3\sqrt{x} + \sqrt{2}x$

(iv) $x - \frac{4}{x}$

(v) $x^{12} + y^3 + t^{50}$

Sol:

(i) $3x^2 - 4x + 15$ is a polynomial of one variable x .

(ii) $y^2 + 2\sqrt{3}$ is a polynomial of one variable y .

(iii) $3\sqrt{x} + \sqrt{2}x$ is not a polynomial as the exponents of $3\sqrt{x}$ is not a positive integer.

(iv) $x - \frac{4}{x}$ is not a polynomial as the exponent of $\frac{-4}{x}$ is not a positive integer.

(v) $x^{12} + y^3 + t^{50}$ is a polynomial of three variables x, y, t .

2. Write the coefficient of x^2 in each of the following:

(i) $17 - 2x + 7x^2$

(ii) $9 - 12x + x^3$

(iii) $\frac{\pi}{6}x^2 - 3x + 4$

(iv) $\sqrt{3}x - 7$

Sol:

Coefficient of x^2 in

(i) $17 - 2x + 7x^2$ is 7

(ii) $9 - 12x + x^3$ is 0

(iii) $\frac{\pi}{6}x^2 - 3x + 4$ is $\frac{\pi}{6}$

(iv) $\sqrt{3}x - 7$ is 0

3. Write the degrees of each of the following polynomials:

(i) $7x^3 + 4x^2 - 3x + 12$

(ii) $12 - x + 2x^3$

(iii) $5y - \sqrt{2}$

(iv) $7 = 7 \times x^0$

(v) 0

Sol:

Degree of polynomial

(i) $7x^3 + 4x^2 - 3x + 12$ is 3

(ii) $12 - x + 2x^3$ is 3

(iii) $5y - \sqrt{2}$ is 1

(iv) $7 = 7 \times x^0$ is 0

(v) 0 is un defined.

4. Classify the following polynomials as linear, quadratic, cubic and biquadratic polynomials:

(i) $x + x^2 + 4$

(ii) $3x - 2$

(iii) $2x + x^2$

(iv) $3y$

(v) $t^2 + 1$

(vi) $7t^4 + 4t^3 + 3t - 2$

Sol:

Given polynomial

(i) $x + x^2 + 4$ is quadratic as degree of polynomial is 2.

(ii) $3x - 2$ is linear as degree of polynomial is 1.

(iii) $2x + x^2$ is quadratic as degree of polynomial is 2.

(iv) $3y$ is linear as degree of polynomial is 1.

(v) $t^2 + 1$ is quadratic as degree of polynomial is 2.

(vi) $7t^4 + 4t^3 + 3t - 2$ is bi-quadratic as degree of polynomial is 4.

5. Classify the following polynomials as polynomials in one-variable, two variables etc:

(i) $x^2 - xy + 7y^2$

(ii) $x^2 - 2tx + 7t^2 - x + t$

(iii) $t^3 - 3t^2 + 4t - 5$

(iv) $xy + yz + zx$

Sol:

- (i) $x^2 - xy + 7y^2$ is a polynomial in two variables x, y .
- (ii) $x^2 - 2tx + 7t^2 - x + t$ is a polynomial in 2 variables x, t .
- (iii) $t^3 - 3t^2 + 4t - 5$ is a polynomial in 1 variables t .
- (iv) $xy + yz + zx$ is a polynomial in 3 variables x, y, z .

6. Identify polynomials in the following:

- (i) $f(x) = 4x^3 - x^2 - 3x + 7$
- (ii) $g(x) = 2x^3 - 3x^2 + \sqrt{x} - 1$
- (iii) $p(x) = \frac{2}{3}x^2 - \frac{7}{4}x + 9$.
- (iv) $q(x) = 2x^2 - 3x + \frac{4}{x} + 2$
- (v) $h(x) = x^4 - x^{\frac{3}{2}} + x - 1$
- (vi) $f(x) = 2 + \frac{3}{x} + 4x$

Sol:

- (i) $f(x) = 4x^3 - x^2 - 3x + 7$ is a polynomial
 - (ii) $g(x) = 2x^3 - 3x^2 + \sqrt{x} - 1$ is not a polynomial as exponent of x in \sqrt{x} is not a positive integer.
 - (iii) $p(x) = \frac{2}{3}x^2 - \frac{7}{4}x + 9$ is a polynomial as all the exponents are positive integers.
 - (iv) $q(x) = 2x^2 - 3x + \frac{4}{x} + 2$ is not a polynomial as exponent of x in $\frac{4}{x}$ is not a positive integer.
 - (v) $h(x) = x^4 - x^{\frac{3}{2}} + x - 1$ is not a polynomial as exponent of x in $-x^{\frac{3}{2}}$ is not a positive integer.
 - (vi) $f(x) = 2 + \frac{3}{x} + 4x$ is not a polynomial as exponent of x in $\frac{3}{x}$ is not a positive integer.
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7. Identify constant, linear, quadratic and cubic polynomials from the following polynomials:

(i) $f(x) = 0$

(ii) $g(x) = 2x^3 - 7x + 4$

(iii) $h(x) = -3x + \frac{1}{2}$

(iv) $p(x) = 2x^2 - x + 4$

(v) $q(x) = 4x + 3$

(vi) $r(x) = 3x^2 + 4x^2 + 5x - 7$

Sol:

Given polynomial

(i) $f(x) = 0$ is a constant polynomial as 0 is a constant

(ii) $g(x) = 2x^3 - 7x + 4$ is a cubic polynomial as degree of the polynomial is 3.

(iii) $h(x) = -3x + \frac{1}{2}$ is a linear polynomial as degree of the polynomial is 1.

(iv) $p(x) = 2x^2 - x + 4$ is a quadratic as the degree of the polynomial is 2.

(v) $q(x) = 4x + 3$ is a linear polynomial as the degree of the polynomial is 1.

(vi) $r(x) = 3x^2 + 4x^2 + 5x - 7$ is a cubic polynomial as the degree is 3.

8. Give one example each of a binomial of degree 35, and of a monomial of degree 100.

Sol:

Example of a binomial with degree 35 is $7x^{35} - 5$

Example of a monomial with degree 100 is $2t^{100}$

Exercise – 6.2

1. If $f(x) = 2x^3 - 13x^2 + 17x + 12$, find (i) $f(2)$ (ii) $f(-3)$ (iii) $f(0)$

Sol:

We have

$$f(x) = 2x^3 - 13x^2 + 17x + 12$$

$$\begin{aligned} \text{(i)} \quad f(2) &= 2 \times (2)^3 - 13 \times (2)^2 + 17 \times (2) + 12 \\ &= (2 \times 8) - (13 \times 4) + (17 \times 2) + 12 \\ &= 16 - 52 + 34 + 12 = 10 \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad f(-3) &= 2 \times (-3)^3 - 13 \times (-3)^2 + 17 \times (-3) + 12 \\ &= 2 \times (-27) - 13 \times (9) + 17 \times (-3) + 12 \\ &= -54 - 117 - 51 + 12 = -210 \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad f(0) &= 2 \times (0)^3 - 13 \times (0)^2 + 17 \times (0) + 12 \\ &= 0 - 0 + 0 + 12 = 12 \end{aligned}$$

2. Verify whether the indicated numbers are zeroes of the polynomials corresponding to them in the following cases:

$$\text{(i)} \quad f(x) = 3x + 1, x = -\frac{1}{3}$$

$$\text{(ii)} \quad f(x) = x^2 - 1, x = 1, -1$$

$$\text{(iii)} \quad g(x) = 3x^2 - 2, x = \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}$$

$$\text{(iv)} \quad p(x) = x^3 - 6x^2 + 11x - 6, x = 1, 2, 3$$

$$\text{(v)} \quad f(x) = 5x - \pi, x = \frac{4}{5}$$

$$\text{(vi)} \quad f(x) = x^2 \text{ and } x = 0$$

$$\text{(vii)} \quad f(x) = lx + m, x = -\frac{m}{l}$$

$$\text{(viii)} \quad f(x) = 2x + 1, x = \frac{1}{2}$$

Sol:

$$\text{(i)} \quad f(x) = 3x + 1, x = -\frac{1}{3}$$

We have

$$f(x) = 3x + 1$$

$$\text{Put } x = -\frac{1}{3} \Rightarrow f\left(-\frac{1}{3}\right) = 3 \times \left(-\frac{1}{3}\right) + 1 = -1 + 1 = 0$$

$$\therefore x = -\frac{1}{3} \text{ is a root of } f(x) = 3x + 1$$

$$(ii) \quad f(x) = x^2 - 1, x = 1, -1$$

$$\text{We have } f(x), x^2 - 1$$

$$\text{Put } x = 1 \text{ and } x = -1$$

$$\Rightarrow f(1) = (1)^2 - 1 \text{ and } f(-1) = (-1)^2 - 1$$

$$= 1 - 1 = 0 \qquad = 1 - 1 = 0$$

$$\therefore x = 1, -1 \text{ are the roots of } f(x) = x^2 - 1$$

$$(iii) \quad g(x) = 3x^2 - 2, x = \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}$$

$$\text{We have } g(x) = 3x^2 - 2$$

$$\text{Put } x = \frac{2}{\sqrt{3}} \text{ and } x = -\frac{2}{\sqrt{3}}$$

$$\Rightarrow g\left(\frac{2}{\sqrt{3}}\right) = 3\left(\frac{2}{\sqrt{3}}\right)^2 - 2 \text{ and } g\left(-\frac{2}{\sqrt{3}}\right) = 3\left(-\frac{2}{\sqrt{3}}\right)^2 - 2$$

$$= 3 \times \frac{4}{3} - 2 \qquad = 3\left(\frac{4}{3}\right) - 2$$

$$= 4 - 2 = 2 \neq 0 \qquad = 4 - 2 = 2 \neq 0$$

$$\therefore x = \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}} \text{ are not roots of } g(x) = 3x^2 - 2$$

$$(iv) \quad p(x) = x^3 - 6x^2 + 11x - 6, x = 1, 2, 3$$

$$\text{Put } x = 1 \Rightarrow p(1) = (1)^3 - 6(1)^2 + 11(1) - 6 = 1 - 6 + 11 - 6 = 0$$

$$x = 2 \Rightarrow p(2) = (2)^3 - 6(2)^2 + 11(2) - 6 = 8 - 24 + 22 - 6 = 0$$

$$x = 3 \Rightarrow p(3) = 3^3 - 6(3^2) + 11(3) - 6 = 27 - 54 + 33 - 6 = 0$$

$$\therefore x = 1, 2, 3 \text{ are roots of } p(x) = x^3 - 6x^2 + 11x - 6$$

$$(v) \quad \text{We know } f(x) = 5x - \pi, x = \frac{4}{5}$$

$$\text{Put } x = \frac{4}{5} \Rightarrow f\left(\frac{4}{5}\right) = 5 \times \frac{4}{5} - \pi = 4 - \pi \neq 0$$

$$\therefore x = \frac{4}{5} \text{ is not a root of } f(x) = 5x - \pi$$

(vi) We have $f(x) = x^2$ and $x = 0$

$$\text{Put } x = 0 \Rightarrow f(0) = (0)^2 = 0$$

$$\therefore x = 0 \text{ is a root of } f(x) = x^2$$

(vii) $f(x) = lx + m$ and $x = -\frac{m}{l}$

$$\text{Put } x = -\frac{m}{l} \Rightarrow f\left(-\frac{m}{l}\right) = l \times \left(-\frac{m}{l}\right) + m = -m + m = 0$$

$$\therefore x = -\frac{m}{l} \text{ is a root of } f(x) = lx + m$$

(viii) $f(x) = 2x + 1, x = \frac{1}{2}$

$$\text{Put } x = \frac{1}{2} \Rightarrow f\left(\frac{1}{2}\right) = 2 \times \left(\frac{1}{2}\right) + 1 = 1 + 1 = 2 \neq 0$$

$$\therefore x = \frac{1}{2} \text{ is not a root of } f(x) = 2x + 1$$

3. If $x = 2$ is a root of the polynomial $f(x) = 2x^2 - 3x + 7a$, find the value of a .

Sol:

$$\text{We have } f(x) = 2x^2 - 3x + 7a$$

$$\text{Put } x = 2 \Rightarrow f(2) = 2(2)^2 - 3(2) + 7a$$

$$= 2 \times 4 - 3 \times 2 + 7a = 8 - 6 + 7a$$

$$= 2 + 7a$$

$$\text{Given } x = 2 \text{ is a root of } f(x) = 2x^2 - 3x + 7a$$

$$\Rightarrow f(2) = 0$$

$$\therefore 2 + 7a = 0$$

$$\Rightarrow 7a = -2 \Rightarrow \boxed{a = -\frac{2}{7}}$$

4. If $x = -\frac{1}{2}$ is a zero of the polynomial $p(x) = 8x^3 - ax^2 - x + 2$, find the value of a .

Sol:

$$\text{We have } p(x) = 8x^3 - ax^2 - x + 2$$

$$\text{Put } x = -\frac{1}{2}$$

$$\Rightarrow P\left(-\frac{1}{2}\right) = 8 \times \left(-\frac{1}{2}\right)^3 - ax \left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 2$$

$$= 8 \times \frac{-1}{8} - a \times \frac{1}{4} + \frac{1}{2} + 2$$

$$= -1 - \frac{a}{4} + \frac{1}{2} + 2$$

$$= \frac{3}{2} - \frac{a}{4}$$

Given that $x = \frac{-1}{2}$ is a root of $p(x)$

$$\Rightarrow P\left(\frac{-1}{2}\right) = 0$$

$$\therefore \frac{3}{2} - \frac{a}{4} = 0 \Rightarrow \frac{a}{4} = \frac{3}{2} \Rightarrow a = \frac{3}{2} \times 4$$

$$\Rightarrow \boxed{a=6}$$

5. If $x = 0$ and $x = -1$ are the roots of the polynomial $f(x) = 2x^3 - 3x^2 + ax + b$, find the value of a and b .

Sol:

We have $f(x) = 2x^3 - 3x^2 + ax + b$

Put

$$x = 0 \Rightarrow f(0) = 2 \times (0)^3 - 3(0)^2 + a(0) + b = 0 - 0 + 0 + b = b$$

$$x = -1 \Rightarrow f(-1) = 2 \times (-1)^3 - 3 \times (-1)^2 + a(-1) + b = 2 \times (-1) - 3 \times (1) - a + b$$

$$= -2 - 3 - a + b$$

$$= -5 - a + b$$

Since $x = 0$ and $x = -1$ are roots of $f(x)$

$$\Rightarrow f(0) = 0 \text{ and } f(-1) = 0$$

$$\Rightarrow b = 0 \quad \Rightarrow -5 - a + b = 0$$

$$\boxed{b=0} \text{ and } a - b = -5$$

$$\Rightarrow a - 0 = -5$$

$$\Rightarrow \boxed{a=-5}$$

$$\therefore a = -5 \text{ and } b = 0$$

6. Find the integral roots of the polynomial $f(x) = x^3 + 6x^2 + 11x + 6$.

Sol:

We have

$$f(x) = x^3 + 6x^2 + 11x + 6$$

Clearly, $f(x)$ is a polynomial with integer coefficient and the coefficient of the highest degree term i.e., the leading coefficients is 1.

Therefore, integer roots of $f(x)$ are limited to the integer factors of 6, which are $\pm 1, \pm 2, \pm 3, \pm 6$

We observe that

$$f(-1) = (-1)^3 + 6(-1)^2 + 11(-1) + 6 = -1 + 6 - 11 + 6 = 0$$

$$f(-2) = (-2)^3 + 6(-2)^2 + 11(-2) + 6 = -8 + 24 - 22 + 6 = 0$$

$$f(-3) = (-3)^3 + 6(-3)^2 + 11(-3) + 6 = -27 + 54 - 33 + 6 = 0$$

\therefore Hence, integral roots of $f(x)$ are $-1, -2, -3$.

7. Find rational roots of the polynomial $f(x) = 2x^3 + x^2 - 7x - 6$

Sol:

We have

$$f(x) = 2x^3 + x^2 - 7x - 6$$

Clearly, $f(x)$ is a cubic polynomial with integer coefficient. If $\frac{b}{c}$ is a rational roots in lowest terms, then the value of b are limited to the factors of 6 which $\pm 1, \pm 2, \pm 3, \pm 6$ and values of c are limited to the factors of 2 which are $\pm 1, \pm 2$.

Hence, the possible rational roots of $f(x)$ are

$$\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$$

We observe that

$$f(-1) = 2(-1)^3 + (-1)^2 - 7(-1) - 6 = -2 + 1 + 7 - 6 = 0$$

$$f(2) = 2(2)^3 + (2)^2 - 7(2) - 6 = 16 + 4 - 14 - 6 = 0$$

$$f\left(-\frac{3}{2}\right) = 2\left(-\frac{3}{2}\right)^3 + \left(-\frac{3}{2}\right)^2 - 7\left(\frac{3}{2}\right) - 6 = -\frac{27}{4} + \frac{9}{4} + \frac{21}{2} - 6 = 0$$

\therefore Hence, $-1, 2, -\frac{3}{2}$ are the rational roots of $f(x)$

Exercise– 6.3

In each of the following, using the remainder theorem, find the remainder when $f(x)$ is divided by $g(x)$ and verify the result by actual division: (1 – 8)

1. $f(x) = x^3 + 4x^2 - 3x + 10$, $g(x) = x + 4$

Sol:

We have $f(x) = x^3 + 4x^2 - 3x + 10$ and $g(x) = x + 4$

Therefore, by remainder theorem when $f(x)$ is divided by $g(x) = x - (-4)$, the remainder is equal to $f(-4)$

Now, $f(x) = x^3 + 4x^2 - 3x + 10$

$$\Rightarrow f(-4) = (-4)^3 + (-4)^2 - 3(-4) + 10$$

$$= -64 + 4 \times 16 + 12 + 10$$

$$= -64 + 64 + 12 + 10 = 22$$

Hence, required remainder is 22.

2. $f(x) = 4x^4 - 3x^3 - 2x^2 + x - 7$, $g(x) = x - 1$

Sol:

We have

$$f(x) = 4x^4 - 3x^3 - 2x^2 + x - 7 \text{ and } g(x) = x - 1$$

Therefore by remainder theorem when $f(x)$ is divided by $g(x) = x - 1$, the remainder is equal to $f(+1)$

Now, $f(x) = 4x^4 - 3x^3 - 2x^2 + x - 7$

$$\Rightarrow f(1) = 4(+1)^4 - 3(+1)^3 - 2(+1)^2 + (+1) - 7$$

$$= 4 \times 1 - 3(+1) - 2(1) + 1 - 7$$

$$= 4 - 3 - 2 + 1 - 7 = -7$$

Hence, required remainder is -7

3. $f(x) = 2x^4 - 6x^3 + 2x^2 - x + 2$, $g(x) = x + 2$

Sol:

We have

$$f(x) = 2x^4 - 6x^3 + 2x^2 - x + 2 \text{ and } g(x) = x + 2$$

Therefore, by remainder theorem when $f(x)$ is divide by $g(x) = x - (-2)$, the remainder is equal to $f(-2)$

$$\text{Now, } f(x) = 2x^4 - 6x^3 + 2x^2 - x + 2$$

$$\Rightarrow f(-2) = 2(-2)^4 - 6(-2)^3 + 2(-2)^2 - (-2) + 2$$

$$= 2 \times 16 - 6 \times (-8) + 2 \times 4 + 2 + 2$$

$$= 32 + 48 + 8 + 4 = 92$$

Hence, required remainder is 92.

4. $f(x) = 4x^3 - 12x^2 + 14x - 3$, $g(x) = 2x - 1$

Sol:

We have

$$f(x) = 4x^3 - 12x^2 + 14x - 3 \text{ and } g(x) = 2x - 1$$

Therefore, by remainder theorem when $f(x)$ is divide by $g(x) = 2\left(x - \frac{1}{2}\right)$, the remainder

is equal to $f\left(\frac{1}{2}\right)$

$$\text{Now, } f(x) = 4x^3 - 12x^2 + 14x - 3$$

$$\Rightarrow f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 + 14\left(\frac{1}{2}\right) - 3$$

$$= \left(\cancel{4} \times \frac{1}{\cancel{8}_2}\right) - \left(\cancel{12}^3 \times \frac{1}{\cancel{4}}\right) + \left(\cancel{14}^7 \times \frac{1}{\cancel{2}}\right) - 3$$

$$= \frac{1}{2} - 3 + 7 - 3 = \frac{1}{2} + 1 = \frac{3}{2}$$

Hence, required remainder is $\frac{3}{2}$.

5. $f(x) = x^3 - 6x^2 + 2x - 4$, $g(x) = 1 - 2x$

Sol:

We have

$$f(x) = x^3 - 6x^2 + 2x - 4 \text{ and } g(x) = 1 - 2x$$

Therefore, by remainder theorem when $f(x)$ is divided by $g(x) = -2\left(x - \frac{1}{2}\right)$, the

remainder is equal to $f\left(\frac{1}{2}\right)$

$$\text{Now, } f(x) = x^3 - 6x^2 + 2x - 4$$

$$\Rightarrow f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 4$$

$$= \frac{1}{8} - \left(\frac{3}{2} + \frac{1}{2}\right) + 2 \times \frac{1}{2} - 4$$

$$= \frac{1}{8} - \frac{3}{2} + 1 - 4 = -\frac{35}{8}$$

Hence, the required remainder is $-\frac{35}{8}$

6. $f(x) = x^4 - 3x^2 + 4$, $g(x) = x - 2$

Sol:

We have

$$f(x) = x^4 - 3x^2 + 4 \text{ and } g(x) = x - 2$$

Therefore, by remainder theorem when $f(x)$ is divided by $g(x) = x - 2$, the remainder is equal to $f(2)$

$$\text{Now, } f(x) = x^4 - 3x^2 + 4$$

$$\Rightarrow f(2) = 2^4 - 3(2)^2 + 4$$

$$= 16 - (3 \times 4) + 4 = 16 - 12 + 4 = 20 - 12 = 8$$

Hence, required remainder is 8.

7. $f(x) = 9x^3 - 3x^2 + x - 5$, $g(x) = x - \frac{2}{3}$

Sol:

$$\text{We have } f(x) = 9x^3 - 3x^2 + x - 5 \text{ and } g(x) = x - \frac{2}{3}$$

Therefore, by remainder theorem when $f(x)$ is divided by $g(x) = x - \frac{2}{3}$, the remainder is

$$\text{equal to } f\left(\frac{2}{3}\right)$$

$$\text{Now, } f(x) = 9x^3 - 3x^2 + x - 5$$

$$\Rightarrow f\left(\frac{2}{3}\right) = 9\left(\frac{2}{3}\right)^3 - 3\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right) - 5$$

$$= \left(\cancel{9} \times \frac{8}{\cancel{27}_3} \right) - \left(\cancel{3} \times \frac{4}{\cancel{9}_3} \right) + \frac{2}{3} - 5$$

$$= \frac{8}{3} - \frac{4}{3} + \frac{2}{3} - 5 = \frac{6}{3} - 5 = 2 - 5 = -3$$

Hence, the required remainder is -3.

8. $f(x) = 3x^4 + 2x^3 - \frac{x^2}{3} - \frac{x}{9} + \frac{2}{27}$, $g(x) = x + \frac{2}{3}$

Sol:

We have

$$f(x) = 3x^4 + 2x^3 - \frac{x^2}{3} - \frac{x}{9} + \frac{2}{27} \text{ and } g(x) = x + \frac{2}{3}$$

Therefore, by remainder theorem when $f(x)$ is divided by $g(x) = x - \left(-\frac{2}{3}\right)$, the remainder is equal to $f\left(-\frac{2}{3}\right)$

$$\text{Now, } f(x) = 3x^4 + 2x^3 - \frac{x^2}{3} - \frac{x}{9} + \frac{2}{27}$$

$$\Rightarrow f\left(-\frac{2}{3}\right) = 3 \times \left(\frac{-2}{3}\right)^4 + 2 \times \left(\frac{-2}{3}\right)^3 - \frac{\left(\frac{-2}{3}\right)^2}{3} - \frac{\left(\frac{-2}{3}\right)}{9} + \frac{2}{27}$$

$$= \cancel{3} \times \frac{16}{\cancel{81}_{27}} + 2 \times \frac{-8}{27} - \frac{4}{9 \times 3} - \left(\frac{-2}{3 \times 9}\right) + \frac{2}{27}$$

$$= \frac{16}{27} - \frac{16}{27} - \frac{4}{27} + \frac{2}{27} + \frac{2}{27}$$

$$= \frac{16 - 16 - 4 + 2 + 2}{27} = \frac{0}{27} = 0$$

Hence, required remainder is 0.

