Preliminaries

Overview This chapter reviews the main things you need to know to start calculus. The topics include the real number system, Cartesian coordinates in the plane, straight lines, parabolas, circles, functions, and trigonometry.

1 Real Numbers and the Real Line

This section reviews real numbers, inequalities, intervals, and absolute values.

Real Numbers and the Real Line

Much of calculus is based on properties of the real number system. **Real numbers** are numbers that can be expressed as decimals, such as

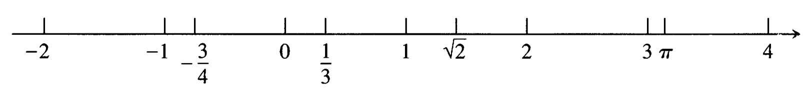
$$-\frac{3}{4} = -0.75000\ldots$$

$$\frac{1}{3} = 0.33333\ldots$$

$$\sqrt{2} = 1.4142\ldots$$

The dots ... in each case indicate that the sequence of decimal digits goes on forever.

The real numbers can be represented geometrically as points on a number line called the **real line**.



The symbol ℝ donotes either the real number system or, equivalently, the real line.

Properties of Real Numbers

The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. The algebraic properties say that the real numbers can be added, subtracted, multiplied, and divided (except by 0) to produce more real numbers under the usual rules of arithmetic. *You can never divide by* 0.

The order properties of real numbers are summarized in the following list.

The symbol $\Rightarrow$ means "implies."

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \dfrac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \dfrac{1}{b} < \dfrac{1}{a}$

Notice the rules for multiplying an inequality by a number. Multiplying by a positive number preserves the inequality; multiplying by a negative number reverses the inequality. Also, reciprocation reverses the inequality for numbers of the same sign.

The completeness property of the real number system is deeper and harder to define precisely. Roughly speaking, it says that there are enough real numbers to "complete" the real number line, in the sense that there are no "holes" or "gaps" in it. Many of the theorems of calculus would fail if the real number system were not complete, and the nature of the connection is important. The topic is best saved for a more advanced course, however, and we will not pursue it.

Subsets of $\mathbb{R}$

We distinguish three special subsets of real numbers.

1. The **natural numbers**, namely $1, 2, 3, 4, \ldots$
2. The **integers**, namely $0, \pm 1, \pm 2, \pm 3, \ldots$
3. The **rational numbers**, namely the numbers that can be expressed in the form of a fraction m/n, where m and n are integers and $n \neq 0$. Examples are
$$\frac{1}{3}, \quad -\frac{4}{9}, \quad \frac{200}{13}, \quad \text{and} \quad 57 = \frac{57}{1}.$$

The rational numbers are precisely the real numbers with decimal expansions that are either

a) terminating (ending in an infinite string of zeros), for example,
$$\frac{3}{4} = 0.75000\ldots = 0.75 \qquad \text{or}$$

b) repeating (ending with a block of digits that repeats over and over), for example
$$\frac{23}{11} = 2.090909\ldots = 2.\overline{09}.$$

The bar indicates the block of repeating digits.

The set of rational numbers has all the algebraic and order properties of the real numbers but lacks the completeness property. For example, there is no rational number whose square is 2; there is a "hole" in the rational line where $\sqrt{2}$ should be.

Real numbers that are not rational are called **irrational numbers**. They are characterized by having nonterminating and nonrepeating decimal expansions. Examples are π, $\sqrt{2}$, $\sqrt[3]{5}$, and $\log_{10} 3$.

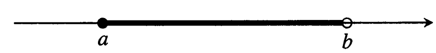
Intervals

A subset of the real line is called an **interval** if it contains at least two numbers and contains all the real numbers lying between any two of its elements. For example, the set of all real numbers x such that $x > 6$ is an interval, as is the set of all x such that $-2 \le x \le 5$. The set of all nonzero real numbers is not an interval; since 0 is absent, the set fails to contain every real number between -1 and 1 (for example).

Geometrically, intervals correspond to rays and line segments on the real line, along with the real line itself. Intervals of numbers corresponding to line segments are **finite intervals**; intervals corresponding to rays and the real line are **infinite intervals**.

A finite interval is said to be **closed** if it contains both of its endpoints, **half-open** if it contains one endpoint but not the other, and **open** if it contains neither endpoint. The endpoints are also called **boundary points**; they make up the interval's **boundary**. The remaining points of the interval are **interior points** and together make up what is called the interval's **interior**.

Table 1 Types of intervals

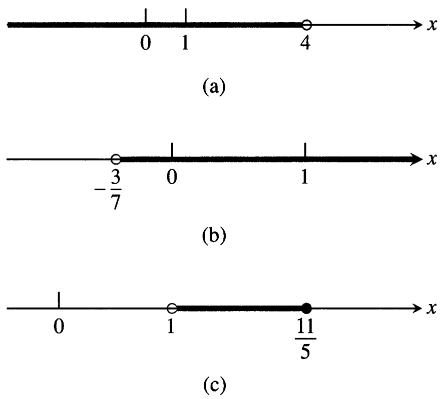
	Notation	Set	Graph
Finite:	(a, b)	$\{x \mid a < x < b\}$	
	$[a, b]$	$\{x \mid a \le x \le b\}$	
	$[a, b)$	$\{x \mid a \le x < b\}$	
	$(a, b]$	$\{x \mid a < x \le b\}$	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	
	$[a, \infty)$	$\{x \mid x \ge a\}$	
	$(-\infty, b)$	$\{x \mid x < b\}$	
	$(-\infty, b]$	$\{x \mid x \le b\}$	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

Solving Inequalities

The process of finding the interval or intervals of numbers that satisfy an inequality in x is called **solving** the inequality.

EXAMPLE 1 Solve the following inequalities and graph their solution sets on the real line.

a) $2x - 1 < x + 3$ **b)** $-\dfrac{x}{3} < 2x + 1$ **c)** $\dfrac{6}{x-1} \geq 5$



1 Solutions for Example 1.

Solution

a)

$$\begin{aligned} 2x - 1 &< x + 3 && \\ 2x &< x + 4 && \text{Add 1 to both sides.} \\ x &< 4 && \text{Subtract } x \text{ from both sides.} \end{aligned}$$

The solution set is the interval $(-\infty, 4)$ (Fig. 1a).

b)

$$\begin{aligned} -\frac{x}{3} &< 2x + 1 && \\ -x &< 6x + 3 && \text{Multiply both sides by 3.} \\ 0 &< 7x + 3 && \text{Add } x \text{ to both sides.} \\ -3 &< 7x && \text{Subtract 3 from both sides.} \\ -\frac{3}{7} &< x && \text{Divide by 7.} \end{aligned}$$

The solution set is the interval $(-3/7, \infty)$ (Fig. 1b).

c) The inequality $6/(x-1) \geq 5$ can hold only if $x > 1$, because otherwise $6/(x-1)$ is undefined or negative. Therefore, the inequality will be preserved if we multiply both sides by $(x-1)$, and we have

$$\begin{aligned} \frac{6}{x-1} &\geq 5 && \\ 6 &\geq 5x - 5 && \text{Multiply both sides by } (x-1). \\ 11 &\geq 5x && \text{Add 5 to both sides.} \\ \frac{11}{5} &\geq x. && \text{Or } x \leq \frac{11}{5}. \end{aligned}$$

The solution set is the half-open interval $(1, 11/5]$ (Fig. 1c). ❑

Absolute Value

The **absolute value** of a number x, denoted by $|x|$, is defined by the formula

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

EXAMPLE 2 $|3| = 3$, $|0| = 0$, $|-5| = -(-5) = 5$, $\big|-|a|\big| = |a|$ ❑

Notice that $|x| \geq 0$ for every real number x, and $|x| = 0$ if and only if $x = 0$.

Since the symbol $\sqrt{a}$ always denotes the *nonnegative* square root of a, an alternate definition of $|x|$ is

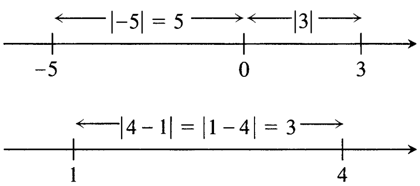
$$|x| = \sqrt{x^2}.$$

It is important to remember that $\sqrt{a^2} = |a|$. Do not write $\sqrt{a^2} = a$ unless you already know that $a \geq 0$.

Geometrically, $|x|$ represents the distance from x to the origin 0 on the real line. More generally (Fig. 2)

$$|x - y| = \text{the distance between } x \text{ and } y.$$

The absolute value has the following properties.



2 Absolute values give distances between points on the number line.

Absolute Value Properties

1. $|-a| = |a|$	A number and its negative have the same absolute value.
2. $|ab| = |a||b|$	The absolute value of a product is the product of the absolute values.
3. $\left\|\dfrac{a}{b}\right\| = \dfrac{|a|}{|b|}$	The absolute value of a quotient is the quotient of the absolute values.
4. $|a + b| \leq |a| + |b|$	**The triangle inequality** The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

If a and b differ in sign, then $|a + b|$ is less than $|a| + |b|$. In all other cases, $|a + b|$ equals $|a| + |b|$.

Notice that absolute value bars in expressions like $|-3 + 5|$ also work like parentheses: We do the arithmetic inside *before* taking the absolute value.

EXAMPLE 3

$$|-3 + 5| = |2| = 2 < |-3| + |5| = 8$$
$$|3 + 5| = |8| = |3| + |5|$$
$$|-3 - 5| = |-8| = 8 = |-3| + |-5|$$

❑

EXAMPLE 4 Solve the equation $|2x - 3| = 7$.

Solution The equation says that $2x - 3 = \pm 7$, so there are two possibilities:

$2x - 3 = 7$	$2x - 3 = -7$	Equivalent equations without absolute values
$2x = 10$	$2x = -4$	Solve as usual.
$x = 5$	$x = -2$	

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$. ❑

Inequalities Involving Absolute Values

The inequality $|a| < D$ says that the distance from a to 0 is less than D. Therefore, a must lie between D and $-D$.

The symbol $\Leftrightarrow$ means "if and only if," or "implies and is implied by."

Intervals and Absolute Values

If D is any positive number, then

$$|a| < D \quad \Leftrightarrow \quad -D < a < D, \tag{1}$$

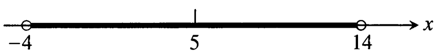
$$|a| \le D \quad \Leftrightarrow \quad -D \le a \le D. \tag{2}$$

EXAMPLE 5 Solve the inequality $|x-5| < 9$ and graph the solution set on the real line.

Solution

$$\begin{aligned} |x-5| &< 9 \\ -9 < x-5 &< 9 && \text{Eq. (1)} \\ -9+5 < x &< 9+5 && \text{Add 5 to each part to isolate } x. \\ -4 < x &< 14 \end{aligned}$$

The solution set is the open interval $(-4, 14)$ (Fig. 3). ❑



3 The solution set of the inequality $|x-5| < 9$ is the interval $(-4, 14)$ graphed here (Example 5).

EXAMPLE 6 Solve the inequality $\left|5 - \dfrac{2}{x}\right| < 1$.

Solution We have

$$\begin{aligned} \left|5-\frac{2}{x}\right| < 1 \quad &\Leftrightarrow \quad -1 < 5 - \frac{2}{x} < 1 && \text{Eq. (1)} \\ &\Leftrightarrow \quad -6 < -\frac{2}{x} < -4 && \text{Subtract 5.} \\ &\Leftrightarrow \quad 3 > \frac{1}{x} > 2 && \text{Multiply by } -\frac{1}{2}. \\ &\Leftrightarrow \quad \frac{1}{3} < x < \frac{1}{2}. && \text{Take reciprocals.} \end{aligned}$$

Notice how the various rules for inequalities were used here. Multiplying by a negative number reverses the inequality. So does taking reciprocals in an inequality in which both sides are positive. The original inequality holds if and only if $(1/3) < x < (1/2)$. The solution set is the open interval $(1/3, 1/2)$. ❑

EXAMPLE 7 Solve the inequality and graph the solution set:

a) $|2x-3| \le 1$ **b)** $|2x-3| \ge 1$

Solution

a)

$$\begin{aligned} |2x-3| &\le 1 \\ -1 \le 2x-3 &\le 1 && \text{Eq. (2)} \\ 2 \le 2x &\le 4 && \text{Add 3.} \\ 1 \le x &\le 2 && \text{Divide by 2.} \end{aligned}$$

The solution set is the closed interval $[1, 2]$ (Fig. 4a).

4 Graphs of the solution sets (a) $[1, 2]$ and (b) $(-\infty, 1] \cup [2, \infty)$ in Example 7.

Union and intersection

Notice the use of the symbol $\cup$ to denote the union of intervals. A number lies in the **union** of two sets if it lies in either set. Similarly we use the symbol $\cap$ to denote intersection. A number lies in the **intersection** $I \cap J$ of two sets if it lies in *both* sets I and J. For example, $[1, 3) \cap [2, 4] = [2, 3)$.

b)

$$|2x - 3| \geq 1$$

$$\begin{array}{lcll}
2x - 3 \geq 1 & \text{or} & -(2x - 3) \geq 1 & \\
2x - 3 \geq 1 & \text{or} & 2x - 3 \leq -1 & \text{Multiply second inequality by } -1. \\
x - \frac{3}{2} \geq \frac{1}{2} & \text{or} & x - \frac{3}{2} \leq -\frac{1}{2} & \text{Divide by 2.} \\
x \geq 2 & \text{or} & x \leq 1 & \text{Add } \frac{3}{2}.
\end{array}$$

The solution set is $(-\infty, 1] \cup [2, \infty)$ (Fig. 4b). ❑

Exercises 1

Decimal Representations

1. Express 1/9 as a repeating decimal, using a bar to indicate the repeating digits. What are the decimal representations of 2/9? 3/9? 8/9?

2. Express 1/11 as a repeating decimal, using a bar to indicate the repeating digits. What are the decimal representations of 2/11? 3/11? 9/11?

Inequalities

3. If $2 < x < 6$, which of the following statements about x are necessarily true, and which are not necessarily true?

   a) $0 < x < 4$ **b)** $0 < x - 2 < 4$
   c) $1 < \frac{x}{2} < 3$ **d)** $\frac{1}{6} < \frac{1}{x} < \frac{1}{2}$
   e) $1 < \frac{6}{x} < 3$ **f)** $|x - 4| < 2$
   g) $-6 < -x < 2$ **h)** $-6 < -x < -2$

4. If $-1 < y - 5 < 1$, which of the following statements about y are necessarily true, and which are not necessarily true?

   a) $4 < y < 6$ **b)** $-6 < y < -4$
   c) $y > 4$ **d)** $y < 6$
   e) $0 < y - 4 < 2$ **f)** $2 < \frac{y}{2} < 3$
   g) $\frac{1}{6} < \frac{1}{y} < \frac{1}{4}$ **h)** $|y - 5| < 1$

In Exercises 5–12, solve the inequalities and graph the solution sets.

5. $-2x > 4$
6. $8 - 3x \geq 5$
7. $5x - 3 \leq 7 - 3x$
8. $3(2 - x) > 2(3 + x)$
9. $2x - \frac{1}{2} \geq 7x + \frac{7}{6}$
10. $\frac{6 - x}{4} < \frac{3x - 4}{2}$
11. $\frac{4}{5}(x - 2) < \frac{1}{3}(x - 6)$
12. $-\frac{x + 5}{2} \leq \frac{12 + 3x}{4}$

Absolute Value

Solve the equations in Exercises 13–18.

13. $|y| = 3$
14. $|y - 3| = 7$
15. $|2t + 5| = 4$
16. $|1 - t| = 1$
17. $|8 - 3s| = \frac{9}{2}$
18. $\left|\frac{s}{2} - 1\right| = 1$

Solve the inequalities in Exercises 19–34, expressing the solution sets as intervals or unions of intervals. Also, graph each solution set on the real line.

19. $|x| < 2$
20. $|x| \leq 2$
21. $|t - 1| \leq 3$
22. $|t + 2| < 1$
23. $|3y - 7| < 4$
24. $|2y + 5| < 1$
25. $\left|\frac{z}{5} - 1\right| \leq 1$
26. $\left|\frac{3}{2}z - 1\right| \leq 2$
27. $\left|3 - \frac{1}{x}\right| < \frac{1}{2}$
28. $\left|\frac{2}{x} - 4\right| < 3$
29. $|2s| \geq 4$
30. $|s + 3| \geq \frac{1}{2}$
31. $|1 - x| > 1$
32. $|2 - 3x| > 5$
33. $\left|\frac{r + 1}{2}\right| \geq 1$
34. $\left|\frac{3r}{5} - 1\right| > \frac{2}{5}$

Quadratic Inequalities

Solve the inequalities in Exercises 35–42. Express the solution sets as intervals or unions of intervals and graph them. Use the result $\sqrt{a^2} = |a|$ as appropriate.

35. $x^2 < 2$
36. $4 \leq x^2$
37. $4 < x^2 < 9$
38. $\frac{1}{9} < x^2 < \frac{1}{4}$
39. $(x - 1)^2 < 4$
40. $(x + 3)^2 < 2$
41. $x^2 - x < 0$
42. $x^2 - x - 2 \geq 0$

Theory and Examples

43. Do not fall into the trap $|-a| = a$. For what real numbers a is this equation true? For what real numbers is it false?

44. Solve the equation $|x-1| = 1-x$.

45. *A proof of the triangle inequality.* Give the reason justifying each of the numbered steps in the following proof of the triangle inequality.

$$\begin{aligned}
|a+b|^2 &= (a+b)^2 && (1)\\
&= a^2+2ab+b^2 \\
&\le a^2+2|a||b|+b^2 && (2)\\
&\le |a|^2+2|a||b|+|b|^2 && (3)\\
&= (|a|+|b|)^2 \\
|a+b| &\le |a|+|b| && (4)
\end{aligned}$$

46. Prove that $|ab| = |a||b|$ for any numbers a and b.

47. If $|x| \le 3$ and $x > -1/2$, what can you say about x?

48. Graph the inequality $|x|+|y| \le 1$.

49. GRAPHER

a) Graph the functions $f(x) = x/2$ and $g(x) = 1+(4/x)$ together to identify the values of x for which

$$\frac{x}{2} > 1 + \frac{4}{x}.$$

b) Confirm your findings in (a) algebraically.

50. GRAPHER

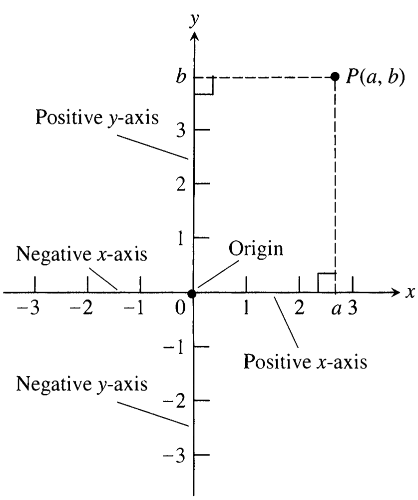
a) Graph the functions $f(x) = 3/(x-1)$ and $g(x) = 2/(x+1)$ together to identify the values of x for which

$$\frac{3}{x-1} < \frac{2}{x+1}.$$

b) Confirm your findings in (a) algebraically.

2 Coordinates, Lines, and Increments

This section reviews coordinates and lines and discusses the notion of increment.



5 Cartesian coordinates.

Cartesian Coordinates in the Plane

The positions of all points in the plane can be measured with respect to two perpendicular real lines in the plane intersecting in the 0-point of each (Fig. 5). These lines are called **coordinate axes** in the plane. On the horizontal x-axis, numbers are denoted by x and increase to the right. On the vertical y-axis, numbers are denoted by y and increase upward. The point where x and y are both 0 is the **origin** of the coordinate system, often denoted by the letter O.

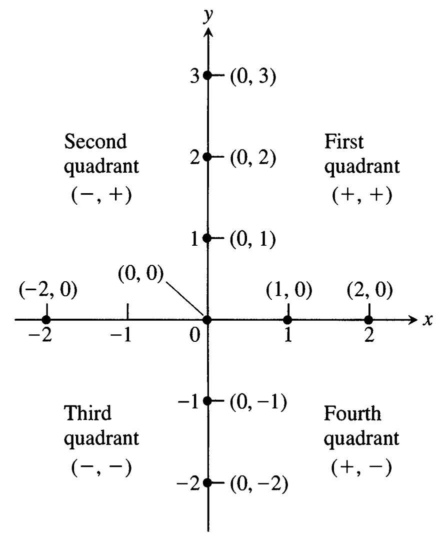
If P is any point in the plane, we can draw lines through P perpendicular to the two coordinate axes. If the lines meet the x-axis at a and the y-axis at b, then a is the **x-coordinate** of P, and b is the **y-coordinate**. The ordered pair (a, b) is the point's **coordinate pair**. The x-coordinate of every point on the y-axis is 0. The y-coordinate of every point on the x-axis is 0. The origin is the point $(0, 0)$.

The origin divides the x-axis into the **positive x-axis** to the right and the **negative x-axis** to the left. It divides the y-axis into the **positive** and **negative y-axis** above and below. The axes divide the plane into four regions called **quadrants**, numbered counterclockwise as in Fig. 6.

A Word About Scales

When we plot data in the coordinate plane or graph formulas whose variables have different units of measure, we do not need to use the same scale on the two axes. If we plot time vs. thrust for a rocket motor, for example, there is no reason to place the mark that shows 1 sec on the time axis the same distance from the origin as the mark that shows 1 lb on the thrust axis.

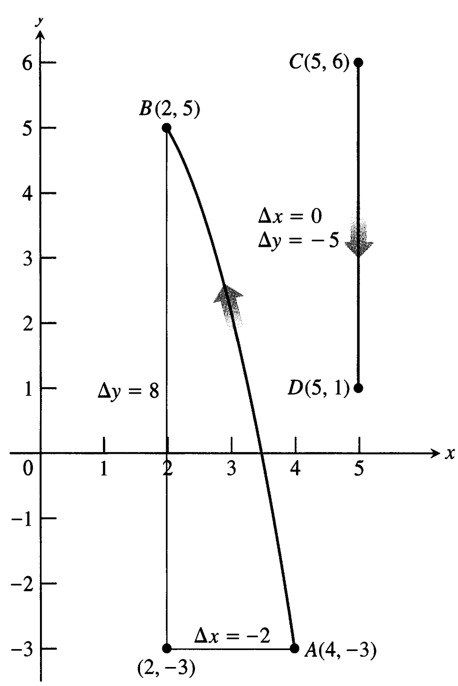
When we graph functions whose variables do not represent physical measurements and when we draw figures in the coordinate plane to study their geometry and trigonometry, we try to make the scales on the axes identical. A vertical unit



6 The points on the axes all have coordinate pairs, but we usually label them with single numbers. Notice the coordinate sign patterns in the quadrants.

of distance then looks the same as a horizontal unit. As on a surveyor's map or a scale drawing, line segments that are supposed to have the same length will look as if they do and angles that are supposed to be congruent will look congruent.

Computer displays and calculator displays are another matter. The vertical and horizontal scales on machine-generated graphs usually differ, and there are corresponding distortions in distances, slopes, and angles. Circles may look like ellipses, rectangles may look like squares, right angles may appear to be acute or obtuse, and so on. Circumstances like these require us to take extra care in interpreting what we see. High-quality computer software usually allows you to compensate for such scale problems by adjusting the *aspect ratio* (ratio of vertical to horizontal scale). Some computer screens also allow adjustment within a narrow range. When you use a grapher, try to make the aspect ratio 1, or close to it.



7 Coordinate increments may be positive, negative, or zero.

Increments and Distance

When a particle moves from one point in the plane to another, the net changes in its coordinates are called *increments*. They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point.

EXAMPLE 1 In going from the point $A(4, -3)$ to the point $B(2, 5)$ (Fig. 7), the increments in the x- and y-coordinates are

$$\Delta x = 2 - 4 = -2, \qquad \Delta y = 5 - (-3) = 8. \qquad \square$$

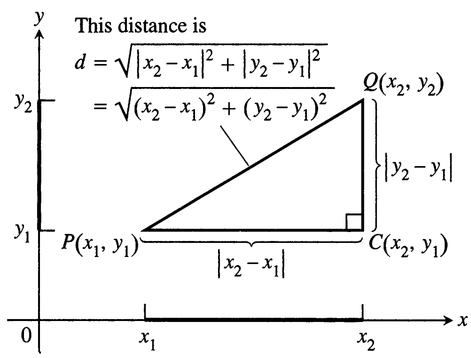
Definition

An **increment** in a variable is a net change in that variable. If x changes from x_1 to x_2, the increment in x is

$$\Delta x = x_2 - x_1.$$

EXAMPLE 2 From $C(5, 6)$ to $D(5, 1)$ (Fig. 7) the coordinate increments are

$$\Delta x = 5 - 5 = 0, \qquad \Delta y = 1 - 6 = -5. \qquad \square$$



8 To calculate the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$, apply the Pythagorean theorem to triangle *PCQ*.

The distance between points in the plane is calculated with a formula that comes from the Pythagorean theorem (Fig. 8).

Distance Formula for Points in the Plane

The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 3

a) The distance between $P(-1, 2)$ and $Q(3, 4)$ is

$$\sqrt{(3 - (-1))^2 + (4 - 2)^2} = \sqrt{(4)^2 + (2)^2} = \sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}.$$

b) The distance from the origin to $P(x, y)$ is

$$\sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}.$$

❑

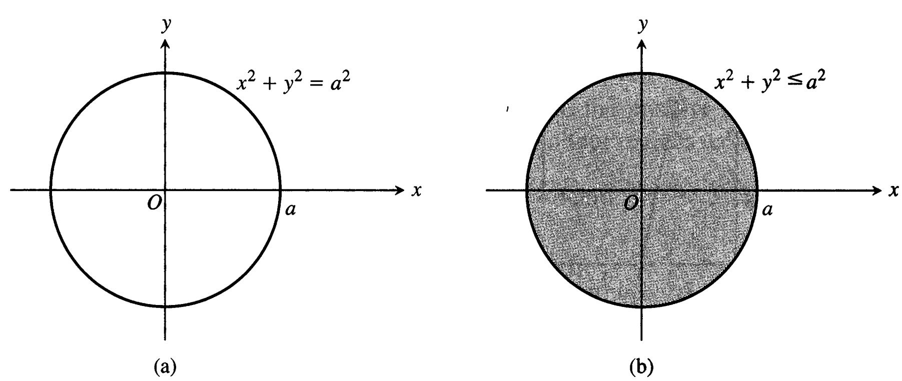
Graphs

The graph of an equation or inequality involving the variables x and y is the set of all points $P(x, y)$ whose coordinates satisfy the equation or inequality.

EXAMPLE 4 *Circles centered at the origin*

a) If $a > 0$, the equation $x^2 + y^2 = a^2$ represents all points $P(x, y)$ whose distance from the origin is $\sqrt{x^2 + y^2} = \sqrt{a^2} = a$. These points lie on the circle of radius a centered at the origin. This circle is the graph of the equation $x^2 + y^2 = a^2$ (Fig. 9a).

b) Points (x, y) whose coordinates satisfy the inequality $x^2 + y^2 \leq a^2$ all have distance $\leq a$ from the origin. The graph of the inequality is therefore the circle of radius a centered at the origin together with its interior (Fig. 9b).



9 Graphs of (a) the equation and (b) the inequality in Example 4.

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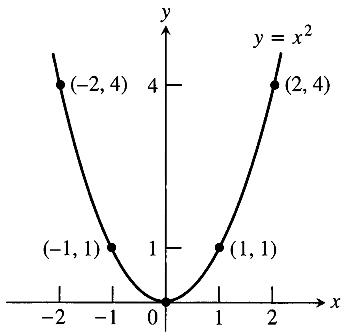
The circle of radius 1 unit centered at the origin is called the **unit circle**.



10 The parabola $y = x^2$.

EXAMPLE 5 Consider the equation $y = x^2$. Some points whose coordinates satisfy this equation are $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(2, 4)$, and $(-2, 4)$. These points (and all others satisfying the equation) make up a smooth curve called a parabola (Fig. 10). ❑

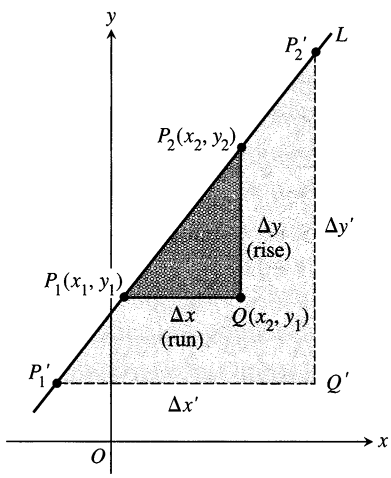
Straight Lines

Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane, we call the increments $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ the **run** and the **rise**, respectively, between P_1 and P_2. Two such points always determine a unique straight line (usually called simply a line) passing through them both. We call the line P_1P_2.

Any nonvertical line in the plane has the property that the ratio

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

has the same value for every choice of the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line (Fig. 11).



11 Triangles P_1QP_2 and $P_1'Q'P_2'$ are similar, so

$$\frac{\Delta y'}{\Delta x'} = \frac{\Delta y}{\Delta x} = m.$$

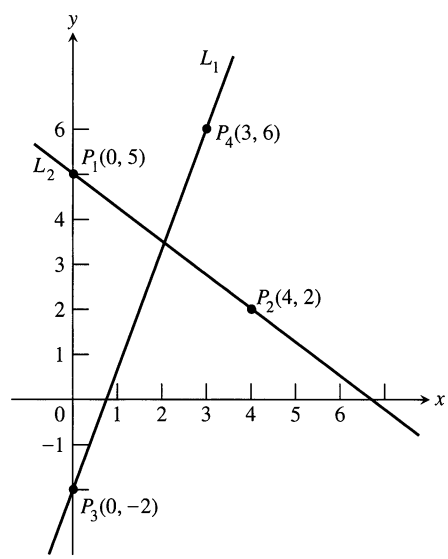
Definition

The constant

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

is the **slope** of the nonvertical line P_1P_2.

The slope tells us the direction (uphill, downhill) and steepness of a line. A line with positive slope rises uphill to the right; one with negative slope falls downhill to the right (Fig. 12). The greater the absolute value of the slope, the more rapid the rise or fall. The slope of a vertical line is *undefined.* Since the run Δx is zero for a vertical line, we cannot form the ratio *m.*



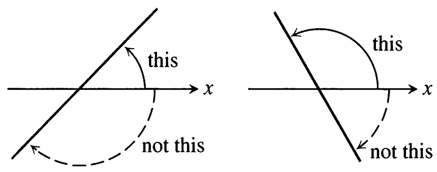
12 The slope of L_1 is

$$m = \frac{\Delta y}{\Delta x} = \frac{6 - (-2)}{3 - 0} = \frac{8}{3}.$$

That is, y increases 8 units every time x increases 3 units. The slope of L_2 is

$$m = \frac{\Delta y}{\Delta x} = \frac{2 - 5}{4 - 0} = \frac{-3}{4}.$$

That is, y decreases 3 units every time x increases 4 units.

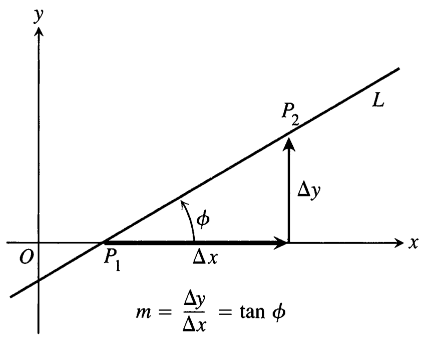


13 Angles of inclination are measured counterclockwise from the x-axis.

The direction and steepness of a line can also be measured with an angle. The **angle of inclination (inclination)** of a line that crosses the x-axis is the smallest counterclockwise angle from the x-axis to the line (Fig. 13). The inclination of a horizontal line is 0°. The inclination of a vertical line is 90°. If ϕ (the Greek letter phi) is the inclination of a line, then $0 \leq \phi < 180°$.

The relationship between the slope m of a nonvertical line and the line's inclination ϕ is shown in Fig. 14:

$$\boxed{m = \tan\phi.}$$



14 The slope of a nonvertical line is the tangent of its angle of inclination.

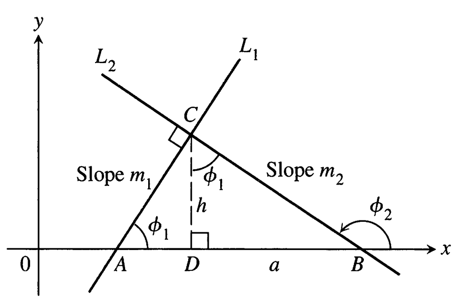
Parallel and Perpendicular Lines

Lines that are parallel have equal angles of inclination. Hence, they have the same slope (if they are not vertical). Conversely, lines with equal slopes have equal angles of inclination and so are parallel.

If two nonvertical lines L_1 and L_2 are perpendicular, their slopes m_1 and m_2 satisfy $m_1 m_2 = -1$, so each slope is the *negative reciprocal* of the other:

$$m_1 = -\frac{1}{m_2}, \qquad m_2 = -\frac{1}{m_1}.$$

The argument goes like this: In the notation of Fig. 15, $m_1 = \tan\phi_1 = a/h$, while $m_2 = \tan\phi_2 = -h/a$. Hence, $m_1 m_2 = (a/h)(-h/a) = -1$.

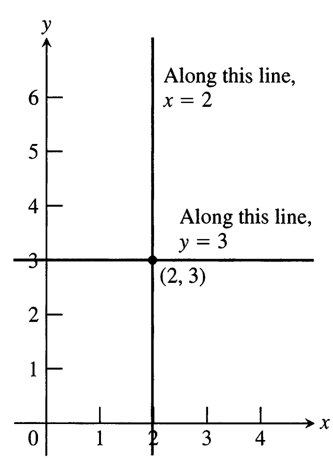


15 ΔADC is similar to ΔCDB. Hence ϕ_1 is also the upper angle in ΔCDB. From the sides of ΔCDB, we read $\tan\phi_1 = a/h$.

Equations of Lines

Straight lines have relatively simple equations. All points on the *vertical line* through the point a on the x-axis have x-coordinates equal to a. Thus, $x = a$ is an equation for the vertical line. Similarly, $y = b$ is an equation for the *horizontal line* meeting the y-axis at b.

EXAMPLE 6 The vertical and horizontal lines through the point $(2, 3)$ have equations $x = 2$ and $y = 3$, respectively (Fig. 16). ❑



16 The standard equations for the vertical and horizontal lines through (2, 3) are $x = 2$ and $y = 3$.

We can write an equation for a nonvertical straight line L if we know its slope m and the coordinates of one point $P_1(x_1, y_1)$ on it. If $P(x, y)$ is *any* other point on L, then

$$\frac{y - y_1}{x - x_1} = m,$$

so that

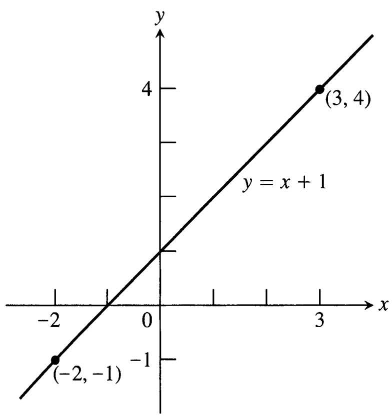
$$y - y_1 = m(x - x_1) \qquad \text{or} \qquad y = y_1 + m(x - x_1).$$

Definition

The equation

$$y = y_1 + m(x - x_1)$$

is the **point–slope equation** of the line that passes through the point (x_1, y_1) and has slope m.



16 The line in Example 8.

EXAMPLE 7 Write an equation for the line through the point (2, 3) with slope $-3/2$.

Solution We substitute $x_1 = 2$, $y_1 = 3$, and $m = -3/2$ into the point–slope equation and obtain

$$y = 3 - \frac{3}{2}(x - 2), \qquad \text{or} \qquad y = -\frac{3}{2}x + 6.$$

❑

EXAMPLE 8 Write an equation for the line through $(-2, -1)$ and $(3, 4)$.

Solution The line's slope is

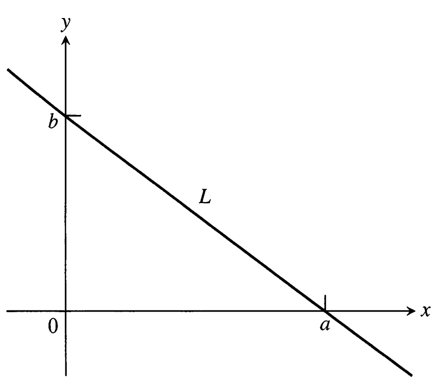
$$m = \frac{-1 - 4}{-2 - 3} = \frac{-5}{-5} = 1.$$

We can use this slope with either of the two given points in the point–slope equation:

With $(x_1, y_1) = (-2, -1)$	With $(x_1, y_1) = (3, 4)$
$y = -1 + 1 \cdot (x - (-2))$	$y = 4 + 1 \cdot (x - 3)$
$y = -1 + x + 2$	$y = 4 + x - 3$
$y = x + 1$	$y = x + 1$

Same result

Either way, $y = x + 1$ is an equation for the line (Fig. 17). ❑



17 Line L has x-intercept a and y-intercept b.

The y-coordinate of the point where a nonvertical line intersects the y-axis is called the ***y*-intercept** of the line. Similarly, the ***x*-intercept** of a nonhorizontal line is the x-coordinate of the point where it crosses the x-axis (Fig. 18). A line with slope m and y-intercept b passes through the point $(0, b)$, so it has equation

$$y = b + m(x - 0), \qquad \text{or, more simply,} \qquad y = mx + b.$$

Definition

The equation

$$y = mx + b$$

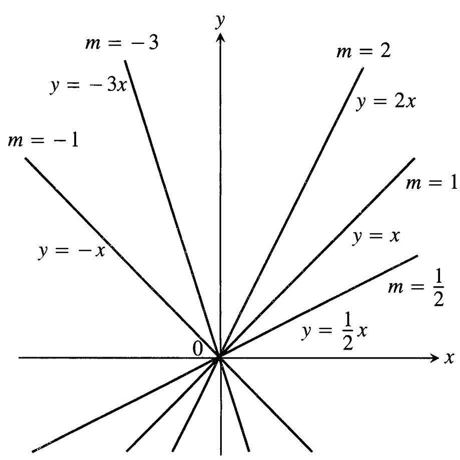
is called the **slope–intercept equation** of the line with slope m and y-intercept b.

EXAMPLE 9 The line $y = 2x - 5$ has slope 2 and y-intercept -5.

The equation

$$Ax + By = C \qquad (A \text{ and } B \text{ not both } 0)$$

is called the **general linear equation** in x and y because its graph always represents a line and every line has an equation in this form (including lines with undefined slope).



19 The line $y = mx$ has slope m and passes through the origin.

EXAMPLE 10 Find the slope and y-intercept of the line $8x + 5y = 20$.

Solution Solve the equation for y to put it in slope–intercept form. Then read the slope and y-intercept from the equation:

$$\begin{aligned} 8x + 5y &= 20 \\ 5y &= -8x + 20 \\ y &= -\frac{8}{5}x + 4. \end{aligned}$$

The slope is $m = -8/5$. The y-intercept is $b = 4$.

EXAMPLE 11 *Lines through the origin*

Lines with equations of the form $y = mx$ have y-intercept 0 and so pass through the origin. Several examples are shown in Fig. 19.

Applications—The Importance of Lines and Slopes

Light travels along lines, as do bodies falling from rest in a planet's gravitational field or coasting under their own momentum (like a hockey puck gliding across the ice). We often use the equations of lines (called **linear equations**) to study such motions.

Many important quantities are related by linear equations. Once we know that a relationship between two variables is linear, we can find it from any two pairs of corresponding values just as we find the equation of a line from the coordinates of two points.

Slope is important because it gives us a way to say how steep something is (roadbeds, roofs, stairs). The notion of slope also enables us to describe how rapidly things are changing. For this reason it will play an important role in calculus.

EXAMPLE 12 *Celsius vs. Fahrenheit*

Fahrenheit temperature (F) and Celsius temperature (C) are related by a linear equation of the form $F = mC + b$. The freezing point of water is $F = 32°$ or $C = 0°$, while the boiling point is $F = 212°$ or $C = 100°$. Thus

$$32 = 0m + b, \qquad \text{and} \qquad 212 = 100m + b,$$

so $b = 32$ and $m = (212 - 32)/100 = 9/5$. Therefore,

$$F = \frac{9}{5}C + 32, \qquad \text{or} \qquad C = \frac{5}{9}(F - 32).$$

❑

Exercises 2

Increments and Distance

In Exercises 1–4, a particle moves from A to B in the coordinate plane. Find the increments Δx and Δy in the particle's coordinates. Also find the distance from A to B.

1. $A(-3, 2)$, $B(-1, -2)$ **2.** $A(-1, -2)$, $B(-3, 2)$

3. $A(-3.2, -2)$, $B(-8.1, -2)$ **4.** $A(\sqrt{2}, 4)$, $B(0, 1.5)$

Describe the graphs of the equations in Exercises 5–8.

5. $x^2 + y^2 = 1$ **6.** $x^2 + y^2 = 2$

7. $x^2 + y^2 \le 3$ **8.** $x^2 + y^2 = 0$

Slopes, Lines, and Intercepts

Plot the points in Exercises 9–12 and find the slope (if any) of the line they determine. Also find the common slope (if any) of the lines perpendicular to line AB.

9. $A(-1, 2)$, $B(-2, -1)$ **10.** $A(-2, 1)$, $B(2, -2)$

11. $A(2, 3)$, $B(-1, 3)$ **12.** $A(-2, 0)$, $B(-2, -2)$

In Exercises 13–16, find an equation for (a) the vertical line and (b) the horizontal line through the given point.

13. $(-1, 4/3)$ **14.** $(\sqrt{2}, -1.3)$

15. $(0, -\sqrt{2})$ **16.** $(-\pi, 0)$

In Exercises 17–30, write an equation for each line described.

17. Passes through $(-1, 1)$ with slope -1

18. Passes through $(2, -3)$ with slope $1/2$

19. Passes through $(3, 4)$ and $(-2, 5)$

20. Passes through $(-8, 0)$ and $(-1, 3)$

21. Has slope $-5/4$ and y-intercept 6

22. Has slope $1/2$ and y-intercept -3

23. Passes through $(-12, -9)$ and has slope 0

24. Passes through $(1/3, 4)$ and has no slope

25. Has y-intercept 4 and x-intercept -1

26. Has y-intercept -6 and x-intercept 2

27. Passes through $(5, -1)$ and is parallel to the line $2x + 5y = 15$

28. Passes through $(-\sqrt{2}, 2)$ parallel to the line $\sqrt{2}x + 5y = \sqrt{3}$

29. Passes through (4, 10) and is perpendicular to the line $6x - 3y = 5$

30. Passes through (0, 1) and is perpendicular to the line $8x - 13y = 13$

In Exercises 31–34, find the line's x- and y-intercepts and use this information to graph the line.

31. $3x + 4y = 12$ **32.** $x + 2y = -4$

33. $\sqrt{2}x - \sqrt{3}y = \sqrt{6}$ **34.** $1.5x - y = -3$

35. Is there anything special about the relationship between the lines $Ax + By = C_1$ and $Bx - Ay = C_2$ $(A \ne 0, B \ne 0)$? Give reasons for your answer.

36. Is there anything special about the relationship between the lines $Ax + By = C_1$ and $Ax + By = C_2$ $(A \ne 0, B \ne 0)$? Give reasons for your answer.

Increments and Motion

37. A particle starts at $A(-2, 3)$ and its coordinates change by increments $\Delta x = 5$, $\Delta y = -6$. Find its new position.

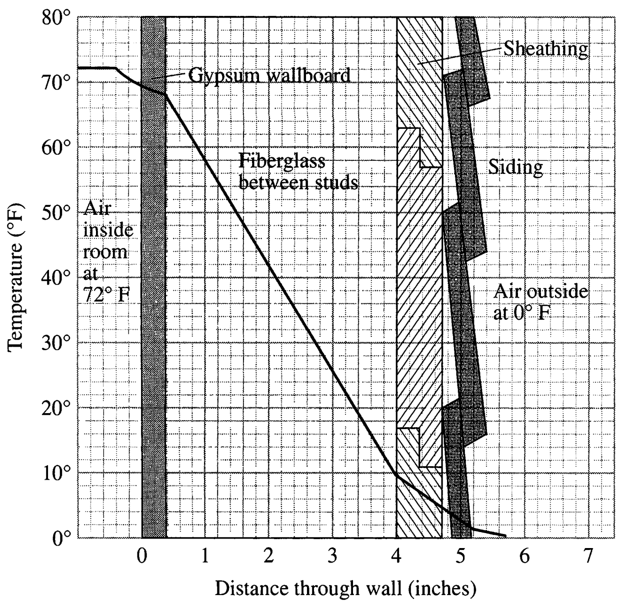
38. A particle starts at $A(6, 0)$ and its coordinates change by increments $\Delta x = -6$, $\Delta y = 0$. Find its new position.

39. The coordinates of a particle change by $\Delta x = 5$ and $\Delta y = 6$ as it moves from $A(x, y)$ to $B(3, -3)$. Find x and y.

40. A particle started at $A(1, 0)$, circled the origin once counterclockwise, and returned to $A(1, 0)$. What were the net changes in its coordinates?

Applications

41. *Insulation.* By measuring slopes in Fig. 20, estimate the temperature change in degrees per inch for (a) the gypsum wallboard; (b) the fiberglass insulation; (c) the wood sheathing. (Graphs can shift in printing, so your answers may differ slightly from those in the back of the book.)



20 The temperature changes in the wall in Exercises 41 and 42. (Source: *Differentiation,* by W. U. Walton et al., Project CALC, Education Development Center, Inc., Newton, Mass. [1975], p. 25.)

