CHAPTER 2

Derivatives

OVERVIEW In Chapter 1 we defined the slope of a curve at a point as the limit of secant slopes. This limit, called a derivative, measures the rate at which a function changes and is one of the most important ideas in calculus. Derivatives are used widely in science, economics, medicine, and computer science to calculate velocity and acceleration, to explain the behavior of machinery, to estimate the drop in water levels as water is pumped out of a tank, and to predict the consequences of making errors in measurements. Finding derivatives by evaluating limits can be lengthy and difficult. In this chapter we develop techniques to make calculating derivatives easier.

2.1 The Derivative of a Function

At the end of Chapter 1, we defined the slope of a curve $y = f(x)$ at the point where $x = x_0$ to be

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We called this limit, when it existed, the derivative of f at x_0. In this section, we investigate the derivative as a *function* derived from f by considering the limit at each point of f's domain.

Definition

The **derivative** of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

The domain of f', the set of points in the domain of f for which the limit exists, may be smaller than the domain of f. If $f'(x)$ exists, we say that f **has a derivative (is differentiable)** at x.

Notation

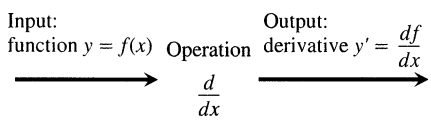
There are many ways to denote the derivative of a function $y = f(x)$. Besides $f'(x)$, the most common notations are these:

y'	"y prime"	Nice and brief but does not name the independent variable
$\dfrac{dy}{dx}$	"dy dx"	Names the variables and uses d for derivative
$\dfrac{df}{dx}$	"df dx"	Emphasizes the function's name
$\dfrac{d}{dx}f(x)$	"ddx of $f(x)$"	Emphasizes the idea that differentiation is an operation performed on f (Fig. 2.1)
$D_x f$	"dx of f"	A common operator notation
$\dot{y}$	"y dot"	One of Newton's notations, now common for time derivatives

We also read dy/dx as "the derivative of y with respect to x," and df/dx and $(d/dx)f(x)$ as "the derivative of f with respect to x."

Why all these notations?

The "prime" notations y' and f' come from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. Each has its own strengths and weaknesses.



2.1 Flow diagram for the operation of taking a derivative with respect to x.

Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation.** Examples 2 and 3 of Section 1.6 illustrate the process for the functions $y = mx + b$ and $y = 1/x$. Example 2 shows that

$$\frac{d}{dx}(mx + b) = m.$$

In Example 3, we see that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Here are two more examples.

Steps for Calculating $f'(x)$ from the Definition of Derivative

1. Write expressions for $f(x)$ and $f(x+h)$.
2. Expand and simplify the difference quotient
$$\frac{f(x+h) - f(x)}{h}.$$
3. Using the simplified quotient, find $f'(x)$ by evaluating the limit
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

EXAMPLE 1

a) Differentiate $f(x) = \dfrac{x}{x-1}$.

b) Where does the curve $y = f(x)$ have slope -1?

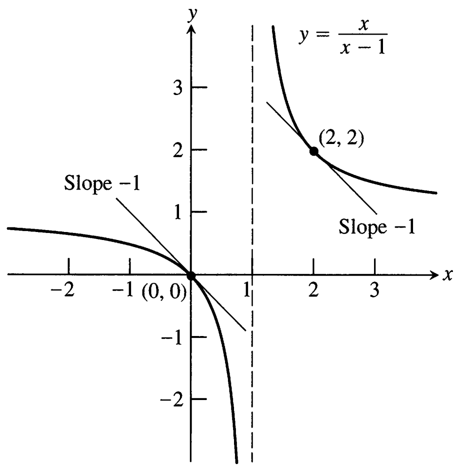
Solution

a) We take the three steps listed in the margin.

Step 1: Here we have $f(x) = \dfrac{x}{x-1}$

and

$$f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$



2.2 $y' = -1$ at $x = 0$ and $x = 2$.

Step 2: $$\frac{f(x+h)-f(x)}{h} = \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h}$$

$$= \frac{1}{h}\cdot\frac{(x+h)(x-1)-x(x+h-1)}{(x+h-1)(x-1)}$$

$\frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd}$

$$= \frac{1}{h}\cdot\frac{-h}{(x+h-1)(x-1)}, \text{ and}$$

Step 3: $$f'(x) = \lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}.$$

b) The slope of $y = f(x)$ will be -1 provided

$$-\frac{1}{(x-1)^2} = -1.$$

This equation is equivalent to $(x-1)^2 = 1$, so $x = 2$ or $x = 0$ (Fig. 2.2). ❑

EXAMPLE 2

a) Find the derivative of $y = \sqrt{x}$ for $x > 0$.
b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution

You will often need to know the derivative of $\sqrt{x}$ for $x > 0$:

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Try to remember it.

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a) **Step 1:** $f(x) = \sqrt{x}$ and $f(x+h) = \sqrt{x+h}$

Step 2: $$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

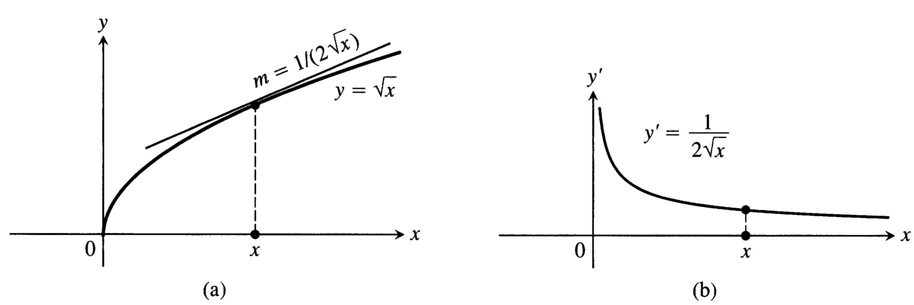
Multiply by $\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$$= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$

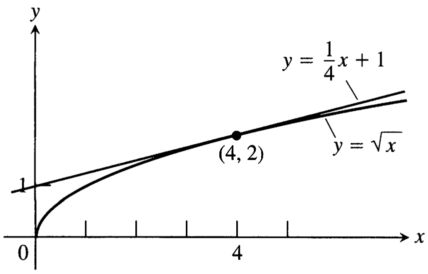
$$= \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

Step 3: $$f'(x) = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

See Fig. 2.3.



2.3 The graphs of (a) $y = \sqrt{x}$ and (b) $y' = 1/(2\sqrt{x}), x > 0$ (Example 2). The function is defined at $x = 0$, but its derivative is not.



2.4 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating *dy/dx* at $x = 4$ (Example 2).

b) The slope of the curve at $x = 4$ is

$$\left.\frac{dy}{dx}\right|_{x=4} = \left.\frac{1}{2\sqrt{x}}\right|_{x=4} = \frac{1}{4}.$$

The tangent is the line through the point (4, 2) with slope 1/4 (Fig. 2.4).

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1$$

❑

Graphing f' from Estimated Values

When we measure the values of a function $y = f(x)$ in the laboratory or in the field (pressure vs. temperature, say, or population vs. time) we usually connect the data points with lines or curves to picture the graph of f. We can often make a reasonable plot of f' by estimating slopes on this graph. The following examples show how this is done and what can be learned from the process.

EXAMPLE 3 *Medicine*

On April 23, 1988, the human-powered airplane *Daedalus* flew a record-breaking 119 km from Crete to the island of Santorini in the Aegean Sea, southeast of mainland Greece. During the 6-h endurance tests before the flight, researchers monitored the prospective pilots' blood-sugar concentrations. The concentration graph for one of the athlete-pilots is shown in Fig. 2.5(a), where the concentration in milligrams/deciliter is plotted against time in hours.

The graph is made of line segments connecting data points. The constant slope of each segment gives an estimate of the derivative of the concentration between measurements. We calculated the slope of each segment from the coordinate grid and plotted the derivative as a step function in Fig. 2.5(b). To make the plot for the first hour, for instance, we observed that the concentration increased from about 79 mg/dL to 93 mg/dL. The net increase was $\Delta y = 93 - 79 = 14$ mg/dL. Dividing this by $\Delta t = 1$ h gave the rate of change as

$$\frac{\Delta y}{\Delta t} = \frac{14}{1} = 14 \text{ mg/dL per h.}$$

❑

The symbol for evaluation

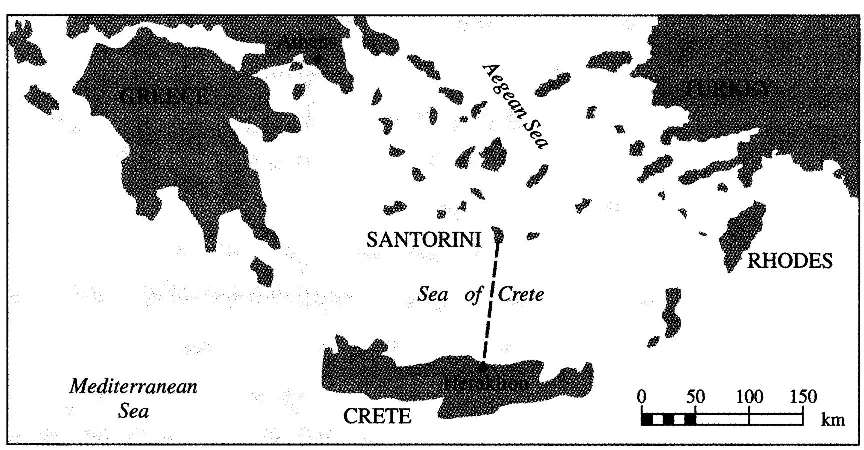
In addition to

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

the value of the derivative of $y = f(x)$ with respect to x at $x = a$ can be denoted in the following ways:

$$y'\,|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}.$$

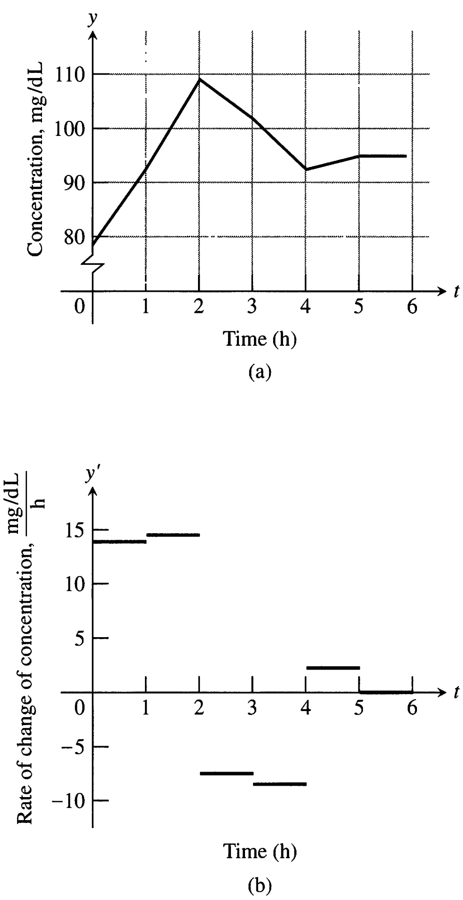
Here the symbol $|_{x=a}$, called an **evaluation symbol,** tells us to evaluate the expression to its left at $x = a$.



Daedalus's flight path on April 23, 1988.

Notice that we can make no estimate of the concentration's rate of change at times $t = 1, 2, \ldots, 5$, where the graph we have drawn for the concentration has a corner and no slope. The derivative step function is not defined at these times.

When we have so many data that the graph we get by connecting the data points resembles a smooth curve, we may wish to plot the derivative as a smooth curve. The next example shows how this is done.



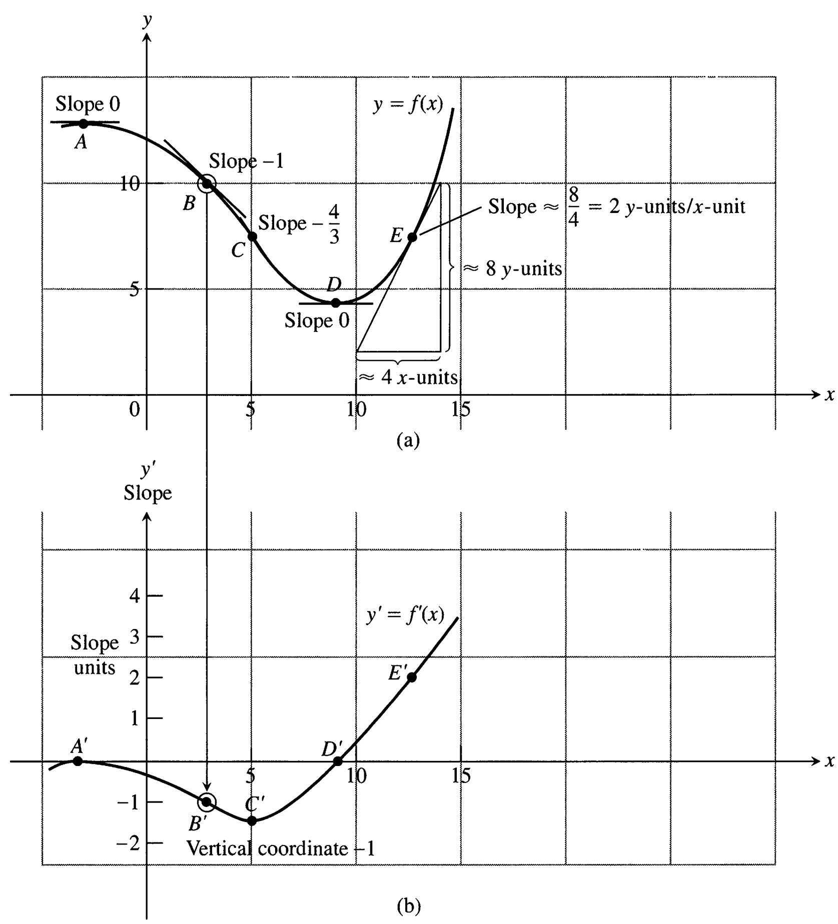
2.5 (a) The sugar concentration in the blood of a *Daedalus* pilot during a 6-h preflight endurance test. (b) The derivative of the pilot's blood-sugar concentration shows how rapidly the concentration rose and fell during various portions of the test. (Source: *The Daedalus Project: Physiological Problems and Solutions* by Ethan R. Nadel and Steven R. Bussolari, *American Scientist,* Vol. 76, No. 4, July–August 1988, p. 358.)

EXAMPLE 4 Graph the derivative of the function $y = f(x)$ in Fig. 2.6(a).

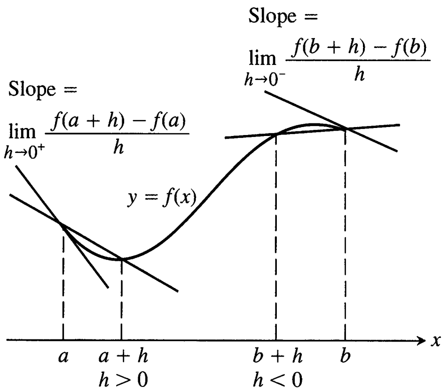
Solution We draw a pair of axes, marking the horizontal axis in x-units and the vertical axis in y'-units (Fig. 2.6b). Next we sketch tangents to the graph of f at frequent intervals and use their slopes to estimate the values of $y' = f'(x)$ at these points. We plot the corresponding (x, y') pairs and connect them with a smooth curve.

From the graph of $y' = f'(x)$ we see at a glance

1. where f's rate of change is positive, negative, or zero;
2. the rough size of the growth rate at any x and its size in relation to the size of $f(x)$;
3. where the rate of change itself is increasing or decreasing. ❑



2.6 We made the graph of $y' = f'(x)$ in (b) by plotting slopes from the graph of $y = f(x)$ in (a). The vertical coordinate of B' is the slope at B, and so on. The graph of $y' = f'(x)$ is a visual record of how the slope of f changes with x.



2.7 Derivatives at endpoints are one-sided limits.

Differentiable on an Interval; One-sided Derivatives

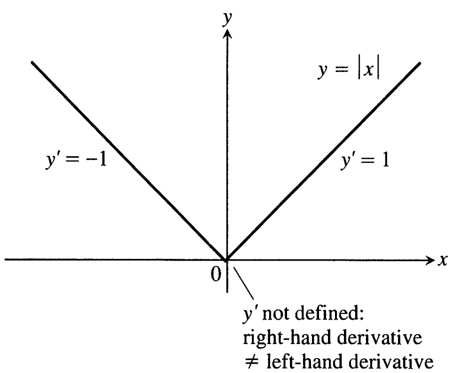
A function $y = f(x)$ is **differentiable** on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \qquad \textbf{Right-hand derivative at } \boldsymbol{a}$$

$$\lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} \qquad \textbf{Left-hand derivative at } \boldsymbol{b}$$

exist at the endpoints (Fig. 2.7).

Right-hand and left-hand derivatives may be defined at any point of a function's domain. The usual relation between one-sided and two-sided limits holds for these derivatives. Because of Theorem 5, Section 1.4, a function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.



2.8 Not differentiable at the origin.

EXAMPLE 5 The function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$. To the right of the origin,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1. \qquad \frac{d}{dx}(mx+b) = m$$

To the left,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1$$

(Fig. 2.8). There can be no derivative at the origin because the one-sided derivatives differ there:

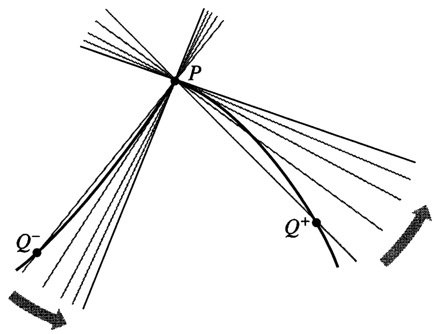
$$\begin{aligned}
\text{Right-hand derivative of } |x| \text{ at zero} &= \lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} \\
&= \lim_{h\to 0^+} \frac{h}{h} \qquad |h| = h \text{ when } h > 0 \\
&= \lim_{h\to 0^+} 1 = 1 \\
\text{Left-hand derivative of } |x| \text{ at zero} &= \lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} \\
&= \lim_{h\to 0^-} \frac{-h}{h} \qquad |h| = -h \text{ when } h < 0 \\
&= \lim_{h\to 0^-} -1 = -1.
\end{aligned}$$

❑

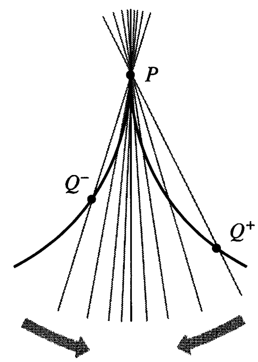
When Does a Function *Not* Have a Derivative at a Point?

A function has a derivative at a point x_0 if the slopes of the secant lines through $P(x_0, f(x_0))$ and a nearby point Q on the graph approach a limit as Q approaches P. Whenever the secants fail to take up a limiting position or become vertical as Q approaches P, the derivative does not exist. A function whose graph is otherwise smooth will fail to have a derivative at a point where the graph has

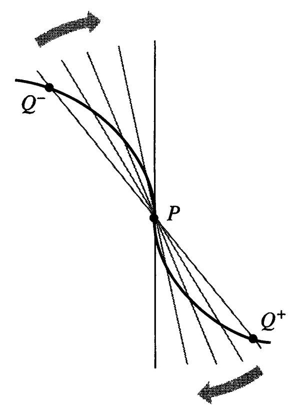
1. a *corner,* where the one-sided derivatives differ



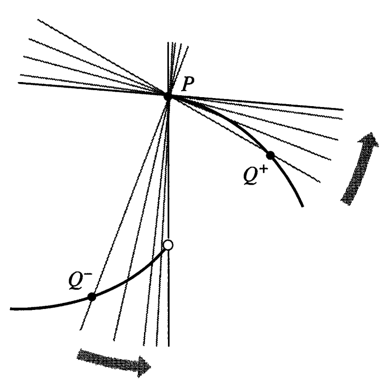
2. a *cusp,* where the slope of PQ approaches ∞ from one side and $-\infty$ from the other

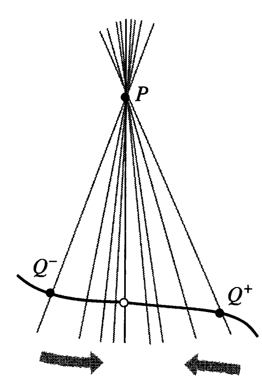


3. a *vertical tangent,* where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$)



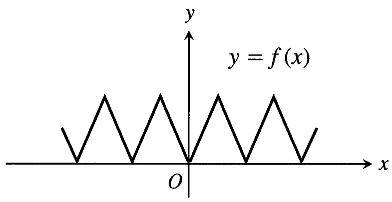
4. a *discontinuity.*





How rough can the graph of a continuous function be?

The absolute value function fails to be differentiable at a single point. Using a similar idea, we can use a sawtooth graph to define a continuous function that fails to have a derivative at infinitely many points.



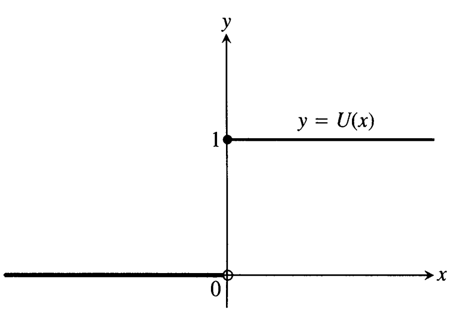
But can a continuous function fail to have a derivative at *every* point?

The answer, surprisingly enough, is yes, as Karl Weierstrass (1815–1897) found in 1872. One of his formulas (there are many like it) was

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \cos(9^n \pi x),$$

a formula that expresses f as an infinite sum of cosines with increasingly higher frequencies. By adding wiggles to wiggles infinitely many times, so to speak, the formula produces a graph that is too bumpy in the limit to have a tangent anywhere.

Continuous curves that fail to have a tangent anywhere play a useful role in chaos theory, in part because there is no way to assign a finite length to such a curve. We will see what length has to do with derivatives when we get to Section 5.5.



2.9 The unit step function does not have the intermediate value property and cannot be the derivative of a function on the real line.

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

Theorem 1

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof Given that $f'(c)$ exists, we must show that $\lim_{x\to c} f(x) = f(c)$, or, equivalently, that $\lim_{h\to 0} f(c+h) = f(c)$. If $h \neq 0$, then

$$\begin{aligned} f(c+h) &= f(c) + (f(c+h) - f(c)) \\ &= f(c) + \frac{f(c+h) - f(c)}{h} \cdot h. \end{aligned}$$

Now take limits as $h \to 0$. By Theorem 1 of Section 1.2,

$$\begin{aligned} \lim_{h\to 0} f(c+h) &= \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h\to 0} h \\ &= f(c) + f'(c) \cdot 0 \\ &= f(c) + 0 \\ &= f(c). \end{aligned}$$

❑

Similar arguments with one-sided limits show that if f has a derivative from one side (right or left) at $x = c$, then f is continuous from that side at $x = c$.

Caution The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous, as we saw in Example 5.

The Intermediate Value Property of Derivatives

Not every function can be some function's derivative, as we see from the following theorem.

Theorem 2

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

Theorem 2 (which we will not prove) says that a function cannot *be* a derivative on an interval unless it has the intermediate value property there (Fig. 2.9). The question of when a function is a derivative is one of the central questions in all calculus, and Newton's and Leibniz's answer to this question revolutionized the world of mathematics. We will see what their answer was when we reach Chapter 4.

Exercises 2.1

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2;\quad f'(-3), f'(0), f'(1)$
2. $F(x) = (x-1)^2 + 1;\quad F'(-1), F'(0), F'(2)$
3. $g(t) = \dfrac{1}{t^2};\quad g'(-1), g'(2), g'(\sqrt{3})$
4. $k(z) = \dfrac{1-z}{2z};\quad k'(-1), k'(1), k'(\sqrt{2})$
5. $p(\theta) = \sqrt{3\theta};\quad p'(1), p'(3), p'(2/3)$
6. $r(s) = \sqrt{2s+1};\quad r'(0), r'(1), r'(1/2)$

In Exercises 7–12, find the indicated derivatives.

7. $\dfrac{dy}{dx}$ if $y = 2x^3$
8. $\dfrac{dr}{ds}$ if $r = \dfrac{s^3}{2} + 1$
9. $\dfrac{ds}{dt}$ if $s = \dfrac{t}{2t+1}$
10. $\dfrac{dv}{dt}$ if $v = t - \dfrac{1}{t}$
11. $\dfrac{dp}{dq}$ if $p = \dfrac{1}{\sqrt{q+1}}$
12. $\dfrac{dz}{dw}$ if $z = \dfrac{1}{\sqrt{3w-2}}$

Slopes and Tangent Lines

In Exercises 13–16, differentiate the functions and find the slope of the tangent line at the given value of the independent variable.

13. $f(x) = x + \dfrac{9}{x},\quad x = -3$
14. $k(x) = \dfrac{1}{2+x},\quad x = 2$
15. $s = t^3 - t^2,\quad t = -1$
16. $y = (x+1)^3,\quad x = -2$

In Exercises 17–18, differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

17. $y = f(x) = \dfrac{8}{\sqrt{x-2}},\quad (x, y) = (6, 4)$
18. $w = g(z) = 1 + \sqrt{4-z},\quad (z, w) = (3, 2)$

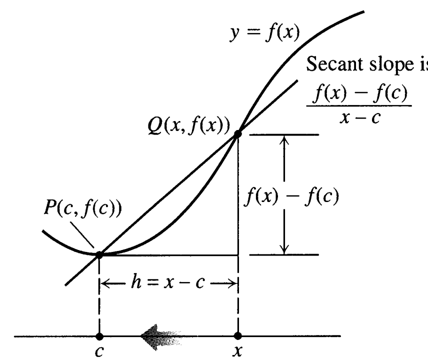
In Exercises 19–22, find the values of the derivatives.

19. $\left.\dfrac{ds}{dt}\right|_{t=-1}$ if $s = 1 - 3t^2$
20. $\left.\dfrac{dy}{dx}\right|_{x=\sqrt{3}}$ if $y = 1 - \dfrac{1}{x}$
21. $\left.\dfrac{dr}{d\theta}\right|_{\theta=0}$ if $r = \dfrac{2}{\sqrt{4-\theta}}$
22. $\left.\dfrac{dw}{dz}\right|_{z=4}$ if $w = z + \sqrt{z}$

An Alternative Formula for Calculating Derivatives

The formula for the secant slope whose limit leads to the derivative depends on how the points involved are labeled. In the notation of Fig. 2.10, the secant slope is $(f(x) - f(c))/(x - c)$ and the slope of the curve at P is

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}.$$



Derivative of f at c is

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

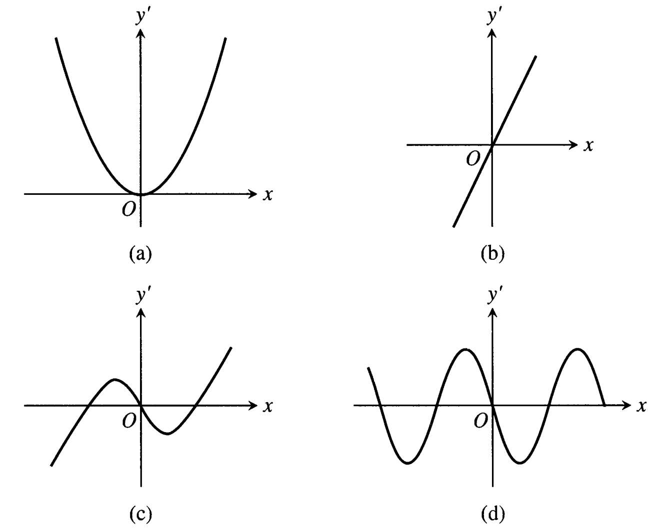
2.10 The way we write the difference quotient for the derivative of a function f depends on how we label the points involved.

The use of this formula simplifies some derivative calculations. Use it in Exercises 23–26 to find the derivative of the function at the given value of c.

23. $f(x) = \dfrac{1}{x+2},\quad c = -1$
24. $f(x) = \dfrac{1}{(x-1)^2},\quad c = 2$
25. $g(t) = \dfrac{t}{t-1},\quad c = 3$
26. $k(s) = 1 + \sqrt{s},\quad c = 9$

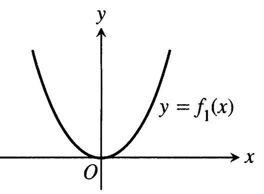
Graphs

Match the functions graphed in Exercises 27–30 with the derivatives graphed in Fig. 2.11.



2.11 The derivative graphs for Exercises 27–30.

27.



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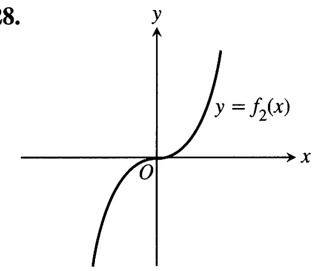

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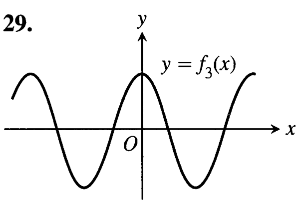

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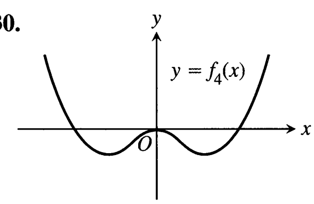
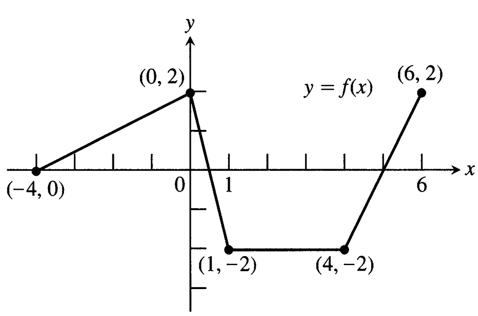


31. a) The graph in Fig. 2.12 is made of line segments joined end to end. At which points of the interval $[-4, 6]$ is f' not defined? Give reasons for your answer.



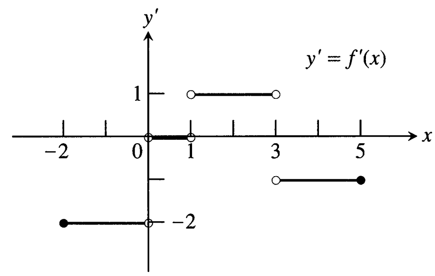
2.12 The graph for Exercise 31.

b) Graph the derivative of f. Call the vertical axis the y'-axis. The graph should show a step function.

