CHAPTER 3

Applications of Derivatives

OVERVIEW This chapter shows how to draw conclusions from derivatives. We use derivatives to find extreme values of functions, to predict and analyze the shapes of graphs, to find replacements for complicated formulas, to determine how sensitive formulas are to errors in measurement, and to find the zeros of functions numerically. The key to many of these accomplishments is the Mean Value Theorem, a theorem whose corollaries provide the gateway to integral calculus in Chapter 4.

3.1 Extreme Values of Functions

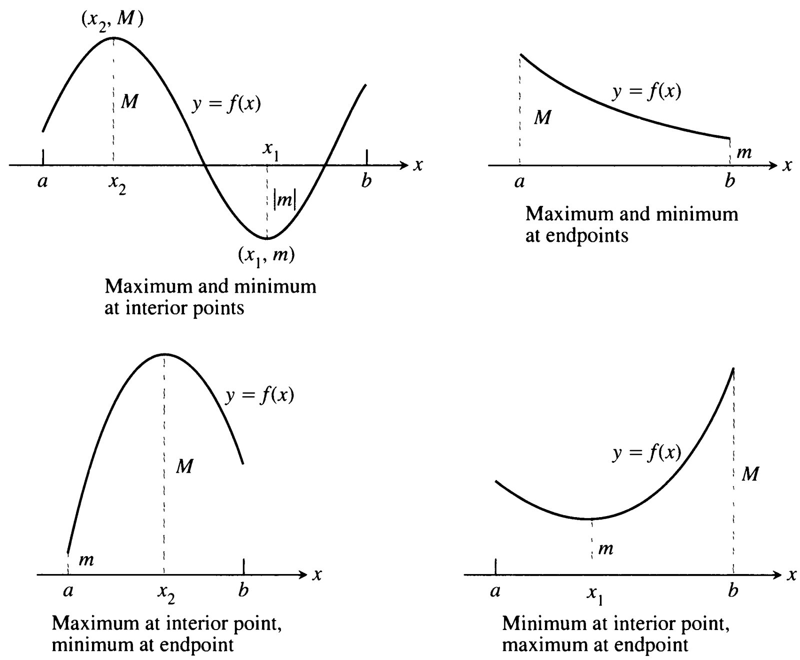
This section shows how to locate and identify extreme values of continuous functions.

The Max-Min Theorem

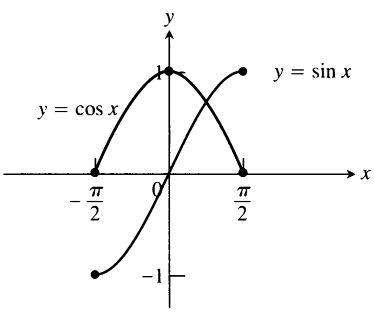
A function that is continuous at every point of a closed interval has an absolute maximum and an absolute minimum value on the interval. We always look for these values when we graph a function, and we will see the role they play in problem solving (this chapter) and in the development of the integral calculus (Chapters 4 and 5).

Theorem 1
The Max-Min Theorem for Continuous Functions
If f is continuous at every point of a closed interval I, then f assumes both an absolute maximum value M and an absolute minimum value m somewhere in I. That is, there are numbers x_1 and x_2 in I with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in I (Fig. 3.1 on the following page).

The proof of Theorem 1 requires a detailed knowledge of the real number system and we will not give it here.



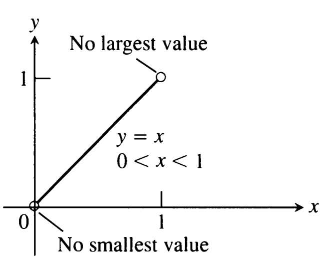
3.1 Typical arrangements of a continuous function's absolute maxima and minima on a closed interval [*a*, *b*].



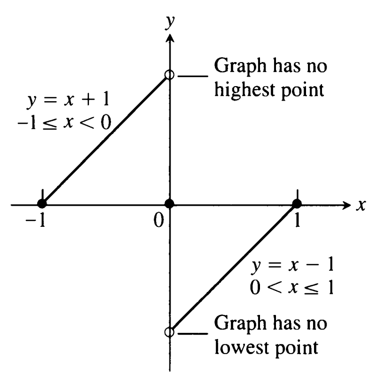
3.2 Figure for Example 1.

EXAMPLE 1 On $[-\pi/2, \pi/2]$, $f(x) = \cos x$ takes on a maximum value of 1 (once) and a minimum value of 0 (twice). The function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Fig. 3.2). ❑

As Figs. 3.3 and 3.4 show, the requirements that the interval be closed and the function continuous are key ingredients of Theorem 1. Without them, the conclusion of the theorem need not hold.



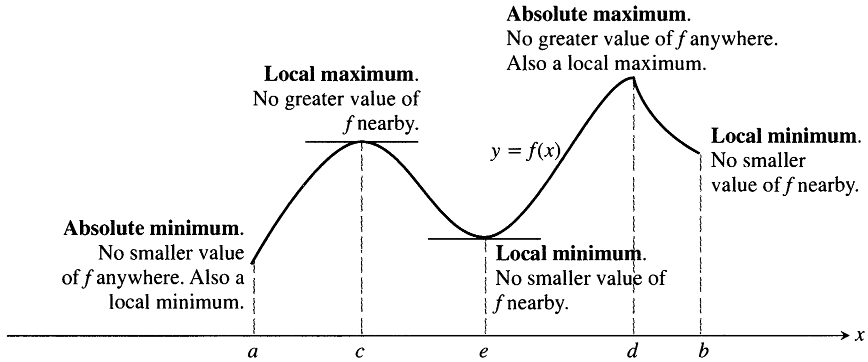
3.3 On an open interval, a continuous function need not have either a maximum or a minimum value. The function $f(x) = x$ has neither a largest nor a smallest value on (0, 1).



3.4 Even a single point of discontinuity can keep a function from having either a maximum or a minimum value on a closed interval. The function

$$y = \begin{cases} x + 1, & -1 \le x < 0 \\ 0, & x = 0 \\ x - 1, & 0 < x \le 1 \end{cases}$$

is continuous at every point of $[-1,1]$ except $x = 0$, yet its graph over $[-1,1]$ has neither a highest nor a lowest point.



3.5 How to classify maxima and minima.

Local vs. Absolute (Global) Extrema

Figure 3.5 shows a graph with five extreme points. The function's absolute minimum occurs at a even though at e the function's value is smaller than at any other point *nearby.* The curve rises to the left and falls to the right around c, making $f(c)$ a maximum locally. The function attains its absolute maximum at d.

Definition
Absolute Extreme Values

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

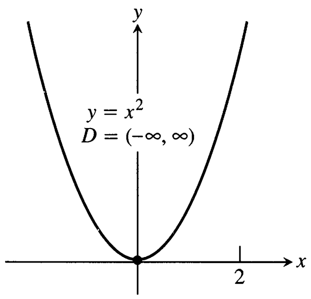
$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

Absolute maximum and minimum values are called absolute **extrema** (plural of the Latin *extremum*). Absolute extrema are also called **global** extrema.

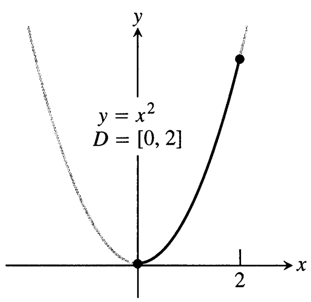
Functions with the same defining rule can have different extrema, depending on the domain.

EXAMPLE 2 (See Fig. 3.6.)

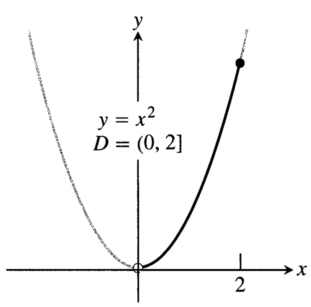
	Function rule	Domain D	Absolute extrema on D (if any)
a)	$y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
b)	$y = x^2$	$[0, 2]$	Absolute maximum of $(2)^2 = 4$ at $x = 2$. Absolute minimum of 0 at $x = 0$.
c)	$y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
d)	$y = x^2$	$(0, 2)$	No absolute extrema. ❑



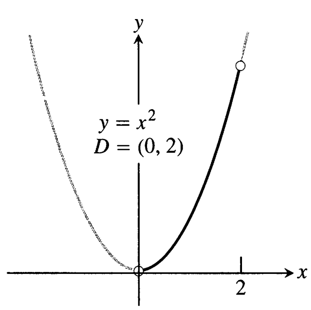
(a) abs min only



(b) abs max and min



(c) abs max only



(d) no abs max or min

3.6 Graphs for Example 2.

Definition
Local Extreme Values

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \le f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \ge f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

We can extend the definitions of local extrema to the endpoints of intervals by defining f to have a **local maximum** or **local minimum** value *at an endpoint* c if the appropriate inequality holds for all x in some half-open interval in its domain containing c. In Fig. 3.5, the function f has local maxima at c and d and local minima at a, e, and b.

An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, *a list of all local maxima will automatically include the absolute maximum if there is one*. Similarly, *a list of all local minima will include the absolute minimum if there is one*.

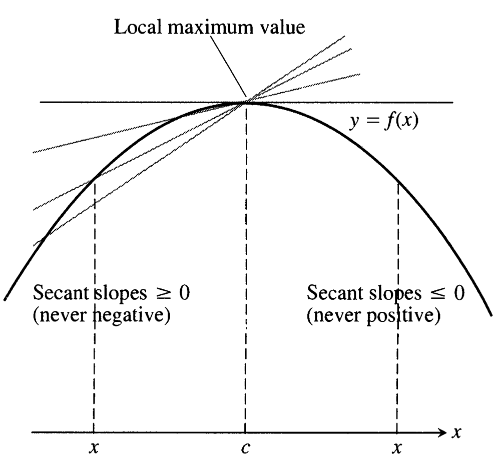
Finding Extrema

The next theorem explains why we usually need to investigate only a few values to find a function's extrema.