42. *Insulation.* According to Fig. 20, which of the materials in Exercise 41 is the best insulator? the poorest? Explain.

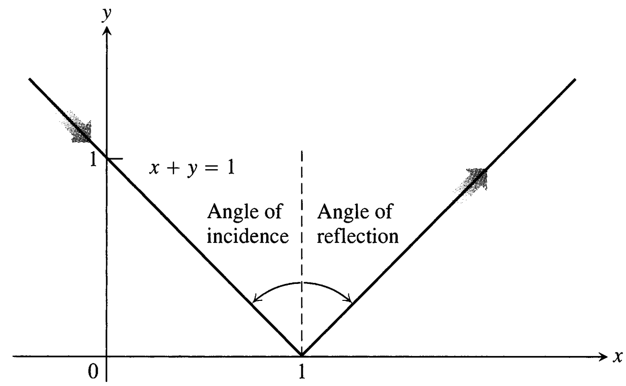
43. *Pressure under water.* The pressure p experienced by a diver under water is related to the diver's depth d by an equation of the form $p = kd + 1$ (k a constant). At the surface, the pressure is 1 atmosphere. The pressure at 100 meters is about 10.94 atmospheres. Find the pressure at 50 meters.

44. *Reflected light.* A ray of light comes in along the line $x + y = 1$ from the second quadrant and reflects off the x-axis (Fig. 21). The angle of incidence is equal to the angle of reflection. Write an equation for the line along which the departing light travels.

45. *Fahrenheit vs. Celsius.* In the FC-plane, sketch the graph of the equation

$$C = \frac{5}{9}(F - 32)$$

linking Fahrenheit and Celsius temperatures (Example 12). On the same graph sketch the line $C = F$. Is there a temperature at which a Celsius thermometer gives the same numerical reading as a Fahrenheit thermometer? If so, find it.



21 The path of the light ray in Exercise 44. Angles of incidence and reflection are measured from the perpendicular.

46. *The Mt. Washington Cog Railway.* Civil engineers calculate the slope of roadbed as the ratio of the distance it rises or falls to the distance it runs horizontally. They call this ratio the **grade** of the roadbed, usually written as a percentage. Along the coast, commercial railroad grades are usually less than 2%. In the mountains, they may go as high as 4%. Highway grades are usually less than 5%.

The steepest part of the Mt. Washington Cog Railway in New Hampshire has an exceptional 37.1% grade. Along this part of the track, the seats in the front of the car are 14 ft above those in the rear. About how far apart are the front and rear rows of seats?

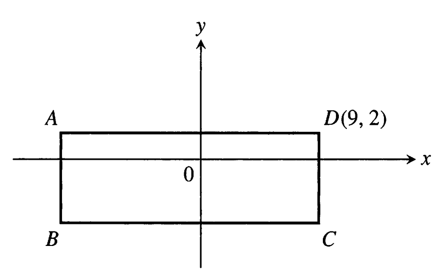
Theory and Examples

47. By calculating the lengths of its sides, show that the triangle with vertices at the points $A(1, 2)$, $B(5, 5)$, and $C(4, -2)$ is isosceles but not equilateral.

48. Show that the triangle with vertices $A(0, 0)$, $B(1, \sqrt{3})$, and $C(2, 0)$ is equilateral.

49. Show that the points $A(2, -1)$, $B(1, 3)$, and $C(-3, 2)$ are vertices of a square, and find the fourth vertex.

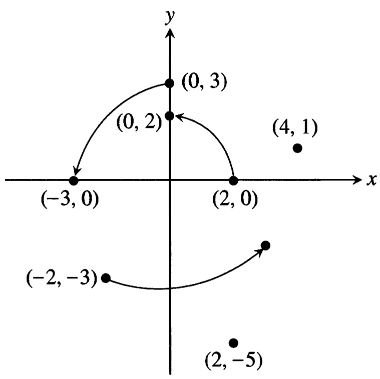
50. The rectangle shown here has sides parallel to the axes. It is three times as long as it is wide, and its perimeter is 56 units. Find the coordinates of the vertices A, B, and C.



51. Three different parallelograms have vertices at $(-1, 1)$, $(2, 0)$, and $(2, 3)$. Sketch them and find the coordinates of the fourth vertex of each.

52. A 90° rotation counterclockwise about the origin takes (2, 0) to (0, 2), and (0, 3) to (−3, 0), as shown in Fig. 22. Where does it take each of the following points?

a) (4, 1) **b)** (−2, −3) **c)** (2, −5)
d) $(x, 0)$ **e)** $(0, y)$ **f)** (x, y)
g) What point is taken to (10, 3)?



22 The points moved by the 90° rotation in Exercise 52.

53. For what value of k is the line $2x + ky = 3$ perpendicular to the line $4x + y = 1$? For what value of k are the lines parallel?

54. Find the line that passes through the point (1, 2) and through the point of intersection of the two lines $x + 2y = 3$ and $2x - 3y = -1$.

55. Show that the point with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

is the midpoint of the line segment joining $P(x_1, y_1)$ to $Q(x_2, y_2)$.

56. *The distance from a point to a line.* We can find the distance from a point $P(x_0, y_0)$ to a line $L\colon Ax + By = C$ by taking the following steps (there is a somewhat faster method in Section 10.5):

1. Find an equation for the line M through P perpendicular to L.
2. Find the coordinates of the point Q in which M and L intersect.
3. Find the distance from P to Q.

Use these steps to find the distance from P to L in each of the following cases.

a) $P(2, 1)$, $L\colon y = x + 2$
b) $P(4, 6)$, $L\colon 4x + 3y = 12$
c) $P(a, b)$, $L\colon x = -1$
d) $P(x_0, y_0)$, $L\colon Ax + By = C$

3 Functions

Functions are the major tools for describing the real world in mathematical terms. This section reviews the notion of function and discusses some of the functions that arise in calculus.

Functions

The temperature at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). The interest paid on a cash investment depends on the length of time the investment is held. In each case, the value of one variable quantity, which we might call y, depends on the value of another variable quantity, which we might call x. Since the value of y is completely determined by the value of x, we say that y is a function of x.

The letters used for variable quantities may come from what is being described. When we study circles, we usually call the area A and the radius r. Since $A = \pi r^2$, we say that A is a function of r. The equation $A = \pi r^2$ is a *rule* that tells how to calculate a *unique* (single) output value of A for each possible input value of the radius r.

The set of all possible input values for the radius is called the **domain** of the function. The set of all output values of the area is the **range** of the function. Since circles cannot have negative radii or areas, the domain and range of the circle area function are both the interval $[0, \infty)$, consisting of all nonnegative real numbers.

The domain and range of a mathematical function can be any sets of objects; they do not have to consist of numbers. Most of the domains and ranges we will encounter in this book, however, will be sets of real numbers.

Leonhard Euler (1707–1783)

Leonhard Euler, the dominant mathematical figure of his century and the most prolific mathematician who ever lived, was also an astronomer, physicist, botanist, chemist, and expert in Oriental languages. He was the first scientist to give the function concept the prominence in his work that it has in mathematics today. Euler's collected books and papers fill 70 volumes. His introductory algebra text, written originally in German (Euler was Swiss), is still read in English translation.

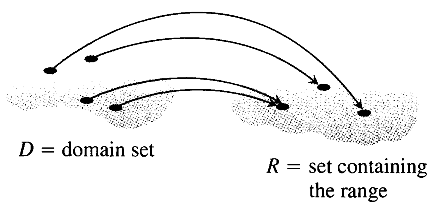
In calculus we often want to refer to a generic function without having any particular formula in mind. Euler invented a symbolic way to say "y is a function of x" by writing

$$y = f(x) \qquad (\text{"}y \text{ equals } f \text{ of } x\text{"})$$

In this notation, the symbol f represents the function. The letter x, called the **independent variable,** represents an input value from the domain of f, and y, the **dependent variable,** represents the corresponding output value $f(x)$ in the range of f. Here is the formal definition of *function.*

Definition

A **function** from a set D to a set R is a rule that assigns a *unique* element $f(x)$ in R to each element x in D.



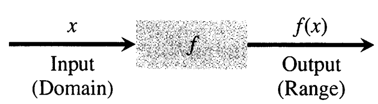
23 A function from a set D to a set R assigns a unique element of R to each element in D.

In this definition, $D = D(f)$ (read "D of f") is the domain of the function f and R is a set *containing* the range of f. See Fig. 23.

Think of a function f as a kind of machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Fig. 24).

In this book we will usually define functions in one of two ways:

1. by giving a formula such as $y = x^2$ that uses a dependent variable y to denote the value of the function, or
2. by giving a formula such as $f(x) = x^2$ that defines a function symbol f to name the function.



24 A "machine" diagram for a function.

Strictly speaking, we should call the function f and not $f(x)$, as the latter denotes the value of the function at the point x. However, as is common usage, we will often refer to the function as $f(x)$ in order to name the variable on which f depends.

It is sometimes convenient to use a single letter to denote both a function and its dependent variable. For instance, we might say that the area A of a circle of radius r is given by the function $A(r) = \pi r^2$.

Evaluation

As we said earlier, most of the functions in this book will be **real-valued functions** of a **real variable,** functions whose domains and ranges are sets of real numbers. We evaluate such functions by substituting particular values from the domain into the function's defining rule to calculate the corresponding values in the range.

EXAMPLE 1 The volume V of a ball (solid sphere) of radius r is given by the function

$$V(r) = \frac{4}{3}\pi r^3.$$

The volume of a ball of radius 3 m is

$$V(3) = \frac{4}{3}\pi(3)^3 = 36\pi \text{ m}^3.$$

❑

EXAMPLE 2 Suppose that the function F is defined for all real numbers t by the formula

$$F(t) = 2(t-1) + 3.$$

Evaluate F at the input values 0, 2, $x + 2$, and $F(2)$.

Solution In each case we substitute the given input value for t into the formula for F:

$$\begin{aligned} F(0) &= 2(0-1) + 3 = -2 + 3 = 1 \\ F(2) &= 2(2-1) + 3 = 2 + 3 = 5 \\ F(x+2) &= 2(x+2-1) + 3 = 2x + 5 \\ F(F(2)) &= F(5) = 2(5-1) + 3 = 11. \end{aligned}$$ ❑

The Domain Convention

When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly, the domain is assumed to be the largest set of x-values for which the formula gives real y-values. This is the function's so-called **natural domain.** If we want the domain to be restricted in some way, we must say so.

The domain of the function $y = x^2$ is understood to be the entire set of real numbers. The formula gives a real y-value for every real number x. If we want to restrict the domain to values of x greater than or equal to 2, we must write "$y = x^2$, $x \geq 2$."

Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$. The range of $y = x^2$, $x \geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. In symbols, the range is $\{x^2 | x \geq 2\}$ or $\{y | y \geq 4\}$ or $[4, \infty)$.

Most of the functions we encounter will have domains that are either intervals or unions of intervals.

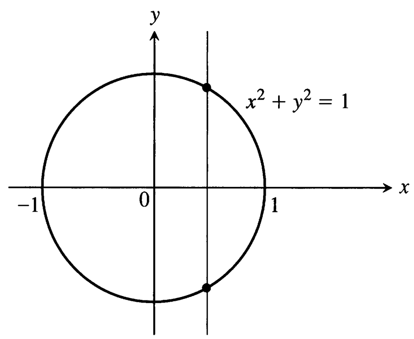
EXAMPLE 3

Function	**Domain (x)**	**Range (y)**
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = \dfrac{1}{x}$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$

The formula $y = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Beyond this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. *We cannot divide any number by zero.* The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is precisely the set of all nonzero real numbers.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).



25 This circle is not the graph of a function $y = f(x)$; it fails the vertical line test.

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all square roots of nonnegative numbers. ❑

Graphs of Functions

The **graph** of a function f is the graph of the equation $y = f(x)$. It consists of the points in the Cartesian plane whose coordinates (x, y) are input–output pairs for f.

Not every curve you draw is the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so no *vertical line* can intersect the graph of a function more than once. Thus, a circle cannot be the graph of a function since some vertical lines intersect the circle twice (Fig. 25). If a is in the domain of a function f, then the vertical line $x = a$ will intersect the graph of f in the single point $(a, f(a))$.

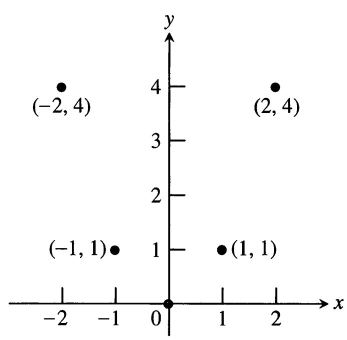
EXAMPLE 4 Graph the function $y = x^2$ over the interval $[-2, 2]$.

Solution

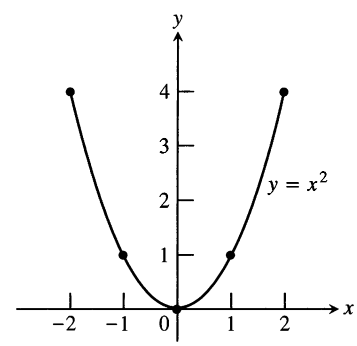
Step 1: Make a table of xy-pairs that satisfy the function rule, in this case the equation $y = x^2$.

x	$y = x^2$
−2	4
−1	1
0	0
1	1
2	4

Step 2: Plot the points (x, y) whose coordinates appear in the table.



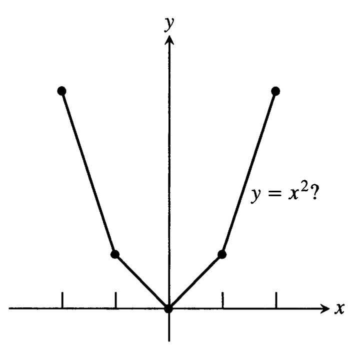
Step 3: Draw a smooth curve through the plotted points. Label the curve with its equation.

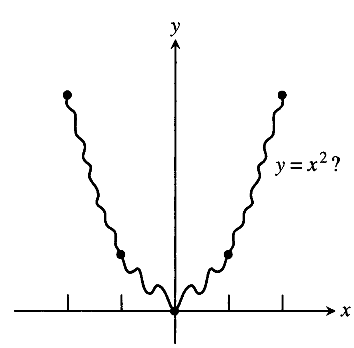


❑

How do we know that the graph of $y = x^2$ doesn't look like one of these curves?

Computers and graphing calculators graph functions in much this way—by stringing together plotted points—and the same question arises.

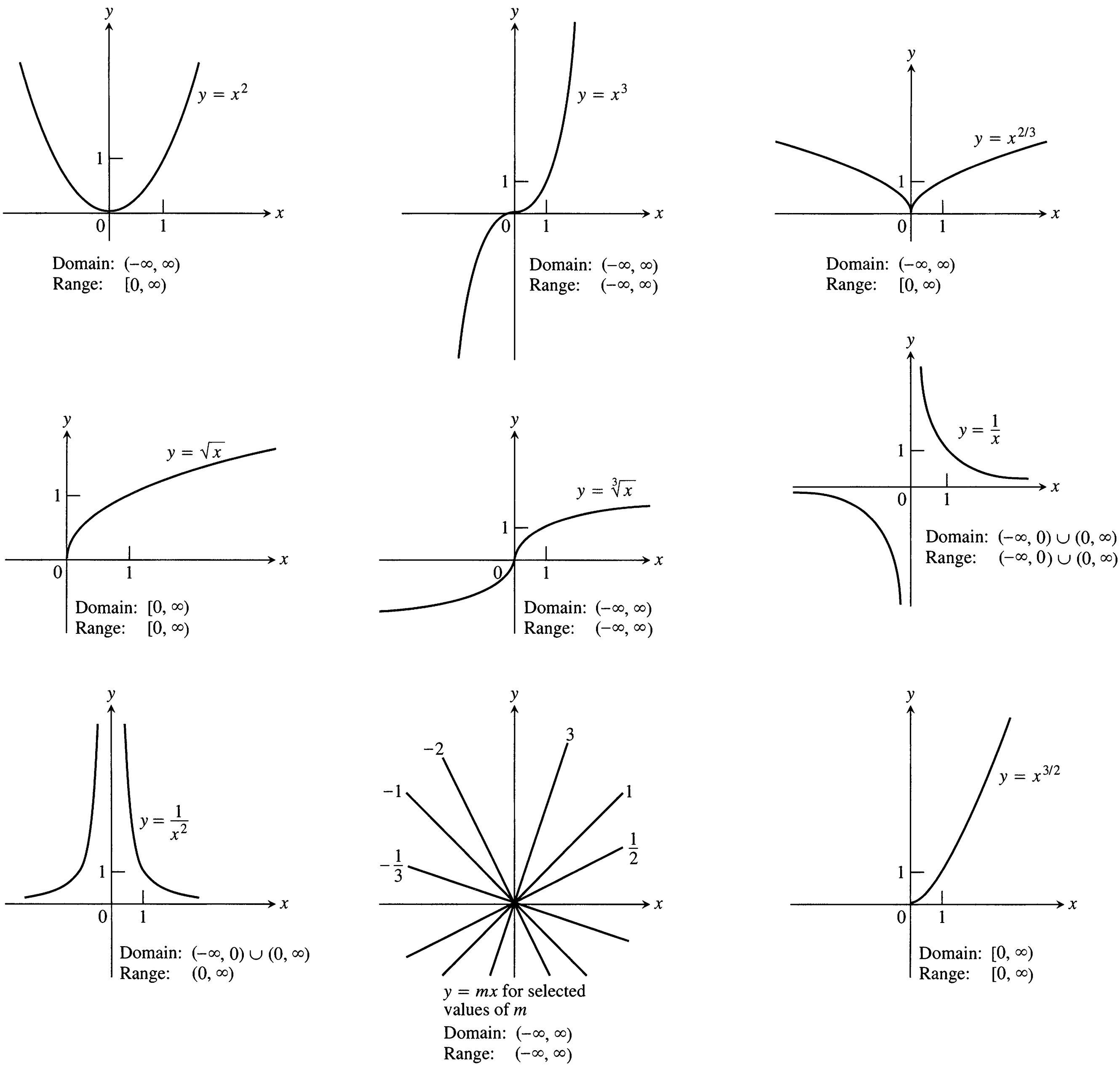




To find out, we could plot more points. But how would we then connect *them*? The basic question still remains: How do we know for sure what the graph looks like between the points we plot? The answer lies in calculus, as we will see in Chapter 3. There we will use a marvelous mathematical tool called the *derivative* to find a curve's shape between plotted points. Meanwhile we will have to settle for plotting points and connecting them as best we can.

Figure 26 shows the graphs of several functions frequently encountered in calculus. It is a good idea to learn the shapes of these graphs so that you can recognize them or sketch them when the need arises.

26 Useful graphs.



Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g, we define functions $f+g$, $f-g$, and fg by the formulas

$$(f+g)(x) = f(x)+g(x)$$
$$(f-g)(x) = f(x)-g(x)$$
$$(fg)(x) = f(x)g(x).$$

At any point of $D(f)\cap D(g)$ at which $g(x)\neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{(where } g(x)\neq 0\text{)}.$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE 5

Function	Formula	Domain
f	$f(x)=\sqrt{x}$	$[0,\infty)$
g	$g(x)=\sqrt{1-x}$	$(-\infty,1]$
$3g$	$3g(x)=3\sqrt{1-x}$	$(-\infty,1]$
$f+g$	$(f+g)(x)=\sqrt{x}+\sqrt{1-x}$	$[0,1]=D(f)\cap D(g)$
$f-g$	$(f-g)(x)=\sqrt{x}-\sqrt{1-x}$	$[0,1]$
$g-f$	$(g-f)(x)=\sqrt{1-x}-\sqrt{x}$	$[0,1]$
$f\cdot g$	$(f\cdot g)(x)=f(x)g(x)=\sqrt{x(1-x)}$	$[0,1]$
f/g	$\frac{f}{g}(x)=\frac{f(x)}{g(x)}=\sqrt{\frac{x}{1-x}}$	$[0,1)$ ($x=1$ excluded)
g/f	$\frac{g}{f}(x)=\frac{g(x)}{f(x)}=\sqrt{\frac{1-x}{x}}$	$(0,1]$ ($x=0$ excluded)

❑

Composite Functions

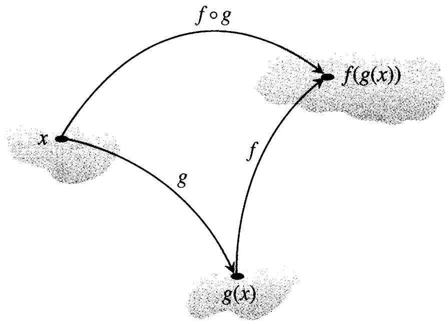
Composition is another method for combining functions.

Definition

If f and g are functions, the **composite** function $f\circ g$ ("f circle g") is defined by

$$(f\circ g)(x) = f(g(x)).$$

The domain of $f\circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.



27 The relation of $f \circ g$ to g and f.

The definition says that two functions can be composed when the range of the first lies in the domain of the second (Fig. 27). To find $(f \circ g)(x)$, we *first find* $g(x)$ and *second find* $f(g(x))$.

To evaluate the composite function $g \circ f$ (when defined), we reverse the order, finding $f(x)$ first and then $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g.

The functions $f \circ g$ and $g \circ f$ are usually quite different.

EXAMPLE 6 If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

a) $(f \circ g)(x)$ **b)** $(g \circ f)(x)$ **c)** $(f \circ f)(x)$ **d)** $(g \circ g)(x)$.

Solution

	Composite	**Domain**
a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x+1) + 1 = x + 2$	$\mathbb{R}$ or $(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, if $x \geq -1$. ❑

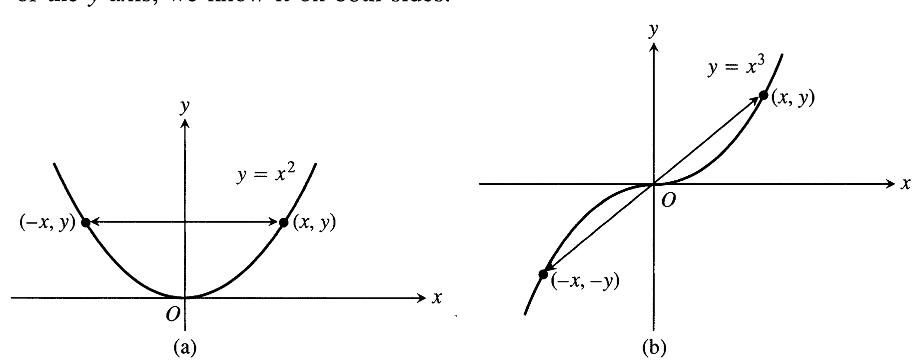
Even Functions and Odd Functions—Symmetry

A function $y = f(x)$ is **even** if $f(-x) = f(x)$ for every number x in the domain of f. Notice that this implies that both x and $-x$ must be in the domain of f. The function $f(x) = x^2$ is even because $f(-x) = (-x)^2 = x^2 = f(x)$.

The graph of an even function $y = f(x)$ is symmetric about the y-axis. Since $f(-x) = f(x)$, the point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Fig. 28a). Once we know the graph on one side of the y-axis, we automatically know it on the other side.

A function $y = f(x)$ is **odd** if $f(-x) = -f(x)$ for every number x in the domain of f. Again, both x and $-x$ must lie in the domain of f. The function $f(x) = x^3$ is odd because $f(-x) = (-x)^3 = -x^3 = -f(x)$.

The graph of an odd function $y = f(x)$ is symmetric about the origin. Since $f(-x) = -f(x)$, the point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Fig. 28b). Here again, once we know the graph of f on one side of the y-axis, we know it on both sides.



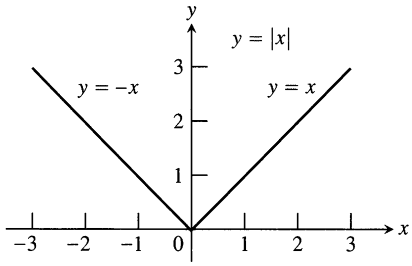
28 (a) Symmetry about the y-axis. If (x, y) is on the graph, so is $(-x, y)$. (b) Symmetry about the origin. If (x, y) is on the graph, so is $(-x, -y)$.

Piecewise Defined Functions

Sometimes a function uses different formulas on different parts of its domain. One example is the absolute value function

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

whose graph is given in Fig. 29. Here are some examples.

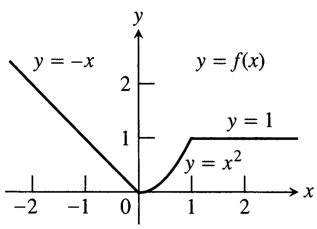


29 The absolute value function.

EXAMPLE 7 The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

is defined on the entire real line but has values given by different formulas depending on the position of x (Fig. 30). ❑



30 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 7).

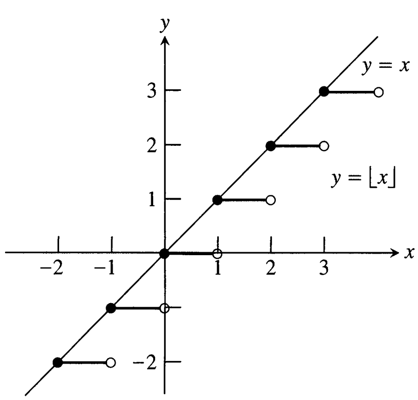
EXAMPLE 8 *The greatest integer function*

The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$, or, in some books, $[x]$ or $[[x]]$. Figure 31 shows the graph. Observe that

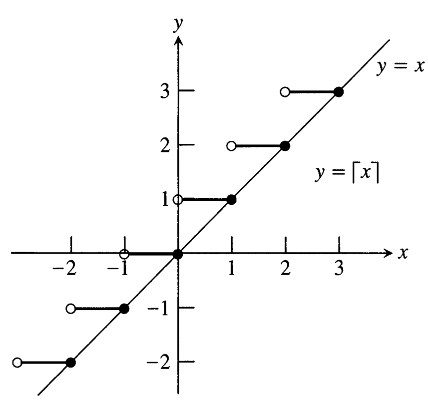
$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1 \quad \lfloor -2 \rfloor = -2.$$ ❑

EXAMPLE 9 *The least integer function*

The function whose value at any number x is the *smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Figure 32 shows the graph. For positive values of x, this function might represent, for example, the cost of parking x hours in a parking lot which charges \$1 for each hour or part of an hour.



31 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x.



32 The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for x. ❑

Exercises 3

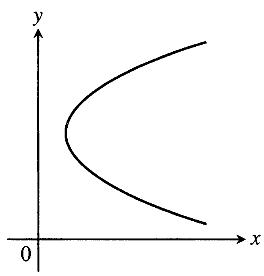
Functions

In Exercises 1–6, find the domain and range of each function.

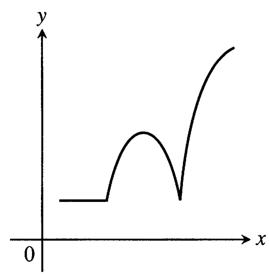
1. $f(x) = 1 + x^2$
2. $f(x) = 1 - \sqrt{x}$
3. $F(t) = \dfrac{1}{\sqrt{t}}$
4. $F(t) = \dfrac{1}{1 + \sqrt{t}}$
5. $g(z) = \sqrt{4 - z^2}$
6. $g(z) = \dfrac{1}{\sqrt{4 - z^2}}$

In Exercises 7 and 8, which of the graphs are graphs of functions of x, and which are not? Give reasons for your answers.

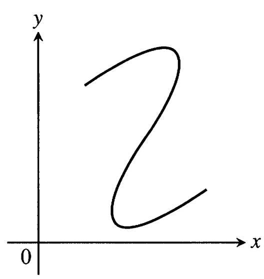
7. a)



b)



8. a)



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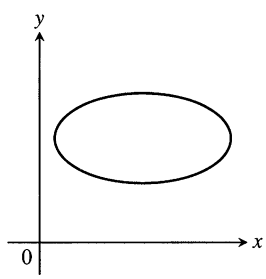


Finding Formulas for Functions

9. Express the area and perimeter of an equilateral triangle as a function of the triangle's side length x.
10. Express the side length of a square as a function of the length d of the square's diagonal. Then express the area as a function of the diagonal length.
11. Express the edge length of a cube as a function of the cube's diagonal length d. Then express the surface area and volume of the cube as a function of the diagonal length.
12. A point P in the first quadrant lies on the graph of the function $f(x) = \sqrt{x}$. Express the coordinates of P as functions of the slope of the line joining P to the origin.

Functions and Graphs

Graph the functions in Exercises 13–24. What symmetries, if any, do the graphs have? Use the graphs in Fig. 26 for guidance, as needed.

13. $y = -x^3$
14. $y = -\dfrac{1}{x^2}$
15. $y = -\dfrac{1}{x}$
16. $y = \dfrac{1}{|x|}$
17. $y = \sqrt{|x|}$
18. $y = \sqrt{-x}$
19. $y = x^3/8$
20. $y = -4\sqrt{x}$
21. $y = -x^{3/2}$
22. $y = (-x)^{3/2}$
23. $y = (-x)^{2/3}$
24. $y = -x^{2/3}$
25. Graph the following equations and explain why they are not graphs of functions of x.

    a) $|y| = x$ b) $y^2 = x^2$
26. Graph the following equations and explain why they are not graphs of functions of x.

    a) $|x| + |y| = 1$ b) $|x + y| = 1$

Even and Odd Functions

In Exercises 27–38, say whether the function is even, odd, or neither.

27. $f(x) = 3$
28. $f(x) = x^{-5}$
29. $f(x) = x^2 + 1$
30. $f(x) = x^2 + x$
31. $g(x) = x^3 + x$
32. $g(x) = x^4 + 3x^2 - 1$
33. $g(x) = \dfrac{1}{x^2 - 1}$
34. $g(x) = \dfrac{x}{x^2 - 1}$
35. $h(t) = \dfrac{1}{t - 1}$
36. $h(t) = |t^3|$
37. $h(t) = 2t + 1$
38. $h(t) = 2|t| + 1$

Sums, Differences, Products, and Quotients

In Exercises 39 and 40, find the domains and ranges of $f, g, f + g$, and $f \cdot g$.

39. $f(x) = x, \quad g(x) = \sqrt{x - 1}$
40. $f(x) = \sqrt{x + 1}, \quad g(x) = \sqrt{x - 1}$

In Exercises 41 and 42, find the domains and ranges of $f, g, f/g$, and g/f.

41. $f(x) = 2, \quad g(x) = x^2 + 1$
42. $f(x) = 1, \quad g(x) = 1 + \sqrt{x}$

Composites of Functions

43. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

a) $f(g(0))$ **b)** $g(f(0))$
c) $f(g(x))$ **d)** $g(f(x))$
e) $f(f(-5))$ **f)** $g(g(2))$
g) $f(f(x))$ **h)** $g(g(x))$

44. If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find the following.

a) $f(g(1/2))$ **b)** $g(f(1/2))$
c) $f(g(x))$ **d)** $g(f(x))$
e) $f(f(2))$ **f)** $g(g(2))$
g) $f(f(x))$ **h)** $g(g(x))$

45. If $u(x) = 4x - 5$, $v(x) = x^2$, and $f(x) = 1/x$, find formulas for the following.

a) $u(v(f(x)))$ **b)** $u(f(v(x)))$
c) $v(u(f(x)))$ **d)** $v(f(u(x)))$
e) $f(u(v(x)))$ **f)** $f(v(u(x)))$

46. If $f(x) = \sqrt{x}$, $g(x) = x/4$, and $h(x) = 4x - 8$, find formulas for the following.

a) $h(g(f(x)))$ **b)** $h(f(g(x)))$
c) $g(h(f(x)))$ **d)** $g(f(h(x)))$
e) $f(g(h(x)))$ **f)** $f(h(g(x)))$

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 47 and 48 as a composite involving one or more of f, g, h, and j.

47. a) $y = \sqrt{x} - 3$ **b)** $y = 2\sqrt{x}$
c) $y = x^{1/4}$ **d)** $y = 4x$
e) $y = \sqrt{(x-3)^3}$ **f)** $y = (2x - 6)^3$

48. a) $y = 2x - 3$ **b)** $y = x^{3/2}$
c) $y = x^9$ **d)** $y = x - 6$
e) $y = 2\sqrt{x - 3}$ **f)** $y = \sqrt{x^3 - 3}$

49. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a)	$x - 7$	$\sqrt{x}$	
b)	$x + 2$	$3x$	
c)		$\sqrt{x - 5}$	$\sqrt{x^2 - 5}$
d)	$\dfrac{x}{x-1}$	$\dfrac{x}{x-1}$	
e)		$1 + \dfrac{1}{x}$	x
f)	$\dfrac{1}{x}$		x

50. *A magic trick.* You may have heard of a magic trick that goes like this: Take any number. Add 5. Double the result. Subtract 6. Divide by 2. Subtract 2. Now tell me your answer, and I'll tell you what you started with.

Pick a number and try it.

You can see what is going on if you let x be your original number and follow the steps to make a formula $f(x)$ for the number you end up with.

Piecewise Defined Functions

Graph the functions in Exercises 51–54.

51. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

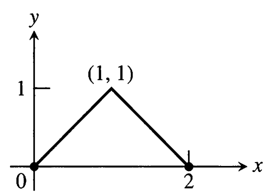
52. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

53. $F(x) = \begin{cases} 3 - x, & x \le 1 \\ 2x, & x > 1 \end{cases}$

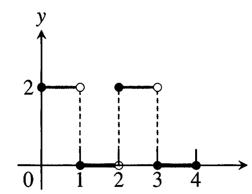
54. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \le x \end{cases}$

55. Find a formula for each function graphed.

a)

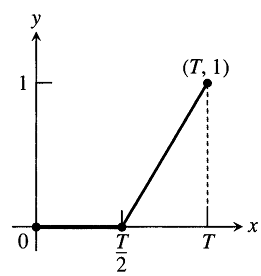


b)



56. Find a formula for each function graphed.

a)



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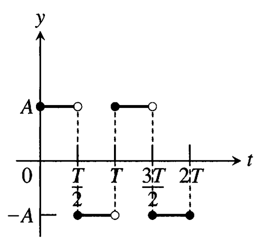


The Greatest and Least Integer Functions

57. For what values of x is (a) $\lfloor x \rfloor = 0$? (b) $\lceil x \rceil = 0$?

58. What real numbers x satisfy the equation $\lfloor x \rfloor = \lceil x \rceil$?

59. Does $\lceil -x \rceil = -\lfloor x \rfloor$ for all real x? Give reasons for your answer.

60. Graph the function

$$f(x) = \begin{cases} \lfloor x \rfloor, & x \ge 0 \\ \lceil x \rceil, & x < 0 \end{cases}$$

Why is $f(x)$ called the *integer part* of x?

Even and Odd Functions

61. Assume that f is an even function, g is an odd function, and both f and g are defined on the entire real line $\mathbb{R}$. Which of the following (where defined) are even? odd?

a) fg b) f/g c) g/f
d) $f^2 = ff$ e) $g^2 = gg$ f) $f \circ g$
g) $g \circ f$ h) $f \circ f$ i) $g \circ g$

62. Can a function be both even and odd? Give reasons for your answer.

Grapher

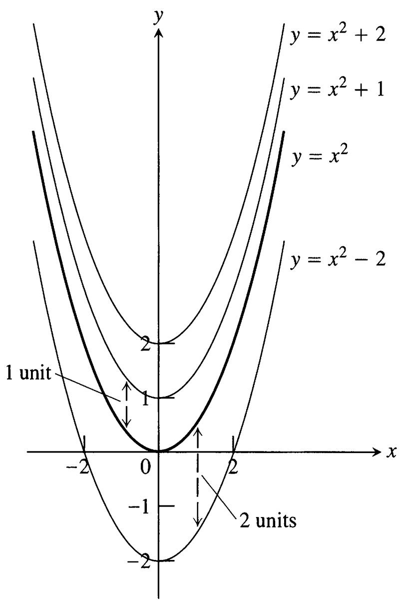
63. (*Continuation of Example 5.*) Graph the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$ together with their (a) sum, (b) product, (c) two differences, (d) two quotients.

64. Let $f(x) = x - 7$ and $g(x) = x^2$. Graph f and g together with $f \circ g$ and $g \circ f$.

4

Shifting Graphs

This section shows how to change an equation to shift its graph up or down or to the right or left. Knowing about this can help us spot familiar graphs in new locations. It can also help us graph unfamiliar equations more quickly. We practice mostly with circles and parabolas (because they make useful examples in calculus), but the methods apply to other curves as well. We will revisit parabolas and circles in Chapter 9.



33 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f.

How to Shift a Graph

To shift the graph of a function $y = f(x)$ straight up, we add a positive constant to the right-hand side of the formula $y = f(x)$.

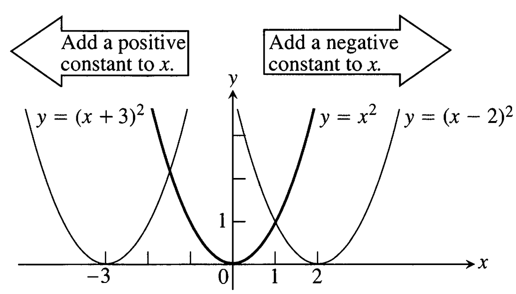
EXAMPLE 1 Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Fig. 33). ❑

To shift the graph of a function $y = f(x)$ straight down, we add a negative constant to the right-hand side of the formula $y = f(x)$.

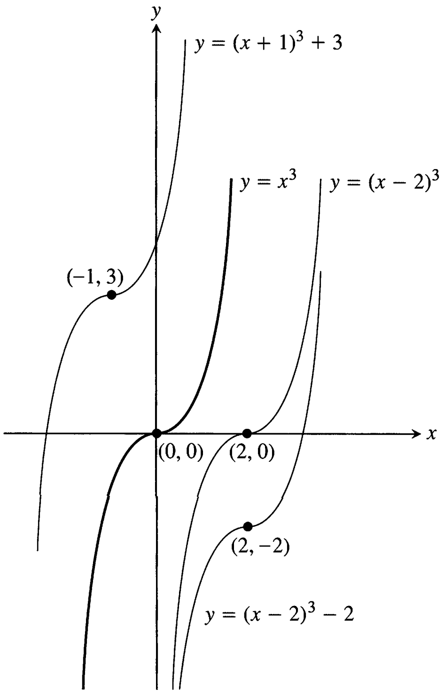
EXAMPLE 2 Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Fig. 33). ❑

To shift the graph of $y = f(x)$ to the left, we add a positive constant to x.

EXAMPLE 3 Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left (Fig. 34). ❑



34 To shift the graph of $y = x^2$ to the left, we add a positive constant to x. To shift the graph to the right, we add a negative constant to x.



35 The graph of $y = x^3$ shifted to three new positions in the xy-plane.

To shift the graph of $y = f(x)$ to the right, we add a negative constant to x.

EXAMPLE 4 Adding -2 to x in $y = x^2$ to get $y = (x - 2)^2$ shifts the graph 2 units to the right (Fig. 34). ❑

Shift Formulas

VERTICAL SHIFTS

$y - k = f(x)$ or	Shifts the graph *up* k units if $k > 0$
$y = f(x) + k$	Shifts it *down* $\lvert k \rvert$ units if $k < 0$

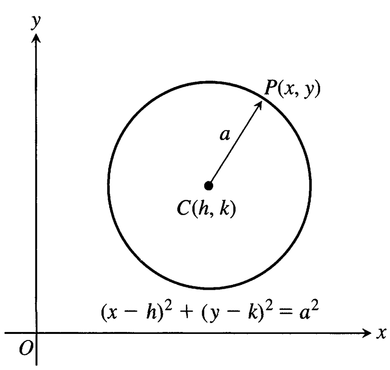
HORIZONTAL SHIFTS

$y = f(x - h)$	Shifts the graph *right* h units *if* $h > 0$
	Shifts it *left* $\lvert h \rvert$ units if $h < 0$

EXAMPLE 5 The graph of $y = (x - 2)^3 - 2$ is the graph of $y = x^3$ shifted 2 units to the right and 2 units down. The graph of $y = (x + 1)^3 + 3$ is the graph of $y = x^3$ shifted 1 unit to the left and 3 units up (Fig. 35). ❑

Equations for Circles

A **circle** is the set of points in a plane whose distance from a given fixed point in the plane is constant (Fig. 36). The fixed point is the **center** of the circle; the constant distance is the **radius**. We saw in Section 2, Example 4, that the circle of radius a centered at the origin has equation $x^2 + y^2 = a^2$. If we shift the circle to place its center at the point (h, k), its equation becomes $(x - h)^2 + (y - k)^2 = a^2$.



36 A circle of radius a in the xy-plane, with center at (h, k).

The Standard Equation for the Circle of Radius *a* Centered at the Point (*h*, *k*)

$$(x - h)^2 + (y - k)^2 = a^2 \qquad \textbf{(1)}$$

EXAMPLE 6 If the circle $x^2 + y^2 = 25$ is shifted 2 units to the left and 3 units up, its new equation is $(x + 2)^2 + (y - 3)^2 = 25$. As Eq. (1) says it should be, this is the equation of the circle of radius 5 centered at $(h, k) = (-2, 3)$. ❑

EXAMPLE 7 The standard equation for the circle of radius 2 centered at (3, 4) is

$$(x - 3)^2 + (y - 4)^2 = (2)^2$$

or

$$(x - 3)^2 + (y - 4)^2 = 4.$$

There is no need to square out the x- and y-terms in this equation. In fact, it is better not to do so. The present form reveals the circle's center and radius. ❑

EXAMPLE 8 Find the center and radius of the circle

$$(x-1)^2+(y+5)^2=3.$$

Solution Comparing

$$(x-h)^2+(y-k)^2=a^2$$

with

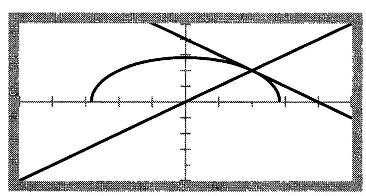
$$(x-1)^2+(y+5)^2=3$$

shows that $h=1, k=-5$, and $a=\sqrt{3}$. The center is the point $(h,k)=(1,-5)$; the radius is $a=\sqrt{3}$. ❑

Technology *Square Windows* We use the term "square window" when the units or scalings on both axes are the same. In a square window graphs are true in shape. They are distorted in a nonsquare window.

The term square window does not refer to the shape of the graphic display. Graphing calculators usually have rectangular displays. The displays of Computer Algebra Systems are usually square. When a graph is displayed, the x-unit may differ from the y-unit in order to fit the graph in the display, resulting in a distorted picture. The graphing window can be made square by shrinking or stretching the units on one axis to match the scale on the other, giving the true graph. Many systems have built-in functions to make the window "square." If yours does not, you will have to do some calculations and set the window size manually to get a square window, or bring to your viewing some foreknowledge of the true picture.

On your graphing utility, compare the perpendicular lines $y_1=x$ and $y_2=-x+4$ in a square window and a nonsquare one such as $[-10,10]$ by $[10,10]$. Graph the semicircle $y=\sqrt{8-x^2}$ in the same windows.

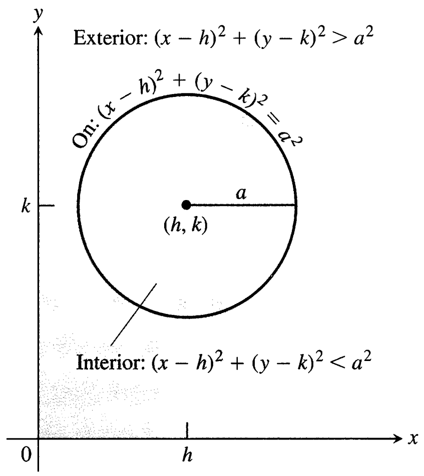


Two perpendicular lines and a semicircle graphed distorted by a rectangular window.

If an equation for a circle is not in standard form, we can find the circle's center and radius by first converting the equation to standard form. The algebraic technique for doing so is *completing the square* (see inside front cover).

EXAMPLE 9 Find the center and radius of the circle

$$x^2+y^2+4x-6y-3=0.$$



37 The interior and exterior of the circle $(x-h)^2 + (y-k)^2 = a^2$.

Solution We convert the equation to standard form by completing the squares in x and y:

$$x^2 + y^2 + 4x - 6y - 3 = 0$$

Start with the given equation.

$$(x^2 + 4x \quad) + (y^2 - 6y \quad) = 3$$

Gather terms. Move the constant to the right-hand side.

$$\left(x^2 + 4x + \left(\frac{4}{2}\right)^2\right) + \left(y^2 - 6y + \left(\frac{-6}{2}\right)^2\right) = 3 + \left(\frac{4}{2}\right)^2 + \left(\frac{-6}{2}\right)^2$$

Add the square of half the coefficient of x to each side of the equation. Do the same for y. The parenthetical expressions on the left-hand side are now perfect squares.

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) = 3 + 4 + 9$$

$$(x+2)^2 + (y-3)^2 = 16$$

Write each quadratic as a squared linear expression.

With the equation now in standard form, we read off the center's coordinates and the radius: $(h, k) = (-2, 3)$ and $a = 4$. ❑

Interior and Exterior

The points that lie inside the circle $(x-h)^2 + (y-k)^2 = a^2$ are the points less than a units from (h, k). They satisfy the inequality

$$(x-h)^2 + (y-k)^2 < a^2.$$

They make up the region we call the **interior** of the circle (Fig. 37).