32. *Recovering a function from its derivative*

a) Use the following information to graph the function f over the closed interval $[-2, 5]$.

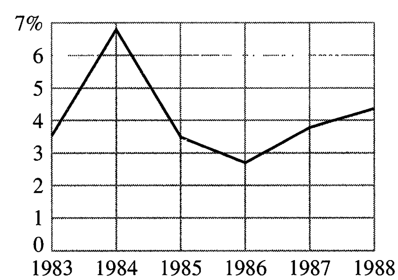
i) The graph of f is made of closed line segments joined end to end.
ii) The graph starts at the point $(-2, 3)$.
iii) The derivative of f is the step function in Fig. 2.13.



2.13 The derivative graph for Exercise 32.

b) Repeat part (a) assuming that the graph starts at $(-2, 0)$ instead of $(-2, 3)$.

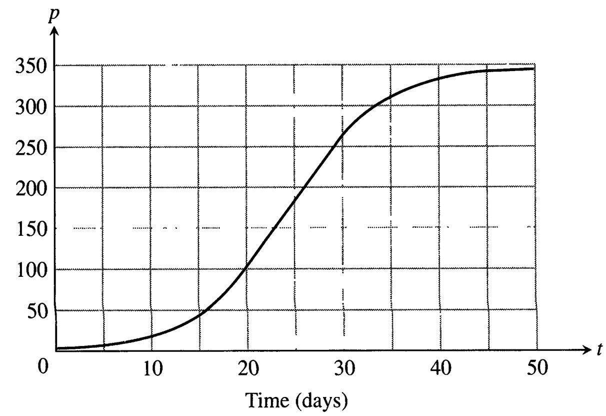
33. *Growth in the economy.* The graph in Fig. 2.14 shows the average annual percentage change $y = f(t)$ in the U.S. gross national product (GNP) for the years 1983–1988. Graph dy/dt (where defined). (Source: *Statistical Abstracts of the United States,* 110th Edition, U.S. Department of Commerce, p. 427.)



2.14 The graph for Exercise 33.

34. *Fruit flies. (Continuation of Example 3, Section 1.1.)* Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

a) Use the graphical technique of Example 4 to graph the derivative of the fruit fly population introduced in Section 1.1. The graph of the population is reproduced here as Fig. 2.15. What units should be used on the horizontal and vertical axes for the derivative's graph?

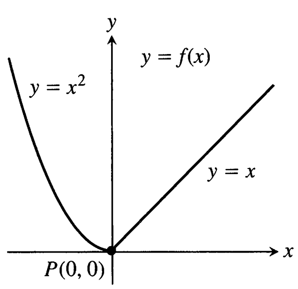


2.15 The graph for Exercise 34.

b) During what days does the population seem to be increasing fastest? slowest?

Compare the right-hand and left-hand derivatives to show that the functions in Exercises 35–38 are not differentiable at the point *P*.

35.



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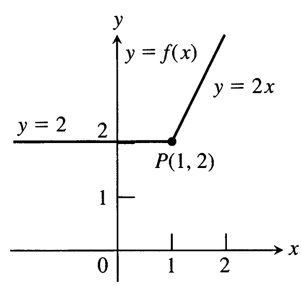

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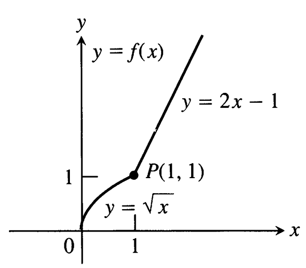

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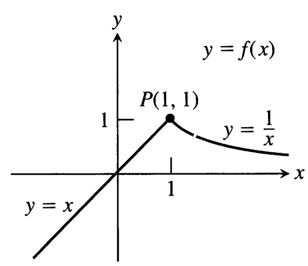
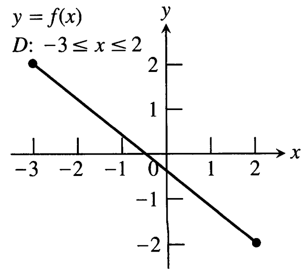


Each figure in Exercises 39–44 shows the graph of a function over a closed interval *D*. At what domain points does the function appear to be

a) differentiable?
b) continuous but not differentiable?
c) neither continuous nor differentiable?

Give reasons for your answers.

39.



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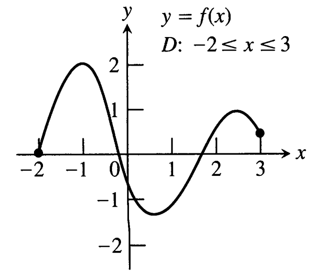

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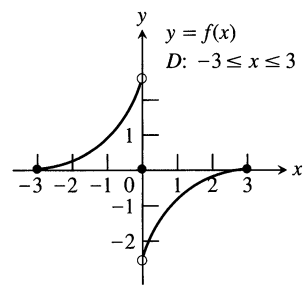

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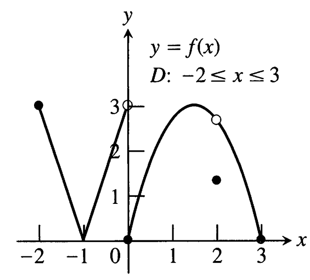

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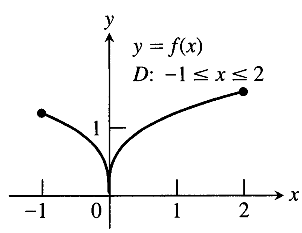

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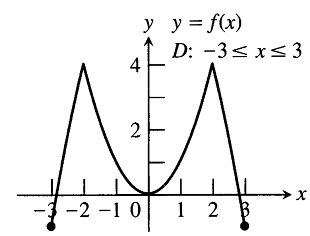


Theory and Examples

In Exercises 45–48,

a) Find the derivative $y' = f'(x)$ of the given function $y = f(x)$.
b) Graph $y = f(x)$ and $y' = f'(x)$ side by side using separate sets of coordinate axes, and answer the following questions.
c) For what values of *x*, if any, is y' positive? zero? negative?
d) Over what intervals of *x*-values, if any, does the function $y = f(x)$ increase as *x* increases? decrease as *x* increases? How is this related to what you found in (c)? (We will say more about this relationship in Chapter 3.)

45. $y = -x^2$

46. $y = -1/x$

47. $y = x^3/3$

48. $y = x^4/4$

49. Does the curve $y = x^3$ ever have a negative slope? If so, where? Give reasons for your answer.

50. Does the curve $y = 2\sqrt{x}$ have any horizontal tangents? If so, where? Give reasons for your answer.

51. Does the parabola $y = 2x^2 - 13x + 5$ have a tangent whose slope is -1? If so, find an equation for the line and the point of tangency. If not, why not?

52. Does any tangent to the curve $y = \sqrt{x}$ cross the x-axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?

53. Does any function differentiable on $(-\infty, \infty)$ have $y = \lfloor x \rfloor$ as its derivative? Give reasons for your answer.

54. Graph the derivative of $f(x) = |x|$. Then graph $y = (|x| - 0)/(x - 0) = |x|/x$. What can you conclude?

55. Does knowing that a function $f(x)$ is differentiable at $x = x_0$ tell you anything about the differentiability of the function $-f$ at $x = x_0$? Give reasons for your answer.

56. Does knowing that a function $g(t)$ is differentiable at $t = 7$ tell you anything about the differentiability of the function $3g$ at $t = 7$? Give reasons for your answer.

57. Suppose that functions $g(t)$ and $h(t)$ are defined for all values of t and that $g(0) = h(0) = 0$. Can $\lim_{t\to 0}(g(t))/(h(t))$ exist? If it does exist, must it equal zero? Give reasons for your answers.

58. **a)** Let $f(x)$ be a function satisfying $|f(x)| \le x^2$ for $-1 \le x \le 1$. Show that f is differentiable at $x = 0$ and find $f'(0)$.

b) Show that

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$ and find $f'(0)$.

Grapher Explorations

59. Graph $y = 1/(2\sqrt{x})$ in a window that has $0 \le x \le 2$. Then, on the same screen, graph

$$y = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

for $h = 1, 0.5, 0.1$. Then try $h = -1, -0.5, -0.1$. Explain what is going on.

60. Graph $y = 3x^2$ in a window that has $-2 \le x \le 2, 0 \le y \le 3$. Then, on the same screen, graph

$$y = \frac{(x+h)^3 - x^3}{h}$$

for $h = 2, 1, 0.2$. Then try $h = -2, -1, -0.2$. Explain what is going on.

61. *Weierstrass's nowhere differentiable continuous function.* The sum of the first eight terms of the Weierstrass function $f(x) = \sum_{n=0}^{\infty}(2/3)^n \cos(9^n \pi x)$ is

$$g(x) = \cos(\pi x) + \left(\frac{2}{3}\right)^1 \cos(9\pi x) + \left(\frac{2}{3}\right)^2 \cos(9^2 \pi x) + \left(\frac{2}{3}\right)^3 \cos(9^3 \pi x) + \cdots + \left(\frac{2}{3}\right)^7 \cos(9^7 \pi x).$$

Graph this sum. Zoom in several times. How wiggly and bumpy is this graph? Specify a viewing window in which the displayed portion of the graph is smooth.

CAS Explorations and Projects

Use a CAS to perform the following steps for the functions in Exercises 62–67.

a) Plot $y = f(x)$ to see that function's global behavior.

b) Define the difference quotient q at a general point x, with general stepsize h.

c) Take the limit as $h \to 0$. What formula does this give?

d) Substitute the value $x = x_0$ and plot the function together with its tangent line at that point.

e) Substitute various values for x larger and smaller than x_0 into the formula obtained in part (c). Do the numbers make sense with your picture?

f) Graph the formula obtained in part (c). What does it mean when its values are negative? zero? positive? Does this make sense with your plot from part (a)? Give reasons for your answer.

62. $f(x) = x^3 + x^2 - x, \quad x_0 = 1$

63. $f(x) = x^{1/3} + x^{2/3}, \quad x_0 = 1$

64. $f(x) = \dfrac{4x}{x^2 + 1}, \quad x_0 = 2$

65. $f(x) = \dfrac{x - 1}{3x^2 + 1}, \quad x_0 = -1$

66. $f(x) = \sin 2x, \quad x_0 = \pi/2$

67. $f(x) = x^2 \cos x, \quad x_0 = \pi/4$

2.2

Differentiation Rules

This section shows how to differentiate functions without having to apply the definition each time.

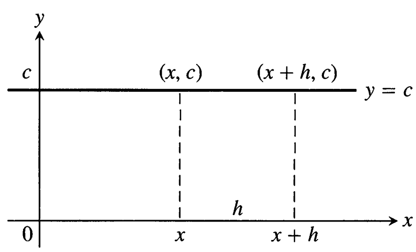
Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is zero.

Rule 1 Derivative of a Constant

If c is constant, then $\frac{d}{dx}c = 0$.

EXAMPLE 1 $\frac{d}{dx}(8) = 0, \quad \frac{d}{dx}\left(-\frac{1}{2}\right) = 0, \quad \frac{d}{dx}\left(\sqrt{3}\right) = 0$ ❑



2.16 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

Proof of Rule 1 We apply the definition of derivative to $f(x) = c$, the function whose outputs have the constant value c (Fig. 2.16). At every value of x, we find that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$ ❑

The next rule tells how to differentiate x^n if n is a positive integer.

Rule 2 Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

To apply the Power Rule, we subtract 1 from the original exponent (n) and multiply the result by n.

EXAMPLE 2

f	x	x^2	x^3	x^4	$\cdots$
f'	1	$2x$	$3x^2$	$4x^3$	$\cdots$

❑

Proof of Rule 2 If $f(x) = x^n$, then $f(x+h) = (x+h)^n$. Since n is a positive integer, we can use the fact that

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

to simplify the difference quotient for f. Taking $x+h=a$ and $x=b$, we have $a-b=h$. Thus

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^n - x^n}{h}\\ &= \frac{(h)[(x+h)^{n-1}+(x+h)^{n-2}x+\cdots+(x+h)x^{n-2}+x^{n-1}]}{h}\\ &= \underbrace{(x+h)^{n-1}+(x+h)^{n-2}x+\cdots+(x+h)x^{n-2}+x^{n-1}}_{n \text{ terms, each with limit } x^{n-1} \text{ as } h\to 0}.\end{aligned}$$

Hence

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = nx^{n-1}.$$ ❑

The next rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

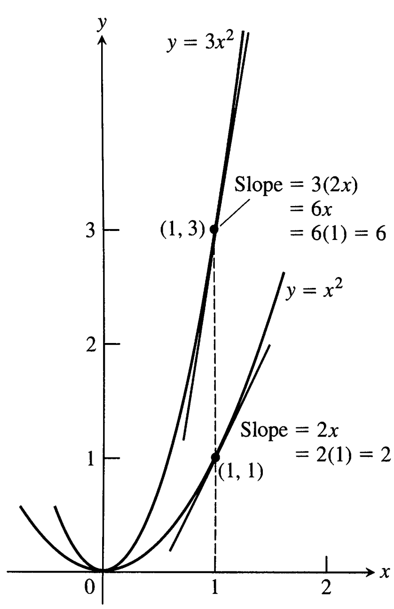
Rule 3 The Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

In particular, if n is a positive integer, then

$$\frac{d}{dx}(cx^n) = cn\,x^{n-1}.$$



2.17 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinates triples the slope (Example 3).

EXAMPLE 3 The derivative formula

$$\frac{d}{dx}(3x^2) = 3\cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3 (Fig. 2.17). ❑

EXAMPLE 4 *A useful special case*

The derivative of the negative of a differentiable function is the negative of the function's derivative. Rule 3 with $c=-1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1\cdot u) = -1\cdot\frac{d}{dx}(u) = -\frac{du}{dx}.$$ ❑

Proof of Rule 3

$$\begin{aligned}\frac{d}{dx}cu &= \lim_{h\to 0}\frac{cu(x+h)-cu(x)}{h} && \text{Derivative definition with } f(x)=cu(x)\\ &= c\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} && \text{Limit property}\\ &= c\frac{du}{dx} && u \text{ is differentiable.}\end{aligned}$$ ❑

The next rule says that the derivative of the sum of two differentiable functions is the sum of their derivatives.

Denoting functions by *u* and *v*

The functions we are working with when we need a differentiation formula are likely to be denoted by letters like f and g. When we apply the formula, we do not want to find it using these same letters in some other way. To guard against this, we denote the functions in differentiation rules by letters like u and v that are not likely to be already in use.

Rule 4 The Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Combining the Sum Rule with the Constant Multiple Rule gives the equivalent **Difference Rule,** which says that the derivative of a *difference* of differentiable functions is the difference of their derivatives.

$$\frac{d}{dx}(u-v) = \frac{d}{dx}[u+(-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

The Sum Rule also extends to sums of more than two functions, as long as there are only finitely many functions in the sum. If $u_1, u_2, \ldots, u_n$ are differentiable at x, then so is $u_1 + u_2 + \cdots + u_n$, and

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.$$

EXAMPLE 5

a) $y = x^4 + 12x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x)$$

$$= 4x^3 + 12$$

b) $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0$$

$$= 3x^2 + \frac{8}{3}x - 5$$

❑

Notice that we can differentiate any polynomial term by term, the way we differentiated the polynomials in Example 5.

Proof of Rule 4 We apply the definition of derivative to $f(x) = u(x) + v(x)$:

$$\frac{d}{dx}[u(x)+v(x)] = \lim_{h\to 0} \frac{[u(x+h)+v(x+h)] - [u(x)+v(x)]}{h}$$

$$= \lim_{h\to 0}\left[\frac{u(x+h)-u(x)}{h} + \frac{v(x+h)-v(x)}{h}\right]$$

$$= \lim_{h\to 0}\frac{u(x+h)-u(x)}{h} + \lim_{h\to 0}\frac{v(x+h)-v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}.$$

❑

Proof by mathematical induction

Many formulas can be shown to hold for every positive integer n greater than or equal to some lowest integer n_0 by applying an axiom called the *mathematical induction principle.* A proof using this axiom is called a *proof by mathematical induction* or a *proof by induction.* The steps in proving a formula by induction are

1. Check that it holds for $n = n_0$.
2. Prove that if it holds for any positive integer $n = k \geq n_0$, then it holds for $n = k + 1$.

Once these steps are completed, the axiom says, we know that the formula holds for all $n \geq n_0$. For more mathematical induction, see Appendix 1.

Proof of the Sum Rule for Sums of More Than Two Functions We prove the statement

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}$$

by mathematical induction. The statement is true for $n = 2$, as was just proved. This is step 1 of the induction proof.

Step 2 is to show that if the statement is true for any positive integer $n = k$, where $k \geq n_0 = 2$, then it is also true for $n = k + 1$. So suppose that

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_k) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx}. \quad (1)$$

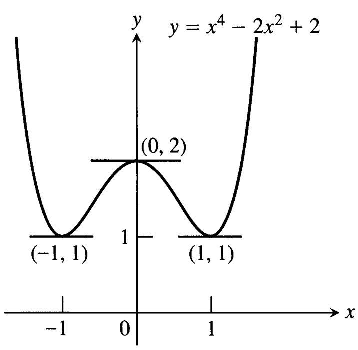
Then

$$\frac{d}{dx}\underbrace{(u_1 + u_2 + \cdots + u_k}_{\text{Call the function defined by this sum } u.} + \underbrace{u_{k+1})}_{\text{Call this function } v.}$$

$$= \frac{d}{dx}(u_1 + u_2 + \cdots + u_k) + \frac{du_{k+1}}{dx} \qquad \text{Rule 4 for } \frac{d}{dx}(u+v)$$

$$= \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx} + \frac{du_{k+1}}{dx}. \qquad \text{Eq. (1)}$$

With these steps verified, the mathematical induction principle now guarantees the Sum Rule for every integer $n \geq 2$. ❑



2.18 The curve $y = x^4 - 2x^2 + 2$ and its horizontal tangents (Example 6).

EXAMPLE 6 Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Solution The horizontal tangents, if any, occur where the slope dy/dx is zero. To find these points, we

1. Calculate dy/dx: $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x$
2. Solve the equation $\dfrac{dy}{dx} = 0$ for x:
$$\begin{aligned} 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ x &= 0, 1, -1 \end{aligned}$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1$, and -1. The corresponding points on the curve are (0, 2), (1, 1) and (−1, 1). See Fig. 2.18. ❑

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \qquad \text{while} \qquad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product of two functions is the sum of *two* products, as we now explain.

Rule 5 The Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The derivative of the product uv is u times the derivative of v plus v times the derivative of u. In *prime notation,* $(uv)' = uv' + vu'$.

Proof of Rule 5

$$\frac{d}{dx}(uv) = \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of u and v, we subtract and add $u(x+h)v(x)$ in the numerator:

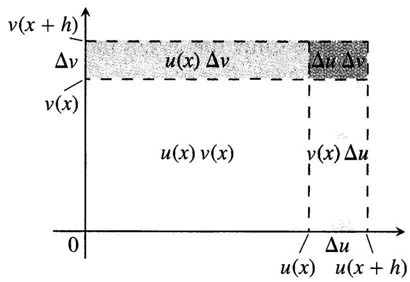
$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}\\ &= \lim_{h\to 0}\left[u(x+h)\frac{v(x+h)-v(x)}{h} + v(x)\frac{u(x+h)-u(x)}{h}\right]\\ &= \lim_{h\to 0} u(x+h)\cdot\lim_{h\to 0}\frac{v(x+h)-v(x)}{h} + v(x)\cdot\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}.\end{aligned}$$

As h approaches zero, $u(x+h)$ approaches $u(x)$ because u, being differentiable at x, is continuous at x. The two fractions approach the values of dv/dx at x and du/dx at x. In short,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$ ❑

Picturing the product rule

If $u(x)$ and $v(x)$ are positive and increase when x increases, and if $h > 0$,



the total shaded area in the picture is

$$u(x+h)v(x+h) - u(x)v(x) = u(x+h)\Delta v + v(x+h)\Delta u - \Delta u\Delta v.$$

Dividing both sides of this equation by h gives

$$\frac{u(x+h)v(x+h) - u(x)v(x)}{h} = u(x+h)\frac{\Delta v}{h} + v(x+h)\frac{\Delta u}{h} - \Delta u\frac{\Delta v}{h}.$$

As $h \to 0^+$, $\Delta u \cdot \frac{\Delta v}{h} \to 0 \cdot \frac{dv}{dx} = 0$, leaving

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

EXAMPLE 7 Find the derivative of $y = (x^2+1)(x^3+3)$.

Solution From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned}\frac{d}{dx}[(x^2+1)(x^3+3)] &= (x^2+1)(3x^2) + (x^3+3)(2x)\\ &= 3x^4 + 3x^2 + 2x^4 + 6x\\ &= 5x^4 + 3x^2 + 6x.\end{aligned}$$ ❑

Example 7 can be done as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial. We now check:

$$y = (x^2+1)(x^3+3) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

This is in agreement with our first calculation.

There are times, however, when the Product Rule *must* be used. In the following example, we have only numerical values to work with.

EXAMPLE 8 Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if

$$u(2) = 3, \qquad u'(2) = -4, \qquad v(2) = 1, \qquad \text{and} \qquad v'(2) = 2.$$

Solution From the Product Rule, in the form

$$y' = (uv)' = uv' + vu',$$

we have

$$\begin{aligned} y'(2) &= u(2)v'(2) + v(2)u'(2) \\ &= (3)(2) + (1)(-4) = 6 - 4 = 2. \end{aligned}$$

❑

Quotients

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two functions is not the quotient of their derivatives. What happens instead is this:

Rule 6 The Quotient Rule

If u and v are differentiable at x, and $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Proof of Rule 6

$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h} \\ &= \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)} \end{aligned}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of u and v, we subtract and add $v(x)u(x)$ in the numerator. We then get

$$\begin{aligned} \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)} \\ &= \lim_{h\to 0} \frac{v(x)\dfrac{u(x+h) - u(x)}{h} - u(x)\dfrac{v(x+h) - v(x)}{h}}{v(x+h)v(x)}. \end{aligned}$$

Taking the limit in the numerator and denominator now gives the Quotient Rule.

❑

EXAMPLE 9 Find the derivative of $y = \dfrac{t^2 - 1}{t^2 + 1}$.

Solution We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$$\begin{aligned}\frac{dy}{dt} &= \frac{(t^2+1)\cdot 2t - (t^2-1)\cdot 2t}{(t^2+1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}\\ &= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2}\\ &= \frac{4t}{(t^2+1)^2}.\end{aligned}$$

❑

The Power Rule for Negative Integers

The Power Rule for negative integers is the same as the rule for positive integers.

> **Rule 7 Power Rule for Negative Integers**
>
> If n is a negative integer and $x \neq 0$, then
>
> $$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Proof of Rule 7 The proof uses the Quotient Rule in a clever way. If n is a negative integer, then $n = -m$ where m is a positive integer. Hence, $x^n = x^{-m} = 1/x^m$ and

$$\begin{aligned}\frac{d}{dx}(x^n) &= \frac{d}{dx}\left(\frac{1}{x^m}\right)\\ &= \frac{x^m \cdot \frac{d}{dx}(1) - 1\cdot \frac{d}{dx}(x^m)}{(x^m)^2} \qquad \text{Quotient Rule with } u = 1 \text{ and } v = x^m\\ &= \frac{0 - mx^{m-1}}{x^{2m}} \qquad \text{Since } m > 0,\ \frac{d}{dx}(x^m) = mx^{m-1}\\ &= -mx^{-m-1}\\ &= nx^{n-1}. \qquad \text{Since } -m = n\end{aligned}$$

❑

EXAMPLE 10

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$$

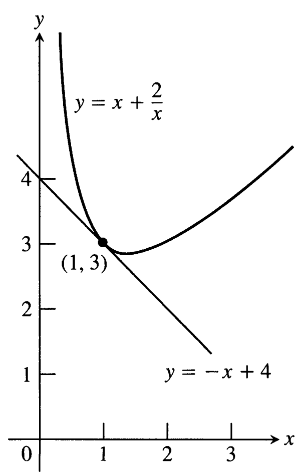
$$\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$$

❑

EXAMPLE 11 Find an equation for the tangent to the curve

$$y = x + \frac{2}{x}$$

at the point (1, 3) (Fig. 2.19).



2.19 The tangent to the curve $y = x + (2/x)$ at (1, 3). The curve has a third-quadrant portion not shown here. We will see how to graph functions like this in Chapter 3.

Solution The slope of the curve is

$$\frac{dy}{dx} = \frac{d}{dx}(x) + 2\frac{d}{dx}\left(\frac{1}{x}\right) = 1 + 2\left(-\frac{1}{x^2}\right) = 1 - \frac{2}{x^2}.$$

The slope at $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \left[1 - \frac{2}{x^2}\right]_{x=1} = 1 - 2 = -1.$$

The line through (1, 3) with slope $m = -1$ is

$$y - 3 = (-1)(x - 1) \qquad \text{Point–slope equation}$$
$$y = -x + 1 + 3$$
$$y = -x + 4.$$

❑

Choosing Which Rules to Use

The choice of which rules to use in solving a differentiation problem can make a difference in how much work you have to do. Here is an example.

EXAMPLE 12 Rather than using the Quotient Rule to find the derivative of

$$y = \frac{(x-1)(x^2-2x)}{x^4},$$

expand the numerator and divide by x^4:

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\frac{dy}{dx} = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4}$$
$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$

❑

Second and Higher Order Derivatives

The derivative $y' = dy/dx$ is the **first (first order) derivative** of y with respect to x. This derivative may itself be a differentiable function of x; if so, its derivative

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

is called the **second (second order) derivative** of y with respect to x.

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$ is the **third (third order) derivative** of y with respect to x. The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)}$$

denoting the ***n*th (*n*th order) derivative** of y with respect to x, for any positive integer n.

Notice that

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

does not mean multiplication. It means "the derivative of the derivative."

How to read the symbols for derivatives

y'	"y prime"
y''	"y double prime"
$\dfrac{d^2y}{dx^2}$	"d squared y dx squared"
y'''	"y triple prime"
$y^{(n)}$	"y super n"
$\dfrac{d^ny}{dx^n}$	"d to the n of y by dx to the n"

EXAMPLE 13 The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

The function has derivatives of all orders, the fifth and later derivatives all being zero. ❑

Exercises 2.2

Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$

2. $y = x^2 + x + 8$

3. $s = 5t^3 - 3t^5$

4. $w = 3z^7 - 7z^3 + 21z^2$

5. $y = \dfrac{4x^3}{3} - x$

6. $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} + \dfrac{x}{4}$

7. $w = 3z^{-2} - \dfrac{1}{z}$

8. $s = -2t^{-1} + \dfrac{4}{t^2}$

9. $y = 6x^2 - 10x - 5x^{-2}$

10. $y = 4 - 2x - x^{-3}$

11. $r = \dfrac{1}{3s^2} - \dfrac{5}{2s}$

12. $r = \dfrac{12}{\theta} - \dfrac{4}{\theta^3} + \dfrac{1}{\theta^4}$

In Exercises 13–16, find y' (a) by applying the Product Rule and (b) by multiplying the factors to produce a sum of simpler terms to differentiate.

13. $y = (3 - x^2)(x^3 - x + 1)$

14. $y = (x - 1)(x^2 + x + 1)$

15. $y = (x^2 + 1)\left(x + 5 + \dfrac{1}{x}\right)$

16. $y = \left(x + \dfrac{1}{x}\right)\left(x - \dfrac{1}{x} + 1\right)$

Find the derivatives of the functions in Exercises 17–28.

17. $y = \dfrac{2x + 5}{3x - 2}$

18. $z = \dfrac{2x + 1}{x^2 - 1}$

19. $g(x) = \dfrac{x^2 - 4}{x + 0.5}$

20. $f(t) = \dfrac{t^2 - 1}{t^2 + t - 2}$

21. $v = (1 - t)(1 + t^2)^{-1}$

22. $w = (2x - 7)^{-1}(x + 5)$

23. $f(s) = \dfrac{\sqrt{s} - 1}{\sqrt{s} + 1}$

24. $u = \dfrac{5x + 1}{2\sqrt{x}}$

25. $v = \dfrac{1 + x - 4\sqrt{x}}{x}$

26. $r = 2\left(\dfrac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$

27. $y = \dfrac{1}{(x^2 - 1)(x^2 + x + 1)}$

28. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

Find the derivatives of all orders of the functions in Exercises 29 and 30.

29. $y = \dfrac{x^4}{2} - \dfrac{3}{2}x^2 - x$

30. $y = \dfrac{x^5}{120}$

Find the first and second derivatives of the functions in Exercises 31–38.

31. $y = \dfrac{x^3 + 7}{x}$

32. $s = \dfrac{t^2 + 5t - 1}{t^2}$

33. $r = \dfrac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$

34. $u = \dfrac{(x^2 + x)(x^2 - x + 1)}{x^4}$

35. $w = \left(\dfrac{1 + 3z}{3z}\right)(3 - z)$

36. $w = (z + 1)(z - 1)(z^2 + 1)$

37. $p = \left(\dfrac{q^2 + 3}{12q}\right)\left(\dfrac{q^4 - 1}{q^3}\right)$

38. $p = \dfrac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$

Using Numerical Values

39. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

a) $\dfrac{d}{dx}(uv)$ **b)** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ **c)** $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d)** $\dfrac{d}{dx}(7v - 2u)$

40. Suppose u and v are differentiable functions of x and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

a) $\dfrac{d}{dx}(uv)$ **b)** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ **c)** $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ **d)** $\dfrac{d}{dx}(7v - 2u)$

Slopes and Tangents

41. a) Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point $(2, 1)$.

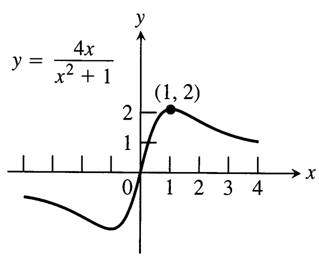
b) What is the smallest slope on the curve? At what point on the curve does the curve have this slope?

c) Find equations for the tangents to the curve at the points where the slope of the curve is 8.