Theorem 2
The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$



3.7 A curve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

Proof To show that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative. The only number that is neither positive nor negative is zero, so that is what $f'(c)$ must be.

To begin, suppose that f has a local maximum value at $x = c$ (Fig. 3.7) so that $f(x) - f(c) \le 0$ for all values of x near enough to c. Since c is an interior point of f's domain, $f'(c)$ is defined by the two-sided limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$. When we examine these limits separately, we find that

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0. \qquad \text{Because } (x - c) > 0 \text{ and } f(x) \le f(c) \qquad (1)$$

Similarly,

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0. \qquad \text{Because } (x - c) < 0 \text{ and } f(x) \leq f(c) \qquad (2)$$

Together, (1) and (2) imply $f'(c) = 0$.

This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \geq f(c)$, which reverses the inequalities in (1) and (2). ❑

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. Hence the only places where a function f can possibly have an extreme value (local or global) are

1. interior points where $f' = 0$,
2. interior points where f' is undefined,
3. endpoints of the domain of f.

The following definition helps us to summarize.

Definition

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Summary

The only domain points where a function can assume extreme values are critical points and endpoints.

Most quests for extreme values call for finding the absolute extrema of a continuous function on a closed interval. Theorem 1 assures us that such values exist; Theorem 2 tells us that they are taken on only at critical points and endpoints. These points are often so few in number that we can simply list them and calculate the corresponding function values to see what the largest and smallest are.

How to Find the Absolute Extrema of a Continuous Function f on a Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

EXAMPLE 3 Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution The function is differentiable over its entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = 4$

$f(1) = 1$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. ❑

EXAMPLE 4 Find the absolute extrema values of $g(t) = 8t - t^4$ on $[-2, 1]$.

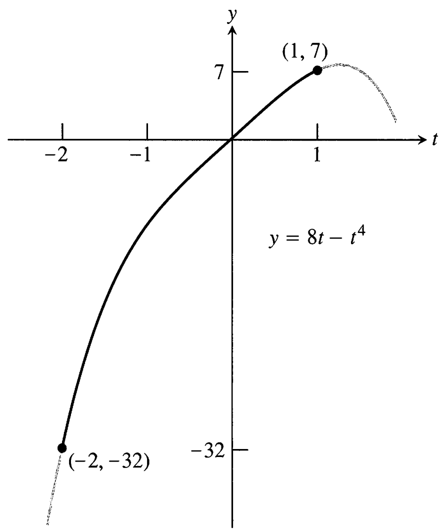
Solution The function is differentiable on its entire domain, so the only critical points occur where $g'(t) = 0$. Solving this equation gives

$$\begin{aligned} 8 - 4t^3 &= 0 \\ t^3 &= 2 \\ t &= 2^{1/3}, \end{aligned}$$

a point not in the given domain. The function's local extrema therefore occur at the endpoints, where we find

$$\begin{aligned} g(-2) &= -32 \qquad \text{(Absolute minimum)} \\ g(1) &= 7. \qquad \text{(Absolute maximum)} \end{aligned}$$

See Fig. 3.8. ❑



3.8 The extreme values of $g(t) = 8t - t^4$ on $[-2, 1]$ (Example 4).

EXAMPLE 5 Find the absolute extrema of $h(x) = x^{2/3}$ on $[-2, 3]$.

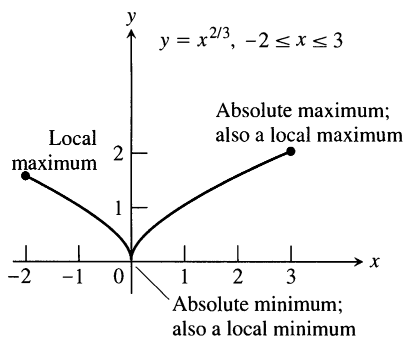
Solution The first derivative

$$h'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

has no zeros but is undefined at $x = 0$. The values of h at this one critical point and at the endpoints $x = -2$ and $x = 3$ are

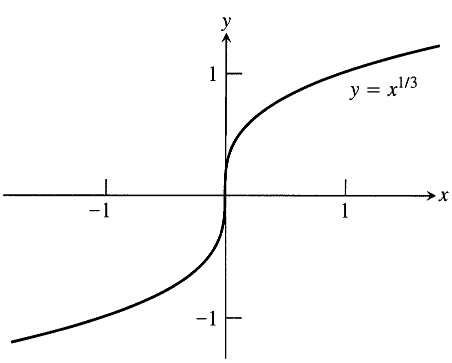
$$\begin{aligned} h(0) &= 0 \\ h(-2) &= (-2)^{2/3} = 4^{1/3} \\ h(3) &= (3)^{2/3} = 9^{1/3}. \end{aligned}$$

The absolute maximum value is $9^{1/3}$, assumed at $x = 3$; the absolute minimum is 0, assumed at $x = 0$ (Fig. 3.9). ❑

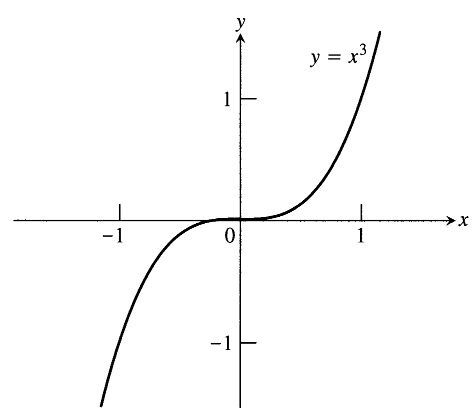


3.9 The extreme values of $h(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 5).

While a function's extrema can occur only at critical points and endpoints, not every critical point or endpoint signals the presence of an extreme value. Figures 3.10 and 3.11 illustrate this for interior points, and Exercise 34 asks you for a function that fails to assume an extreme value at an endpoint of its domain.



3.10 $f(x) = x^{1/3}$ has no extremum at $x = 0$, even though $f'(x) = (1/3)x^{-2/3}$ is undefined at $x = 0$.



3.11 $g(x) = x^3$ has no extremum at $x = 0$ even though $g'(x) = 3x^2$ is zero at $x = 0$.

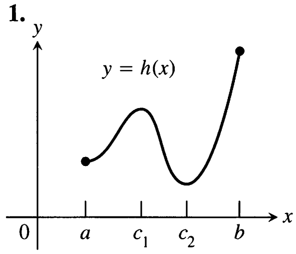
As we will see in Section 3.3, we can determine the behavior of a function f at a critical point c by further examining f', but we must look beyond what f' does at c itself.

Exercises 3.1

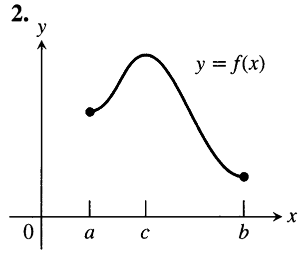
Finding Extrema from Graphs

In Exercises 1–6, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Theorem 1.

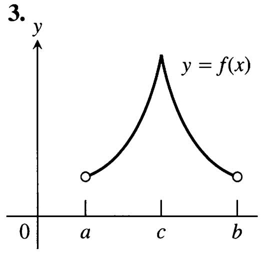
1.



2.



3.



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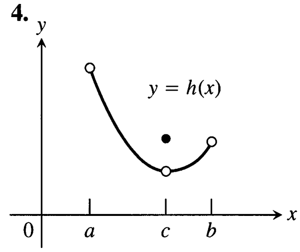
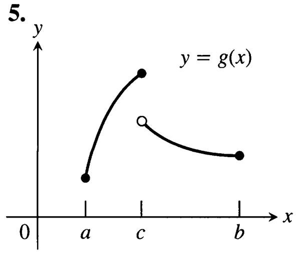
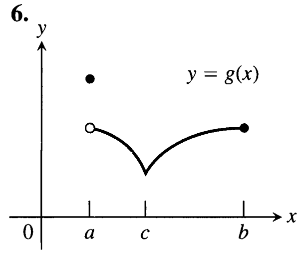


5.



6.



Absolute Extrema on Closed Intervals

In Exercises 7–22, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

7. $f(x) = \frac{2}{3}x - 5, \quad -2 \le x \le 3$

8. $f(x) = -x - 4, \quad -4 \le x \le 1$

9. $f(x) = x^2 - 1, \quad -1 \le x \le 2$

10. $f(x) = 4 - x^2, \quad -3 \le x \le 1$

11. $F(x) = -\frac{1}{x^2}, \quad 0.5 \le x \le 2$

12. $F(x) = -\frac{1}{x}, \quad -2 \le x \le -1$

13. $h(x) = \sqrt[3]{x}, \quad -1 \le x \le 8$

14. $h(x) = -3x^{2/3}, \quad -1 \le x \le 1$

15. $g(x) = \sqrt{4 - x^2}, \quad -2 \le x \le 1$

16. $g(x) = -\sqrt{5 - x^2}, \quad -\sqrt{5} \le x \le 0$

17. $f(\theta) = \sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{5\pi}{6}$

18. $f(\theta) = \tan\theta, \quad -\frac{\pi}{3} \le \theta \le \frac{\pi}{4}$

19. $g(x) = \csc x, \quad \frac{\pi}{3} \le x \le \frac{2\pi}{3}$

20. $g(x) = \sec x, \quad -\frac{\pi}{3} \le x \le \frac{\pi}{6}$

21. $f(t) = 2 - |t|, \quad -1 \le t \le 3$

22. $f(t) = |t - 5|, \quad 4 \le t \le 7$

In Exercises 23–26, find the function's absolute maximum and minimum values and say where they are assumed.