The circle's **exterior** consists of the points that lie more than a units from (h, k). These points satisfy the inequality

$$(x-h)^2 + (y-k)^2 > a^2.$$

EXAMPLE 10

Inequality	Region
$x^2 + y^2 < 1$	Interior of the unit circle
$x^2 + y^2 \leq 1$	Unit circle plus its interior
$x^2 + y^2 > 1$	Exterior of the unit circle
$x^2 + y^2 \geq 1$	Unit circle plus its exterior

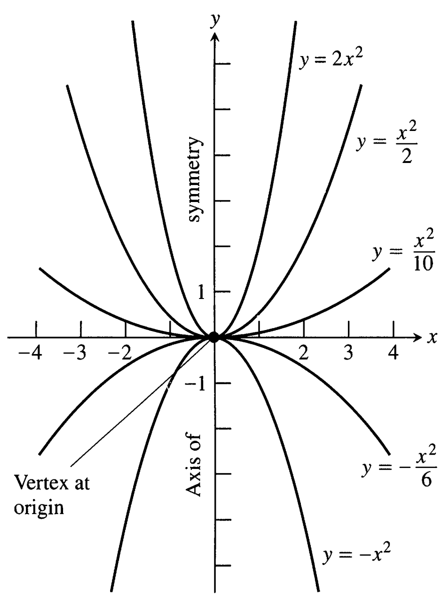
❑

Parabolic Graphs

The graph of an equation like $y = 3x^2$ or $y = -5x^2$ that has the form

$$y = ax^2$$

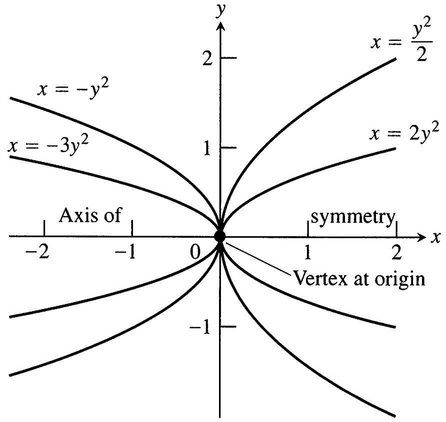
is a **parabola** whose **axis** (axis of symmetry) is the y-axis. The parabola's **vertex** (point where the parabola and axis cross) lies at the origin. The parabola opens upward if $a > 0$ and downward if $a < 0$. The larger the value of $|a|$, the narrower the parabola (Fig. 38).



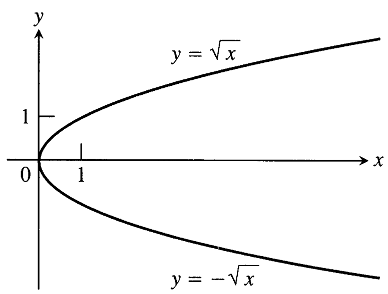
38 Besides determining the direction in which the parabola $y = ax^2$ opens, the number a is a scaling factor. The parabola widens as a approaches zero and narrows as $|a|$ becomes large.

If we interchange x and y in the formula $y = ax^2$, we obtain the equation

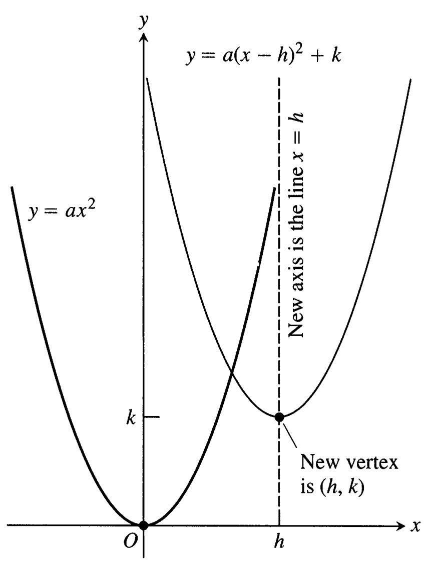
$$x = ay^2.$$



39 The parabola $x = ay^2$ is symmetric about the x-axis. It opens to the right if $a > 0$ and to the left if $a < 0$.



40 The graphs of the functions $y = \sqrt{x}$ and $y = -\sqrt{x}$ join at the origin to make the graph of the equation $x = y^2$ (Example 11).



41 The parabola $y = ax^2$, $a > 0$, shifted h units to the right and k units up.

With x and y now reversed, the graph is a parabola whose axis is the x-axis and whose vertex lies at the origin (Fig. 39).

EXAMPLE 11 The formula $x = y^2$ gives x as a function of y but does *not* give y as a function of x. If we solve for y, we find that $y = \pm\sqrt{x}$. For each positive value of x we get *two* values of y instead of the required single value.

When taken separately, the formulas $y = \sqrt{x}$ and $y = -\sqrt{x}$ do define functions of x. Each formula gives exactly one value of y for each possible value of x. The graph of $y = \sqrt{x}$ is the upper half of the parabola $x = y^2$. The graph of $y = -\sqrt{x}$ is the lower half (Fig. 40). ❑

The Quadratic Equation $y = ax^2 + bx + c$, $a \neq 0$

To shift the parabola $y = ax^2$ horizontally, we rewrite the equation as

$$y = a(x - h)^2.$$

To shift it vertically as well, we change the equation to

$$y - k = a(x - h)^2. \tag{2}$$

The combined shifts place the vertex at the point (h, k) and the axis along the line $x = h$ (Fig. 41).

Normally there would be no point in multiplying out the right-hand side of Eq. (2). In this case, however, we can learn something from doing so because the resulting equation, when rearranged, takes the form

$$y = ax^2 + bx + c. \tag{3}$$

This tells us that the graph of every equation of the form $y = ax^2 + bx + c, a \neq 0$, is the graph of $y = ax^2$ shifted somewhere else. Why? Because the steps that take us from Eq. (2) to Eq. (3) can be reversed to take us from (3) back to (2). The curve $y = ax^2 + bx + c$ has the same shape and orientation as the curve $y = ax^2$.

The axis of the parabola $y = ax^2 + bx + c$ turns out to be the line $x = -b/(2a)$. The y-intercept, $y = c$, is obtained by setting $x = 0$.

The Graph of $y = ax^2 + bx + c$, $a \neq 0$

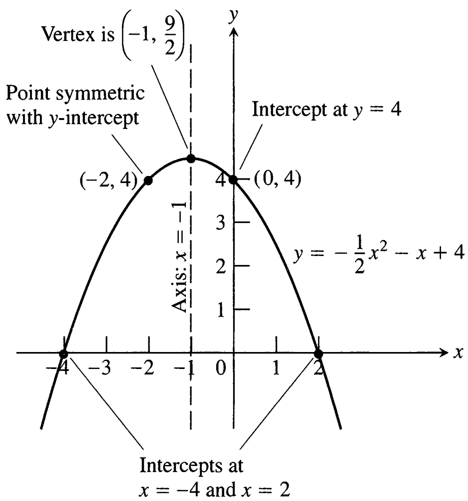
The graph of the equation $y = ax^2 + bx + c, a \neq 0$, is a parabola. The parabola opens upward if $a > 0$ and downward if $a < 0$. The axis is the line

$$x = -\frac{b}{2a}. \tag{4}$$

The vertex of the parabola is the point where the axis and parabola intersect. Its x-coordinate is $x = -b/2a$; its y-coordinate is found by substituting $x = -b/2a$ in the parabola's equation.

EXAMPLE 12 *Graphing a parabola*

Graph the equation $y = -\frac{1}{2}x^2 - x + 4$.



42 The parabola in Example 12.

Solution We take the following steps.

Step 1: *Compare the equation with $y = ax^2 + bx + c$ to identify a, b, and c.*

$$a = -\frac{1}{2}, \qquad b = -1, \qquad c = 4$$

Step 2: *Find the direction of opening.* Down, because $a < 0$.

Step 3: *Find the axis and vertex.* The axis is the line

$$x = -\frac{b}{2a} = -\frac{(-1)}{2(-1/2)} = -1, \qquad \text{Eq. (4)}$$

so the x-coordinate of the vertex is -1. The y-coordinate is

$$y = -\frac{1}{2}(-1)^2 - (-1) + 4 = \frac{9}{2}.$$

The vertex is $(-1, 9/2)$.

Step 4: *Find the x-intercepts (if any).*

$$-\frac{1}{2}x^2 - x + 4 = 0 \qquad \text{Set } y = 0 \text{ in the parabola's equation.}$$

$$x^2 + 2x - 8 = 0 \qquad \text{Solve as usual.}$$

$$(x - 2)(x + 4) = 0$$

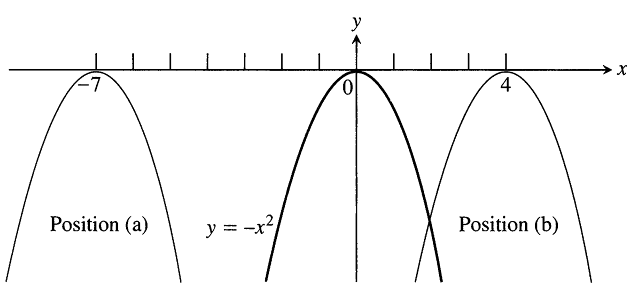
$$x = 2, \qquad x = -4$$

Step 5: *Sketch the graph.* We plot points, sketch the axis (lightly), and use what we know about symmetry and the direction of opening to complete the graph (Fig. 42). ❑

Exercises 4

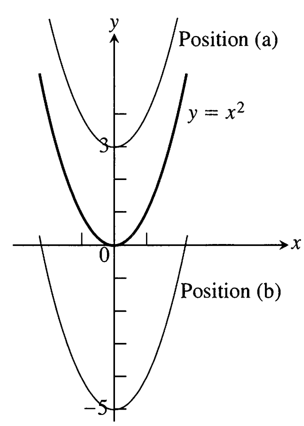
Shifting Graphs

1. Figure 43 shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.



43 The parabolas in Exercise 1.

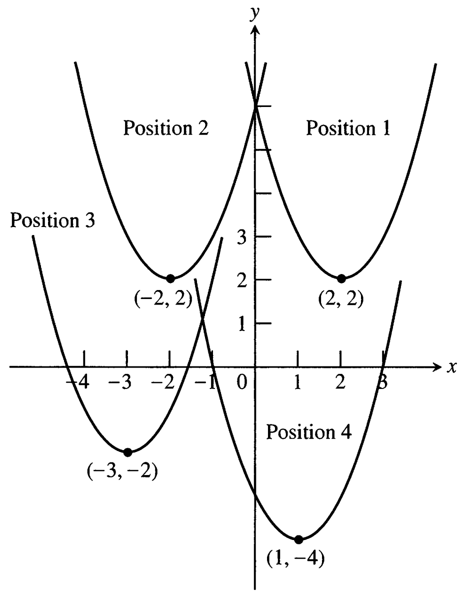
2. Figure 44 shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.



44 The parabolas in Exercise 2.

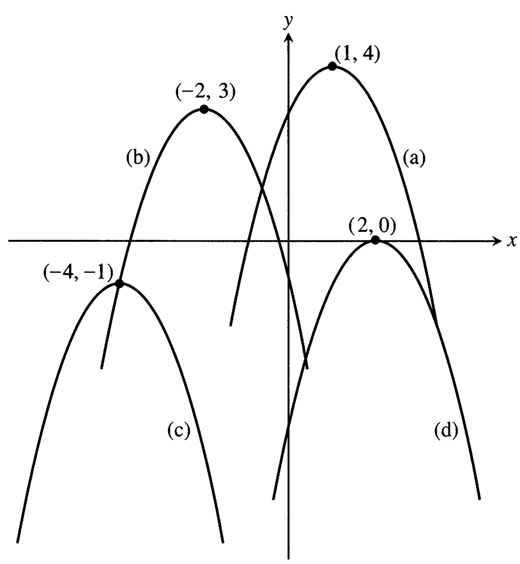
3. Match the equations listed in (a)–(d) to the graphs in Fig. 45.

a) $y = (x-1)^2 - 4$ **b)** $y = (x-2)^2 + 2$
c) $y = (x+2)^2 + 2$ **d)** $y = (x+3)^2 - 2$



45 The parabolas in Exercise 3.

4. Figure 46 shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



46 The parabolas in Exercise 4.

Exercises 5–16 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation. Use the graphs in Fig. 26 for reference as needed.

5. $x^2 + y^2 = 49$ Down 3, left 2

6. $x^2 + y^2 = 25$ Up 3, left 4

7. $y = x^3$ Left 1, down 1

8. $y = x^{2/3}$ Right 1, down 1

9. $y = \sqrt{x}$ Left 0.81

10. $y = -\sqrt{x}$ Right 3

11. $y = 2x - 7$ Up 7

12. $y = \frac{1}{2}(x+1) + 5$ Down 5, right 1

13. $x = y^2$ Left 1

14. $x = -3y^2$ Up 2, right 3

15. $y = 1/x$ Up 1, right 1

16. $y = 1/x^2$ Left 2, down 1

Graph the functions in Exercises 17–36. Use the graphs in Fig. 26 for reference as needed.

17. $y = \sqrt{x+4}$

18. $y = \sqrt{9-x}$

19. $y = |x-2|$

20. $y = |1-x| - 1$

21. $y = 1 + \sqrt{x-1}$

22. $y = 1 - \sqrt{x}$

23. $y = (x+1)^{2/3}$

24. $y = (x-8)^{2/3}$

25. $y = 1 - x^{2/3}$

26. $y + 4 = x^{2/3}$

27. $y = \sqrt[3]{x-1} - 1$

28. $y = (x+2)^{3/2} + 1$

29. $y = \dfrac{1}{x-2}$

30. $y = \dfrac{1}{x} - 2$

31. $y = \dfrac{1}{x} + 2$

32. $y = \dfrac{1}{x+2}$

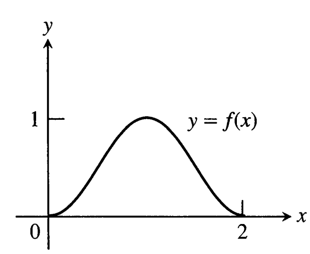
33. $y = \dfrac{1}{(x-1)^2}$

34. $y = \dfrac{1}{x^2} - 1$

35. $y = \dfrac{1}{x^2} + 1$

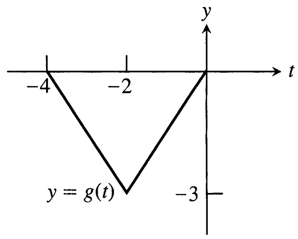
36. $y = \dfrac{1}{(x+1)^2}$

37. The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.



a) $f(x) + 2$ **b)** $f(x) - 1$
c) $2f(x)$ **d)** $-f(x)$
e) $f(x+2)$ **f)** $f(x-1)$
g) $f(-x)$ **h)** $-f(x+1) + 1$

38. The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions, and sketch their graphs.



a) $g(-t)$ **b)** $-g(t)$

c) $g(t)+3$ **d)** $1-g(t)$

e) $g(-t+2)$ **f)** $g(t-2)$

g) $g(1-t)$ **h)** $-g(t-4)$

Circles

In Exercises 39–44, find an equation for the circle with the given center $C(h, k)$ and radius a. Then sketch the circle in the xy-plane. Include the circle's center in your sketch. Also, label the circle's x- and y-intercepts, if any, with their coordinate pairs.

39. $C(0, 2), \quad a = 2$

40. $C(-3, 0), \quad a = 3$

41. $C(-1, 5), \quad a = \sqrt{10}$

42. $C(1, 1), \quad a = \sqrt{2}$

43. $C(-\sqrt{3}, -2), \quad a = 2$

44. $C(3, 1/2), \quad a = 5$

Graph the circles whose equations are given in Exercises 45–50. Label each circle's center and intercepts (if any) with their coordinate pairs.

45. $x^2 + y^2 + 4x - 4y + 4 = 0$

46. $x^2 + y^2 - 8x + 4y + 16 = 0$

47. $x^2 + y^2 - 3y - 4 = 0$

48. $x^2 + y^2 - 4x - (9/4) = 0$

49. $x^2 + y^2 - 4x + 4y = 0$

50. $x^2 + y^2 + 2x = 3$

Parabolas

Graph the parabolas in Exercises 51–58. Label the vertex, axis, and intercepts in each case.

51. $y = x^2 - 2x - 3$

52. $y = x^2 + 4x + 3$

53. $y = -x^2 + 4x$

54. $y = -x^2 + 4x - 5$

55. $y = -x^2 - 6x - 5$

56. $y = 2x^2 - x + 3$

57. $y = \frac{1}{2}x^2 + x + 4$

58. $y = -\frac{1}{4}x^2 + 2x + 4$

59. Graph the parabola $y = x - x^2$. Then find the domain and range of $f(x) = \sqrt{x - x^2}$.

60. Graph the parabola $y = 3 - 2x - x^2$. Then find the domain and range of $g(x) = \sqrt{3 - 2x - x^2}$.

Inequalities

Describe the regions defined by the inequalities and pairs of inequalities in Exercises 61–68.

61. $x^2 + y^2 > 7$

62. $x^2 + y^2 < 5$

63. $(x-1)^2 + y^2 \le 4$

64. $x^2 + (y-2)^2 \ge 4$

65. $x^2 + y^2 > 1, \quad x^2 + y^2 < 4$

66. $x^2 + y^2 \le 4, \quad (x+2)^2 + y^2 \le 4$

67. $x^2 + y^2 + 6y < 0, \quad y > -3$

68. $x^2 + y^2 - 4x + 2y > 4, \quad x > 2$

69. Write an inequality that describes the points that lie inside the circle with center $(-2, 1)$ and radius $\sqrt{6}$.

70. Write an inequality that describes the points that lie outside the circle with center $(-4, 2)$ and radius 4.

71. Write a pair of inequalities that describe the points that lie inside or on the circle with center (0, 0) and radius $\sqrt{2}$, and on or to the right of the vertical line through (1, 0).

72. Write a pair of inequalities that describe the points that lie outside the circle with center (0, 0) and radius 2, and inside the circle that has center (1, 3) and passes through the origin.

Shifting Lines

73. The line $y = mx$, which passes through the origin, is shifted vertically and horizontally to pass through the point (x_0, y_0). Find an equation for the new line. (This equation is called the line's *point–slope equation.*)

74. The line $y = mx$ is shifted vertically to pass through the point $(0, b)$. What is the new line's equation?

Intersecting Lines, Circles, and Parabolas

In Exercises 75–82, graph the two equations and find the points in which the graphs intersect.

75. $y = 2x, \quad x^2 + y^2 = 1$

76. $x + y = 1, \quad (x-1)^2 + y^2 = 1$

77. $y - x = 1, \quad y = x^2$

78. $x + y = 0, \quad y = -(x-1)^2$

79. $y = -x^2, \quad y = 2x^2 - 1$

80. $y = \frac{1}{4}x^2, \quad y = (x-1)^2$

81. $x^2 + y^2 = 1, \quad (x-1)^2 + y^2 = 1$

82. $x^2 + y^2 = 1, \quad x^2 + y = 1$

CAS Explorations and Projects

In Exercises 83–86, you will explore graphically what happens to the graph of $y = f(ax)$ as you change the value of the constant a. Use a CAS or computer grapher to perform the following steps.

a) Plot the function $y = f(x)$ together with the function $y = f(ax)$ for $a = 2, 3$, and 10 over the specified interval. Describe what happens to the graph as a increases through positive values.

b) Plot the function $y = f(x)$ and $y = f(ax)$ for the negative values $a = -2, -3$. What happens to the graph in this situation?

c) Plot the function $y = f(x)$ and $y = f(ax)$ for the fractional values $a = 1/2, 1/3, 1/4$. Describe what happens to the graph when $|a| < 1$.

83. $f(x) = \dfrac{5x}{x^2 + 4}, \quad [-10, 10]$

84. $f(x) = \dfrac{2x(x-1)}{x^2 + 1}, \quad [-3, 2]$

85. $f(x) = \dfrac{x+1}{2x^2 + 1}, \quad [-2, 2]$

86. $f(x) = \dfrac{x^4 - 4x^3 + 10}{x^2 + 4}, \quad [-1, 4]$

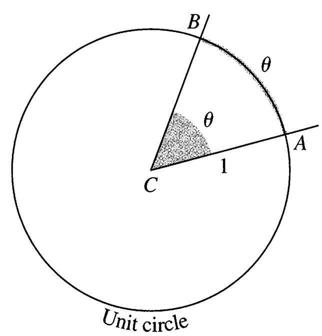
5 Trigonometric Functions

This section reviews radian measure, trigonometric functions, periodicity, and basic trigonometric identities.

Radian Measure

In navigation and astronomy, angles are measured in degrees, but in calculus it is best to use units called radians because of the way they simplify later calculations (Section 2.4).

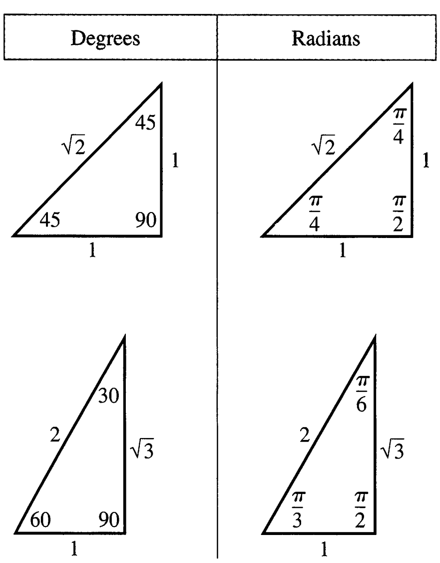
Let ACB be a central angle in a **unit circle** (circle of radius 1), as in Fig. 47.



47 The radian measure of angle *ACB* is the length of the arc *AB*.

The **radian measure** $\boldsymbol{\theta}$ of angle ACB is defined to be the length of the circular arc AB. Since the circumference of the circle is 2π and one complete revolution of a circle is 360°, the relation between radians and degrees is given by the following equation.

$$\pi \text{ radians} = 180°$$

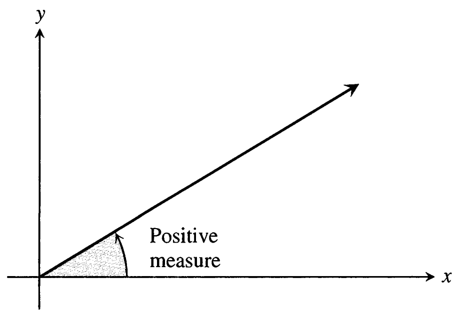


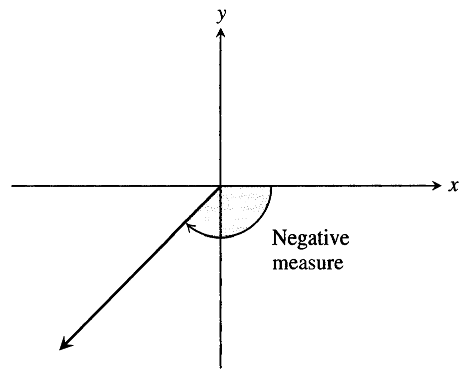
48 The angles of two common triangles, in degrees and radians.

EXAMPLE 1 *Conversions (Fig. 48)*

Convert 45° to radians: $\quad 45 \cdot \dfrac{\pi}{180} = \dfrac{\pi}{4}$ rad

Convert $\dfrac{\pi}{6}$ rad to degrees: $\quad \dfrac{\pi}{6} \cdot \dfrac{180}{\pi} = 30°$ ❑





49 Angles in standard position in the *xy*-plane.

Conversion formulas

1 degree $= \dfrac{\pi}{180}$ (≈ 0.02) radians

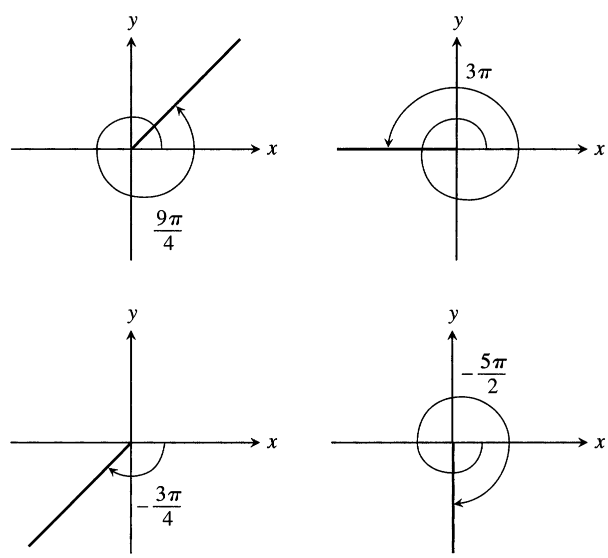
Degrees to radians: multiply by $\dfrac{\pi}{180}$

1 radian $= \dfrac{180}{\pi}$ (≈ 57) degrees

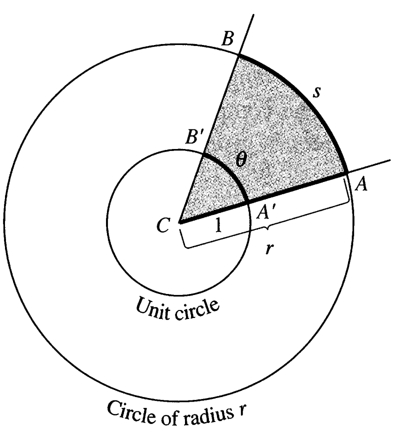
Radians to degrees: multiply by $\dfrac{180}{\pi}$

An angle in the xy-plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive x-axis (Fig. 49). Angles measured counterclockwise from the positive x-axis are assigned positive measures; angles measured clockwise are assigned negative measures.

When angles are used to describe counterclockwise rotations, our measurements can go arbitrarily far beyond 2π radians or 360°. Similarly, angles describing clockwise rotations can have negative measures of all sizes (Fig. 50).

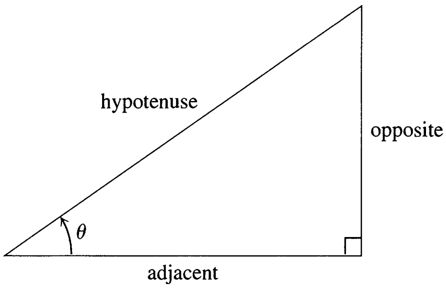


50 Nonzero radian measures can be positive or negative.



51 The radian measure of angle *ACB* is the length θ of arc $A'B'$ on the unit circle centered at *C*. The value of θ can be found from any other circle as s/r.

There is a useful relationship between the length s of an arc AB on a circle of radius r and the radian measure θ of the angle the arc subtends at the circle's center C (Fig. 51). If we draw a unit circle with the same center C, the arc $A'B'$ cut by the angle will have length θ, by the definition of radian measure. From the similarity of the circular sectors ACB and $A'CB'$, we then have $s/r = \theta/1$.

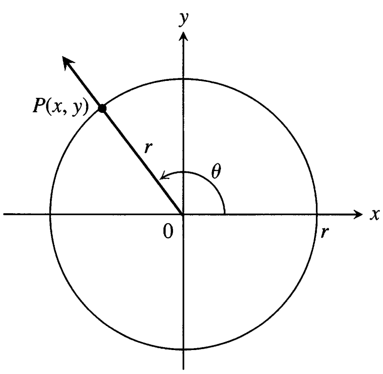


$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

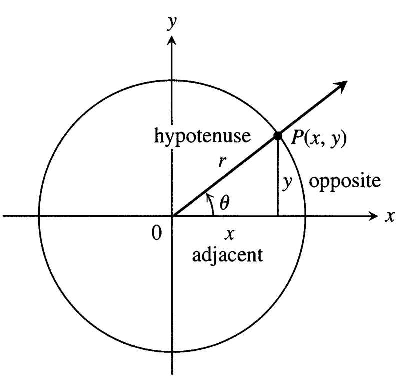
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

52 Trigonometric ratios of an acute angle.



53 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.



54 The new and old definitions agree for acute angles.

Radian Measure and Arc Length

$$\frac{s}{r} = \theta, \qquad \text{or} \qquad s = r\theta$$

Notice that these equalities hold precisely because we are measuring the angle in radians.

Angle Convention: Use Radians

From now on in this book it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. When you do calculus, keep your calculator in radian mode.

EXAMPLE 2 Consider a circle of radius 8. (a) Find the central angle subtended by an arc of length 2π on the circle. (b) Find the length of an arc subtending a central angle of $3\pi/4$.

Solution

a) $\theta = \dfrac{s}{r} = \dfrac{2\pi}{8} = \dfrac{\pi}{4}$ **b)** $s = r\theta = 8\left(\dfrac{3\pi}{4}\right) = 6\pi$ ❑

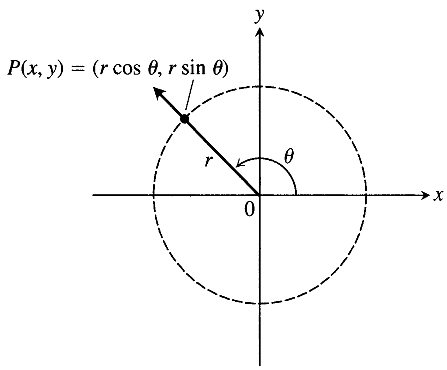
The Six Basic Trigonometric Functions

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Fig. 52). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r. We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Fig. 53).

Sine:	$\sin\theta = \dfrac{y}{r}$	Cosecant:	$\csc\theta = \dfrac{r}{y}$
Cosine:	$\cos\theta = \dfrac{x}{r}$	Secant:	$\sec\theta = \dfrac{r}{x}$
Tangent:	$\tan\theta = \dfrac{y}{x}$	Cotangent:	$\cot\theta = \dfrac{x}{y}$

These extended definitions agree with the right-triangle definitions when the angle is acute (Fig. 54).

As you can see, $\tan\theta$ and $\sec\theta$ are not defined if $x = 0$. This means they are



55 The Cartesian coordinates of a point in the plane expressed in terms of r and θ.

not defined if θ is $\pm\pi/2,\ \pm 3\pi/2, \ldots$. Similarly, $\cot\theta$ and $\csc\theta$ are not defined for values of θ for which $y = 0$, namely $\theta = 0,\ \pm\pi,\ \pm 2\pi, \ldots$.

Notice also the following definitions, whenever the quotients are defined.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

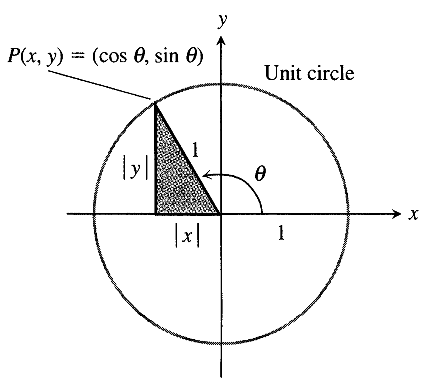
The coordinates of any point $P(x, y)$ in the plane can now be expressed in terms of the point's distance from the origin and the angle that ray OP makes with the positive x-axis (Fig. 55). Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

Values of Trigonometric Functions

If the circle in Fig. 53 has radius $r = 1$, the equations defining $\sin\theta$ and $\cos\theta$ become

$$\cos\theta = x, \qquad \sin\theta = y.$$



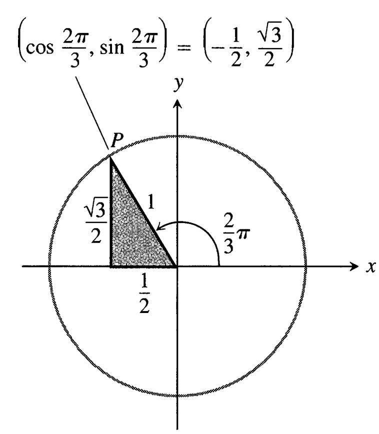
56 The acute reference triangle for an angle θ.

We can then calculate the values of the cosine and sine directly from the coordinates of P, if we happen to know them, or indirectly from the acute reference triangle made by dropping a perpendicular from P to the x-axis (Fig. 56). We read the magnitudes of x and y from the triangle's sides. The signs of x and y are determined by the quadrant in which the triangle lies.

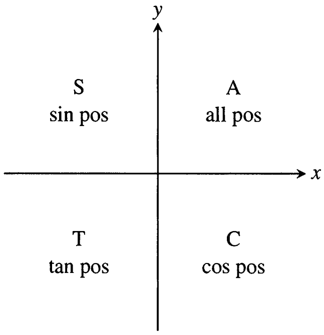
EXAMPLE 3 Find the sine and cosine of $2\pi/3$ radians.

Solution

Step 1: Draw the angle in standard position in the unit circle and write in the lengths of the sides of the reference triangle (Fig. 57).



57 The triangle for calculating the sine and cosine of $2\pi/3$ radians (Example 3).



58 The CAST rule.

Step 2: Find the coordinates of the point P where the angle's terminal ray cuts the circle:

$$\cos\frac{2\pi}{3} = x\text{-coordinate of } P = -\frac{1}{2}$$

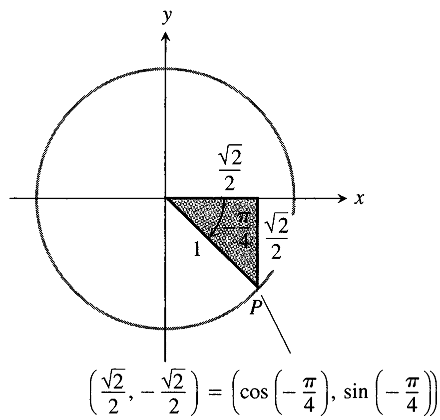
$$\sin\frac{2\pi}{3} = y\text{-coordinate of } P = \frac{\sqrt{3}}{2}. \qquad \square$$

A useful rule for remembering when the basic trigonometric functions are positive and negative is the CAST rule (Fig. 58).

EXAMPLE 4 Find the sine and cosine of $-\pi/4$ radians.

Solution

Step 1: Draw the angle in standard position in the unit circle and write in the lengths of the sides of the reference triangle (Fig. 59).



59 The triangle for calculating the sine and cosine of $-\pi/4$ radians (Example 4).

Step 2: Find the coordinates of the point P where the angle's terminal ray cuts the circle:

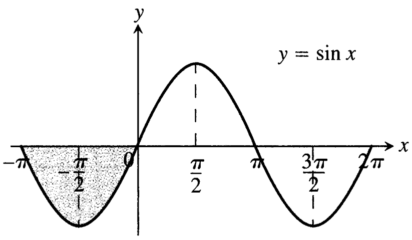
$$\cos\left(-\frac{\pi}{4}\right) = x\text{-coordinate of } P = \frac{\sqrt{2}}{2},$$

$$\sin\left(-\frac{\pi}{4}\right) = y\text{-coordinate of} P = -\frac{\sqrt{2}}{2}. \qquad \square$$

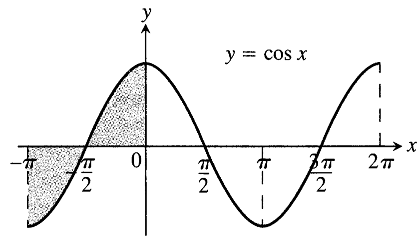
Calculations similar to those in Examples 3 and 4 allow us to fill in Table 2.

Table 2 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

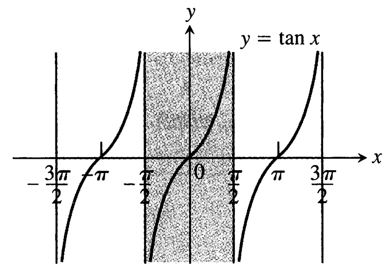
Degrees	−180	−135	−90	−45	0	30	45	60	90	135	180
θ (radians)	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$3\pi/4$	π
$\sin\theta$	0	$-\sqrt{2}/2$	−1	$-\sqrt{2}/2$	0	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{2}/2$	0
$\cos\theta$	−1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0	$-\sqrt{2}/2$	−1
$\tan\theta$	0	1		−1	0	$\sqrt{3}/3$	1	$\sqrt{3}$		−1	0



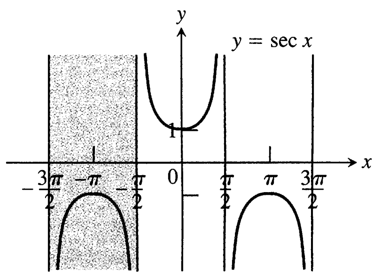
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$



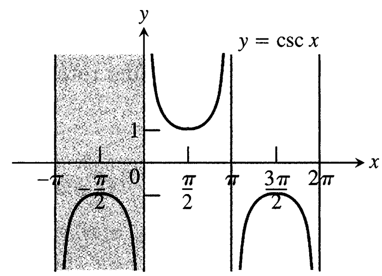
Domain: $(-\infty, \infty)$
Range: $[-1, 1]$



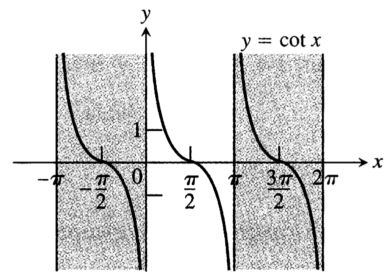
Domain: All real numbers except odd integer multiples of $\pi/2$
Range: $(-\infty, \infty)$



Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $(-\infty, -1] \cup [1, \infty)$



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $(-\infty, -1] \cup [1, \infty)$



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $(-\infty, \infty)$

60 The graphs of the six basic trigonometric functions as functions of radian measure. Each function's periodicity shows clearly in its graph.

Graphs

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ. See Fig. 60.

Periodicity

When an angle of measure x and an angle of measure $x + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric values. For example, $\cos(x + 2\pi) = \cos x$. Functions like the trigonometric functions whose values repeat at regular intervals are called periodic.

Definition

A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for all x. The smallest such value of p is the **period** of f.

Periods of trigonometric functions

Period π:

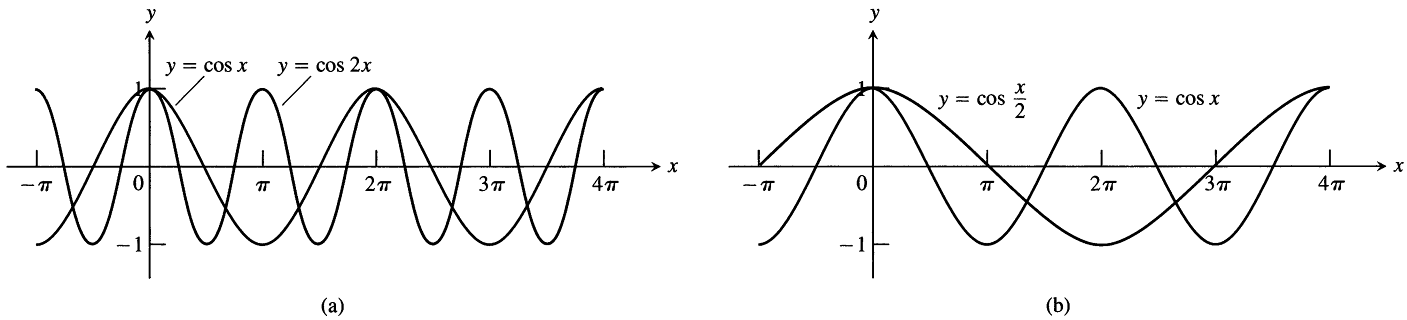
$$\tan(x + \pi) = \tan x$$
$$\cot(x + \pi) = \cot x$$

Period 2π:

$$\sin(x + 2\pi) = \sin x$$
$$\cos(x + 2\pi) = \cos x$$
$$\sec(x + 2\pi) = \sec x$$
$$\csc(x + 2\pi) = \csc x$$

As we can see in Fig. 60, the tangent and cotangent functions have period $p = \pi$. The other four functions have period 2π.

Figure 61 shows graphs of $y = \cos 2x$ and $y = \cos(x/2)$ plotted against the graph of $y = \cos x$. Multiplying x by a number greater than 1 speeds up a trigonometric function (increases the frequency) and shortens its period. Multiplying x by a positive number less than 1 slows a trigonometric function down and lengthens its period.



61 (a) Shorter period: $\cos 2x$. (b) Longer period: $\cos(x/2)$

The importance of periodic functions stems from the fact that much of the behavior we study in science is periodic. Brain waves and heartbeats are periodic, as are household voltage and electric current. The electromagnetic field that heats food in a microwave oven is periodic, as are cash flows in seasonal businesses and the behavior of rotational machinery. The seasons are periodic—so is the weather. The phases of the moon are periodic, as are the motions of the planets. There is strong evidence that the ice ages are periodic, with a period of 90,000–100,000 years.

If so many things are periodic, why limit our discussion to trigonometric functions? The answer lies in a surprising and beautiful theorem from advanced calculus that says that every periodic function we want to use in mathematical modeling can be written as an algebraic combination of sines and cosines. Thus, once we learn the calculus of sines and cosines, we will know everything we need to know to model the mathematical behavior of periodic phenomena.

Even vs. Odd

The symmetries in the graphs in Fig. 60 reveal that the cosine and secant functions are even and the other four functions are odd:

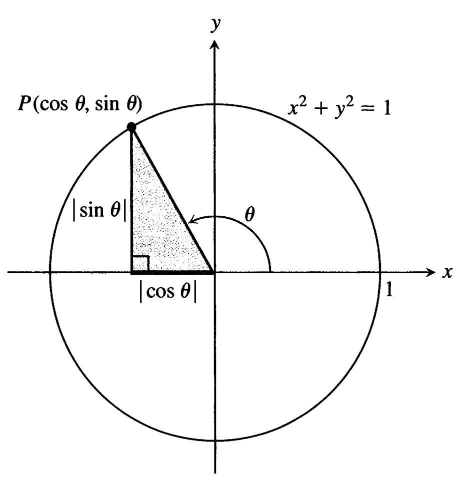
Even	**Odd**
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

Identities

Applying the Pythagorean theorem to the reference right triangle we obtain by dropping a perpendicular from the point $P(\cos\theta, \sin\theta)$ on the unit circle to the x-axis (Fig. 62) gives

$$\cos^2\theta + \sin^2\theta = 1. \tag{2}$$

This equation, true for all values of θ, is probably the most frequently used identity in trigonometry.



62 The reference triangle for a general angle θ.

Dividing Eq. (2) in turn by $\cos^2\theta$ and $\sin^2\theta$ gives the identities

$$1+\tan^2\theta = \sec^2\theta,$$
$$1+\cot^2\theta = \csc^2\theta.$$

You may recall the following identities from an earlier course.

All the trigonometric identities you will need in this book derive from Eqs. (2) and (3).

Angle Sum Formulas

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A+B) = \sin A\cos B + \cos A\sin B \tag{3}$$

These formulas hold for all angles A and B. There are similar formulas for $\cos(A-B)$ and $\sin(A-B)$ (Exercises 35 and 36).

Substituting θ for both A and B in the angle sum formulas gives two more useful identities:

Instead of memorizing Eqs. (3) you might find it helpful to remember Eqs. (4), and then recall where they came from.

Double-angle Formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta \tag{4}$$

Additional formulas come from combining the equations

$$\cos^2\theta + \sin^2\theta = 1, \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta.$$

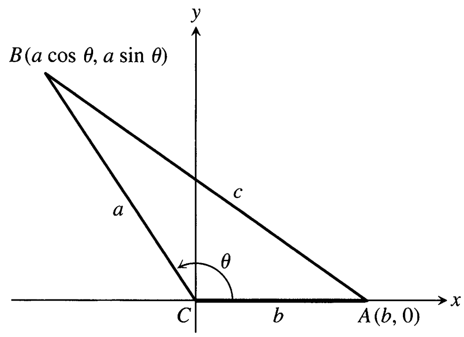
We add the two equations to get $2\cos^2\theta = 1+\cos 2\theta$ and subtract the second from the first to get $2\sin^2\theta = 1-\cos 2\theta$.

Additional Double-angle Formulas

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \tag{5}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \tag{6}$$

When θ is replaced by $\theta/2$ in Eqs. (5) and (6), the resulting formulas are called **half-angle** formulas. Some books refer to Eqs. (5) and (6) by this name as well.



63 The square of the distance between A and B gives the law of cosines.

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{7}$$

This equation is called the **law of cosines.**

We can see why the law holds if we introduce coordinate axes with the origin at C and the positive x-axis along one side of the triangle, as in Fig. 63. The coordinates of A are $(b, 0)$; the coordinates of B are $(a\cos\theta,\ a\sin\theta)$. The square of the distance between A and B is therefore

$$\begin{aligned} c^2 &= (a\cos\theta - b)^2 + (a\sin\theta)^2 \\ &= a^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + b^2 - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned}$$

Combining these equalities gives the law of cosines.

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos\theta = 0$ and $c^2 = a^2 + b^2$.

Exercises 5

Radians, Degrees, and Circular Arcs

1. On a circle of radius 10 m, how long is an arc that subtends a central angle of (a) $4\pi/5$ radians? (b) $110°$?
2. A central angle in a circle of radius 8 is subtended by an arc of length 10π. Find the angle's radian and degree measures.
3. CALCULATOR You want to make an $80°$ angle by marking an arc on the perimeter of a 12-in.-diameter disk and drawing lines from the ends of the arc to the disk's center. To the nearest tenth of an inch, how long should the arc be?
4. CALCULATOR If you roll a 1-m-diameter wheel forward 30 cm over level ground, through what angle will the wheel turn? Answer in radians (to the nearest tenth) and degrees (to the nearest degree).

Evaluating Trigonometric Functions

5. Copy and complete the table of function values shown on the following page. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

6. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

7. $\sin x = \frac{3}{5}, \quad x \text{ in } \left[\frac{\pi}{2}, \pi\right]$

8. $\tan x = 2, \quad x \text{ in } \left[0, \frac{\pi}{2}\right]$

9. $\cos x = \frac{1}{3}, \quad x \text{ in } \left[-\frac{\pi}{2}, 0\right]$

10. $\cos x = -\frac{5}{13}, \quad x \text{ in } \left[\frac{\pi}{2}, \pi\right]$

11. $\tan x = \frac{1}{2}, \quad x \text{ in } \left[\pi, \frac{3\pi}{2}\right]$

12. $\sin x = -\frac{1}{2}, \quad x \text{ in } \left[\pi, \frac{3\pi}{2}\right]$

Graphing Trigonometric Functions

Graph the functions in Exercises 13–22. What is the period of each function?