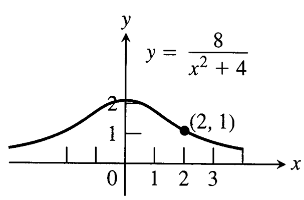
42. a) Find equations for the horizontal tangents to the curve $y = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangents at the points of tangency.

b) What is the smallest slope on the curve? At what point on the curve does the curve have this slope? Find an equation for the line that is perpendicular to the curve's tangent at this point.

43. Find the tangents to *Newton's Serpentine* (graphed here) at the origin and the point $(1, 2)$.



44. Find the tangent to the *Witch of Agnesi* (graphed here) at the point $(2, 1)$. There is a nice story about the name of this curve in the marginal note on Agnesi in Section 9.4.



45. The curve $y = ax^2 + bx + c$ passes through the point $(1, 2)$ and is tangent to the line $y = x$ at the origin. Find a, b, and c.

46. The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent line at the point $(1, 0)$. Find a, b, and c.

47. a) Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1, 0)$.

b) GRAPHER Graph the curve and tangent line together. The tangent intersects the curve at another point. Use ZOOM and TRACE to estimate the point's coordinates.

c) GRAPHER Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent simultaneously (SOLVER key).

48. a) Find an equation for the line that is tangent to the curve $y = x^3 - 6x^2 + 5x$ at the origin.

b) GRAPHER Graph the curve and tangent together. The tangent intersects the curve at another point. Use ZOOM and TRACE to estimate the point's coordinates.

c) GRAPHER Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent simultaneously (SOLVER key).

Physical Applications

49. *Pressure and volume.* If the gas in a closed container is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

in which a, b, n, and R are constants. Find dP/dV.

50. *The body's reaction to medicine.* The reaction of the body to a dose of medicine can sometimes be represented by an equation of the form

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right),$$

where C is a positive constant and M is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

Find dR/dM. This derivative, as a function of M, is called the sensitivity of the body to the medicine. In Section 3.6, we will see how to find the amount of medicine to which the body is most sensitive. (Source: *Some Mathematical Models in Biology*, Revised Edition, R. M Thrall, J. A. Mortimer, K. R. Rebman, R. F. Baum, eds., December 1967, PB-202 364, p. 221; distributed by NTIS, U.S. Department of Commerce.)

Theory and Examples

51. Suppose that the function v in the Product Rule has a constant value c. What does the Product Rule then say? What does this say about the Constant Multiple Rule?

52. *The Reciprocal Rule*

a) The **Reciprocal Rule** says that at any point where the function $v(x)$ is differentiable and different from zero,

$$\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{dv}{dx}.$$

Show that the Reciprocal Rule is a special case of the Quotient Rule.

b) Show that the Reciprocal Rule and the Product Rule together imply the Quotient Rule.

53. *Another proof of the Power Rule for positive integers.* Use the algebra formula

$$x^n - c^n = (x - c)(x^{n-1} + x^{n-2}c + \cdots + xc^{n-2} + c^{n-1})$$

together with the derivative formula

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

from Exercises 2.1 to show that $(d/dx)(x^n) = nx^{n-1}$.

54. *Generalizing the Product Rule.* The Product Rule gives the formula

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

for the derivative of the product uv of two differentiable functions of x.

a) What is the analogous formula for the derivative of the product uvw of *three* differentiable functions of x?

b) What is the formula for the derivative of the product $u_1u_2u_3u_4$ of *four* differentiable functions of x?

c) What is the formula for the derivative of a product $u_1u_2u_3 \cdots u_n$ of a finite number n of differentiable functions of x?

55. *Rational Powers*

a) Find $\frac{d}{dx}(x^{3/2})$ by writing $x^{3/2}$ as $x \cdot x^{1/2}$ and using the Product Rule. Express your answer as a rational number times a rational power of x. Work parts (b) and (c) by a similar method.

b) Find $\frac{d}{dx}(x^{5/2})$.

c) Find $\frac{d}{dx}(x^{7/2})$.

d) What patterns do you see in your answers to (a), (b), and (c)? Rational powers are one of the topics in Section 2.6.

2.3 Rates of Change

In this section we examine some applications in which derivatives are used to represent and interpret the rates at which things change in the world around us. It is natural to think of change in terms of dependence on time, such as the position, velocity, and acceleration of a moving object, but there is no need to be so restrictive. Change with respect to variables other than time can be treated in the same way. For example, a physician may want to know how small changes in dosage can affect the body's response to a drug. An economist may want to study how investment changes with respect to variations in interest rates. These questions can all be expressed in terms of the rate of change of a function with respect to a variable.

Average and Instantaneous Rates of Change

We start by recalling the concept of average rate of change of a function over an interval, introduced in Section 1.1. The derivative of the function is the limit of this average rate as the length of the interval goes to zero.

Definitions

The **average rate of change** of a function $f(x)$ with respect to x over the interval from x_0 to $x_0 + h$ is

$$\text{Average rate of change} = \frac{f(x_0 + h) - f(x_0)}{h}.$$

The **(instantaneous) rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

It is conventional to use the word *instantaneous* even when x does not represent time. The word is, however, frequently omitted. When we say *rate of change,* we mean *instantaneous rate of change.*

EXAMPLE 1 The area A of a circle is related to its diameter by the equation

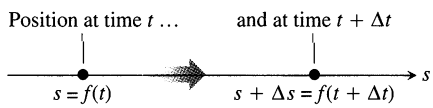
$$A = \frac{\pi}{4} D^2.$$

How fast is the area changing with respect to the diameter when the diameter is 10 m?

Solution The (instantaneous) rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4} 2D = \frac{\pi D}{2}.$$

When $D = 10$ m, the area is changing at rate $(\pi/2)10 = 5\pi$ m^2/m. This means that a small change ΔD m in the diameter would result in a change of about $5\pi \Delta D$ m^2 in the area of the circle. ❑



2.20 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$.

Motion Along a Line—Displacement, Velocity, Speed, and Acceleration

Suppose that an object is moving along a coordinate line (say an s-axis) so that we know its position s on that line as a function of time t:

$$s = f(t).$$

The **displacement** of the object over the time interval from t to $t + \Delta t$ (Fig. 2.20) is

$$\Delta s = f(t + \Delta t) - f(t),$$

and the **average velocity** of the object over that time interval is

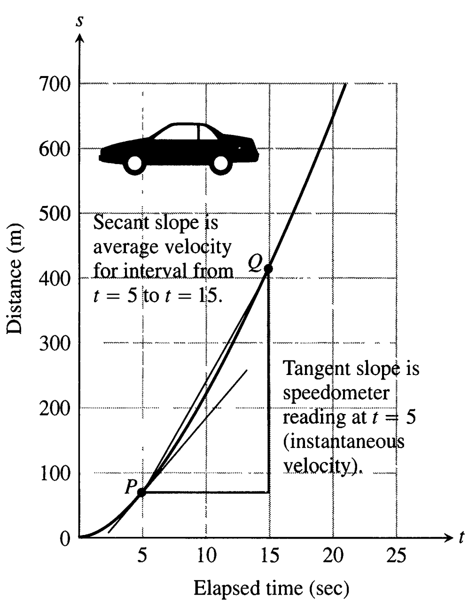
$$v_{\mathrm{av}} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

To find the body's velocity at the exact instant t, we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt shrinks to zero. This limit is the derivative of f with respect to t.

Definition

The **(instantaneous) velocity** is the derivative of the position function $s = f(t)$ with respect to time. At time t the velocity is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

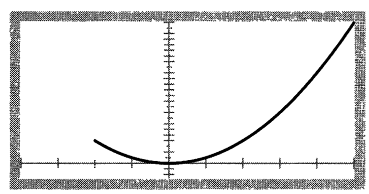


2.21 The time-to-distance data for Example 2.

EXAMPLE 2 Figure 2.21 shows a distance–time graph of a 1994 Ford Mustang Cobra. The slope of the secant PQ is the average velocity for the 10-sec interval from $t = 5$ to $t = 15$ sec, in this case 35.5 m/sec or 128 km/h. The slope of the tangent at P is the speedometer reading at $t = 5$ sec, about 20 m/sec or 72 km/h. The car's top speed is 220 km/h (about 137 mph). (Source: *Car & Driver,* April 1994.) ❑

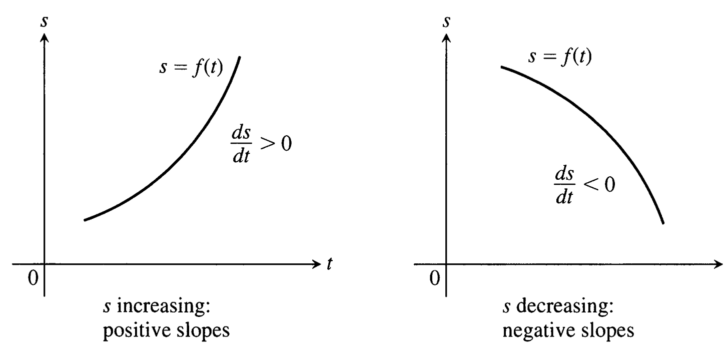
Technology *Parametric Functions* To graph curves $y = f(x)$, where y is a function of x, your graphing utility should be set in *function mode*. Not all curves can be represented in that mode, so most graphing utilities have a *parametric mode* as well. In this mode you plot the points $(x(t), y(t))$ whose coordinates are functions of the varying "time" parameter t. Thus you can think of the curve as the path of a moving particle as it changes its (x, y) position over time (see Section 9.4). A curve $y = f(x)$ can be graphed in parametric mode using the equations $x = t$, $y = f(t)$. Set your graphing utility to parametric mode and try the following equations.

Relation	**Parametrization**
$y = x^2$ (y a function of x)	$x(t) = t, \quad y(t) = t^2, \quad -\infty < t < \infty$
$x^2 + y^2 = 4$ (y not a function of x)	$x(t) = 2\cos t, \quad y(t) = 2\sin t,$ $0 \le t \le 2\pi$



The parabola $x(t) = t$, $y(t) = t^2$, for $t \geq -2$

Besides telling us how fast the object is moving, the velocity also tells us in what direction it is moving. When the object is moving forward (s increasing) the velocity is positive; when the body is moving backward (s decreasing) the velocity is negative (Fig. 2.22).



2.22 $v = ds/dt$ is positive when s increases and negative when s decreases.

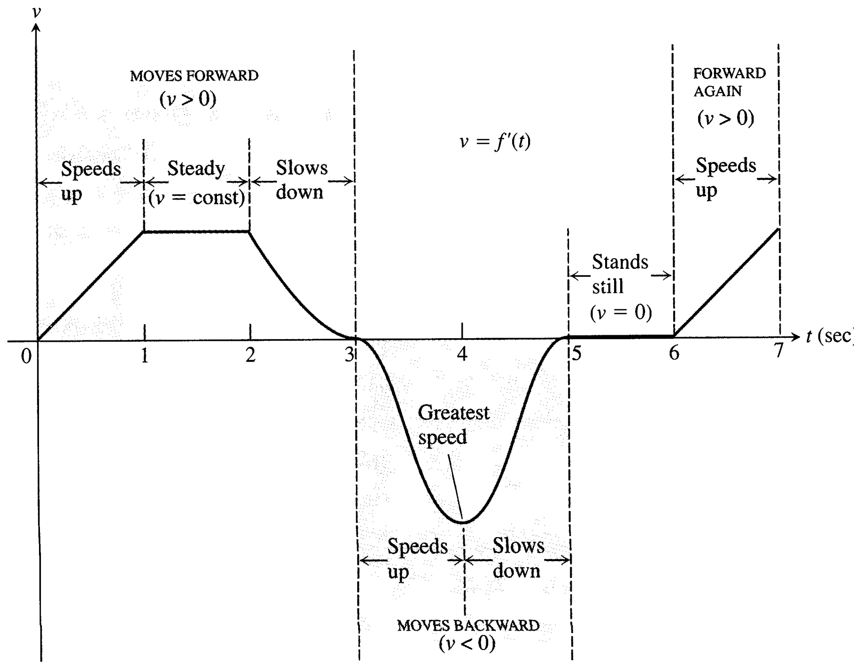
If we drive to a friend's house and back at 30 mph, say, the speedometer will show 30 on the way over but it will not show -30 on the way back, even though our distance from home is decreasing. The speedometer always shows speed, which is the absolute value of velocity. Speed measures the rate of forward progress regardless of direction.

Definition

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

EXAMPLE 3 Figure 2.23 shows the velocity $v = f'(t)$ of a particle moving on a coordinate line. The particle moves forward for the first 3 seconds, moves backward for the next 2 seconds, stands still for a second, and moves forward again. Notice that the particle achieves its greatest speed at time $t = 4$, while moving backward.



2.23 The velocity graph for Example 3.

❑

The rate at which a body's velocity changes is called the body's acceleration. The acceleration measures how quickly the body picks up or loses speed.

Definition

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

We can illustrate all this with free fall. As we mentioned at the beginning of Chapter 1, near the surface of the earth all bodies fall with the same constant acceleration. When air resistance is absent or insignificant and the only force acting on a falling body is the force of gravity, we call the way the body falls **free fall.** The mathematical description of this type of motion captured the imagination of many great scientists, including Aristotle, Galileo, and Newton. Experimental and theoretical investigations revealed that the distance a body released from rest falls in time t is proportional to the square of the amount of time it has fallen. We express this by saying that

$$s = \frac{1}{2}gt^2,$$

where s is distance and g is the acceleration due to Earth's gravity. This equation holds in a vacuum, where there is no air resistance, but it closely models the fall of dense, heavy objects, such as rocks or steel tools, for the first few seconds of their fall, before air resistance starts to slow them down.

The value of g in the equation $s = (1/2)gt^2$ depends on the units used to measure t and s. With t in seconds (the usual unit), we have the following values:

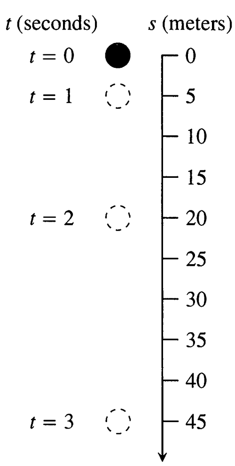
Free-Fall Equations (Earth)

English units: $g = 32\dfrac{\text{ft}}{\text{sec}^2}$, $s = \dfrac{1}{2}(32)t^2 = 16t^2$ (s in feet)

Metric units: $g = 9.8\dfrac{\text{m}}{\text{sec}^2}$, $s = \dfrac{1}{2}(9.8)t^2 = 4.9t^2$ (s in meters)

The abbreviation ft/sec^2 is read "feet per second squared" or "feet per second per second," and m/sec^2 is read "meters per second squared."

This description allows us to answer many questions concerning the position and velocity of a falling object.



2.24 A ball bearing falling from rest (Example 4).

EXAMPLE 4 Figure 2.24 shows the free fall of a heavy ball bearing released from rest at time $t = 0$ sec.

a) How many meters does the ball fall in the first 2 sec?
b) What is its velocity, speed, and acceleration then?

Solution

a) The metric free-fall equation is $s = 4.9t^2$. During the first 2 sec, the ball falls

$$s(2) = 4.9(2)^2 = 19.6 \text{ m}.$$

b) At any time t, *velocity* is the derivative of displacement:

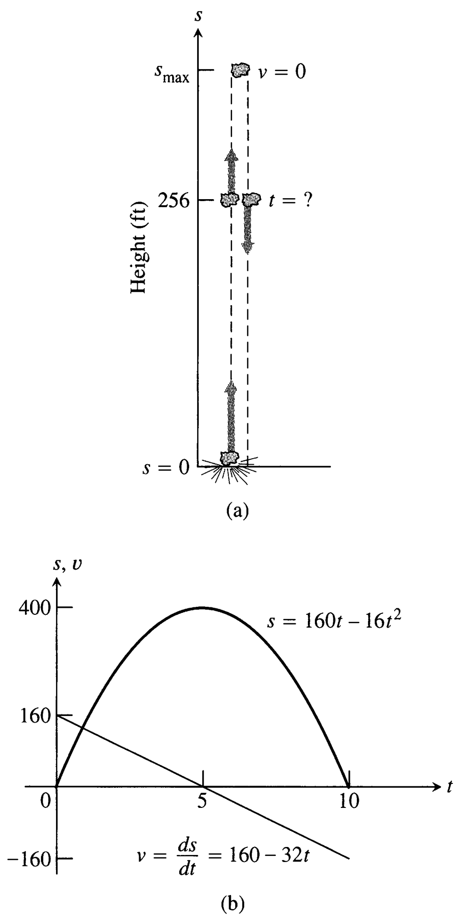
$$v(t) = s'(t) = \frac{d}{dt}(4.9t^2) = 9.8t.$$

At $t = 2$, the velocity is

$$v(2) = 19.6 \text{ m/sec}$$

in the downward (increasing s) direction. The *speed* at $t = 2$ is

$$\text{speed} = |v(2)| = 19.6 \text{ m/sec}.$$



2.25 (a) The rock in Example 5. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: it is a plot of height vs. time. The slope is the rock's velocity.

The *acceleration* at any time t is

$$a(t) = v'(t) = s''(t) = 9.8 \text{ m/sec}^2.$$

At $t = 2$, the acceleration is 9.8 m/sec^2. ❑

EXAMPLE 5 A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) (Fig. 2.25a). It reaches a height of $s = 160t - 16t^2$ ft after t sec.

a) How high does the rock go?
b) What is the velocity and speed of the rock when it is 256 ft above the ground on the way up? on the way down?
c) What is the acceleration of the rock at any time t during its flight (after the blast)?
d) When does the rock hit the ground again?

Solution

a) In the coordinate system we have chosen, s measures height from the ground up, so the velocity is positive on the way up and negative on the way down. The instant the rock is at its highest point is the one instant during the flight when the velocity is 0. Therefore, to find the maximum height, all we need to do is to find when $v = 0$ and evaluate s at this time.

At any time t, the velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \text{ ft/sec}.$$

The velocity is zero when

$$160 - 32t = 0, \qquad \text{or} \qquad t = 5 \text{ sec}.$$

The rock's height at $t = 5$ sec is

$$s_{max} = s(5) = 160(5) - 16(5)^2 = 800 - 400 = 400 \text{ ft}.$$

See Fig. 2.25(b).

b) To find the rock's velocity at 256 ft on the way up and again on the way down, we find the two values of t for which

$$s(t) = 160t - 16t^2 = 256.$$

To solve this equation we write

$$16t^2 - 160t + 256 = 0$$
$$16(t^2 - 10t + 16) = 0$$
$$(t - 2)(t - 8) = 0$$
$$t = 2 \text{ sec}, \quad t = 8 \text{ sec}.$$

The rock is 256 ft above the ground 2 sec after the explosion and again 8 sec after the explosion. The rock's velocities at these times are

$$v(2) = 160 - 32(2) = 160 - 64 = 96 \text{ ft/sec},$$
$$v(8) = 160 - 32(8) = 160 - 256 = -96 \text{ ft/sec}.$$

At both instants, the rock's speed is 96 ft/sec.

c) At any time during its flight following the explosion, the rock's acceleration is

$$a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \text{ ft/sec}^2.$$

The acceleration is always downward. When the rock is rising, it is slowing down; when it is falling, it is speeding up.

d) The rock hits the ground at the positive time t for which $s = 0$. The equation $160t - 16t^2 = 0$ factors to give $16t(10 - t) = 0$, so it has solutions $t = 0$ and $t = 10$. At $t = 0$ the blast occurred and the rock was thrown upward. It returned to the ground 10 seconds later. ❑

Technology *Simulation of Motion on a Vertical Line* The parametric equations

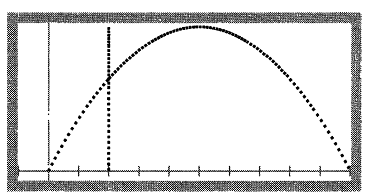
$$x(t) = c, \qquad y(t) = f(t)$$

will illuminate pixels along the vertical line $x = c$. If $f(t)$ denotes the height of a moving body at time t, graphing $(x(t), y(t)) = (c, f(t))$ will simulate the actual motion. Try it for the rock in Example 5 with $x(t) = 2$, say, and $y(t) = 160t - 16t^2$, in dot mode with tStep $= 0.1$. Why does the spacing of the dots vary? Why does the grapher seem to stop after it reaches the top? (Try the plots for $0 \le t \le 5$ and $5 \le t \le 10$ separately.)

For a second experiment, plot the parametric equations

$$x(t) = t, \qquad y(t) = 160t - 16t^2$$

together with the vertical line simulation of the motion, again in dot mode. Use what you know about the behavior of the rock from the calculations of Example 5 to select a window size that will display all the interesting behavior.



$\begin{cases} x(t) = 2 \\ y(t) = 160t - 16t^2 \end{cases}$ and $\begin{cases} x(t) = t \\ y(t) = 160t - 16t^2 \end{cases}$ in dot mode

Sensitivity to Change

When a small change in x produces a large change in the value of a function $f(x)$, we say that the function is relatively **sensitive** to changes in x. The derivative $f'(x)$ is a measure of the sensitivity to change at x.

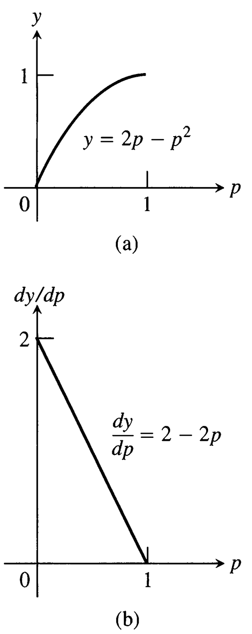
EXAMPLE 6 *Sensitivity to change*

The Austrian monk Gregor Johann Mendel (1822–1884), working with garden peas and other plants, provided the first scientific explanation of hybridization. His careful records showed that if p (a number between 0 and 1) is the frequency of the gene for smooth skin in peas (dominant) and $(1 - p)$ is the frequency of the gene for wrinkled skin in peas, then the proportion of smooth-skinned peas in the population at large is

$$y = 2p(1 - p) + p^2 = 2p - p^2.$$

Why peas wrinkle

British geneticists have recently discovered that the wrinkling trait comes from an extra piece of DNA that prevents the gene that directs starch synthesis from functioning properly. With the plant's starch conversion impaired, sucrose and water build up in the young seeds. As the seeds mature, they lose much of this water, and the shrinkage leaves them wrinkled.



2.26 (a) The graph of $y = 2p - p^2$, describing the proportion of smooth-skinned peas. (b) The graph of dy/dp.

The graph of y versus p in Fig. 2.26(a) suggests that the value of y is more sensitive to a change in p when p is small than when p is large. Indeed, this is borne out by the derivative graph in Fig. 2.26(b), which shows that dy/dp is close to 2 when p is near 0 and close to 0 when p is near 1.

We will say more about sensitivity in Section 3.7. ❑

Derivatives in Economics

Engineers use the terms *velocity* and *acceleration* to refer to the derivatives of functions describing motion. Economists, too, have a specialized vocabulary for rates of change and derivatives. They call them *marginals.*

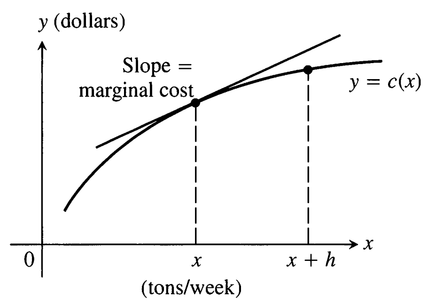
In a manufacturing operation, the *cost of production* $c(x)$ is a function of x, the number of units produced. The *marginal cost of production* is the rate of change of cost (c) with respect to level of production (x), so it is dc/dx.

For example, let $c(x)$ represent the dollars needed to produce x tons of steel in one week. It costs more to produce $x + h$ units, and the cost difference, divided by h, is the average increase in cost per ton per week:

$$\frac{c(x+h)-c(x)}{h} = \begin{array}{l}\text{average increase in cost/ton/wk}\\ \text{to produce the next } h \text{ tons of steel}\end{array}$$

The limit of this ratio as $h \to 0$ is the *marginal cost* of producing more steel when the current production level is x tons (Fig. 2.27):

$$\frac{dc}{dx} = \lim_{h\to 0} \frac{c(x+h)-c(x)}{h} = \text{marginal cost of production.}$$

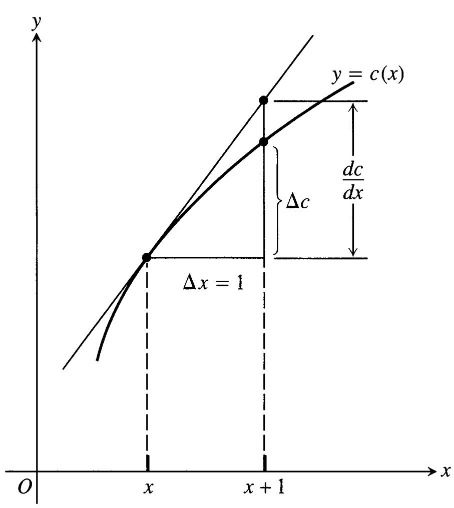


2.27 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons is $c(x + h) - c(x)$.

Sometimes the marginal cost of production is loosely defined to be the extra cost of producing one unit:

$$\frac{\Delta c}{\Delta x} = \frac{c(x+1)-c(x)}{1},$$

which is approximately the value of dc/dx at x. To see why this is an acceptable approximation, observe that if the slope of c does not change quickly near x, then the difference quotient will be close to its limit, the derivative dc/dx, even if $\Delta x = 1$ (Fig. 2.28). In practice, the approximation works best for large values of x.



2.28 The marginal cost dc/dx is approximately the extra cost Δc of producing $\Delta x = 1$ more unit.

EXAMPLE 7 *Marginal cost*

Suppose it costs

$$c(x) = x^3 - 6x^2 + 15x$$

Choosing functions to illustrate economics

In case you are wondering why economists use polynomials of low degree to illustrate complicated phenomena like cost and revenue, here is the rationale: While formulas for real phenomena are rarely available in any given instance, the theory of economics can still provide valuable guidance. The functions about which theory speaks can often be illustrated with low degree polynomials on relevant intervals. Cubic polynomials provide a good balance between being easy to work with and being complicated enough to illustrate important points.

dollars to produce x radiators when 8 to 30 radiators are produced. Your shop currently produces 10 radiators a day. About how much extra will it cost to produce one more radiator a day?

Solution The cost of producing one more radiator a day when 10 are produced is about $c'(10)$:

$$c'(x) = \frac{d}{dx}(x^3 - 6x^2 + 15x) = 3x^2 - 12x + 15$$

$$c'(10) = 3(100) - 12(10) + 15 = 195.$$

The additional cost will be about \$195. ❑

EXAMPLE 8 *Marginal tax rate*

To get some feel for the language of marginal rates, consider marginal tax rates. If your marginal income tax rate is 28% and your income increases by \$1,000, you can expect to have to pay an extra \$280 in income taxes. This does not mean that you pay 28% of your entire income in taxes. It just means that at your current income level I, the rate of increase of taxes T with respect to income is $dT/dI = 0.28$. You will pay \$0.28 out of every extra dollar you earn in taxes. Of course, if you earn a lot more, you may land in a higher tax bracket and your marginal rate will increase. ❑

EXAMPLE 9 *Marginal revenue*

If

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x thousand candy bars, $5 \le x \le 20$, the marginal revenue when x thousand are sold is

$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x) = 3x^2 - 6x + 12.$$

As with marginal cost, the marginal revenue function estimates the increase in revenue that will result from selling one additional unit. If you currently sell 10 thousand candy bars a week, you can expect your revenue to increase by about

$$r'(10) = 3(100) - 6(10) + 12 = \$252$$

if you increase sales to 11 thousand bars a week. ❑

Exercises 2.3

Motion Along a Coordinate Line

Exercises 1–6 give the position $s = f(t)$ of a body moving on a coordinate line for $a \le t \le b$, with s in meters and t in seconds.

a) Find the body's displacement and average velocity for the given time interval.

b) Find the body's speed and acceleration at the endpoints of the interval.

c) When during the interval does the body change direction (if ever)?

1. $s = 0.8t^2, \quad 0 \le t \le 10$ (free fall on the moon)

2. $s = 1.86t^2, \quad 0 \le t \le 0.5$ (free fall on Mars)

3. $s = -t^3 + 3t^2 - 3t, \quad 0 \le t \le 3$

4. $s = (t^4/4) - t^3 + t^2, \quad 0 \le t \le 2$

5. $s = \frac{25}{t^2} - \frac{5}{t}, \quad 1 \le t \le 5$

6. $s = \frac{25}{t+5}, \quad -4 \le t \le 0$

7. At time t, the position of a body moving along the s-axis is $s = t^3 - 6t^2 + 9t$ m. (a) Find the body's acceleration each time the velocity is zero. (b) Find the body's speed each time the acceleration is zero. (c) Find the total distance traveled by the body from $t = 0$ to $t = 2$.

8. At time $t \ge 0$, the velocity of a body moving along the s-axis is $v = t^2 - 4t + 3$. (a) Find the body's acceleration each time the velocity is zero. (b) When is the body moving forward? moving backward? (c) When is the body's velocity increasing? decreasing?

Free-Fall Applications

9. The equations for free fall at the surfaces of Mars and Jupiter (s in meters, t in seconds) are $s = 1.86t^2$ on Mars, $s = 11.44t^2$ on Jupiter. How long would it take a rock falling from rest to reach a velocity of 27.8 m/sec (about 100 km/h) on each planet?

10. A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ meters in t seconds.
   a) Find the rock's velocity and acceleration at time t. (The acceleration in this case is the acceleration of gravity on the moon.)
   b) How long does it take the rock to reach its highest point?
   c) How high does the rock go?
   d) How long does it take the rock to reach half its maximum height?
   e) How long is the rock aloft?

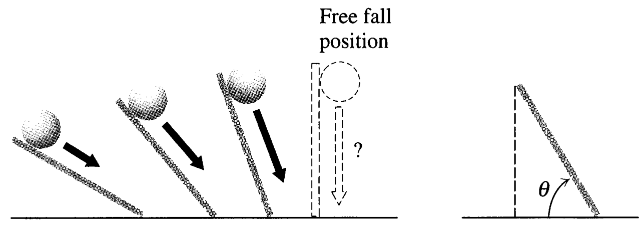
11. On Earth, in the absence of air, the rock in Exercise 10 would reach a height of $s = 24t - 4.9t^2$ meters in t seconds.
   a) Find the rock's velocity and acceleration at time t.
   b) How long would it take the rock to reach its highest point?
   c) How high would the rock go?
   d) How long would it take the rock to reach half its maximum height?
   e) How long would the rock be aloft?