23. $f(x) = x^{4/3}, \quad -1 \le x \le 8$

24. $f(x) = x^{5/3}, \quad -1 \le x \le 8$

25. $g(\theta) = \theta^{3/5}, \quad -32 \le \theta \le 1$

26. $h(\theta) = 3\theta^{2/3}, \quad -27 \le \theta \le 8$

Local Extrema in the Domain

In Exercises 27 and 28, find the values of any local maxima and minima the functions may have on the given domains, and say where they are assumed. Which extrema, if any, are absolute for the given domain?

27. **a)** $f(x) = x^2 - 4, \quad -2 \le x \le 2$
b) $g(x) = x^2 - 4, \quad -2 \le x < 2$
c) $h(x) = x^2 - 4, \quad -2 < x < 2$
d) $k(x) = x^2 - 4, \quad -2 \le x < \infty$
e) $l(x) = x^2 - 4, \quad 0 < x < \infty$

28. **a)** $f(x) = 2 - 2x^2, \quad -1 \leq x \leq 1$
b) $g(x) = 2 - 2x^2, \quad -1 < x \leq 1$
c) $h(x) = 2 - 2x^2, \quad -1 < x < 1$
d) $k(x) = 2 - 2x^2, \quad -\infty < x \leq 1$
e) $l(x) = 2 - 2x^2, \quad -\infty < x < 0$

Theory and Examples

29. The function $f(x) = |x|$ has an absolute minimum value at $x = 0$ even though f is not differentiable at $x = 0$. Is this consistent with Theorem 2? Give reasons for your answer.

30. Why can't the conclusion of Theorem 2 be expected to hold if c is an endpoint of the function's domain?

31. If an even function $f(x)$ has a local maximum value at $x = c$, can anything be said about the value of f at $x = -c$? Give reasons for your answer.

32. If an odd function $g(x)$ has a local minimum value at $x = c$, can anything be said about the value of g at $x = -c$? Give reasons for your answer.

33. We know how to find the extreme values of a continuous function $f(x)$ by investigating its values at critical points and endpoints. But what if there *are* no critical points or endpoints? What happens then? Do such functions really exist? Give reasons for your answers.

34. Give an example of a function defined on [0, 1] that has neither a local maximum nor a local minimum value at 0.

CAS Explorations and Projects

In Exercises 35–40, you will use a CAS to help find the absolute extrema of the given function over the specified closed interval. Perform the following steps:

a) Plot the function over the interval to see general behavior there.
b) Find the interior points where $f' = 0$. (In some exercises you may have to use the numerical equation solver to approximate a solution.) You may want to plot f' as well.
c) Find the interior points where f' does not exist.
d) Evaluate the function at all points found in parts (b) and (c) and at the endpoints of the interval.
e) Find the function's absolute extreme values on the interval and identify where they occur.

35. $f(x) = x^4 - 8x^2 + 4x + 2, \quad \left[-\frac{20}{25}, \frac{64}{25}\right]$

36. $f(x) = -x^4 + 4x^3 - 4x + 1, \quad \left[-\frac{3}{4}, 3\right]$

37. $f(x) = x^{2/3}(3 - x), \quad [-2, 2]$

38. $f(x) = 2 + 2x - 3x^{2/3}, \quad \left[-1, \frac{10}{3}\right]$

39. $f(x) = \sqrt{x} + \cos x, \quad [0, 2\pi]$

40. $f(x) = x^{3/4} - \sin x + \frac{1}{2}, \quad [0, 2\pi]$

3.2 The Mean Value Theorem

If a body falls freely from rest near the surface of the earth, its position t seconds into the fall is $s = 4.9t^2$ m. From this we deduce that the body's velocity and acceleration are $v = ds/dt = 9.8t$ m/sec and $a = d^2s/dt^2 = 9.8$ m/sec^2. But suppose we started with the body's acceleration. Could we work backward to find its velocity and displacement functions?

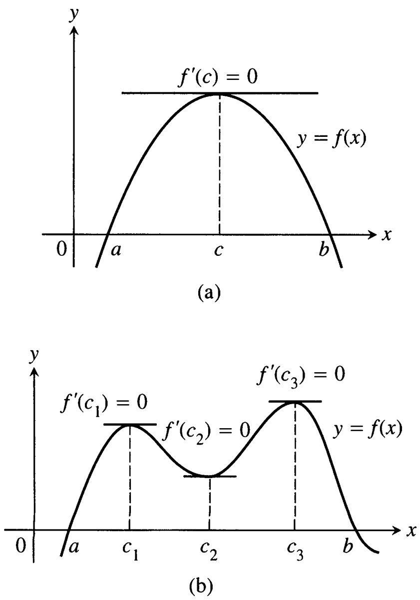
What we are really asking here is what functions can have a given derivative. More generally, we might ask what kind of function can have a particular *kind* of derivative. What kind of function has a positive derivative, for instance, or a negative derivative, or a derivative that is always zero? We answer these questions by applying corollaries of the Mean Value Theorem.

When the French mathematician Michel Rolle published his theorem in 1691, his goal was to show that between every two zeros of a polynomial function there always lies a zero of the polynomial we now know to be the function's derivative. (The modern version of the theorem is not restricted to polynomials.)

Rolle distrusted the new methods of calculus, however, and spent a great deal of time and energy denouncing their use and attacking l'Hôpital's all too popular (he felt) calculus book. It is ironic that Rolle is known today only for his inadvertent contribution to a field he tried to suppress.

Rolle's Theorem

There is strong geometric evidence that between any two points where a differentiable curve crosses the x-axis there is a point on the curve where the tangent is horizontal. A 300-year-old theorem of Michel Rolle (1652–1719) assures us that this is indeed the case.



3.12 Rolle's theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses the x-axis. It may have just one (a), or it may have more (b).

Theorem 3

Rolle's Theorem

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If

$$f(a) = f(b) = 0,$$

then there is at least one number c in (a, b) at which

$$f'(c) = 0.$$

See Fig. 3.12.

Proof Being continuous, f assumes absolute maximum and minimum values on $[a, b]$. These can occur only

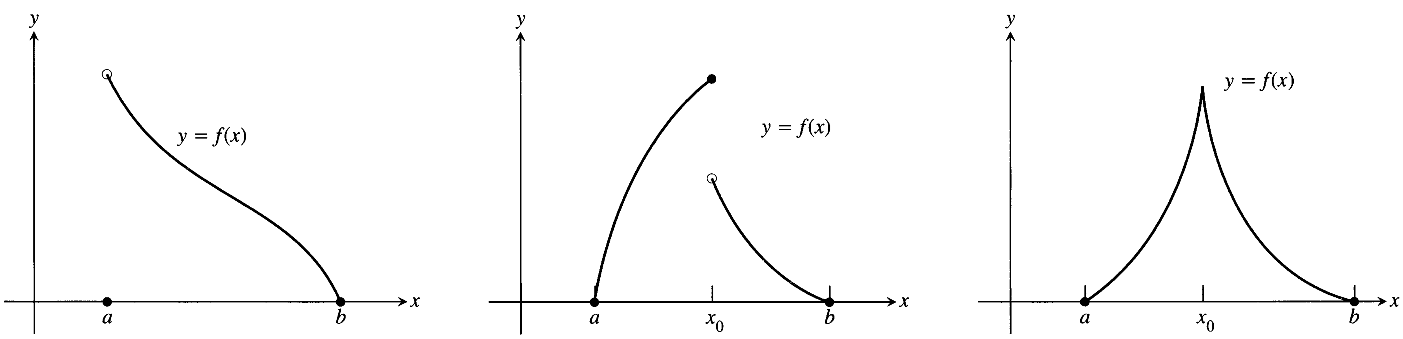
1. at interior points where f' is zero,
2. at interior points where f' does not exist,
3. at the endpoints of the function's domain, in this case a and b.

By hypothesis, f has a derivative at every interior point. That rules out (2), leaving us with interior points where $f' = 0$ and with the two endpoints a and b.

If either the maximum or the minimum occurs at a point c inside the interval, then $f'(c) = 0$ by Theorem 2 in Section 3.1, and we have found a point for Rolle's theorem.

If both maximum and minimum are at a or b, then f is constant, $f' = 0$, and c can be taken anywhere in the interval. This completes the proof. ❑

The hypotheses of Theorem 3 are essential. If they fail at even one point, the graph may not have a horizontal tangent (Fig. 3.13).