13. $\sin 2x$

14. $\sin(x/2)$

15. $\cos \pi x$

16. $\cos \frac{\pi x}{2}$

17. $-\sin \frac{\pi x}{3}$

18. $-\cos 2\pi x$

19. $\cos\left(x - \frac{\pi}{2}\right)$

20. $\sin\left(x + \frac{\pi}{2}\right)$

21. $\sin\left(x - \frac{\pi}{4}\right) + 1$

22. $\cos\left(x + \frac{\pi}{4}\right) - 1$

Graph the functions in Exercises 23–26 in the ts-plane (t-axis horizontal, s-axis vertical). What is the period of each function? What symmetries do the graphs have?

23. $s = \cot\ 2t$

24. $s = -\tan\ \pi t$

25. $s = \sec\left(\frac{\pi t}{2}\right)$

26. $s = \csc\left(\frac{t}{2}\right)$

27. GRAPHER
   a) Graph $y = \cos x$ and $y = \sec x$ together for $-3\pi/2 \le x \le 3\pi/2$. Comment on the behavior of $\sec x$ in relation to the signs and values of $\cos x$.
   b) Graph $y = \sin x$ and $y = \csc x$ together for $-\pi \le x \le 2\pi$. Comment on the behavior of $\csc x$ in relation to the signs and values of $\sin x$.

28. GRAPHER Graph $y = \tan x$ and $y = \cot x$ together for $-7 \le x \le 7$. Comment on the behavior of $\cot x$ in relation to the signs and values of $\tan x$.

29. Graph $y = \sin x$ and $y = \lfloor \sin x \rfloor$ together. What are the domain and range of $\lfloor \sin x \rfloor$?

30. Graph $y = \sin x$ and $y = \lceil \sin x \rceil$ together. What are the domain and range of $\lceil \sin x \rceil$?

Additional Trigonometric Identities

Use the angle sum formulas to derive the identities in Exercises 31–36.

31. $\cos\left(x - \frac{\pi}{2}\right) = \sin x$

32. $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$

33. $\sin\left(x + \frac{\pi}{2}\right) = \cos x$

34. $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

35. $\cos(A - B) = \cos A \cos B + \sin A \sin B$

36. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

37. What happens if you take $B = A$ in the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$? Does the result agree with something you already know?

38. What happens if you take $B = 2\pi$ in the angle sum formulas? Do the results agree with something you already know?

Using the Angle Sum Formulas

In Exercises 39–42, express the given quantity in terms of $\sin x$ and $\cos x$.

39. $\cos(\pi + x)$

40. $\sin(2\pi - x)$

41. $\sin\left(\frac{3\pi}{2} - x\right)$

42. $\cos\left(\frac{3\pi}{2} + x\right)$

43. Evaluate $\sin \frac{7\pi}{12}$ as $\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$.

44. Evaluate $\cos \dfrac{11\pi}{12}$ as $\cos\left(\dfrac{\pi}{4} + \dfrac{2\pi}{3}\right)$.

45. Evaluate $\cos \dfrac{\pi}{12}$.

46. Evaluate $\sin \dfrac{5\pi}{12}$.

Using the Double-angle Formulas

Find the function values in Exercises 47–50.

47. $\cos^2 \dfrac{\pi}{8}$

48. $\cos^2 \dfrac{\pi}{12}$

49. $\sin^2 \dfrac{\pi}{12}$

50. $\sin^2 \dfrac{\pi}{8}$

Theory and Examples

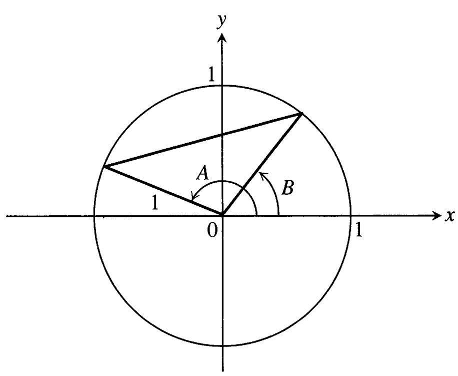
51. *The tangent sum formula.* The standard formula for the tangent of the sum of two angles is

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Derive the formula.

52. *(Continuation of Exercise 51.)* Derive a formula for $\tan(A - B)$.

53. Apply the law of cosines to the triangle in the accompanying figure to derive the formula for $\cos(A - B)$.



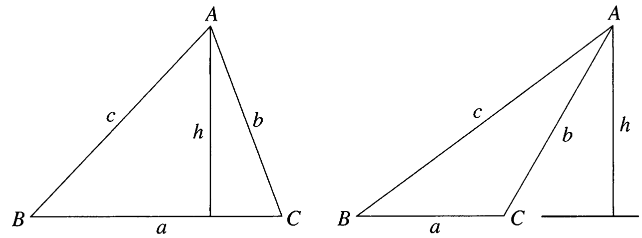
54. When applied to a figure similar to the one in Exercise 53, the law of cosines leads directly to the formula for $\cos(A + B)$. What is that figure and how does the derivation go?

55. CALCULATOR A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$. Find the length of side c.

56. CALCULATOR A triangle has sides $a = 2$ and $b = 3$ and angle $C = 40°$. Find the length of side c.

57. *The law of sines.* The **law of sines** says that if a, b, and c are the sides opposite the angles A, B, and C in a triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



Use the accompanying figures and the identity $\sin(\pi - \theta) = \sin\theta$, if required, to derive the law.

58. CALCULATOR A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$ (as in Exercise 55). Find the sine of angle B using the law of sines.

59. CALCULATOR A triangle has side $c = 2$ and angles $A = \pi/4$ and $B = \pi/3$. Find the length a of the side opposite A.

60. *The approximation $\sin x \approx x$.* It is often useful to know that, when x is measured in radians, $\sin x \approx x$ for numerically small values of x. In Section 3.7, we will see why the approximation holds. The approximation error is less than 1 in 5000 if $|x| < 0.1$.

a) With your grapher in radian mode, graph $y = \sin x$ and $y = x$ together in a viewing window about the origin. What do you see happening as x nears the origin?

b) With your grapher in degree mode, graph $y = \sin x$ and $y = x$ together about the origin again. How is the picture different from the one obtained with radian mode?

c) *A quick radian mode check.* Is your calculator in radian mode? Evaluate $\sin x$ at a value of x near the origin, say $x = 0.1$. If $\sin x \approx x$, the calculator is in radian mode; if not, it isn't. Try it.

General Sine Curves

Figure 64 on the following page shows the graph of a **general sine function** of the form

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

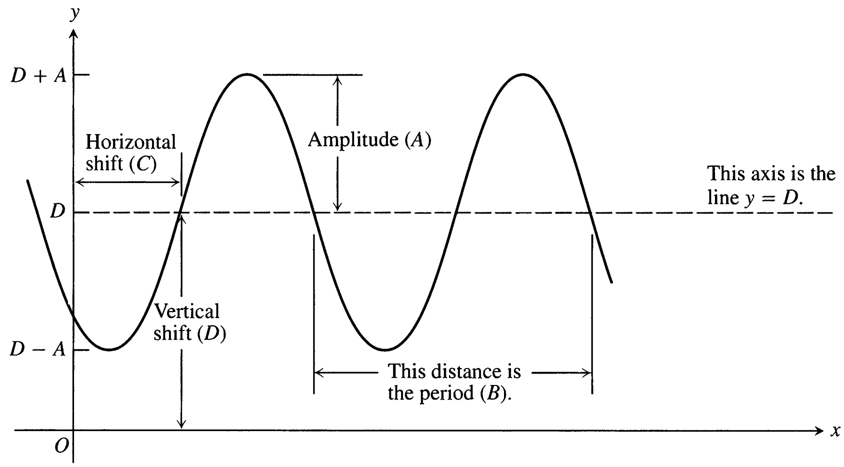
where $|A|$ is the *amplitude,* $|B|$ is the *period,* C is the *horizontal shift,* and D is the *vertical shift.* Identify A, B, C, and D for the sine functions in Exercises 61–64 and sketch their graphs.

61. $y = 2\sin(x + \pi) - 1$

62. $y = \dfrac{1}{2}\sin(\pi x - \pi) + \dfrac{1}{2}$

63. $y = -\dfrac{2}{\pi}\sin\left(\dfrac{\pi}{-2}t\right) + \dfrac{1}{\pi}$

64. $y = \dfrac{L}{2\pi}\sin\dfrac{2\pi t}{L}, \quad L > 0$



64 The general sine curve

$$y = A \sin [(2\pi/B)(x - C)] + D,$$

shown for *A, B, C,* and *D* positive.

The Trans-Alaska Pipeline

The builders of the Trans-Alaska Pipeline used insulated pads to keep the heat from the hot oil in the pipeline from melting the permanently frozen soil beneath. To design the pads, it was necessary to take into account the variation in air temperature throughout the year. Figure 65 shows how we can use a general sine function, defined in the introduction to Exercises 61–64, to represent temperature data. The data points in the figure are plots of the mean air temperature for Fairbanks, Alaska, based on records of the National Weather Service from 1941 to 1970. The sine function used to fit the data is

$$f(x) = 37 \sin\left(\frac{2\pi}{365}(x - 101)\right) + 25,$$

where f is temperature in degrees Fahrenheit and x is the number of the day counting from the beginning of the year. The fit is remarkably good.

65. *Temperature in Fairbanks, Alaska.* Find the (a) amplitude, (b) period, (c) horizontal shift, and (d) vertical shift of the general sine function

$$f(x) = 37 \sin\left(\frac{2\pi}{365}(x - 101)\right) + 25.$$

66. *Temperature in Fairbanks, Alaska.* Use the equation in Exercise 65 to approximate the answers to the following questions about the temperature in Fairbanks, Alaska, shown in Fig. 65. Assume that the year has 365 days.

a) What are the highest and lowest mean daily temperatures shown?

b) What is the average of the highest and lowest mean daily temperatures shown? Why is this average the vertical shift of the function?

CAS Explorations and Projects

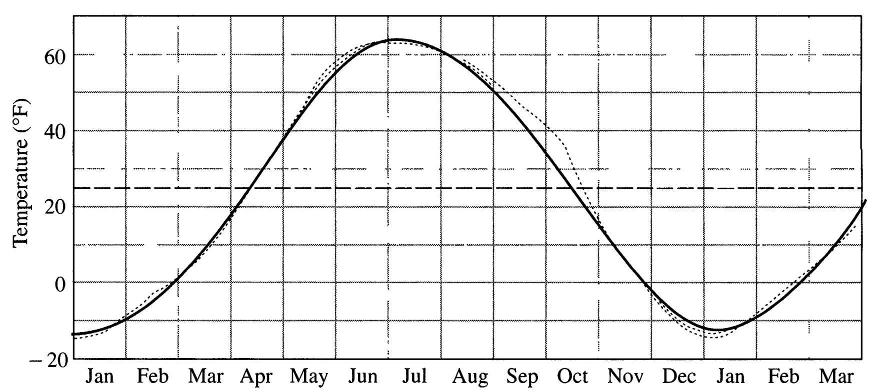
In Exercises 67–70, you will explore graphically the general sine function

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D$$

as you change the values of the constants A, B, C, and D. Use a CAS or computer grapher to perform the steps in the exercises.

67. *The period B.* Set the constants $A = 3, C = D = 0$.

a) Plot $f(x)$ for the values $B = 1,\ 3,\ 2\pi,\ 5\pi$ over the interval



65 Normal mean air temperature at Fairbanks, Alaska, plotted as data points. The approximating sine function is

$$f(x) = 37 \sin\left(\frac{2\pi}{365}(x - 101)\right) + 25.$$

(Source: "Is the Curve of Temperature Variation a Sine Curve?" by B. M. Lando and C. A. Lando, *The Mathematics Teacher,* 7:6, Fig. 2, p. 535 [September 1977].)

$-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as the period increases.

b) What happens to the graph for negative values of B? Try it with $B = -3$ and $B = -2\pi$.

68. *The horizontal shift C.* Set the constants $A = 3$, $B = 6$, $D = 0$.

a) Plot $f(x)$ for the values $C = 0$, 1, and 2 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as C increases through positive values.

b) What happens to the graph for negative values of C?

c) What smallest positive value should be assigned to C so the graph exhibits no horizontal shift? Confirm your answer with a plot.

69. *The vertical shift D.* Set the constants $A = 3$, $B = 6$, $C = 0$.

a) Plot $f(x)$ for the values $D = 0$, 1, and 3 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as D increases through positive values.

b) What happens to the graph for negative values of D?

70. *The amplitude A.* Set the constants $B = 6$, $C = D = 0$.

a) Describe what happens to the graph of the general sine function as A increases through positive values. Confirm your answer by plotting $f(x)$ for the values $A = 1$, 5, and 9.

b) What happens to the graph for negative values of A?

PRELIMINARIES — QUESTIONS TO GUIDE YOUR REVIEW

1. What are the order properties of the real numbers? How are they used in solving inequalities?
2. What is a number's absolute value? Give examples. How are $|-a|$, $|ab|$, $|a/b|$, and $|a+b|$ related to $|a|$ and $|b|$?
3. How are absolute values used to describe intervals or unions of intervals? Give examples.
4. How do you find the distance between two points in the coordinate plane?
5. How can you write an equation for a line if you know the coordinates of two points on the line? the line's slope and the coordinates of one point on the line? the line's slope and y-intercept? Give examples.
6. What are the standard equations for lines perpendicular to the coordinate axes?
7. How are the slopes of mutually perpendicular lines related? What about parallel lines? Give examples.
8. When a line is not vertical, what is the relation between its slope and its angle of inclination?
9. What is a function? Give examples. How do you graph a real-valued function of a real variable?
10. Name some typical algebraic and trigonometric functions and draw their graphs.
11. What is an even function? an odd function? What geometric properties do the graphs of such functions have? What advantage can we take of this? Give an example of a function that is neither even nor odd. What, if anything, can you say about sums, products, quotients, and composites involving even and odd functions?
12. If f and g are real-valued functions, how are the domains of $f+g$, $f-g$, fg, and f/g related to the domains of f and g? Give examples.
13. When is it possible to compose one function with another? Give examples of composites and their values at various points. Does the order in which functions are composed ever matter?
14. How do you change the equation $y = f(x)$ to shift its graph up or down? to the left or right? Give examples.
15. Describe the steps you would take to graph the circle $x^2 + y^2 + 4x - 6y + 12 = 0$.
16. If a, b, and c are constants and $a \ne 0$, what can you say about the graph of the equation $y = ax^2 + bx + c$? In particular, how would you go about sketching the curve $y = 2x^2 + 4x$?
17. What inequality describes the points in the coordinate plane that lie inside the circle of radius a centered at the point (h, k)? that lie inside or on the circle? that lie outside the circle? that lie outside or on the circle?
18. What is radian measure? How do you convert from radians to degrees? degrees to radians?
19. Graph the six basic trigonometric functions. What symmetries do the graphs have?
20. How can you sometimes find the values of trigonometric functions from triangles? Give examples.
21. What is a periodic function? Give examples. What are the periods of the six basic trigonometric functions?
22. Starting with the identity $\cos^2\theta + \sin^2\theta = 1$ and the formulas for $\cos(A+B)$ and $\sin(A+B)$, show how a variety of other trigonometric identities may be derived.

PRELIMINARIES — PRACTICE EXERCISES

Geometry

1. A particle in the plane moved from $A(-2, 5)$ to the y-axis in such a way that Δy equaled $3\,\Delta x$. What were the particle's new coordinates?
2. a) Plot the points $A(8, 1)$, $B(2, 10)$, $C(-4, 6)$, $D(2, -3)$, and $E(14/3, 6)$.
   b) Find the slopes of the lines *AB, BC, CD, DA, CE,* and *BD.*
   c) Do any four of the five points *A, B, C, D,* and *E* form a parallelogram?
   d) Are any three of the five points collinear? How do you know?
   e) Which of the lines determined by the five points pass through the origin?
3. Do the points $A(6, 4)$, $B(4, -3)$, and $C(-2, 3)$ form an isosceles triangle? a right triangle? How do you know?
4. Find the coordinates of the point on the line $y = 3x + 1$ that is equidistant from (0, 0) and (−3, 4).

Functions and Graphs

5. Express the area and circumference of a circle as functions of the circle's radius. Then express the area as a function of the circumference.
6. Express the radius of a sphere as a function of the sphere's surface area. Then express the surface area as a function of the volume.
7. A point P in the first quadrant lies on the parabola $y = x^2$. Express the coordinates of P as functions of the angle of inclination of the line joining P to the origin.
8. A hot-air balloon rising straight up from a level field is tracked by a range finder located 500 ft from the point of lift-off. Express the balloon's height as a function of the angle the line from the range finder to the balloon makes with the ground.

Composition with absolute values. In Exercises 9–14, graph f_1 and f_2 together. Then describe how applying the absolute value function before applying f_1 affects the graph.

	$f_1(x)$	$f_2(x) = f_1(\lvert x\rvert)$
9.	x	$\lvert x\rvert$
10.	x^3	$\lvert x\rvert^3$
11.	x^2	$\lvert x\rvert^2$
12.	$\dfrac{1}{x}$	$\dfrac{1}{\lvert x\rvert}$
13.	$\sqrt{x}$	$\sqrt{\lvert x\rvert}$
14.	$\sin x$	$\sin \lvert x\rvert$

Composition with absolute values. In Exercises 15–20, graph g_1 and g_2 together. Then describe how taking absolute values after applying g_1 affects the graph.

	$g_1(x)$	$g_2(x) = \lvert g_1(x)\rvert$
15.	x^3	$\lvert x^3\rvert$
16.	$\sqrt{x}$	$\lvert\sqrt{x}\rvert$
17.	$\dfrac{1}{x}$	$\left\lvert\dfrac{1}{x}\right\rvert$
18.	$4 - x^2$	$\lvert 4 - x^2\rvert$
19.	$x^2 + x$	$\lvert x^2 + x\rvert$
20.	$\sin x$	$\lvert \sin x\rvert$

Trigonometry

In Exercises 21–24, sketch the graph of the given function. What is the period of the function?

21. $y = \cos 2x$
22. $y = \sin \dfrac{x}{2}$
23. $y = \sin \pi x$
24. $y = \cos \dfrac{\pi x}{2}$
25. Sketch the graph $y = 2\cos\left(x - \dfrac{\pi}{3}\right)$.
26. Sketch the graph $y = 1 + \sin\left(x + \dfrac{\pi}{4}\right)$.

In Exercises 27–30, *ABC* is a right triangle with the right angle at *C*. The sides opposite angles *A, B,* and *C* are *a, b,* and *c,* respectively.

27. a) Find a and b if $c = 2$, $B = \pi/3$.
    b) Find a and c if $b = 2$, $B = \pi/3$.
28. a) Express a in terms of A and c.
    b) Express a in terms of A and b.
29. a) Express a in terms of B and b.
    b) Express c in terms of A and a.
30. a) Express $\sin A$ in terms of a and c.
    b) Express $\sin A$ in terms of b and c.
31. CALCULATOR Two guy wires stretch from the top T of a vertical pole to points B and C on the ground, where C is 10 m closer to the base of the pole than is B. If wire BT makes an angle of 35° with the horizontal, and wire CT makes an angle of 50° with the horizontal, how high is the pole?
32. CALCULATOR Observers at positions A and B 2 km apart simultaneously measure the angle of elevation of a weather balloon to

be 40° and 70°, respectively. If the balloon is directly above a point on the line segment between A and B, find the height of the balloon.

33. Express $\sin 3x$ in terms of $\sin x$ and $\cos x$.

34. Express $\cos 3x$ in terms of $\sin x$ and $\cos x$.

35. a) GRAPHER Graph the function $f(x) = \sin x + \cos(x/2)$.
b) What appears to be the period of this function?
c) Confirm your finding in (b) algebraically.

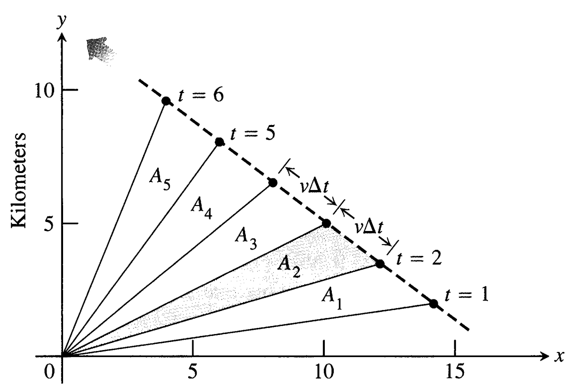
36. a) GRAPHER Graph $f(x) = \sin(1/x)$.
b) What are the domain and range of f?
c) Is f periodic? Give reasons for your answer.

PRELIMINARIES

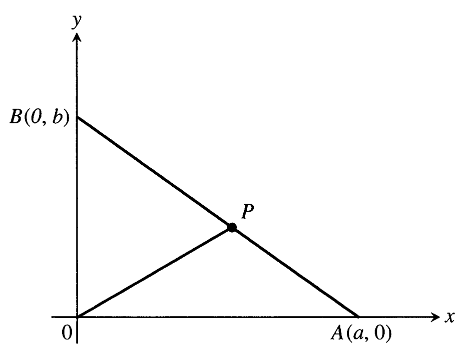
ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Geometry

1. An object's center of mass moves at a constant velocity v along a straight line past the origin. The accompanying figure shows the coordinate system and the line of motion. The dots show positions that are 1 sec apart. Why are the areas $A_1, A_2, \ldots, A_5$ in the figure all equal? As in Kepler's equal area law (see Section 11.5), the line that joins the object's center of mass to the origin sweeps out equal areas in equal times.



2. a) Find the slope of the line from the origin to the midpoint P of side AB in the triangle in the accompanying figure $(a, b > 0)$.



b) When is OP perpendicular to AB?

Functions and Graphs

3. Are there two functions f and g such that $f \circ g = g \circ f$? Give reasons for your answer.

4. Are there two functions f and g with the following property? The graphs of f and g are not straight lines but the graph of $f \circ g$ *is* a straight line. Give reasons for your answer.

5. If $f(x)$ is odd, can anything be said of $g(x) = f(x) - 2$? What if f is even instead? Give reasons for your answer.

6. If $g(x)$ is an odd function defined for all values of x, can anything be said about $g(0)$? Give reasons for your answer.

7. Graph the equation $|x| + |y| = 1 + x$.

8. Graph the equation $y + |y| = x + |x|$.

Trigonometry

In Exercises 9–14, ABC is an arbitrary triangle with sides a, b, and c opposite angles A, B, and C, respectively.

9. Find b if $a = \sqrt{3}$, $A = \pi/3$, $B = \pi/4$.

10. Find $\sin B$ if $a = 4, b = 3, A = \pi/4$.

11. Find $\cos A$ if $a = 2, b = 2, c = 3$.

12. Find c if $a = 2, b = 3, C = \pi/4$.

13. Find $\sin B$ if $a = 2, b = 3, c = 4$.

14. Find $\sin C$ if $a = 2, b = 4, c = 5$.

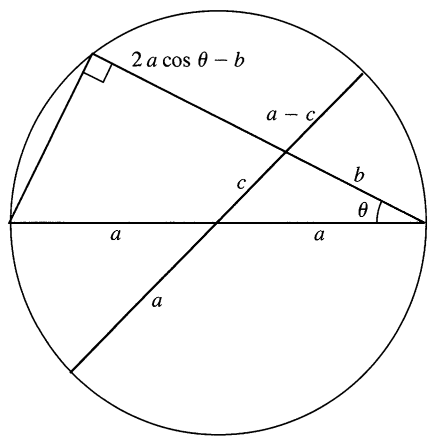
Derivations and Proofs

15. Prove the following identities.

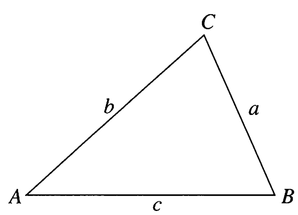
a) $$\frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$$

b) $$\frac{1 - \cos x}{1 + \cos x} = \tan^2 \frac{x}{2}$$

16. Explain the following "proof without words" of the law of cosines. (Source: "Proof without Words: The Law of Cosines," Sidney H. Kung, *Mathematics Magazine,* Vol. 63, No. 5, Dec. 1990, p. 342.)



17. Show that the area of triangle ABC is given by $(1/2)ab\sin C = (1/2)bc\sin A = (1/2)ca\sin B$.



* **18.** Show that the area of triangle ABC is given by $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$ is the semi-perimeter of the triangle.*

19. *Properties of inequalities.* If a and b are real numbers, we say that a **is less than** b and write $a < b$ if (and only if) $b - a$ is positive. Use this definition to prove the following properties of inequalities.

If a, b, and c are real numbers, then:

1. $a < b \implies a + c < b + c$
2. $a < b \implies a - c < b - c$
3. $a < b$ and $c > 0 \implies ac < bc$
4. $a < b$ and $c < 0 \implies bc < ac$
   (Special case: $a < b \implies -b < -a$)
5. $a > 0 \implies \dfrac{1}{a} > 0$
6. $0 < a < b \implies \dfrac{1}{b} < \dfrac{1}{a}$
7. $a < b < 0 \implies \dfrac{1}{b} < \dfrac{1}{a}$

20. *Properties of absolute values.* Prove the following properties of absolute values of real numbers.

a) $|-a| = |a|$

b) $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

21. Prove that the following inequalities hold for any real numbers a and b.

a) $|a| < |b|$ if and only if $a^2 < b^2$
b) $|a - b| \geq ||a| - |b||$

22. *Generalizing the triangle inequality.* Prove by mathematical induction that the following inequalities hold for any n real numbers $a_1, a_2, \ldots, a_n$. (Mathematical induction is reviewed in Appendix 1.)

a) $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$
b) $|a_1 + a_2 + \cdots + a_n| \geq |a_1| - |a_2| - \cdots - |a_n|$

23. Show that if f is both even and odd, then $f(x) = 0$ for every x in the domain of f.

24. a) *Even-odd decompositions.* Let f be a function whose domain is symmetric about the origin, that is, $-x$ belongs to the domain whenever x does. Show that f is the sum of an even function and an odd function:

$$f(x) = E(x) + O(x),$$

where E is an even function and O is an odd function. (*Hint:* Let $E(x) = (f(x) + f(-x))/2$. Show that $E(-x) = E(x)$, so that E is even. Then show that $O(x) = f(x) - E(x)$ is odd.)

b) *Uniqueness.* Show that there is only one way to write f as the sum of an even and an odd function. (*Hint:* One way is given in part (a). If also $f(x) = E_1(x) + O_1(x)$ where E_1 is even and O_1 is odd, show that $E - E_1 = O_1 - O$. Then use Exercise 23 to show that $E = E_1$ and $O = O_1$.)

Grapher Explorations—Effects of Parameters

25. What happens to the graph of $y = ax^2 + bx + c$ as

a) a changes while b and c remain fixed?
b) b changes (a and c fixed, $a \neq 0$)?
c) c changes (a and b fixed, $a \neq 0$)?

26. What happens to the graph of $y = a(x+b)^3 + c$ as

a) a changes while b and c remain fixed?
b) b changes (a and c fixed, $a \neq 0$)?
c) c changes (a and b fixed, $a \neq 0$)?

27. Find all values of the slope of the line $y = mx + 2$ for which the x-intercept exceeds $1/2$.

*Asterisk denotes more challenging problem.

CHAPTER 1

Limits and Continuity

OVERVIEW The concept of limit of a function is one of the fundamental ideas that distinguishes calculus from algebra and trigonometry.

In this chapter we develop the limit, first intuitively and then formally. We use limits to describe the way a function f varies. Some functions vary continuously; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump or vary erratically. We also use limits to define tangent lines to graphs of functions. This geometric application leads at once to the important concept of derivative of a function. The derivative, which we investigate thoroughly in Chapter 2, quantifies the way a function's values change.

1.1 Rates of Change and Limits

In this section we introduce two rates of change, speed and population growth. This leads to the main idea of the section, the idea of limit.

Speed

A moving body's **average speed** over any particular time interval is the amount of distance covered during the interval divided by the length of the interval.

Free fall

Near the surface of the earth, all bodies fall with the same constant acceleration. The distance a body falls after it is released from rest is a constant multiple of the square of the time elapsed. At least, that is what happens when the body falls in a vacuum, where there is no air to slow it down. The square-of-time rule also holds for dense, heavy objects like rocks, ball bearings, and steel tools during the first few seconds of their fall through air, before their velocities build up to where air resistance begins to matter. When air resistance is absent or insignificant and the only force acting on a falling body is the force of gravity, we call the way the body falls *free fall.*

EXAMPLE 1 A rock falls from the top of a 150-ft cliff. What is its average speed (a) during the first 2 sec of fall? (b) during the 1-sec interval between second 1 and second 2?

Solution Physical experiments show that a solid object dropped from rest to fall freely near the surface of the earth will fall

$$y = 16t^2 \text{ ft}$$

during the first t sec. The average speed of the rock during a given time interval is the change in distance, Δy, divided by the length of the time interval, Δt.

a) For the first 2 sec: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2-0} = 32\frac{\text{ft}}{\text{sec}}$$

b) From second 1 to second 2: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2-1} = 48\frac{\text{ft}}{\text{sec}}$$ ❑

Table 1.1 Average speeds over short time intervals

Average speed: $\dfrac{\Delta y}{\Delta t} = \dfrac{16(t_0+h)^2 - 16{t_0}^2}{h}$

Length of time interval h	**Average speed over interval of length h starting at $t_0 = 1$**	**Average speed over interval of length h starting at $t_0 = 2$**
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

EXAMPLE 2 Find the speed of the rock at $t = 1$ and $t = 2$ sec.

Solution We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$, having length $\Delta t = h$, as

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0+h)^2 - 16{t_0}^2}{h}.$$

We cannot use this formula to calculate the "instantaneous" speed at t_0 by substituting $h = 0$, because we cannot divide by zero. But we *can* use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$. When we do so, we see a pattern (Table 1.1).

The average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 32 as the length of the interval decreases. This suggests that the rock is falling at a speed of 32 ft/sec at $t_0 = 1$ sec. Similarly, the rock's speed at $t_0 = 2$ sec would appear to be 64 ft/sec. ❑

Average Rates of Change and Secant Lines

Given an arbitrary function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in value of y, $\Delta y = f(x_2) - f(x_1)$, by the length of the interval $\Delta x = x_2 - x_1 = h$ over which the change occurred.

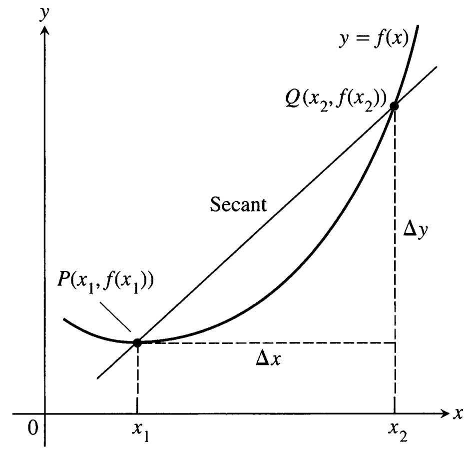
Definition

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}.$$

Geometrically, an average rate of change is a secant slope.

Notice that the average rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ (Fig. 1.1). In geometry, a line joining two points of a curve is called a **secant** to the curve. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant PQ.



1.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Experimental biologists often want to know the rates at which populations grow under controlled laboratory conditions.

EXAMPLE 3 *The average growth rate of a laboratory population*

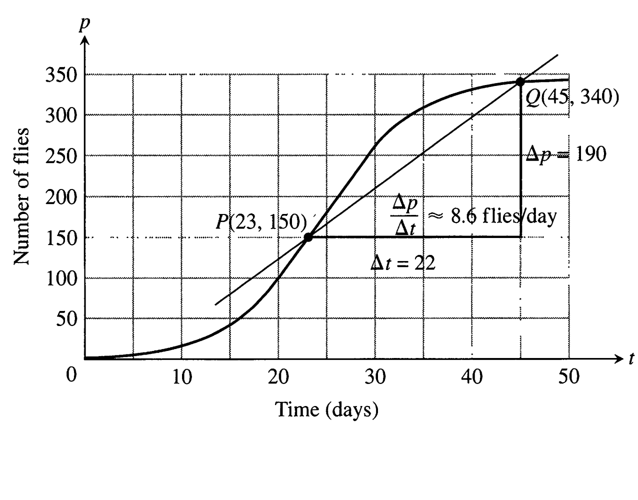
Figure 1.2 shows how a population of fruit flies *(Drosophila)* grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time, and the points joined by a smooth curve. Find the average growth rate from day 23 to day 45.

Solution There were 150 flies on day 23 and 340 flies on day 45. Thus the number of flies increased by $340 - 150 = 190$ in $45 - 23 = 22$ days. The average rate of change of the population from day 23 to day 45 was

$$\text{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day.}$$

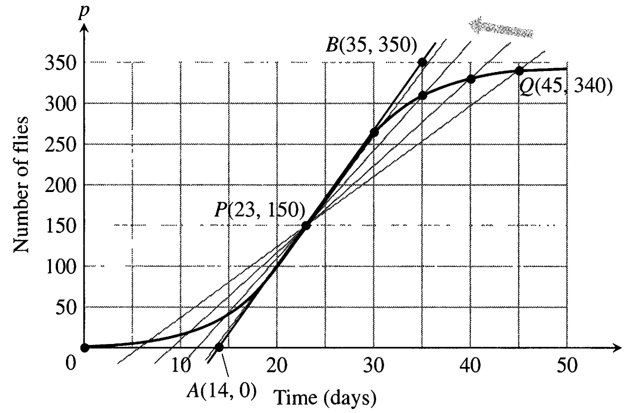
This average is the slope of the secant through the points P and Q on the graph in Fig. 1.2. ❑

The average rate of change from day 23 to day 45 calculated in Example 3 does not tell us how fast the population was changing on day 23 itself. For that we need to examine time intervals closer to the day in question.



1.2 Growth of a fruit fly population in a controlled experiment. (Source: *Elements of Mathematical Biology* by A. J. Lotka, 1956, Dover, New York, p. 69.)

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\dfrac{340-150}{45-23} \approx 8.6$
(40, 330)	$\dfrac{330-150}{40-23} \approx 10.6$
(35, 310)	$\dfrac{310-150}{35-23} \approx 13.3$
(30, 265)	$\dfrac{265-150}{30-23} \approx 16.4$



1.3 The positions and slopes of four secants through the point P on the fruit fly graph.

EXAMPLE 4 How fast was the number of flies in the population of Example 3 growing on day 23 itself?

Solution To answer this question, we examine the average rates of change over increasingly short time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secants from P to Q, for a sequence of points Q approaching P along the curve (Fig. 1.3).

The values in the table show that the secant slopes rise from 8.6 to 16.4 as the t-coordinate of Q decreases from 45 to 30, and we would expect the slopes to rise slightly higher as t continued on toward 23. Geometrically, the secants rotate about P and seem to approach the red line in the figure, a line that goes through P in the same direction that the curve goes through P. We will see that this line is called the *tangent* to the curve at P. Since the line appears to pass through the points (14, 0) and (35, 350), it has slope

$$\frac{350-0}{35-14} = 16.7 \text{ flies/day (approximately).}$$

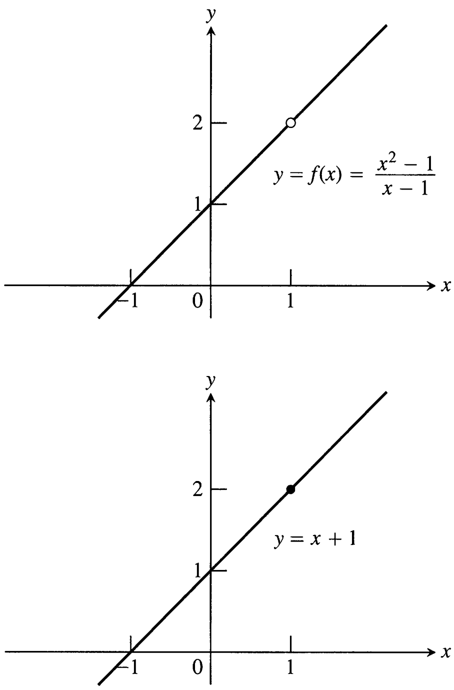
On day 23 the population was increasing at a rate of about 16.7 flies/day. ❑

The rates at which the rock in Example 2 was falling at the instants $t = 1$ and $t = 2$ and the rate at which the population in Example 4 was changing on day $t = 23$ are called *instantaneous rates of change.* As the examples suggest, we find instantaneous rates as limiting values of average rates. In Example 4, we also pictured the tangent line to the population curve on day 23 as a limiting position of secant lines. Instantaneous rates and tangent lines, intimately connected, appear in many other contexts. To talk about the two constructively, and to understand the connection further, we need to investigate the process by which we determine limiting values, or *limits,* as we will soon call them.

Limits of Function Values

Before we give a definition of limit, let us look at another example.

EXAMPLE 5 How does the function $f(x) = \dfrac{x^2-1}{x-1}$ behave near $x = 1$?



1.4 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined.

Solution The given formula defines f for all real numbers x except $x = 1$ (we cannot divide by zero). For any $x \neq 1$ we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \quad \text{for} \quad x \neq 1.$$

The graph of f is thus the line $y = x + 1$ with one point removed, namely the point $(1, 2)$. This removed point is shown as a "hole" in Fig. 1.4. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ *as close as we want* to 2 by choosing x *close enough* to 1 (Table 1.2).

We say that $f(x)$ approaches arbitrarily close to 2 as x approaches 1, or, more simply, $f(x)$ approaches the *limit* 2 as x approaches 1. We write this as

$$\lim_{x \to 1} f(x) = 2, \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2. \qquad \square$$

Table 1.2 The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2.

Values of x below and above 1	$f(x) = \dfrac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

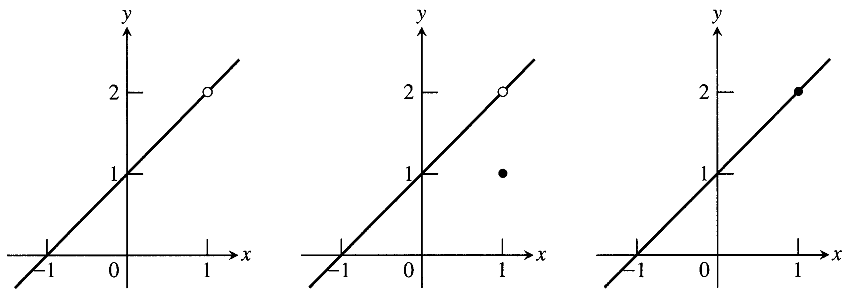
Definition

Informal Definition of Limit

Let $f(x)$ be defined on an open interval about x_0, *except possibly at x_0 itself.* If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and we write

$$\lim_{x \to x_0} f(x) = L.$$

This definition is "informal" because phrases like *arbitrarily close* and *sufficiently close* are imprecise; their meaning depends on the context. To a machinist manufacturing a piston, *close* may mean *within a few thousandths of an inch.* To an astronomer studying distant galaxies, *close* may mean *within a few thousand light-years.* The definition is clear enough, however, to enable us to recognize and evaluate limits of specific functions. We will need the more precise definition of Section 1.3, however, when we set out to prove theorems about limits.



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

1.5 $\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} h(x) = 2.$

EXAMPLE 6 The existence of a limit as $x \to x_0$ does not depend on how the function may be defined at x_0. The function f in Fig 1.5 has limit 2 as $x \to 1$ even though f is not defined at $x = 1$. The function g has limit 2 as $x \to 1$ even though $2 \neq g(1)$. The function h is the only one whose limit as $x \to 1$ equals its value at $x = 1$. For h we have $\lim_{x \to 1} h(x) = h(1)$. This kind of equality of limit and function value is special, and we will return to it in Section 1.5. ❑

Sometimes $\lim_{x \to x_0} f(x)$ can be evaluated by calculating $f(x_0)$. This holds, for example, whenever $f(x)$ is an algebraic combination of polynomials and trigonometric functions for which $f(x_0)$ is defined. (We will say more about this in Sections 1.2 and 1.5.)

EXAMPLE 7

a) $\lim_{x \to 2} (4) = 4$

b) $\lim_{x \to -13} (4) = 4$

c) $\lim_{x \to 3} x = 3$

d) $\lim_{x \to 2} (5x - 3) = 10 - 3 = 7$

e) $\lim_{x \to -2} \dfrac{3x + 4}{x + 5} = \dfrac{-6 + 4}{-2 + 5} = -\dfrac{2}{3}$ ❑

EXAMPLE 8

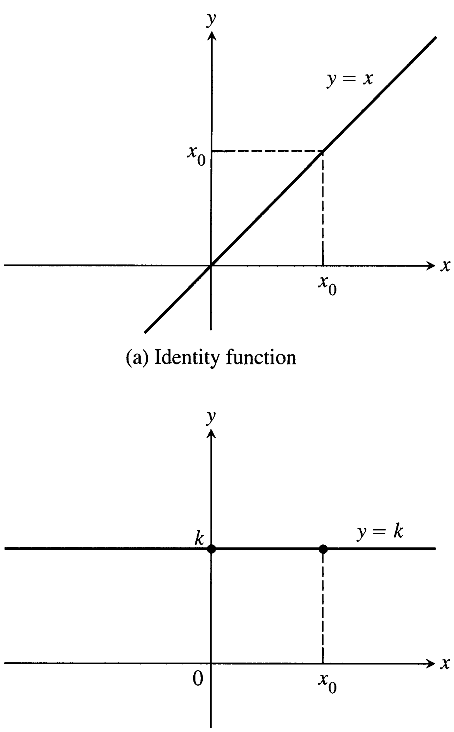
a) If f is the **identity function** $f(x) = x$, then for any value of x_0 (Fig. 1.6a),

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0.$$

b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0 (Fig. 1.6b),

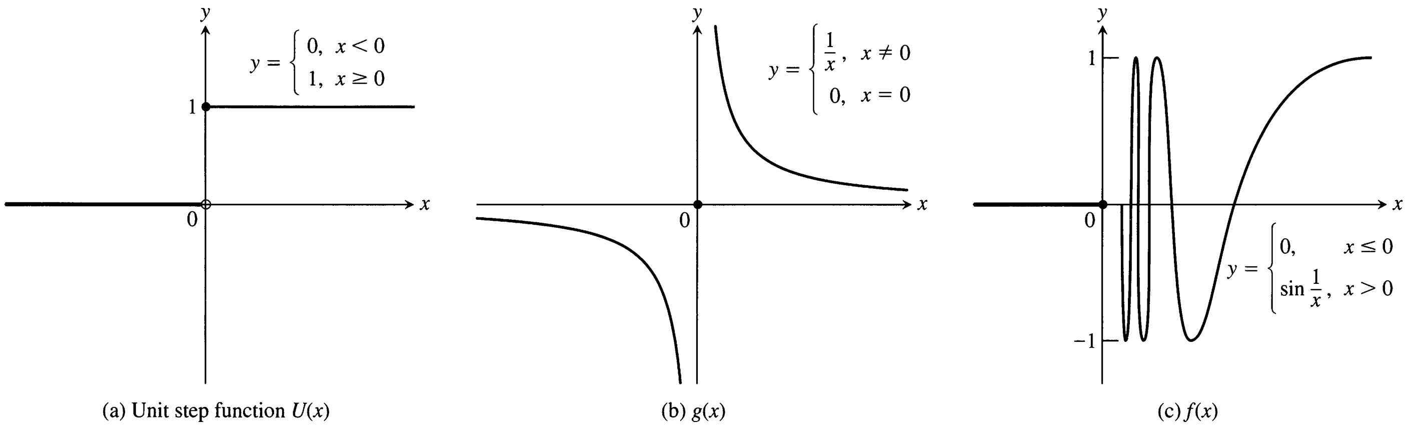
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k$$ ❑

Some ways that limits can fail to exist are illustrated in Fig. 1.7 and described in the next example.



(b) Constant function

1.6 The functions in Example 8.



1.7 The functions in Example 9.

EXAMPLE 9 *A function may fail to have a limit at a point in its domain.*

Discuss the behavior of the following functions as $x \to 0$.

a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

b) $g(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\dfrac{1}{x}, & x > 0 \end{cases}$

Solution

a) It jumps: The **unit step function** $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \to 0$ (Fig. 1.7a).

b) It grows too large: $g(x)$ has no limits as $x \to 0$ because the values of g grow arbitrarily large in absolute value as $x \to 0$ and do not stay close to *any* real number (Fig. 1.7b).

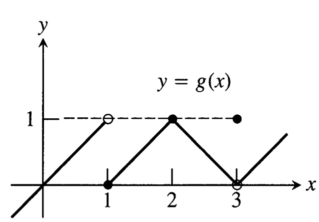
c) It oscillates too much: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between $+1$ and -1 in every open interval containing 0. The values do not stay close to any one number as $x \to 0$ (Fig. 1.7c). ❑

Exercises 1.1

Limits from Graphs

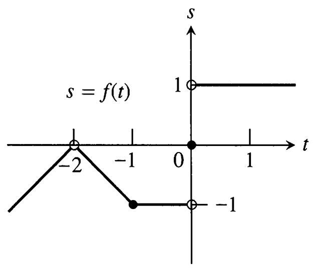
1. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

a) $\lim_{x\to 1} g(x)$ **b)** $\lim_{x\to 2} g(x)$ **c)** $\lim_{x\to 3} g(x)$

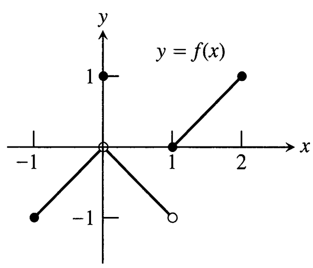


2. For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

a) $\lim_{t\to -2} f(t)$ **b)** $\lim_{t\to -1} f(t)$ **c)** $\lim_{t\to 0} f(t)$



3. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a) $\lim_{x\to 0} f(x)$ exists

b) $\lim_{x\to 0} f(x) = 0$

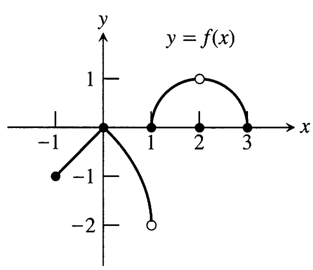
c) $\lim_{x\to 0} f(x) = 1$

d) $\lim_{x\to 1} f(x) = 1$

e) $\lim_{x\to 1} f(x) = 0$

f) $\lim_{x\to x_0} f(x)$ exists at every point x_0 in $(-1, 1)$

4. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a) $\lim_{x\to 2} f(x)$ does not exist

b) $\lim_{x\to 2} f(x) = 2$

c) $\lim_{x\to 1} f(x)$ does not exist

d) $\lim_{x\to x_0} f(x)$ exists at every point x_0 in $(-1, 1)$

e) $\lim_{x\to x_0} f(x)$ exists at every point x_0 in $(1, 3)$

Existence of Limits

In Exercises 5 and 6, explain why the limits do not exist.