12. Explorers on a small airless planet used a spring gun to launch a ball bearing vertically upward from the surface at a launch velocity of 15 m/sec. Because the acceleration of gravity at the planet's surface was g_s m/sec^2, the explorers expected the ball bearing to reach a height of $s = 15t - (1/2)g_s t^2$ meters t seconds later. The ball bearing reached its maximum height 20 sec after being launched. What was the value of g_s?

13. A 45-caliber bullet fired straight up from the surface of the moon would reach a height of $s = 832t - 2.6t^2$ feet after t seconds. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ feet after t seconds. How long will the bullet be aloft in each case? How high would the bullet go?

14. *(Continuation of Exercise 13.)* On Jupiter, in the absence of air, the bullet's height would be $s = 832t - 37.53t^2$ feet after t seconds. On Mars it would be $s = 832t - 6.1t^2$ feet after t seconds. How high would the bullet go in each case?

15. *Galileo's free-fall formula.* Galileo developed a formula for a body's velocity during free fall by rolling balls from rest down increasingly steep inclined planks and looking for a limiting formula that would predict a ball's behavior when the plank was vertical and the ball fell freely (part a of the accompanying figure). He found that, for any given angle of the plank, the ball's velocity t seconds into the motion was a constant multiple of t. That is, the velocity was given by a formula of the form $v = kt$. The value of the constant k depended on the inclination of the plank.



In modern notation (part b of the figure), with distance in meters and time in seconds, what Galileo determined by experiment was that, for any given angle θ, the ball's velocity t seconds into the roll was

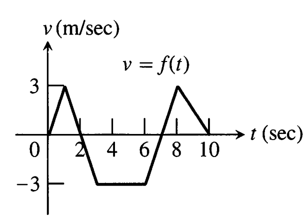
$$v = 9.8(\sin\theta)t \text{ m/sec.}$$

   a) What is the equation for the ball's velocity during free fall?
   b) Building on your work in (a), what constant acceleration does a freely falling body experience near the surface of the earth?

16. *Free fall from the tower of Pisa.* Had Galileo dropped a cannonball from the tower of Pisa, 179 ft above the ground, the ball's height aboveground t seconds into the fall would have been $s = 179 - 16t^2$.
   a) What would have been the ball's velocity, speed, and acceleration at time t?
   b) About how long would it have taken the ball to hit the ground?
   c) What would have been the ball's velocity at the moment of impact?

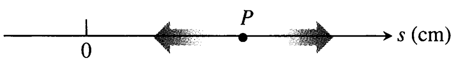
Conclusions About Motion from Graphs

17. The accompanying figure shows the velocity $v = ds/dt = f(t)$ (m/sec) of a body moving along a coordinate line.

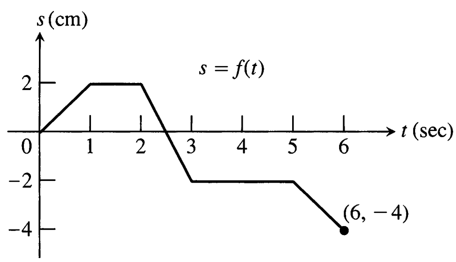


a) When does the body reverse direction?
b) When (approximately) is the body moving at a constant speed?
c) Graph the body's speed for $0 \le t \le 10$.
d) Graph the acceleration, where defined.

18. A particle P moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of P as a function of time t.



(a)

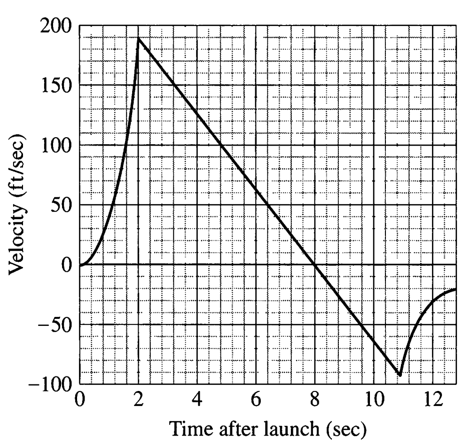


(b)

a) When is P moving to the left? moving to the right? standing still?
b) Graph the particle's velocity and speed (where defined).

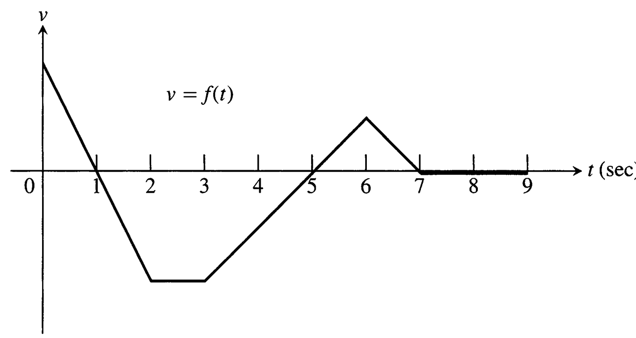
19. When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts down. The parachute slows the rocket to keep it from breaking when it lands.

The figure here shows velocity data from the flight of the model rocket. Use the data to answer the following.



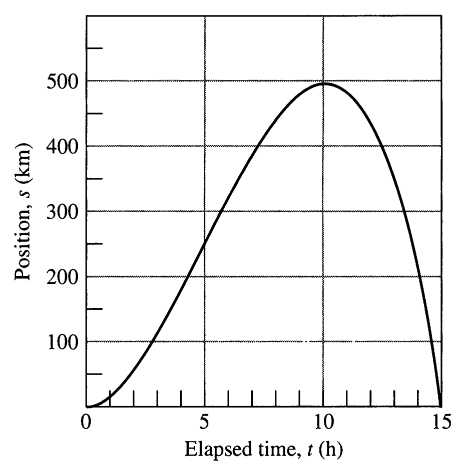
a) How fast was the rocket climbing when the engine stopped?
b) For how many seconds did the engine burn?
c) When did the rocket reach its highest point? What was its velocity then?
d) When did the parachute pop out? How fast was the rocket falling then?
e) How long did the rocket fall before the parachute opened?
f) When was the rocket's acceleration greatest?
g) When was the acceleration constant? What was its value then (to the nearest integer)?

20. The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a coordinate line.



a) When does the particle move forward? move backward? speed up? slow down?
b) When is the particle's acceleration positive? negative? zero?
c) When does the particle move at its greatest speed?
d) When does the particle stand still for more than an instant?

21. The graph here shows the position s of a truck traveling on a highway. The truck starts at $t = 0$ and returns 15 hours later at $t = 15$.



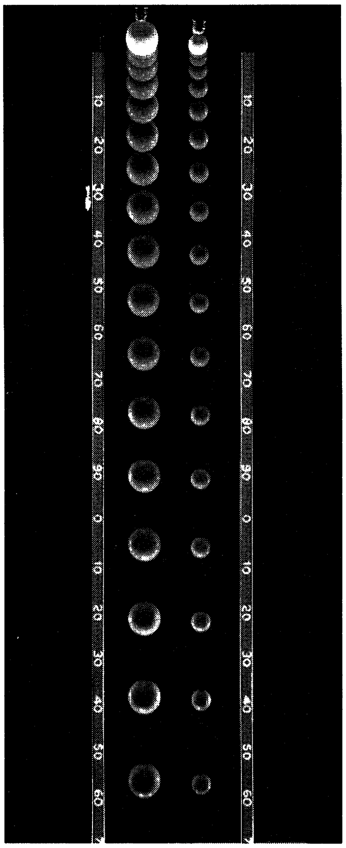
a) Use the technique described in Section 2.1, Example 4, to graph the truck's velocity $v = ds/dt$ for $0 \le t \le 15$. Then repeat the process, with the velocity curve, to graph the truck's acceleration dv/dt.
b) Suppose $s = 15t^2 - t^3$. Graph ds/dt and d^2s/dt^2 and compare your graphs with those in (a).

22. The multiflash photograph in Fig. 2.29 on the following page shows two balls falling from rest. The vertical rulers are marked in centimeters. Use the equation $s = 490t^2$ (the free-fall equation for s in centimeters and t in seconds) to answer the following questions.

a) How long did it take the balls to fall the first 160 cm? What was their average velocity for the period?

b) How fast were the balls falling when they reached the 160-cm mark? What was their acceleration then?

c) About how fast was the light flashing (flashes per second)?



2.29 Two balls falling from rest (Exercise 22).

23. The graphs in Fig. 2.30 show the position s, velocity $v = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time t. Which graph is which? Give reasons for your answers.

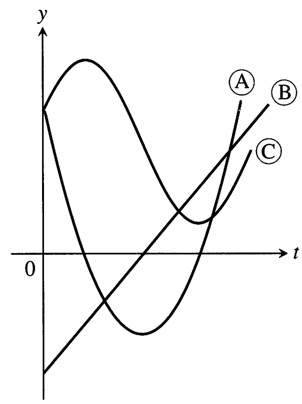
24. The graphs in Fig. 2.31 show the position s, the velocity $v = ds/dt$, and the acceleration $a = d^2s/dt^2$ of a body moving along the coordinate line as functions of time t. Which graph is which? Give reasons for your answers.

Economics

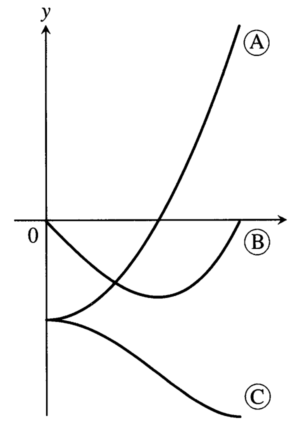
25. *Marginal cost.* Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

a) Find the average cost per machine of producing the first 100 washing machines.

b) Find the marginal cost when 100 washing machines are produced.



2.30 The graphs for Exercise 23.



2.31 The graphs for Exercise 24.

c) Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

26. *Marginal revenue.* Suppose the revenue from selling x custom-made office desks is

$$r(x) = 2000\left(1 - \frac{1}{x+1}\right)$$

dollars.

a) Find the marginal revenue when x desks are produced.

b) Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing production from 5 desks a week to 6 desks a week.

c) Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?

Additional Applications

27. When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while, but then stopped growing and began to decline. The size of the population at time t (hours) was $b = 10^6 + 10^4t - 10^3t^2$. Find the growth rates at (a) $t = 0$; (b) $t = 5$; and (c) $t = 10$ hours.

28. The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?

29. It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth y of fluid in the tank t hours after the valve is opened is given by the formula

$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ m.}$$

a) Find the rate dy/dt (m/h) at which the tank is draining at time t.
b) When is the fluid level in the tank falling fastest? slowest? What are the values of dy/dt at these times?
c) GRAPHER Graph y and dy/dt together and discuss the behavior of y in relation to the signs and values of dy/dt.

30. The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius.
a) At what rate does the volume change with respect to the radius when $r = 2$ ft?
b) By approximately how much does the volume increase when the radius changes from 2 to 2.2 ft?

31. Suppose that the distance an aircraft travels along a runway before takeoff is given by $D = (10/9)t^2$, where D is measured in meters from the starting point and t is measured in seconds from the time the brakes are released. If the aircraft will become airborne when its speed reaches 200 km/hr, how long will it take to become airborne, and what distance will it travel in that time?

32. *Volcanic lava fountains.* Although the November 1959 Kilauea Iki eruption on the island of Hawaii began with a line of fountains along the wall of the crater, activity was later confined to a single vent in the crater's floor, which at one point shot lava 1900 ft straight into the air (a world record). What was the lava's exit velocity in feet per second? in miles per hour? (*Hint:* If v_0 is the exit velocity of a particle of lava, its height t seconds later will be $s = v_0 t - 16t^2$ feet. Begin by finding the time at which $ds/dt = 0$. Neglect air resistance.)

Grapher Explorations

Exercises 33–36 give the position function $s = f(t)$ of a body moving along the s-axis as a function of time t. Graph f together with the velocity function $v(t) = ds/dt = f'(t)$ and the acceleration function $a(t) = d^2s/dt^2 = f''(t)$. Comment on the body's behavior in relation to the signs and values v and a. Include in your commentary such topics as the following.

a) When is the body momentarily at rest?
b) When does it move to the left (down) or to the right (up)?
c) When does it change direction?
d) When does it speed up and slow down?
e) When is it moving fastest (highest speed)? slowest?
f) When is it farthest from the axis origin?

33. $s = 200t - 16t^2,\quad 0 \le t \le 12.5$ (A heavy object fired straight up from the earth's surface at 200 ft/sec)

34. $s = t^2 - 3t + 2,\quad 0 \le t \le 5$

35. $s = t^3 - 6t^2 + 7t,\quad 0 \le t \le 4$

36. $s = 4 - 7t + 6t^2 - t^3,\quad 0 \le t \le 4$

2.4 Derivatives of Trigonometric Functions

Trigonometric functions are important because so many of the phenomena we want information about are periodic (electromagnetic fields, heart rhythms, tides, weather). A surprising and beautiful theorem from advanced calculus says that every periodic function we are likely to use in mathematical modeling can be written as an algebraic combination of sines and cosines, so the derivatives of sines and cosines play a key role in describing important changes. This section shows how to differentiate the six basic trigonometric functions.

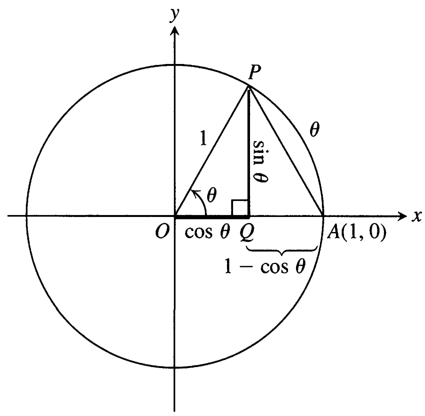
Some Special Limits

Our first step is to establish some inequalities and limits. It is assumed throughout that angles are measured in radians.

Theorem 3

If θ is measured in radians, then

$$-|\theta| < \sin\theta < |\theta| \qquad \text{and} \qquad -|\theta| < 1 - \cos\theta < |\theta|.$$



2.32 From the geometry of this figure, drawn for $\theta > 0$, we get the inequality $\sin^2\theta + (1-\cos\theta)^2 < \theta^2$.

Proof To establish these inequalities, we picture θ as an angle in standard position (Fig. 2.32). The circle in the figure is a unit circle, so $|\theta|$ equals the length of the circular arc AP. The length of line segment AP is therefore less than $|\theta|$.

Triangle APQ is a right triangle with sides of length

$$QP = |\sin\theta|, \qquad AQ = 1 - \cos\theta.$$

From the Pythagorean theorem and the fact that $AP < |\theta|$, we get

$$\sin^2\theta + (1-\cos\theta)^2 = (AP)^2 < \theta^2. \tag{1}$$

The terms on the left side of Eq. (1) are both positive, so each is smaller than their sum and hence is less than θ^2:

$$\sin^2\theta < \theta^2 \qquad \text{and} \qquad (1-\cos\theta)^2 < \theta^2.$$

By taking square roots, we can see that this is equivalent to saying that

$$|\sin\theta| < |\theta| \qquad \text{and} \qquad |1-\cos\theta| < |\theta|$$

or

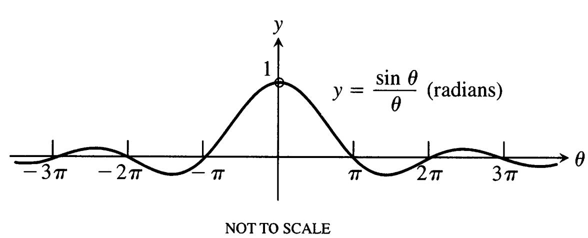
$$-|\theta| < \sin\theta < |\theta| \qquad \text{and} \qquad -|\theta| < 1-\cos\theta < |\theta|.$$ ❑

EXAMPLE 1 Show that $\sin\theta$ and $\cos\theta$ are continuous at $\theta = 0$. That is,

$$\lim_{\theta\to 0} \sin\theta = 0 \qquad \text{and} \qquad \lim_{\theta\to 0} \cos\theta = 1.$$

Solution As $\theta \to 0$, both $|\theta|$ and $-|\theta|$ approach 0. The values of the limits therefore follow immediately from Theorem 3 and the Sandwich Theorem. ❑

The function $f(\theta) = (\sin\theta)/\theta$ graphed in Fig. 2.33 appears to have a removable discontinuity at $\theta = 0$. As the figure suggests, $\lim_{\theta\to 0} f(\theta) = 1$.



2.33 The graph of $f(\theta) = (\sin\theta)/\theta$.

Theorem 4

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{2}$$

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with values of θ that are positive and less than $\pi/2$ (Fig. 2.34). Notice that

$$\text{Area } \Delta OAP < \text{area sector } OAP < \text{area } \Delta OAT.$$

We can express these areas in terms of θ as follows:

Equation (3) is where radian measure comes in: The area of sector OAP is $\theta/2$ only if θ is measured in radians.

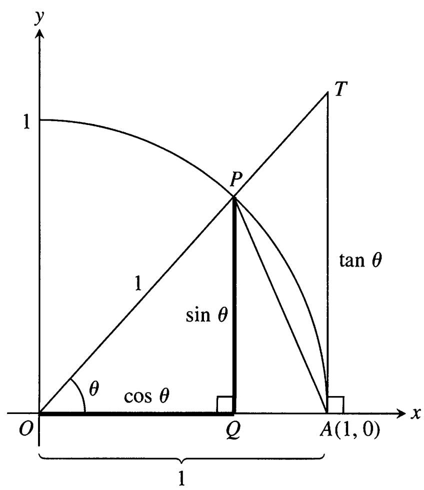
$$\text{Area } \Delta OAP = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta$$

$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2} \tag{3}$$

$$\text{Area } \Delta OAT = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta,$$

so

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$



2.34 The figure for the proof of Theorem 4. $TA/OA = \tan\theta$, but $OA = 1$, so $TA = \tan\theta$.

This last inequality will go the same way if we divide all three terms by the positive number $(1/2)\sin\theta$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

We next take reciprocals, which reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$, the Sandwich Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$$

Finally, observe that $\sin\theta$ and θ are both *odd functions.* Therefore, $f(\theta) = (\sin\theta)/\theta$ is an *even function,* with a graph symmetric about the y-axis (see Fig. 2.33). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta},$$

so $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ by Theorem 5 of Section 1.4. ❑

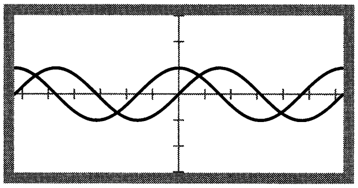
Theorem 4 can be combined with limit rules and known trigonometric identities to yield other trigonometric limits.

EXAMPLE 2 Show that $\lim_{h\to 0}\dfrac{\cos h - 1}{h} = 0.$

Solution Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned}\lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h}\\ &= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta \qquad \text{Let } \theta = h/2.\\ &= -(1)(0) = 0.\end{aligned}$$

❑



$y_1 = \sin x, -2\pi \le x \le 2\pi$
$y_2 = d(y_1)/dx, -2\pi \le x \le 2\pi$

Technology *Conjectures Based on Grapher Images* What you see in the window of a graphing utility can suggest conjectures, sometimes rather strongly. Graph the functions

$y_1 = \sin x$

$y_2 = d(y_1)/dx$ (This is computed by a built-in differentiation utility.)

Does the graph of y_2 look familiar? What function do you think it is? Test your conjecture by adding the function's graph to the screen.

The Derivative of the Sine

To calculate the derivative of $y = \sin x$, we combine the limits in Example 2 and Theorem 4 with the addition formula

$$\sin(x+h) = \sin x \cos h + \cos x \sin h. \quad \textbf{(4)}$$

We have

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Derivative definition} \\
&= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} && \text{Eq. (4)} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \left(\cos x \cdot \frac{\sin h}{h}\right) \\
&= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 && \text{Example 2 and Theorem 4} \\
&= \cos x.
\end{aligned}$$

In short, the derivative of the sine is the cosine.

$$\frac{d}{dx}(\sin x) = \cos x$$

EXAMPLE 3

a) $y = x^2 - \sin x$:

$$\begin{aligned}
\frac{dy}{dx} &= 2x - \frac{d}{dx}(\sin x) && \text{Difference Rule} \\
&= 2x - \cos x
\end{aligned}$$

b) $y = x^2 \sin x$:

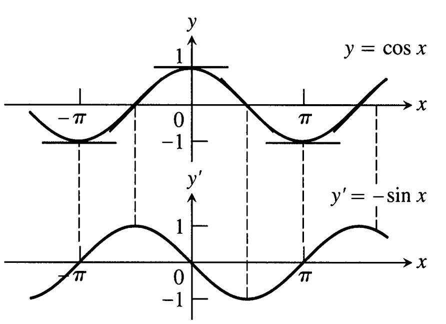
$$\begin{aligned}
\frac{dy}{dx} &= x^2 \frac{d}{dx}(\sin x) + 2x \sin x && \text{Product Rule} \\
&= x^2 \cos x + 2x \sin x
\end{aligned}$$

Radian measure in calculus

In case you are wondering why calculus uses radian measure when the rest of the world seems to use degrees, the answer lies in the argument that the derivative of the sine is the cosine. The derivative of $\sin x$ is $\cos x$ *only* if x is measured in radians. The argument requires that when h is a small increment in x,

$$\lim_{h\to 0} (\sin h)/h = 1.$$

This is true only for radian measure, as we saw during the proof of Theorem 4. You will see what the degree-mode derivatives of the sine and cosine are if you do Exercise 76.



2.35 The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

c) $y = \dfrac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$

$$= \frac{x\cos x - \sin x}{x^2}$$

❑

The Derivative of the Cosine

With the help of the addition formula,

$$\cos(x+h) = \cos x \cos h - \sin x \sin h, \tag{5}$$

we have

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \qquad \text{Derivative definition}$$

$$= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \qquad \text{Eq. (5)}$$

$$= \lim_{h\to 0} \frac{\cos x\,(\cos h - 1) - \sin x \sin h}{h}$$

$$= \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \cdot \frac{\sin h}{h}$$

$$= \cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \cos x \cdot 0 - \sin x \cdot 1 \qquad \text{Example 2 and Theorem 4}$$

$$= -\sin x.$$

In short, the derivative of the cosine is the negative of the sine.

$$\frac{d}{dx}(\cos x) = -\sin x$$

Figure 2.35 shows another way to visualize this result.

EXAMPLE 4

a) $y = 5x + \cos x$

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x) \qquad \text{Sum Rule}$$

$$= 5 - \sin x$$

b) $y = \sin x \cos x$

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) \qquad \text{Product Rule}$$

$$= \sin x\,(-\sin x) + \cos x\,(\cos x)$$

$$= \cos^2 x - \sin^2 x$$

c)
$$y = \frac{\cos x}{1 - \sin x}$$

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$

$$= \frac{(1 - \sin x)(- \sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

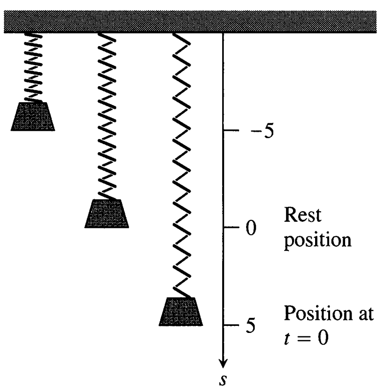
$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$

$$= \frac{1}{1 - \sin x}$$

❑

Simple Harmonic Motion

The motion of a body bobbing up and down on the end of a spring is an example of *simple harmonic motion.* The next example describes a case in which there are no opposing forces like friction or buoyancy to slow the motion down.



2.36 The body in Example 5.

EXAMPLE 5 A body hanging from a spring (Fig. 2.36) is stretched 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5 \cos t.$$

What are its velocity and acceleration at time t?

Solution We have

Position: $\quad s = 5 \cos t$

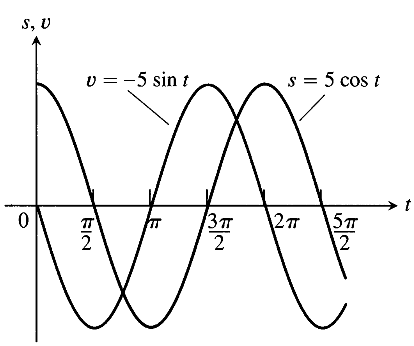
Velocity: $\quad v = \frac{ds}{dt} = \frac{d}{dt}(5 \cos t) = 5\frac{d}{dt}(\cos t) = -5 \sin t$

Acceleration: $\quad a = \frac{dv}{dt} = \frac{d}{dt}(-5 \sin t) = -5\frac{d}{dt}(\sin t) = -5 \cos t.$

Here is what we can learn from these equations:

1. As time passes, the body moves up and down between $s = 5$ and $s = -5$ on the s-axis. The amplitude of the motion is 5. The period of the motion is 2π, the period of $\cos t$.
2. The function $\sin t$ attains its greatest magnitude (1) when $\cos t = 0$, as the graphs of the sine and cosine show (Fig. 2.37). Hence, the body's speed, $|v| = 5|\sin t|$, is greatest every time $\cos t = 0$, i.e., every time the body passes its rest position.

   The body's speed is zero when $\sin t = 0$. This occurs at the endpoints of the interval of motion, when $\cos t = \pm 1$.
3. The acceleration, $a = -5 \cos t$, is zero only at the rest position, where the cosine is zero. When the body is anywhere else, the spring is either pulling on it or pushing on it. The acceleration is greatest in magnitude at the points farthest from the origin, where $\cos t = \pm 1$. ❑



2.37 The graphs of the position and velocity of the body in Example 5.

Jerk

A sudden change in acceleration is called a "jerk." When a ride in a car or a bus is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt. Jerk is what spills your soft drink. The derivative responsible for jerk is d^3s/dt^3.

Definition

Jerk is the derivative of acceleration. If a body's position at time t is $s = f(t)$, the body's jerk at time t is

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Recent tests have shown that motion sickness comes from accelerations whose changes in magnitude or direction take us by surprise. Keeping an eye on the road helps us to see the changes coming. A driver is less likely to become sick than a passenger reading in the backseat.

EXAMPLE 6

a) The jerk of the constant acceleration of gravity ($g = 32$ ft/sec^2) is zero:

$$j = \frac{d}{dt}(g) = 0.$$

We don't experience motion sickness if we are just sitting around.

b) The jerk of the simple harmonic motion in Example 5 is

$$j = \frac{da}{dt} = \frac{d}{dt}(-5\cos t)$$
$$= 5\sin t.$$

It has its greatest magnitude when $\sin t = \pm 1$, not at the extremes of the displacement but at the origin, where the acceleration changes direction and sign. ❑

The Derivatives of the Other Basic Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x} \qquad \sec x = \frac{1}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x} \qquad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas.

Notice the minus signs in the derivative formulas for the cofunctions.

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad (6) \qquad\qquad \frac{d}{dx}(\sec x) = \sec x \tan x \qquad (7)$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad (8) \qquad\qquad \frac{d}{dx}(\csc x) = -\csc x \cot x \qquad (9)$$

To show how a typical calculation goes, we derive Eq. (6). The other derivations are left to Exercises 67 and 68.

EXAMPLE 7 Find dy/dx if $y = \tan x$.

Solution

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \qquad \text{Quotient Rule}$$

$$= \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

❑

EXAMPLE 8 Find y'' if $y = \sec x$.

Solution

$$y = \sec x$$

$$y' = \sec x \tan x \qquad \text{Eq. (7)}$$

$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \qquad \text{Product Rule}$$

$$= \sec x\,(\sec^2 x) + \tan x\,(\sec x \tan x)$$

$$= \sec^3 x + \sec x \tan^2 x$$

❑

EXAMPLE 9

a) $\frac{d}{dx}(3x + \cot x) = 3 + \frac{d}{dx}(\cot x) = 3 - \csc^2 x$

b) $\frac{d}{dx}\left(\frac{2}{\sin x}\right) = \frac{d}{dx}(2\csc x) = 2\frac{d}{dx}(\csc x)$

$$= 2(-\csc x \cot x) = -2\csc x \cot x$$

❑

Continuity of Trigonometric Functions

Since the six basic trigonometric functions are differentiable throughout their domains they are also continuous throughout their domains by Theorem 1, Section 2.1. This means that $\sin x$ and $\cos x$ are continuous for all x, that $\sec x$ and $\tan x$ are continuous except when x is a nonzero integer multiple of $\pi/2$, and that $\csc x$ and $\cot x$ are continuous except when x is an integer multiple of π. For each function, $\lim_{x\to c} f(x) = f(c)$ whenever $f(c)$ is defined. As a result, we can calculate the limits of many algebraic combinations and composites of trigonometric functions by direct substitution.

EXAMPLE 10

$$\lim_{x\to 0} \frac{\sqrt{2+\sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2+\sec 0}}{\cos(\pi - \tan 0)} = \frac{\sqrt{2+1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$ ❑

Other Limits Calculated with Theorem 4

The equation $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ holds no matter how θ may be expressed:

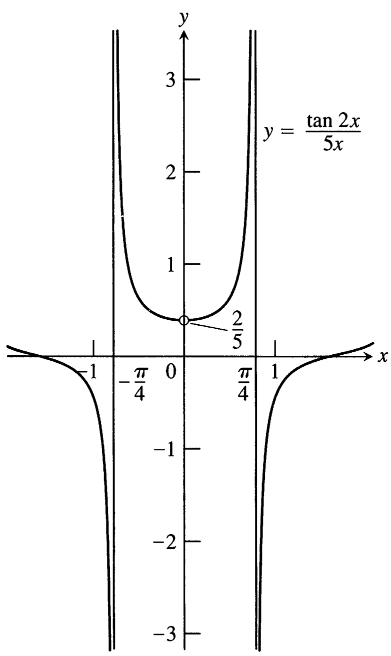
$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \theta = x; \qquad \lim_{x\to 0} \frac{\sin 7x}{7x} = 1, \quad \theta = 7x;$$

As $x \to 0$, $\theta \to 0$ As $x \to 0$, $\theta \to 0$

$$\lim_{x\to 0} \frac{\sin(2/3)x}{(2/3)x} = 1, \quad \theta = (2/3)x$$

As $x \to 0$, $\theta \to 0$

Knowing this helps us calculate related limits involving angles in radian measure.