9. If the polynomials $2x^3 + ax^2 + 3x - 5$ and $x^3 + x^2 - 4x + a$ leave the same remainder when divided by $x - 2$, find the value of a .

Sol:

Let $p(x) = 2x^3 + ax^2 + 3x - 5$ and $q(x) = x^3 + x^2 - 4x + a$ be the given polynomials

The remainders when $p(x)$ and $q(x)$ are divided by $(x - 2)$ are $p(2)$ and $q(2)$ respectively.

By the given condition we have

$$p(2) = q(2)$$

$$\Rightarrow 2(2)^3 + a(2)^2 + 3(2) - 5 = 2^3 + 2^2 - 4(2) + a$$

$$\Rightarrow 16 + 4a + 6 - 5 = 8 + 4 - 8 + a$$

$$\Rightarrow 3a + 13 = 0 \Rightarrow 3a = -13 \Rightarrow \boxed{a = \frac{-13}{3}}$$

10. The polynomials $ax^3 + 3x^2 - 3$ and $2x^3 - 5x + a$ when divided by $(x - 4)$ leave the remainders R_1 and R_2 respectively. Find the values of a in each of the following cases, if
(i) $R_1 = R_2$ (ii) $R_1 + R_2 = 0$ (iii) $2R_1 - R_2 = 0$.

Sol:

Let $p(x) = ax^3 + 3x^2 - 3$ and $q(x) = 2x^3 - 5x + a$ be the given polynomials.

Now,

R_1 = Remainder when $p(x)$ is divided by $x - 4$

$$\Rightarrow R_1 = p(4)$$

$$\Rightarrow R_1 = a(4)^3 + 3(4)^2 - 3 \quad \left[\because p(x) = ax^3 + 3x^2 - 3 \right]$$

$$\Rightarrow R_1 = 64a + 48 - 3$$

$$\Rightarrow \boxed{R_1 = 64a + 45}$$

And,

R_2 = Remainder when $q(x)$ is divided by $x - 4$

$$\Rightarrow R_2 = q(4)$$

$$\Rightarrow R_2 = q(4)^3 - 5(4) + a \quad \left[\because q(x) = 2x^3 - 5x + a \right]$$

$$\Rightarrow R_2 = 128 - 20 + a$$

$$\Rightarrow \boxed{R_2 = 108 + a}$$

- (i) Given condition is $R_1 = R_2$

$$\Rightarrow 64a + 45 = 108 + a$$

$$\Rightarrow 63a - 63 = 0 \Rightarrow 63a = 63 \Rightarrow \boxed{a = 1}$$

- (ii) Given condition is $R_1 + R_2 = 0$

$$\Rightarrow 64a + 45 + 108 + a = 0$$

$$\Rightarrow 65a + 153 = 0 \Rightarrow 65a = -153 \Rightarrow \boxed{a = \frac{-153}{65}}$$

$$(iii) \text{ Given condition is } 2R_1 - R_2 = 0$$

$$\Rightarrow 2(64a + 45) - (108 + a) = 0$$

$$\Rightarrow 128a + 90 - 108 - a = 0$$

$$\Rightarrow 127a - 18 = 0 \Rightarrow 127a = 18 \Rightarrow a = \frac{18}{127}$$

11. If the polynomials $ax^3 + 3x^2 - 13$ and $2x^3 - 5x + a$, when divided by $(x - 2)$ leave the same remainder, find the value of a .

Sol:

Let $p(x) = ax^3 + 3x^2 - 13$ and $q(x) = 2x^3 - 5x + a$ be the given polynomials

The remainders when $p(x)$ and $q(x)$ are divided by $(x - 2)$ are $p(2)$ and $q(2)$.

By the given condition we have

$$p(2) = q(2)$$

$$\Rightarrow a(2)^3 + 3(2)^2 - 13 = 2(2)^3 - 5(2) + a$$

$$\Rightarrow 8a + 12 - 13 = 16 - 10 + a$$

$$\Rightarrow 7a - 7 = 0 \Rightarrow 7a = 7 \Rightarrow a = \frac{7}{7} \Rightarrow a = 1$$

(i) $x \Rightarrow x - 0$

By remainder theorem, required remainder is equal to $f(0)$

$$\text{Now, } f(x) = x^3 + 3x^2 + 3x + 1$$

$$\Rightarrow f(0) = 0^3 + 3 \times 0^2 + 3 \times 0 + 1 = 0 + 0 + 0 + 1 = 1$$

Hence, required remainder is 1.

(ii) $x + \pi \Rightarrow x - (-\pi)$

By remainder theorem, required remainder is equal to $f(-\pi)$

$$\text{Now, } f(x) = x^3 + 3x^2 + 3x + 1$$

$$\Rightarrow f(-\pi) = (-\pi)^3 + 3(-\pi)^2 + 3(-\pi) + 1$$

$$= -\pi^3 + 3\pi^2 - 3\pi + 1$$

Hence, required remainder is $-\pi^3 + 3\pi^2 - 3\pi + 1$.

(iii) $5 + 2x \Rightarrow 2\left(x - \left(-\frac{5}{2}\right)\right)$

By remainder theorem, required remainder is equal to $f\left(-\frac{5}{2}\right)$.

$$\text{Now, } f(x) = x^3 + 3x^2 + 3x + 1$$

$$\begin{aligned}
 \Rightarrow f\left(-\frac{5}{2}\right) &= \left(-\frac{5}{2}\right)^3 + 3\left(-\frac{5}{2}\right)^2 + 3\left(-\frac{5}{2}\right) + 1 \\
 &= \frac{-125}{8} + \frac{3 \times 25}{4} + \frac{3 \times -5}{2} + 1 \\
 &= \frac{-125}{8} + \frac{75}{4} - \frac{15}{2} + 1 \\
 &= \frac{-27}{8}
 \end{aligned}$$

Hence, required remainder is $\frac{-27}{8}$.

Exercise – 6.4

In each of the following, use factor theorem to find whether polynomial $g(x)$ is a factor of polynomial $f(x)$ or, not: (1 – 7)

1. $f(x) = x^3 - 6x^2 + 11x - 6$; $g(x) = x - 3$

Sol:

We have $f(x) = x^3 - 6x^2 + 11x - 6$ and $g(x) = x - 3$

In order to find whether polynomial $g(x) = x - 3$ is a factor of $f(x)$, it is sufficient to show that $f(3) = 0$

Now, $f(x) = x^3 - 6x^2 + 11x - 6$

$$\Rightarrow f(3) = 3^3 - 6(3)^2 + 11(3) - 6$$

$$= 27 - 54 + 33 - 6 = 60 - 60 = 0$$

Hence, $g(x)$ is a factor of $f(x)$

2. $f(x) = 3x^4 + 17x^3 + 9x^2 - 7x - 10$; $g(x) = x + 5$

Sol:

We have $f(x) = 3x^4 + 17x^3 + 9x^2 - 7x - 10$ and $g(x) = x + 5$

In order to find whether $g(x) = x - (-5)$ is a factor of $f(x)$ or not, it is sufficient to show that $f(-5) = 0$

Now, $f(x) = 3x^4 + 17x^3 + 9x^2 - 7x - 10$

$$\Rightarrow f(-5) = 3(-5)^4 + 17(-5)^3 + 9(-5)^2 - 7(-5) - 10$$

$$\begin{aligned} &= 3 \times 625 + 17 \times (-125) + 9 \times 25 + 35 - 10 \\ &= 1875 - 2125 + 225 + 35 - 10 \\ &= 0 \end{aligned}$$

Hence, $g(x)$ is a factor of $f(x)$

3. $f(x) = x^5 + 3x^4 - x^3 - 3x^2 + 5x + 15$, $g(x) = x + 3$

Sol:

We have $f(x) = x^5 + 3x^4 - x^3 - 3x^2 + 5x + 15$ and $g(x) = x + 3$

In order to find whether $g(x) = x - (-3)$ is a factor of $f(x)$ or not, it is sufficient to prove that $f(-3) = 0$

$$\begin{aligned} \text{Now, } f(x) &= x^5 + 3x^4 - x^3 - 3x^2 + 5x + 15 \\ \Rightarrow f(-3) &= (-3)^5 + 3(-3)^4 - 3(-3)^2 + 5(-3) + 15 \\ &= -243 + 243 - (-27) - 3(9) + 5(-3) + 15 \\ &= -243 + 243 + 27 - 27 - 15 + 15 \\ &= 0 \end{aligned}$$

Hence, $g(x)$ is a factor of $f(x)$

4. $f(x) = x^3 - 6x^2 - 19x + 84$, $g(x) = x - 7$

Sol:

We have $f(x) = x^3 - 6x^2 - 19x + 84$ and $g(x) = x - 7$

In order to find whether $g(x) = x - 7$ is a factor of $f(x)$ or not, it is sufficient to show that $f(7) = 0$

$$\begin{aligned} \text{Now, } f(x) &= x^3 - 6x^2 - 19x + 84 \\ \Rightarrow f(7) &= 7^3 - 6(7)^2 - 19(7) + 84 \\ &= 343 - 294 - 133 + 84 = 427 - 427 \\ &= 0 \end{aligned}$$

Hence $g(x)$ is a factor $f(x)$

5. $f(x) = 3x^3 + x^2 - 20x + 12$ and $g(x) = 3x - 2$

Sol:

We have

$$f(x) = 3x^3 + x^2 - 20x + 12 \text{ and } g(x) = 3x - 2$$

In order to find whether $g(x) = 3\left(x - \frac{2}{3}\right)$ is a factor of $f(x)$ or not, it is sufficient to prove

$$\text{that } f\left(\frac{2}{3}\right) = 0$$

$$\text{Now, } f(x) = 3x^3 + x^2 - 20x + 12$$

$$\Rightarrow f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^2 - 20\left(\frac{2}{3}\right) + 12$$

$$= 3 \times \frac{8}{\cancel{27}_9} + \frac{4}{9} = \frac{40}{3} + 12$$

$$= \frac{12}{9} - \frac{40}{3} + 12 = \frac{12 - 20 + 108}{9} = \frac{120 - 120}{9} = 0$$

Hence $g(x) = 3x - 2$ is a factor of $f(x)$

6. $f(x) = 2x^3 - 9x^2 + x + 12$, $g(x) = 3 - 2x$

Sol:

We have $f(x) = 2x^3 - 9x^2 + x + 12$ and $g(x) = 3 - 2x$

In order to find whether $g(x) = 3 - 2x = -2\left(x - \frac{3}{2}\right)$ is a factor of $f(x)$ or not, it is sufficient

$$\text{to prove that } f\left(\frac{3}{2}\right) = 0$$

$$\text{Now, } f(x) = 2x^3 - 9x^2 + x + 12$$

$$\Rightarrow f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right) + 12$$

$$= \cancel{2} \times \frac{27}{\cancel{8}_4} - 9 \times \frac{9}{4} + \frac{3}{2} + 12$$

$$= \frac{27}{4} - \frac{81}{4} + \frac{3}{2} + 12 = \frac{27 - 81 + 6 + 48}{4} = \frac{81 - 81}{4} = 0$$

Hence $g(x) = 3 - 2x$ is a factor of $f(x)$

7. $f(x) = x^3 - 6x^2 + 11x - 6$, $g(x) = x^2 - 3x + 2$

Sol:

We have

$$f(x) = x^3 - 6x^2 + 11x - 6 \text{ and } g(x) = x^2 - 3x + 2$$

$$\Rightarrow g(x) = x^2 - 3x + 2 = (x-1)(x-2)$$

Clearly, $(x-1)$ and $(x-2)$ are factors of $g(x)$

In order to find whether $g(x) = (x-1)(x-2)$ is a factor of $f(x)$ or not, it is sufficient to prove that $(x-1)$ and $(x-2)$ are factors of $f(x)$.

i.e., we should prove that $f(1) = 0$ and $f(2) = 0$

$$\text{Now, } f(x) = x^3 - 6x^2 + 11x - 6$$

$$\Rightarrow f(1) = 1^3 - 6(1)^2 + 11(1) - 6 = 1 - 6 + 11 - 6 = 12 - 12 = 0$$

$$\Rightarrow f(2) = 2^3 - 6(2)^2 + 11(2) - 6 = 8 - 24 + 22 - 6 = 30 - 30 = 0$$

$\therefore (x-1)$ and $(x-2)$ are factors of $f(x)$

$$\Rightarrow g(x) = (x-1)(x-2) \text{ is factor of } f(x)$$

8. Show that $(x-2)$, $(x+3)$ and $(x-4)$ are factors of $x^3 - 3x^2 - 10x + 24$.

Sol:

Let $f(x) = x^3 - 3x^2 - 10x + 24$ be the given polynomial.