5. $\lim_{x\to 0} \dfrac{x}{|x|}$

6. $\lim_{x\to 1} \dfrac{1}{x-1}$

7. Suppose that a function $f(x)$ is defined for all real values of x except $x = x_0$. Can anything be said about the existence of $\lim_{x\to x_0} f(x)$? Give reasons for your answer.

8. Suppose that a function $f(x)$ is defined for all x in $[-1, 1]$. Can anything be said about the existence of $\lim_{x\to 0} f(x)$? Give reasons for your answer.

9. If $\lim_{x\to 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of f at $x = 1$? Explain.

10. If $f(1) = 5$, must $\lim_{x\to 1} f(x)$ exist? If it does, then must $\lim_{x\to 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x\to 1} f(x)$? Explain.

Calculator/Grapher Exercises—Estimating Limits

11. Let $f(x) = (x^2 - 9)/(x + 3)$.

a) CALCULATOR Make a table of the values of f at the points $x = -3.1, -3.01, -3.001$, and so on as far as your calculator can go. Then estimate $\lim_{x\to -3} f(x)$. What estimate do you arrive at if you evaluate f at $x = -2.9, -2.99, -2.999, \ldots$ instead?

b) GRAPHER Support your conclusions in (a) by graphing f near $x_0 = -3$ and using ZOOM and TRACE to estimate y-values on the graph as $x \to -3$.

c) Find $\lim_{x\to -3} f(x)$ algebraically.

12. Let $g(x) = (x^2 - 2)/(x - \sqrt{2})$.

a) CALCULATOR Make a table of the values of g at the points $x = 1.4, 1.41, 1.414$, and so on through successive decimal approximations of $\sqrt{2}$. Estimate $\lim_{x\to\sqrt{2}} g(x)$.

b) GRAPHER Support your conclusion in (a) by graphing g near $x_0 = \sqrt{2}$ and using ZOOM and TRACE to estimate y-values on the graph as $x \to \sqrt{2}$.

c) Find $\lim_{x\to\sqrt{2}} g(x)$ algebraically.

13. Let $G(x) = (x + 6)/(x^2 + 4x - 12)$.

a) CALCULATOR Make a table of the values of G at $x = -5.9, -5.99, -5.999, \ldots$. Then estimate $\lim_{x\to -6} G(x)$. What estimate do you arrive at if you evaluate G at $x = -6.1, -6.01, -6.001, \ldots$ instead?

b) GRAPHER Support your conclusions in (a) by graphing G and using ZOOM and TRACE to estimate y-values on the graph as $x \to -6$.

c) Find $\lim_{x\to -6} G(x)$ algebraically.

14. Let $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$.

a) CALCULATOR Make a table of the values of h at $x = 2.9, 2.99, 2.999$, and so on. Then estimate $\lim_{x\to 3} h(x)$. What estimate do you arrive at if you evaluate h at $x = 3.1, 3.01, 3.001, \ldots$ instead?

b) GRAPHER Support your conclusions in (a) by graphing h near $x_0 = 3$ and using ZOOM and TRACE to estimate y-values on the graph as $x \to 3$.

c) Find $\lim_{x\to 3} h(x)$ algebraically.

15. Let $f(x) = (x^2 - 1)/(|x| - 1)$.

a) CALCULATOR Make tables of the values of f at values of x that approach $x_0 = -1$ from above and below. Then estimate $\lim_{x\to -1} f(x)$.

b) GRAPHER Support your conclusion in (a) by graphing f near $x_0 = -1$ and using ZOOM and TRACE to estimate y-values on the graph as $x \to -1$.

c) Find $\lim_{x\to -1} f(x)$ algebraically.

16. Let $F(x) = (x^2 + 3x + 2)/(2 - |x|)$.

a) CALCULATOR Make tables of values of F at values of x that approach $x_0 = -2$ from above and below. Then estimate $\lim_{x\to -2} F(x)$.

b) GRAPHER Support your conclusion in (a) by graphing F near $x_0 = -2$ and using ZOOM and TRACE to estimate y-values on the graph as $x \to -2$.

c) Find $\lim_{x\to -2} F(x)$ algebraically.

17. Let $g(\theta) = (\sin\theta)/\theta$.

a) CALCULATOR Make tables of values of g at values of θ that approach $\theta_0 = 0$ from above and below. Then estimate $\lim_{\theta\to 0} g(\theta)$.

b) GRAPHER Support your conclusion in (a) by graphing g near $\theta_0 = 0$.

18. Let $G(t) = (1 - \cos t)/t^2$.

a) CALCULATOR Make tables of values of G at values of t that approach $t_0 = 0$ from above and below. Then estimate $\lim_{t\to 0} G(t)$.

b) GRAPHER Support your conclusion in (a) by graphing G near $t_0 = 0$.

19. Let $f(x) = x^{1/(1-x)}$.

a) CALCULATOR Make tables of values of f at values of x that approach $x_0 = 1$ from above and below. Does f appear to have a limit as $x \to 1$? If so, what is it? If not, why not?

b) GRAPHER Support your conclusions in (a) by graphing f near $x_0 = 1$.

20. Let $f(x) = (3^x - 1)/x$.

a) CALCULATOR Make tables of values of f at values of x that approach $x_0 = 0$ from above and below. Does f appear to have a limit as $x \to 0$? If so, what is it? If not, why not?

b) GRAPHER Support your conclusions in (a) by graphing f near $x_0 = 0$.

Limits by Substitution

In Exercises 21–28, find the limits by substitution. *Support your answers with a grapher or calculator if available.*

21. $\lim_{x\to 2} 2x$

22. $\lim_{x\to 0} 2x$

23. $\lim_{x\to 1/3} (3x - 1)$

24. $\lim_{x\to 1} \dfrac{-1}{(3x-1)}$

25. $\lim_{x\to -1} 3x(2x - 1)$

26. $\lim_{x\to -1} \dfrac{3x^2}{2x-1}$

27. $\lim_{x\to \pi/2} x\sin x$

28. $\lim_{x\to \pi} \dfrac{\cos x}{1-\pi}$

Average Rates of Change

In Exercises 29–34, find the average rate of change of the function over the given interval or intervals.

29. $f(x) = x^3 + 1$;
(a) $[2, 3]$, (b) $[-1, 1]$

30. $g(x) = x^2$;
(a) $[-1, 1]$, (b) $[-2, 0]$

31. $h(t) = \cot t$;
(a) $[\pi/4, 3\pi/4]$, (b) $[\pi/6, \pi/2]$

32. $g(t) = 2 + \cos t$;
(a) $[0, \pi]$, (b) $[-\pi, \pi]$

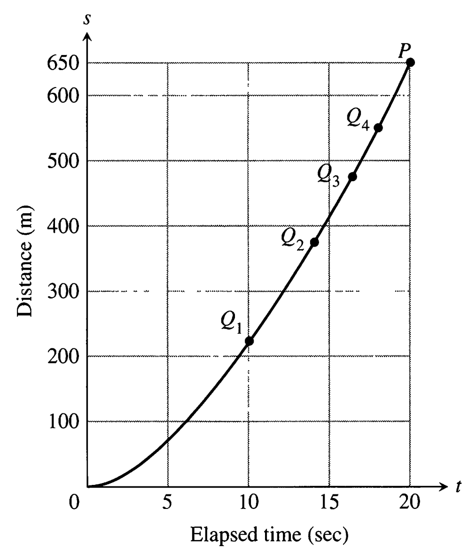
33. $R(\theta) = \sqrt{4\theta + 1}$; $[0, 2]$

34. $P(\theta) = \theta^3 - 4\theta^2 + 5\theta$; $[1, 2]$

35. Figure 1.8 shows the time-to-distance graph for a 1994 Ford Mustang Cobra accelerating from a standstill.

a) Estimate the slopes of secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in order in a table. What are the appropriate units for these slopes?

b) Then estimate the Cobra's speed at time $t = 20$ sec.

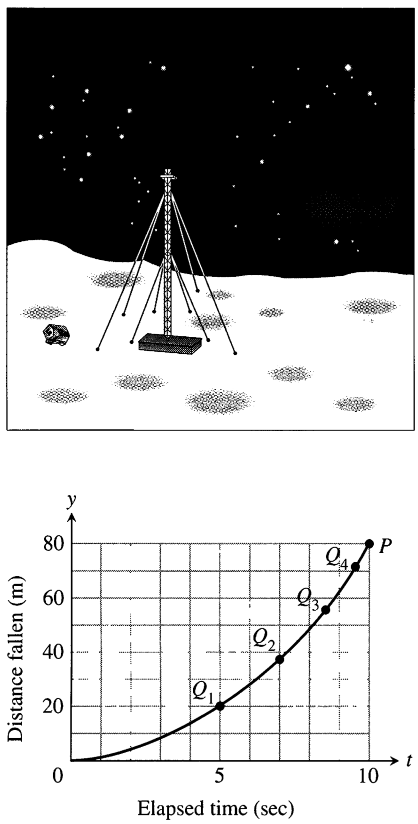


1.8 The time-to-distance graph for Exercise 35.

36. Figure 1.9 shows the plot of distance fallen (m) vs. time for a wrench that fell from the top platform of a communications mast on the moon to the station roof 80 m below.

a) Estimate the slopes of the secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in a table like the one in Fig. 1.3.

b) About how fast was the wrench going when it hit the roof?



1.9 The time-to-distance graph for Exercise 36.

37. CALCULATOR The profits of a small company for each of the first five years of its operation are given in the following table:

Year	Profit in \$1000s
1990	6
1991	27
1992	62
1993	111
1994	174

a) Plot points representing the profit as a function of year, and join them by as smooth a curve as you can.

b) What is the average rate of increase of the profits between 1992 and 1994?

c) Use your graph to estimate the rate at which the profits were changing in 1992.

38. CALCULATOR Make a table of values for the function $F(x) = (x+2)/(x-2)$ at the points $x = 2, x = 11/10, x = 101/100, x = 1001/1000, x = 10001/10000$, and $x = 1$.

a) Find the average rate of change of $F(x)$ over the intervals $[1, x]$ for each $x \neq 1$ in your table.

b) Extending the table if necessary, try to determine the rate of change of $F(x)$ at $x = 1$.

39. CALCULATOR Let $g(x) = \sqrt{x}$ for $x \geq 0$.

a) Find the average rate of change of $g(x)$ with respect to x over the intervals $[1, 2]$, $[1, 1.5]$, and $[1, 1+h]$.

b) Make a table of values of the average rate of change of g with respect to x over the interval $[1, 1+h]$ for some values of h approaching zero, say $h = 0.1, 0.01, 0.001, 0.0001, 0.00001$, and 0.000001.

c) What does your table indicate is the rate of change of $g(x)$ with respect to x at $x = 1$?

d) Calculate the limit as h approaches zero of the average rate of change of $g(x)$ with respect to x over the interval $[1, 1+h]$.

40. CALCULATOR Let $f(t) = 1/t$ for $t \neq 0$.

a) Find the average rate of change of f with respect to t over the intervals (i) from $t = 2$ to $t = 3$, and (ii) from $t = 2$ to $t = T$.

b) Make a table of values of the average rate of change of f with respect to t over the interval $[2, T]$, for some values of T approaching 2, say $T = 2.1, 2.01, 2.001, 2.0001, 2.00001$, and 2.000001.

c) What does your table indicate is the rate of change of f with respect to t at $t = 2$?

d) Calculate the limit as T approaches 2 of the average rate of change of f with respect to t over the interval from 2 to T. You will have to do some algebra before you can substitute $T = 2$.

CAS Explorations and Projects

In Exercises 41–46, use a CAS to perform the following steps:

a) Plot the function near the point x_0 being approached.

b) From your plot guess the value of the limit.

c) Evaluate the limit symbolically. How close was your guess?

41. $\lim_{x\to 2} \dfrac{x^4 - 16}{x - 2}$

42. $\lim_{x\to -1} \dfrac{x^3 - x^2 - 5x - 3}{(x+1)^2}$

43. $\lim_{x\to 0} \dfrac{\sqrt[3]{1+x} - 1}{x}$

44. $\lim_{x\to 3} \dfrac{x^2 - 9}{\sqrt{x^2+7} - 4}$

45. $\lim_{x\to 0} \dfrac{1 - \cos x}{x \sin x}$

46. $\lim_{x\to 0} \dfrac{2x^2}{3 - 3\cos x}$

1.2

Rules for Finding Limits

This section presents theorems for calculating limits. The first three let us build on the results of Example 8 in the preceding section to find limits of polynomials, rational functions, and powers. The fourth prepares for calculations later in the text.

Limits of Powers and Algebraic Combinations

Theorem 1
Properties of Limits

The following rules hold if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ (L and M real numbers).

1. *Sum Rule:* $\lim_{x\to c} [f(x) + g(x)] = L + M$
2. *Difference Rule:* $\lim_{x\to c} [f(x) - g(x)] = L - M$
3. *Product Rule:* $\lim_{x\to c} f(x) \cdot g(x) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to c} kf(x) = kL \quad \text{(any number } k\text{)}$
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If m and n are integers, then
$$\lim_{x\to c} [f(x)]^{m/n} = L^{m/n},$$
provided $L^{m/n}$ is a real number.

In words, the formulas in Theorem 1 say:

1. The limit of the sum of two functions is the sum of their limits.
2. The limit of the difference of two functions is the difference of their limits.
3. The limit of the product of two functions is the product of their limits.
4. The limit of a constant times a function is that constant times the limit of the function.
5. The limit of the quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.
6. The limit of any rational power of a function is that power of the limit of the function, provided the latter is a real number.

We will prove the Sum Rule in Section 1.3. Rules 2–5 are proved in Appendix 2. Rule 6 is proved in more advanced texts.

EXAMPLE 1 Find $\displaystyle\lim_{x\to c} \frac{x^3 + 4x^2 - 3}{x^2 + 5}$.

Solution Starting with the limits $\lim_{x\to c} x = c$ and $\lim_{x\to c} k = k$ from Section 1.1, Example 8, and combining them using various parts of Theorem 1, we obtain:

a) $\lim_{x\to c} x^2 = \left(\lim_{x\to c} x\right)\left(\lim_{x\to c} x\right) = c \cdot c = c^2$ — Product or Power

b) $\lim_{x\to c} (x^2+5) = \lim_{x\to c} x^2 + \lim_{x\to c} 5 = c^2 + 5$ — Sum and (a)

c) $\lim_{x\to c} 4x^2 = 4\lim_{x\to c} x^2 = 4c^2$ — Constant Multiple and (a)

d) $\lim_{x\to c} (4x^2-3) = \lim_{x\to c} 4x^2 - \lim_{x\to c} 3 = 4c^2 - 3$ — Difference and (c)

e) $\lim_{x\to c} x^3 = \left(\lim_{x\to c} x^2\right)\left(\lim_{x\to c} x\right) = c^2 \cdot c = c^3$ — Product and (a), or Power

f) $\lim_{x\to c} (x^3+4x-3) = \lim_{x\to c} x^3 + \lim_{x\to c} (4x^2-3)$ — Sum

$= c^3 + 4c^2 - 3$ — (d) and (e)

g) $\lim_{x\to c} \dfrac{x^3+4x^2-3}{x^2+5} = \dfrac{\lim_{x\to c} (x^3+4x^2-3)}{\lim_{x\to c} (x^2+5)}$ — Quotient

$= \dfrac{c^3+4c^2-3}{c^2+5}$ — (f) and (b) ❑

EXAMPLE 2 Find $\lim_{x\to -2} \sqrt{4x^2-3}$.

Solution

$$\lim_{x\to -2} \sqrt{4x^2-3} = \sqrt{4(-2)^2-3}$$ Example 1(d) and Power Rule with $n = 1/2$

$$= \sqrt{16-3}$$

$$= \sqrt{13}$$ ❑

Two consequences of Theorem 1 further simplify the task of calculating limits of polynomials and rational functions. To evaluate the limit of a polynomial function as x approaches c, merely substitute c for x in the formula for the function. To evaluate the limit of a rational function as x approaches a point c *at which the denominator is not zero,* substitute c for x in the formula for the function.

Theorem 2

Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

Theorem 3

Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 3

$$\lim_{x\to -1} \frac{x^3+4x^2-3}{x^2+5} = \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5} = \frac{0}{6} = 0.$$

This is the limit in Example 1 with $c=-1$, now done in one step. ❑

Identifying common factors

It can be shown that if $Q(x)$ is a polynomial and $Q(c)=0$, then $(x-c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x=c$, then $(x-c)$ is a common factor.

Eliminating Zero Denominators Algebraically

Theorem 3 applies only when the denominator of the rational function is not zero at the limit point c. If the denominator is zero, canceling common factors in the numerator and denominator will sometimes reduce the fraction to one whose denominator is no longer zero at c. When this happens, we can find the limit by substitution in the simplified fraction.

EXAMPLE 4 *Canceling a common factor*

Evaluate $\lim_{x\to 1} \dfrac{x^2+x-2}{x^2-x}$.

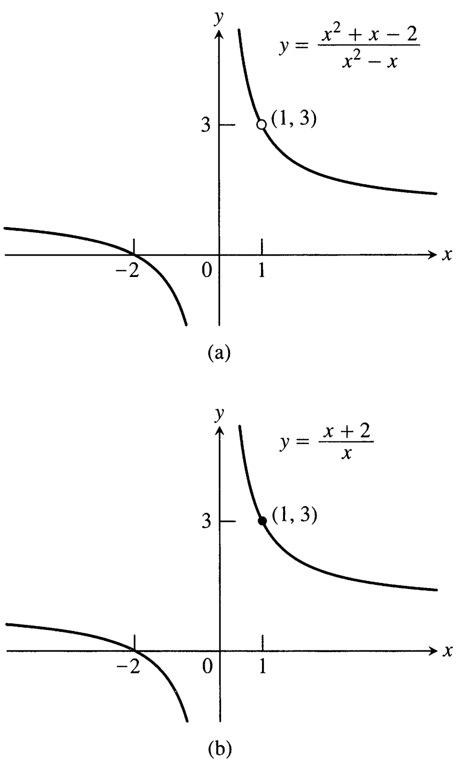
Solution We cannot just substitute $x=1$, because it makes the denominator zero. However, we can factor the numerator and denominator and cancel the common factor to obtain

$$\frac{x^2+x-2}{x^2-x} = \frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x}, \quad \text{if } x \neq 1.$$

Thus

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3.$$

See Fig. 1.10. ❑



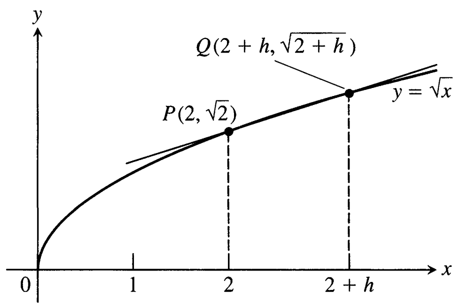
1.10 The graph of $f(x) = (x^2+x-2)/(x^2-x)$ in (a) is the same as the graph of $g(x) = (x+2)/x$ in (b) except at $x=1$, where f is undefined. The functions have the same limit as $x \to 1$.

EXAMPLE 5 *Creating and canceling a common factor*

Find $\lim_{h\to 0} \dfrac{\sqrt{2+h}-\sqrt{2}}{h}$.

Solution We cannot find the limit by substituting $h=0$, and the numerator and denominator do not have obvious factors. However, we can create a common factor in the numerator by multiplying it (and the denominator) by the so-called *conjugate expression* $\sqrt{2+h}+\sqrt{2}$, obtained by changing the sign between the square roots:

$$\begin{aligned}\frac{\sqrt{2+h}-\sqrt{2}}{h} &= \frac{\sqrt{2+h}-\sqrt{2}}{h} \cdot \frac{\sqrt{2+h}+\sqrt{2}}{\sqrt{2+h}+\sqrt{2}} \\ &= \frac{2+h-2}{h(\sqrt{2+h}+\sqrt{2})} \\ &= \frac{h}{h(\sqrt{2+h}+\sqrt{2})} && \text{We have created a common factor of } h \ldots \\ &= \frac{1}{\sqrt{2+h}+\sqrt{2}} && \ldots \text{which we cancel.}\end{aligned}$$



1.11 The limit of the slope of secant PQ as $Q \to P$ along the curve is $1/(2\sqrt{2})$ (Example 5).

Therefore,

$$\lim_{h \to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} = \lim_{h \to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}}$$

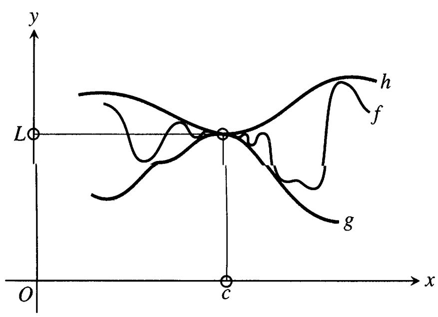
$$= \frac{1}{\sqrt{2+0} + \sqrt{2}} \qquad \text{The denominator is no longer 0 at } h = 0\text{, so we can substitute.}$$

$$= \frac{1}{2\sqrt{2}}.$$

Notice that the fraction $(\sqrt{2+h} - \sqrt{2})/h$ is the slope of the secant through the point $P(2, \sqrt{2})$ and the point $Q(2+h, \sqrt{2+h})$ nearby on the curve $y = \sqrt{x}$. Figure 1.11 shows the secant for $h > 0$. Our calculation shows that the limiting value of this slope as $Q \to P$ along the curve from either side is $1/(2\sqrt{2})$. ❑

The Sandwich Theorem

The following theorem will enable us to calculate a variety of limits in subsequent chapters. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions g and h that have the same limit L at a point c. Being trapped between the values of two functions that approach L, the values of f must also approach L (Fig. 1.12). You will find a proof in Appendix 2.



1.12 The graph of f is sandwiched between the graphs of g and h.

Theorem 4
The Sandwich Theorem

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

EXAMPLE 6 Given that

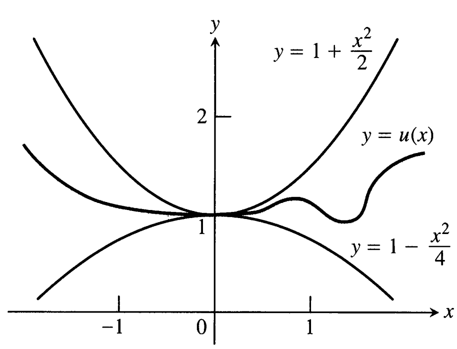
$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \quad \text{for all } x \ne 0,$$

find $\lim_{x \to 0} u(x)$.

Solution Since

$$\lim_{x \to 0} (1 - (x^2/4)) = 1 \quad \text{and} \quad \lim_{x \to 0} (1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x \to 0} u(x) = 1$ (Fig. 1.13). ❑



1.13 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$.

EXAMPLE 7 Show that if $\lim_{x \to c} |f(x)| = 0$, then $\lim_{x \to c} f(x) = 0$.

Solution Since $-|f(x)| \le f(x) \le |f(x)|$, and $-|f(x)|$ and $|f(x)|$ both have limit 0 as x approaches c, $\lim_{x \to c} f(x) = 0$ by the Sandwich Theorem. ❑

Exercises 1.2

Limit Calculations

Find the limits in Exercises 1–16.

1. $\lim_{x\to -7} (2x+5)$
2. $\lim_{x\to 12} (10-3x)$
3. $\lim_{x\to 2} (-x^2+5x-2)$
4. $\lim_{x\to -2} (x^3-2x^2+4x+8)$
5. $\lim_{t\to 6} 8(t-5)(t-7)$
6. $\lim_{s\to 2/3} 3s(2s-1)$
7. $\lim_{x\to 2} \dfrac{x+3}{x+6}$
8. $\lim_{x\to 5} \dfrac{4}{x-7}$
9. $\lim_{y\to -5} \dfrac{y^2}{5-y}$
10. $\lim_{y\to 2} \dfrac{y+2}{y^2+5y+6}$
11. $\lim_{x\to -1} 3(2x-1)^2$
12. $\lim_{x\to -4} (x+3)^{1984}$
13. $\lim_{y\to -3} (5-y)^{4/3}$
14. $\lim_{z\to 0} (2z-8)^{1/3}$
15. $\lim_{h\to 0} \dfrac{3}{\sqrt{3h+1}+1}$
16. $\lim_{h\to 0} \dfrac{5}{\sqrt{5h+4}+2}$

Find the limits in Exercises 17–30.

17. $\lim_{x\to 5} \dfrac{x-5}{x^2-25}$
18. $\lim_{x\to -3} \dfrac{x+3}{x^2+4x+3}$
19. $\lim_{x\to -5} \dfrac{x^2+3x-10}{x+5}$
20. $\lim_{x\to 2} \dfrac{x^2-7x+10}{x-2}$
21. $\lim_{t\to 1} \dfrac{t^2+t-2}{t^2-1}$
22. $\lim_{t\to -1} \dfrac{t^2+3t+2}{t^2-t-2}$
23. $\lim_{x\to -2} \dfrac{-2x-4}{x^3+2x^2}$
24. $\lim_{y\to 0} \dfrac{5y^3+8y^2}{3y^4-16y^2}$
25. $\lim_{u\to 1} \dfrac{u^4-1}{u^3-1}$
26. $\lim_{v\to 2} \dfrac{v^3-8}{v^4-16}$
27. $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$
28. $\lim_{x\to 4} \dfrac{4x-x^2}{2-\sqrt{x}}$
29. $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2}$
30. $\lim_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

Using Limit Rules

31. Suppose $\lim_{x\to 0} f(x)=1$ and $\lim_{x\to 0} g(x)=-5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 0} \frac{2f(x)-g(x)}{(f(x)+7)^{2/3}} = \frac{\lim_{x\to 0}(2f(x)-g(x))}{\lim_{x\to 0}(f(x)+7)^{2/3}} \quad \text{(a)}$$

$$= \frac{\lim_{x\to 0} 2f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0}(f(x)+7)\right)^{2/3}} \quad \text{(b)}$$

$$= \frac{2\lim_{x\to 0} f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0} f(x) + \lim_{x\to 0} 7\right)^{2/3}} \quad \text{(c)}$$

$$= \frac{(2)(1)-(-5)}{(1+7)^{2/3}} = \frac{7}{4}$$

32. Let $\lim_{x\to 1} h(x)=5$, $\lim_{x\to 1} p(x)=1$, and $\lim_{x\to 1} r(x)=2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 1} \frac{\sqrt{5h(x)}}{p(x)(4-r(x))} = \frac{\lim_{x\to 1}\sqrt{5h(x)}}{\lim_{x\to 1}(p(x)(4-r(x)))} \quad \text{(a)}$$

$$= \frac{\sqrt{\lim_{x\to 1} 5h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1}(4-r(x))\right)} \quad \text{(b)}$$

$$= \frac{\sqrt{5\lim_{x\to 1} h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1} 4 - \lim_{x\to 1} r(x)\right)} \quad \text{(c)}$$

$$= \frac{\sqrt{(5)(5)}}{(1)(4-2)} = \frac{5}{2}$$

33. Suppose $\lim_{x\to c} f(x)=5$ and $\lim_{x\to c} g(x)=-2$. Find
   a) $\lim_{x\to c} f(x)g(x)$
   b) $\lim_{x\to c} 2f(x)g(x)$
   c) $\lim_{x\to c} (f(x)+3g(x))$
   d) $\lim_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

34. Suppose $\lim_{x\to 4} f(x)=0$ and $\lim_{x\to 4} g(x)=-3$. Find
   a) $\lim_{x\to 4} (g(x)+3)$
   b) $\lim_{x\to 4} xf(x)$
   c) $\lim_{x\to 4} (g(x))^2$
   d) $\lim_{x\to 4} \dfrac{g(x)}{f(x)-1}$

35. Suppose $\lim_{x\to b} f(x)=7$ and $\lim_{x\to b} g(x)=-3$. Find
   a) $\lim_{x\to b} (f(x)+g(x))$
   b) $\lim_{x\to b} f(x)\cdot g(x)$
   c) $\lim_{x\to b} 4g(x)$
   d) $\lim_{x\to b} f(x)/g(x)$

36. Suppose that $\lim_{x\to -2} p(x)=4$, $\lim_{x\to -2} r(x)=0$, and $\lim_{x\to -2} s(x)=-3$. Find
   a) $\lim_{x\to -2} (p(x)+r(x)+s(x))$
   b) $\lim_{x\to -2} p(x)\cdot r(x)\cdot s(x)$
   c) $\lim_{x\to -2} (-4p(x)+5r(x))/s(x)$

Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

occur frequently in calculus. In Exercises 37–42, evaluate this limit for the given value of x and function f.

37. $f(x)=x^2,\quad x=1$

38. $f(x)=x^2,\quad x=-2$

39. $f(x)=3x-4,\quad x=2$

40. $f(x)=1/x,\quad x=-2$

41. $f(x)=\sqrt{x},\quad x=7$

42. $f(x)=\sqrt{3x+1},\quad x=0$

Using the Sandwich Theorem

43. If $\sqrt{5-2x^2}\le f(x)\le\sqrt{5-x^2}$ for $-1\le x\le 1$, find $\lim_{x\to 0} f(x)$.

44. If $2-x^2\le g(x)\le 2\cos x$ for all x, find $\lim_{x\to 0} g(x)$.

45. a) It can be shown that the inequalities

$$1-\frac{x^2}{6}<\frac{x\sin x}{2-2\cos x}<1$$

hold for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x\to 0}\frac{x\sin x}{2-2\cos x}?$$

Give reasons for your answer.

b) GRAPHER Graph

$$y=1-(x^2/6),\ y=(x\sin x)/(2-2\cos x),\text{ and } y=1$$

together for $-2\le x\le 2$. Comment on the behavior of the graphs as $x\to 0$.

46. a) Suppose that the inequalities

$$\frac{1}{2}-\frac{x^2}{24}<\frac{1-\cos x}{x^2}<\frac{1}{2}$$

hold for values of x close to zero. (They do, as you will see in Section 8.10.) What, if anything, does this tell you about

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}?$$

Give reasons for your answer.

b) GRAPHER Graph the equations $y=(1/2)-(x^2/24)$, $y=(1-\cos x)/x^2$, and $y=1/2$ together for $-2\le x\le 2$. Comment on the behavior of the graphs as $x\to 0$.

Theory and Examples

47. If $x^4\le f(x)\le x^2$ for x in $[-1,1]$ and $x^2\le f(x)\le x^4$ for $x<-1$ and $x>1$, at what points c do you automatically know $\lim_{x\to c} f(x)$? What can you say about the value of the limit at these points?

48. Suppose that $g(x)\le f(x)\le h(x)$ for all $x\ne 2$ and suppose that

$$\lim_{x\to 2} g(x)=\lim_{x\to 2} h(x)=-5.$$

Can we conclude anything about the values of f, g, and h at $x=2$? Could $f(2)=0$? Could $\lim_{x\to 2} f(x)=0$? Give reasons for your answers.

49. If $\displaystyle\lim_{x\to 4}\frac{f(x)-5}{x-2}=1$, find $\displaystyle\lim_{x\to 4} f(x)$.

50. If $\displaystyle\lim_{x\to -2}\frac{f(x)}{x^2}=1$, find (a) $\displaystyle\lim_{x\to -2} f(x)$ and (b) $\displaystyle\lim_{x\to -2}\frac{f(x)}{x}$.

51. a) If $\displaystyle\lim_{x\to 2}\frac{f(x)-5}{x-2}=3$, find $\displaystyle\lim_{x\to 2} f(x)$.

b) If $\displaystyle\lim_{x\to 2}\frac{f(x)-5}{x-2}=4$, find $\displaystyle\lim_{x\to 2} f(x)$.

52. If $\displaystyle\lim_{x\to 0}\frac{f(x)}{x^2}=1$, find (a) $\displaystyle\lim_{x\to 0} f(x)$ and (b) $\displaystyle\lim_{x\to 0}\frac{f(x)}{x}$.

53. a) GRAPHER Graph $g(x)=x\sin(1/x)$ to estimate $\lim_{x\to 0} g(x)$, zooming in on the origin as necessary.

b) Confirm your estimate in (a) with a proof.

54. a) GRAPHER Graph $h(x)=x^2\cos(1/x^3)$ to estimate $\lim_{x\to 0} h(x)$, zooming in on the origin as necessary.

b) Confirm your estimate in (a) with a proof.

1.3 Target Values and Formal Definitions of Limits

In this section we give a formal definition of the limit introduced in the previous two sections. We replace vague phrases like "gets arbitrarily close" in the informal definition with specific conditions that can be applied to any particular example. To do this we first examine how to control the input of a function to ensure that the output is kept within preset bounds.

Keeping Outputs near Target Values

We sometimes need to know what input values x will result in output values of the function $y=f(x)$ near a particular target value. How near depends on the context.

A gas station attendant, asked for \$5.00 worth of gas, will try to pump a volume of gas worth \$5.00 to the nearest cent. An automobile mechanic grinding a 3.385-in. cylinder will not let the bore exceed this value by more than 0.002 in. A pharmacist making ointments will measure ingredients to the nearest milligram.

EXAMPLE 1 *Controlling a linear function*

How close to $x_0 = 4$ must we hold the input x to be sure that the output $y = 2x - 1$ lies within 2 units of $y_0 = 7$?

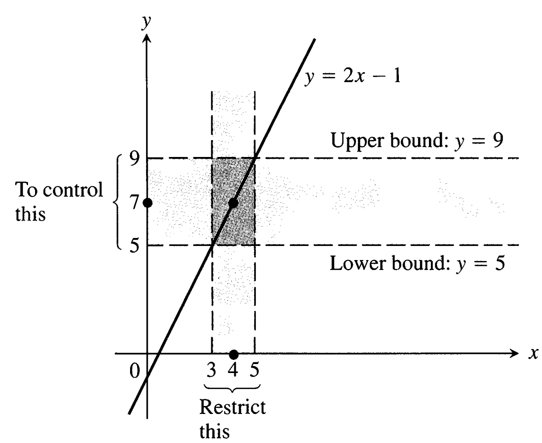
Solution We are asked: For what values of x is $|y - 7| < 2$? To find the answer we first express $|y - 7|$ in terms of x:

$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: What values of x satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality:

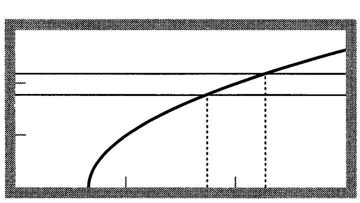
$$\begin{aligned} |2x - 8| &< 2 \\ -2 < 2x - 8 &< 2 \\ 6 < 2x &< 10 \\ 3 < x &< 5 \\ -1 < x - 4 &< 1. \end{aligned}$$

Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Fig. 1.14).



1.14 Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$.

❑

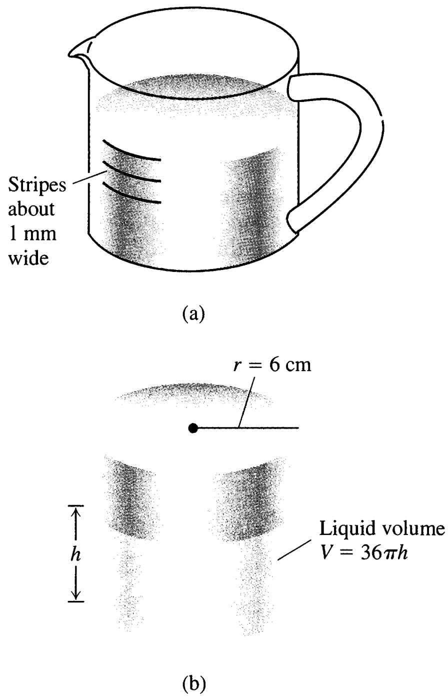


[0, 3] by [0, 3]

Keeping x between 1.75 and 2.28 will keep y between 1.8 and 2.2.

Technology *Target Values* You can experiment with target values on a graphing utility. Graph the function together with a target interval defined by horizontal lines above and below the proposed limit. Adjust the range or use zoom until the function's behavior inside the target interval is clear. Then observe what happens when you try to find an interval of x-values that will keep the function values within the target interval. (See also Exercises 7–14 and CAS Exercises 61–64.)

For example, try this for $f(x) = \sqrt{3x - 2}$ and the target interval (1.8, 2.2) on the y-axis. That is, graph $y_1 = f(x)$ and the lines $y_2 = 1.8$, $y_3 = 2.2$. Then try the target intervals (1.98, 2.02) and (1.9998, 2.0002).



1.15 A 1-L measuring cup (a), modeled as a right circular cylinder (b) of radius $r = 6$ cm (Example 2).

EXAMPLE 2 *Why the stripes on a 1-liter kitchen measuring cup are about a millimeter wide*

The interior of a typical 1-L measuring cup is a right circular cylinder of radius 6 cm (Fig. 1.15). The volume of water we put in the cup is therefore a function of the level h to which the cup is filled, the formula being

$$V = \pi 6^2 h = 36\pi h.$$

How closely must we measure h to measure out 1 L of water (1000 cm^3) with an error of no more than 1% (10 cm^3)?

Solution We want to know in what interval to hold values of h to make V satisfy the inequality

$$|V - 1000| = |36\pi h - 1000| \le 10.$$

To find out, we solve the inequality:

$$|36\pi h - 1000| \le 10$$

$$-10 \le 36\pi h - 1000 \le 10$$

$$990 \le 36\pi h \le 1010$$

$$\frac{990}{36\pi} \le h \le \frac{1010}{36\pi}$$

$$\underset{\text{rounded up, to be safe}}{8.8} \le h \le \underset{\text{rounded down, to be safe}}{8.9}$$

The interval in which we should hold h is about $8.9 - 8.8 = 0.1$ cm wide (1 mm). With stripes 1 mm wide, we can expect to measure a liter of water with an accuracy of 1%, which is more than enough accuracy for cooking. ❑

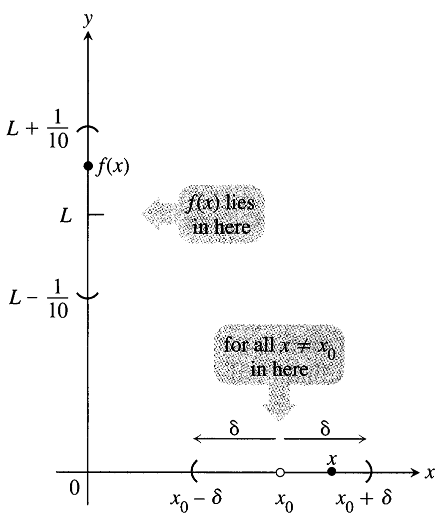
The Precise Definition of Limit

In a target-value problem, we determine how close to hold a variable x to a particular value x_0 to ensure that the outputs $f(x)$ of some function lie within a prescribed interval about a target value L. To show that the limit of $f(x)$ as $x \to x_0$ actually equals L, we must be able to show that the gap between $f(x)$ and L can be made less than *any prescribed error,* no matter how small, by holding x close enough to x_0.

Suppose we are watching the values of a function $f(x)$ as x approaches x_0 (without taking on the value of x_0 itself). Certainly we want to be able to say that $f(x)$ stays within one-tenth of a unit of L as soon as x stays within some distance δ of x_0 (Fig. 1.16). But that in itself is not enough, because as x continues on its course toward x_0, what is to prevent $f(x)$ from jittering about within the interval from $L - 1/10$ to $L + 1/10$ without tending toward L?

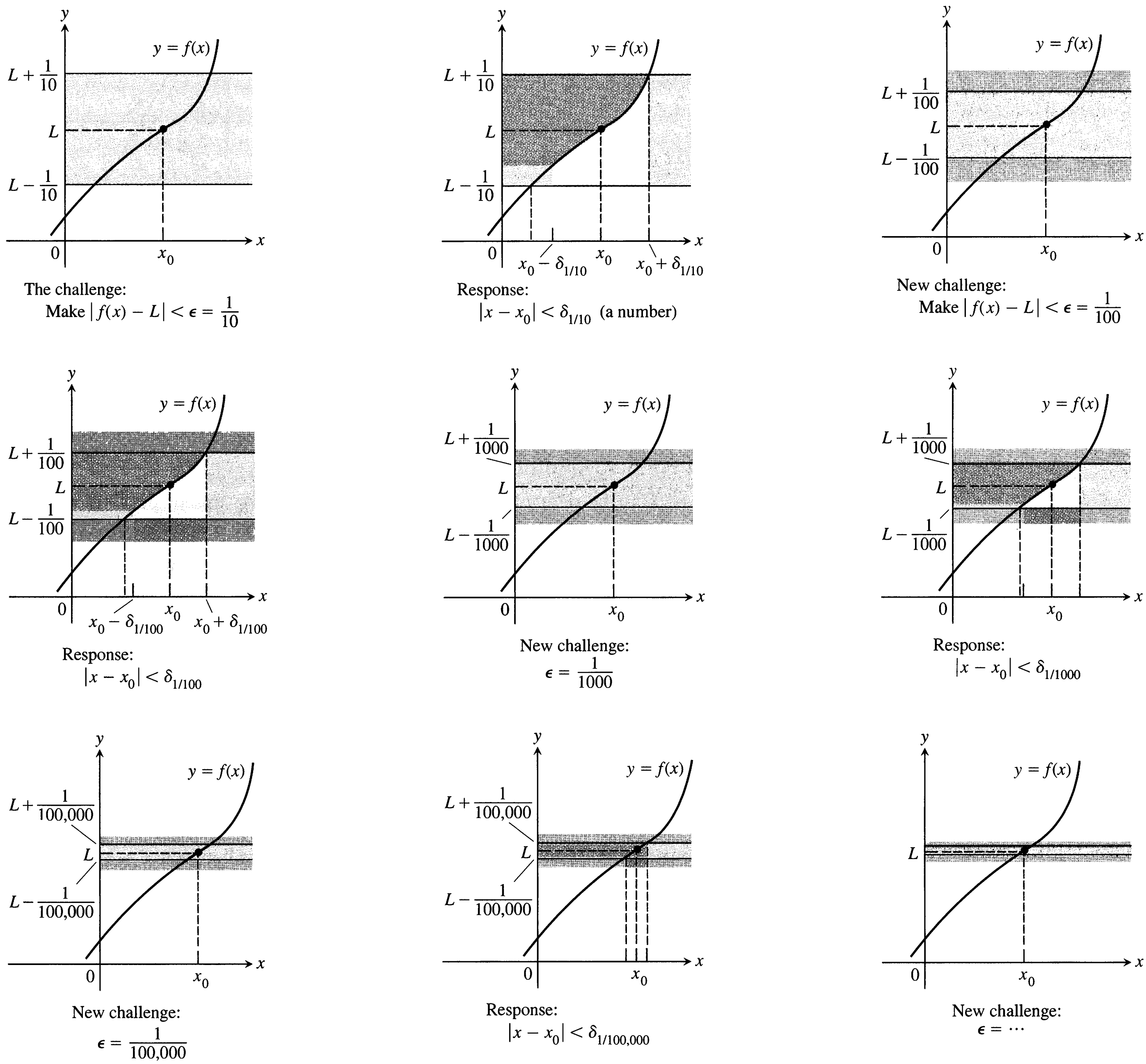
We can be told that the error can be no more than 1/100 or 1/1000 or 1/100,000. Each time, we find a new δ-interval about x_0 so that keeping x within that interval satisfies the new error tolerance. And each time the possibility exists that $f(x)$ jitters away from L at the last minute.

The following figures illustrate the problem. You can think of this as a quarrel between a skeptic and a scholar. The skeptic presents ϵ-challenges to prove that

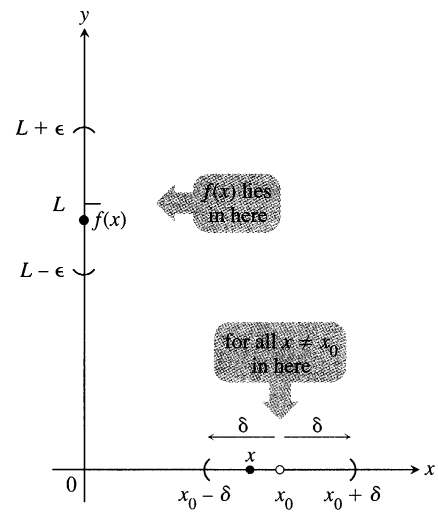


1.16 A preliminary stage in the development of the definition of limit.

the limit does not exist or, more precisely, that there is room for doubt, and the scholar answers every challenge with a δ-interval around x_0.



How do we stop this seemingly endless series of challenges and responses? By proving that for every error tolerance ϵ that the challenger can produce, we can find, calculate, or conjure a matching distance δ that keeps x "close enough" to x_0 to keep $f(x)$ within that tolerance of L (Fig. 1.17 on the following page).



1.17 The relation of δ and ϵ in the definition of limit.

Here, at last, is a mathematical way to say that the closer x gets to x_0, the closer $y = f(x)$ gets to L.