2.38 The graph of $y = (\tan 2x)/5x$ steps across the y-axis at $y = 2/5$ (Example 11).

EXAMPLE 11

a) $$\lim_{x\to 0} \frac{\sin 2x}{5x} = \lim_{x\to 0} \frac{(2/5)\cdot \sin 2x}{(2/5)\cdot 5x}$$

Eq. (2) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by 2/5.

$$= \frac{2}{5} \lim_{x\to 0} \frac{\sin 2x}{2x}$$

Now Eq. (2) applies

$$= \frac{2}{5}(1) = \frac{2}{5}$$

b) $$\lim_{x\to 0} \frac{\tan 2x}{5x} = \lim_{x\to 0} \left(\frac{\sin 2x}{5x} \cdot \frac{1}{\cos 2x}\right)$$

$\tan 2x = \dfrac{\sin 2x}{\cos 2x}$

$$= \left(\lim_{x\to 0} \frac{\sin 2x}{5x}\right)\left(\lim_{x\to 0} \frac{1}{\cos 2x}\right)$$

$$= \left(\frac{2}{5}\right)\left(\frac{1}{\cos 0}\right) = \frac{2}{5}$$

Part (a) and continuity of $\cos x$

See Fig. 2.38. ❑

Applications

The occurrence of the function $(\sin x)/x$ in calculus is not an isolated event. The function arises in such diverse fields as quantum physics (where it appears in solutions of the wave equation) and electrical engineering (in signal analysis and signal filter design) as well as in the mathematical fields of differential equations and probability theory.

EXAMPLE 12

$$\lim_{t \to (\pi/2)} \frac{\sin\left(t - \frac{\pi}{2}\right)}{t - \frac{\pi}{2}} \qquad \text{Set } \theta = t - (\pi/2). \text{ Then } \theta \to 0 \text{ as } t \to (\pi/2).$$

$$= \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

❑

Exercises 2.4

Derivatives

In Exercises 1–12, find dy/dx.

1. $y = -10x + 3\cos x$

2. $y = \frac{3}{x} + 5\sin x$

3. $y = \csc x - 4\sqrt{x} + 7$

4. $y = x^2 \cot x - \frac{1}{x^2}$

5. $y = (\sec x + \tan x)(\sec x - \tan x)$

6. $y = (\sin x + \cos x)\sec x$

7. $y = \frac{\cot x}{1 + \cot x}$

8. $y = \frac{\cos x}{1 + \sin x}$

9. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$

10. $y = \frac{\cos x}{x} + \frac{x}{\cos x}$

11. $y = x^2 \sin x + 2x\cos x - 2\sin x$

12. $y = x^2 \cos x - 2x\sin x - 2\cos x$

In Exercises 13–16, find ds/dt.

13. $s = \tan t - t$

14. $s = t^2 - \sec t + 1$

15. $s = \frac{1 + \csc t}{1 - \csc t}$

16. $s = \frac{\sin t}{1 - \cos t}$

In Exercises 17–20, find $dr/d\theta$.

17. $r = 4 - \theta^2 \sin\theta$

18. $r = \theta\sin\theta + \cos\theta$

19. $r = \sec\theta\csc\theta$

20. $r = (1 + \sec\theta)\sin\theta$

In Exercises 21–24, find dp/dq.

21. $p = 5 + \frac{1}{\cot q}$

22. $p = (1 + \csc q)\cos q$

23. $p = \frac{\sin q + \cos q}{\cos q}$

24. $p = \frac{\tan q}{1 + \tan q}$

25. Find y'' if (a) $y = \csc x$, (b) $y = \sec x$.

26. Find $y^{(4)} = d^4y/dx^4$ if (a) $y = -2\sin x$, (b) $y = 9\cos x$.

Limits

Find the limits in Exercises 27–32.

27. $\lim_{x \to 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right)$

28. $\lim_{x \to -\pi/6} \sqrt{1 + \cos(\pi \csc x)}$

29. $\lim_{x \to 0} \sec\left[\cos x + \pi\tan\left(\frac{\pi}{4\sec x}\right) - 1\right]$

30. $\lim_{x \to 0} \sin\left(\frac{\pi + \tan x}{\tan x - 2\sec x}\right)$

31. $\lim_{t \to 0} \tan\left(1 - \frac{\sin t}{t}\right)$

32. $\lim_{\theta \to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right)$

Find the limits in Exercises 33–48.

33. $\lim_{\theta \to 0} \frac{\sin\sqrt{2}\theta}{\sqrt{2}\theta}$

34. $\lim_{t \to 0} \frac{\sin kt}{t}$ (k constant)

35. $\lim_{y \to 0} \frac{\sin 3y}{4y}$

36. $\lim_{h \to 0^-} \frac{h}{\sin 3h}$

37. $\lim_{x \to 0} \frac{\tan 2x}{x}$

38. $\lim_{t \to 0} \frac{2t}{\tan t}$

39. $\lim_{x \to 0} \frac{x\csc 2x}{\cos 5x}$

40. $\lim_{x \to 0} 6x^2(\cot x)(\csc 2x)$

41. $\lim_{x \to 0} \frac{x + x\cos x}{\sin x\cos x}$

42. $\lim_{x \to 0} \frac{x^2 - x + \sin x}{2x}$

43. $\lim_{t \to 0} \frac{\sin(1 - \cos t)}{1 - \cos t}$

44. $\lim_{h \to 0} \frac{\sin(\sin h)}{\sin h}$

45. $\lim_{\theta \to 0} \frac{\sin\theta}{\sin 2\theta}$

46. $\lim_{x \to 0} \frac{\sin 5x}{\sin 4x}$

47. $\lim_{x \to 0} \dfrac{\tan 3x}{\sin 8x}$

48. $\lim_{y \to 0} \dfrac{\sin 3y \cot 5y}{y \cot 4y}$

Tangent Lines

In Exercises 49–52, graph the curves over the given intervals, together with their tangents at the given values of x. Label each curve and tangent with its equation.

49. $y = \sin x, \quad -3\pi/2 \le x \le 2\pi$
$x = -\pi,\ 0,\ 3\pi/2$

50. $y = \tan x, \quad -\pi/2 < x < \pi/2$
$x = -\pi/3,\ 0,\ \pi/3$

51. $y = \sec x, \quad -\pi/2 < x < \pi/2$
$x = -\pi/3,\ \pi/4$

52. $y = 1 + \cos x, \quad -3\pi/2 \le x \le 2\pi$
$x = -\pi/3,\ 3\pi/2$

Do the graphs of the functions in Exercises 53–56 have any horizontal tangents in the interval $0 \le x \le 2\pi$? If so, where? If not, why not? You may want to visualize your findings by graphing the functions with a grapher.

53. $y = x + \sin x$

54. $y = 2x + \sin x$

55. $y = x - \cot x$

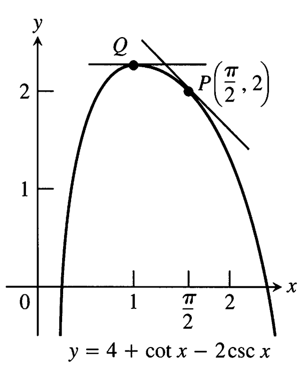
56. $y = x + 2\cos x$

57. Find all points on the curve $y = \tan x, -\pi/2 < x < \pi/2$, where the tangent line is parallel to the line $y = 2x$. Sketch the curve and tangent(s) together, labeling each with its equation.

58. Find all points on the curve $y = \cot x, 0 < x < \pi$, where the tangent line is parallel to the line $y = -x$. Sketch the curve and tangent(s) together, labeling each with its equation.

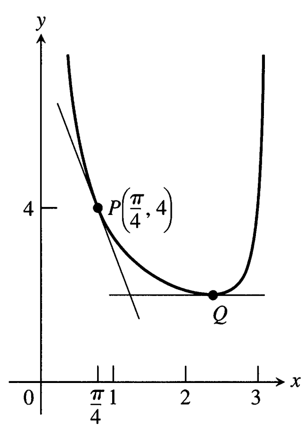
In Exercises 59 and 60, find an equation for (a) the tangent to the curve at P and (b) the horizontal tangent to the curve at Q.

59.



$y = 4 + \cot x - 2\csc x$
(Generated by Mathematica)

60.



$y = 1 + \sqrt{2}\csc x + \cot x$
(Generated by Mathematica)

Simple Harmonic Motion

The equations in Exercises 61 and 62 give the position $s = f(t)$ of a body moving on a coordinate line (s in meters, t in seconds). Find the body's velocity, speed, acceleration, and jerk at time $t = \pi/4$ sec.

61. $s = 2 - 2\sin t$

62. $s = \sin t + \cos t$

Theory and Examples

63. Is there a value of c that will make

$$f(x) = \begin{cases} \dfrac{\sin^2 3x}{x^2}, & x \ne 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$? Give reasons for your answer.

64. Is there a value of b that will make

$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \ge 0 \end{cases}$$

continuous at $x = 0$? differentiable at $x = 0$? Give reasons for your answers.

65. Find $\dfrac{d^{999}}{dx^{999}}(\cos x)$

66. Find $\dfrac{d^{725}}{dx^{725}}(\sin x)$

67. Derive the formula for the derivative with respect to x of

a) $\sec x$ **b)** $\csc x$.

68. Derive the formula for the derivative with respect to x of $\cot x$.

Grapher Explorations

69. Graph $y = \cos x$ for $-\pi \le x \le 2\pi$. On the same screen, graph

$$y = \frac{\sin(x + h) - \sin x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1. Then, in a new window, try $h = -1, -0.5$, and -0.3. What happens as $h \to 0^+$? as $h \to 0^-$? What phenomenon is being illustrated here?

70. Graph $y = -\sin x$ for $-\pi \le x \le 2\pi$. On the same screen, graph

$$y = \frac{\cos(x + h) - \cos x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1. Then, in a new window, try $h = -1, -0.5$, and -0.3. What happens as $h \to 0^+$? as $h \to 0^-$? What phenomenon is being illustrated here?

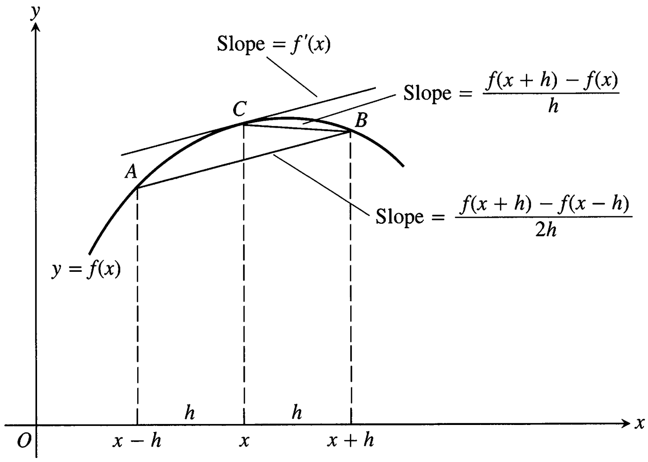
71. *Centered difference quotients.* The **centered difference quotient**

$$\frac{f(x + h) - f(x - h)}{2h}$$

is used to approximate $f'(x)$ in numerical work because (1) its limit as $h \to 0$ equals $f'(x)$ when $f'(x)$ exists, and (2) it usually gives a better approximation of $f'(x)$ for a given value of h than Fermat's difference quotient

$$\frac{f(x + h) - f(x)}{h}.$$

See the figure below.



a) To see how rapidly the centered difference quotient for $f(x) = \sin x$ converges to $f'(x) = \cos x$, graph $y = \cos x$ together with

$$y = \frac{\sin(x+h) - \sin(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 69 for the same values of h.

b) To see how rapidly the centered difference quotient for $f(x) = \cos x$ converges to $f'(x) = -\sin x$, graph $y = -\sin x$ together with

$$y = \frac{\cos(x+h) - \cos(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 70 for the same values of h.

72. *A caution about centered difference quotients. (Continuation of Exercise 71.)* The quotient

$$\frac{f(x+h) - f(x-h)}{2h}$$

may have a limit as $h \to 0$ when f has no derivative at x. As a case in point, take $f(x) = |x|$ and calculate

$$\lim_{h \to 0} \frac{|0+h| - |0-h|}{2h}.$$

As you will see, the limit exists even though $f(x) = |x|$ has no derivative at $x = 0$.

73. Graph $y = \tan x$ and its derivative together on $(-\pi/2, \pi/2)$. Does the graph of the tangent function appear to have a smallest slope? a largest slope? Is the slope ever negative? Give reasons for your answers.

74. Graph $y = \cot x$ and its derivative together for $0 < x < \pi$. Does the graph of the cotangent function appear to have a smallest slope? a largest slope? Is the slope ever positive? Give reasons for your answers.

75. Graph $y = (\sin x)/x$, $y = (\sin 2x)/x$, and $y = (\sin 4x)/x$ together over the interval $-2 \le x \le 2$. Where does each graph appear to cross the y-axis? Do the graphs really intersect the axis? What would you expect the graphs of $y = (\sin 5x)/x$ and $y = (\sin(-3x))/x$ to do as $x \to 0$? Why? What about the graph of $y = (\sin kx)/x$ for other values of k? Give reasons for your answers.

76. *Radians vs. degrees.* What happens to the derivatives of $\sin x$ and $\cos x$ if x is measured in degrees instead of radians? To find out, take the following steps.

a) With your graphing calculator or computer grapher *in degree mode,* graph

$$f(h) = \frac{\sin h}{h}$$

and estimate $\lim_{h \to 0} f(h)$. Compare your estimate with $\pi/180$. Is there any reason to believe the limit *should* be $\pi/180$?

b) With your grapher still in degree mode, estimate

$$\lim_{h \to 0} \frac{\cos h - 1}{h}.$$

c) Now go back to the derivation of the formula for the derivative of $\sin x$ in the text and carry out the steps of the derivation using degree-mode limits. What formula do you obtain for the derivative?

d) Work through the derivation of the formula for the derivative of $\cos x$ using degree-mode limits. What formula do you obtain for the derivative?

e) The disadvantages of the degree-mode formulas become apparent as you start taking derivatives of higher order. Try it. What are the second and third degree-mode derivatives of $\sin x$ and $\cos x$?

2.5 The Chain Rule

We now know how to differentiate $\sin x$ and $x^2 - 4$, but how do we differentiate a composite like $\sin(x^2 - 4)$? The answer is, with the Chain Rule, which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points. The Chain Rule is probably the most widely used differentiation rule in mathematics. This section describes the rule and how to use it. We begin with examples.

EXAMPLE 1 The function $y = 6x - 10 = 2(3x - 5)$ is the composite of the functions $y = 2u$ and $u = 3x - 5$. How are the derivatives of these three functions related?

Solution We have

$$\frac{dy}{dx} = 6, \qquad \frac{dy}{du} = 2, \qquad \frac{du}{dx} = 3.$$

Since $6 = 2 \cdot 3$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

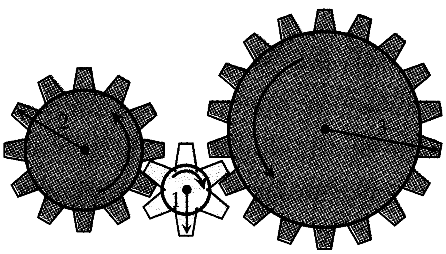
❑

Is it an accident that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}?$$

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. For $y = f(u)$ and $u = g(x)$, if y changes twice as fast as u and u changes three times as fast as x, then we expect y to change six times as fast as x. This is much like the effect of a multiple gear train (Fig. 2.39).

Let us try this again on another function.



2.39 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ and $u = 3x$, so $y = 3x/2$. Thus $dy/du = 1/2$, $du/dx = 3$, and $dy/dx = 3/2 = (dy/du)(du/dx)$.

EXAMPLE 2

$$y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$$

is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned}\frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \\ &= 36x^3 + 12x\end{aligned}$$

and

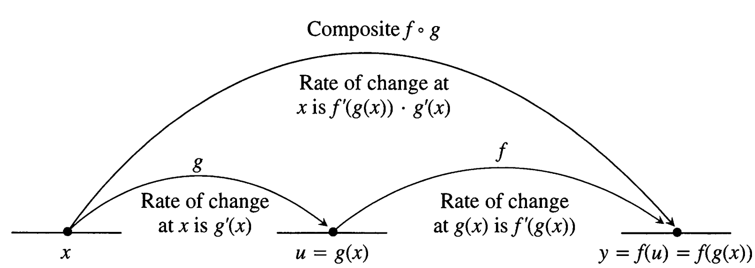
$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1) \\ &= 36x^3 + 12x.\end{aligned}$$

Once again,

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}.$$

❑

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. This is known as the Chain Rule (Fig. 2.40).



2.40 Rates of change multiply: the derivative of $f \circ g$ at x is the derivative of f at the point $g(x)$ times the derivative of g at x.

Theorem 5
The Chain Rule
If $f(u)$ is differentiable at the point $u = g(x)$, and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x). \tag{1}$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \tag{2}$$

where dy/du is evaluated at $u = g(x)$.

It would be tempting to try to prove the Chain Rule by writing

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

and taking the limit as $\Delta x \to 0$. This would work if we knew that Δu, the change in u, was nonzero, but we do not know this. A small change in x could conceivably produce no change in u. The proof requires a different approach, using ideas in Section 3.7. We will return to it when the time comes.

EXAMPLE 3 Find the derivative of $y = \sqrt{x^2 + 1}$.

Solution Here $y = f(g(x))$, where $f(u) = \sqrt{u}$ and $g(x) = x^2 + 1$. Since the derivatives of f and g are

$$f'(u) = \frac{1}{2\sqrt{u}} \quad \text{and} \quad g'(x) = 2x,$$

the Chain Rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)\\ &= \frac{1}{2\sqrt{g(x)}} \cdot g'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot (2x)\\ &= \frac{x}{\sqrt{x^2+1}}.\end{aligned}$$

❑

The "Outside-Inside" Rule

It sometimes helps to think about the Chain Rule the following way. If $y = f(g(x))$, Eq. (2) tells us that

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x). \tag{3}$$

In words, Eq. (3) says: To find dy/dx, differentiate the "outside" function f and leave the "inside" $g(x)$ alone; then multiply by the derivative of the inside.

EXAMPLE 4

$$\frac{d}{dx}\underbrace{\sin}_{\text{outside}}\underbrace{(x^2+x)}_{\text{inside}} = \underbrace{\cos}_{\text{derivative of the outside}}\underbrace{(x^2+x)}_{\text{inside left alone}} \cdot \underbrace{(2x+1)}_{\text{derivative of the inside}}$$

❑

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative. Here is an example.

EXAMPLE 5 Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution

$$\begin{aligned} g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) \\ &= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\ &= \sec^2(5 - \sin 2t) \cdot \left(0 - (\cos 2t) \cdot \frac{d}{dt}(2t)\right) && \text{Derivative of } 5 - \sin u \text{ with } u = 2t \\ &= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\ &= -2(\cos 2t)\sec^2(5 - \sin 2t) \end{aligned}$$

❑

Differentiation Formulas That Include the Chain Rule

Many of the differentiation formulas you will encounter in your scientific work already include the Chain Rule.

If f is a differentiable function of u, and u is a differentiable function of x, then substituting $y = f(u)$ in the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}. \tag{4}$$

For example, if u is a differentiable function of x, n is an integer, and $y = u^n$, then the Chain Rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{du}(u^n) \cdot \frac{du}{dx} \\ &= nu^{n-1}\frac{du}{dx}. && \text{Differentiating } u^n \text{ with respect to } u \text{ itself gives } nu^{n-1}. \end{aligned}$$

Power Chain Rule

If $u(x)$ is a differentiable function and n is an integer, then u^n is differentiable and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \tag{5}$$

$\sin^n x$ is short for $(\sin x)^n$, $\quad n \neq -1$.

EXAMPLE 6

a) $\dfrac{d}{dx}\sin^5 x = 5\sin^4 x\dfrac{d}{dx}(\sin x)$ Eq. (5) with $u = \sin x$, $n = 5$

$= 5\sin^4 x\cos x$

b) $\dfrac{d}{dx}(2x+1)^{-3} = -3(2x+1)^{-4}\dfrac{d}{dx}(2x+1)$ Eq. (5) with $u = 2x+1$, $n = -3$

$= -3(2x+1)^{-4}(2)$

$= -6(2x+1)^{-4}$

c) $\dfrac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6\dfrac{d}{dx}(5x^3 - x^4)$ Eq. (5) with $u = 5x^3 - x^4$, $n = 7$

$= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3)$

$= 7(5x^3 - x^4)^6(15x^2 - 4x^3)$

d) $\dfrac{d}{dx}\left(\dfrac{1}{3x-2}\right) = \dfrac{d}{dx}(3x-2)^{-1}$ Eq. (5) with $u = 3x-2$, $n = -1$

$= -1(3x-2)^{-2}\dfrac{d}{dx}(3x-2)$

$= -1(3x-2)^{-2}(3)$

$= -\dfrac{3}{(3x-2)^2}$

In part (d) we could also have found the derivative with the Quotient Rule. ❑

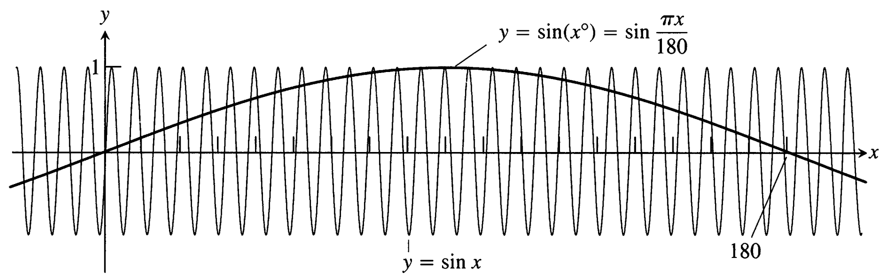
EXAMPLE 7 *Radians vs. degrees*

It is important to remember that the formulas for the derivatives of $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees. The Chain Rule brings new understanding to the difference between the two. Since $180° = \pi$ radians, $x° = \pi x/180$ radians. By the Chain Rule,

$$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°).$$

See Fig. 2.41. Similarly, the derivative of $\cos(x°)$ is $-(\pi/180)\sin(x°)$.

The factor $\pi/180$, annoying in the first derivative, would compound with repeated differentiation. We see at a glance the compelling reason for the use of radian measure. ❑



2.41 $\sin(x°)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$.

⁂ Melting Ice Cubes

In mathematics, we tend to use letters like f, g, x, y, and u for functions and variables. However, other fields use letters like V, for volume, and s, for side, that come from the names of the things being modeled. The letters in the Chain Rule then change too, as in the next example.

EXAMPLE 8 *The melting ice cube*

How long will it take an ice cube to melt?

Solution As with all applications to science, we start with a mathematical model. We assume that the cube retains its cubical shape as it melts. We call its side length s, so its volume is $V = s^3$. We assume that V and s are differentiable functions of time t. We assume also that the cube's volume decreases at a rate that is proportional to its surface area. This latter assumption seems reasonable enough when we think that the melting takes place at the surface: Changing the amount of surface changes the amount of ice exposed to melt. In mathematical terms,

$$\frac{dV}{dt} = -k(6s^2), \qquad k > 0.$$

The minus sign indicates that the volume is decreasing. We assume that the proportionality factor k is constant. (It probably depends on many things, however, such as the relative humidity of the surrounding air, the air temperature, and the incidence or absence of sunlight, to name only a few.)

Finally, we need at least one more piece of information: How long will it take a specific percentage of the ice cube to melt? We have nothing to guide us unless we make one or more observations, but now let us assume a particular set of conditions in which the cube lost 1/4 of its volume during the first hour. (You could use letters instead of particular numbers: say $n\%$ in r hours. Then your answer would be in terms of n and r.)

Mathematically, we now have the following problem.

Given: $V = s^3$ and $\dfrac{dV}{dt} = -k(6s^2)$

$V = V_0$ when $t = 0$

$V = (3/4)V_0$ when $t = 1$ h

Find: The value of t when $V = 0$

We apply the Chain Rule to differentiate $V = s^3$ with respect to t:

$$\frac{dV}{dt} = 3s^2 \frac{ds}{dt}.$$

We set this equal to the given rate, $-k(6s^2)$, to get

$$3s^2 \frac{ds}{dt} = -6ks^2$$

$$\frac{ds}{dt} = -2k.$$

The side length is *decreasing* at the constant rate of $2k$ units per hour. Thus, if the initial length of the cube's side is s_0, the length of its side one hour later is $s_1 = s_0 - 2k$. This equation tells us that

$$2k = s_0 - s_1.$$

The melting time is the value of t that makes $2kt = s_0$. Hence,

$$t_{\text{melt}} = \frac{s_0}{2k} = \frac{s_0}{s_0 - s_1} = \frac{1}{1 - (s_1/s_0)}.$$

But

$$\frac{s_1}{s_0} = \frac{\left(\frac{3}{4}V_0\right)^{1/3}}{(V_0)^{1/3}} = \left(\frac{3}{4}\right)^{1/3} \approx 0.91.$$

Therefore,

$$t_{\text{melt}} = \frac{1}{1 - 0.91} \approx 11 \text{ h}.$$

If 1/4 of the cube melts in 1 h, it will take about 10 h more for the rest of it to melt. ❑

If we were natural scientists interested in testing the assumptions on which our mathematical model is based, our next step would be to run a number of experiments and compare their outcomes with the model's predictions. One practical application might lie in analyzing the proposal to tow large icebergs from polar waters to offshore locations near southern California, where the melting ice could provide fresh water. As a first approximation, we might imagine the iceberg to be a large cube or rectangular solid, or perhaps a pyramid. We will say more about mathematical modeling in Section 4.2.

Exercises 2.5

Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$

2. $y = 2u^3, \quad u = 8x - 1$

3. $y = \sin u, \quad u = 3x + 1$

4. $y = \cos u, \quad u = -x/3$

5. $y = \cos u, \quad u = \sin x$

6. $y = \sin u, \quad u = x - \cos x$

7. $y = \tan u, \quad u = 10x - 5$

8. $y = -\sec u, \quad u = x^2 + 7x$

In Exercises 9–18, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

9. $y = (2x+1)^5$
10. $y = (4-3x)^9$
11. $y = \left(1 - \frac{x}{7}\right)^{-7}$
12. $y = \left(\frac{x}{2} - 1\right)^{-10}$
13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$
14. $y = \left(\frac{x}{5} + \frac{1}{5x}\right)^5$
15. $y = \sec(\tan x)$
16. $y = \cot\left(\pi - \frac{1}{x}\right)$
17. $y = \sin^3 x$
18. $y = 5\cos^{-4} x$

Find the derivatives of the functions in Exercises 19–38.

19. $p = \sqrt{3-t}$
20. $q = \sqrt{2r - r^2}$
21. $s = \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t$
22. $s = \sin\left(\frac{3\pi t}{2}\right) + \cos\left(\frac{3\pi t}{2}\right)$
23. $r = (\csc\theta + \cot\theta)^{-1}$
24. $r = -(\sec\theta + \tan\theta)^{-1}$
25. $y = x^2\sin^4 x + x\cos^{-2} x$
26. $y = \frac{1}{x}\sin^{-5} x - \frac{x}{3}\cos^3 x$
27. $y = \frac{1}{21}(3x-2)^7 + \left(4 - \frac{1}{2x^2}\right)^{-1}$
28. $y = (5-2x)^{-3} + \frac{1}{8}\left(\frac{2}{x} + 1\right)^4$
29. $y = (4x+3)^4(x+1)^{-3}$
30. $y = (2x-5)^{-1}(x^2-5x)^6$
31. $h(x) = x\tan(2\sqrt{x}) + 7$
32. $k(x) = x^2\sec\left(\frac{1}{x}\right)$
33. $f(\theta) = \left(\frac{\sin\theta}{1+\cos\theta}\right)^2$
34. $g(t) = \left(\frac{1+\cos t}{\sin t}\right)^{-1}$
35. $r = \sin(\theta^2)\cos(2\theta)$
36. $r = \sec\sqrt{\theta}\tan\left(\frac{1}{\theta}\right)$
37. $q = \sin\left(\frac{t}{\sqrt{t+1}}\right)$
38. $q = \cot\left(\frac{\sin t}{t}\right)$

In Exercises 39–48, find dy/dt.

39. $y = \sin^2(\pi t - 2)$
40. $y = \sec^2 \pi t$
41. $y = (1 + \cos 2t)^{-4}$
42. $y = (1 + \cot(t/2))^{-2}$
43. $y = \sin(\cos(2t-5))$
44. $y = \cos\left(5\sin\left(\frac{t}{3}\right)\right)$
45. $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$
46. $y = \frac{1}{6}\left(1 + \cos^2(7t)\right)^3$
47. $y = \sqrt{1 + \cos(t^2)}$
48. $y = 4\sin\left(\sqrt{1+\sqrt{t}}\right)$

Find y'' in Exercises 49–52.

49. $y = \left(1 + \frac{1}{x}\right)^3$
50. $y = (1 - \sqrt{x})^{-1}$
51. $y = \frac{1}{9}\cot(3x-1)$
52. $y = 9\tan\left(\frac{x}{3}\right)$

Finding Numerical Values of Derivatives

In Exercises 53–58, find the value of $(f \circ g)'$ at the given value of x.

53. $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$
54. $f(u) = 1 - \frac{1}{u}, \quad u = g(x) = \frac{1}{1-x}, \quad x = -1$
55. $f(u) = \cot\frac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$
56. $f(u) = u + \frac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = 1/4$
57. $f(u) = \frac{2u}{u^2+1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$
58. $f(u) = \left(\frac{u-1}{u+1}\right)^2, \quad u = g(x) = \frac{1}{x^2} - 1, \quad x = -1$

59. Suppose that functions f and g and their derivatives with respect to x have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

Find the derivatives with respect to x of the following combinations at the given value of x.

a) $2f(x), \quad x = 2$
b) $f(x) + g(x), \quad x = 3$
c) $f(x) \cdot g(x), \quad x = 3$
d) $f(x)/g(x), \quad x = 2$
e) $f(g(x)), \quad x = 2$
f) $\sqrt{f(x)}, \quad x = 2$
g) $1/g^2(x), \quad x = 3$
h) $\sqrt{f^2(x) + g^2(x)}, \quad x = 2$

60. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Find the derivatives with respect to x of the following combinations at the given value of x,

a) $5f(x) - g(x), \quad x = 1$
b) $f(x)g^3(x), \quad x = 0$
c) $\frac{f(x)}{g(x)+1}, \quad x = 1$
d) $f(g(x)), \quad x = 0$
e) $g(f(x)), \quad x = 0$
f) $(x^{11} + f(x))^{-2}, \quad x = 1$
g) $f(x + g(x)), \quad x = 0$

61. Find ds/dt when $\theta = 3\pi/2$ if $s = \cos\theta$ and $d\theta/dt = 5$.

62. Find dy/dt when $x = 1$ if $y = x^2 + 7x - 5$ and $dx/dt = 1/3$.

Choices in Composition

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 63 and 64.