In order to prove that $(x-2)$, $(x+3)$, $(x-4)$ are factors of $f(x)$, it is sufficient to prove that $f(2) = 0$, $f(-3) = 0$ and $f(4) = 0$ respectively.

$$\text{Now } f(x) = x^3 - 3x^2 - 10x + 24$$

$$\Rightarrow f(2) = 2^3 - 3(2)^2 - 10(2) + 24 = 8 - 12 - 20 + 24 = 0$$

$$\Rightarrow f(-3) = (-3)^3 - 3(-3)^2 - 10(-3) + 24 = -27 - 27 + 30 + 24 = 0$$

$$\Rightarrow f(4) = 4^3 - 3(4)^2 - 10(4) + 24 = 64 - 48 - 40 + 24 = 0$$

Hence, $(x-2)$, $(x+3)$ and $(x-4)$ are factors of the given polynomial.

9. Show that $(x+4)$, $(x-3)$ and $(x-7)$ are factors $x^3 - 6x^2 - 19x + 84$

Sol:

Let $f(x) = x^3 - 6x^2 - 19x + 84$ be the given polynomial

In order to prove that $(x+4)$, $(x-3)$ and $(x-7)$ are factors of $f(x)$, it is sufficient to prove that $f(-4) = 0$, $f(3) = 0$ and $f(7) = 0$ respectively

$$\text{Now, } f(x) = x^3 - 6x^2 - 19x + 84$$

$$\Rightarrow f(-4) = (-4)^3 - 6(-4)^2 - 19(-4) + 84 = -64 - 96 + 76 + 84 = 0$$

$$\Rightarrow f(3) = (3)^3 - 6(3)^2 - 19(3) + 84 = 27 - 54 - 57 + 84 = 0$$

$$\Rightarrow f(7) = 7^3 - 6(7)^2 - 19(7) + 84 = 343 - 294 - 133 + 84 = 0$$

Hence, $(x+4)$, $(x-3)$ and $(x-7)$ are factors of the given polynomial $x^3 - 6x^2 - 19x + 84$.

10. For what value of a is $(x-5)$ a factor of $x^3 - 3x^2 + ax - 10$?

Sol:

Let $f(x) = x^3 - 3x^2 + ax - 10$ be the given polynomial

From factor theorem,

If $(x-5)$ is a factor of $f(x)$ then $f(5) = 0$

Now, $f(x) = x^3 - 3x^2 + ax - 10$

$$\Rightarrow f(5) = 5^3 - 3(5)^2 + a(5) - 10 = 0$$

$$\Rightarrow 125 - 3(25) + 5a - 10 = 0$$

$$\Rightarrow 5a + 40 = 0$$

$$\Rightarrow 5a = -40$$

$$\Rightarrow \boxed{a = -8}$$

Hence $(x-5)$ is a factor of $f(x)$ if $a = -8$

11. Find the value of a such that $(x-4)$ is a factor of $5x^3 - 7x^2 - ax - 28$.

Sol:

Let $f(x) = 5x^3 - 7x^2 - ax - 28$ be the given polynomial from factor theorem, if $(x-4)$ is a factor of $f(x)$ then $f(4) = 0$

$$\Rightarrow f(4) = 0$$

$$\Rightarrow 5(4)^3 - 7(4)^2 - a(4) - 28 = 0$$

$$\Rightarrow 5 \times 64 - 7 \times 16 - 4a - 28 = 0$$

$$\Rightarrow 320 - 112 - 4a - 28 = 0$$

$$\Rightarrow 180 - 4a = 0$$

$$\Rightarrow 4a = 180$$

$$\Rightarrow a = \frac{180}{4} = 45$$

Hence $(x-4)$ is a factor of $f(x)$ when $a = 45$

12. For what value of a , if $x + 2$ is a factor of factor of $4x^4 + 2x^3 - 3x^2 + 8x + 5a$.

Sol:

Let $f(x) = 4x^4 + 2x^3 - 3x^2 + 8x + 5a$ be the given polynomial

From factor theorem if $(x + 2)$ is a factor of $f(x)$ then $f(-2) = 0$

Now, $f(x) = 4x^4 + 2x^3 - 3x^2 + 8x + 5a$

$$\Rightarrow f(-2) = 0$$

$$\Rightarrow 4(-2)^4 + 2(-2)^3 - 3(-2)^2 + 8(-2) + 5a = 0$$

$$\Rightarrow 64 - 16 - 12 - 16 + 5a = 0 \Rightarrow 5a + 20 = 0$$

$$\Rightarrow 5a = 20$$

$$\Rightarrow \boxed{a = -4}$$

Hence $(x + 2)$ is a factor of $f(x)$ when $a = -4$

13. Find the value of k if $x - 3$ is a factor of $k^2x^3 - kx^2 + 3kx - k$.

Sol:

Let $f(x) = k^2x^3 - kx^2 + 3kx - k$ be the given polynomial from factor theorem if $(x - 3)$ is a factor of $f(x)$ then $f(3) = 0$

$$\Rightarrow k^2(3)^3 - k(3)^2 + 3k(3) - k = 0$$

$$\Rightarrow 27k^2 - 9k + 9k - k = 0$$

$$\Rightarrow 27k^2 - k = 0 \Rightarrow k(27k - 1) = 0$$

$$\Rightarrow k = 0 \text{ and } 27k - 1 = 0 \Rightarrow k = \frac{1}{27}$$

Hence, $(x - 3)$ is a factor of $f(x)$ when $k = 0$ or $k = \frac{1}{27}$

14. Find the values of a and b , if $x^2 - 4$ is a factor of $ax^4 + 2x^3 - 3x^2 + bx - 4$.

Sol:

Let $f(x) = ax^4 + 2x^3 - 3x^2 + bx - 4$ and $g(x) = x^2 - 4$

We have $g(x) = x^2 - 4 = (x - 2)(x + 2)$

Given $g(x)$ is a factor of $f(x)$.

$\Rightarrow (x - 2)$ and $(x + 2)$ are factors of $f(x)$

From factor theorem,

If $(x - 2)$ and $(x + 2)$ are factors of $f(x)$ then $f(2) = 0$ and $f(-2) = 0$ respectively

$$\Rightarrow f(2) = 0 \Rightarrow a(2)^4 + 2(2)^3 - 3(2)^2 + b(2) - 4 = 0$$

$$\Rightarrow 16a + 16 - 12 + 2b - 4 = 0$$

$$\Rightarrow 16a + 2b = 0 \Rightarrow 2(8a + b) = 0$$

$$\Rightarrow \boxed{8a + b = 0} \quad \dots\dots\dots(1)$$

Similarly $f(-2) = 0 \Rightarrow a(-2)^4 + 2(-2)^3 - 3(-2)^2 + b(-2) - 4 = 0$

$$\Rightarrow 16a - 16 - 12 - 2b - 4 = 0$$

$$\Rightarrow 16a - 2b - 32 = 0 \Rightarrow 2(8a + b) = 32$$

$$\Rightarrow \boxed{8a - b = 16} \quad \dots\dots\dots(2)$$

Adding equation (1) and (2)

$$8a + b + 8a - b = 16 \Rightarrow 16a = 16 \Rightarrow \boxed{a = 1}$$

Put $a = 1$ in equation (1)

$$\Rightarrow 8 \times 1 + b = 0 \Rightarrow \boxed{b = -8}$$

Hence, $a = 1$ and $b = -8$

15. Find α and β , if $x + 1$ and $x + 2$ are factors of $x^3 + 3x^2 - 2\alpha x + \beta$.

Sol:

Let $f(x) = x^3 + 3x^2 - 2\alpha x + \beta$ be the given polynomial from factor theorem, if $(x + 1)$ and

$(x + 2)$ are factors of $f(x)$ then $f(-1) = 0$ and $f(-2) = 0$

$$\Rightarrow f(-1) = 0 \Rightarrow (-1)^3 + 3(-1)^2 - 2\alpha(-1) + \beta = 0$$

$$\Rightarrow -1 + 3 + 2\alpha + \beta = 0 \Rightarrow 2\alpha + \beta + 2 = 0 \quad \dots\dots\dots(1)$$

Similarly,

$$f(-2) = 0 \Rightarrow (-2)^3 + 3(-2)^2 - 2\alpha(-2) + \beta = 0$$

$$\Rightarrow -8 + 12 + 4\alpha + \beta = 0 \Rightarrow 4\alpha + \beta + 4 = 0 \quad \dots\dots\dots(2)$$

Subtract equation (1) from (2)

$$\Rightarrow 4\alpha + \beta + 4 - (2\alpha + \beta + 2) = 0 - 0$$

$$\Rightarrow 4\alpha + \beta + 4 - 2\alpha - \beta - 2 = 0$$

$$\Rightarrow 2\alpha + 2 = 0 \Rightarrow 2\alpha = -2 \Rightarrow \boxed{\alpha = -1}$$

Put $\alpha = -1$ in equation (1)

$$\Rightarrow 2(-1) + \beta + 2 = 0 \Rightarrow -2 + \beta + 2 = 0 \Rightarrow \boxed{\beta = 0}$$

Hence, $\alpha = -1$ and $\beta = 0$

16. Find the values of p and q so that $x^4 + px^3 + 2x^2 - 3x + q$ is divisible by $(x^2 - 1)$

Sol:

Let $f(x) = x^4 + px^3 + 2x^2 - 3x + q$ be the given polynomial.

and let $g(x) = x^2 - 1 = (x-1)(x+1)$

Clearly, $(x-1)$ and $(x+1)$ are factors of $g(x)$

Given $g(x)$ is a factor of $f(x)$

$\Rightarrow (x-1)$ and $(x+1)$ are factors of $f(x)$

From factor theorem,

If $(x-1)$ and $(x+1)$ are factors of $f(x)$ then $f(1) = 0$ and $f(-1) = 0$ respectively

$$\Rightarrow f(1) = 0 \Rightarrow 1^4 + p(1)^3 + 2(1)^2 - 3(1) + q = 0$$

$$\Rightarrow 1 + p + 2 - 3 + q = 0 \Rightarrow p + q = 0 \quad \dots\dots(1)$$

$$\Rightarrow f(-1) = 0 \Rightarrow (-1)^4 + p(-1)^3 + 2(-1)^2 - 3(-1) + q = 0$$

$$\Rightarrow 1 + (-p) + 2 + 3 + q = 0 \Rightarrow q - p + 6 = 0 \quad \dots\dots(2)$$

Adding equation (1) and (2)

$$\Rightarrow p + q + q - p + 6 = 0 \Rightarrow 2q + 6 = 0 \Rightarrow 2q = -6 \Rightarrow \boxed{q = -3}$$

Put $q = -3$ in equation (1)

$$\Rightarrow p - 3 = 0 \Rightarrow \boxed{p = 3}$$

Hence $x^2 - 1$ is divisible by $f(x)$ when $p = 3, q = -3$

17. Find the values of a and b so that $(x + 1)$ and $(x - 1)$ are factors of $x^4 + ax^3 - 3x^2 + 2x + b$.

Sol:

Let $f(x) = x^4 + ax^3 - 3x^2 + 2x + b$ be the given polynomial.