Definition
A Formal Definition of Limit

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the limit L as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.$$

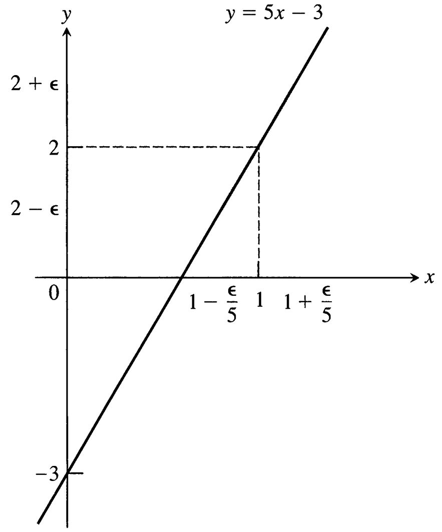
The Weierstrass definition

The concepts of limit and continuity (and, indeed, real number and function) did not enter mathematics overnight with the great discoveries of Sir Isaac Newton (1642–1727) and Baron Gottfried Wilhelm Leibniz (1646–1716). Mathematicians had an imperfect understanding of these fundamental ideas even as late as the last century. Definitions of the limit given by French mathematician Augustin-Louis Cauchy (1789–1857) and others referred to variables "approaching indefinitely" a fixed value and frequently made use of "infinitesimals," quantities that become infinitely small but not zero. The now accepted ϵ-δ definition of limit was formulated by German mathematician Karl Weierstrass (1815–1897) in the middle of the nineteenth century as part of his attempt to put mathematical analysis on a sound logical foundation.

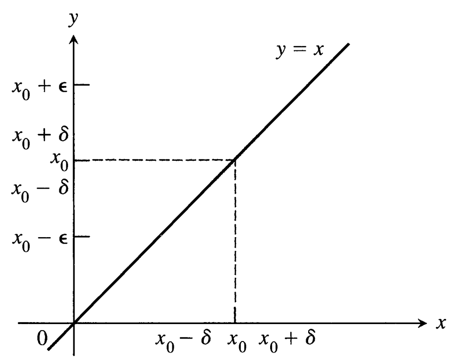
To return to the idea of target values, suppose you are machining a generator shaft to a close tolerance. You may try for diameter L, but since nothing is perfect, you must be satisfied with a diameter $f(x)$ somewhere between $L - \epsilon$ and $L + \epsilon$. The δ is the measure of how accurate your control setting for x must be to guarantee this degree of accuracy in the diameter of the shaft. Notice that as the tolerance for error becomes stricter, you may have to adjust δ. That is, the value of δ, how tight your control setting must be, depends on the value of ϵ, the error tolerance.

Examples: Testing the Definition

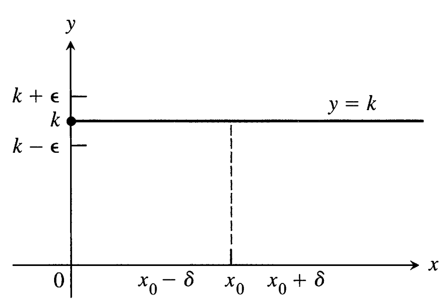
The formal definition of limit does not tell how to find the limit of a function, but it enables us to verify that a suspected limit is correct. The following examples show how the definition can be used to verify limit statements for specific functions. (The first two examples correspond to parts of Examples 7 and 8 in Section 1.1.) However, the real purpose of the definition is not to do calculations like this, but rather to prove general theorems so that the calculation of specific limits can be simplified.



1.18 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 3).



1.19 For the function $f(x) = x$, we find that $0 < |x - x_0| < \delta$ will guarantee $|f(x) - x_0| < \epsilon$ whenever $\delta \le \epsilon$ (Example 4a).



1.20 For the function $f(x) = k$, we find that $|f(x) - k| < \epsilon$ for any positive δ (Example 4b).

EXAMPLE 3 Show that $\lim_{x\to 1} (5x - 3) = 2$.

Solution Set $x_0 = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\epsilon > 0$ we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $x_0 = 1$, that is, if

$$0 < |x - 1| < \delta,$$

then $f(x)$ is within distance ϵ of $L = 2$, that is

$$|f(x) - 2| < \epsilon.$$

We find δ by working backwards from the ϵ-inequality:

$$\begin{aligned} |(5x - 3) - 2| = |5x - 5| &< \epsilon \\ 5|x - 1| &< \epsilon \\ |x - 1| &< \epsilon/5 \end{aligned}$$

Thus we can take $\delta = \epsilon/5$ (Fig. 1.18). If $0 < |x - 1| < \delta = \epsilon/5$, then

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < 5(\epsilon/5) = \epsilon.$$

This proves that $\lim_{x\to 1} (5x - 3) = 2$.

The value of $\delta = \epsilon/5$ is not the only value that will make $0 < |x - 1| < \delta$ imply $|5x - 5| < \epsilon$. Any smaller positive δ will do as well. The definition does not ask for a "best" positive δ, just one that will work. ❑

EXAMPLE 4 *Two important limits*

Verify: (a) $\lim\limits_{x\to x_0} x = x_0$ (b) $\lim\limits_{x\to x_0} k = k$ (k constant).

Solution

a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \text{implies} \qquad |x - x_0| < \epsilon.$$

The implication will hold if δ equals ϵ or any smaller positive number (Fig. 1.19). This proves that $\lim_{x\to x_0} x = x_0$.

b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \text{implies} \qquad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive number for δ and the implication will hold (Fig. 1.20). This proves that $\lim_{x\to x_0} k = k$. ❑

Finding Deltas Algebraically for Given Epsilons

In Examples 3 and 4, the interval of values about x_0 for which $|f(x) - L|$ was less than ϵ was symmetric about x_0 and we could take δ to be half the length of the interval. When such symmetry is absent, as it usually is, we can take δ to be the distance from x_0 to the interval's nearer endpoint.

EXAMPLE 5 For the limit $\lim_{x\to 5} \sqrt{x - 1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$. That is, find a $\delta > 0$ such that for all x

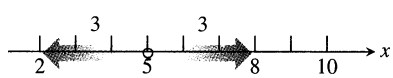
$$0 < |x - 5| < \delta \implies |\sqrt{x - 1} - 2| < 1.$$

Solution We organize the search into two steps. First we solve the inequality $|\sqrt{x-1}-2| < 1$ to find an interval (a, b) about $x_0 = 5$ on which the inequality holds for all $x \neq x_0$. Then we find a value of $\delta > 0$ that places the interval $5 - \delta < x < 5 + \delta$ (centered at $x_0 = 5$) inside the interval (a, b).

Step 1: *Solve the inequality* $|\sqrt{x-1}-2| < 1$ *to find an interval about* $x_0 = 5$ *on which the inequality holds for all* $x \neq x_0$.

$$\begin{aligned} |\sqrt{x-1}-2| &< 1 \\ -1 < \sqrt{x-1}-2 &< 1 \\ 1 < \sqrt{x-1} &< 3 \\ 1 < x-1 &< 9 \\ 2 < x &< 10 \end{aligned}$$

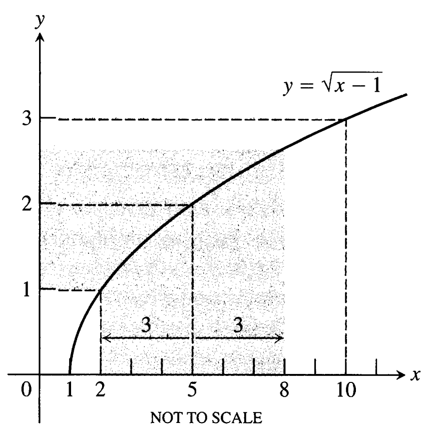
The inequality holds for all x in the open interval (2, 10), so it holds for all $x \neq 5$ in this interval as well.



1.21 An open interval of radius 3 about $x_0 = 5$ will lie inside the open interval (2, 10).

Step 2: *Find a value of* $\delta > 0$ *that places the centered interval* $5 - \delta < x < 5 + \delta$ *inside the interval* (2, 10). The distance from 5 to the nearer endpoint of (2, 10) is 3 (Fig. 1.21). If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x - 5| < \delta$ will automatically place x between 2 and 10 to make $|\sqrt{x-1}-2| < 1$ (Fig. 1.22):

$$0 < |x-5| < 3 \quad \Longrightarrow \quad |\sqrt{x-1}-2| < 1. \qquad \square$$



1.22 The function and intervals in Example 5.

How to Find a δ for a Given f, L, x_0, and $\epsilon > 0$ Algebraically

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

Step 1 *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) about x_0 on which the inequality holds for all $x \neq x_0$.

Step 2 *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

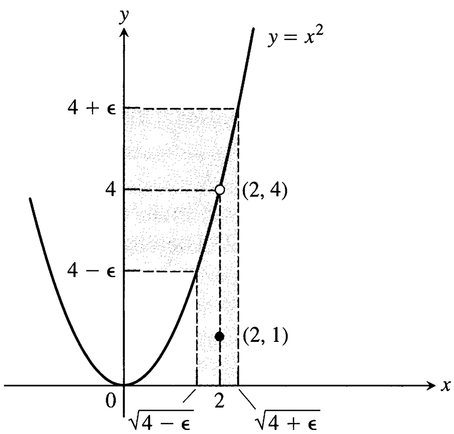
EXAMPLE 6 Prove that $\lim_{x\to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2. \end{cases}$$

Solution Our task is to show that given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x

$$0 < |x - 2| < \delta \quad \Longrightarrow \quad |f(x) - 4| < \epsilon.$$

Step 1: *Solve the inequality* $|f(x) - 4| < \epsilon$ *to find an open interval about* $x_0 = 2$ *on which the inequality holds for all* $x \neq x_0$.



1.23 The function in Example 6.

For $x \neq x_0 = 2$, we have $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \epsilon$:

$$|x^2 - 4| < \epsilon$$
$$-\epsilon < x^2 - 4 < \epsilon$$
$$4 - \epsilon < x^2 < 4 + \epsilon$$
$$\sqrt{4 - \epsilon} < |x| < \sqrt{4 + \epsilon} \qquad \text{Assumes } \epsilon < 4\text{; see below.}$$
$$\sqrt{4 - \epsilon} < x < \sqrt{4 + \epsilon}. \qquad \text{An open interval about } x_0 = 2 \text{ that solves the inequality}$$

The inequality $|f(x) - 4| < \epsilon$ holds for all $x \neq 2$ in the open interval $(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon})$ (Fig. 1.23).

Step 2: *Find a value of* $\delta > 0$ *that places the centered interval* $(2 - \delta, 2 + \delta)$ *inside the interval* $(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon})$.

Take δ to be the distance from $x_0 = 2$ to the nearer endpoint of $(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon})$. In other words, take $\delta = \min\ \{2 - \sqrt{4 - \epsilon}, \sqrt{4 + \epsilon} - 2\}$, the *minimum* (the smaller) of the two numbers $2 - \sqrt{4 - \epsilon}$ and $\sqrt{4 + \epsilon} - 2$. If δ has this or any smaller positive value, the inequality $0 < |x - 2| < \delta$ will automatically place x between $\sqrt{4 - \epsilon}$ and $\sqrt{4 + \epsilon}$ to make $|f(x) - 4| < \epsilon$. For all x,

$$0 < |x - 2| < \delta \qquad \Longrightarrow \qquad |f(x) - 4| < \epsilon.$$

This completes the proof.

Why was it all right to assume $\epsilon < 4$? Because, in finding a δ such that for all x, $0 < |x - 2| < \delta$ implied $|f(x) - 4| < \epsilon < 4$, we found a δ that would work for any larger ϵ as well.

Finally, notice the freedom we gained in letting $\delta = \min\ \{2 - \sqrt{4 - \epsilon}, \sqrt{4 + \epsilon} - 2\}$. We did not have to spend time deciding which, if either, number was the smaller of the two. We just let δ represent the smaller and went on to finish the argument. ❑

Using the Definition to Prove Theorems

We do not usually rely on the formal definition of limit to verify specific limits such as those in the preceding examples. Rather we appeal to general theorems about limits, in particular the theorems of Section 1.2. The definition is used to prove these theorems. As an example, we prove part 1 of Theorem 1, the Sum Rule.

EXAMPLE 7 *Proving the rule for the limit of a sum*

Given that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, prove that

$$\lim_{x \to c} (f(x) + g(x)) = L + M.$$

Solution Let $\epsilon > 0$ be given. We want to find a positive number δ such that for all x

$$0 < |x - c| < \delta \qquad \Longrightarrow \qquad |f(x) + g(x) - (L + M)| < \epsilon.$$

Regrouping terms, we get

$$|f(x) + g(x) - (L + M)| = |(f(x) - L) + (g(x) - M)|$$
$$\leq |f(x) - L| + |g(x) - M|. \qquad \text{Triangle Inequality: } |a + b| \leq |a| + |b|$$

Since $\lim_{x \to c} f(x) = L$, there exists a number $\delta_1 > 0$ such that for all x

$$0 < |x - c| < \delta_1 \quad \Longrightarrow \quad |f(x) - L| < \epsilon/2.$$

Similarly, since $\lim_{x \to c} g(x) = M$, there exists a number $\delta_2 > 0$ such that for all x

$$0 < |x - c| < \delta_2 \quad \Longrightarrow \quad |g(x) - M| < \epsilon/2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2. If $0 < |x - c| < \delta$ then $|x - c| < \delta_1$, so $|f(x) - L| < \epsilon/2$, and $|x - c| < \delta_2$, so $|g(x) - M| < \epsilon/2$. Therefore

$$|f(x) + g(x) - (L + M)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\lim_{x \to c} (f(x) + g(x)) = L + M$. ❑

Exercises 1.3

Centering Intervals About a Point

In Exercises 1–6, sketch the interval (a, b) on the x-axis with the point x_0 inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta \Longrightarrow a < x < b$.

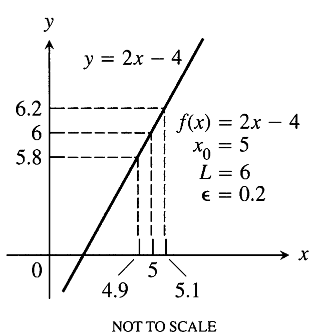
1. $a = 1, \quad b = 7, \quad x_0 = 5$
2. $a = 1, \quad b = 7, \quad x_0 = 2$
3. $a = -7/2, \quad b = -1/2, \quad x_0 = -3$
4. $a = -7/2, \quad b = -1/2, \quad x_0 = -3/2$
5. $a = 4/9, \quad b = 4/7, \quad x_0 = 1/2$
6. $a = 2.7591, \quad b = 3.2391, \quad x_0 = 3$

Finding Deltas Graphically

In Exercises 7–14, use the graphs to find a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - L| < \epsilon.$$

7.



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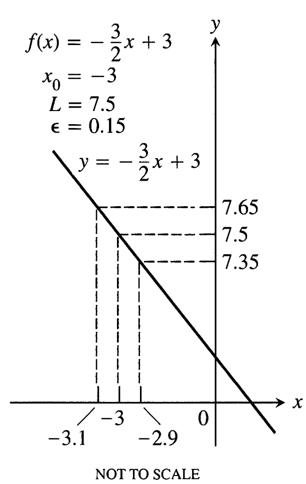

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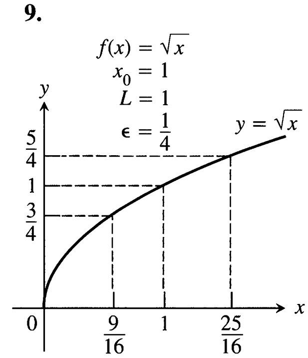

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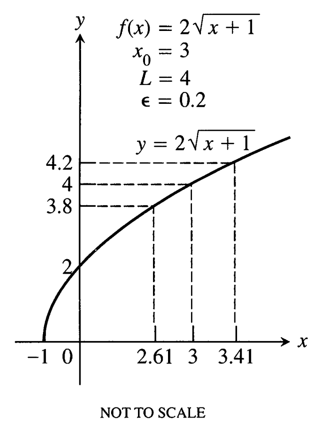

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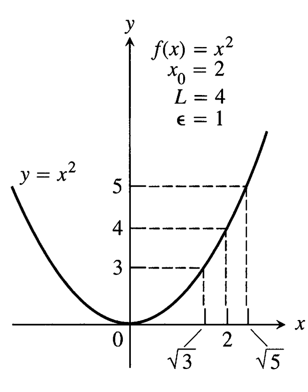

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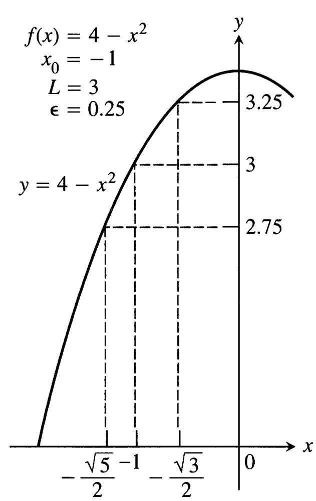

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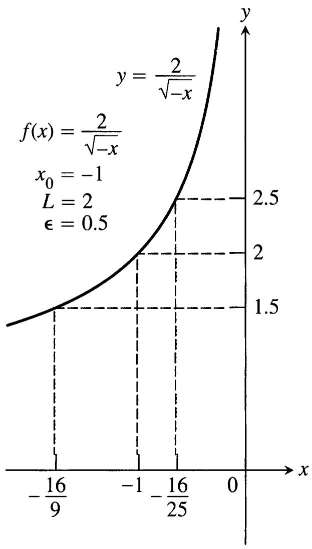

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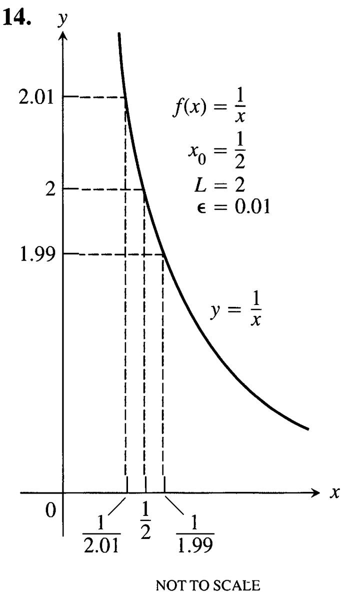


Finding Deltas Algebraically

Each of Exercises 15–30 gives a function $f(x)$ and numbers L, x_0, and $\epsilon > 0$. In each case, find an open interval about x_0 on which the inequality $|f(x) - L| < \epsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - x_0| < \delta$ the inequality $|f(x) - L| < \epsilon$ holds.

15. $f(x) = x + 1, \quad L = 5, \quad x_0 = 4, \quad \epsilon = 0.01$

16. $f(x) = 2x - 2, \quad L = -6, \quad x_0 = -2, \quad \epsilon = 0.02$

17. $f(x) = \sqrt{x + 1}, \quad L = 1, \quad x_0 = 0, \quad \epsilon = 0.1$

18. $f(x) = \sqrt{x}, \quad L = 1/2, \quad x_0 = 1/4, \quad \epsilon = 0.1$

19. $f(x) = \sqrt{19 - x}, \quad L = 3, \quad x_0 = 10, \quad \epsilon = 1$

20. $f(x) = \sqrt{x - 7}, \quad L = 4, \quad x_0 = 23, \quad \epsilon = 1$

21. $f(x) = 1/x, \quad L = 1/4, \quad x_0 = 4, \quad \epsilon = 0.05$

22. $f(x) = x^2, \quad L = 3, \quad x_0 = \sqrt{3}, \quad \epsilon = 0.1$

23. $f(x) = x^2, \quad L = 4, \quad x_0 = -2, \quad \epsilon = 0.5$

24. $f(x) = 1/x, \quad L = -1, \quad x_0 = -1, \quad \epsilon = 0.1$

25. $f(x) = x^2 - 5, \quad L = 11, \quad x_0 = 4, \quad \epsilon = 1$

26. $f(x) = 120/x, \quad L = 5, \quad x_0 = 24, \quad \epsilon = 1$

27. $f(x) = mx, \quad m > 0, \quad L = 2m, \quad x_0 = 2, \quad \epsilon = 0.03$

28. $f(x) = mx, \quad m > 0, \quad L = 3m, \quad x_0 = 3, \quad \epsilon = c > 0$

29. $f(x) = mx + b, \quad m > 0, \quad L = (m/2) + b, \quad x_0 = 1/2,$
$\epsilon = c > 0$

30. $f(x) = mx + b, \quad m > 0, \quad L = m + b, \quad x_0 = 1, \quad \epsilon = 0.05$

More on Formal Limits

Each of Exercises 31–36 gives a function $f(x)$, a point x_0, and a positive number ϵ. Find $L = \lim_{x \to x_0} f(x)$. Then find a number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

31. $f(x) = 3 - 2x, \quad x_0 = 3, \quad \epsilon = 0.02$

32. $f(x) = -3x - 2, \quad x_0 = -1, \quad \epsilon = 0.03$

33. $f(x) = \dfrac{x^2 - 4}{x - 2}, \quad x_0 = 2, \quad \epsilon = 0.05$

34. $f(x) = \dfrac{x^2 + 6x + 5}{x + 5}, \quad x_0 = -5, \quad \epsilon = 0.05$

35. $f(x) = \sqrt{1 - 5x}, \quad x_0 = -3, \quad \epsilon = 0.5$

36. $f(x) = 4/x, \quad x_0 = 2, \quad \epsilon = 0.4$

Prove the limit statements in Exercises 37–50.

37. $\lim_{x \to 4} (9 - x) = 5$

38. $\lim_{x \to 3} (3x - 7) = 2$

39. $\lim_{x \to 9} \sqrt{x - 5} = 2$

40. $\lim_{x \to 0} \sqrt{4 - x} = 2$

41. $\lim_{x \to 1} f(x) = 1 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$

42. $\lim_{x \to -2} f(x) = 4 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq -2 \\ 1, & x = -2 \end{cases}$

43. $\lim_{x \to 1} \dfrac{1}{x} = 1$

44. $\lim_{x \to \sqrt{3}} \dfrac{1}{x^2} = \dfrac{1}{3}$

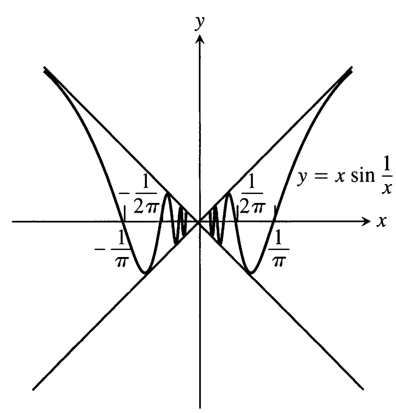
45. $\lim_{x \to -3} \dfrac{x^2 - 9}{x + 3} = -6$

46. $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2$

47. $\lim_{x \to 1} f(x) = 2 \quad \text{if} \quad f(x) = \begin{cases} 4 - 2x, & x < 1 \\ 6x - 4, & x \geq 1 \end{cases}$

48. $\lim_{x \to 0} f(x) = 0 \quad \text{if} \quad f(x) = \begin{cases} 2x, & x < 0 \\ x/2, & x \geq 0 \end{cases}$

49. $\lim_{x \to 0} x \sin \dfrac{1}{x} = 0$



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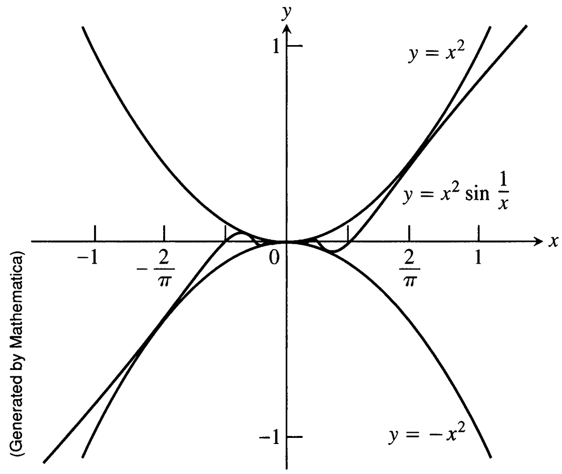
50. $\lim_{x\to 0} x^2 \sin\dfrac{1}{x} = 0$



Theory and Examples

51. Define what it means to say that $\lim_{x\to 2} f(x) = 5$.

52. Define what it means to say that $\lim_{x\to 0} g(x) = k$.

53. *A wrong statement about limits.* Show by example that the following statement is wrong.

> The number L is the limit of $f(x)$ as x approaches x_0 if $f(x)$ gets closer to L as x approaches x_0.

Explain why the function in your example does not have the given value of L as a limit as $x \to x_0$.

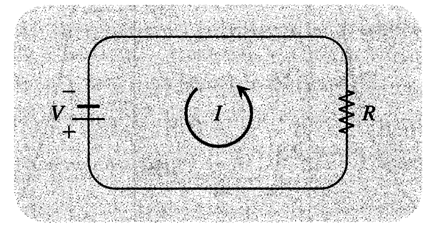
54. *Another wrong statement about limits.* Show by example that the following statement is wrong.

> The number L is the limit of $f(x)$ as x approaches x_0 if, given any $\epsilon > 0$, there exists a value of x for which $|f(x) - L| < \epsilon$.

Explain why the function in your example does not have the given value of L as a limit as $x \to x_0$.

55. *Grinding engine cylinders.* Before contracting to grind engine cylinders to a cross-section area of 9 in^2, you need to know how much deviation from the ideal cylinder diameter of $x_0 = 3.385$ in. you can allow and still have the area come within 0.01 in^2 of the required 9 in^2. To find out, you let $A = \pi(x/2)^2$ and look for the interval in which you must hold x to make $|A - 9| \le 0.01$. What interval do you find?

56. *Manufacturing electrical resistors.* Ohm's law for electrical circuits like the one shown in Fig. 1.24 states that $V = RI$. In this equation, V is a constant voltage, I is the current in amperes, and R is the resistance in ohms. Your firm has been asked to supply the resistors for a circuit in which V will be 120 volts and I is to be 5 ± 0.1 amp. In what interval does R have to lie for I to be within 0.1 amp of the target value $I_0 = 5$?



1.24 The circuit in Exercise 56.

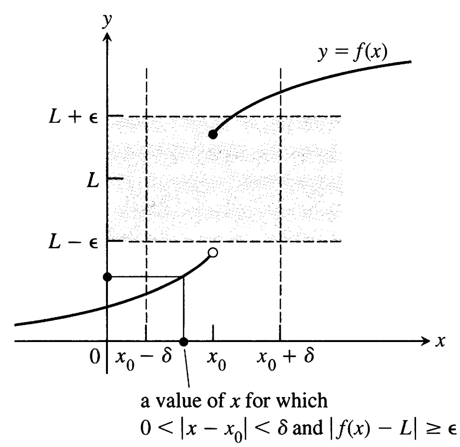
When Is a Number *L Not* the Limit of *f(x)* as $x \to x_0$?

We can prove that $\lim_{x\to x_0} f(x) \ne L$ by providing an $\epsilon > 0$ such that no possible $\delta > 0$ satisfies the condition

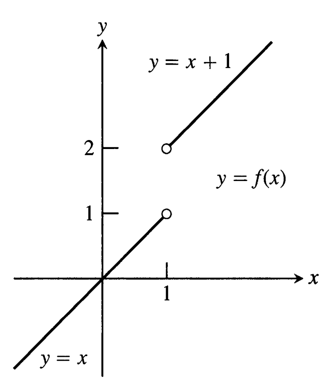
$$\text{For all } x,\quad 0 < |x - x_0| < \delta \quad\Longrightarrow\quad |f(x) - L| < \epsilon.$$

We accomplish this for our candidate ϵ by showing that for each $\delta > 0$ there exists a value of x such that

$$0 < |x - x_0| < \delta \quad\text{and}\quad |f(x) - L| \ge \epsilon.$$



57. Let $f(x) = \begin{cases} x, & x < 1 \\ x + 1, & x > 1. \end{cases}$



a) Let $\epsilon = 1/2$. Show that no possible $\delta > 0$ satisfies the following condition:

For all x, $0 < |x - 1| < \delta \quad \Longrightarrow \quad |f(x) - 2| < 1/2$.

That is, for each $\delta > 0$ show that there is a value of x such that

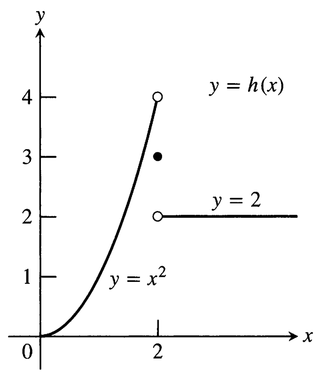
$$0 < |x - 1| < \delta \quad \text{and} \quad |f(x) - 2| \geq 1/2.$$

This will show that $\lim_{x \to 1} f(x) \neq 2$.

b) Show that $\lim_{x \to 1} f(x) \neq 1$.

c) Show that $\lim_{x \to 1} f(x) \neq 1.5$.

58. Let $h(x) = \begin{cases} x^2, & x < 2 \\ 3, & x = 2 \\ 2, & x > 2. \end{cases}$



Show that

a) $\lim_{x \to 2} h(x) \neq 4$

b) $\lim_{x \to 2} h(x) \neq 3$

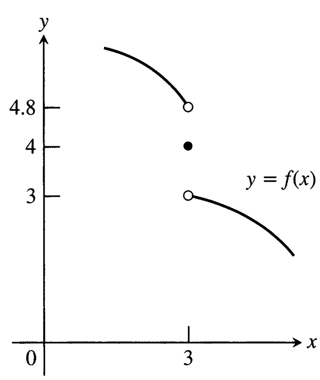
c) $\lim_{x \to 2} h(x) \neq 2$

59. For the function graphed here, show that

a) $\lim_{x \to 3} f(x) \neq 4$

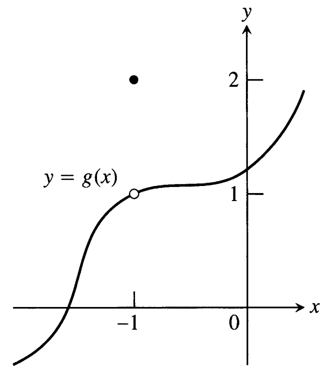
b) $\lim_{x \to 3} f(x) \neq 4.8$

c) $\lim_{x \to 3} f(x) \neq 3$



60. a) For the function graphed here, show that $\lim_{x \to -1} g(x) \neq 2$.

b) Does $\lim_{x \to -1} g(x)$ appear to exist? If so, what is the value of the limit? If not, why not?



CAS Explorations and Projects

In Exercises 61–66, you will further explore finding deltas graphically. Use a CAS to perform the following steps:

a) Plot the function $y = f(x)$ near the point x_0 being approached.

b) Guess the value of the limit L and then evaluate the limit symbolically to see if you guessed correctly.

c) Using the value $\epsilon = 0.2$, graph the banding lines $y_1 = L - \epsilon$ and $y_2 = L + \epsilon$ together with the function f near x_0.

d) From your graph in part (c), estimate a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - L| < \epsilon.$$

Test your estimate by plotting f, y_1, and y_2 over the interval $0 < |x - x_0| < \delta$. For your viewing window use $x_0 - 2\delta \leq x \leq x_0 + 2\delta$ and $L - 2\epsilon \leq y \leq L + 2\epsilon$. If any function values lie outside the interval $[L - \epsilon, L + \epsilon]$, your choice of δ was too large. Try again with a smaller estimate.

e) Repeat parts (c) and (d) successively for $\epsilon = 0.1, 0.05$, and 0.001.

61. $f(x) = \dfrac{x^4 - 81}{x - 3}, \quad x_0 = 3$

62. $f(x) = \dfrac{5x^3 + 9x^2}{2x^5 + 3x^2}, \quad x_0 = 0$

63. $f(x) = \dfrac{\sin 2x}{3x}, \quad x_0 = 0$

64. $f(x) = \dfrac{x(1 - \cos x)}{x - \sin x}, \quad x_0 = 0$

65. $f(x) = \dfrac{\sqrt[3]{x} - 1}{x - 1}, \quad x_0 = 1$

66. $f(x) = \dfrac{3x^2 - (7x + 1)\sqrt{x} + 5}{x - 1}, \quad x_0 = 1$

1.4 Extensions of the Limit Concept

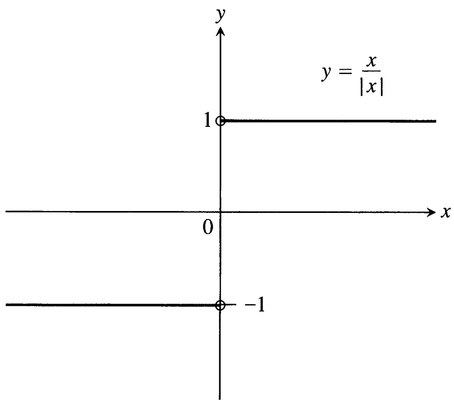
In this section we extend the concept of limit to

1. *one-sided limits,* which are limits as x approaches a from the left-hand side or the right-hand side only,
2. *infinite limits,* which are not really limits at all, but provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large, positive or negative.

One-Sided Limits

To have a limit L as x approaches a, a function f must be defined on *both sides* of a, and its values $f(x)$ must approach L as x approaches a from either side. Because of this, ordinary limits are sometimes called **two-sided** limits.

It is possible for a function to approach a limiting value as x approaches a from only one side, either from the right or from the left. In this case we say that f has a **one-sided** (either right-hand or left-hand) limit at a. The function $f(x) = x/|x|$ graphed in Fig. 1.25 has limit 1 as x approaches zero from the right, and limit -1 as x approaches zero from the left.



1.25 Different right-hand and left-hand limits at the origin.

Definition

Informal Definition of Right-hand and Left-hand Limits

Let $f(x)$ be defined on an interval (a, b) where $a < b$. If $f(x)$ approaches arbitrarily close to L as x approaches a from within that interval, then we say that f has **right-hand limit** L at a, and we write

$$\lim_{x \to a^+} f(x) = L.$$

Let $f(x)$ be defined on an interval (c, a) where $c < a$. If $f(x)$ approaches arbitrarily close to M as x approaches a from within the interval (c, a), then we say that f has **left-hand limit** M at a, and we write

$$\lim_{x \to a^-} f(x) = M.$$

For the function $f(x) = x/|x|$ in Fig. 1.25, we have

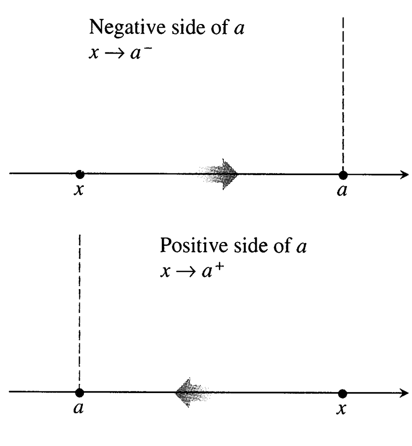
$$\lim_{x \to 0^+} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 0^-} f(x) = -1.$$

The "+" and "−"

The significance of the signs in the notation for one-sided limits is this:

$\boldsymbol{x \longrightarrow a^-}$ means x approaches a from the negative side of a, through values less than a.

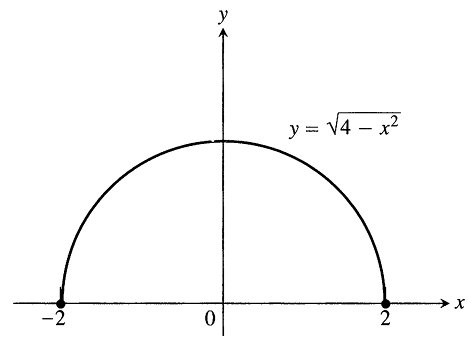
$\boldsymbol{x \longrightarrow a^+}$ means x approaches a from the positive side of a, through values greater than a.



A function cannot have an ordinary limit at an endpoint of its domain, but it can have a one-sided limit.

EXAMPLE 1 The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle in Fig. 1.26. We have

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \qquad \text{and} \qquad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$



1.26 $\lim_{x\to 2^-} \sqrt{4-x^2} = 0$, $\lim_{x\to -2^+} \sqrt{4-x^2} = 0$.

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have ordinary two-sided limits at either -2 or 2. ❑

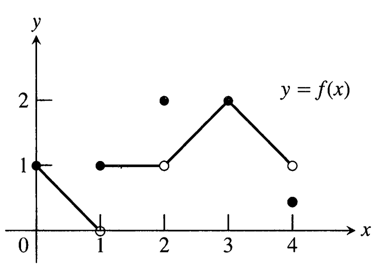
One-sided limits have all the limit properties listed in Theorem 1, Section 1.2. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as does the Sandwich Theorem.

The connection between one-sided and two-sided limits is stated in the following theorem (proved at the end of this section).

Theorem 5
One-sided vs. Two-sided Limits

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there, and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$



1.27 Graph of the function in Example 2.

EXAMPLE 2 All of the following statements about the function graphed in Figure 1.27 are true.

At $x = 0$: $\lim_{x\to 0^+} f(x) = 1$,
$\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$ do not exist. (The function is not defined to the left of $x = 0$.)

At $x = 1$: $\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$,
$\lim_{x\to 1^+} f(x) = 1$,
$\lim_{x\to 1} f(x)$ does not exist. (The right- and left-hand limits are not equal.)

At $x = 2$: $\lim_{x\to 2^-} f(x) = 1$,
$\lim_{x\to 2^+} f(x) = 1$,
$\lim_{x\to 2} f(x) = 1$ even though $f(2) = 2$.

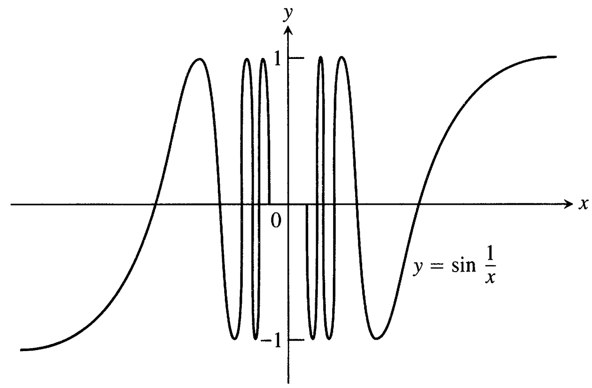
At $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$

At $x = 4$: $\lim_{x\to 4^-} f(x) = 1$ even though $f(4) \neq 1$,
$\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4} f(x)$ do not exist. (The function is not defined to the right of $x = 4$.)

At every other point a in $[0, 4]$, $f(x)$ has limit $f(a)$. ❑

In the examples so far in this section, the functions that failed to have a limit at some point at least had one existing one-sided limit there. The function in the following example has neither a left-hand limit nor a right-hand limit at $x = 0$ even though it is defined everywhere except at $x = 0$.

EXAMPLE 3 Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side (Fig. 1.28).



1.28 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 3).

Solution As x approaches zero, its reciprocal, $1/x$, grows without bound and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$. ❑

Infinite Limits

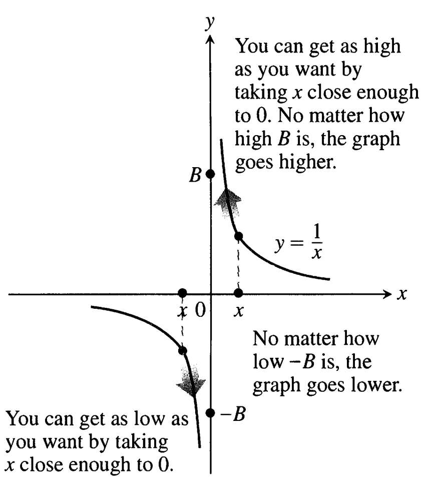
Let us look closely at the function $f(x) = 1/x$ that drives the sine in Example 3. As $x \to 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still (Fig. 1.29). Thus, f has no limit as $x \to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$. We write

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty.$$

In writing this, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞, for there is no such number. Rather, we are saying that $\lim_{x\to 0^+} (1/x)$ *does not exist because* $1/x$ *becomes arbitrarily large and positive as* $x \to 0^+$.

As $x \to 0^-$, the values of $f(x) = 1/x$ become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$. (See Fig. 1.29.) We write

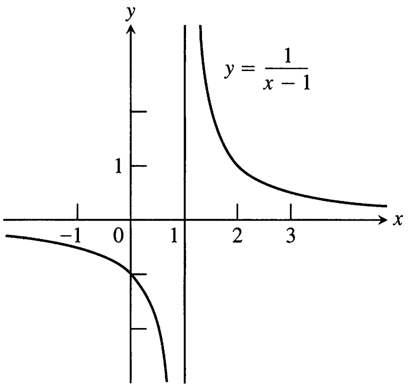
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$



1.29 One-sided infinite limits:

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There *is* no real number $-\infty$. We are describing the behavior of a function whose limit as $x \to 0^-$ *does not exist because its values become arbitrarily large and negative.*



1.30 Near $x = 1$, the function $y = 1/(x-1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right.

EXAMPLE 4 *One-sided infinite limits*

Find $\displaystyle\lim_{x\to 1^+} \frac{1}{x-1}$ and $\displaystyle\lim_{x\to 1^-} \frac{1}{x-1}$.

Geometric Solution The graph of $y = 1/(x-1)$ is the graph of $y = 1/x$ shifted 1 unit to the right (Fig. 1.30). Therefore, $y = 1/(x-1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty \quad \text{and} \quad \lim_{x\to 1^-} \frac{1}{x-1} = -\infty.$$

Analytic Solution Think about the number $x-1$ and its reciprocal. As $x \to 1^+$, we have $(x-1) \to 0^+$ and $1/(x-1) \to \infty$. As $x \to 1^-$, we have $(x-1) \to 0^-$ and $1/(x-1) \to -\infty$. ❑

EXAMPLE 5 *Two-sided infinite limits*

Discuss the behavior of

a) $f(x) = \dfrac{1}{x^2}$ near $x = 0$,

b) $g(x) = \dfrac{1}{(x+3)^2}$ near $x = -3$.

Solution

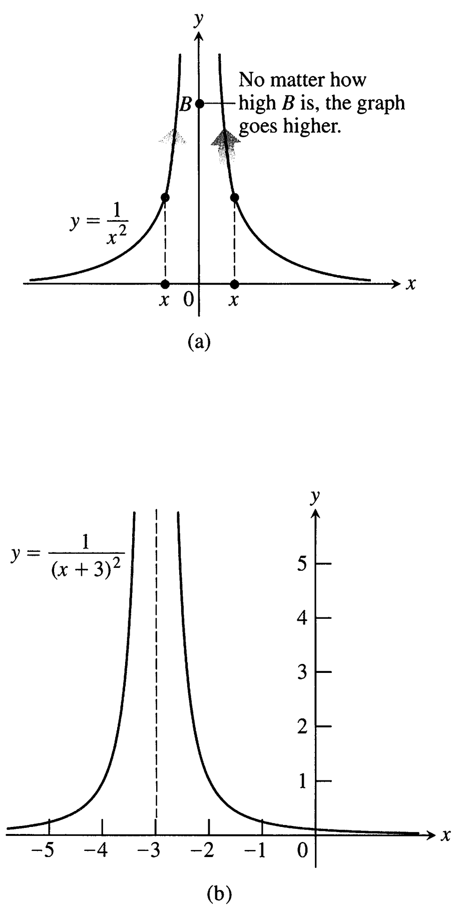
a) As x approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large (Fig. 1.31a):

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = \infty.$$

b) The graph of $g(x) = 1/(x+3)^2$ is the graph of $f(x) = 1/x^2$ shifted 3 units to the left (Fig. 1.31b). Therefore, g behaves near -3 exactly the way f behaves near 0.

$$\lim_{x\to -3} g(x) = \lim_{x\to -3} \frac{1}{(x+3)^2} = \infty.$$ ❑

The function $y = 1/x$ shows no consistent behavior as $x \to 0$. We have $1/x \to \infty$ if $x \to 0^+$, but $1/x \to -\infty$ if $x \to 0^-$. All we can say about $\lim_{x\to 0}(1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x\to 0}(1/x^2) = \infty$.



1.31 The graphs of the functions in Example 5.

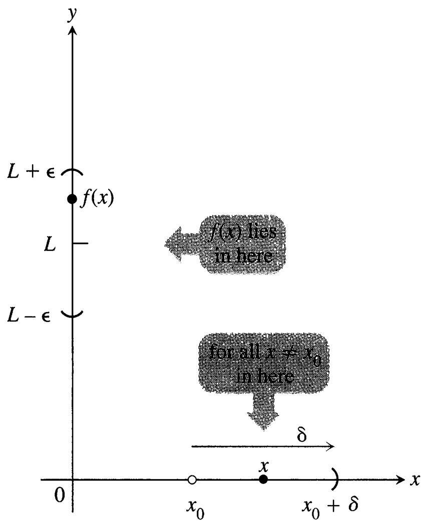
EXAMPLE 6 *Rational functions can behave in various ways near zeros of their denominators.*

a) $\displaystyle\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = 0$

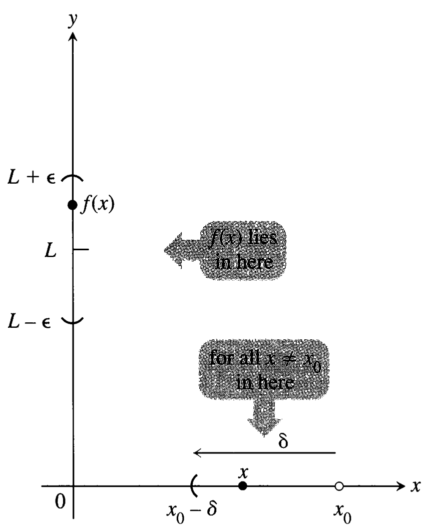
b) $\displaystyle\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

c) $\displaystyle\lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$

The values are negative for $x > 2$, x near 2.