63. Find dy/dx if $y = x$ by using the Chain Rule with y as a composite of

a) $y = (u/5) + 7$ and $u = 5x - 35$
b) $y = 1 + (1/u)$ and $u = 1/(x - 1)$.

64. Find dy/dx if $y = x^{3/2}$ by using the Chain Rule with y as a composite of

a) $y = u^3$ and $u = \sqrt{x}$
b) $y = \sqrt{u}$ and $u = x^3$.

Tangents and Slopes

65. **a)** Find the tangent to the curve $y = 2\tan(\pi x/4)$ at $x = 1$.
b) What is the smallest value the slope of the curve can ever have on the interval $-2 < x < 2$? Give reasons for your answer.

66. **a)** Find equations for the tangents to the curves $y = \sin 2x$ and $y = -\sin(x/2)$ at the origin. Is there anything special about how the tangents are related? Give reasons for your answer.
b) Can anything be said about the tangents to the curves $y = \sin mx$ and $y = -\sin(x/m)$ at the origin (m a constant $\neq 0$)? Give reasons for your answer.
c) For a given m, what are the largest values the slopes of the curves $y = \sin mx$ and $y = -\sin(x/m)$ can ever have? Give reasons for your answer.
d) The function $y = \sin x$ completes one period on the interval $[0, 2\pi]$, the function $y = \sin 2x$ completes two periods, the function $y = \sin(x/2)$ completes half a period, and so on. Is there any relation between the number of periods $y = \sin mx$ completes on $[0, 2\pi]$ and the slope of the curve $y = \sin mx$ at the origin? Give reasons for your answer.

Theory, Examples, and Applications

67. *Running machinery too fast.* Suppose that a piston is moving straight up and down and that its position at time t seconds is

$$s = A\cos(2\pi bt),$$

with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why machinery breaks when you run it too fast.)

68. *Temperatures in Fairbanks, Alaska.* The graph in Fig. 2.42 shows the average Fahrenheit temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day x is

$$y = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25.$$

a) On what day is the temperature increasing the fastest?
b) About how many degrees per day is the temperature increasing when it is increasing at its fastest?

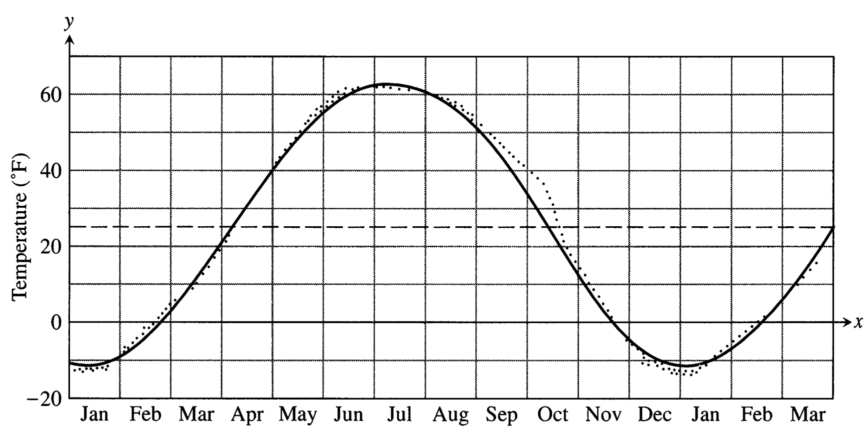
69. The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ sec.

70. Suppose the velocity of a falling body is $v = k\sqrt{s}$ m/sec (k a constant) at the instant the body has fallen s meters from its starting point. Show that the body's acceleration is constant.

71. The velocity of a heavy meteorite entering the earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is s kilometers from the earth's center. Show that the meteorite's acceleration is inversely proportional to s^2.

72. A particle moves along the x-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

73. *Temperature and the period of a pendulum.* For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple



2.42 Normal mean air temperatures at Fairbanks, Alaska, plotted as data points. The approximating sine function is

$$f(x) = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25$$

(Exercise 68).

pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L. In symbols, with u being temperature and k the proportionality constant

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

74. Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composites

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

75. Suppose that $u = g(x)$ is differentiable at $x = 1$ and that $y = f(u)$ is differentiable at $u = g(1)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, can we conclude anything about the tangent to the graph of g at $x = 1$ or the tangent to the graph of f at $u = g(1)$? Give reasons for your answer.

76. Suppose $u = g(x)$ is differentiable at $x = -5$, $y = f(u)$ is differentiable at $u = g(-5)$, and $(f \circ g)'(-5)$ is negative. What, if anything, can be said about the values of $g'(-5)$ and $f'(g(-5))$?

Using the Chain Rule, show that the power rule $(d/dx)x^n = nx^{n-1}$ holds for the functions x^n in Exercises 77 and 78.

77. $x^{1/4} = \sqrt{\sqrt{x}}$

78. $x^{3/4} = \sqrt{x\sqrt{x}}$

Grapher Explorations

79. *The derivative of* $\sin 2x$. Graph the function $y = 2\cos 2x$ for $-2 \le x \le 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x+h) - \sin 2x}{h}$$

for $h = 1.0, 0.5$, and 0.2. Experiment with other values of h, including negative values. What do you see happening as $h \to 0$? Explain this behavior.

80. *The derivative of* $\cos(x^2)$. Graph $y = -2x\sin(x^2)$ for $-2 \le x \le 3$. Then, on the same screen, graph

$$y = \frac{\cos[(x+h)^2] - \cos(x^2)}{h}$$

for $h = 1.0, 0.7$, and 0.3. Experiment with other values of h. What do you see happening as $h \to 0$? Explain this behavior.

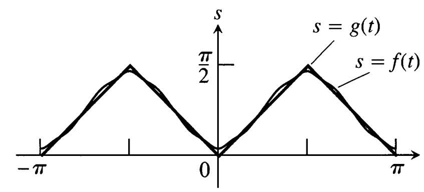
CAS Explorations and Projects

81. As Fig. 2.43 shows, the trigonometric "polynomial"

$$s = f(t) = 0.78540 - 0.63662\cos 2t - 0.07074\cos 6t - 0.02546\cos 10t - 0.01299\cos 14t$$

gives a good approximation of the sawtooth function $s = g(t)$ on the interval $[-\pi, \pi]$. How well does the derivative of f approximate the derivative of g at the points where dg/dt is defined? To find out, carry out the following steps.

a) Graph dg/dt (where defined) over $[-\pi, \pi]$.
b) Find df/dt.
c) Graph df/dt. Where does the approximation of dg/dt by df/dt seem to be best? least good? Approximations by trigonometric polynomials are important in the theories of heat and oscillation, but we must not expect too much of them, as we see in the next exercise.



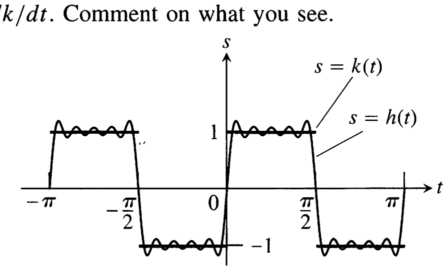
2.43 The approximation of a sawtooth function by a trigonometric "polynomial" (Exercise 81).

82. (*Continuation of Exercise 81.*) In Exercise 81, the trigonometric polynomial $f(t)$ that approximated the sawtooth function $g(t)$ on $[-\pi, \pi]$ had a derivative that approximated the derivative of the sawtooth function. It is possible, however, for a trigonometric polynomial to approximate a function in a reasonable way without its derivative approximating the function's derivative at all well. As a case in point, the "polynomial"

$$s = h(t) = 1.2732\sin 2t + 0.4244\sin 6t + 0.25465\sin 10t + 0.18186\sin 14t + 0.14147\sin 18t$$

graphed in Fig. 2.44 approximates the step function $s = k(t)$ shown there. Yet the derivative of h is nothing like the derivative of k.

a) Graph dk/dt (where defined) over $[-\pi, \pi]$.
b) Find dh/dt.
c) Graph dh/dt to see how badly the graph fits the graph of dk/dt. Comment on what you see.



2.44 The approximation of a step function by a trigonometric "polynomial" (Exercise 82).

2.6

Implicit Differentiation and Rational Exponents

When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate in the usual way, we may still be able to find dy/dx by *implicit differentiation.* This section describes the technique and uses it to extend the Power Rule for differentiation to include all rational exponents.

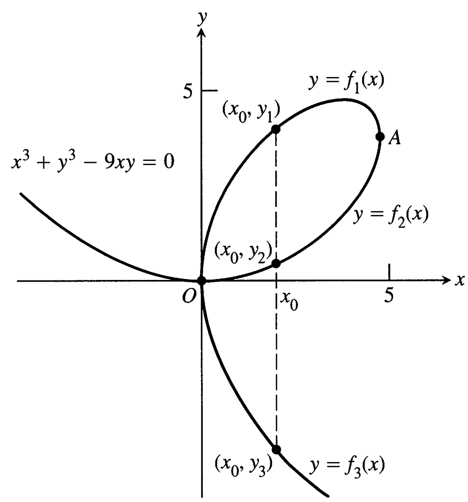
When are the functions defined by $F(x, y) = 0$ differentiable?

When may we expect the functions of x defined by an equation of the form $F(x, y) = 0$, where $F(x, y)$ denotes an expression in x and y, to be differentiable? A theorem in advanced calculus guarantees this to be the case if F is continuous (in a sense to be described in Chapter 12) and the first derivatives of F with respect to each variable, with the other held constant, are continuous, and the derivative with respect to y is nonzero. The functions you will encounter in this section all meet these criteria.

Implicit Differentiation

The graph of the equation $x^3 + y^3 - 9xy = 0$ (Fig. 2.45) has a well-defined slope at nearly every point because it is the union of the graphs of the functions $y = f_1(x)$, $y = f_2(x)$, and $y = f_3(x)$, which are differentiable except at O and A. But how do we find the slope when we cannot conveniently solve the equation to find the functions? The answer is to treat y as a differentiable function of x and differentiate both sides of the equation with respect to x, using the differentiation rules for powers, sums, products, and quotients and the Chain Rule. Then solve for dy/dx in terms of x and y *together* to obtain a formula that calculates the slope at any point (x, y) on the graph from the values of x and y.

The process by which we find dy/dx is called **implicit differentiation.** The phrase derives from the fact that the equation $x^3 + y^3 - 9xy = 0$ defines the functions f_1, f_2, and f_3 that give the graph's slope *implicitly* (i.e., hidden inside the equation), without giving us *explicit* formulas to work with.



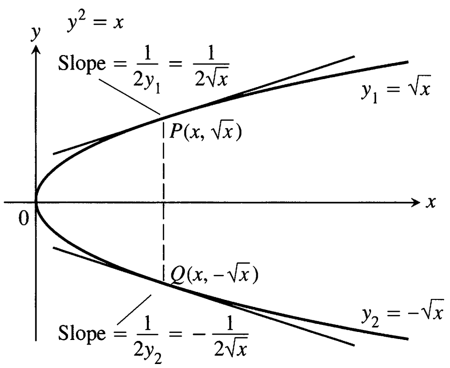
2.45 The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of x. However, the curve can be divided into separate arcs that *are* the graphs of functions of x. This particular curve, called a *folium,* dates to Descartes in 1638.

EXAMPLE 1 Find dy/dx if $y^2 = x$.

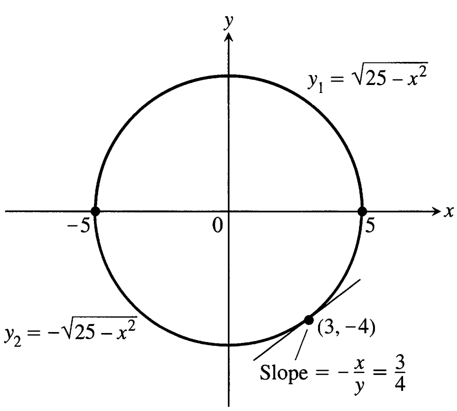
Solution The equation $y^2 = x$ defines two differentiable functions of x that we can actually find, namely $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ (Fig. 2.46). We know how to calculate the derivative of each of these for $x > 0$:

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}}, \qquad \text{and} \qquad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

But suppose we knew only that the equation $y^2 = x$ defined y as one or more differentiable functions of x for $x > 0$ without knowing exactly what these functions were. Could we still find dy/dx?



2.46 The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of x on the interval $x \geq 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for y.



2.47 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through (3, −4).

Solving polynomial equations in x and y

The quadratic formula enables us to solve a second degree equation like $y^2 - 2xy + 3x^2 = 0$ for y in terms of x. There are somewhat more complicated formulas for solving equations of degree three and four. But there are no general formulas for solving equations of degree five or higher. Finding slopes on curves defined by such equations usually requires implicit differentiation.

The answer is yes. To find dy/dx we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating $y = f(x)$ as a differentiable function of x:

$$y^2 = x$$

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

The Chain Rule gives $\frac{d}{dx}y^2 = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y\frac{dy}{dx}$.

This one formula gives the derivatives we calculated for *both* of the explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}}, \qquad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = -\frac{1}{2\sqrt{x}}.$$ ❑

EXAMPLE 2 Find the slope of circle $x^2 + y^2 = 25$ at the point (3, −4).

Solution The circle is not the graph of a single function of x. Rather it is the combined graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Fig. 2.47). The point (3, −4) lies on the graph of y_2, so we can find the slope by calculating explicitly:

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25 - x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25 - 9}} = \frac{3}{4}. \qquad (1)$$

But we can also solve the problem more easily by differentiating the given equation of the circle implicitly with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at (3, −4) is $\left.-\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}.$

Notice that unlike the slope formula in Eq. (1), which applies only to points below the x-axis, the formula $dy/dx = -x/y$ applies everywhere the circle has a slope. Notice also that the derivative involves *both* variables x and y, not just the independent variable x. ❑

To calculate the derivatives of other implicitly defined functions, we proceed as in Examples 1 and 2: We treat y as a differentiable implicit function of x and apply the usual rules to differentiate both sides of the defining equation.

EXAMPLE 3 Find dy/dx if $2y = x^2 + \sin y$.

Implicit Differentiation Takes Four Steps

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Factor out dy/dx.
4. Solve for dy/dx by dividing.

Solution

$$2y = x^2 + \sin y$$

$$\frac{d}{dx}(2y) = \frac{d}{dx}(x^2 + \sin y) \qquad \text{Differentiate both sides with respect to } x \ldots$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dy}(\sin y)$$

$$2\frac{dy}{dx} = 2x + \cos y\frac{dy}{dx} \qquad \ldots\text{treating } y \text{ as a function of } x \text{ and using the Chain Rule.}$$

$$2\frac{dy}{dx} - \cos y\frac{dy}{dx} = 2x \qquad \text{Collect terms with } dy/dx \ldots$$

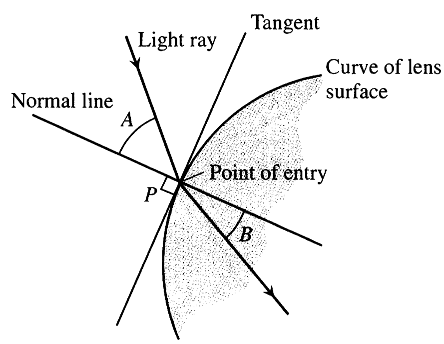
$$(2 - \cos y)\frac{dy}{dx} = 2x \qquad \ldots\text{and factor out } dy/dx.$$

$$\frac{dy}{dx} = \frac{2x}{2 - \cos y} \qquad \text{Solve for } dy/dx \text{ by dividing.}$$

❑

Lenses, Tangents, and Normal Lines

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles A and B in Fig. 2.48). This line is called the *normal* to the surface at the point of entry. In a profile view of a lens like the one in Fig. 2.48, the normal is the line perpendicular to the tangent to the profile curve at the point of entry.



2.48 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

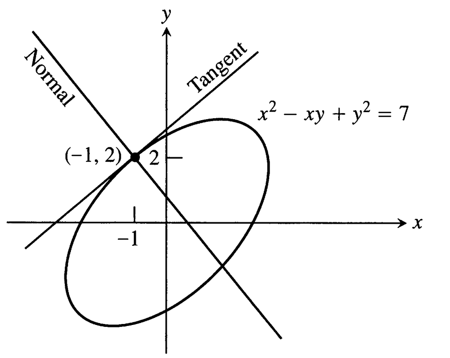
The word *normal*

When analytic geometry was developed in the seventeenth century, European scientists still wrote about their work and ideas in Latin, the one language that all educated Europeans could read and understand. The word *normalis,* which scholars used for "perpendicular" in Latin, became *normal* when they discussed geometry in English.

Definition

A line is **normal** to a curve at a point if it is perpendicular to the curve's tangent there. The line is called the **normal** to the curve at that point.

The profiles of lenses are often described by quadratic curves. When they are, we can use implicit differentiation to find the tangents and normals.

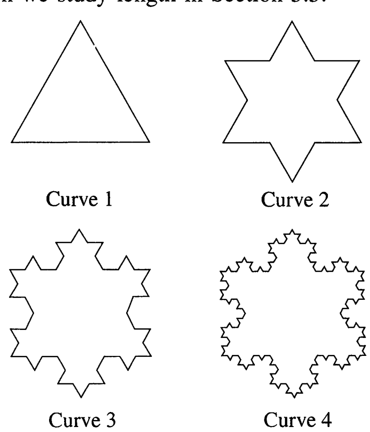


2.49 The graph of $x^2 - xy + y^2 = 7$ is an ellipse. Example 4 shows how to find equations for the tangent and normal lines at the point $(-1, 2)$.

Helga von Koch's snowflake curve (1904)

Start with an equilateral triangle, calling it curve 1. On the middle third of each side, build an equilateral triangle pointing outward. Then erase the interiors of the old middle thirds. Call the expanded curve curve 2. Now put equilateral triangles, again pointing outward, on the middle thirds of the sides of curve 2. Erase the interiors of the old middle thirds to make curve 3. Repeat the process, as shown, to define an infinite sequence of plane curves. The limit curve of the sequence is Koch's snowflake curve.

The snowflake curve is too rough to have a tangent at any point. In other words, the equation $F(x, y) = 0$ defining the curve does not define y as a differentiable function of x or x as a differentiable function of y at any point. We will encounter the snowflake again when we study length in Section 5.5.



EXAMPLE 4 Find the tangent and normal to the curve $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$ (Fig. 2.49).

Solution We first use implicit differentiation to find dy/dx:

$$x^2 - xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7) \qquad \text{Differentiate both sides with respect to } x, \ldots$$

$$2x - \left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + 2y\frac{dy}{dx} = 0 \qquad \ldots \text{ treating } xy \text{ as a product and } y \text{ as a function of } x.$$

$$(2y - x)\frac{dy}{dx} = y - 2x \qquad \text{Collect terms.}$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}. \qquad \text{Solve for } dy/dx.$$

We then evaluate the derivative at $(x, y) = (-1, 2)$ to obtain

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = \left.\frac{y - 2x}{2y - x}\right|_{(-1,2)} = \frac{2 - 2(-1)}{2(2) - (-1)} = \frac{4}{5}.$$

The tangent to the curve at $(-1, 2)$ is the line

$$y = 2 + \frac{4}{5}(x - (-1))$$

$$y = \frac{4}{5}x + \frac{14}{5}.$$

The normal to the curve at $(-1, 2)$ is

$$y = 2 - \frac{5}{4}(x - (-1))$$

$$y = -\frac{5}{4}x + \frac{3}{4}.$$

❑

Using Implicit Differentiation to Find Derivatives of Higher Order

Implicit differentiation can also produce derivatives of higher order.

EXAMPLE 5 Find d^2y/dx^2 if $2x^3 - 3y^2 = 7$.

Solution To start, we differentiate both sides of the equation with respect to x to find $y' = dy/dx$:

$$2x^3 - 3y^2 = 7$$

$$\frac{d}{dx}(2x^3) - \frac{d}{dx}(3y^2) = \frac{d}{dx}(7)$$

$$6x^2 - 6yy' = 0$$

$$x^2 - yy' = 0$$

$$y' = \frac{x^2}{y} \qquad (\text{if } y \neq 0).$$

We differentiate the equation $x^2 - yy' = 0$ again to find y'':

$$\frac{d}{dx}(x^2 - yy') = \frac{d}{dx}(0)$$

$$2x - y'y' - yy'' = 0 \qquad \text{Product Rule with } u = y,\ v = y'$$

$$yy'' = 2x - (y')^2$$

$$y'' = \frac{2x}{y} - \frac{(y')^2}{y} \qquad (y \neq 0).$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y:

$$y'' = \frac{2x}{y} - \frac{(x^2/y)^2}{y} = \frac{2x}{y} - \frac{x^4}{y^3} \qquad (y \neq 0).$$

❑

Rational Powers of Differentiable Functions

We know that the Power Rule

$$\frac{d}{dx}x^n = nx^{n-1} \tag{2}$$

holds when n is an integer. We can now show that it holds when n is any rational number.

Theorem 6
Power Rule for Rational Powers

If n is a rational number, then x^n is differentiable at every interior point x of the domain of x^{n-1}, and

$$\frac{d}{dx}x^n = nx^{n-1}. \tag{3}$$

Proof Let p and q be integers with $q > 0$ and suppose that $y = \sqrt[q]{x^p} = x^{p/q}$. Then

$$y^q = x^p.$$

This equation is an algebraic combination of powers of x and y, so the advanced theorem we mentioned at the beginning of the section assures us that y is a differentiable function of x. Since p and q are integers (for which we already have the Power Rule), we can differentiate both sides of the equation implicitly with respect to x and obtain

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}. \tag{4}$$

If $y \neq 0$, we can then divide both sides of Eq. (4) by qy^{q-1} to solve for dy/dx, obtaining

$$\frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}} \qquad \text{Eq. (4) divided by } qy^{q-1}$$

$$= \frac{p}{q} \cdot \frac{x^{p-1}}{(x^{(p/q)})^{q-1}} \qquad y = x^{p/q}$$

$$= \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-p/q}} \qquad \frac{p}{q}(q-1) = p - \frac{p}{q}$$

$$= \frac{p}{q} \cdot x^{(p-1)-(p-p/q)} \qquad \text{A law of exponents}$$

$$= \frac{p}{q} \cdot x^{(p/q)-1}.$$

This proves the rule. ❑

EXAMPLE 6

a) $\dfrac{d}{dx}(x^{1/2}) = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$ Eq. (3) with $n = \frac{1}{2}$

function defined for $x \geq 0$ — derivative defined only for $x > 0$

b) $\dfrac{d}{dx}(x^{1/5}) = \dfrac{1}{5}x^{-4/5}$ Eq. (3) with $n = \frac{1}{5}$

function defined for all x — derivative not defined at $x = 0$ ❑

A version of the Power Rule with a built-in application of the Chain Rule states that if n is a rational number, u is differentiable at x, and $(u(x))^{n-1}$ is defined, then u^n is differentiable at x, and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \qquad (5)$$

EXAMPLE 7

a) $\dfrac{d}{dx}(1-x^2)^{1/4} = \dfrac{1}{4}(1-x^2)^{-3/4}(-2x)$ Eq. (5) with $u = 1 - x^2$ and $n = 1/4$

function defined on $[-1, 1]$

$$= \frac{-x}{2(1-x^2)^{3/4}}$$

derivative defined only on $(-1, 1)$

b) $$\frac{d}{dx}(\cos x)^{-1/5} = -\frac{1}{5}(\cos x)^{-6/5}\frac{d}{dx}(\cos x)$$

$$= -\frac{1}{5}(\cos x)^{-6/5}(-\sin x)$$

$$= \frac{1}{5}\sin x(\cos x)^{-6/5}$$ ❑

Exercises 2.6

Derivatives of Rational Powers

Find dy/dx in Exercises 1–10.

1. $y = x^{9/4}$
2. $y = x^{-3/5}$
3. $y = \sqrt[3]{2x}$
4. $y = \sqrt[4]{5x}$
5. $y = 7\sqrt{x+6}$
6. $y = -2\sqrt{x-1}$
7. $y = (2x+5)^{-1/2}$
8. $y = (1-6x)^{2/3}$
9. $y = x(x^2+1)^{1/2}$
10. $y = x(x^2+1)^{-1/2}$

Find the first derivatives of the functions in Exercises 11–18.

11. $s = \sqrt[7]{t^2}$
12. $r = \sqrt[4]{\theta^{-3}}$
13. $y = \sin\left[(2t+5)^{-2/3}\right]$
14. $z = \cos\left[(1-6t)^{2/3}\right]$
15. $f(x) = \sqrt{1-\sqrt{x}}$
16. $g(x) = 2(2x^{-1/2}+1)^{-1/3}$
17. $h(\theta) = \sqrt[3]{1+\cos(2\theta)}$
18. $k(\theta) = (\sin(\theta+5))^{5/4}$

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 19–32.

19. $x^2y + xy^2 = 6$
20. $x^3 + y^3 = 18xy$
21. $2xy + y^2 = x + y$
22. $x^3 - xy + y^3 = 1$
23. $x^2(x-y)^2 = x^2 - y^2$
24. $(3xy+7)^2 = 6y$
25. $y^2 = \dfrac{x-1}{x+1}$
26. $x^2 = \dfrac{x-y}{x+y}$
27. $x = \tan y$
28. $x = \sin y$
29. $x + \tan(xy) = 0$
30. $x + \sin y = xy$
31. $y \sin\left(\dfrac{1}{y}\right) = 1 - xy$
32. $y^2 \cos\left(\dfrac{1}{y}\right) = 2x + 2y$

Find $dr/d\theta$ in Exercises 33–36.

33. $\theta^{1/2} + r^{1/2} = 1$
34. $r - 2\sqrt{\theta} = \dfrac{3}{2}\theta^{2/3} + \dfrac{4}{3}\theta^{3/4}$
35. $\sin(r\theta) = \dfrac{1}{2}$
36. $\cos r + \cos\theta = r\theta$

Higher Derivatives

In Exercises 37–42, use implicit differentiation to find dy/dx and then d^2y/dx^2.

37. $x^2 + y^2 = 1$
38. $x^{2/3} + y^{2/3} = 1$
39. $y^2 = x^2 + 2x$
40. $y^2 - 2x = 1 - 2y$
41. $2\sqrt{y} = x - y$
42. $xy + y^2 = 1$
43. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2, 2)$.
44. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

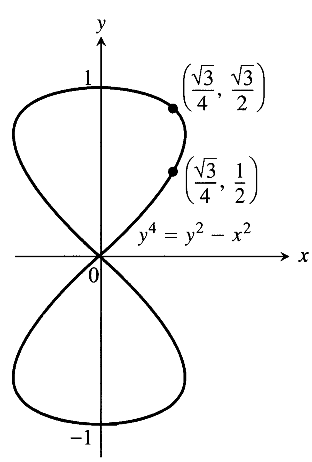
Slopes, Tangents, and Normals

In Exercises 45 and 46, find the slope of the curve at the given points.

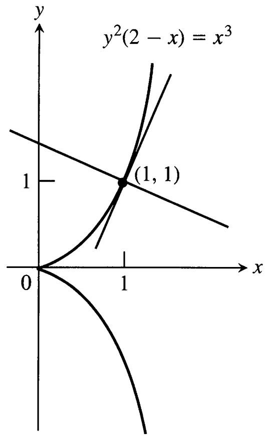
45. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$
46. $(x^2+y^2)^2 = (x-y)^2$ at $(1, 0)$ and $(1, -1)$

In Exercises 47–56, verify that the given point is on the curve and find the lines that are (a) tangent and (b) normal to the curve at the given point.

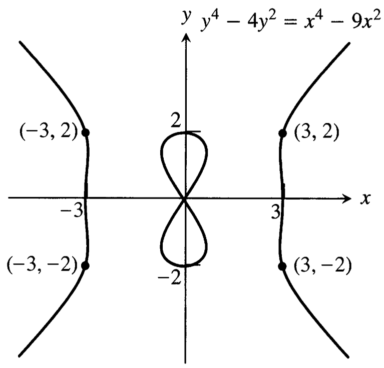
47. $x^2 + xy - y^2 = 1, \quad (2, 3)$
48. $x^2 + y^2 = 25, \quad (3, -4)$
49. $x^2y^2 = 9, \quad (-1, 3)$
50. $y^2 - 2x - 4y - 1 = 0, \quad (-2, 1)$
51. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0, \quad (-1, 0)$
52. $x^2 - \sqrt{3}xy + 2y^2 = 5, \quad (\sqrt{3}, 2)$
53. $2xy + \pi \sin y = 2\pi, \quad (1, \pi/2)$
54. $x \sin 2y = y \cos 2x, \quad (\pi/4, \pi/2)$
55. $y = 2\sin(\pi x - y), \quad (1, 0)$
56. $x^2 \cos^2 y - \sin y = 0, \quad (0, \pi)$
57. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?
58. Find points on the curve $x^2 + xy + y^2 = 7$ (a) where the tangent is parallel to the x-axis and (b) where the tangent is parallel to the y-axis. In the latter case, dy/dx is not defined, but dx/dy is. What value does dx/dy have at these points?
59. *The eight curve.* Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.



60. *The cissoid of Diocles (from about 200 B.C.).* Find equations for the tangent and normal to the cissoid of Diocles $y^2(2-x)=x^3$ at $(1, 1)$.



61. *The devil's curve (Gabriel Cramer [the Cramer of Cramer's rule], 1750).* Find the slopes of the devil's curve $y^4-4y^2=x^4-9x^2$ at the four indicated points.



62. *The folium of Descartes.* (See Fig. 2.45.)

a) Find the slope of the folium of Descartes, $x^3+y^3-9xy=0$ at the points (4, 2) and (2, 4).

b) At what point other than the origin does the folium have a horizontal tangent?

c) Find the coordinates of the point A in Fig. 2.45, where the folium has a vertical tangent.