From factor theorem; if $(x+1)$ and $(x-1)$ are factors of $f(x)$ then $f(-1) = 0$ and $f(1) = 0$ respectively.

$$\Rightarrow f(-1) = 0 \Rightarrow (-1)^4 + a(-1)^3 - 3(-1)^2 + 2(-1) + b = 0$$

$$\Rightarrow 1 - a - 3 - 2 + b = 0 \Rightarrow b - a - 4 = 0 \quad \dots\dots(1)$$

$$\Rightarrow f(1) = 0 \Rightarrow (1)^4 + a(1)^3 - 3(1)^2 + 2(1) + b = 0$$

$$\Rightarrow 1 + a - 3 + 2 + b = 0 \Rightarrow a + b = 0 \quad \dots\dots(2)$$

Adding equation (1) and (2)

$$\Rightarrow b - a - 4 + a + b = 0 + 0$$

$$\Rightarrow 2b - 4 = 0 \Rightarrow 2b = 4 \Rightarrow b = \frac{4}{2} \Rightarrow \boxed{b = 2}$$

Substitute $b = 2$ in equation (2)

$$\Rightarrow a + 2 = 0 \Rightarrow \boxed{a = -2}$$

Hence, $a = -2$ and $b = 2$

18. If $x^3 + ax^2 - bx + 10$ is divisible by $x^2 - 3x + 2$, find the values of a and b .

Sol:

Let $f(x) = x^3 + ax^2 - bx + 10$ and $g(x) = x^2 - 3x + 2$ be the given polynomials.

$$\text{We have } g(x) = x^2 - 3x + 2 = (x - 2)(x - 1)$$

\Rightarrow Clearly, $(x - 1)$ and $(x - 2)$ are factors of $g(x)$

Given that $f(x)$, is divisible by $g(x)$

$\Rightarrow g(x)$ is a factor of $f(x)$

$\Rightarrow (x - 2)$ and $(x - 1)$ are factors of $f(x)$

From factor theorem,

If $(x - 1)$ and $(x - 2)$ are factors of $f(x)$ then $f(1) = 0$ and $f(2) = 0$ respectively.

$$\Rightarrow f(1) = 0 \Rightarrow (1)^3 + a(1)^2 - b(1) + 10 = 0$$

$$\Rightarrow 1 + a - b + 10 = 0 \Rightarrow a - b + 11 = 0 \quad \dots\dots(1)$$

$$\Rightarrow f(2) = 0 \Rightarrow (2)^3 + a(2)^2 - b(2) + 10 = 0$$

$$\Rightarrow 8 + 4a - 2b + 10 = 0$$

$$\Rightarrow 4a - 2b + 18 = 0$$

$$\Rightarrow 2(2a - b + 9) = 0$$

$$\Rightarrow 2a - b + 9 = 0 \quad \dots\dots(2)$$

Subtract equation (1) from (2)

$$\Rightarrow 2a - b + 9 - (a - b + 11) = 0 - 0$$

$$\Rightarrow 2a - b + 9 - a + b - 11 = 0 \Rightarrow a - 2 = 0 \Rightarrow \boxed{a = 2}$$

Put $a = 2$ in equation (1)

$$\Rightarrow a - b + 11 = 0 \Rightarrow 2 - b + 11 = 0 \Rightarrow 13 - b = 0 \Rightarrow \boxed{b = 13}$$

Hence, $a = 2$ and $b = 13$

$\therefore x^3 + ax^2 - bx + 10$ is divisible by $x^2 - 3x + 2$ when $a = 2$ and $b = 13$

19. If both $x + 1$ and $x - 1$ are factors of $ax^3 + x^2 - 2x + b$, find the values of a and b .

Sol:

Let $f(x) = ax^3 + x^2 - 2x + b$ be the given polynomial.

Given $(x+1)$ and $(x-1)$ are factor of $f(x)$.

From factor theorem,

If $(x+1)$ and $(x-1)$ are factors of $f(x)$ then $f(-1) = 0$ and $f(1) = 0$ respectively.

$$\Rightarrow f(-1) = 0 \Rightarrow a(-1)^3 + (-1)^2 - 2(-1) + b = 0$$

$$\Rightarrow -a + 1 + 2 + b = 0 \Rightarrow b - a + 3 = 0 \quad \dots\dots(1)$$

$$\Rightarrow f(1) = 0 \Rightarrow a(1)^3 + (1)^2 - 2(1) + b = 0$$

$$\Rightarrow a + 1 - 2 + b = 0 \Rightarrow b + a - 1 = 0 \quad \dots\dots(2)$$

Adding equation (1) and (2)

$$\Rightarrow b - a + 3 + b + a - 1 = 0 + 0$$

$$\Rightarrow 2b + 2 = 0 \Rightarrow 2b = -2 \Rightarrow \boxed{b = -1}$$

Put $b = -1$ in equation (1)

$$\Rightarrow -1 - a + 3 = 0 \Rightarrow 2 - a = 0 \Rightarrow \boxed{a = 2}$$

Hence the values of a, b are $2, -1$ respectively.

20. What must be added to $x^3 - 3x^2 - 12x + 19$ so that the result is exactly divisibly by $x^2 + x - 6$?

Sol:

Let $p(x) = x^3 - 3x^2 - 12x + 19$ and $q(x) = x^2 + x - 6$.

By division algorithm, when $p(x)$ is divided by $q(x)$, the remainder is a linear expression in x .

So, let $r(x) = ax + b$ is added to $p(x)$ so that $p(x) + r(x)$ is divisible by $q(x)$.

Let $f(x) = p(x) + r(x)$

$$\Rightarrow f(x) = x^3 - 3x^2 - 12x + 19 + ax + b$$

$$\Rightarrow \boxed{f(x) = x^3 - 3x^2 + x(a - 12) + b + 19}$$

We have,

$$q(x) = x^2 + x - 6 = (x + 3)(x - 2)$$

Clearly, $q(x)$ is divisible by $(x - 2)$ and $(x + 3)$

i.e., $(x - 2)$ and $(x + 3)$ are factors of $q(x)$

We have,

$f(x)$ is divisible by $q(x)$

$\Rightarrow (x-2)$ and $(x+3)$ are factors of $f(x)$

From factors theorem,

If $(x-2)$ and $(x+3)$ are factors of $f(x)$ then $f(2) = 0$ and $f(-3) = 0$ respectively.

$$\Rightarrow f(2) = 0 \Rightarrow 2^3 - 3(2)^2 + 2(a-12) + b + 19 = 0$$

$$\Rightarrow 8 - 12 + 2a - 24 + b + 19 = 0$$

$$\Rightarrow 2a + b - 9 = 0 \quad \dots\dots\dots(1)$$

Similarly

$$f(-3) = 0 \Rightarrow (-3)^3 - 3(-3)^2 + (-3)(a-12) + b + 19 = 0$$

$$\Rightarrow -27 - 27 - 3a + 36 + b + 19 = 0$$

$$\Rightarrow b - 3a + 1 = 0 \quad \dots\dots\dots(2)$$

Subtract equation (1) from (2)

$$b - 3a + 1 - (2a + b - 9) = 0 - 0$$

$$\Rightarrow b - 3a + 1 - 2a - b + 9 = 0$$

$$\Rightarrow -5a + 10 = 0 \Rightarrow 5a = 10 \Rightarrow \boxed{a = 2}$$

Put $a = 2$ in equation (2)

$$\Rightarrow b - 3 \times 2 + 1 = 0 \Rightarrow b - 6 + 1 = 0 \Rightarrow b - 5 = 0 \Rightarrow \boxed{b = 5}$$

$$\therefore r(x) = ax + b \Rightarrow \boxed{r(x) = 2x + 5}$$

Hence, $x^3 - 3x^2 - 12x + 19$ is divisible by $x^2 + x - 6$ when $2x + 5$ is added to it.

21. What must be subtracted from $x^3 - 6x^2 - 15x + 80$ so that the result is exactly divisible by $x^2 + x - 12$?

Sol:

Let $p(x) = x^3 - 6x^2 - 15x + 80$ and $q(x) = x^2 + x - 12$

By division algorithm, when $p(x)$ is divided by $q(x)$ the remainder is a linear expression in x .

So, let $r(x) = ax + b$ is subtracted from $p(x)$, So that $p(x) - r(x)$ is divisible by $q(x)$

Let $f(x) = p(x) - r(x)$

Clearly, $(3x-2)$ and $(x+3)$ are factors of $q(x)$

Therefore, $f(x)$ will be divisible by $q(x)$ if $(3x-2)$ and $(x+3)$ are factors of $f(x)$

i.e., from factor theorem,

$$f\left(\frac{2}{3}\right) = 0 \text{ and } f(-3) = 0 \quad \left[\because 3x-2=0 \Rightarrow x=\frac{2}{3} \text{ and } x+3=0 \Rightarrow x=-3 \right]$$

$$\begin{aligned} \Rightarrow f\left(\frac{2}{3}\right) &= 3\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^2 + \frac{2}{3}(a-22) + b + 9 = 0 \\ \Rightarrow 3 \times \frac{8}{27} + \frac{4}{9} + \frac{2}{3}a - \frac{44}{3} + b + 9 &= 0 \\ \Rightarrow \frac{12}{9} + \frac{2}{3}a - \frac{44}{3} + b + 9 &= 0 \\ \Rightarrow \frac{12 + 6a - 132 + 9b + 81}{9} &= 0 \\ \Rightarrow 6a + 9b - 39 &= 0 \\ \Rightarrow 3(2a + 3b - 13) = 0 &\Rightarrow 2a + 3b - 13 = 0 \quad \dots\dots(1) \end{aligned}$$

Similarly,

$$\begin{aligned} f(-3) = 0 &\Rightarrow 3(-3)^3 + (-3)^2 + (-3)(a-22) + b + 9 = 0 \\ \Rightarrow -81 + 9 - 3a + 66 + b + 9 &= 0 \\ \Rightarrow b - 3a + 3 &= 0 \\ \Rightarrow 3(b - 3a + 3) = 0 &\Rightarrow 3b - 9a + 9 = 0 \quad \dots\dots(2) \end{aligned}$$

Subtract equation (1) from (2)

$$\begin{aligned} \Rightarrow 3b - 9a + 9 - (2a + 3b - 13) &= 0 - 0 \\ \Rightarrow 3b - 9a + 9 - 2a - 3b + 13 &= 0 \end{aligned}$$

Put $a = 4$ in equation (2)

$$\begin{aligned} \Rightarrow 4 \times 4 - b - 20 &= 0 \\ \Rightarrow 16 - b - 20 = 0 &\Rightarrow -b - 4 = 0 \Rightarrow \boxed{b = -4} \end{aligned}$$

Putting the value of a and b in $r(x) = ax + b$,

$$\text{We get } \boxed{r(x) = 4x - 4}$$

Hence, $p(x)$ is divisible by $q(x)$, if $r(x) = 4x - 4$ is subtracted from it.

22. What must be added to $3x^3 + x^2 - 22x + 9$ so that the result is exactly divisible by $3x^2 + 7x - 6$?

Sol:

$$\text{Let } p(x) = 3x^3 + x^2 - 22x + 9 \text{ and } q(x) = 3x^2 + 7x - 6$$

By division algorithm,

When $p(x)$ is divided by $q(x)$, the remainder is a linear equation in x .

So, let $r(x) = ax + b$ is added to $p(x)$, so that $p(x) + r(x)$ is divisible by $q(x)$

$$\text{Let } f(x) = p(x) + r(x)$$

$$\Rightarrow f(x) = 3x^3 + x^2 - 22x + 9 + (ax + b)$$

$$\Rightarrow \boxed{f(x) = 3x^3 + x^2 + x(a - 22) + b + 9}$$

We have,

$$q(x) = 3x^2 + 7x - 6 = 3x^2 + 9x - 2x - 6 = 3x(x + 3) - 2(x + 3)$$

$$= (3x - 2)(x + 3)$$

$$= f(x) = x^3 - 6x^2 - 15x + 80 - (ax + b)$$

$$\Rightarrow \boxed{f(x) = x^3 - 6x^2 - x(a + 15) + 80 - b}$$

We have,

$$q(x) = x^2 + x - 12 = x^2 + 4x - 3x - 12 = x(x + 4) - 3(x + 4)$$

$$= (x - 3)(x + 4)$$

Clearly, $(x - 3)$ and $(x + 4)$ are factors of $q(x)$.