1.32 Diagram for the definition of right-hand limit.



1.33 Diagram for the definition of left-hand limit.

d) $$\lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$$ The values are positive for $x < 2$, x near 2.

e) $$\lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)} \text{ does not exist.}$$ See (c) and (d).

f) $$\lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$$

In parts (a) and (b) the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancellation still leaves a zero in the denominator. ❑

Precise Definitions of One-sided Limits

The formal definition of two-sided limit in Section 1.3 is readily modified for one-sided limits.

Definitions

Right-hand Limit

We say that $f(x)$ has right-hand limit L at x_0, and write

$$\lim_{x\to x_0^+} f(x) = L \qquad \text{(See Fig. 1.32)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon. \tag{1}$$

Left-hand Limit

We say that f has left-hand limit L at x_0, and write

$$\lim_{x\to x_0^-} f(x) = L \qquad \text{(See Fig. 1.33)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \epsilon. \tag{2}$$

The Relation Between One- and Two-sided Limits

If we subtract x_0 from the δ-inequalities in implications (1) and (2), we can see the logical relation between the one-sided limits just defined and the two-sided limit defined in Section 1.3. For right-hand limits, subtracting x_0 gives

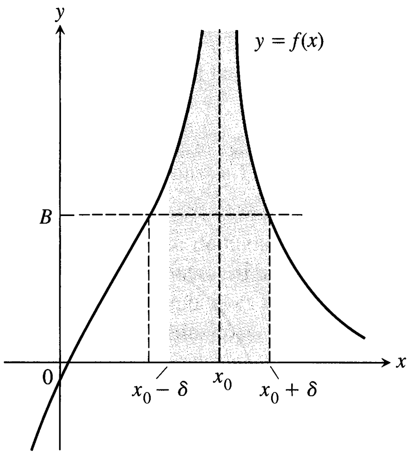
$$0 < x - x_0 < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon; \tag{3}$$

for left-hand limits we get

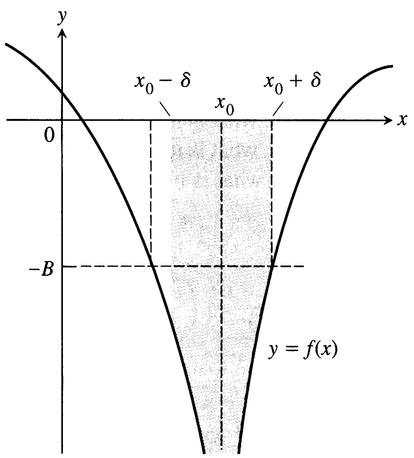
$$-\delta < x - x_0 < 0 \quad \Rightarrow \quad |f(x) - L| < \epsilon. \tag{4}$$

Together, (3) and (4) say the same thing as

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon, \tag{5}$$



1.34 $\lim_{x \to x_0} f(x) = \infty.$



1.35 $\lim_{x \to x_0} f(x) = -\infty.$

the implication required for two-sided limit. Thus, f has limit L at x_0 if and only if f has right-hand limit L and left-hand limit L at x_0.

Precise Definitions of Infinite Limits

Instead of requiring $f(x)$ to lie arbitrarily close to a finite number L for all x sufficiently close to x_0, the definitions of infinite limits require $f(x)$ to lie arbitrarily far from the origin. Except for this change, the language is identical with what we have seen before. Figures 1.34 and 1.35 accompany these definitions.

Definitions

Infinite Limits

1. We say that $f(x)$ approaches infinity as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) > B.$$

2. We say that $f(x)$ approaches minus infinity as x approaches x_0, and write

$$\lim_{x \to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

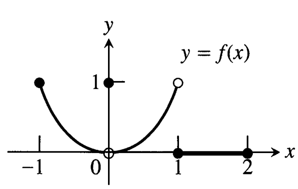
$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) < -B.$$

The precise definitions of one-sided infinite limits at x_0 are similar and are stated in the exercises.

Exercises 1.4

Finding Limits Graphically

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a) $\lim_{x \to -1^+} f(x) = 1$

b) $\lim_{x \to 0^-} f(x) = 0$

c) $\lim_{x \to 0^-} f(x) = 1$

d) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

e) $\lim_{x \to 0} f(x)$ exists

f) $\lim_{x \to 0} f(x) = 0$

g) $\lim_{x \to 0} f(x) = 1$

h) $\lim_{x \to 1} f(x) = 1$

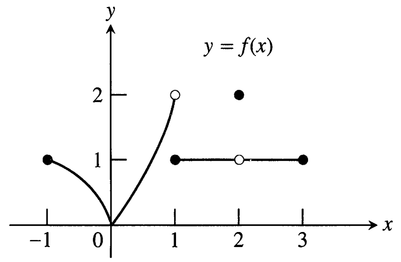
i) $\lim_{x \to 1} f(x) = 0$

j) $\lim_{x \to 2^-} f(x) = 2$

k) $\lim_{x \to -1^-} f(x)$ does not exist.

l) $\lim_{x \to 2^+} f(x) = 0$

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a) $\lim_{x\to -1^+} f(x) = 1$

b) $\lim_{x\to 2} f(x)$ does not exist.

c) $\lim_{x\to 2} f(x) = 2$

d) $\lim_{x\to 1^-} f(x) = 2$

e) $\lim_{x\to 1^+} f(x) = 1$

f) $\lim_{x\to 1} f(x)$ does not exist.

g) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

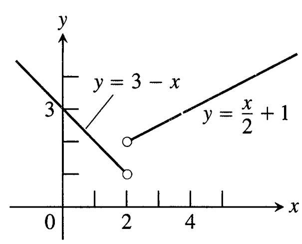
h) $\lim_{x\to c} f(x)$ exists at every c in the open interval $(-1, 1)$.

i) $\lim_{x\to c} f(x)$ exists at every c in the open interval $(1, 3)$.

j) $\lim_{x\to -1^-} f(x) = 0$

k) $\lim_{x\to 3^+} f(x)$ does not exist.

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2 \end{cases}$



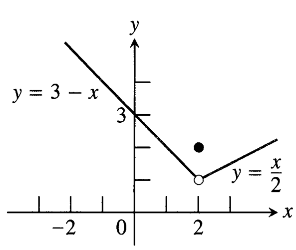
a) Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$.

b) Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c) Find $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$.

d) Does $\lim_{x\to 4} f(x)$ exist? If so, what is it? If not, why not?

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \dfrac{x}{2}, & x > 2. \end{cases}$



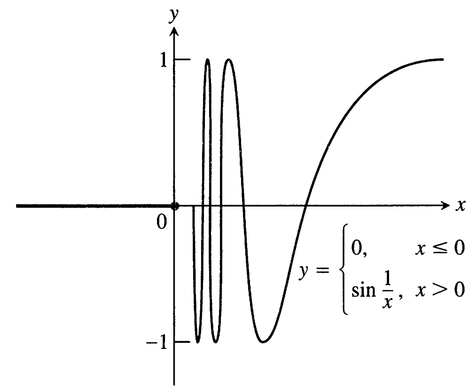
a) Find $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $f(2)$.

b) Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c) Find $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$.

d) Does $\lim_{x\to -1} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin \dfrac{1}{x}, & x > 0. \end{cases}$

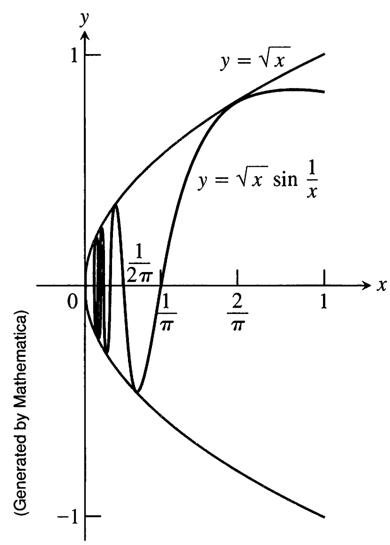


a) Does $\lim_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

6. Let $g(x) = \sqrt{x}\sin(1/x)$.



a) Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b) Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c) Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

7. a) Graph $f(x) = \begin{cases} x^3, & x \ne 1 \\ 0, & x = 1. \end{cases}$

b) Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

c) Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

8. a) Graph $f(x) = \begin{cases} 1 - x^2, & x \neq 1 \\ 2, & x = 1. \end{cases}$

b) Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

c) Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

a) What are the domain and range of f?

b) At what points c, if any, does $\lim_{x\to c} f(x)$ exist?

c) At what points does only the left-hand limit exist?

d) At what points does only the right-hand limit exist?

9. $f(x) = \begin{cases} \sqrt{1-x^2} & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2 \\ 2 & \text{if } x = 2 \end{cases}$

10. $f(x) = \begin{cases} x & \text{if } -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x < -1, \text{ or } x > 1 \end{cases}$

Finding Limits Algebraically

Find the limits in Exercises 11–20.

11. $\lim\limits_{x\to -0.5^-} \sqrt{\dfrac{x+2}{x+1}}$

12. $\lim\limits_{x\to 1^+} \sqrt{\dfrac{x-1}{x+2}}$

13. $\lim\limits_{x\to -2^+} \left(\dfrac{x}{x+1}\right)\left(\dfrac{2x+5}{x^2+x}\right)$

14. $\lim\limits_{x\to 1^-} \left(\dfrac{1}{x+1}\right)\left(\dfrac{x+6}{x}\right)\left(\dfrac{3-x}{7}\right)$

15. $\lim\limits_{h\to 0^+} \dfrac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}$

16. $\lim\limits_{h\to 0^-} \dfrac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}$

17. a) $\lim\limits_{x\to -2^+} (x+3)\dfrac{|x+2|}{x+2}$ **b)** $\lim\limits_{x\to -2^-} (x+3)\dfrac{|x+2|}{x+2}$

18. a) $\lim\limits_{x\to 1^+} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$ **b)** $\lim\limits_{x\to 1^-} \dfrac{\sqrt{2x}(x-1)}{|x-1|}$

19. a) $\lim\limits_{\theta\to 3^+} \dfrac{\lfloor\theta\rfloor}{\theta}$ **b)** $\lim\limits_{\theta\to 3^-} \dfrac{\lfloor\theta\rfloor}{\theta}$

20. a) $\lim\limits_{t\to 4^+} (t - \lfloor t\rfloor)$ **b)** $\lim\limits_{t\to 4^-} (t - \lfloor t\rfloor)$

Infinite Limits

Find the limits in Exercises 21–32.

21. $\lim\limits_{x\to 0^+} \dfrac{1}{3x}$

22. $\lim\limits_{x\to 0^-} \dfrac{5}{2x}$

23. $\lim\limits_{x\to 2^-} \dfrac{3}{x-2}$

24. $\lim\limits_{x\to 3^+} \dfrac{1}{x-3}$

25. $\lim\limits_{x\to -8^+} \dfrac{2x}{x+8}$

26. $\lim\limits_{x\to -5^-} \dfrac{3x}{2x+10}$

27. $\lim\limits_{x\to 7} \dfrac{4}{(x-7)^2}$

28. $\lim\limits_{x\to 0} \dfrac{-1}{x^2(x+1)}$

29. a) $\lim\limits_{x\to 0^+} \dfrac{2}{3x^{1/3}}$ **b)** $\lim\limits_{x\to 0^-} \dfrac{2}{3x^{1/3}}$

30. a) $\lim\limits_{x\to 0^+} \dfrac{2}{x^{1/5}}$ **b)** $\lim\limits_{x\to 0^-} \dfrac{2}{x^{1/5}}$

31. $\lim\limits_{x\to 0} \dfrac{4}{x^{2/5}}$

32. $\lim\limits_{x\to 0} \dfrac{1}{x^{2/3}}$

Find the limits in Exercises 33–36.

33. $\lim\limits_{x\to (\pi/2)^-} \tan x$

34. $\lim\limits_{x\to (-\pi/2)^+} \sec x$

35. $\lim\limits_{\theta\to 0^-} (1+\csc\theta)$

36. $\lim\limits_{\theta\to 0} (2-\cot\theta)$

Additional Calculations

Find the limits in Exercises 37–42.

37. $\lim \dfrac{1}{x^2-4}$ as

a) $x\to 2^+$ **b)** $x\to 2^-$

c) $x\to -2^+$ **d)** $x\to -2^-$

38. $\lim \dfrac{x}{x^2-1}$ as

a) $x\to 1^+$ **b)** $x\to 1^-$

c) $x\to -1^+$ **d)** $x\to -1^-$

39. $\lim \left(\dfrac{x^2}{2}-\dfrac{1}{x}\right)$ as

a) $x\to 0^+$ **b)** $x\to 0^-$

c) $x\to \sqrt[3]{2}$ **d)** $x\to -1$

40. $\lim \dfrac{x^2-1}{2x+4}$ as

a) $x\to -2^+$ **b)** $x\to -2^-$

c) $x\to 1^+$ **d)** $x\to 0^-$

41. $\lim \dfrac{x^2-3x+2}{x^3-2x^2}$ as

a) $x\to 0^+$ **b)** $x\to 2^+$

c) $x\to 2^-$ **d)** $x\to 2$

e) What, if anything, can be said about the limit as $x\to 0$?

42. $\lim \dfrac{x^2-3x+2}{x^3-4x}$ as

a) $x\to 2^+$ **b)** $x\to -2^+$

c) $x\to 0^-$ **d)** $x\to 1^+$

e) What, if anything, can be said about the limit as $x\to 0$?

Find the limits in Exercises 43–46.

43. $\lim \left(2-\dfrac{3}{t^{1/3}}\right)$ as

a) $t\to 0^+$ **b)** $t\to 0^-$

44. $\lim \left(\frac{1}{t^{3/5}} + 7\right)$ as

a) $t \to 0^+$
b) $t \to 0^-$

45. $\lim \left(\frac{1}{x^{2/3}} + \frac{2}{(x-1)^{2/3}}\right)$ as

a) $x \to 0^+$
b) $x \to 0^-$
c) $x \to 1^+$
d) $x \to 1^-$

46. $\lim \left(\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}\right)$ as

a) $x \to 0^+$
b) $x \to 0^-$
c) $x \to 1^+$
d) $x \to 1^-$

Theory and Examples

47. Once you know $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ at an interior point of the domain of f, do you then know $\lim_{x\to a} f(x)$? Give reasons for your answer.

48. If you know that $\lim_{x\to c} f(x)$ exists, can you find its value by calculating $\lim_{x\to c^+} f(x)$? Give reasons for your answer.

49. Suppose that f is an odd function of x. Does knowing that $\lim_{x\to 0^+} f(x) = 3$ tell you anything about $\lim_{x\to 0^-} f(x)$? Give reasons for your answer.

50. Suppose that f is an even function of x. Does knowing that $\lim_{x\to 2^-} f(x) = 7$ tell you anything about either $\lim_{x\to -2^-} f(x)$ or $\lim_{x\to -2^+} f(x)$? Give reasons for your answer.

Formal Definitions of One-sided Limits

51. Given $\epsilon > 0$, find an interval $I = (5, 5+\delta)$, $\delta > 0$, such that if x lies in I, then $\sqrt{x-5} < \epsilon$. What limit is being verified and what is its value?

52. Given $\epsilon > 0$, find an interval $I = (4-\delta, 4)$, $\delta > 0$, such that if x lies in I, then $\sqrt{4-x} < \epsilon$. What limit is being verified and what is its value?

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 53 and 54.

53. $\lim\limits_{x\to 0^-} \frac{x}{|x|} = -1$

54. $\lim\limits_{x\to 2^+} \frac{x-2}{|x-2|} = 1$

55. Find (a) $\lim_{x\to 400^+} \lfloor x \rfloor$ and (b) $\lim_{x\to 400^-} \lfloor x \rfloor$; then use limit definitions to verify your findings. (c) Based on your conclusions in (a) and (b), can anything be said about $\lim_{x\to 400} \lfloor x \rfloor$? Give reasons for your answers.

56. Let $f(x) = \begin{cases} x^2 \sin(1/x), & x < 0 \\ \sqrt{x}, & x > 0. \end{cases}$

Find (a) $\lim_{x\to 0^+} f(x)$ and (b) $\lim_{x\to 0^-} f(x)$; then use limit definitions to verify your findings. (c) Based on your conclusions in (a) and (b), can anything be said about $\lim_{x\to 0} f(x)$? Give reasons for your answer.

The Formal Definition of Infinite Limit

Use formal definitions to prove the limit statements in Exercises 57–60.

57. $\lim\limits_{x\to 0} \frac{1}{x^2} = \infty$

58. $\lim\limits_{x\to 0} \frac{-1}{x^2} = -\infty$

59. $\lim\limits_{x\to 3} \frac{-2}{(x-3)^2} = -\infty$

60. $\lim\limits_{x\to -5} \frac{1}{(x+5)^2} = \infty$

Formal Definitions of Infinite One-sided Limits

61. Here is the definition of **infinite right-hand limit.**

> We say that $f(x)$ approaches infinity as x approaches x_0 from the right, and write
>
> $$\lim_{x\to x_0^+} f(x) = \infty,$$
>
> if, for every positive real number B, there exists a corresponding number $\delta > 0$ such that for all x
>
> $$x_0 < x < x_0 + \delta \quad \Rightarrow \quad f(x) > B.$$

Modify the definition to cover the following cases.

a) $\lim\limits_{x\to x_0^-} f(x) = \infty$
b) $\lim\limits_{x\to x_0^+} f(x) = -\infty$
c) $\lim\limits_{x\to x_0^-} f(x) = -\infty$

Use the formal definitions from Exercise 61 to prove the limit statements in Exercises 62–67.

62. $\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$

63. $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$

64. $\lim\limits_{x\to 2^-} \frac{1}{x-2} = -\infty$

65. $\lim\limits_{x\to 2^+} \frac{1}{x-2} = \infty$

66. $\lim\limits_{x\to 1^+} \frac{1}{1-x^2} = -\infty$

67. $\lim\limits_{x\to 1^-} \frac{1}{1-x^2} = \infty$

1.5 Continuity

When we plot function values generated in the laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the times we did not measure. In doing so, we are assuming that we are working with a continuous function, a function whose outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between.

So many physical processes proceed continuously that throughout the eighteenth and nineteenth centuries it rarely occurred to anyone to look for any other kind of behavior. It came as quite a surprise when the physicists of the 1920s discovered that the vibrating atoms in a hydrogen molecule can oscillate only at discrete energy levels, that light comes in particles, and that, when heated, atoms emit light at discrete frequencies and not in continuous spectra. As a result of these and other discoveries, and because of the heavy use of discrete functions in computer science and statistics, the issue of continuity has become one of practical as well as theoretical importance.

In this section, we define continuity, show how to tell whether a function is continuous at a given point, and examine the intermediate value property of continuous functions.

Continuity at a Point

In practice, most functions of a real variable have domains that are intervals or unions of separate intervals, and it is natural to restrict our study of continuity to functions with these domains. This leaves us with only three kinds of points to consider: **interior points** (points that lie in an open interval in the domain), **left endpoints,** and **right endpoints.**

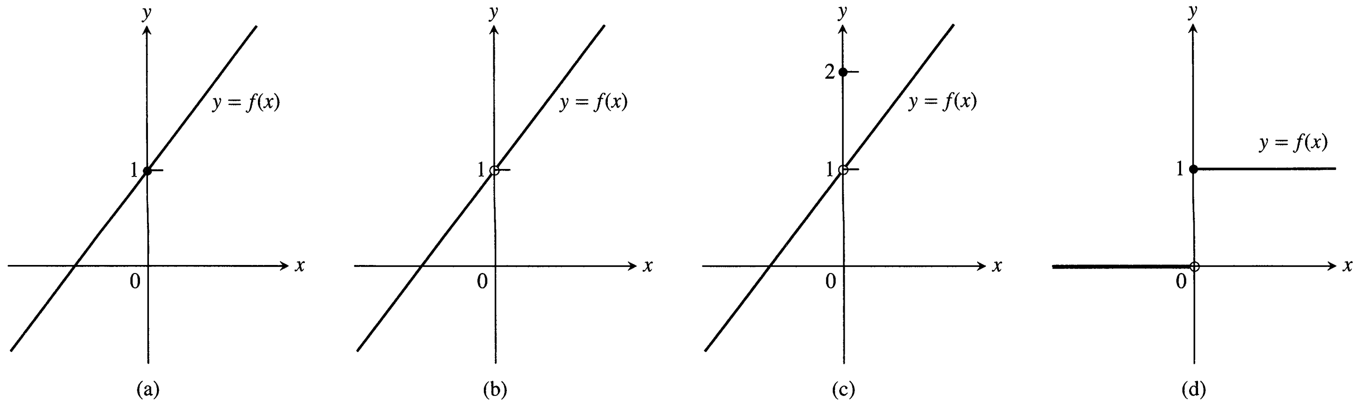
Definition

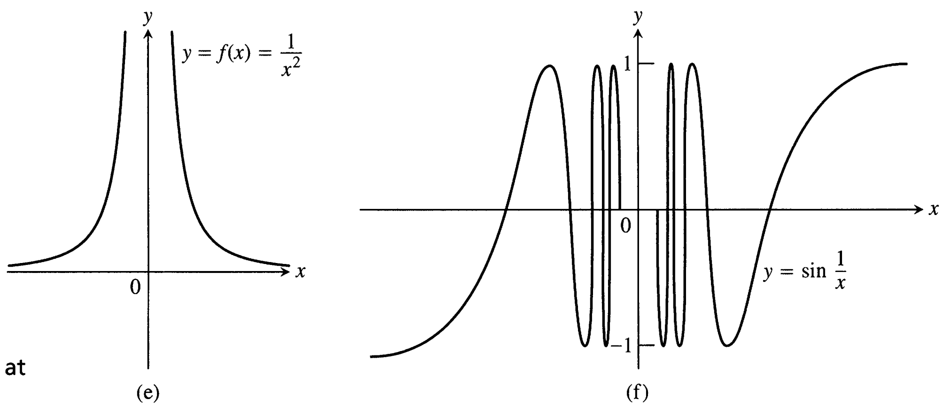
A function f is **continuous at an interior point** $x = c$ of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

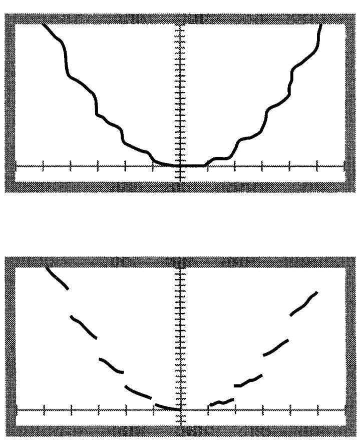
In Fig. 1.36 on the following page, the first function is continuous at $x = 0$. The function in (b) would be continuous if it had $f(0) = 1$. The function in (c) would be continuous if $f(0)$ were 1 instead of 2. The discontinuities in (b) and (c) are **removable.** Each function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in parts (d)–(f) of Fig. 1.36 are more serious: $\lim_{x \to 0} f(x)$ does not exist and there is no way to improve the situation by changing f at 0. The step function in (d) has a **jump discontinuity:** the one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in (e) has an **infinite discontinuity.** Jumps and infinite discontinuities are the ones most frequently encountered, but there are others. The function in (f) is discontinuous at the origin because it oscillates too much to have a limit as $x \to 0$.





1.36 The function in (a) is continuous at $x = 0$; the functions in (b)–(f) are not.

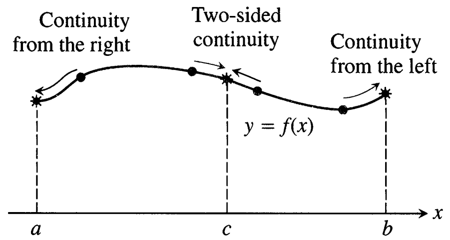


a) $y_1 = x$*int x incorrectly graphed in connected mode.
b) $y_1 = x$*int x correctly graphed in dot mode.

Technology *Deceptive Pictures* A graphing utility (calculator or Computer Algebra System—CAS*) plots a graph much as you do when plotting by hand: by plotting points, or *pixels*, and then connecting them in succession. The resulting picture may be misleading when points on opposite sides of a point of discontinuity in the graph are incorrectly connected. To avoid incorrect connections some systems allow you to use a "dot mode," which plots only the points. Dot mode, however, may not reveal enough information to portray the true behavior of the graph. Try the following four functions on your graphing device. If you can, plot them in both "connected" and "dot" modes.

$y_1 = x^*\text{int } x$	at $x = 2$	jump discontinuity
$y_2 = \sin \dfrac{1}{x}$	at $x = 0$	oscillating discontinuity
$y_3 = \dfrac{1}{x-2}$	at $x = 2$	infinite discontinuity
$y_4 = \dfrac{x^2-2}{x-\sqrt{2}}$	at $x = 2$	removable discontinuity

*Rhymes with class.



1.37 Continuity at points *a*, *b*, and *c*.

Continuity at endpoints is defined by taking one-sided limits.

Definition

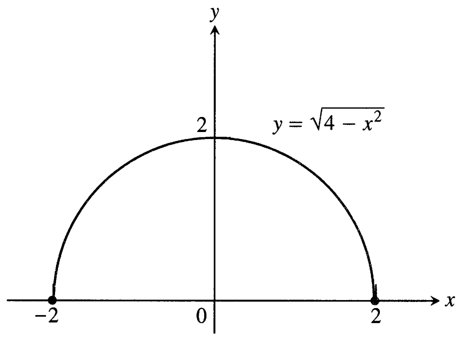
A function f is **continuous at a left endpoint** $x = a$ of its domain if

$$\lim_{x\to a^+} f(x) = f(a)$$

and **continuous at a right endpoint** $x = b$ of its domain if

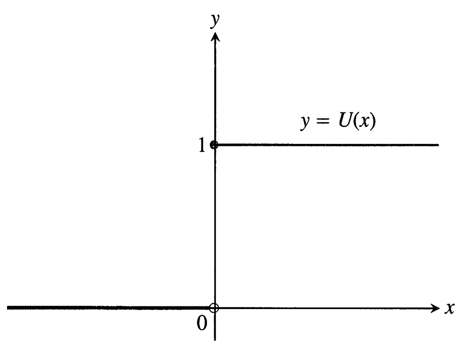
$$\lim_{x\to b^-} f(x) = f(b).$$

In general, a function f is **right-continuous (continuous from the right)** at a point $x = c$ in its domain if $\lim_{x\to c^+} f(x) = f(c)$. It is **left-continuous (continuous from the left)** at c if $\lim_{x\to c^-} f(x) = f(c)$. Thus, a function is continuous at a left endpoint a of its domain if it is right-continuous at a and continuous at a right endpoint b of its domain if it is left-continuous at b. A function is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c (Fig. 1.37).



1.38 Continuous at every domain point.

EXAMPLE 1 The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain, $[-2, 2]$ (Fig. 1.38). This includes $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous. ❑



1.39 Right-continuous at the origin.

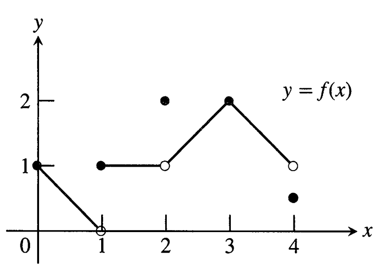
EXAMPLE 2 The unit step function $U(x)$, graphed in Fig. 1.39, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. ❑

We summarize continuity at a point in the form of a test.

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1.	$f(c)$ exists	(c lies in the domain of f)
2.	$\lim_{x\to c} f(x)$ exists	(f has a limit as $x \to c$)
3.	$\lim_{x\to c} f(x) = f(c)$	(the limit equals the function value)



1.40 This function, defined on the closed interval [0, 4], is discontinuous at $x = 1$, 2, and 4. It is continuous at all other points of its domain.

For one-sided continuity and continuity at an endpoint, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

EXAMPLE 3 Consider the function $y = f(x)$ in Fig. 1.40, whose domain is the closed interval $[0, 4]$. Discuss the continuity of f at $x = 0, 1, 2, 3$, and 4.

Solution The continuity test gives the following results:

a) f is continuous at $x = 0$ because

i) $f(0)$ exists ($f(0) = 1$),
ii) $\lim_{x\to 0^+} f(x) = 1$ (the right-hand limit exists at this left endpoint),
iii) $\lim_{x\to 0^+} f(x) = f(0)$ (the limit equals the function value).

b) f is discontinuous at $x = 1$ because $\lim_{x\to 1} f(x)$ does not exist. Part 2 of the test fails: f has different right- and left-hand limits at the interior point $x = 1$. However, f *is* right-continuous at $x = 1$ because

i) $f(1)$ exists ($f(1) = 1$),
ii) $\lim_{x\to 1^+} f(x) = 1$ (the right-hand limit exists at $x = 1$),
iii) $\lim_{x\to 1^+} f(x) = f(1)$ (the right-hand limit equals the function value).

c) f is discontinuous at $x = 2$ because $\lim_{x\to 2} f(x) \neq f(2)$. Part 3 of the test fails.

d) f is continuous at $x = 3$ because

i) $f(3)$ exists ($f(3) = 2$),
ii) $\lim_{x\to 3} f(x) = 2$ (the limit exists at $x = 2$),
iii) $\lim_{x\to 3} f(x) = f(3)$ (the limit equals the function value).

e) f is discontinuous at the right endpoint $x = 4$ because $\lim_{x\to 4^-} f(x) \neq f(4)$. The right-endpoint version of Part 3 of the test fails. ❑

Rules of Continuity

It follows from Theorem 1 in Section 1.2 that if two functions are continuous at a point, then various algebraic combinations of those functions are continuous at that point.

Theorem 6
Continuity of Algebraic Combinations

If functions f and g are continuous at $x = c$, then the following functions are continuous at $x = c$:

1. $f + g$ and $f - g$
2. fg
3. kf, where k is any number
4. f/g (provided $g(c) \neq 0$)
5. $(f(x))^{m/n}$ (provided $f(x))^{m/n}$ is defined on an interval containing c, and m and n are integers)

As a consequence, polynomials and rational functions are continuous at every point where they are defined.

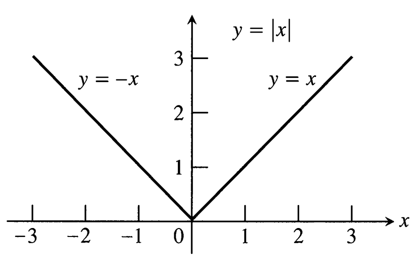
Theorem 7
Continuity of Polynomials and Rational Functions

Every polynomial is continuous at every point of the real line. Every rational function is continuous at every point where its denominator is different from zero.

EXAMPLE 4 The functions $f(x) = x^4 + 20$ and $g(x) = 5x(x-2)$ are continuous at every value of x. The function

$$r(x) = \frac{f(x)}{g(x)} = \frac{x^4 + 20}{5x(x-2)}$$

is continuous at every value of x except $x = 0$ and $x = 2$, where the denominator is 0. ❑



1.41 The sharp corner does not prevent the function from being continuous at the origin (Example 5).

EXAMPLE 5 *Continuity of f(x) = |x|*

The function $f(x) = |x|$ is continuous at every value of x (Fig. 1.41). If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x \to 0} |x| = 0 = |0|$. ❑

EXAMPLE 6 *Continuity of trigonometric functions*

We will show in Chapter 2 that the functions $\sin x$ and $\cos x$ are continuous at every value of x. Accordingly, the quotients

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

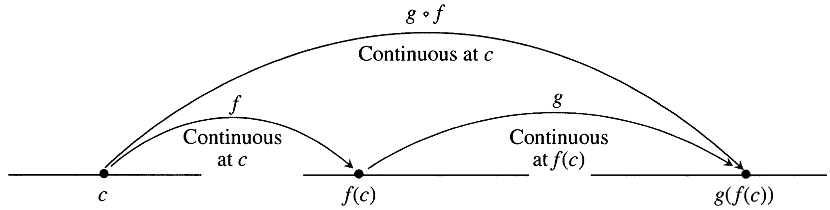
$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

are continuous at every point where they are defined. ❑

Theorem 8 tells us that continuity is preserved under the operation of composition.

Theorem 8
Continuity of Composites

If f is continuous at c, and g is continuous at $f(c)$, then $g \circ f$ is continuous at c (see Fig. 1.42).



1.42 The continuity of composites.

The continuity of composites holds for any finite number of functions. The only requirement is that each function be continuous where it is applied. For an outline of the proof of Theorem 8, see Exercise 6 in Appendix 2.

EXAMPLE 7 The following functions are continuous everywhere on their respective domains.

a) $y = \sqrt{x}$ — Theorems 6 and 7 (rational power of a polynomial)

b) $y = \sqrt{x^2 - 2x - 5}$ — Theorems 6 and 7, or (a) plus Theorems 7 and 8 (power of a polynomial or composition with the square root)

c) $y = \dfrac{x\cos(x^{2/3})}{1 + x^4}$ — Theorems 6, 7, and 8 (power, composite, product, polynomial, and quotient)

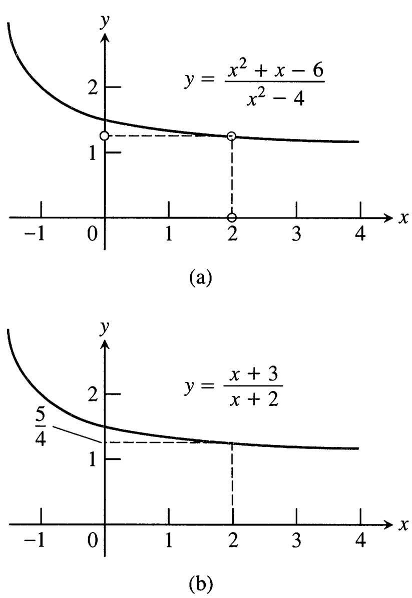
d) $y = \left|\dfrac{x - 2}{x^2 - 2}\right|$ — Theorems 7 and 8 (composite of absolute value and a rational function) ❑

Continuous Extension to a Point

As we saw in Section 1.2, a rational function may have a limit even at a point where its denominator is zero. If $f(c)$ is not defined, but $\lim_{x\to c} f(x) = L$ exists, we can define a new function $F(x)$ by the rule

$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ L & \text{if } x = c. \end{cases}$$

The function F is continuous at $x = c$. It is called the **continuous extension** of f to $x = c$. For rational functions f, continuous extensions are usually found by canceling common factors.



1.43 (a) The graph of

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}$$

and (b) the graph of its continuous extension

$$F(x) = \frac{x+3}{x+2} = \begin{cases} \dfrac{x^2 + x - 6}{x^2 - 4}, & x \neq 2 \\ \dfrac{5}{4}, & x = 2 \end{cases}$$

(Example 8).

EXAMPLE 8 Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}$$

has a continuous extension to $x = 2$, and find that extension.

Solution Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{x+3}{x+2}.$$

The function

$$F(x) = \frac{x+3}{x+2}$$

is equal to $f(x)$ for $x \neq 2$, but is also continuous at $x = 2$, having there the value of $5/4$. Thus F is the continuous extension of f to $x = 2$, and

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x\to 2} f(x) = \frac{5}{4}.$$

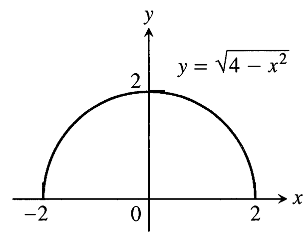
The graph of f is shown in Fig. 1.43. The continuous extension F has the same graph except with no hole at $(2, 5/4)$. ❑

Continuity on Intervals

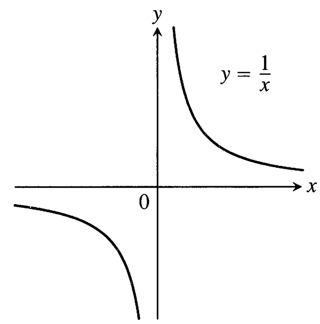
A function is called **continuous** if it is continuous everywhere in its domain. A function that is not continuous throughout its entire domain may still be continuous when restricted to particular intervals within the domain.

A function f is said to be **continuous on an interval** I in its domain if $\lim_{x\to c} f(x) = f(c)$ at every interior point c and if the appropriate one-sided limits equal the function values at any endpoints I may contain. A function continuous on an interval I is automatically continuous on any interval contained in I. Polynomials are continuous on every interval, and rational functions are continuous on every interval on which they are defined.

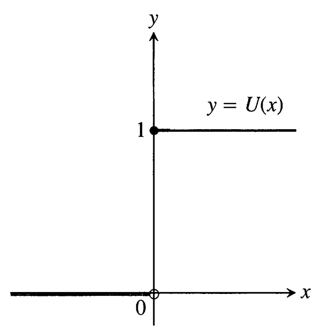
EXAMPLE 9 *Functions continuous on intervals*



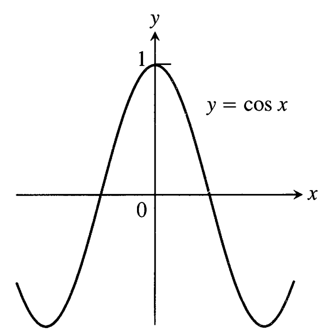
(a) Continuous on [−2, 2]



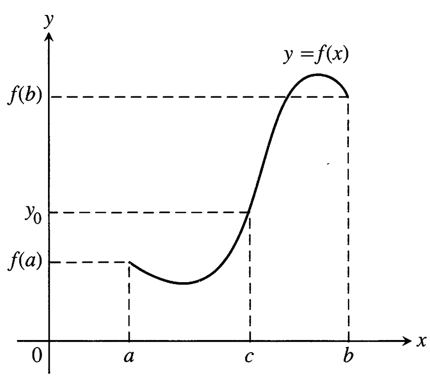
(b) Continuous on (−∞, 0) and (0, ∞)



(c) Continuous on (−∞, 0) and [0, ∞)



(d) Continuous on (−∞, ∞) ❑



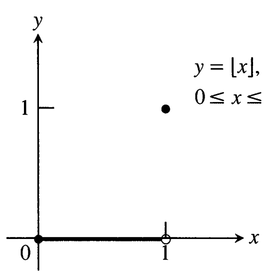
1.44 The function *f*, being continuous on [*a*, *b*], takes on every value between *f*(*a*) and *f*(*b*).

Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. One of these is the intermediate value property. A function is said to have the **intermediate value property** if it never takes on two values without taking on all the values in between.

Theorem 9

The Intermediate Value Theorem

Suppose $f(x)$ is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between $f(a)$ and $f(b)$, there exists a number c between a and b such that $f(c) = y_0$ (Fig. 1.44).



1.45 The function $f(x) = \lfloor x \rfloor, 0 \le x \le 1$, does not take on any value between $f(0) = 0$ and $f(1) = 1$.

The proof of the Intermediate Value Theorem depends on the completeness property of the real number system and can be found in more advanced texts.

The continuity of f on I is essential to the theorem. If f is discontinuous at even one point of I, the theorem's conclusion may fail, as it does for the function graphed in Fig. 1.45.

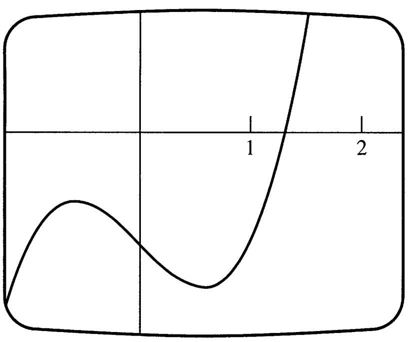
A Consequence for Graphing: Connectivity Theorem 9 is the reason the graph of a function continuous on an interval I cannot have any breaks. It will be **connected,** a single, unbroken curve, like the graph of $\sin x$. It will not have jumps like the graph of the greatest integer function $\lfloor x \rfloor$ or separate branches like the graph of $1/x$.

The Consequence for Root Finding We call a solution of the equation $f(x) = 0$ a **root** or **zero** of the function f. The Intermediate Value Theorem tells us that if f is continuous, then any interval on which f changes sign must contain a zero of the function.

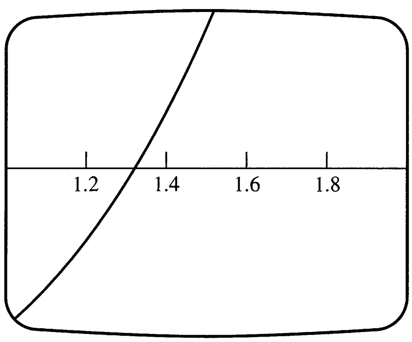
This observation is the basis of the way we solve equations of the form $f(x) = 0$ with a graphing calculator or computer grapher (when f is continuous). The solutions are the x-intercepts of the graph of f. We graph the function $y = f(x)$ over a large interval to see roughly where its zeros are. Then we zoom in on the intersection points one at a time to estimate their coordinates. Figure 1.46 shows a typical sequence of steps in a graphical solution of the equation $x^3 - x - 1 = 0$.

Graphical procedures for solving equations and finding zeros of functions, while instructive, are relatively slow. We usually get faster results from numerical methods, as you will see in Section 3.8.

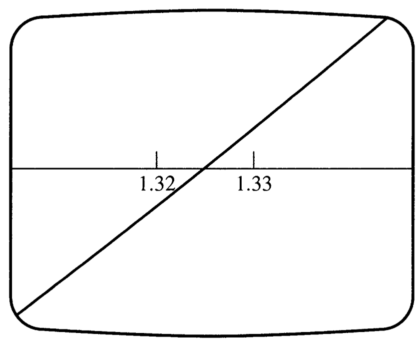
1.46 A graphical solution of the equation $x^3 - x - 1 = 0$. We graph the function $f(x) = x^3 - x - 1$ and, with successive screen enlargements, estimate the coordinates of the point where the graph crosses the x-axis.



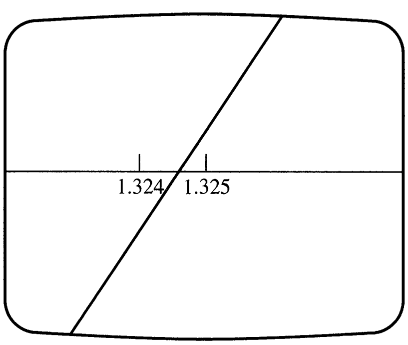
First we make a graph with a relatively large scale. It reveals a root (zero) between $x = 1$ and $x = 2$.



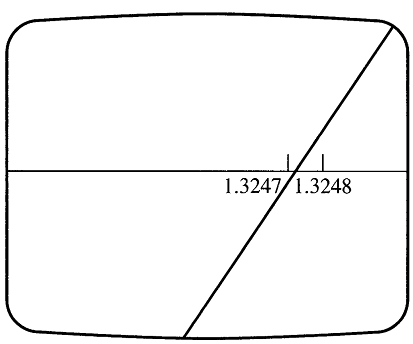
We change the viewing window to $1 \le x \le 2, -1 \le y \le 1$. We now see that the root lies between 1.3 and 1.4.



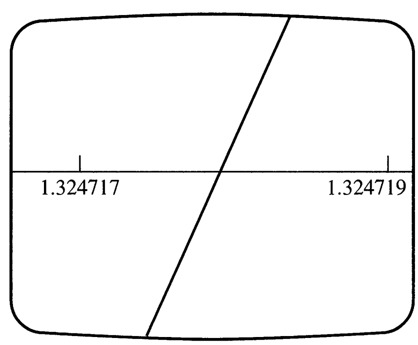
We change the window to $1.3 \le x \le 1.35, -0.1 \le y \le 0.1$. The root lies between 1.32 and 1.33.



We change the window to $1.32 \le x \le 1.33, -0.01 \le y \le 0.01$. The root lies between 1.324 and 1.325.



We change the window to $1.324 \le x \le 1.325, -0.001 \le y \le 0.001$. The root lies between 1.3247 and 1.3248.



After two more enlargements, we arrive at a screen that shows the root to be approximately 1.324718.

EXAMPLE 10 Is any real number exactly 1 less than its cube?

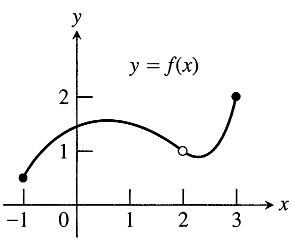
Solution This is the question that gave rise to the equation we just solved. Any such number must satisfy the equation $x = x^3 - 1$ or $x^3 - x - 1 = 0$. Hence, we are looking for a zero of $f(x) = x^3 - x - 1$. By trial, we find that $f(1) = -1$ and $f(2) = 5$ and conclude from Theorem 9 that there is at least one number in $[1, 2]$ where f is zero. So, yes, there is a number that is 1 less than its cube, and we just estimated its value graphically to be about 1.3247 18. ❑

Exercises 1.5

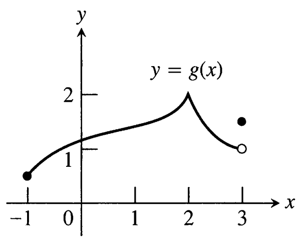
Continuity from Graphs

In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?

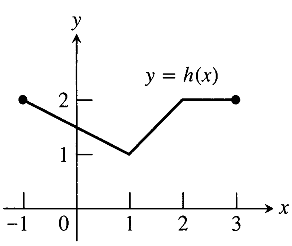
1.



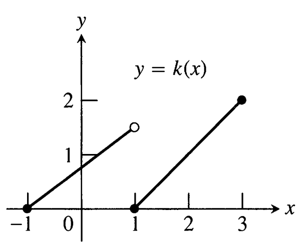
2.



3.



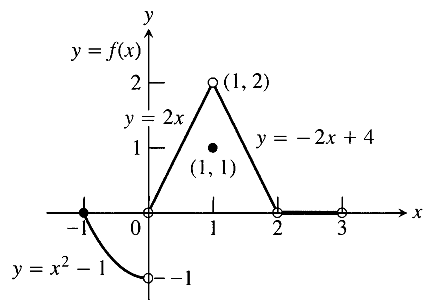
4.



Exercises 5–10 are about the function

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in Fig. 1.47.