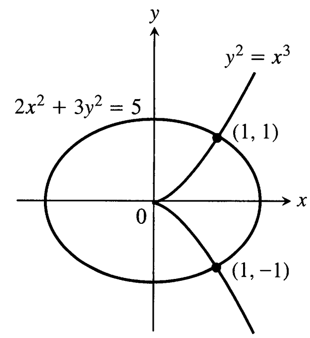
Theory and Examples

63. Which of the following could be true if $f''(x)=x^{-1/3}$?

a) $f(x)=\frac{3}{2}x^{2/3}-3$ **b)** $f(x)=\frac{9}{10}x^{5/3}-7$

c) $f'''(x)=-\frac{1}{3}x^{-4/3}$ **d)** $f'(x)=\frac{3}{2}x^{2/3}+6$

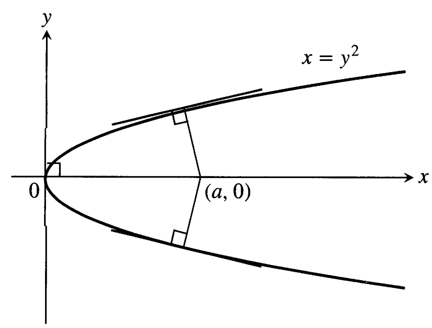
64. Is there anything special about the tangents to the curves $2x^2+3y^2=5$ and $y^2=x^3$ at the points $(1, \pm 1)$? Give reasons for your answer.



65. The line that is normal to the curve $x^2+2xy-3y^2=0$ at $(1, 1)$ intersects the curve at what other point?

66. Find the normals to the curve $xy+2x-y=0$ that are parallel to the line $2x+y=0$.

67. Show that if it is possible to draw these three normals from the point $(a, 0)$ to the parabola $x=y^2$ shown here, then a must be greater than $1/2$. One of the normals is the x-axis. For what value of a are the other two normals perpendicular?



68. What is the geometry behind the restrictions on the domains of the derivatives in Example 6 and Example 7(a)?

In Exercises 69 and 70 find both dy/dx (treating y as a function of x) and dx/dy (treating x as a function of y). How do dy/dx and dx/dy seem to be related? Can you explain the relationship geometrically in terms of the graphs?

69. $xy^3+x^2y=6$ **70.** $x^3+y^2=\sin^2 y$

Grapher Explorations

71. a) Given that $x^4+4y^2=1$, find dy/dx two ways: (1) by solving for y and differentiating the resulting functions in the usual way and (2) by implicit differentiation. Do you get the same result each way?

b) Solve the equation $x^4 + 4y^2 = 1$ for y and graph the resulting functions together to produce a complete graph of the equation $x^4 + 4y^2 = 1$. Then add the graphs of the first derivatives of these functions to your display. Could you have predicted the general behavior of the derivative graphs from looking at the graph of $x^4 + 4y^2 = 1$? Could you have predicted the general behavior of the graph of $x^4 + 4y^2 = 1$ by looking at the derivative graphs? Give reasons for your answers.

72. a) Given that $(x-2)^2 + y^2 = 4$, find dy/dx two ways: (1) by solving for y and differentiating the resulting functions with respect to x and (2) by implicit differentiation. Do you get the same result each way?

b) Solve the equation $(x-2)^2 + y^2 = 4$ for y and graph the resulting functions together to produce a complete graph of the equation $(x-2)^2 + y^2 = 4$. Then add the graphs of the functions' first derivatives to your picture. Could you have predicted the general behavior of the derivative graphs from looking at the graph of $(x-2)^2 + y^2 = 4$? Could you have predicted the general behavior of the graph of $(x-2)^2 + y^2 = 4$ by looking at the derivative graphs? Give reasons for your answers.

CAS Explorations and Projects

Use a CAS to perform the following steps in Exercises 73–80.

a) Plot the equation with the implicit plotter of CAS. Check to see that the given point P satisfies the equation.

b) Using implicit differentiation find a formula for the derivative dy/dx and evaluate it at the given point P.

c) Use the slope found in part (b) to define the equation of the tangent line to the curve at P. Then plot the implicit curve and tangent line together on a single graph.

73. $x^3 - xy + y^3 = 7, \quad P(2, 1)$

74. $x^5 + y^3x + yx^2 + y^4 = 4, \quad P(1, 1)$

75. $y^2 + y = \dfrac{2+x}{1-x}, \quad P(0, 1)$

76. $y^3 + \cos xy = x^2, \quad P(1, 0)$

77. $x + \tan\left(\dfrac{y}{x}\right) = 2, \quad P\left(1, \dfrac{\pi}{4}\right)$

78. $xy^3 + \tan(x + y) = 1, \quad P\left(\dfrac{\pi}{4}, 0\right)$

79. $2y^2 + (xy)^{1/3} = x^2 + 2, \quad P(1, 1)$

80. $x\sqrt{1+2y} + y = x^2, \quad P(1, 0)$

2.7 Related Rates of Change

How rapidly will the fluid level inside a vertical cylindrical storage tank drop if we pump the fluid out at the rate of 3000 L/min?

A question like this asks us to calculate a rate that we cannot measure directly from a rate that we can. To do so, we write an equation that relates the variables involved and differentiate it to get an equation that relates the rate we seek to the rate we know.

EXAMPLE 1 *Pumping out a tank*

How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L/min?

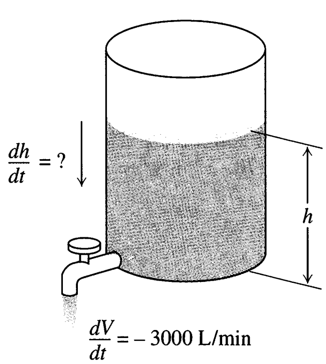
Solution We draw a picture of a partially filled vertical cylindrical tank, calling its radius r and the height of the fluid h (Fig. 2.50). Call the volume of the fluid V.

As time passes, the radius remains constant, but V and h change. We think of V and h as differentiable functions of time and use t to represent time. We are told that

$$\frac{dV}{dt} = -3000. \qquad \text{We pump out at the rate of 3000 L/min. The rate is negative because the volume is decreasing.}$$

We are asked to find

$$\frac{dh}{dt}. \qquad \text{How fast will the fluid level drop?}$$



2.50 The cylindrical tank in Example 1.

Reminder

Rates of change are represented by derivatives. If a quantity is increasing, its derivative with respect to time is positive; if a quantity is decreasing, its derivative is negative.

To find dh/dt, we first write an equation that relates h to V. The equation depends on the units chosen for V, r, and h. With V in liters and r and h in meters, the appropriate equation for the cylinder's volume is

$$V = 1000\pi r^2 h$$

because a cubic meter contains 1000 liters.

Since V and h are differentiable functions of t, we can differentiate both sides of the equation $V = 1000\pi r^2 h$ with respect to t to get an equation that relates dh/dt to dV/dt:

$$\frac{dV}{dt} = 1000\pi r^2 \frac{dh}{dt}. \qquad r \text{ is a constant.}$$

We substitute the known value $dV/dt = -3000$ and solve for dh/dt:

$$\frac{dh}{dt} = \frac{-3000}{1000\pi r^2} = -\frac{3}{\pi r^2}. \tag{1}$$

The fluid level will drop at the rate of $3/(\pi r^2)$ m/min. ❑

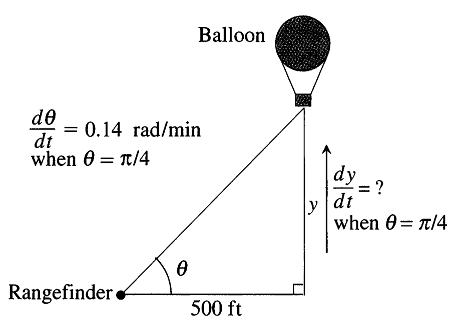
Equation (1) shows how the rate at which the fluid level drops depends on the tank's radius. If r is small, dh/dt will be large; if r is large, dh/dt will be small.

If $r = 1$ m: $\quad \frac{dh}{dt} = -\frac{3}{\pi} \approx -0.95 \text{ m/min} = -95 \text{ cm/min}$

If $r = 10$ m: $\quad \frac{dh}{dt} = -\frac{3}{100\pi} \approx -0.0095 \text{ m/min} = -0.95 \text{ cm/min}$

EXAMPLE 2 *A rising balloon*

A hot-air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?



2.51 The balloon in Example 2.

Solution We answer the question in six steps.

Step 1: *Draw a picture and name the variables and constants* (Fig. 2.51). The variables in the picture are

θ = the angle the range finder makes with the ground (radians)

y = the height of the balloon (feet).

We let t represent time and assume θ and y to be differentiable functions of t.

The one constant in the picture is the distance from the range finder to the lift-off point (500 ft). There is no need to give it a special symbol.

Step 2: *Write down the additional numerical information.*

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \qquad \text{when} \qquad \theta = \frac{\pi}{4}$$

Step 3: *Write down what we are asked to find.* We want dy/dt when $\theta = \pi/4$.

Step 4: *Write an equation that relates the variables y and θ.*

$$\frac{y}{500} = \tan\theta, \qquad \text{or} \qquad y = 500\tan\theta$$

Step 5: *Differentiate with respect to t using the Chain Rule.* The result tells how dy/dt (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 500 \sec^2 \theta \frac{d\theta}{dt}$$

Step 6: *Evaluate with* $\theta = \pi/4$ *and* $d\theta/dt = 0.14$ *to find* dy/dt.

$$\frac{dy}{dt} = 500(\sqrt{2})^2(0.14) = (1000)(0.14) = 140 \qquad \sec\frac{\pi}{4} = \sqrt{2}$$

At the moment in question, the balloon is rising at the rate of 140 ft/min. ❑

Strategy for Solving Related Rate Problems

1. *Draw a picture and name the variables and constants.* Use t for time. Assume all variables are differentiable functions of t.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables.* You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variable whose rate you know.
5. *Differentiate with respect to t.* Then express the rate you want in terms of the rate and variables whose values you know.
6. *Evaluate.* Use known values to find the unknown rate.

EXAMPLE 3 *A highway chase*

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solution We carry out the steps of the basic strategy.

Step 1: *Picture and variables.* We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the southbound highway (Fig. 2.52). We let t represent time and set

x = position of car at time t,

y = position of cruiser at time t,

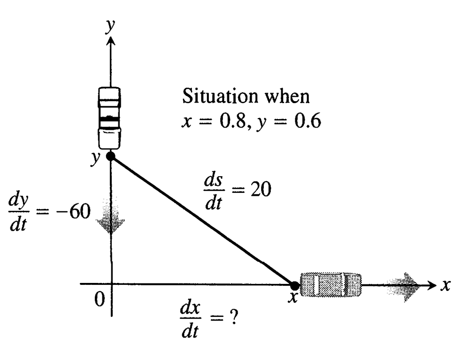
s = distance between car and cruiser at time t.

We assume x, y, and s to be differentiable functions of t.

Step 2: *Numerical information.* At the instant in question,

$$x = 0.8 \text{ mi}, \qquad y = 0.6 \text{ mi}, \qquad \frac{dy}{dt} = -60 \text{ mph}, \qquad \frac{ds}{dt} = 20 \text{ mph}.$$

(dy/dt is negative because y is decreasing.)



2.52 Figure for Example 3.

Step 3: *To find:* $\dfrac{dx}{dt}$

Step 4: *How the variables are related:* $s^2 = x^2 + y^2$ Pythagorean theorem
(The equation $s = \sqrt{x^2 + y^2}$ would also work.)

Step 5: *Differentiate with respect to t.*

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \qquad \text{Chain Rule}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

Step 6: *Evaluate, with* $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, *and solve for* dx/dt.

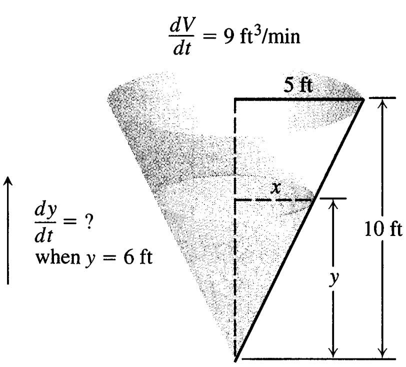
$$20 = \underbrace{\frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}}_{1}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right)$$

$$20 = 0.8\frac{dx}{dt} - 36$$

$$\frac{dx}{dt} = \frac{20 + 36}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph. ❑

EXAMPLE 4 Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?



2.53 The conical tank in Example 4.

Solution We carry out the steps of the basic strategy.

Step 1: *Picture and variables.* We draw a picture of a partially filled conical tank (Fig. 2.53). The variables in the problem are

V = volume (ft^3) of water in the tank at time t (min),

x = radius (ft) of the surface of the water at time t,

y = depth (ft) of water in the tank at time t.

We assume V, x, and y to be differentiable functions of t. The constants are the dimensions of the tank.

Step 2: *Numerical information.* At the time in question,

$$y = 6 \text{ ft}, \qquad \frac{dV}{dt} = 9 \text{ ft}^3/\text{min}.$$

Step 3: *To find:* $\dfrac{dy}{dt}$.

Step 4: *How the variables are related.*

$$V = \frac{1}{3}\pi x^2 y \qquad \text{Cone volume formula} \qquad (2)$$

This equation involves x as well as V and y. Because no information is given about x and dx/dt at the time in question, we need to eliminate x. Using similar triangles (Fig. 2.53) gives us a way to express x in terms of y:

$$\frac{x}{y} = \frac{5}{10}, \quad \text{or} \quad x = \frac{y}{2}.$$

Therefore,

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3. \tag{3}$$

Step 5: *Differentiate with respect to t.* We differentiate Eq. (3), getting

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3y^2 \frac{dy}{dt} = \frac{\pi}{4}y^2 \frac{dy}{dt}. \tag{4}$$

We then solve for dy/dt to express the rate we want (dy/dt) in terms of the rate we know (dV/dt):

$$\frac{dy}{dt} = \frac{4}{\pi y^2}\frac{dV}{dt}.$$

Step 6: *Evaluate, with* $y = 6$ *and* $dV/dt = 9$.

$$\frac{dy}{dt} = \frac{4}{\pi(6)^2} \cdot 9 = \frac{1}{\pi} \approx 0.32 \text{ ft/min}$$

At the moment in question, the water level is rising at about 0.32 ft/min. ❑

Exercises 2.7

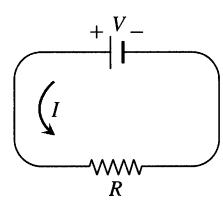
1. Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t. Write an equation that relates dA/dt to dr/dt.

2. Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of t. Write an equation that relates dS/dt to dr/dt.

3. The radius r and height h of a right circular cylinder are related to the cylinder's volume V by the formula $V = \pi r^2 h$.
   a) How is dV/dt related to dh/dt if r is constant?
   b) How is dV/dt related to dr/dt if h is constant?
   c) How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

4. The radius r and height h of a right circular cone are related to the cone's volume V by the equation $V = (1/3)\pi r^2 h$.
   a) How is dV/dt related to dh/dt if r is constant?
   b) How is dV/dt related to dr/dt if h is constant?
   c) How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

5. *Changing voltage.* The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of 1/3 amp/sec. Let t denote time in seconds.



   a) What is the value of dV/dt?
   b) What is the value of dI/dt?
   c) What equation relates dR/dt to dV/dt and dI/dt?
   d) Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing, or decreasing?

6. The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current i (amperes) by the equation $P = Ri^2$.
   a) How are dP/dt, dR/dt, and di/dt related if none of P, R, and i are constant?
   b) How is dR/dt related to di/dt if P is constant?

7. Let x and y be differentiable functions of t and let $s = \sqrt{x^2 + y^2}$

be the distance between the points $(x, 0)$ and $(0, y)$ in the xy-plane.

a) How is ds/dt related to dx/dt if y is constant?
b) How is ds/dt related to dx/dt and dy/dt if neither x nor y is constant?
c) How is dx/dt related to dy/dt if s is constant?

8. If x, y, and z are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$.

a) Assuming that x, y, and z are differentiable functions of t, how is ds/dt related to dx/dt, dy/dt, and dz/dt?
b) How is ds/dt related to dy/dt and dz/dt if x is constant?
c) How are dx/dt, dy/dt, and dz/dt related if s is constant?

9. The area A of a triangle with sides of lengths a and b enclosing an angle of measure θ is

$$A = \frac{1}{2}ab\sin\theta.$$

a) How is dA/dt related to $d\theta/dt$ if a and b are constant?
b) How is dA/dt related to $d\theta/dt$ and da/dt if only b is constant?
c) How is dA/dt related to $d\theta/dt$, da/dt, and db/dt if none of a, b, and θ are constant?

10. *Heating a plate.* When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 50 cm?

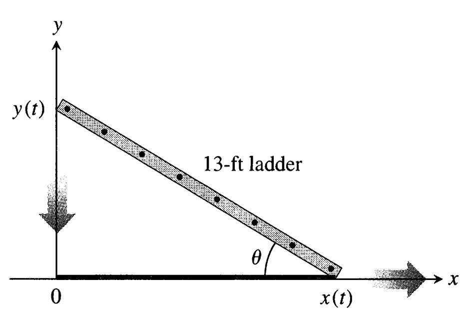
11. *Changing dimensions in a rectangle.* The length l of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $l = 12$ cm and $w = 5$ cm, find the rates of change of (a) the area, (b) the perimeter, and (c) the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

12. *Changing dimensions in a rectangular box.* Suppose that the edge lengths x, y, and z of a closed rectangular box are changing at the following rates:

$$\frac{dx}{dt} = 1 \text{ m/sec}, \qquad \frac{dy}{dt} = -2 \text{ m/sec}, \qquad \frac{dz}{dt} = 1 \text{ m/sec}.$$

Find the rates at which the box's (a) volume, (b) surface area, and (c) diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

13. *A sliding ladder.* A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.



a) How fast is the top of the ladder sliding down the wall then?
b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?
c) At what rate is the angle θ between the ladder and the ground changing then?

14. *Commercial air traffic.* Two commercial airplanes are flying at 40,000 ft along straight-line courses that intersect at right angles. Plane A is approaching the intersection point at a speed of 442 knots (nautical miles per hour; a nautical mile is 2000 yd). Plane B is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when A is 5 nautical miles from the intersection point and B is 12 nautical miles from the intersection point?

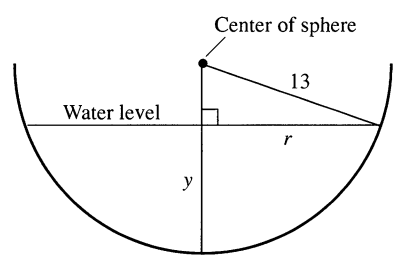
15. *Flying a kite.* A girl flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?

16. *Boring a cylinder.* The mechanics at Lincoln Automotive are reboring a 6-in.-deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.?

17. *A growing sand pile.* Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the (a) height and (b) radius changing when the pile is 4 m high? Answer in cm/min.

18. *A draining conical reservoir.* Water is flowing at the rate of 50 m^3/min from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m. (a) How fast is the water level falling when the water is 5 m deep? (b) How fast is the radius of the water's surface changing then? Answer in cm/min.

19. *A draining hemispherical reservoir.* Water is flowing at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y units deep.



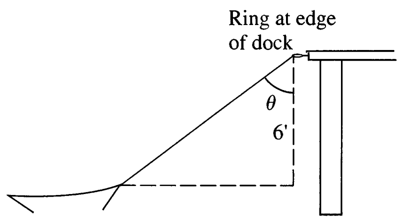
a) At what rate is the water level changing when the water is 8 m deep?
b) What is the radius r of the water's surface when the water is y m deep?

c) At what rate is the radius r changing when the water is 8 m deep?

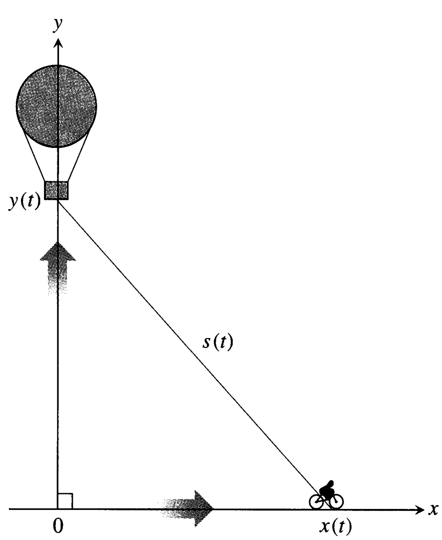
20. *A growing raindrop.* Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.

21. *The radius of an inflating balloon.* A spherical balloon is inflated with helium at the rate of 100π ft^3/min. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing?

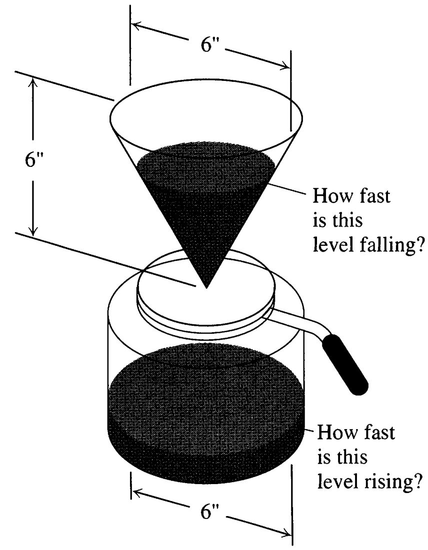
22. *Hauling in a dinghy.* A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. The rope is hauled in at the rate of 2 ft/sec. (a) How fast is the boat approaching the dock when 10 ft of rope are out? (b) At what rate is angle θ changing then (see the figure)?



23. *A balloon and a bicycle.* A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance between the bicycle and balloon increasing 3 sec later?



24. *Making coffee.* Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min. (a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep? (b) How fast is the level in the cone falling then?



25. *Cardiac output.* In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würtzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 liters a minute. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where Q is the number of milliliters of CO_2 you exhale in a minute and D is the difference between the CO_2 concentration (ml/L) in the blood pumped to the lungs and the CO_2 concentration in the blood returning from the lungs. With $Q = 233$ ml/min and $D = 97 - 56 = 41$ ml/L,

$$y = \frac{233 \text{ ml/min}}{41 \text{ ml/L}} \approx 5.68 \text{ L/min},$$

fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. What is happening to the cardiac output?

26. *Cost, revenue, and profit.* A company can manufacture x items at a cost of $c(x)$ dollars, a sales revenue of $r(x)$ dollars, and a profit of $p(x) = r(x) - c(x)$ dollars (everything in thousands). Find dc/dt, dr/dt, and dp/dt for the following values of x and dx/dt.

a) $r(x) = 9x$, $c(x) = x^3 - 6x^2 + 15x$, and $dx/dt = 0.1$ when $x = 2$

b) $r(x) = 70x$, $c(x) = x^3 - 6x^2 + 45/x$, and $dx/dt = 0.05$ when $x = 1.5$

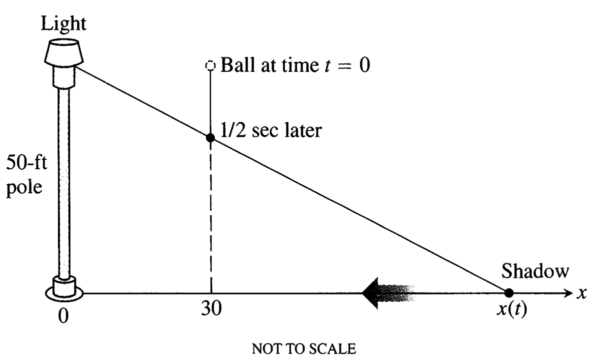
27. *Moving along a parabola.* A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (measured in meters) increases at a steady 10 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ m?

28. *Moving along another parabola.* A particle moves from right to left along the parabola $y = \sqrt{-x}$ in such a way that its x-coordinate (measured in meters) decreases at the rate of 8 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = -4$?

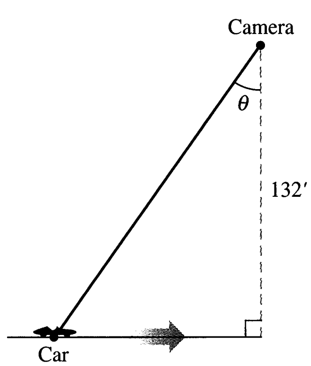
29. *Motion in the plane.* The coordinates of a particle in the metric xy-plane are differentiable functions of time t with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

30. *A moving shadow.* A man 6 ft tall walks at the rate of 5 ft/sec toward a streetlight that is 16 ft above the ground. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing when he is 10 ft from the base of the light?

31. *Another moving shadow.* A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light. How fast is the shadow of the ball moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ ft in t sec.)



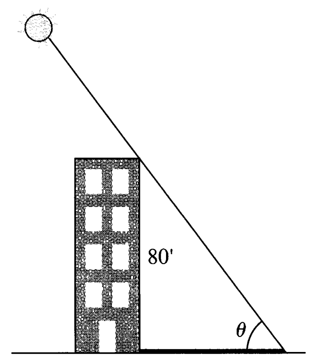
32. You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mph (264 ft/sec). How fast will your camera angle θ be changing when the car is right in front of you? A half second later?



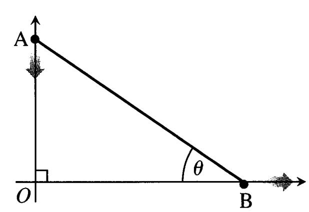
33. *A melting ice layer.* A spherical iron ball 8 in. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of 10 in^3/min, how fast is the thickness of the ice decreasing when it is 2 in. thick? How fast is the outer surface area of ice decreasing?

34. *Highway patrol.* A highway patrol plane flies 3 mi above a level, straight road at a steady 120 mi/h. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi the line-of-sight distance is decreasing at the rate of 160 mi/h. Find the car's speed along the highway.

35. *A building's shadow.* On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of 0.27°/min. At what rate is the shadow decreasing? (Remember to use radians. Express your answer in inches per minute, to the nearest tenth.)



36. *Walkers.* A and B are walking on straight streets that meet at right angles. A approaches the intersection at 2 m/sec; B moves away from the intersection 1 m/sec. At what rate is the angle θ changing when A is 10 m from the intersection and B is 20 m from the intersection? Express your answer in degrees per second to the nearest degree.

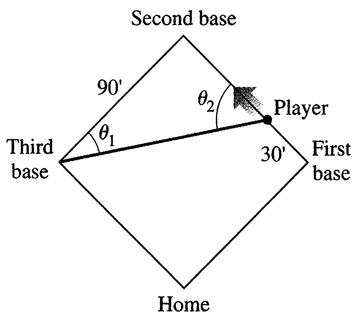


37. A baseball diamond is a square 90 ft on a side. A player runs from first base to second at a rate of 16 ft/sec.

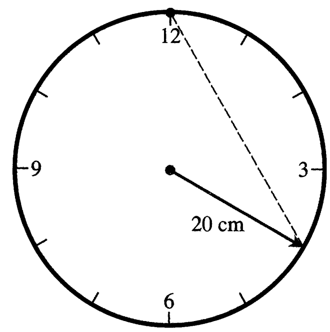
a) At what rate is the player's distance from third base changing when the player is 30 ft from first base?

b) At what rates are angles θ_1 and θ_2 (see the figure) changing at that time?

c) The player slides into second base at the rate of 15 ft/sec. At what rates are angles θ_1 and θ_2 changing as the player touches base?



38. *A second hand.* At what rate is the distance between the tip of the second hand and the 12 o'clock mark changing when the second hand points to 4 o'clock?



39. *Ships.* Two ships are steaming straight away from a point O along routes that make a 120° angle. Ship A moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yd). Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

CHAPTER 2 QUESTIONS TO GUIDE YOUR REVIEW

1. What is the derivative of a function f? How is its domain related to the domain of f? Give examples.
2. What role does the derivative play in defining slopes, tangents, and rates of change?
3. How can you sometimes graph the derivative of a function when all you have is a table of the function's values?
4. What does it mean for a function to be differentiable on an open interval? on a closed interval?
5. How are derivatives and one-sided derivatives related?
6. Describe geometrically when a function typically does *not* have a derivative at a point.
7. How is a function's differentiability at a point related to its continuity there, if at all?
8. Could the unit step function

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

possibly be the derivative of some other function on $[-1, 1]$? Explain.

9. What rules do you know for calculating derivatives? Give some examples.
10. Explain how the three formulas

a) $\dfrac{d}{dx}(x^n) = nx^{n-1}$,

b) $\dfrac{d}{dx}(cu) = c\dfrac{du}{dx}$,

c) $\dfrac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \dfrac{du_1}{dx} + \dfrac{du_2}{dx} + \cdots + \dfrac{du_n}{dx}$

enable us to differentiate any polynomial.

11. What formula do we need, in addition to the three listed in question 10, to differentiate rational functions?
12. What is a second derivative? a third derivative? How many derivatives do the functions you know have? Give examples.
13. What is the relationship between a function's average and instantaneous rates of change? Give an example.
14. How do derivatives arise in the study of motion? What can you learn about a body's motion along a line by examining the derivatives of the body's position function? Give examples.
15. How can derivatives arise in economics?
16. Give examples of still other applications of derivatives.
17. What is the value of $\lim_{\theta \to 0} (\sin\theta)/\theta$? Does it matter whether θ is measured in degrees or radians? Explain.
18. What do the limits $\lim_{h \to 0} (\sin h)/h$ and $\lim_{h \to 0} (\cos h - 1)/h$ have to do with the derivatives of the sine and cosine functions? What *are* the derivatives of these functions?
19. Once you know the derivatives of $\sin x$ and $\cos x$, how can you

find the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$? What *are* the derivatives of these functions?

20. At what points are the six basic trigonometric functions continuous? How do you know?

21. What is the rule for calculating the derivative of a composite of two differentiable functions? How is such a derivative evaluated? Give examples.

22. If u is a differentiable function of x, how do you find $(d/dx)(u^n)$ if n is an integer? if n is a rational number? Give examples.

23. What is implicit differentiation? When do you need it? Give examples.

24. How do related rate problems arise? Give examples.

25. Outline a strategy for solving related rate problems. Illustrate with an example.

CHAPTER 2 PRACTICE EXERCISES

Derivatives of Functions

Find the derivatives of the functions in Exercises 1–36.