(a) Discontinuous at an endpoint

(b) Discontinuous at an interior point

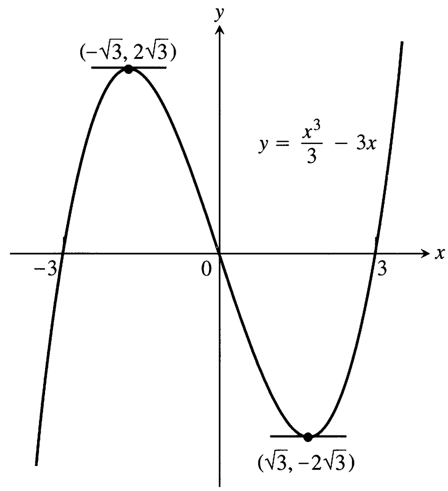
(c) Continuous on $[a, b]$ but not differentiable at some interior point

3.13 No horizontal tangent.

EXAMPLE 1 The polynomial function

$$f(x) = \frac{x^3}{3} - 3x$$

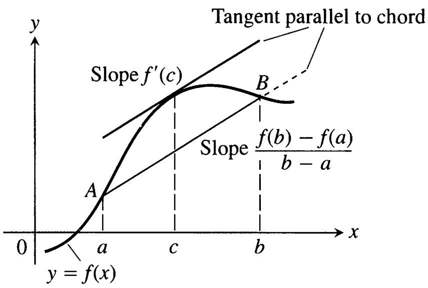
graphed in Fig. 3.14 (on the following page) is continuous at every point of $[-3, 3]$ and is differentiable at every point of $(-3, 3)$. Since $f(-3) = f(3) = 0$, Rolle's



3.14 As predicted by Rolle's theorem, this curve has horizontal tangents between the points where it crosses the x-axis (Example 1).

theorem says that f' must be zero at least once in the open interval between $a = -3$ and $b = 3$. In fact, $f'(x) = x^2 - 3$ is zero twice in this interval, once at $x = -\sqrt{3}$ and again at $x = \sqrt{3}$. ❑

The Mean Value Theorem



3.15 Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

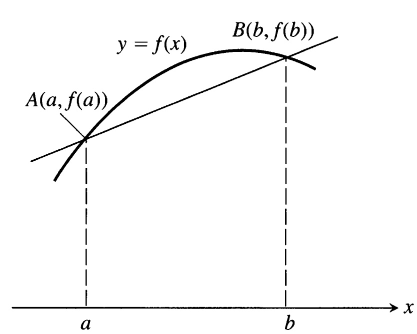
The Mean Value Theorem is a slanted version of Rolle's theorem (Fig. 3.15). There is a point where the tangent is parallel to chord AB.

Theorem 4
The Mean Value Theorem

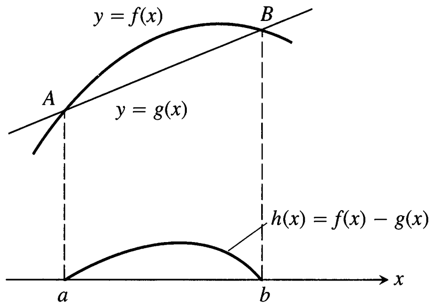
Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

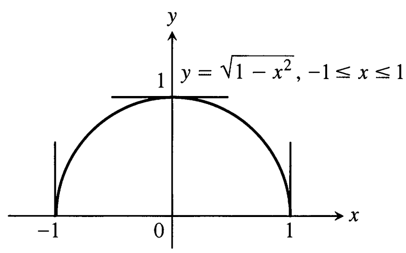
Proof We picture the graph of f as a curve in the plane and draw a line through the points $A(a, f(a))$ and $B(b, f(b))$ (see Fig. 3.16). The line is the graph of the



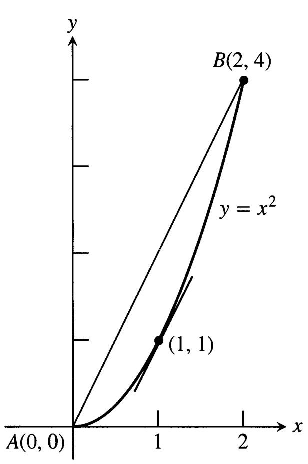
3.16 The graph of f and the chord AB over the interval $[a, b]$.



3.17 The chord AB in Fig. 3.16 is the graph of the function $g(x)$. The function $h(x) = f(x) - g(x)$ gives the vertical distance between the graphs of f and g at x.



3.18 The function $f(x) = \sqrt{1-x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though f is not differentiable at -1 and 1.



3.19 As we find in Example 2, $c = 1$ is where the tangent is parallel to the chord.

function

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \tag{2}$$

(point–slope equation). The vertical difference between the graphs of f and g at x is

$$\begin{aligned} h(x) &= f(x) - g(x) \\ &= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a). \end{aligned} \tag{3}$$

Figure 3.17 shows the graphs of f, g, and h together.

The function h satisfies the hypotheses of Rolle's theorem on $[a, b]$. It is continuous on $[a, b]$ and differentiable on (a, b) because both f and g are. Also, $h(a) = h(b) = 0$ because the graphs of f and g both pass through A and B. Therefore, $h' = 0$ at some point c in (a, b). This is the point we want for Eq. (1).

To verify Eq. (1), we differentiate both sides of Eq. (3) with respect to x and then set $x = c$:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \qquad \text{Derivative of Eq. (3) . . .}$$

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad \text{. . . with } x = c$$

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad h'(c) = 0$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \qquad \text{Rearranged}$$

which is what we set out to prove. ❑

Notice that the hypotheses of the Mean Value Theorem do not require f to be differentiable at either a or b. Continuity at a and b is enough (Fig. 3.18).

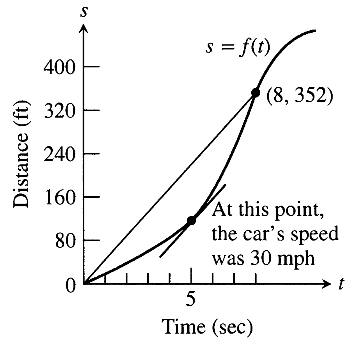
We usually do not know any more about the number c than the theorem tells, which is that c exists. In a few cases we can satisfy our curiosity about the identity of c, as in the next example. However, our ability to identify c is the exception rather than the rule, and the importance of the theorem lies elsewhere.

EXAMPLE 2 The function $f(x) = x^2$ (Fig. 3.19) is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this (exceptional) case we can identify c by solving the equation $2c = 2$ to get $c = 1$. ❑

Physical Interpretations

If we think of the number $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

EXAMPLE 3 If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec. At some point during the acceleration, the Mean Value Theorem says, the speedometer must read exactly 30 mph (44 ft/sec) (Fig. 3.20). ❑



3.20 Distance vs. elapsed time for the car in Example 3.

Corollaries and Some Answers

At the beginning of the section, we asked what kind of function has a zero derivative. The first corollary of the Mean Value Theorem provides the answer.

Corollary 1

Functions with Zero Derivatives Are Constant

If $f'(x) = 0$ at each point of an interval I, then $f(x) = C$ for all x in I, where C is a constant.

We know that if a function f has a constant value on an interval I, then f is differentiable on I and $f'(x) = 0$ for all x in I. Corollary 1 provides the converse.

Proof of Corollary 1 We want to show that f has a constant value on I. We do so by showing that if x_1 and x_2 are any two points in I, then $f(x_1) = f(x_2)$.

Suppose that x_1 and x_2 are two points in I, numbered from left to right so that $x_1 < x_2$. Then f satisfies the hypotheses of the Mean Value Theorem on $[x_1, x_2]$: It is differentiable at every point of $[x_1, x_2]$, and hence continuous at every point as well. Therefore,

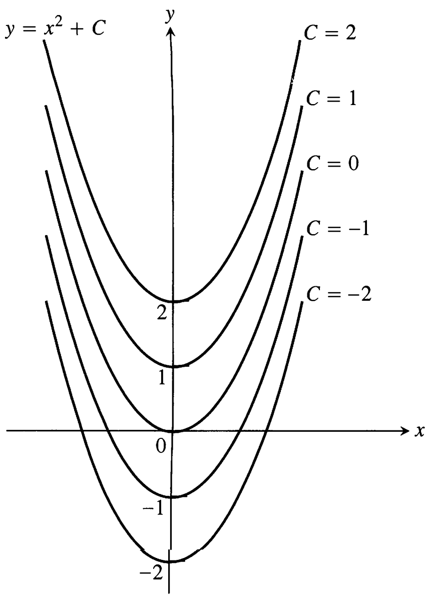
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

at some point c between x_1 and x_2. Since $f' = 0$ throughout I, this equation translates successively into

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0, \quad f(x_2) - f(x_1) = 0, \quad \text{and} \quad f(x_1) = f(x_2).$$

❑

At the beginning of the section, we also asked if we could work backward from the acceleration of a body falling freely from rest to find the body's velocity and displacement functions. The answer is yes, and it is a consequence of the next corollary.



3.21 From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives can differ only by a vertical shift. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for selected values of C.

Corollary 2
Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point of an interval I, then there exists a constant C such that $f(x) = g(x) + C$ for all x in I.

Proof At each point x in I the derivative of the difference function $h = f - g$ is

$$h'(x) = f'(x) - g'(x) = 0.$$

Thus, $h(x) = C$ on I (Corollary 1). That is, $f(x) - g(x) = C$ on I, so $f(x) = g(x) + C$. ❑

Corollary 2 says that functions can have identical derivatives on an interval only if their values on the interval have a constant difference. We know, for instance, that the derivative of $f(x) = x^2$ on $(-\infty, \infty)$ is $2x$. Any other function with derivative $2x$ on $(-\infty, \infty)$ must have the formula $x^2 + C$ for some value of C (Fig. 3.21).

EXAMPLE 4 Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point (0, 2).

Solution Since $f(x)$ has the same derivative as $g(x) = -\cos x$, we know that $f(x) = -\cos x + C$ for some constant C. The value of C can be determined from the condition that $f(0) = 2$ (the graph of f passes through (0, 2)):

$$f(0) = -\cos(0) + C = 2, \qquad \text{so} \qquad C = 3.$$

The formula for f is $f(x) = -\cos x + 3$. ❑

Finding Velocity and Position from Acceleration

Here is how to find the velocity and displacement functions of a body falling freely from rest with acceleration 9.8 m/sec^2.