Therefore, $f(x)$ will be divisible by $q(x)$ if $(x - 3)$ and $(x + 4)$ are factor of $f(x)$.

i.e., from factors theorem,

$$f(3) = 0 \text{ and } f(-4) = 0 \quad [\because x - 3 = 0 \Rightarrow x = 3 \text{ and } x + 4 = 0 \Rightarrow x = -4]$$

$$\Rightarrow f(3) = 0 \Rightarrow (3)^3 - 6(3)^2 - 3(a + 15) + 80 - b = 0$$

$$\Rightarrow 27 - 54 - 3a - 45 + 80 - b = 0$$

$$\Rightarrow 8 - 3a - b = 0 \quad \dots\dots\dots(1)$$

$$\Rightarrow f(-4) = 0 \Rightarrow (-4)^3 - 6(-4)^2 - (4)(a + 15) + 80 - b = 0$$

$$\Rightarrow -64 - 96 - 4a + 60 + 80 - b = 0$$

$$\Rightarrow 4a - b - 20 = 0 \quad \dots\dots\dots(2)$$

Subtract equation (1) from (2)

$$\Rightarrow 4a - b - 20 - (8 - 3a - b) = 0 - 0$$

$$\Rightarrow 4a - b - 20 - 8 + 3a + b = 0$$

$$\Rightarrow 7a - 28 = 0 \Rightarrow 7a = 28 \Rightarrow \boxed{a = 4}$$

$$\Rightarrow -11a + 22 = 0 \Rightarrow 11a = 22 \Rightarrow \boxed{a = 2}$$

Put $a = 2$ in equation (1)

$$\Rightarrow 2 \times 2 + 3b - 13 = 0$$

$$\Rightarrow 4 + 3b - 13 = 0 \Rightarrow 3b - 9 = 0 \Rightarrow 3b = 9 \Rightarrow \boxed{b = 3}$$

Putting the value of a and b in $r(x) = ax + b$,

$$\text{We get, } \boxed{r(x) = 2x + 3}$$

Hence, $3x^3 + x^2 - 22x + 9$ will be divisible by $3x^2 + 7x - 6$, if $2x + 3$ is added to it.

23. If $x - 2$ is a factor of each of the following two polynomials, find the values of a in each case:

(i) $x^3 - 2ax^2 + ax - 1$

(ii) $x^5 - 3x^4 - ax^3 + 3ax^2 + 2ax + 4$

Sol:

- (i) Let $f(x) = x^3 - 2ax^2 + ax - 1$ be the given polynomial.

From factor theorem,

If $(x - 2)$ is a factor of $f(x)$ then $f(2) = 0$ [$\because x - 2 = 0 \Rightarrow x = 2$]

$$\Rightarrow f(2) = 0 \Rightarrow 2^3 - 2a(2)^2 + a(2) - 1 = 0$$

$$\Rightarrow 8 - 8a + 2a - 1 = 0$$

$$\Rightarrow 7 - 6a = 0 \Rightarrow 6a = 7 \Rightarrow \boxed{a = \frac{7}{6}}$$

Hence, $(x - 2)$ is a factor of $f(x)$ when $a = \frac{7}{6}$

- (ii) Let $f(x) = x^5 - 3x^4 - ax^3 + 3ax^2 + 2ax + 4$ be the given polynomial.

From factor theorem

If $x - 2$ is a factor of $f(x)$ then $f(2) = 0$ [$\because x - 2 = 0 \Rightarrow x = 2$]

$$\Rightarrow f(2) = 0 \Rightarrow 2^5 - 3(2)^4 - a(2)^3 + 3a(2)^2 + 2a(2) + 4 = 0$$

$$\Rightarrow 32 - 48 - 8a + 12a + 4a + 4 = 0$$

$$\Rightarrow 8a - 12 = 0 \Rightarrow 8a = 12 \Rightarrow \boxed{a = \frac{3}{2}}$$

Hence, $(x - 2)$ is a factor of $f(x)$ when $a = \frac{3}{2}$

24. In each of the following two polynomials, find the value of a , if $x - a$ is a factor:

(i) $x^6 - ax^5 + x^4 - ax^3 + 3x - a + 2$

(ii) $x^5 - a^2x^3 + 2x + a + 1$

Sol:

- (i) Let $f(x) = x^6 - ax^5 + x^4 - ax^3 + 3x - a + 2$ be the given polynomial.

From factor theorem,

If $(x - a)$ is a factor of $f(x)$ then $f(a) = 0$ [$\because x - a = 0 \Rightarrow x = a$]

$$\Rightarrow f(a) = 0 \Rightarrow a^6 - a(a)^5 + a^4 - a(a)^3 + 3(a) - a + 2 = 0$$

$$\Rightarrow a^6 - a^6 + a^4 - a^4 + 3a - a + 2 = 0$$

$$\Rightarrow 2a + 2 = 0 \Rightarrow 2a = -2 \Rightarrow \boxed{a = -1}$$

Hence, $(x - a)$ is a factor of $f(x)$, if $\boxed{a = -1}$

- (ii) Let $f(x) = x^5 - a^2x^3 + 2x + a + 1$ be the given polynomial.

From factor theorem,

If $(x - a)$ is a factor of $f(x)$ then $f(a) = 0$ [$\because x - a = 0 \Rightarrow x = a$]

$$\Rightarrow f(a) = 0 \Rightarrow a^5 - a^2(a)^3 + 2(a) + a + 1 = 0$$

$$\Rightarrow a^5 - a^5 + 2a + a + 1 = 0$$

$$\Rightarrow 3a + 1 = 0 \Rightarrow 3a = -1 \Rightarrow \boxed{a = -\frac{1}{3}}$$

Hence, $(x - a)$ is a factor of $f(x)$, if $a = -\frac{1}{3}$

25. In each of the following two polynomials, find the value of a , if $x + a$ is a factor:

(i) $x^3 + ax^2 - 2x + a + 4$

(ii) $x^4 - a^2x^2 + 3x - a$

Sol:

- (i) Let $f(x) = x^3 + ax^2 - 2x + a + 4$ be the given polynomial.

From factor theorem,

If $(x + a)$ is a factor of $f(x)$ then $f(-a) = 0$ [$\because x + a = 0 \Rightarrow x = -a$]

$$\Rightarrow f(-a) = 0 \Rightarrow (-a)^3 + a(-a)^2 - 2(-a) + a + 4 = 0$$

$$\Rightarrow -a^3 + a^3 + 2a + a + 4 = 0$$

$$\Rightarrow 3a + 4 = 0 \Rightarrow 3a = -4 \Rightarrow \boxed{a = -\frac{4}{3}}$$

Hence, $(x + a)$ is a factor of $f(x)$, if $a = -\frac{4}{3}$

- (ii) Let $f(x) = x^4 - a^2x^2 + 3x - a$ be the given polynomial

From factor theorem,

If $(x + a)$ is a factor of $f(x)$ then $f(-a) = 0$ [$\because x + a = 0 \Rightarrow x = -a$]

$$\Rightarrow f(-a) = 0 \Rightarrow (-a)^4 - a^2(-a)^2 + 3(-a) - a = 0$$

$$\Rightarrow a^4 - a^4 - 3a - a = 0$$

$$\Rightarrow -3a - a = 0 \Rightarrow -4a = 0 \Rightarrow \boxed{a = 0}$$

Hence, $(x + a)$ is a factor of $f(x)$, if $a = 0$

Exercise – 6.5

Using factor theorem, factorize each of the following polynomials:

1. $x^3 + 6x^2 + 11x + 6$

Sol:

Let $f(x) = x^3 + 6x^2 + 11x + 6$ be the given polynomial.

The constant term in $f(x)$ is 6 and factors of 6 are $\pm 1, \pm 2, \pm 3$ and ± 6

Putting $x = -1$ in $f(x)$, we have

$$f(-1) = (-1)^3 + 6(-1)^2 + 11(-1) + 6 = -1 + 6 - 11 + 6 = 0$$

$\therefore (x+1)$ is a factor of $f(x)$

Similarly, $(x+2)$ and $(x+3)$ are factors of $f(x)$.

Since $f(x)$ is a polynomial of degree 3. So, it cannot have more than three linear factors.

$$\therefore f(x) = k(x+1)(x+2)(x+3)$$

$$\Rightarrow x^3 + 6x^2 + 11x + 6 = k(x+1)(x+2)(x+3)$$

Putting $x = 0$ on both sides, we get

$$0 + 0 + 0 + 6 = k(0+1)(0+2)(0+3)$$

$$6 = k(1)(2)(3)$$

$$\Rightarrow 6 = 6k \Rightarrow \boxed{k = 1}$$

Putting $k = 1$ in $f(x) = k(x+1)(x+2)(x+3)$, we get

$$\boxed{f(x) = (x+1)(x+2)(x+3)}$$

Hence, $x^3 + 6x^2 + 11x + 6 = (x+1)(x+2)(x+3)$

2. $x^3 + 2x^2 - x - 2$

Sol:

Let $f(x) = x^3 + 2x^2 - x - 2$

The constant term in $f(x)$ is equal to -2 and factors of -2 and $\pm 1, \pm 2$.

Putting $x = 1$ in $f(x)$, we have

$$f(1) = 1^3 + 2(1)^2 - 1 - 2 = 0$$

$\therefore (x-1)$ is a factor of $f(x)$

Similarly, $(x+1), (x+2)$ are factors of $f(x)$.

Since $f(x)$ is a polynomial of degree 3. So, it cannot have more than three linear factors.

$$\therefore f(x) = k(x-1)(x+1)(x+2)$$

$$\Rightarrow x^3 + 2x^2 - x - 2 = k(x-1)(x+1)(x+2)$$

Putting $x = 0$ on both sides, we get $0 + 0 - 0 - 2 = k(-1)(+1)(+2)$

$$-2 = -2k \Rightarrow \boxed{k = 1}$$

Putting $k = 1$ in $f(x) = k(x-1)(x+1)(x+2)$, we get

$$\boxed{f(x) = (x-1)(x+1)(x+2)}$$

Hence, $x^3 + 2x^2 - x - 2 = (x-1)(x+1)(x+2)$

3. $x^3 - 6x^2 + 3x + 10$

Sol:

$$\text{Let } f(x) = x^3 - 6x^2 + 3x + 10$$

The constant term in $f(x)$ is equal to 10 and factors of 10 are $\pm 1, \pm 2, \pm 5$ and ± 10

Putting $x = -1$ in $f(x)$, we have

$$f(-1) = (-1)^3 - 6(-1)^2 + 3(-1) + 10 = -1 - 6 - 3 + 10 = 0$$

$$\therefore (x+1) \text{ is a factor of } f(x)$$

Similarly, $(x-2)$ and $(x-5)$ are factors of $f(x)$

Since $f(x)$ is a polynomial of degree 3. So, it cannot have more than three linear factors.

$$\therefore f(x) = k(x+1)(x-2)(x-5)$$

Putting $x = 0$ on both sides, we get

$$\Rightarrow x^3 - 6x^2 + 3x + 10 = k(x+1)(x-2)(x-5)$$

$$0 - 0 + 0 + 10 = k(1)(-2)(-5)$$

$$\Rightarrow 10 = 10k \Rightarrow \boxed{k = 1}$$

Putting $k = 1$ in $f(x) = k(x+1)(x-2)(x-5)$, we get

$$\boxed{f(x) = (x+1)(x-2)(x-5)}$$

Hence, $x^3 - 6x^2 + 3x + 10 = (x+1)(x-2)(x-5)$

4. $x^4 - 7x^3 + 9x^2 + 7x - 10$

Sol:

Let $f(x) = x^4 - 7x^3 + 9x^2 + 7x - 10$

The constant term in $f(x)$ is equal to -10 and factors of -10 are $\pm 1, \pm 2, \pm 5$ and ± 10

Putting $x = 1$ in $f(x)$, we have

$$f(1) = 1^4 - 7(1)^3 + 9(1)^2 + 7(1) - 10 = 1 - 7 + 9 + 7 - 10 = 0$$

$\therefore (x - 1)$ is a factor of $f(x)$

Similarly $(x + 1), (x - 2), (x - 5)$ are also factors of $f(x)$

Since $f(x)$ is a polynomial of degree 4. So, it cannot have more than four linear factors

$$\therefore f(x) = k(x - 1)(x + 1)(x - 2)(x - 5)$$

$$\Rightarrow x^4 - 7x^3 + 9x^2 + 7x - 10 = k(x - 1)(x + 1)(x - 2)(x - 5)$$

Putting $x = 0$ on both sides, we get

$$\Rightarrow 0 - 0 + 0 + 0 - 10 = k(-1)(1)(-2)(-5)$$

$$\Rightarrow -10 = k(-10)$$

$$\Rightarrow \boxed{k = 1}$$

Putting $k = 1$ in $f(x) = k(x - 1)(x + 1)(x - 2)(x - 5)$, we get

$$\boxed{f(x) = (x + 1)(x - 1)(x - 2)(x - 5)}$$

Hence, $x^4 - 7x^3 + 9x^2 + 7x - 10 = (x + 1)(x - 1)(x - 2)(x - 5)$

5. $x^4 - 2x^3 - 7x^2 + 8x + 12$

Sol:

Let $f(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$

The constant term in $f(x)$ is equal to $+12$ and factors of $+12$ are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$ and ± 12

Putting $x = -1$ in $f(x)$, we have

$$f(-1) = (-1)^4 - 2(-1)^3 - 7(-1)^2 + 8(-1) + 12$$

$$= 1 + 2 - 7 - 8 + 12 = 0$$

$\therefore (x + 1)$ is a factor of $f(x)$

Similarly $(x + 2), (x - 2), (x - 3)$ are also factors of $f(x)$

Since $f(x)$ is a polynomial of degree 4. So, it cannot have more than four linear factors.

$$\therefore f(x) = k(x + 1)(x + 2)(x - 2)(x - 3)$$

$$\Rightarrow x^4 - 2x^3 - 7x^2 + 8x + 12 = k(x+1)(x+2)(x-2)(x-3)$$

Putting $x = 0$ on both sides, we get

$$\Rightarrow 0 - 0 - 0 + 0 + 12 = k(1)(2)(-2)(-3)$$

$$\Rightarrow 12 = k(12)$$

$$= \boxed{k=1}$$

Putting $k = 1$ in $f(x) = k(x+1)(x+2)(x-2)(x-3)$, we get

$$\boxed{f(x) = (x+1)(x+2)(x-2)(x-3)}$$

Hence, $x^4 - 2x^3 - 7x^2 + 8x + 12 = (x+1)(x+2)(x-2)(x-3)$

6. $x^4 + 10x^3 + 35x^2 + 50x + 24$

Sol:

Let $f(x) = x^4 + 10x^3 + 35x^2 + 50x + 24$

The constant term in $f(x)$ is equal to +24 and factors of +24 are

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$$

Putting $x = -1$ in $f(x)$, we have

$$f(-1) = (-1)^4 + 10(-1)^3 + 35(-1)^2 + 50(-1) + 24$$

$$= 1 - 10 + 35 - 50 + 24 = 0$$

$\therefore (x+1)$ is a factor of $f(x)$

Similarly, $(x+2)$, $(x+3)$ and $(x+4)$ are also factors of $f(x)$.

Since $f(x)$ is polynomial of degree 4. So, it cannot have more than four linear factors.

$$\therefore f(x) = k(x+1)(x+2)(x+3)(x+4)$$

$$\Rightarrow x^4 + 10x^3 + 35x^2 + 50x + 24 = k(x+1)(x+2)(x+3)(x+4)$$

Putting $x = 0$ on both sides, we get

$$\Rightarrow 0 + 0 + 0 + 0 + 24 = k(1)(2)(3)(4)$$

$$\Rightarrow 24 = 24k \Rightarrow \boxed{k=1}$$

Putting $k = 1$ in $f(x) = k(x+1)(x+2)(x+3)(x+4)$, we get

$$\boxed{f(x) = (x+1)(x+2)(x+3)(x+4)}$$

Hence, $x^4 + 10x^3 + 35x^2 + 50x + 24 = (x+1)(x+2)(x+3)(x+4)$

7. $2x^4 - 7x^3 - 13x^2 + 63x - 45$

Sol:

Let $f(x) = 2x^4 - 7x^3 - 13x^2 + 63x - 45$

The factors of the constant term -45 are $\pm 1, \pm 3, \pm 5, \pm 9, \pm 15$ and ± 45

The factor of the coefficient of x^4 is 2. Hence possible rational roots of $f(x)$ are

$$\pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{9}{2}, \pm \frac{15}{2}, \pm \frac{45}{2}$$

We have,

$$\begin{aligned} f(1) &= 2(1)^4 - 7(1)^3 - 13(1)^2 + 63(1) - 45 \\ &= 2 - 7 - 13 + 63 - 45 = 0 \end{aligned}$$

$$\begin{aligned} \text{And } f(3) &= 2(3)^4 - 7(3)^3 - 13(3)^2 + 63(3) - 45 \\ &= 162 - 189 - 117 + 189 - 45 = 0 \end{aligned}$$

$$\begin{aligned} \text{And } f(3) &= 2(3)^4 - 7(3)^3 - 13(3)^2 + 63(3) - 45 \\ &= 162 - 189 - 117 + 189 - 45 = 0 \end{aligned}$$

So, $(x-1)$ and $(x-3)$ are factors of $f(x)$

$\Rightarrow (x-1)(x-3)$ is also a factor of $f(x)$

$\Rightarrow (x^2 - 4x + 3)$ is a factor of $f(x)$

Let us now divide

$f(x) = 2x^4 - 7x^3 - 13x^2 + 63x - 45$ by $(x^2 - 4x + 3)$ to get the other factors of $f(x)$.

By long division, we have

$$\begin{array}{r} x^2 - 4x + 3 \overline{) 2x^4 - 7x^3 - 13x^2 + 63x - 45} \quad (2x^2 + x - 15) \\ \underline{2x^4 - 8x^3 + 6x^2} \\ - x^3 - 7x^2 + 63x - 45 \\ \underline{x^3 - 4x^2 + 3x} \\ - 3x^2 + 66x - 45 \\ \underline{- 15x^2 + 60x - 45} \\ 6x - 45 \\ \underline{- 15x^2 + 60x - 45} \\ 0 \end{array}$$

$$\therefore 2x^4 - 7x^3 - 13x^2 + 63x - 45 = (x^2 - 4x + 3)(2x^2 + x - 15)$$

$$\Rightarrow 2x^4 - 7x^3 - 13x^2 + 63x - 45 = (x-1)(x-3)(2x^2 + x - 15)$$

Now,

$$2x^2 + x - 15 = 2x^2 + 6x - 5x - 15 = 2x(x+3) - 5(x+3)$$

$$= (2x-5)(x+3)$$

$$\text{Hence } 2x^4 - 7x^3 - 13x^2 + 63x - 45 = (x-1)(x-3)(x+3)(2x-5)$$

8. $3x^3 - x^2 - 3x + 1$

Sol:

$$\text{Let } f(x) = 3x^3 - x^2 - 3x + 1$$

The factors of the constant term +1 is ± 1 .

The factors of the coefficient of x^3 is 3.

Hence possible rational roots of $f(x)$ are $\pm 1, \pm \frac{1}{3}$

We have,

$$f(1) + 3(1)^3 - (1)^2 - 3(1) + 1 = 3 - 1 - 3 + 1 = 0$$

So, $(x-1)$ is a factor of $f(x)$.

Let us now divide $f(x) = 3x^3 - x^2 - 3x + 1$ by $(x-1)$ to get the other factors.

By long division method, we have

$$\begin{array}{r} x-1 \overline{) 3x^3 - x^2 - 3x + 1} \quad (3x^2 + 2x - 1) \\ \underline{3x^3 - 3x^2} \\ - 2x^2 - 3x \\ \underline{2x^2 - 2x} \\ - x + 1 \\ \underline{-x + 1} \\ 0 \end{array}$$

$$\therefore 3x^3 - x^2 - 3x + 1 = (x-1)(3x^2 + 2x - 1)$$

Now,

$$3x^2 + 2x - 1 = 3x^2 + 3x - x - 1 = 3x(x+1) - 1(x+1) = (3x-1)(x+1)$$

$$\text{Hence, } 3x^3 - x^2 - 3x + 1 = (x-1)(x+1)(3x-1)$$

9. $x^3 - 23x^2 + 142x - 120$

Sol:

Let $f(x) = x^3 - 23x^2 + 142x - 120$

The constant term in $f(x)$ is equal to -120 and factors of -120 are

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 8, \pm 10, \pm 12, \pm 15, \pm 20, \pm 24, \pm 30, \pm 40, \pm 60, \pm 120$.

Putting $x = 1$ we have

$$f(1) = 1^3 - 23(1)^2 + 142(1) - 120 = 1 - 23 + 142 - 120 = 0$$

So, $(x - 1)$ is a factor of $f(x)$.

Let us now divide $f(x) = x^3 - 23x^2 + 142x - 120$ by $(x - 1)$ to get the other factors.

By long division, we have

$$x-1 \overline{) x^3 - 23x^2 + 142x - 120} \quad (x^2 - 22x + 120)$$

$$\begin{array}{r} x^3 - x^2 \\ - \quad + \\ \hline -22x^2 + 142x \\ -22x^2 + 22x \\ + \quad - \\ \hline 120x - 120 \\ 120x - 120 \\ - \quad + \\ \hline 0 \end{array}$$

$$\therefore x^3 - 23x^2 + 142x - 120 = (x - 1)(x^2 - 22x + 120)$$

Now,

$$x^2 - 22x + 120 = x^2 - 10x - 12x + 120 = x(x - 10) - 12(x - 10)$$

$$= (x - 12)(x - 10)$$

$$\text{Hence, } x^3 - 23x^2 + 142x - 120 = (x - 1)(x - 10)(x - 12)$$

10. $y^3 - 7y + 6$

Sol:

Let $f(y) = y^3 - 7y + 6$

The constant term in $f(y)$ is $+6$ and factors of $+6$ are $\pm 1, \pm 2, \pm 3$ and ± 6

Putting $y = 1$ we have

$$f(1) = 1^3 - 7(1) + 6 = 1 - 7 + 6 = 0$$

$\therefore (y-1)$ is a factor of $f(y)$

Similarly it can be verified that $(y-2)$ and $(y+3)$ are also factors of $f(y)$

Since $f(y)$ is a polynomial of degree 3. So, it cannot have more than 3 linear factors.

$$\therefore f(y) = k(y-1)(y-2)(y+3)$$

$$\Rightarrow y^3 + 7y + 6 = k(y-1)(y-2)(y+3)$$

Putting $y = 0$ on both sides, we get

$$\Rightarrow 0 - 0 + 6 = k(-1)(-2)(3)$$

$$\Rightarrow 6 = 6k \Rightarrow \boxed{k=1}$$

Putting $k = 1$ in $f(y) = k(y-1)(y-2)(y+3)$, we get

$$\boxed{f(y) = (y-1)(y-2)(y+3)}$$

Hence, $y^3 - 7y + 6 = (y-1)(y-2)(y+3)$

11. $x^3 - 10x^2 - 53x - 42$

Sol:

Let $f(x) = x^3 - 10x^2 - 53x - 42$

The constant term in $f(x)$ is -42 and factors of -42 are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 7, \pm 14, \pm 21, \pm 42$

Putting $x = -1$, we get

$$f(-1) = (-1)^3 - 10(1)^2 - 53(-1) - 42 = -1 - 10 + 53 - 42 = 0$$

So, $(x+1)$ is a factor of $f(x)$

Let us now divide $f(x) = x^3 - 10x^2 - 53x - 42$ by $(x+1)$ to get the other factors.