1. $y = x^5 - 0.125x^2 + 0.25x$

2. $y = 3 - 0.7x^3 + 0.3x^7$

3. $y = x^3 - 3(x^2 + \pi^2)$

4. $y = x^7 + \sqrt{7}x - \dfrac{1}{\pi + 1}$

5. $y = (x+1)^2(x^2+2x)$

6. $y = (2x-5)(4-x)^{-1}$

7. $y = (\theta^2 + \sec\theta + 1)^3$

8. $y = \left(-1 - \dfrac{\csc\theta}{2} - \dfrac{\theta^2}{4}\right)^2$

9. $s = \dfrac{\sqrt{t}}{1+\sqrt{t}}$

10. $s = \dfrac{1}{\sqrt{t} - 1}$

11. $y = 2\tan^2 x - \sec^2 x$

12. $y = \dfrac{1}{\sin^2 x} - \dfrac{2}{\sin x}$

13. $s = \cos^4(1-2t)$

14. $s = \cot^3\left(\dfrac{2}{t}\right)$

15. $s = (\sec t + \tan t)^5$

16. $s = \csc^5(1 - t + 3t^2)$

17. $r = \sqrt{2\theta \sin\theta}$

18. $r = 2\theta\sqrt{\cos\theta}$

19. $r = \sin\sqrt{2\theta}$

20. $r = \sin(\theta + \sqrt{\theta+1})$

21. $y = \dfrac{1}{2}x^2 \csc\dfrac{2}{x}$

22. $y = 2\sqrt{x}\sin\sqrt{x}$

23. $y = x^{-1/2}\sec(2x)^2$

24. $y = \sqrt{x}\csc(x+1)^3$

25. $y = 5\cot x^2$

26. $y = x^2 \cot 5x$

27. $y = x^2\sin^2(2x^2)$

28. $y = x^{-2}\sin^2(x^3)$

29. $s = \left(\dfrac{4t}{t+1}\right)^{-2}$

30. $s = \dfrac{-1}{15(15t-1)^3}$

31. $y = \left(\dfrac{\sqrt{x}}{1+x}\right)^2$

32. $y = \left(\dfrac{2\sqrt{x}}{2\sqrt{x}+1}\right)^2$

33. $y = \sqrt{\dfrac{x^2+x}{x^2}}$

34. $y = 4x\sqrt{x + \sqrt{x}}$

35. $r = \left(\dfrac{\sin\theta}{\cos\theta - 1}\right)^2$

36. $r = \left(\dfrac{1+\sin\theta}{1-\cos\theta}\right)^2$

In Exercises 37–48, find dy/dx.

37. $y = (2x+1)\sqrt{2x+1}$

38. $y = 20(3x-4)^{1/4}(3x-4)^{-1/5}$

39. $y = \dfrac{3}{(5x^2 + \sin 2x)^{3/2}}$

40. $y = (3 + \cos^3 3x)^{-1/3}$

41. $xy + 2x + 3y = 1$

42. $x^2 + xy + y^2 - 5x = 2$

43. $x^3 + 4xy - 3y^{4/3} = 2x$

44. $5x^{4/5} + 10y^{6/5} = 15$

45. $\sqrt{xy} = 1$

46. $x^2y^2 = 1$

47. $y^2 = \dfrac{x}{x+1}$

48. $y^2 = \sqrt{\dfrac{1+x}{1-x}}$

In Exercises 49 and 50, find dp/dq.

49. $p^3 + 4pq - 3q^2 = 2$

50. $q = (5p^2 + 2p)^{-3/2}$

In Exercises 51 and 52, find dr/ds.

51. $r\cos 2s + \sin^2 s = \pi$

52. $2rs - r - s + s^2 = -3$

53. Find d^2y/dx^2 by implicit differentiation:

a) $x^3 + y^3 = 1$

b) $y^2 = 1 - \dfrac{2}{x}$

54. **a)** By differentiating $x^2 - y^2 = 1$ implicitly, show that $dy/dx = x/y$.

b) Then show that $d^2y/dx^2 = -1/y^3$.

Numerical Values of Derivatives

55. Suppose that functions $f(x)$ and $g(x)$ and their first derivatives have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Find the first derivatives of the following combinations at the given value of x.

a) $5f(x) - g(x), \quad x = 1$
b) $f(x)g^3(x), \quad x = 0$
c) $\dfrac{f(x)}{g(x)+1}, \quad x = 1$
d) $f(g(x)), \quad x = 0$
e) $g(f(x)), \quad x = 0$
f) $(x + f(x))^{3/2}, \quad x = 1$
g) $f(x + g(x)), \quad x = 0$

56. Suppose that the function $f(x)$ and its first derivative have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$f'(x)$
0	9	−2
1	−3	1/5

Find the first derivatives of the following combinations at the given value of x.

a) $\sqrt{x}f(x), \quad x = 1$
b) $\sqrt{f(x)}, \quad x = 0$
c) $f\left(\sqrt{x}\right), \quad x = 1$
d) $f(1 - 5\tan x), \quad x = 0$
e) $\dfrac{f(x)}{2 + \cos x}, \quad x = 0$
f) $10\sin\left(\dfrac{\pi x}{2}\right) f^2(x), \quad x = 1$

57. Find the value of dy/dt at $t = 0$ if $y = 3\sin 2x$ and $x = t^2 + \pi$.

58. Find the value of ds/du at $u = 2$ if $s = t^2 + 5t$ and $t = (u^2 + 2u)^{1/3}$.

59. Find the value of dw/ds at $s = 0$ if $w = \sin(\sqrt{r} - 2)$ and $r = 8\sin(s + \pi/6)$.

60. Find the value of dr/dt at $t = 0$ if $r = (\theta^2 + 7)^{1/3}$ and $\theta^2 t + \theta = 1$.

61. If $y^3 + y = 2\cos x$, find the value of d^2y/dx^2 at the point $(0, 1)$.

62. If $x^{1/3} + y^{1/3} = 4$, find d^2y/dx^2 at the point $(8, 8)$.

Derivative Definition

In Exercises 63 and 64, find the derivative using the definition.

63. $f(t) = \dfrac{1}{2t + 1}$

64. $g(x) = 2x^2 + 1$

65. a) Graph the function

$$f(x) = \begin{cases} x^2, & -1 \le x < 0 \\ -x^2, & 0 \le x \le 1. \end{cases}$$

b) Is f continuous at $x = 0$?
c) Is f differentiable at $x = 0$?

Give reasons for your answers.

66. a) Graph the function

$$f(x) = \begin{cases} x, & -1 \le x < 0 \\ \tan x, & 0 \le x \le \pi/4. \end{cases}$$

b) Is f continuous at $x = 0$?
c) Is f differentiable at $x = 0$?

Give reasons for your answers.

67. a) Graph the function

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$$

b) Is f continuous at $x = 1$?
c) Is f differentiable at $x = 1$?

Give reasons for your answers.

68. For what value or values of the constant m, if any, is

$$f(x) = \begin{cases} \sin 2x, & x \le 0 \\ mx, & x > 0 \end{cases}$$

a) continuous at $x = 0$?
b) differentiable at $x = 0$?

Give reasons for your answers.

Slopes, Tangents, and Normals

69. Are there any points on the curve $y = (x/2) + 1/(2x - 4)$ where the slope is $-3/2$? If so, find them.

70. Are there any points on the curve $y = x - 1/(2x)$ where the slope is 3? If so, find them.

71. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis.

72. Find the x- and y-intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.

73. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is

a) perpendicular to the line $y = 1 - (x/24)$;
b) parallel to the line $y = \sqrt{2} - 12x$.

74. Show that the tangents to the curve $y = (\pi \sin x)/x$ at $x = \pi$ and $x = -\pi$ intersect at right angles.

75. Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the normal is parallel to the line $y = -x/2$. Sketch the curve and normals together, labeling each with its equation.

76. Find equations for the tangent and normal to the curve $y = 1 + \cos x$ at the point $(\pi/2, 1)$. Sketch the curve, tangent, and normal together, labeling each with its equation.

77. The parabola $y = x^2 + C$ is to be tangent to the line $y = x$. Find C.

78. Show that the tangent to the curve $y = x^3$ at any point (a, a^3) meets the curve again at a point where the slope is four times the slope at (a, a^3).

79. For what value of c is the curve $y = c/(x + 1)$ tangent to the line through the points (0, 3) and (5, −2)?

80. Show that the normal line at any point of the circle $x^2 + y^2 = a^2$ passes through the origin.

In Exercises 81–86, find equations for the lines that are tangent and normal to the curve at the given point.

81. $x^2 + 2y^2 = 9$, $(1, 2)$

82. $x^3 + y^2 = 2$, $(1, 1)$

83. $xy + 2x - 5y = 2$, $(3, 2)$

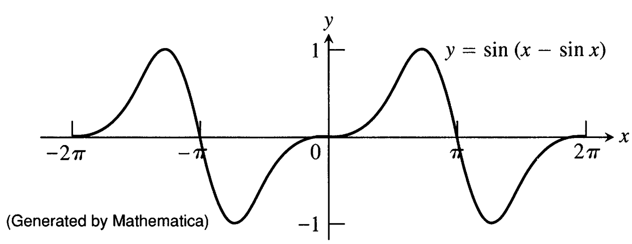
84. $(y - x)^2 = 2x + 4$, $(6, 2)$

85. $x + \sqrt{xy} = 6$, $(4, 1)$

86. $x^{3/2} + 2y^{3/2} = 17$, $(1, 4)$

87. Find the slope of the curve $x^3y^3 + y^2 = x + y$ at the points (1, 1) and (1, −1).

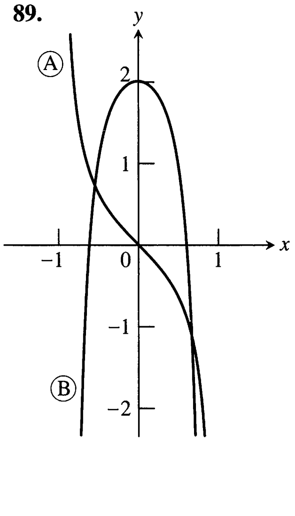
88. The graph below suggests that the curve $y = \sin(x - \sin x)$ might have horizontal tangents at the x-axis. Does it? Give reasons for your answer.



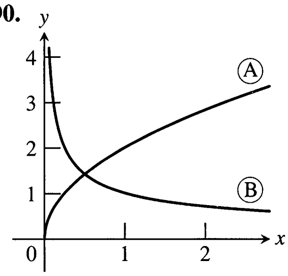
Analyzing Graphs

Each of the figures in Exercises 89 and 90 shows two graphs, the graph of a function $y = f(x)$ together with the graph of its derivative $f'(x)$. Which graph is which? How do you know?

89.

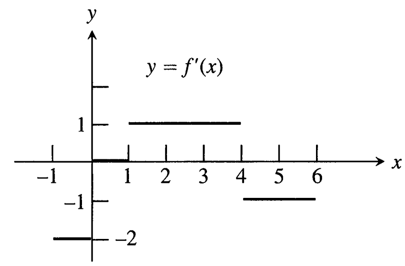


90.



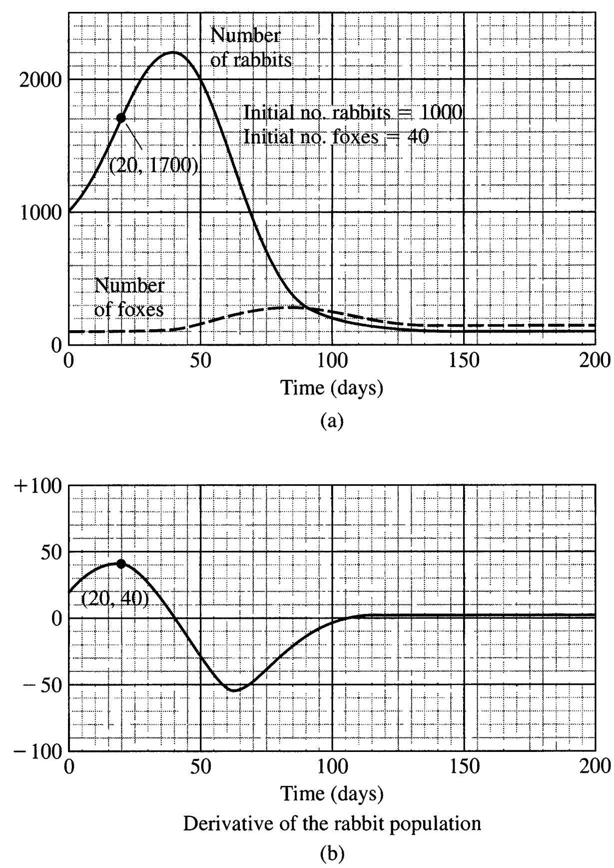
91. Use the following information to graph the function $y = f(x)$ for $-1 \le x \le 6$.

i) The graph of f is made of line segments joined end to end.
ii) The graph starts at the point $(-1, 2)$.
iii) The derivative of f, where defined, agrees with the step function shown here.



92. Repeat Exercise 91, supposing that the graph starts at $(-1, 0)$ instead of $(-1, 2)$.

Exercises 93 and 94 are about the graphs in Fig. 2.54. The graphs in part (a) show the numbers of rabbits and foxes in a small arctic



2.54 Rabbits and foxes in an arctic predator-prey food chain. (Source: *Differentiation* by W. U. Walton et al., Project CALC, Education Development Center, Inc., Newton, Mass, 1975, p. 86.)

population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on the rabbits and, as the number of foxes increases, the rabbit population levels off and then drops. Figure 2.54(b) shows the graph of the derivative of the rabbit population. We made it by plotting slopes, as in Example 4 in Section 2.1.

93. **a)** What is the value of the derivative of the rabbit population in Fig. 2.54 when the number of rabbits is largest? smallest?
b) What is the size of the rabbit population in Fig. 2.54 when its derivative is largest? smallest?

94. In what units should the slopes of the rabbit and fox population curves be measured?

Limits

Find the limits in Exercises 95–104.

95. $\lim_{s\to 0} \dfrac{\sin(s/2)}{s/3}$

96. $\lim_{\theta\to -\pi} \dfrac{\sin^2(\theta+\pi)}{\theta+\pi}$

97. $\lim_{x\to 0} \dfrac{\sin x}{2x^2 - x}$

98. $\lim_{x\to 0} \dfrac{3x - \tan 7x}{2x}$

99. $\lim_{r\to 0} \dfrac{\sin r}{\tan 2r}$

100. $\lim_{\theta\to 0} \dfrac{\sin(\sin\theta)}{\theta}$

101. $\lim_{\theta\to (\pi/2)^-} \dfrac{4\tan^2\theta + \tan\theta + 1}{\tan^2\theta + 5}$

102. $\lim_{\theta\to 0^+} \dfrac{1 - 2\cot^2\theta}{5\cot^2\theta - 7\cot\theta - 8}$

103. $\lim_{x\to 0} \dfrac{x\sin x}{2 - 2\cos x}$

104. $\lim_{\theta\to 0} \dfrac{1 - \cos\theta}{\theta^2}$

Show how to extend the functions in Exercises 105 and 106 to be continuous at the origin.

105. $g(x) = \dfrac{\tan(\tan x)}{\tan x}$

106. $f(x) = \dfrac{\tan(\tan x)}{\sin(\sin x)}$

107. Is there any value of k that will make

$$f(x) = \begin{cases} \dfrac{\sin x}{2x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

continuous at $x = 0$? If so, what is it? Give reasons for your answer.

108. **a)** GRAPHER Graph the function

$$f(x) = \begin{cases} \dfrac{x^2}{\sin^2 2x}, & x \neq 0 \\ c, & x = 0. \end{cases}$$

b) Find a value of c that makes f continuous at $x = 0$. Justify your answer.

Related Rates

109. The total surface area S of a right circular cylinder is related to the base radius r and height h by the equation $S = 2\pi r^2 + 2\pi r h$.
a) How is dS/dt related to dr/dt if h is constant?
b) How is dS/dt related to dh/dt if r is constant?
c) How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?
d) How is dr/dt related to dh/dt if S is constant?

110. The lateral surface area S of a right circular cone is related to the base radius r and height h by the equation $S = \pi r\sqrt{r^2 + h^2}$.
a) How is dS/dt related to dr/dt if h is constant?
b) How is dS/dt related to dh/dt if r is constant?
c) How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?

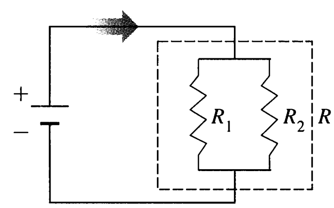
111. The radius of a circle is changing at the rate of $-2/\pi$ m/sec. At what rate is the circle's area changing when $r = 10$ m?

112. The volume of a cube is increasing at the rate of 1200 cm^3/min at the instant its edges are 20 cm long. At what rate are the edges changing at that instant?

113. If two resistors of R_1 and R_2 ohms are connected in parallel in an electric circuit to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

If R_1 is decreasing at the rate of 1 ohm/sec and R_2 is increasing at the rate of 0.5 ohm/sec, at what rate is R changing when $R_1 = 75$ ohms and $R_2 = 50$ ohms?

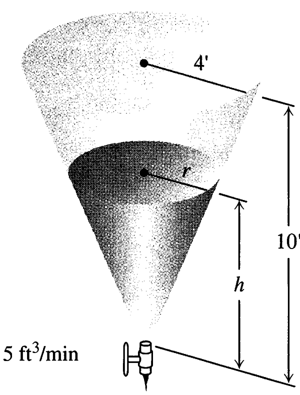


114. The impedance Z (ohms) in a series circuit is related to the resistance R (ohms) and reactance X (ohms) by the equation $Z = \sqrt{R^2 + X^2}$. If R is increasing at 3 ohms/sec and X is decreasing at 2 ohms/sec, at what rate is Z changing when $R = 10$ ohms and $X = 20$ ohms?

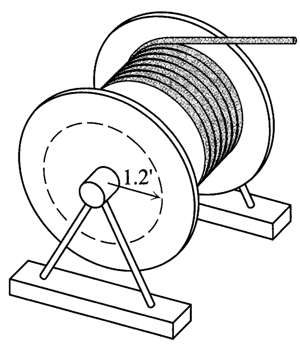
115. The coordinates of a particle moving in the metric xy-plane are differentiable functions of time t with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle approaching the origin as it passes through the point (5, 12)?

116. A particle moves along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second. Find dx/dt when $x = 3$.

117. Water drains from the conical tank shown in Fig. 2.55 at the rate of 5 ft^3/min. (a) What is the relation between the variables h and r in the figure? (b) How fast is the water level dropping when $h = 6$ ft?



2.55 The conical tank in Exercise 117.



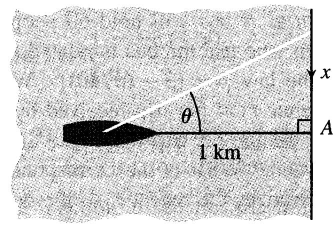
2.56 The television cable in Exercise 118.

118. As television cable is pulled from a large spool to be strung from the telephone poles along a street, it unwinds from the spool in layers of constant radius (see Fig. 2.56). If the truck pulling the cable moves at a steady 6 ft/sec (a touch over 4 mph), use the equation $s = r\theta$ to find how fast (rad/sec) the spool is turning when the layer of radius 1.2 ft is being unwound.

119. The figure below shows a boat 1 km offshore, sweeping the shore with a searchlight. The light turns at a constant rate, $d\theta/dt = -0.6$ rad/sec.

a) How fast is the light moving along the shore when it reaches point A?

b) How many revolutions per minute is 0.6 rad/sec?



120. Points A and B move along the x- and y-axes, respectively, in such a way that the distance r (meters) along the perpendicular from the origin to line AB remains constant. How fast is OA changing, and is it increasing, or decreasing, when $OB = 2r$ and B is moving toward O at the rate of $0.3r$ m/sec?

CHAPTER 2 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

1. An equation like $\sin^2\theta + \cos^2\theta = 1$ is called an **identity** because it holds for all values of θ. An equation like $\sin\theta = 0.5$ is not an identity because it holds only for selected values of θ, not all. If you differentiate both sides of a trigonometric identity in θ with respect to θ, the resulting new equation will also be an identity.

Differentiate the following to show that the resulting equations hold for all θ.

a) $\sin 2\theta = 2\sin\theta\cos\theta$

b) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

2. If the identity $\sin(x+a) = \sin x\cos a + \cos x\sin a$ is differentiated with respect to x, is the resulting equation also an identity? Does this principle apply to the equation $x^2 - 2x - 8 = 0$? Explain.

3. a) Find values for the constants a, b, and c that will make

$$f(x) = \cos x \quad \text{and} \quad g(x) = a + bx + cx^2$$

satisfy the conditions

$$f(0) = g(0), \quad f'(0) = g'(0), \quad \text{and} \quad f''(0) = g''(0).$$

a) Find values for b and c that will make

$$f(x) = \sin(x+a) \quad \text{and} \quad g(x) = b\sin x + c\cos x$$

satisfy the conditions

$$f(0) = g(0) \quad \text{and} \quad f'(0) = g'(0).$$

b) For the determined values of a, b, and c, what happens for the third and fourth derivatives of f and g in each of parts (a) and (b)?

4. a) Show that $y = \sin x$, $y = \cos x$, and $y = a\cos x + b\sin x$ (a and b constants) all satisfy the equation

$$y'' + y = 0.$$

b) How would you modify the functions in (a) to satisfy the equation

$$y'' + 4y = 0?$$

Generalize this result.

5. *An osculating circle.* Find the values of h, k, and a that make the circle $(x-h)^2 + (y-k)^2 = a^2$ tangent to the parabola $y = x^2 + 1$ at the point (1, 2) and that also make the second derivatives d^2y/dx^2 have the same value on both curves there. Circles like this one that are tangent to a curve and have the same second derivative as the curve at the point of tangency are called *osculating circles* (from the Latin *osculari* meaning "to kiss"). We will encounter them again in Chapter 11.

6. *Marginal revenue.* A bus will hold 60 people. The number x of people per trip who use the bus is related to the fare charged (p dollars) by the law $p = [3 - (x/40)]^2$. Write an expression for the total revenue $r(x)$ per trip received by the bus company. What number of people per trip will make the marginal revenue dr/dx equal to zero? What is the corresponding fare? (This is the fare that maximizes the revenue, so the bus company should probably rethink its fare policy.)

7. *Industrial production*

a) Economists often use the expression "rate of growth" in relative rather than absolute terms. For example, let $u = f(t)$ be the number of people in the labor force at time t in a given industry. (We treat this function as though it were differentiable even though it is an integer-valued step function.)

Let $v = g(t)$ be the average production per person in the labor force at time t. The total production is then $y = uv$. If the labor force is growing at the rate of 4% per year ($du/dt = 0.04u$) and the production per worker is growing at the rate of 5% per year ($dv/dt = 0.05v$), find the rate of growth of the total production, y.

b) Suppose that the labor force in (a) is decreasing at the rate of 2% per year while the production per person is increasing at the rate of 3% per year. Is the total production increasing, or is it decreasing, and at what rate?

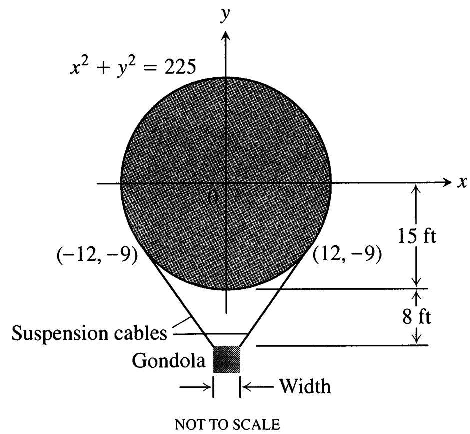
8. The designer of a 30-ft-diameter spherical hot-air balloon wants to suspend the gondola 8 ft below the bottom of the balloon with cables tangent to the surface of the balloon (Fig. 2.57). Two of the cables are shown running from the top edges of the gondola to their points of tangency, $(-12, -9)$ and $(12, -9)$. How wide should the gondola be?

9. *Pisa by parachute.* The accompanying photograph shows Mike McCarthy parachuting from the top of the Tower of Pisa on August 5, 1988. Make a rough sketch to show the shape of the graph of his speed during the jump.

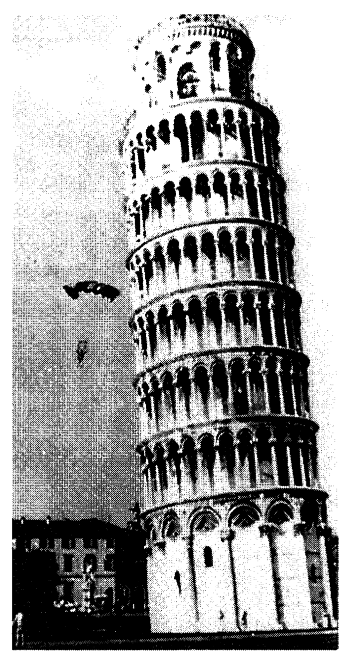
10. The position at time $t \geq 0$ of a particle moving along a coordinate line is

$$s = 10\cos(t + \pi/4).$$

a) What is the particle's starting position ($t = 0$)?



2.57 The balloon and gondola in Exercise 8.



Mike McCarthy of London jumped from the Tower of Pisa and then opened his parachute in what he said was a world record low-level parachute jump of 179 feet. Source: *Boston Globe*, Aug. 6, 1988.

b) What are the points farthest to the left and right of the origin reached by the particle?

c) Find the particle's velocity and acceleration at the points in question (b).

d) When does the particle first reach the origin? What are its velocity, speed, and acceleration then?

11. On Earth, you can easily shoot a paper clip 64 ft straight up into the air with a rubber band. In t seconds after firing, the paper clip is $s = 64t - 16t^2$ ft above your hand.

a) How long does it take the paper clip to reach its maximum height? With what velocity does it leave your hand?

b) On the moon, the same acceleration will send the paper clip to a height of $s = 64t - 2.6t^2$ ft in t seconds. About how long will it take the paper clip to reach its maximum height and how high will it go?

12. At time t sec, the positions of two particles on a coordinate line are $s_1 = 3t^3 - 12t^2 + 18t + 5$ m and $s_2 = -t^3 + 9t^2 - 12t$ m. When do the particles have the same velocities?

13. A particle of constant mass m moves along the x-axis. Its velocity v and position x satisfy the equation

$$\frac{1}{2}m\,(v^2 - v_0{}^2) = \frac{1}{2}k(x_0{}^2 - x^2),$$

where k, v_0, and x_0 are constants. Show that whenever $v \neq 0$,

$$m\,\frac{dv}{dt} = -kx.$$

14. a) Show that if the position x of a moving point is given by a quadratic function of t, $x = At^2 + Bt + C$, then the average velocity over any time interval $[t_1, t_2]$ is equal to the instantaneous velocity at the midpoint of the time interval.

b) What is the geometric significance of the result in (a)?

15. Find all values of the constants m and b for which the function

$$y = \begin{cases} \sin x & \text{for } x < \pi \\ mx + b & \text{for } x \geq \pi, \end{cases}$$

is (a) continuous at $x = \pi$; (b) differentiable at $x = \pi$.

16. Does the function

$$f(x) = \begin{cases} \dfrac{1 - \cos x}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

have a derivative at $x = 0$? Explain.

17. a) For what values of a and b will

$$f(x) = \begin{cases} ax, & x < 2 \\ ax^2 - bx + 3, & x \geq 2 \end{cases}$$

be differentiable for all values of x?

b) Discuss the geometry of the resulting graph of f.

18. a) For what values of a and b will

$$g(x) = \begin{cases} ax + b, & x \leq -1 \\ ax^3 + x + 2b, & x > -1 \end{cases}$$

be differentiable for all values of x?

b) Discuss the geometry of the resulting graph of g.

19. Is there anything special about the derivative of an odd differentiable function of x? Give reasons for your answer.

20. Is there anything special about the derivative of an even differentiable function of x? Give reasons for your answer.

21. *A surprising result.* Suppose that the functions f and g are defined throughout an open interval containing the point x_0, that f is differentiable at x_0, that $f(x_0) = 0$, and that g is continuous at x_0. Show that the product fg is differentiable at x_0. This shows, for example, that while $|x|$ is not differentiable at $x = 0$, the product $x|x|$ *is* differentiable at $x = 0$.

22. *(Continuation of Exercise 21.)* Use the result of Exercise 21 to show that the following functions are differentiable at $x = 0$.

a) $|x|\sin x$ **b)** $x^{2/3}\sin x$ **c)** $\sqrt[3]{x}(1 - \cos x)$

d) $h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

23. Is the derivative of

$$h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

derived at $x = 0$? continuous at $x = 0$? How about the derivative of $k(x) = xh(x)$? Give reasons for your answers.

24. Suppose that a function f satisfies the following conditions for all real values of x and y:

i) $f(x + y) = f(x) \cdot f(y)$;

ii) $f(x) = 1 + xg(x)$, where $\lim_{x \to 0} g(x) = 1$.

Show that the derivative $f'(x)$ exists at every value of x and that $f'(x) = f(x)$.

25. *The generalized product rule.* Use mathematical induction (Appendix 1) to prove that if $y = u_1 u_2 \cdots u_n$ is a finite product of differentiable functions, then y is differentiable on their common domain and

$$\frac{dy}{dx} = \frac{du_1}{dx}u_2 \cdots u_n + u_1\frac{du_2}{dx} \cdots u_n + \cdots + u_1 u_2 \cdots u_{n-1}\frac{du_n}{dx}.$$

26. *Leibniz's rule for higher order derivatives of products.* Leibniz's rule for higher order derivatives of products of differentiable functions says that

a) $$\frac{d^2(uv)}{dx^2} = \frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2},$$

b) $$\frac{d^3(uv)}{dx^3} = \frac{d^3u}{dx^3}v + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3},$$

c) $$\frac{d^n(uv)}{dx^n} = \frac{d^nu}{dx^n}v + n\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{d^{n-k}u}{dx^{n-k}}\frac{d^kv}{dx^k} + \cdots + u\frac{d^nv}{dx^n}.$$

The equations in (a) and (b) are special cases of the equation in (c). Derive the equation in (c) by mathematical induction, using the fact that

$$\binom{m}{k} + \binom{m}{k+1} = \frac{m!}{k!(m-k)!} + \frac{m!}{(k+1)!(m-k-1)!}.$$