We know that $v(t)$ is some function whose derivative is 9.8. We also know that the derivative of $g(t) = 9.8t$ is 9.8. By Corollary 2,

$$v(t) = 9.8t + C \tag{4}$$

for some constant C. Since the body falls from rest, $v(0) = 0$. Thus

$$9.8(0) + C = 0, \qquad \text{and} \qquad C = 0.$$

The velocity function must be $v(t) = 9.8t$. How about the position function $s(t)$?

We know that $s(t)$ is some function whose derivative is $9.8t$. We also know that the derivative of $h(t) = 4.9t^2$ is $9.8t$. By Corollary 2,

$$s(t) = 4.9t^2 + C \tag{5}$$

for some constant C. Since $s(0) = 0$,

$$4.9(0)^2 + C = 0, \qquad \text{and} \qquad C = 0.$$

The position function must be $s(t) = 4.9t^2$.

The ability to find functions from their rates of change is one of the great powers we gain from calculus. As we will see, it lies at the heart of the mathematical developments in Chapter 4. We will continue the story there.

Increasing Functions and Decreasing Functions

At the beginning of the section we asked what kinds of functions have positive derivatives or negative derivatives. The answer, provided by the Mean Value Theorem's third corollary, is this: The only functions with positive derivatives are increasing functions; the only functions with negative derivatives are decreasing functions.

Definitions

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. f **increases** on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
2. f **decreases** on I if $x_1 < x_2 \Rightarrow f(x_2) < f(x_1)$.

Corollary 3
The First Derivative Test for Increasing and Decreasing

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.

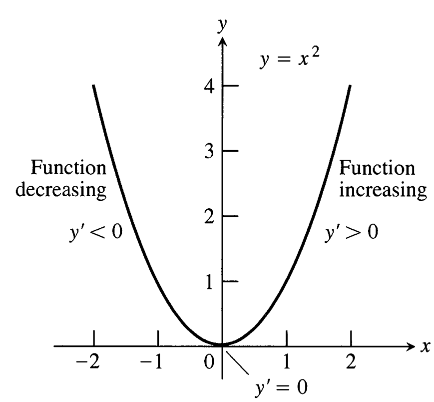
If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

Proof Let x_1 and x_2 be two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \tag{6}$$

for some c between x_1 and x_2. The sign of the right-hand side of Eq. (6) is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore, $f(x_2) > f(x_1)$ if f' is positive on (a, b), and $f(x_2) < f(x_1)$ if f' is negative on (a, b). ❑

EXAMPLE 5 The function $f(x) = x^2$ decreases on $(-\infty, 0)$, where $f'(x) = 2x < 0$. It increases on $(0, \infty)$, where $f'(x) = 2x > 0$ (Fig. 3.22). ❑



3.22 The graph for Example 5.

Exercises 3.2

Finding *c* in the Mean Value Theorem

Find the value or values of c that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–4.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$
2. $f(x) = x^{2/3}, \quad [0, 1]$
3. $f(x) = x + \dfrac{1}{x}, \quad \left[\dfrac{1}{2}, 2\right]$
4. $f(x) = \sqrt{x - 1}, \quad [1, 3]$

Checking and Using Hypotheses

Which of the functions in Exercises 5–8 satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

5. $f(x) = x^{2/3}, \quad [-1, 8]$
6. $f(x) = x^{4/5}, \quad [0, 1]$
7. $f(x) = \sqrt{x(1 - x)}, \quad [0, 1]$
8. $f(x) = \begin{cases} \dfrac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$
9. The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and $x = 1$ and differentiable on (0, 1), but its derivative on (0, 1) is never zero. How can this be? Doesn't Rolle's theorem say the derivative has to be zero somewhere in (0, 1)? Give reasons for your answer.

10. For what values of a, m, and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval [0, 2]?

Roots (Zeros)

11. a) Plot the zeros of each polynomial on a line together with the zeros of its first derivative.
    - i) $y = x^2 - 4$
    - ii) $y = x^2 + 8x + 15$
    - iii) $y = x^3 - 3x^2 + 4 = (x + 1)(x - 2)^2$
    - iv) $y = x^3 - 33x^2 + 216x = x(x - 9)(x - 24)$

    b) Use Rolle's theorem to prove that between every two zeros of $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ there lies a zero of

$$nx^{n-1} + (n - 1)a_{n-1}x^{n-2} + \cdots + a_1.$$

12. Suppose that f'' is continuous on $[a, b]$ and that f has three zeros in the interval. Show that f'' has at least one zero in (a, b). Generalize this result.
13. Show that if $f'' > 0$ throughout an interval $[a, b]$, then f' has at most one zero in $[a, b]$. What if $f'' < 0$ throughout $[a, b]$ instead?
14. Show that a cubic polynomial can have at most three real zeros.

Theory and Examples

15. Show that at some instant during a 2-h automobile trip the car's speedometer reading will equal the average speed for the trip.
16. *Temperature change.* It took 14 sec for a thermometer to rise from −19°C to 100°C when it was taken from a freezer and placed in boiling water. Show that somewhere along the way the mercury was rising at exactly 8.5°C/sec.
17. Suppose that f is differentiable on [0, 1] and that its derivative is never zero. Show that $f(0) \neq f(1)$.
18. Show that $|\sin b - \sin a| \le |b - a|$ for any numbers a and b.
19. Suppose that f is differentiable on $[a, b]$ and that $f(b) < f(a)$. Can you then say anything about the values of f' on $[a, b]$?
20. Suppose that f and g are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between a and b where the tangents to the graphs of f and g are parallel.
21. Let f be differentiable at every value of x and suppose that $f(1) = 1$, that $f' < 0$ on $(-\infty, 1)$, and that $f' > 0$ on $(1, \infty)$.
    - a) Show that $f(x) \ge 1$ for all x.
    - b) Must $f'(1) = 0$? Explain.
22. Let $f(x) = px^2 + qx + r$ be a quadratic function defined on a closed interval $[a, b]$. Show that there is exactly one point c in (a, b) at which f satisfies the conclusion of the Mean Value Theorem.
23. *A surprising graph.* Graph the function

$$f(x) = \sin x \sin(x + 2) - \sin^2(x + 1).$$

What does the graph do? Why does the function behave this way? Give reasons for your answers.

24. If the graphs of two functions $f(x)$ and $g(x)$ start at the same point in the plane and the functions have the same rate of change at every point, do the graphs have to be identical? Give reasons for your answer.
25. a) Show that $g(x) = 1/x$ decreases on every interval in its domain.

b) If the conclusion in (a) is really true, how do you explain the fact that $g(1) = 1$ is actually greater than $g(-1) = -1$?

26. Let f be a function defined on an interval $[a, b]$. What conditions could you place on f to guarantee that

$$\min f' \le \frac{f(b) - f(a)}{b - a} \le \max f',$$

where $\min f'$ and $\max f'$ refer to the minimum and maximum values of f' on $[a, b]$? Give reasons for your answer.

27. CALCULATOR Use the inequalities in Exercise 26 to estimate $f(0.1)$ if $f'(x) = 1/(1 + x^4 \cos x)$ for $0 \le x \le 0.1$ and $f(0) = 1$.

28. CALCULATOR Use the inequalities in Exercise 26 to estimate $f(0.1)$ if $f'(x) = 1/(1 - x^4)$ for $0 \le x \le 0.1$ and $f(0) = 2$.

29. *The geometric mean of a and b.* The **geometric mean** of two positive numbers a and b is the number $\sqrt{ab}$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = 1/x$ on an interval $[a, b]$ of positive numbers is $c = \sqrt{ab}$.

30. *The arithmetic mean of a and b.* The **arithmetic mean** of two numbers a and b is the number $(a + b)/2$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = x^2$ on any interval $[a, b]$ is $c = (a + b)/2$.

Finding Functions from Derivatives

31. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all x. Must $f(x) = 3$ for all x? Give reasons for your answer.

32. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x. Must $f(x) = 2x + 5$ for all x? Give reasons for your answer.

33. Suppose that $f'(x) = 2x$ for all x. Find $f(2)$ if

a) $f(0) = 0$ **b)** $f(1) = 0$ **c)** $f(-2) = 3$.

34. What can be said about functions whose derivatives are constant? Give reasons for your answer.

In Exercises 35–40, find all possible functions with the given derivative.

35. a) $y' = x$ **b)** $y' = x^2$ **c)** $y' = x^3$

36. a) $y' = 2x$
b) $y' = 2x - 1$
c) $y' = 3x^2 + 2x - 1$

37. a) $y' = -\dfrac{1}{x^2}$

b) $y' = 1 - \dfrac{1}{x^2}$

c) $y' = 5 + \dfrac{1}{x^2}$

38. a) $y' = \dfrac{1}{2\sqrt{x}}$

b) $y' = \dfrac{1}{\sqrt{x}}$

c) $y' = 4x - \dfrac{1}{\sqrt{x}}$

39. a) $y' = \sin 2t$ **b)** $y' = \cos \dfrac{t}{2}$

c) $y' = \sin 2t + \cos \dfrac{t}{2}$

40. a) $y' = \sec^2 \theta$ **b)** $y' = \sqrt{\theta}$

c) $y' = \sqrt{\theta} - \sec^2 \theta$

In Exercises 41–44, find the function with the given derivative whose graph passes through the point P.