1.47 The graph for Exercises 5–10.

5. a) Does $f(-1)$ exist?
b) Does $\lim_{x \to -1^+} f(x)$ exist?
c) Does $\lim_{x \to -1^+} f(x) = f(-1)$?
d) Is f continuous at $x = -1$?

6. a) Does $f(1)$ exist?
b) Does $\lim_{x \to 1} f(x)$ exist?
c) Does $\lim_{x \to 1} f(x) = f(1)$?
d) Is f continuous at $x = 1$?

7. a) Is f defined at $x = 2$? (Look at the definition of f.)
b) Is f continuous at $x = 2$?

8. At what values of x is f continuous?

9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

10. To what new value should $f(1)$ be changed to remove the discontinuity?

Applying the Continuity Test

At which points do the functions in the following exercises fail to be continuous? At which points, if any, are the discontinuities removable? not removable? Give reasons for your answers.

11. Exercise 1, Section 1.4 **12.** Exercise 2, Section 1.4

At what points are the functions in Exercises 13–28 continuous?

13. $y = \dfrac{1}{x - 2} - 3x$

14. $y = \dfrac{1}{(x + 2)^2} + 4$

15. $y = \dfrac{x + 1}{x^2 - 4x + 3}$

16. $y = \dfrac{x + 3}{x^2 - 3x - 10}$

17. $y = |x - 1| + \sin x$

18. $y = \dfrac{1}{|x| + 1} - \dfrac{x^2}{2}$

19. $y = \dfrac{\cos x}{x}$

20. $y = \dfrac{x + 2}{\cos x}$

21. $y = \csc 2x$

22. $y = \tan \dfrac{\pi x}{2}$

23. $y = \dfrac{x \tan x}{x^2 + 1}$

24. $y = \dfrac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$

25. $y = \sqrt{2x + 3}$

26. $y = \sqrt[4]{3x - 1}$

27. $y = (2x - 1)^{1/3}$

28. $y = (2 - x)^{1/5}$

Limits of Composite Functions

Find the limits in Exercises 29–34.

29. $\lim\limits_{x \to \pi} \sin(x - \sin x)$

30. $\lim\limits_{t \to 0} \sin\left(\dfrac{\pi}{2} \cos(\tan t)\right)$

31. $\lim\limits_{y \to 1} \sec(y \sec^2 y - \tan^2 y - 1)$

32. $\lim\limits_{x \to 0} \tan\left(\dfrac{\pi}{4} \cos(\sin x^{1/3})\right)$

33. $\lim\limits_{t \to 0} \cos\left(\dfrac{\pi}{\sqrt{19 - 3 \sec 2t}}\right)$

34. $\lim\limits_{x \to \pi/6} \sqrt{\csc^2 x + 5\sqrt{3} \tan x}$

Continuous Extensions

35. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.

36. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.

37. Define $f(1)$ in a way that extends $f(s) = (s^3 - 1)/(s^2 - 1)$ to be continuous at $s = 1$.

38. Define $g(4)$ in a way that extends $g(x) = (x^2 - 16)/(x^2 - 3x - 4)$ to be continuous at $x = 4$.

39. For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x?

40. For what value of b is

$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every x?

Grapher Explorations—Continuous Extension

In Exercises 41–44, graph the function f to see whether it appears to have a continuous extension to the origin. If it does, use TRACE and ZOOM to find a good candidate for the extended function's value at $x = 0$. If the function does not appear to have a continuous extension, can it be extended to be continuous at the origin from the right or from the left? If so, what do you think the extended function's value(s) should be?

41. $f(x) = \dfrac{10^x - 1}{x}$

42. $f(x) = \dfrac{10^{|x|} - 1}{x}$

43. $f(x) = \dfrac{\sin x}{|x|}$

44. $f(x) = (1 + 2x)^{1/x}$

Theory and Examples

45. A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Why does the equation $f(x) = 0$ have at least one solution between $x = 0$ and $x = 1$? Illustrate with a sketch.

46. Explain why the equation $\cos x = x$ has at least one solution.

47. Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

48. Show that the function $F(x) = (x - a)^2(x - b)^2 + x$ takes on the value $(a + b)/2$ for some value of x.

49. If $f(x) = x^3 - 8x + 10$, show that there are values c for which $f(c)$ equals (a) π; (b) $-\sqrt{3}$; (c) 5,000,000.

50. Explain why the following five statements ask for the same information.

a) Find the roots of $f(x) = x^3 - 3x - 1$.

b) Find the x-coordinates of the points where the curve $y = x^3$ crosses the line $y = 3x + 1$.

c) Find all the values of x for which $x^3 - 3x = 1$.

d) Find the x-coordinates of the points where the cubic curve $y = x^3 - 3x$ crosses the line $y = 1$.

e) Solve the equation $x^3 - 3x - 1 = 0$.

51. Give an example of a function $f(x)$ that is continuous for all values of x except $x = 2$, where it has a removable discontinuity. Explain how you know that f is discontinuous at $x = 2$, and how you know the discontinuity is removable.

52. Give an example of a function $g(x)$ that is continuous for all values of x except $x = -1$, where it has a nonremovable discontinuity. Explain how you know that g is discontinuous there and why the discontinuity is not removable.

*** 53.** * *A function discontinuous at every point.*

a) Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point.

b) Is f right-continuous or left-continuous at any point?

54. If functions $f(x)$ and $g(x)$ are continuous for $0 \leq x \leq 1$, could $f(x)/g(x)$ possibly be discontinuous at a point of $[0, 1]$? Give reasons for your answer.

55. If the product function $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, must $f(x)$ and $g(x)$ be continuous at $x = 0$? Give reasons for your answer.

---

*Asterisk denotes a challenging problem.

56. Give an example of functions f and g, both continuous at $x = 0$, for which the composite $f \circ g$ is discontinuous at $x = 0$. Does this contradict Theorem 8? Give reasons for your answer.

57. Is it true that a continuous function that is never zero on an interval never changes sign on that interval? Give reasons for your answer.

58. Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give reasons for your answer.

59. *A fixed point theorem.* Suppose that a function f is continuous on the closed interval [0, 1] and that $0 \le f(x) \le 1$ for every x in [0, 1]. Show that there must exist a number c in [0, 1] such that $f(c) = c$ (c is called a **fixed point** of f).

60. *The sign-preserving property of continuous functions.* Let f be defined on an interval (a, b) and suppose that $f(c) \neq 0$ at some c where f is continuous. Show that there is an interval $(c - \delta,\ c + \delta)$ about c where f has the same sign as $f(c)$. Notice how remarkable this conclusion is. Although f is defined throughout (a, b), it is not required to be continuous at any point except c. That and the condition $f(c) \neq 0$ are enough to make f different from zero (positive or negative) throughout an entire interval.

61. Explain how Theorem 6 follows from Theorem 1 in Section 1.2.

62. Explain how Theorem 7 follows from Theorems 2 and 3 in Section 1.2.

Solving Equations Graphically

Use a graphing calculator or computer grapher to solve the equations in Exercises 63–70.

63. $x^3 - 3x - 1 = 0$

64. $2x^3 - 2x^2 - 2x + 1 = 0$

65. $x(x-1)^2 = 1$ (one root)

66. $x^x = 2$

67. $\sqrt{x} + \sqrt{1+x} = 4$

68. $x^3 - 15x + 1 = 0$ (three roots)

69. $\cos x = x$ (one root). Make sure you are using radian mode.

70. $2 \sin x = x$ (three roots). Make sure you are using radian mode.

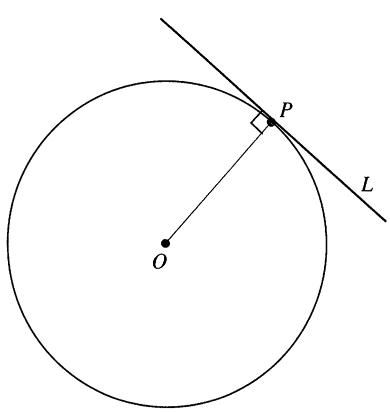
1.6 Tangent Lines

This section continues the discussion of secants and tangents begun in Section 1.1. We calculate limits of secant slopes to find tangents to curves.

What *Is* a Tangent to a Curve?

For circles, tangency is straightforward. A line L is tangent to a circle at a point P if L passes through P perpendicular to the radius at P (Fig. 1.48). Such a line just *touches* the circle. But what does it mean to say that a line L is tangent to some other curve C at a point P? Generalizing from the geometry of the circle, we might say that it means one of the following.

1. L passes through P perpendicular to the line from P to the center of C.
2. L passes through only one point of C, namely P.
3. L passes through P and lies on one side of C only.



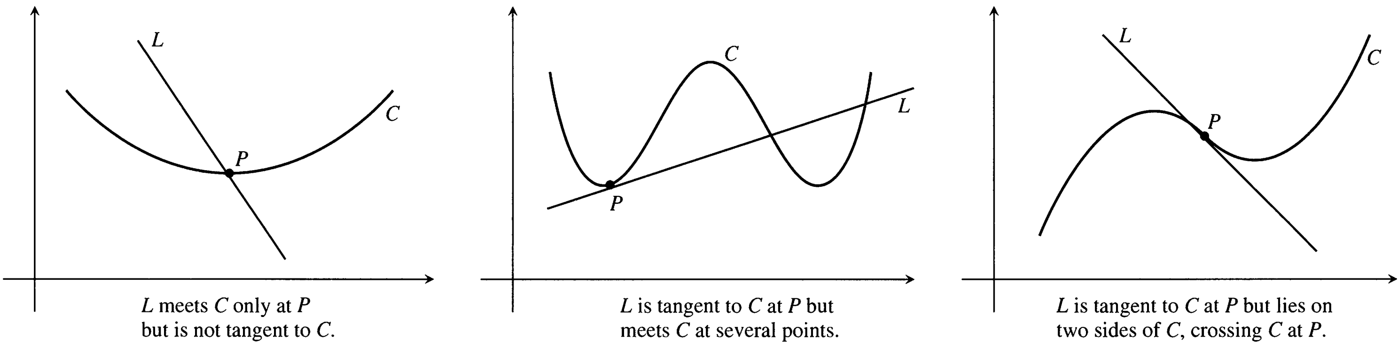
1.48 L is tangent to the circle at P if it passes through P perpendicular to radius OP.

While these statements are valid if C is a circle, none of them work consistently for more general curves. Most curves do not have centers, and a line we may want to call tangent may intersect C at other points or cross C at the point of tangency (Fig. 1.49 on the following page).

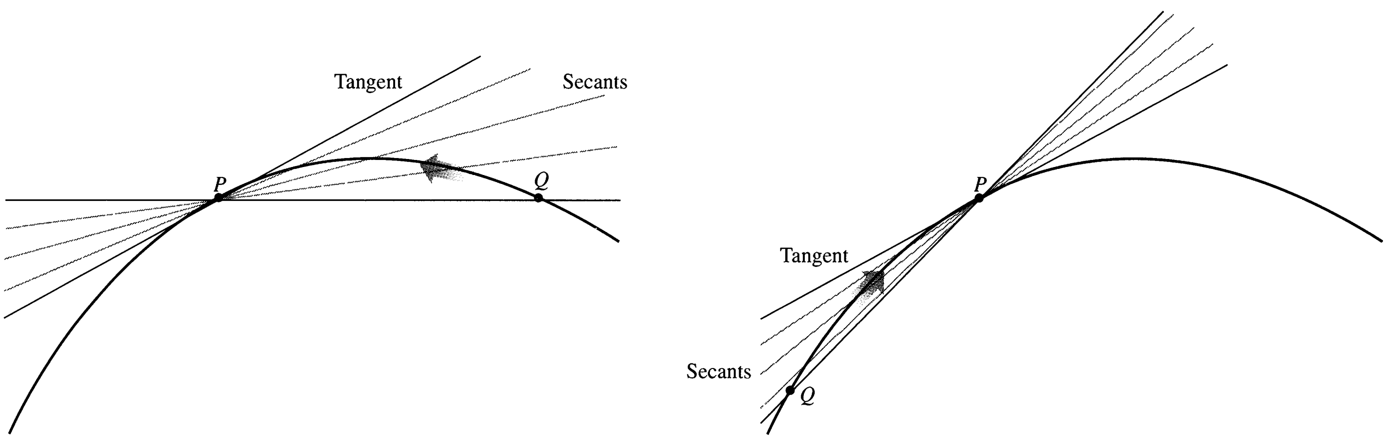
To define tangency for general curves, we need a dynamic approach that takes into account the behavior of the secants through P and nearby points Q as Q moves toward P along the curve (Fig. 1.50 on the following page). It goes like this:

1. We start with what we *can* calculate, namely the slope of the secant PQ.
2. Investigate the limit of the secant slope as Q approaches P along the curve.
3. If the limit exists, take it to be the slope of the curve at P and define the tangent to the curve at P to be the line through P with this slope.

This is what we were doing in the fruit fly example in Section 1.1.



1.49 Exploding myths about tangent lines.



1.50 The dynamic approach to tangency. The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

How do you find a tangent to a curve?

This was the dominant mathematical question of the early seventeenth century and it is hard to overestimate how badly the scientists of the day wanted to know the answer. In optics, the tangent determined the angle at which a ray of light entered a curved lens. In mechanics, the tangent determined the direction of a body's motion at every point along its path. In geometry, the tangents to two curves at a point of intersection determined the angle at which the curves intersected. Descartes went so far as to say that the problem of finding a tangent to a curve was "the most useful and most general problem not only that I know but even that I have any desire to know."

EXAMPLE 1 Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

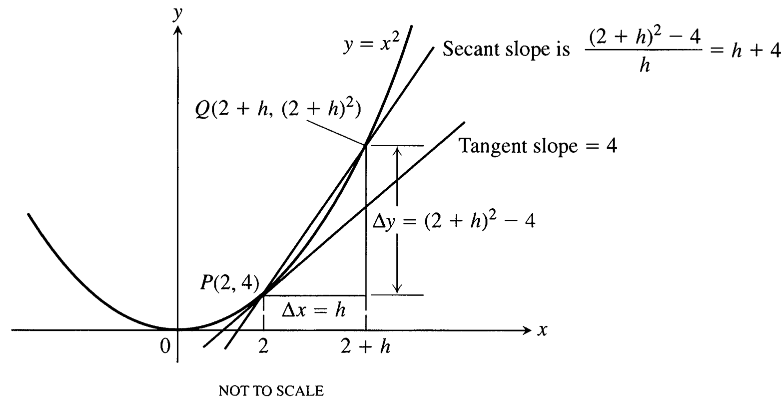
Solution We begin with a secant line through $P(2, 4)$ and $Q(2+h, (2+h)^2)$ nearby. We then write an expression for the slope of the secant PQ and investigate what happens to the slope as Q approaches P along the curve:

$$\begin{aligned}\text{Secant slope} &= \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h} \\ &= \frac{h^2 + 4h}{h} = h + 4.\end{aligned}$$

If $h > 0$, Q lies above and to the right of P, as in Fig. 1.51. If $h < 0$, Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant slope approaches 4:

$$\lim_{h \to 0} (h + 4) = 4.$$

We take 4 to be the parabola's slope at P.



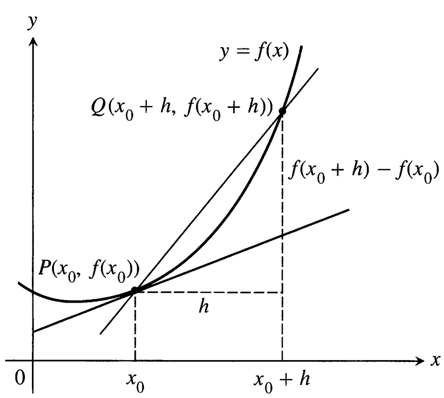
1.51 Diagram for finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ (Example 1).

The tangent to the parabola at P is the line through P with slope 4:

$$y = 4 + 4(x - 2) \qquad \text{Point-slope equation}$$

$$y = 4x - 4.$$

❑



1.52 The tangent slope is
$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Finding a Tangent to the Graph of a Function

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$ we use the same dynamic procedure. We calculate the slope of the secant through P and a point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$ (Fig. 1.52). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

Definitions

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Whenever we make a new definition it is a good idea to try it on familiar objects to be sure it gives the results we want in familiar cases. The next example shows that the new definition of slope agrees with the old definition when we apply it to nonvertical lines.

How to Find the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope
$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
3. If the limit exists, find the tangent line as $y = y_0 + m(x - x_0)$.

EXAMPLE 2 *Testing the definition*

Show that the line $y = mx + b$ is its own tangent at any point $(x_0, mx_0 + b)$.

Solution We let $f(x) = mx + b$ and organize the work into three steps.

Step 1: *Find $f(x_0)$ and $f(x_0 + h)$.*

$$f(x_0) = mx_0 + b$$

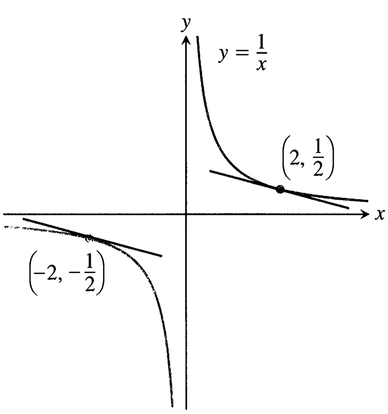
$$f(x_0 + h) = m(x_0 + h) + b = mx_0 + mh + b$$

Pierre de Fermat (1601–1665)

The dynamic approach to tangency, invented by Fermat in 1629, proved to be one of the seventeenth century's major contributions to calculus.

Fermat, a skilled linguist and one of his century's greatest mathematicians, tended to confine his writing to professional correspondence and to papers written for personal friends. He rarely wrote completed descriptions of his work, even for his personal use. His famous "last theorem" (that $a^n + b^n = c^n$ has no positive integer solutions for a, b, and c if n is an integer greater than 2) is known only from a note he jotted in the margin of a book. His name slipped into relative obscurity until the late 1800s, and it was only from a four-volume edition of his works published at the beginning of this century that the true importance of his many achievements became clear.

Besides the work in physics and number theory for which he is best known, Fermat found the areas under curves as limits of sums of rectangle areas (as we do today) and developed a method for finding the centroids of shapes bounded by curves in the plane. The standard formula for the first derivative of a polynomial function, the formulas for calculating arc length and for finding the area of a surface of revolution, and the second derivative test for extreme values of functions can all be found in his papers. We will see what these are as the text continues.



1.53 The two tangent lines to $y = 1/x$ having slope $-1/4$.

Step 2: *Find the slope* $\lim_{h\to 0} (f(x_0+h) - f(x_0))/h$.

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{(mx_0 + mh + b) - (mx_0 + b)}{h}$$
$$= \lim_{h\to 0} \frac{mh}{h} = m$$

Step 3: *Find the tangent line using the point–slope equation.* The tangent line at the point $(x, mx_0 + b)$ is

$$y = (mx_0 + b) + m(x - x_0)$$
$$y = mx_0 + b + mx - mx_0$$
$$y = mx + b. \qquad \square$$

EXAMPLE 3

a) Find the slope of the curve $y = 1/x$ at $x = a$.
b) Where does the slope equal $-1/4$?
c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution

a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\frac{a - (a+h)}{a(a+h)}$$
$$= \lim_{h\to 0} \frac{-h}{ha(a+h)}$$
$$= \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

Notice how we had to keep writing "$\lim_{h\to 0}$" at the beginning of each line until the stage where we could evaluate the limit by substituting $h = 0$.

b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

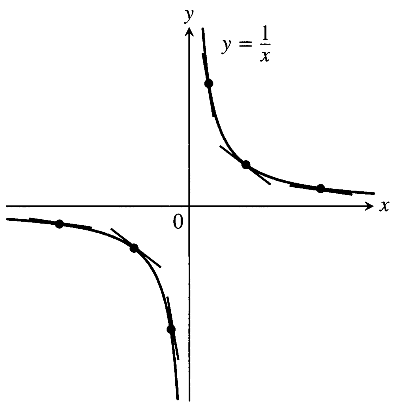
This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Fig. 1.53).

c) Notice that the slope $-1/a^2$ is always negative. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Fig. 1.54). We see this again as $a \to 0^-$. As a moves away from the origin, the slope approaches 0^- and the tangent levels off. $\square$

Rates of Change

The expression

$$\frac{f(x_0+h) - f(x_0)}{h}$$



1.54 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away.

is called the **difference quotient of *f* at x_0**. If the difference quotient has a limit as h approaches zero, that limit is called the **derivative of *f* at x_0**. If we interpret the difference quotient as a secant slope, the derivative gives the slope of the curve and tangent at the point where $x = x_0$. If we interpret the difference quotient as an average rate of change, as we did in Section 1.1, the derivative gives the function's rate of change with respect to x at the point $x = x_0$. The derivative is one of the two most important mathematical objects considered in calculus. We will begin a thorough study of it in Chapter 2.

All of these refer to the same thing.

1. The slope of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative of f at $x = x_0$
5. $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$

EXAMPLE 4 *Instantaneous speed (Continuation of Section 1.1, Examples 1 and 2)*

In Examples 1 and 2 in Section 1.1, we studied the speed of a rock falling freely from rest near the surface of the earth. We knew that the rock fell $y = 16t^2$ feet during the first t seconds, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. Exactly what *was* the rock's speed at this time?

Solution We let $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds was

$$\frac{f(1+h) - f(1)}{h} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h+2).$$

The rock's speed at the instant $t = 1$ was

$$\lim_{h \to 0} 16(h+2) = 16(0+2) = 32 \text{ ft/sec}.$$

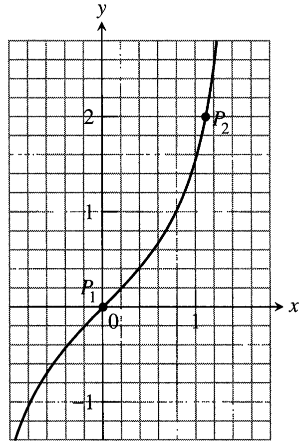
Our original estimate of 32 ft/sec was right. ❑

Exercises 1.6

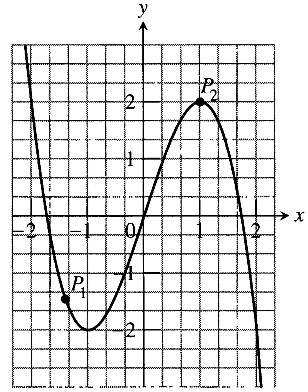
Slopes and Tangent Lines

In Exercises 1–4, use the grid and a straight edge to make a rough estimate of the slope of the curve (in y-units per x-unit) at the points P_1 and P_2. Graphs can shift during a press run, so your estimates may be somewhat different from those in the back of the book.

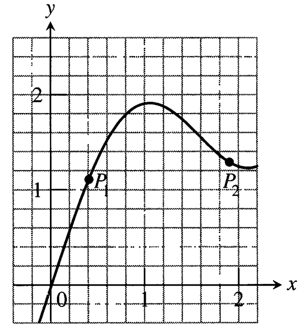
1.



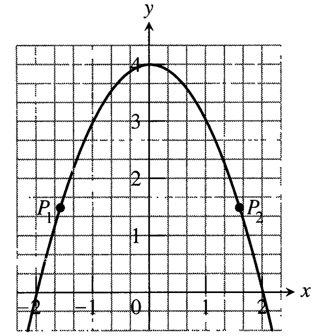
2.



3.



4.



In Exercises 5–10, find an equation for the tangent to the curve at the given point. Then sketch the curve and tangent together.

5. $y = 4 - x^2,\quad (-1, 3)$

6. $y = (x-1)^2 + 1,\quad (1, 1)$

7. $y = 2\sqrt{x},\quad (1, 2)$

8. $y = \dfrac{1}{x^2},\quad (-1, 1)$

9. $y = x^3,\quad (-2, -8)$

10. $y = \dfrac{1}{x^3},\quad \left(-2, -\dfrac{1}{8}\right)$

In Exercises 11–18, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

11. $f(x) = x^2 + 1,\quad (2, 5)$

12. $f(x) = x - 2x^2,\quad (1, -1)$

13. $g(x) = \dfrac{x}{x-2},\quad (3, 3)$

14. $g(x) = \dfrac{8}{x^2},\quad (2, 2)$

15. $h(t) = t^3,\quad (2, 8)$

16. $h(t) = t^3 + 3t,\quad (1, 4)$

17. $f(x) = \sqrt{x},\quad (4, 2)$

18. $f(x) = \sqrt{x+1},\quad (8, 3)$

In Exercises 19–22, find the slope of the curve at the point indicated.

19. $y = 5x^2,\quad x = -1$

20. $y = 1 - x^2,\quad x = 2$

21. $y = \dfrac{1}{x-1},\quad x = 3$

22. $y = \dfrac{x-1}{x+1},\quad x = 0$

Tangent Lines with Specified Slopes

At what points do the graphs of the functions in Exercises 23 and 24 have horizontal tangents?

23. $f(x) = x^2 + 4x - 1$

24. $g(x) = x^3 - 3x$

25. Find equations of all lines having slope -1 that are tangent to the curve $y = 1/(x-1)$.

26. Find an equation of the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

Rates of Change

27. An object is dropped from the top of a 100-m-high tower. Its height aboveground after t seconds is $100 - 4.9t^2$ m. How fast is it falling 2 sec after it is dropped?

28. At t seconds after lift-off, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing after 10 sec?

29. What is the rate of change of the area of a circle ($A = \pi r^2$) with respect to its radius when the radius is $r = 3$?

30. What is the rate of change of the volume of a ball ($V = (4/3)\pi r^3$) with respect to the radius when the radius is $r = 2$?

Testing for Tangents

31. Does the graph of

$$f(x) = \begin{cases} x^2 \sin\ (1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

32. Does the graph of

$$g(x) = \begin{cases} x \sin\ (1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

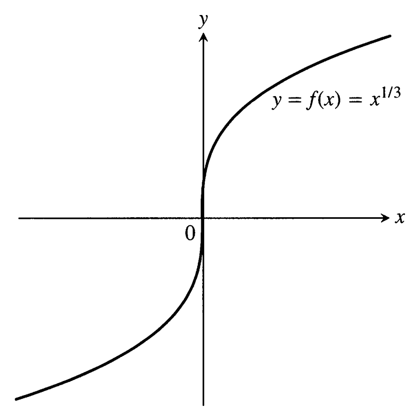
have a tangent at the origin? Give reasons for your answer.

Vertical Tangents

We say that the curve $y = f(x)$ has a **vertical tangent** at the point where $x = x_0$ if $\lim_{h\to 0}\ (f(x_0+h) - f(x_0))/h = \infty$ or $-\infty$.

Vertical tangent at $x = 0$:

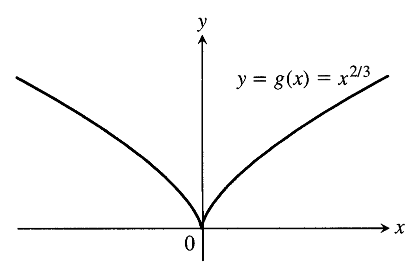
$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^{1/3} - 0}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty$$



No vertical tangent at $x = 0$:

$$\lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{h^{2/3} - 0}{h} = \lim_{h\to 0} \frac{1}{h^{1/3}}$$

does not exist, because the limit is ∞ from the right and $-\infty$ from the left.



33. Does the graph of

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

have a vertical tangent at the origin? Give reasons for your answer.

34. Does the graph of

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

have a vertical tangent at the point (0, 1)? Give reasons for your answer.

Grapher Explorations—Vertical Tangents

a) Graph the curves in Exercises 35–44. Where do the graphs appear to have vertical tangents?

b) Confirm your findings in (a) with limit calculations.

35. $y = x^{2/5}$

36. $y = x^{4/5}$

37. $y = x^{1/5}$

38. $y = x^{3/5}$

39. $y = 4x^{2/5} - 2x$

40. $y = x^{5/3} - 5x^{2/3}$

41. $y = x^{2/3} - (x - 1)^{1/3}$

42. $y = x^{1/3} + (x - 1)^{1/3}$

43. $y = \begin{cases} -\sqrt{|x|}, & x \leq 0 \\ \sqrt{x}, & x > 0 \end{cases}$

44. $y = \sqrt{|4 - x|}$

CAS Explorations and Projects

Use a CAS to perform the following steps for the functions in Exercises 45–48.

a) Plot $y = f(x)$ over the interval $x_0 - \frac{1}{2} \leq x \leq x_0 + 3$.

b) Define the difference quotient q at x_0 as a function of the general step size h.

c) Find the limit of q as $h \to 0$.

d) Define the secant lines $y = f(x_0) + q^*(x - x_0)$ for $h = 3, 2,$ and 1. Graph them together with f and the tangent line over the interval in part (a).

45. $f(x) = x^3 + 2x, \quad x_0 = 0$

46. $f(x) = x + \dfrac{5}{x}, \quad x_0 = 1$

47. $f(x) = x + \sin\ (2x), \quad x_0 = \pi/2$

48. $f(x) = \cos\ x + 4\sin\ (2x), \quad x_0 = \pi$

CHAPTER 1 QUESTIONS TO GUIDE YOUR REVIEW

1. What is the average rate of change of the function $g(t)$ over the interval from $t = a$ to $t = b$? How is it related to a secant line?

2. What limit must be calculated to find the rate of change of a function $g(t)$ at $t = t_0$?

3. Does the existence and value of the limit of a function $f(x)$ as x approaches c ever depend on what happens at $x = c$? Explain, and give examples.

4. What theorems are available for calculating limits? Give examples of how the theorems are used.

5. How are one-sided limits related to limits? How can this relationship sometimes be used to calculate a limit or prove it does not exist? Give examples.

6. How is the problem of controlling the input x of a function f so that the output $y = f(x)$ will be within a certain specified tolerance ϵ of a target value $y_0 = f(x_0)$ related to the problem of proving that f has limit y_0 as $x \to x_0$?

7. What exactly does $\lim_{x \to x_0} f(x) = L$ mean? Give an example in which you find a $\delta > 0$ for a given $f, L, x_0,$ and $\epsilon > 0$ in the formal definition of limit.

8. Give formal definitions of the following statements.

a) $\lim_{x \to 2^-} f(x) = 5$

b) $\lim_{x \to 2^+} f(x) = 5$

c) $\lim_{x \to 2} f(x) = \infty$

d) $\lim_{x \to 2} f(x) = -\infty$

9. What conditions must be satisfied by a function if it is to be continuous at an interior point of its domain? at an endpoint?

10. How can looking at the graph of a function help you tell where the function is continuous?

11. What does it mean for a function to be right-continuous at a point? left-continuous? How are continuity and one-sided continuity related?

12. What can be said about the continuity of polynomials? of rational functions? of trigonometric functions? of rational powers and algebraic combinations of functions? of composites of functions? of absolute values of functions?

13. Under what circumstances can you extend a function $f(x)$ to be continuous at a point $x = c$? Give an example.

14. What does it mean for a function to be continuous on an interval?

15. What does it mean for a function to be continuous? Give examples to illustrate the fact that a function that is not continuous on its entire domain may still be continuous on selected intervals within the domain.

16. What property must a function f that is continuous on an interval $[a, b]$ have? Show by examples that f need not have this property if it is discontinuous at some point of the interval.

17. It is often said that a function is continuous if you can draw its graph without having to lift your pen from the paper. Why is that?

18. What does continuity have to do with solving equations?

19. When is a line tangent to a curve C at a point P?

20. What is the significance of the formula
$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}?$$

CHAPTER 1 PRACTICE EXERCISES

Limit Calculations and Continuity

1. Graph the function
$$f(x)=\begin{cases}1, & x\le -1\\ -x, & -1<x<0\\ 1, & x=0\\ -x, & 0<x<1\\ 1, & x\ge 1.\end{cases}$$
Then discuss, in complete detail, limits, one-sided limits, continuity, and one-sided continuity of f at each of the points $x=-1, 0$, and 1. Are any of the discontinuities removable? Explain.

2. Repeat the instructions of Exercise 1 for
$$f(x)=\begin{cases}0, & x\le -1\\ 1/x, & 0<|x|<1\\ 0, & x=1\\ 1, & x>1.\end{cases}$$

3. Suppose that $f(x)$ and $g(x)$ are defined for all x and that $\lim_{x\to c} f(x)=-7$ and $\lim_{x\to c} g(x)=0$. Find the limit as $x \to c$ of the following functions.

   a) $3f(x)$ b) $(f(x))^2$
   c) $f(x)\cdot g(x)$ d) $\dfrac{f(x)}{g(x)-7}$
   e) $\cos(g(x))$ f) $|f(x)|$

4. Suppose that $f(x)$ and $g(x)$ are defined for all x and that $\lim_{x\to 0} f(x)=1/2$ and $\lim_{x\to 0} g(x)=\sqrt{2}$. Find the limits as $x\to 0$ of the following functions.

   a) $-g(x)$ b) $g(x)\cdot f(x)$
   c) $f(x)+g(x)$ d) $1/f(x)$
   e) $x+f(x)$ f) $\dfrac{f(x)\cdot\cos x}{x-1}$

In Exercises 5 and 6, find the value that $\lim_{x\to 0} g(x)$ must have if the given limit statements hold.

5. $\displaystyle\lim_{x\to 0}\left(\frac{4-g(x)}{x}\right)=1$ 6. $\displaystyle\lim_{x\to -4}\left(x\lim_{x\to 0} g(x)\right)=2$

In Exercises 7–10, find the limit of $g(x)$ as x approaches the indicated value.

7. $\displaystyle\lim_{x\to 0^+}(4g(x))^{1/3}=2$ 8. $\displaystyle\lim_{x\to\sqrt{5}}\frac{1}{x+g(x)}=2$

9. $\displaystyle\lim_{x\to 1}\frac{3x^2+1}{g(x)}=\infty$ 10. $\displaystyle\lim_{x\to -2}\frac{5-x^2}{\sqrt{g(x)}}=0$

In Exercises 11–18, find the limit or explain why it does not exist.

11. $\displaystyle\lim\frac{x^2-4x+4}{x^3+5x^2-14x}$ (a) as $x\to 0$, (b) as $x\to 2$

12. $\displaystyle\lim\frac{x^2+x}{x^5+2x^4+x^3}$ (a) as $x\to 0$, (b) as $x\to -1$

13. $\displaystyle\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}$ 14. $\displaystyle\lim_{x\to a}\frac{x^2-a^2}{x^4-a^4}$

15. $\displaystyle\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$ 16. $\displaystyle\lim_{x\to 0}\frac{(x+h)^2-x^2}{h}$

17. $\displaystyle\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x}$ 18. $\displaystyle\lim_{x\to 0}\frac{(2+x)^3-8}{x}$

19. On what intervals are the following functions continuous?

   a) $f(x)=x^{1/3}$ b) $g(x)=x^{3/4}$
   c) $h(x)=x^{-2/3}$ d) $k(x)=x^{-1/6}$

20. Can $f(x)=x(x^2-1)/|x^2-1|$ be extended to be continuous at $x=1$ or -1? Give reasons for your answers. (Graph the function—you will find the graph interesting.)

Grapher Explorations—Continuous Extensions

In Exercises 21–24, graph the function to see whether it appears to have a continuous extension to the given point a. If it does, use TRACE and ZOOM to find a good candidate for the extended function's value at a. If the function does not appear to have a continuous extension, can it be extended to be continuous from the right or left? If so, what do you think the extended function's value should be?

21. $f(x) = \dfrac{x-1}{x-\sqrt[4]{x}}, \quad a = 1$

22. $g(\theta) = \dfrac{5\cos\theta}{4\theta - 2\pi}, \quad a = \pi/2$

23. $h(t) = (1+|t|)^{1/t}, \quad a = 0$

24. $k(x) = \dfrac{x}{1-2^{|x|}}, \quad a = 0$

Grapher Explorations—Roots

25. Let $f(x) = x^3 - x - 1$.

a) Show that f must have a zero between -1 and 2.

b) Solve the equation $f(x) = 0$ graphically with an error of at most 10^{-8}.

c) It can be shown that the exact value of the solution in (b) is

$$\left(\frac{1}{2}+\frac{\sqrt{69}}{18}\right)^{1/3} + \left(\frac{1}{2}-\frac{\sqrt{69}}{18}\right)^{1/3}.$$

Evaluate this exact answer and compare it with the value determined in (b).

26. Let $f(x) = x^3 - 2x + 2$.

a) Show that f must have a zero between -2 and 0.

b) Solve the equation $f(x) = 0$ graphically with an error of at most 10^{-4}.

c) It can be shown that the exact value of the solution in (b) is

$$\left(\sqrt{\frac{19}{27}}-1\right)^{1/3} - \left(\sqrt{\frac{19}{27}}+1\right)^{1/3}.$$

Evaluate this exact answer and compare it with the value determined in (b).

CHAPTER 1 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. a) If $\lim_{x\to c} f(x) = 5$, must $f(c) = 5$?

b) If $f(c) = 5$, must $\lim_{x\to c} f(x) = 5$?

Give reasons for your answers.

2. Can $\lim_{x\to c} (f(x)/g(x))$ exist even if $\lim_{x\to c} f(c) = 0$ and $\lim_{x\to c} g(x) = 0$? Give reasons for your answer.

3. *Assigning a value to 0^0.* The rules of exponents tell us that $a^0 = 1$ if a is any number different from zero. They also tell us that $0^n = 0$ if n is any positive number.

If we tried to extend these rules to include the case 0^0, we would get conflicting results. The first rule would say $0^0 = 1$ while the second would say $0^0 = 0$.

We are not dealing with a question of right or wrong here. Neither rule applies as it stands, so there is no contradiction. We could, in fact, define 0^0 to have any value we wanted as long as we could persuade others to agree.

What value would you like 0^0 to have? Here are two examples that might help you to decide. (See Exercise 4 for another example.)

a) CALCULATOR Calculate x^x for $x = 0.1, 0.01, 0.001$, and so on as far as your calculator can go. Write down the value you get each time. What pattern do you see?

b) GRAPHER Graph the function $y = x^x$ (as $y = x\hat{\ }x$) for $0 \le x \le 1$. Even though the function is not defined for $x \le 0$, the graph will approach the y-axis from the right. Toward what y-value does it seem to be headed? Zoom in to estimate the value more closely. What do you think it is?

4. *A reason you might want 0^0 to be something other than 0 or 1.* As the number x increases through positive values, the numbers $1/x$ and $1/(\ln\ x)$ both approach zero. What happens to the number

$$f(x) = \left(\frac{1}{x}\right)^{1/(\ln\ x)}$$

as x increases? Here are two ways to find out.

a) CALCULATOR Evaluate f for $x = 10, 100, 1000$, and so on, as far as your calculator can reasonably go. What pattern do you see?

b) GRAPHER Graph f in a variety of graphing windows, including windows that contain the origin. What do you see? Use TRACE to read y-values along the graph. What do you find? Chapter 6 will explain what is going on.

5. *Lorentz contraction.* In relativity theory the length of an object, say a rocket, appears, to an observer, to depend on the speed at which the object is traveling with respect to the observer. If the observer measures the rocket's length as L_0 at rest, then at speed v the rocket's length will appear to be

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}.$$

The Lorentz contraction formula.

Here, $c \approx 3 \times 10^8$ m/sec is the speed of light in a vacuum. What happens to L as v increases? Find $\lim_{v\to c^-} L$. Why was the left-hand limit needed?

6. *Roots of a quadratic equation that is almost linear.* The equation $ax^2 + 2x - 1 = 0$, where a is a constant, has two roots if $a > -1$ and $a \neq 0$, one positive and one negative:

$$r_+(a) = \frac{-1 + \sqrt{1+a}}{a}, \quad r_-(a) = \frac{-1 - \sqrt{1+a}}{a}.$$

   a) What happens to $r_+(a)$ as $a \to 0$? as $a \to -1^+$?
   b) What happens to $r_-(a)$ as $a \to 0$? as $a \to -1^+$?
   c) GRAPHER Support your conclusions by graphing $r_+(a)$ and $r_-(a)$ as functions of a. Describe what you see.
   d) GRAPHER For added support, graph $f(x) = ax^2 + 2x - 1$ simultaneously for $a = 1, 0.5, 0.2, 0.1$, and 0.05.

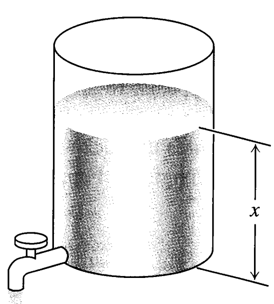
7. If $\lim_{x\to 0^+} f(x) = A$ and $\lim_{x\to 0^-} f(x) = B$, find
   a) $\lim_{x\to 0^+} f(x^3 - x)$
   b) $\lim_{x\to 0^-} f(x^3 - x)$
   c) $\lim_{x\to 0^+} f(x^2 - x^4)$
   d) $\lim_{x\to 0^-} f(x^2 - x^4)$

8. Which of the following statements are true, and which are false? If true, say why; if false, give a counterexample (that is, an example confirming the falsehood).
   a) If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} (f(x) + g(x))$ does not exist.
   b) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a} (f(x) + g(x))$ does not exist.
   c) If f is continuous at a, then so is $|f|$.
   d) If $|f|$ is continuous at a, then so is f.

9. Show that the equation $x + 2\cos x = 0$ has at least one solution.

10. Explain why the function $f(x) = \sin(1/x)$ has no continuous extension to $x = 0$.

11. *Controlling the flow from a draining tank.* Torricelli's law says that if you drain a tank like the one in the figure below, the rate y at which water runs out is a constant times the square root of the water's depth x. The constant depends on the size of the exit valve. Suppose that $y = \sqrt{x}/2$ for a certain tank. You are trying to maintain a fairly constant exit rate by pouring more water into the tank with a hose from time to time. How deep must you keep the water if you want to maintain the exit rate (a) within 0.2 ft^3/min of the rate $y_0 = 1$ ft^3/min? (b) within 0.1 ft^3/min of the rate $y_0 = 1$ ft^3/min?



Exit rate y ft^3/min

12. *Thermal expansion in precise equipment.* As you may know, most metals expand when heated and contract when cooled. The dimensions of a piece of laboratory equipment are sometimes so critical that the temperature in the shop where it is made and the laboratory where it is used must not be allowed to vary. A typical aluminum bar that is 10 cm wide at 70°F will be

$$y = 10 + (t - 70) \times 10^{-4}$$

centimeters wide at a nearby temperature t. Suppose you are using a bar like this in a gravity wave detector, where its width must stay within 0.0005 cm of the ideal 10 cm. How close to $t_0 = 70$°F must you maintain the temperature to ensure that this tolerance is not exceeded?

13. *Antipodal points.* Is there any reason to believe that there is always a pair of antipodal (diametrically opposite) points on the earth's equator where the temperatures are the same? Explain.

14. *Uniqueness of limits.* Show that a function cannot have two different limits at the same point. That is, if $\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} f(x) = L_2$, then $L_1 = L_2$.

In Exercises 15–18, use the formal definition of limit to prove that the function is continuous at x_0.

15. $f(x) = x^2 - 7, \quad x_0 = 1$

16. $g(x) = 1/(2x), \quad x_0 = 1/4$

17. $h(x) = \sqrt{2x - 3}, \quad x_0 = 2$

18. $F(x) = \sqrt{9 - x}, \quad x_0 = 5$

In Exercises 19 and 20, use the formal definition of limit to prove that the function has a continuous extension to the given value of x.

19. $f(x) = \dfrac{x^2 - 1}{x + 1}, \quad x = -1$

20. $g(x) = \dfrac{x^2 - 2x - 3}{2x - 6}, \quad x = 3$

21. *Max {a, b} and min {a, b}.*
   a) Show that the expression

$$\max \{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$$

   equals a if $a \geq b$ and equals b if $b \geq a$. In other words, max (a, b) gives the larger of the two numbers a and b.
   b) Find a similar expression for min $\{a, b\}$, the smaller of a and b.

*22. *A function continuous at only one point.* Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

   a) Show that f is continuous at $x = 0$.
   b) Use the fact that every nonempty open interval of real numbers contains both rational and irrational numbers to show that f is not continuous at any nonzero value of x.

***23.** *Bounded functions.* A real-valued function f is **bounded from above** on a set D if there exists a number N such that $f(x) \leq N$ for all x in D. We call N, when it exists, an **upper bound** for f on D and say that f is bounded from above by N. In a similar manner, we say that f is **bounded from below** on D if there exists a number M such that $f(x) \geq M$ for all x in D. We call M, when it exists, a **lower bound** for f on D and say that f is bounded from below by M. We say that f is **bounded** on D if it is bounded from both above and below.

a) Show that f is bounded on D if and only if there exists a number B such that $|f(x)| \leq B$ for all x in D.

b) Suppose that f is bounded from above by N. Show that if $\lim_{x\to x_0} f(x) = L$, then $L \leq N$.

c) Suppose that f is bounded from below by M. Show that if $\lim_{x\to x_0} f(x) = L$, then $L \geq M$.

***24.** *The Dirichlet ruler function.* If x is a rational number, then x can be written in a unique way as a quotient of integers m/n where $n > 0$ and m and n have no common factors greater than 1. (We say that such a fraction is in *lowest terms.* For example, $6/4$ written in lowest terms is $3/2$.) Let $f(x)$ be defined for all x in the interval $[0, 1]$ by

$$f(x) = \begin{cases} 1/n & \text{if } x = m/n \text{ is a rational number in lowest terms} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

For instance, $f(0) = f(1) = 1$, $f(1/2) = 1/2$, $f(1/3) = f(2/3) = 1/3$, $f(1/4) = f(3/4) = 1/4$, and so on.

a) Show that f is discontinuous at every rational number in $[0, 1]$.

b) Show that f is continuous at every irrational number in $[0, 1]$. (*Hint:* If ϵ is a given positive number, show that there are only finitely many rational numbers r in $[0, 1]$ such that $f(r) \geq \epsilon$.)

c) Sketch the graph of f. Why do you think f is called the "ruler function"?