By long division, we have

$$x+1 \overline{) x^3 - 10x^2 - 53x - 42} \quad (x^2 - 11x - 42)$$

$$x^3 + x^2$$

- -

$$\hline -11x^2 - 53x$$

$$-11x^2 - 11x$$

+ +

$$\hline -42x - 42$$

$$-42x - 42$$

+ +

$$\hline 0$$

$$\therefore x^3 - 10x^2 - 53x - 42 = (x+1)(x^2 - 11x - 42)$$

$$\begin{aligned}\text{Now, } x^2 - 11x - 42 &= x^2 - 14x + 3x - 42 = x(x-14) + 3(x-14) \\ &= (x+3)(x-14)\end{aligned}$$

$$\text{Hence, } x^3 - 10x^2 - 53x - 42 = (x+1)(x+3)(x-14)$$

12. $y^3 - 2y^2 - 29y - 42$

Sol:

$$\text{Let } f(y) = y^3 - 2y^2 - 29y - 42$$

The constant term in $f(y)$ is -42 and factors of -42 are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 7, \pm 14, \pm 21, \pm 42$

Putting $y = -2$ we get

$$\begin{aligned}f(-2) &= (-2)^3 - 2(-2)^2 - 29(-2) - 42 \\ &= -8 - 8 + 58 - 42 = 0\end{aligned}$$

So, $(y+2)$ is a factor of $f(y)$

Let us now divide $f(y) = y^3 - 2y^2 - 29y - 42$ by $(y+2)$ to get the other factors

By long division, we get

$$\begin{array}{r}y+2 \overline{) y^3 - 2y^2 - 29y - 42} \\ \underline{y^3 + 2y^2} \\ -4y^2 - 29y \\ \underline{-4y^2 - 8y} \\ -21y - 42 \\ \underline{-21y - 42} \\ 0\end{array}$$

$$\therefore y^3 - 2y^2 - 29y - 42 = (y+2)(y^2 - 4y - 21)$$

Now,

$$\begin{aligned}y^2 - 4y - 21 &= y^2 - 7y + 3y - 21 = y(y-7) + 3(y-7) \\ &= (y+3)(y-7)\end{aligned}$$

$$\text{Hence, } y^3 - 2y^2 - 29y - 42 = (y+2)(y+3)(y-7)$$

$(y-2)$ to get the other factors.

By long division, we have

$$y-2 \overline{) 2y^3 - 5y^2 - 19y + 42} (2y^2 - y - 21$$

$$\begin{array}{r} 2y^3 - 4y^2 \\ - \quad + \\ \hline \end{array}$$

$$\begin{array}{r} -y^2 - 19y \\ + \quad - \\ \hline \end{array}$$

$$\begin{array}{r} -y^2 - 2y \\ + \quad - \\ \hline \end{array}$$

$$\begin{array}{r} -21y + 42 \\ -21y + 42 \\ + \quad - \\ \hline \end{array}$$

$$\begin{array}{r} -21y + 42 \\ + \quad - \\ \hline \end{array}$$

$$\begin{array}{r} 0 \end{array}$$

$$\therefore 2y^3 - 5y^2 - 19y + 42 = (y-2)(2y^2 - y - 21)$$

$$= (y-2)(y+3)(2y-7)$$

13. $2y^3 - 5y^2 - 19y + 42$

Sol:

$$(y-2)(y+3)(2y-7)$$

14. $x^3 + 13x^2 + 32x + 20$

Sol:

$$\text{Let } f(x) = x^3 + 13x^2 + 32x + 20$$

The constant term in $f(x)$ is 20 and factors of +20 are $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$.

Putting $x = -1$, we get

$$f(-1) = (-1)^3 + 13(-1)^2 + 32(-1) + 20$$

$$= -1 + 13 - 32 + 20 = 0$$

So, $(x+1)$ is a factor of $f(x)$.

Let us now divide $f(x) = x^3 + 13x^2 + 32x + 20$ by $(x+1)$ to get the remaining factors.

By long division, we have

$$x+1 \overline{) x^3 + 13x^2 + 32x + 20} (x^2 + 12x + 20$$

$$\begin{array}{r} x^3 + x^2 \\ - \quad - \\ \hline \end{array}$$

$$\begin{array}{r} 12x^2 + 32x \\ - \quad - \\ \hline \end{array}$$

$$\begin{array}{r} 12x^2 + 32x \\ - \quad - \\ \hline \end{array}$$

$$\begin{array}{r} 20x + 20 \\ - \quad - \\ \hline \end{array}$$

$$\begin{array}{r} 20x + 20 \\ - \quad - \\ \hline \end{array}$$

$$\begin{array}{r} 0 \end{array}$$

$$\therefore x^3 + 13x^2 + 32x + 20 = (x+1)(x^2 + 12x + 20)$$

Now,

$$\begin{aligned} x^2 + 12x + 20 &= x^2 + 10x + 2x + 20 = x(x+10) + 2(x+10) \\ &= (x+2)(x+10) \end{aligned}$$

$$\text{Hence, } x^3 + 13x^2 + 32x + 20 = (x+1)(x+2)(x+10)$$

15. $x^3 - 3x^2 - 9x - 5$

Sol:

$$\text{Let } f(x) = x^3 - 3x^2 - 9x - 5$$

The constant term in $f(x)$ is -5 and factors of -5 are $\pm 1, \pm 5$.

Putting $x = -1$, we get

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) - 5 = -1 - 3 + 9 - 5 = 0$$

So, $(x+1)$ is a factor of $f(x)$.

Let us now divide $f(x) = x^3 - 3x^2 - 9x - 5$ by $(x+1)$ to get the other factors.

By long division, we have

$$\begin{array}{r} x+1 \overline{) x^3 - 3x^2 - 9x - 5} \\ \underline{x^3 + x^2} \\ -4x^2 - 9x \\ \underline{-4x^2 - 4x} \\ -5x - 5 \\ \underline{-5x - 5} \\ 0 \end{array}$$

$$\therefore x^3 - 3x^2 - 9x - 5 = (x+1)(x^2 - 4x - 5)$$

Now,

$$\begin{aligned} (x^2 - 4x - 5) &= x^2 - 5x + x - 5 = x(x-5) + 1(x-5) \\ &= (x+1)(x-5) \end{aligned}$$

$$\text{Hence, } x^3 - 3x^2 - 9x - 5 = (x+1)(x+1)(x-5)$$

$$= (x+1)^2(x-5)$$

16. $2y^3 + y^2 - 2y - 1$

Sol:

Let $f(y) = 2y^3 + y^2 - 2y - 1$

The factors of the constant term of y^3 is 2. Hence possible rational roots are $\pm 1, \pm \frac{1}{2}$.

We have,

$$f(1) = 2(1)^3 + (1)^2 - 2(1) - 1 = 2 + 1 - 2 - 1 = 0$$

So, $(y-1)$ is a factor of $f(y)$

Let us now divide $f(y) = 2y^3 + y^2 - 2y - 1$ by $(y-1)$ to get the other factors.

By long division, we have

$$\begin{array}{r}
 y-1 \overline{) 2y^3 + y^2 - 2y - 1} \quad (2y^2 + 3y + 1 \\
 \underline{2y^3 - 2y^2} \\
 3y^2 - 2y \\
 \underline{3y^2 - 2y} \\
 y - 1 \\
 \underline{y - 1} \\
 0
 \end{array}$$

$$\therefore 2y^3 - y^2 - 2y - 1 = (y-1)(2y^2 + 3y + 1)$$

Now,

$$\begin{aligned}
 2y^2 + 3y + 1 &= 2y^2 + 2y + y + 1 = 2y(y+1) + 1(y+1) \\
 &= (2y+1)(y+1)
 \end{aligned}$$

$$\text{Hence, } 2y^3 + y^2 - 2y - 1 = (y-1)(y+1)(2y+1)$$

17. $x^3 - 2x^2 - x + 2$

Sol:

Let $f(x) = x^3 - 2x^2 - x + 2$

The constant term in $f(x)$ is 2 and factors of 2 are $\pm 1, \pm 2$.

Putting $x = 1$, we have

$$f(1) = 1^3 - 2(1)^2 - 1 + 2 = 1 - 2 - 1 + 2 = 0$$

So, $(x-1)$ is a factor of $f(x)$

Let us now divide $f(x) = x^3 - 2x^2 - x + 2$ by $(x-1)$ to get the remaining factors.

By long division, we have

$$\begin{array}{r}
 x-1 \overline{) x^3 - 2x^2 - x + 2} \quad (x^2 - x - 2) \\
 \underline{x^3 - x^2} \\
 -x^2 - x \\
 \underline{-x^2 + x} \\
 -2x + 2 \\
 \underline{-2x + 2} \\
 0
 \end{array}$$

$$\therefore x^3 - 2x^2 - x + 2 = (x-1)(x^2 - x - 2)$$

Now,

$$\begin{aligned}
 x^2 - x - 2 &= x^2 - 2x + x - 2 = x(x-2) + 1(x-2) \\
 &= (x+1)(x-2)
 \end{aligned}$$

$$\text{Hence } x^3 - 2x^2 - x + 2 = (x-1)(x+2)(x-2)$$

18. Factorize each of the following polynomials:

- (i) $x^3 + 13x^2 + 31x - 45$ given that $x+9$ is a factor
 (ii) $4x^3 + 20x^2 + 33x + 18$ given that $2x+3$ is a factor

Sol:

(i) Let $f(x) = x^3 + 13x^2 + 31x - 45$

Given that $(x+9)$ is a factor of $f(x)$

Let us divide $f(x)$ by $(x+9)$ to get the other factors. By long division, we have

$$\begin{array}{r}
 x+9 \overline{) x^3 + 13x^2 + 31x - 45} \quad (x^2 + 4x - 5) \\
 \underline{x^3 + 9x^2} \\
 4x^2 + 31x \\
 \underline{4x^2 + 36x} \\
 -5x - 45 \\
 \underline{-5x - 45} \\
 0
 \end{array}$$

$$\therefore f(x) = x^3 + 13x^2 + 31x - 45$$

$$\Rightarrow f(x) = (x+9)(x^2 + 4x - 5)$$

Now,

$$x^2 + 4x - 5 = x^2 + 5x - x - 5 = x(x+5) - 1(x+5)$$

$$= (x-1)(x+5)$$

$$\Rightarrow f(x) = (x+9)(x+5)(x-1)$$

$$\therefore x^3 + 13x^2 + 31x - 45 = (x-1)(x+5)(x+9)$$

(ii) Let $f(x) = 4x^3 + 20x^2 + 33x + 18$

Given that $2x+3$ is a factor of $f(x)$

Let us divide $f(x)$ by $(2x+3)$ to get the other factors. By long division, we have

$$\begin{array}{r} 2x+3 \overline{) 4x^3 + 20x^2 + 33x + 18} \quad (2x^2 + 7x + 6 \\ \underline{4x^3 + 6x^2} \\ 14x^2 + 33x \\ \underline{14x^2 + 21x} \\ 12x + 18 \\ \underline{12x + 18} \\ 0 \end{array}$$

Now,

$$4x^3 + 20x^2 + 33x + 18 = (2x+3)(2x^2 + 7x + 6)$$

We have,

$$2x^2 + 7x + 6 = 2x^2 + 4x + 3x + 6 = 2x(x+2) + 3(x+2)$$

$$= (2x+3)(x+2)$$

$$\Rightarrow 4x^3 + 20x^2 + 33x + 18 = (2x+3)(2x+3)(x+2)$$

$$= (2x+3)^2(x+2)$$

Hence, $4x^3 + 20x^2 + 33x + 18 = (x+2)(2x+3)^2$

We have,

$$2x^2 + 7x + 6 = 2x^2 + 4x + 3x + 6 = 2x(x+2) + 3(x+2)$$

$$= (2x+3)(x+2)$$

$$\Rightarrow 4x^3 + 20x^2 + 33x + 18 = (2x+3)(2x+3)(x+2)$$

$$= (2x+3)^2(x+2)$$

Hence, $4x^3 + 20x^2 + 33x + 18 = (x+2)(2x+3)^2$