41. $f'(x) = 2x - 1, \quad P(0, 0)$

42. $g'(x) = \dfrac{1}{x^2} + 2x, \quad P(-1, 1)$

43. $r'(\theta) = 8 - \csc^2 \theta, \quad P\left(\dfrac{\pi}{4}, 0\right)$

44. $r'(t) = \sec t \tan t - 1, \quad P(0, 0)$

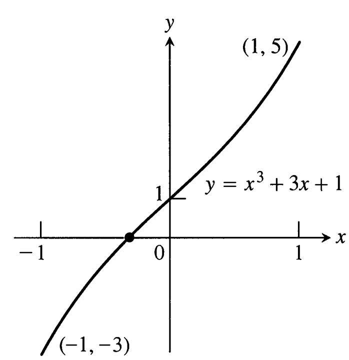
Counting Zeros

When we solve an equation $f(x) = 0$ numerically, we usually want to know beforehand how many solutions to look for in a given interval. With the help of Corollary 3 we can sometimes find out.

Suppose that

1. f is continuous on $[a, b]$ and differentiable on (a, b),
2. $f(a)$ and $f(b)$ have opposite signs,
3. $f' > 0$ on (a, b) or $f' < 0$ on (a, b).

Then f has exactly one zero between a and b: It cannot have more than one because it is either increasing on $[a, b]$ or decreasing on $[a, b]$. Yet it has at least one, by the Intermediate Value Theorem (Section 1.5). For example, $f(x) = x^3 + 3x + 1$ has exactly one zero on $[-1, 1]$ because f is differentiable on $[-1, 1]$, $f(-1) = -3$ and $f(1) = 5$ have opposite signs, and $f'(x) = 3x^2 + 3 > 0$ for all x (Fig. 3.23).



3.23 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here between -1 and 0.

Show that the functions in Exercises 45–52 have exactly one zero in the given interval.

45. $f(x) = x^4 + 3x + 1, \quad [-2, -1]$

46. $f(x) = x^3 + \dfrac{4}{x^2} + 7, \quad (-\infty, 0)$

47. $g(t) = \sqrt{t} + \sqrt{1+t} - 4, \quad (0, \infty)$

48. $g(t) = \dfrac{1}{1-t} + \sqrt{1+t} - 3.1, \quad (-1, 1)$

49. $r(\theta) = \theta + \sin^2\left(\dfrac{\theta}{3}\right) - 8, \quad (-\infty, \infty)$

50. $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}, \quad (-\infty, \infty)$

51. $r(\theta) = \sec\theta - \dfrac{1}{\theta^3} + 5, \quad (0, \pi/2)$

52. $r(\theta) = \tan\theta - \cot\theta - \theta, \quad (0, \pi/2)$

CAS Exploration

53. *Rolle's original theorem*

a) Construct a polynomial $f(x)$ that has zeros at $x = -2, -1, 0, 1,$ and 2.

b) Graph f and its derivative f' together. How is what you see related to Rolle's original theorem? (See the marginal note on Rolle.)

c) Do $g(x) = \sin x$ and its derivative g' illustrate the same phenomenon?

d) How would you state and prove Rolle's original theorem in light of what we know today?

3.3

The First Derivative Test for Local Extreme Values

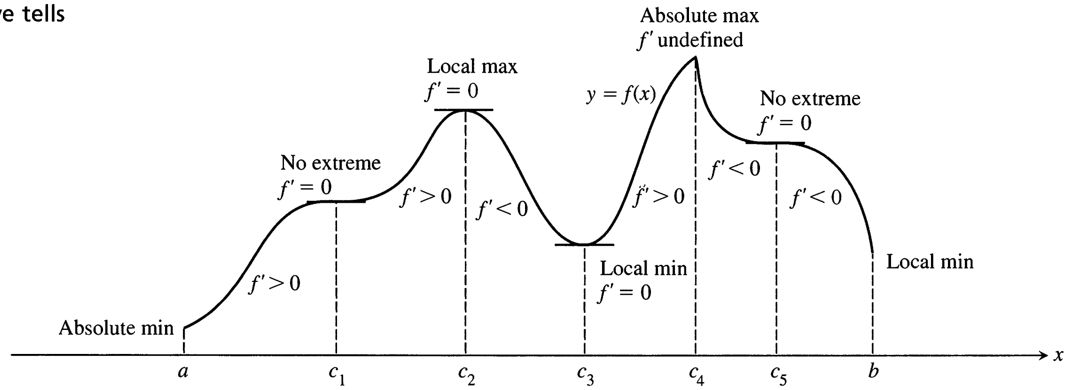
This section shows how to test a function's critical points for the presence of local extreme values.

The Test

As we see once again in Fig. 3.24, a function f may have local extrema at some critical points while failing to have local extrema at others. The key is the sign of f' in the point's immediate vicinity. As x moves from left to right, the values of f increase where $f' > 0$ and decrease where $f' < 0$.

At the points where f has a minimum value, we see that $f' < 0$ on the interval immediately to the left and $f' > 0$ on the interval immediately to the right. (If the point is an endpoint, there is only the interval on the appropriate side to consider.) This means that the curve is falling (values decreasing) on the left of the minimum value and rising (values increasing) on its right. Similarly, at the points where f has a maximum value, $f' > 0$ on the interval immediately to the left and $f' < 0$ on the interval immediately to the right. This means that the curve is rising (values increasing) on the left of the maximum value and falling (values decreasing) on its right.

These observations lead to a test for the presence of local extreme values.



3.24 A function's first derivative tells how the graph rises and falls.

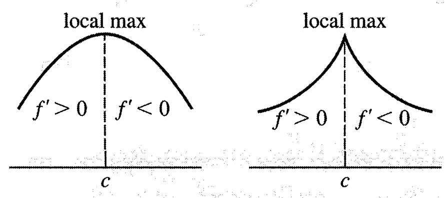
Theorem 5

The First Derivative Test for Local Extreme Values

The following test applies to a continuous function $f(x)$.

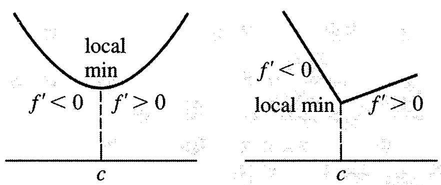
At a critical point c:

1. If f' changes from positive to negative at c ($f' > 0$ for $x < c$ and $f' < 0$ for $x > c$), then f has a local maximum value at c.



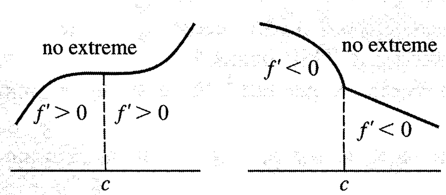
(a) $f'(c) = 0$ (b) $f'(c)$ undefined

2. If f' changes from negative to positive at c ($f' < 0$ for $x < c$ and $f' > 0$ for $x > c$), then f has a local minimum value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

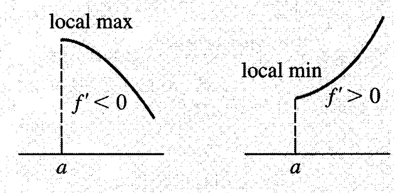
3. If f' does not change sign at c (f' has the same sign on both sides of c), then f has no local extreme value at c.



(a) $f'(c) = 0$ (b) $f'(c)$ undefined

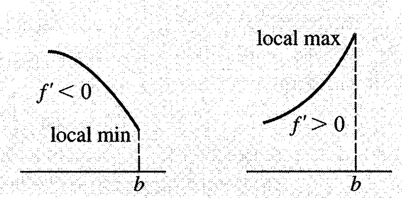
At a left endpoint a:

If $f' < 0$ ($f' > 0$) for $x > a$, then f has a local maximum (minimum) value at a.



At a right endpoint b:

If $f' < 0$ ($f' > 0$) for $x < b$, then f has a local minimum (maximum) value at b.

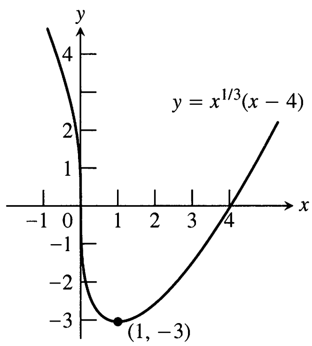


EXAMPLE 1 Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is defined for all real numbers and is continuous (Fig. 3.25).



3.25 The graph of $y = x^{1/3}(x - 4)$ (Example 1).

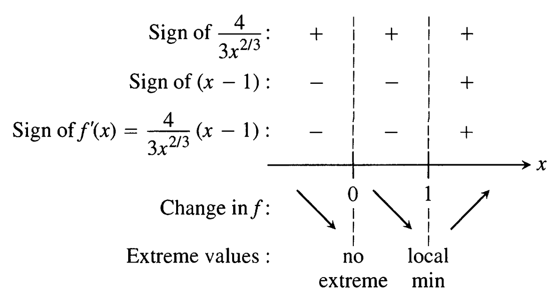
The first derivative

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}(x - 1) = \frac{4(x - 1)}{3x^{2/3}}$$

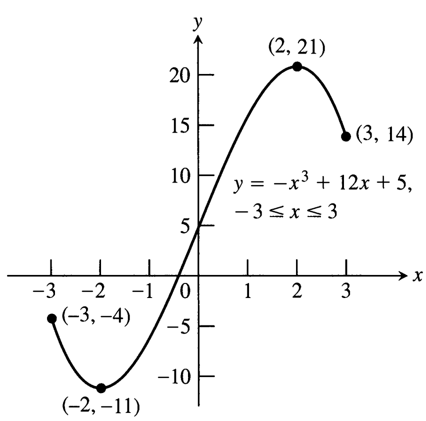
is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in f's domain, so the critical points, $x = 0$ and $x = 1$, are the only places where f might have an extreme value of any kind.

These critical points divide the x-axis into intervals on which f' is either positive or negative. The sign pattern of f' reveals the behavior of f both between and at the critical points. We can display the information in a picture like the following.



To make the picture, we marked the critical points on the x-axis, noted the sign of each factor of f' on the intervals between the points, and "multiplied" the signs of the factors to find the sign of f'. We then applied Corollary 3 of the Mean Value Theorem to determine that f decreases (↘) on $(-\infty, 0)$, decreases on (0, 1), and increases (↗) on $(1, \infty)$. Theorem 5 tells us that f has no extreme at $x = 0$ (f' does not change sign) and that f has a local minimum at $x = 1$ (f' changes from negative to positive).

The value of the local minimum is $f(1) = 1^{1/3}(1 - 4) = -3$. This is also an absolute minimum because the function's values fall toward it from the left and rise away from it on the right. Figure 3.25 shows this value in relation to the function's graph. ❑



3.26 The graph of $g(x) = -x^3 + 12x + 5$, $-3 \le x \le 3$ (Example 2).

EXAMPLE 2 Find the intervals on which

$$g(x) = -x^3 + 12x + 5, \qquad -3 \le x \le 3$$

is increasing and decreasing. Where does the function assume extreme values and what are these values?

Solution The function f is continuous on its domain, $[-3, 3]$ (Fig. 3.26). The first derivative

$$g'(x) = -3x^2 + 12 = -3(x^2 - 4)$$

$$= -3(x + 2)(x - 2),$$

defined at all points of $[-3, 3]$, is zero at $x = -2$ and $x = 2$. These critical points divide the domain of g into intervals on which g' is either positive or negative. We analyze the behavior of g by picturing the sign pattern of g':

	-3 (endpoint) to -2	-2 to 2	2 to 3 (endpoint)
Sign of $-3(x+2)$:	$+$	$-$	$-$
Sign of $(x-2)$:	$-$	$-$	$+$
Sign of $g'(x) = -3(x+2)(x-2)$:	$-$	$+$	$-$
Change in $g(x)$:	↘	↗	↘
Extrema:	-3: local max; -2: local min		2: local max; 3: local min

We conclude that g has local maxima at $x = -3$ and $x = 2$ and local minima at $x = -2$ and $x = 3$. The corresponding values of $g(x) = -x^3 + 12x + 5$ are

Local maxima: $g(-3) = -4, \quad g(2) = 21$

Local minima: $g(-2) = -11, \quad g(3) = 14.$

Since g is defined on a closed interval, we also know that $g(-2)$ is the absolute minimum and $g(2)$ is the absolute maximum. Figure 3.26 shows these values in relation to the function's graph. ❑

Exercises 3.3

Analyzing f Given f'

Answer the following questions about the functions whose derivatives are given in Exercises 1–8:

a) What are the critical points of f?
b) On what intervals is f increasing or decreasing?
c) At what points, if any, does f assume local maximum and minimum values?

1. $f'(x) = x(x-1)$ **2.** $f'(x) = (x-1)(x+2)$

3. $f'(x) = (x-1)^2(x+2)$ **4.** $f'(x) = (x-1)^2(x+2)^2$

5. $f'(x) = (x-1)(x+2)(x-3)$

6. $f'(x) = (x-7)(x+1)(x+5)$

7. $f'(x) = x^{-1/3}(x+2)$ **8.** $f'(x) = x^{-1/2}(x-3)$

Extremes of Given Functions

In Exercises 9–28:

a) Find the intervals on which the function is increasing and decreasing.
b) Then identify the function's local extreme values, if any, saying where they are taken on.
c) Which, if any, of the extreme values are absolute?
d) GRAPHER You may wish to support your findings with a graphing calculator or computer grapher.

9. $g(t) = -t^2 - 3t + 3$ **10.** $g(t) = -3t^2 + 9t + 5$

11. $h(x) = -x^3 + 2x^2$ **12.** $h(x) = 2x^3 - 18x$

13. $f(\theta) = 3\theta^2 - 4\theta^3$ **14.** $f(\theta) = 6\theta - \theta^3$

15. $f(r) = 3r^3 + 16r$ **16.** $h(r) = (r+7)^3$

17. $f(x) = x^4 - 8x^2 + 16$ **18.** $g(x) = x^4 - 4x^3 + 4x^2$

19. $H(t) = \frac{3}{2}t^4 - t^6$ **20.** $K(t) = 15t^3 - t^5$

21. $g(x) = x\sqrt{8 - x^2}$ **22.** $g(x) = x^2\sqrt{5 - x}$

23. $f(x) = \dfrac{x^2 - 3}{x - 2}, \quad x \neq 2$ **24.** $f(x) = \dfrac{x^3}{3x^2 + 1}$

25. $f(x) = x^{1/3}(x+8)$ **26.** $g(x) = x^{2/3}(x+5)$

27. $h(x) = x^{1/3}(x^2 - 4)$ **28.** $k(x) = x^{2/3}(x^2 - 4)$

Extremes on Half-Open Intervals

In Exercises 29–36:

a) Identify the function's local extreme values in the given domain, and say where they are assumed.
b) Which of the extreme values, if any, are absolute?
c) GRAPHER You may wish to support your findings with a graphing calculator or computer grapher.

29. $f(x) = 2x - x^2, \quad -\infty < x \leq 2$

30. $f(x) = (x+1)^2, \quad -\infty < x \leq 0$

31. $g(x) = x^2 - 4x + 4, \quad 1 \le x < \infty$

32. $g(x) = -x^2 - 6x - 9, \quad -4 \le x < \infty$

33. $f(t) = 12t - t^3, \quad -3 \le t < \infty$

34. $f(t) = t^3 - 3t^2, \quad -\infty < t \le 3$

35. $h(x) = \dfrac{x^3}{3} - 2x^2 + 4x, \quad 0 \le x < \infty$

36. $k(x) = x^3 + 3x^2 + 3x + 1, \quad -\infty < x \le 0$

Graphing Calculator or Computer Grapher

In Exercises 37–40:

a) Find the local extrema of each function on the given interval, and say where they are assumed.

b) GRAPHER Graph the function and its derivative together. Comment on the behavior of f in relation to the signs and values of f'.

37. $f(x) = \dfrac{x}{2} - 2\sin\dfrac{x}{2}, \quad 0 \le x \le 2\pi$

38. $f(x) = -2\cos x - \cos^2 x, \quad -\pi \le x \le \pi$

39. $f(x) = \csc^2 x - 2\cot x, \quad 0 < x < \pi$

40. $f(x) = \sec^2 x - 2\tan x, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$

Theory and Examples

Show that the functions in Exercises 41 and 42 have local extreme values at the given values of θ, and say which kind of local extreme the function has.

41. $h(\theta) = 3\cos\dfrac{\theta}{2}, \quad 0 \le \theta \le 2\pi, \quad$ at $\theta = 0$ and $\theta = 2\pi$

42. $h(\theta) = 5\sin\dfrac{\theta}{2}, \quad 0 \le \theta \le \pi, \quad$ at $\theta = 0$ and $\theta = \pi$

43. Sketch the graph of a differentiable function $y = f(x)$ through the point (1, 1) if $f'(1) = 0$ and

a) $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$;
b) $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$;
c) $f'(x) > 0$ for $x \ne 1$;
d) $f'(x) < 0$ for $x \ne 1$.

44. Sketch the graph of a differentiable function $y = f(x)$ that has

a) a local minimum at (1, 1) and a local maximum at (3, 3);
b) a local maximum at (1, 1) and a local minimum at (3, 3);
c) local maxima at (1, 1) and (3, 3);
d) local minima at (1, 1) and (3, 3).

45. Sketch the graph of a continuous function $y = g(x)$ such that

a) $g(2) = 2, \quad 0 < g' < 1$ for $x < 2$, $g'(x) \to 1^-$ as $x \to 2^-$, $-1 < g' < 0$ for $x > 2$, and $g'(x) \to -1^+$ as $x \to 2^+$;
b) $g(2) = 2, \quad g' < 0$ for $x < 2, \quad g'(x) \to -\infty$ as $x \to 2^-$, $g' > 0$ for $x > 2$, and $g'(x) \to \infty$ as $x \to 2^+$.

46. Sketch the graph of a continuous function $y = h(x)$ such that

a) $h(0) = 0$, $-2 \le h(x) \le 2$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$;
b) $h(0) = 0$, $-2 \le h(x) \le 0$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$.

47. As x moves from left to right through the point $c = 2$, is the graph of $f(x) = x^3 - 3x + 2$ rising, or is it falling? Give reasons for your answer.

48. Find the intervals on which the function $f(x) = ax^2 + bx + c$, $a \ne 0$, is increasing and decreasing. Describe the reasoning behind your answer.

3.4 Graphing with *y′* and *y″*

In Section 3.1, we saw the role played by the first derivative in locating a function's extreme values. A function can have extreme values only at the endpoints of its domain and at its critical points. We also saw that critical points do not necessarily yield extreme values. In Section 3.2, we saw that almost all the information about a differentiable function is contained in its derivative. To recover the function completely, the only additional information we need is the value of the function at any one single point. If a function's derivative is $2x$ and the graph passes through the origin, the function must be x^2. If a function's derivative is $2x$ and the graph passes through the point (0, 4), the function must be $x^2 + 4$.

In Section 3.3, we extended our ability to recover information from a function's first derivative by showing how to use it to tell exactly what happens at a critical point. We can tell whether there really is an extreme value there or whether the graph just continues to rise or fall.

In the present section, we show how to determine the way the graph of a function $y = f(x)$ bends or turns. We know that the information must be contained in y', but how do we find it? The answer, for functions that are twice differentiable except perhaps at isolated points, is to differentiate y'. Together y' and y'' tell us the shape of the function's graph. We will see in Chapter 4 how this enables us to sketch solutions of differential equations and initial value problems.

Concavity

As you can see in Fig. 3.27, the curve $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ turn in different ways. As we come in from the left toward the origin along the curve, the curve turns to our right and falls below its tangents. As we leave the origin, the curve turns to our left and rises above its tangents.

To put it another way, the slopes of the tangents decrease as the curve approaches the origin from the left and increase as the curve moves from the origin into the first quadrant.

Definition

The graph of a differentiable function $y = f(x)$ is **concave up** on an interval where y' is increasing and **concave down** on an interval where y' is decreasing.

If $y = f(x)$ has a second derivative, we can apply Corollary 3 of the Mean Value Theorem to conclude that y' increases if $y'' > 0$ and decreases if $y'' < 0$.

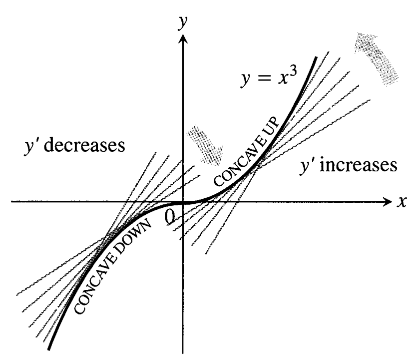
The Second Derivative Test for Concavity

Let $y = f(x)$ be twice differentiable on an interval I.

1. If $y'' > 0$ on I, the graph of f over I is concave up.
2. If $y'' < 0$ on I, the graph of f over I is concave down.

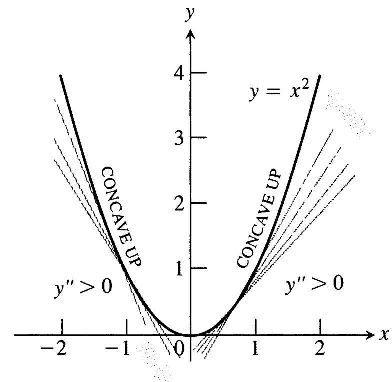
EXAMPLE 1

a) The curve $y = x^3$ (Fig. 3.27) is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.



3.27 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

b) The parabola $y = x^2$ (Fig. 3.28) is concave up on every interval because $y'' = 2 > 0$.



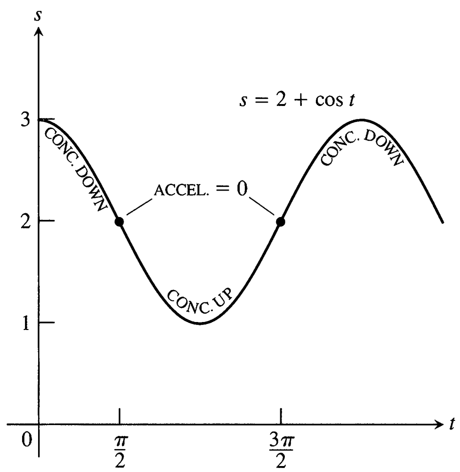
3.28 The graph of $f(x) = x^2$ on any interval is concave up. ❑

Points of Inflection

To study the motion of a body moving along a line, we often graph the body's position as a function of time. One reason for doing so is to reveal where the body's acceleration, given by the second derivative, changes sign. On the graph, these are the points where the concavity changes.

Definition

A point where the graph of a function has a tangent line and where the concavity changes is called a **point of inflection.**



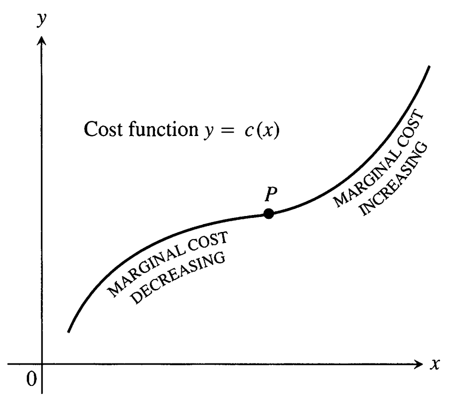
3.29 The motion in Example 2.

Thus a point of inflection on a curve is a point where y'' is positive on one side and negative on the other. At such a point, y'' is either zero (because derivatives have the intermediate value property) or undefined.

On the graph of a twice-differentiable function, $y'' = 0$ at a point of inflection.

EXAMPLE 2 *Simple harmonic motion*

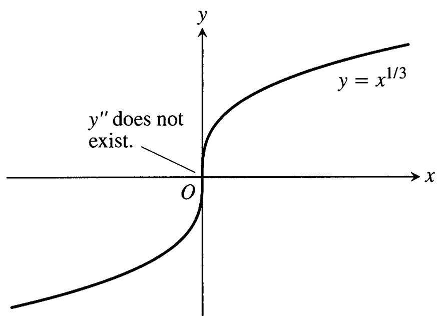
The graph of $s = 2 + \cos t$, $t \geq 0$ (Fig. 3.29), changes concavity at $t = \pi/2, 3\pi/2, \ldots$, where the acceleration $s'' = -\cos t$ is zero. ❑



3.30 The point of inflection on a typical cost curve separates the interval of decreasing marginal cost from the interval of increasing marginal cost. This is the point where the marginal cost is smallest (Example 3).

EXAMPLE 3 *Marginal cost*

Inflection points have applications in some areas of economics. Suppose that $y = c(x)$ is the total cost of producing x units of something (Fig. 3.30). The point of inflection at P is then the point at which the marginal cost (the approximate cost of producing one more unit) changes from decreasing to increasing. ❑



3.31 A point where y'' fails to exist can be a point of inflection.

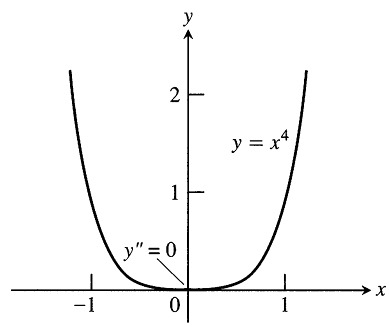
EXAMPLE 4 *An inflection point where y'' does not exist*

The curve $y = x^{1/3}$ has a point of inflection at $x = 0$ (Fig. 3.31), but y'' does not exist there.

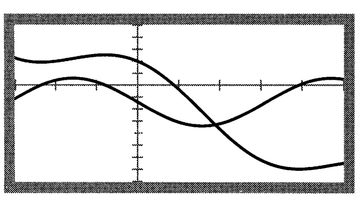
$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}$$ ❑

EXAMPLE 5 *No inflection where $y'' = 0$*

The curve $y = x^4$ has no inflection point at $x = 0$ (Fig. 3.32). Even though $y'' = 12x^2$ is zero there, it does not change sign.



3.32 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there. ❑

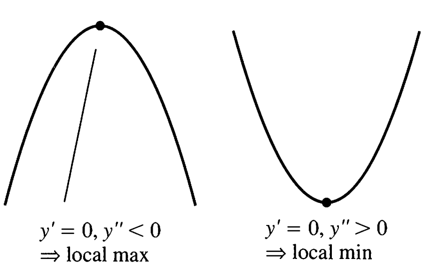


The graph of $y = 2 \cos x - \sqrt{2}\, x$ and its first derivative.

Technology *Graphing a Function with Its Derivatives* When we graph a function $y = f(x)$, it may be difficult to identify the inflection points exactly by zooming in. Try it on the curve $y = 2\cos x - \sqrt{2}x$, $-\pi \le x \le 3\pi/2$. Adding the graph of f' to the display can help to identify inflection points more closely, but the strongest visual evidence comes from graphing f and f'' together. It is interesting to watch all three functions, f, f', and f'', being graphed simultaneously.

The Second Derivative Test for Local Extreme Values

Instead of examining y' for sign changes at a critical point, we can sometimes use the following test to determine the presence of a local extremum.



The Second Derivative Test for Local Extreme Values

If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

Notice that the test requires us to know y'' only at c itself, and not in an interval about c. This makes the test easy to apply. That's the good news. The bad news is that the test is inconclusive if $y'' = 0$ or if y'' does not exist. When this happens, use the first derivative test for local extreme values.

Graphing with y' and y''

We now apply what we have learned to sketch the graphs of functions.

Testing the critical points in Example 6

As a quick test to see if any of the critical points are local extreme values, we could try the second derivative test.

At $x = 3$, $y'' > 0$:
We now know that this point is definitely a local minimum.

At $x = 0$, $y'' = 0$:
Test fails, and so we will need to check the signs of y' to know whether this point gives a local extreme value.

EXAMPLE 6 Graph the function

$$y = x^4 - 4x^3 + 10.$$

Solution

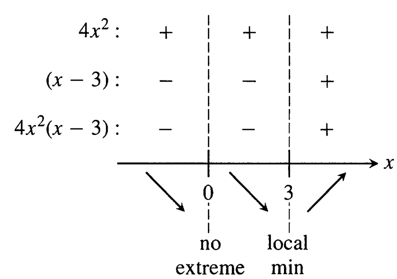
Step 1: *Find y' and y''.*

$$y = x^4 - 4x^3 + 10$$

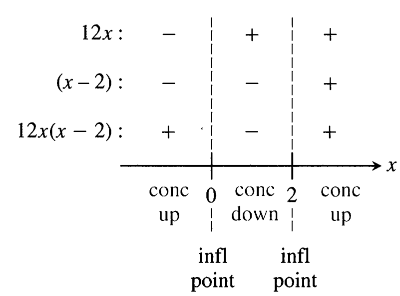
$$y' = 4x^3 - 12x^2 = 4x^2(x - 3)$$ Critical points: $x = 0$, $x = 3$

$$y'' = 12x^2 - 24x = 12x(x - 2)$$ Possible inflection points: $x = 0$, $x = 2$

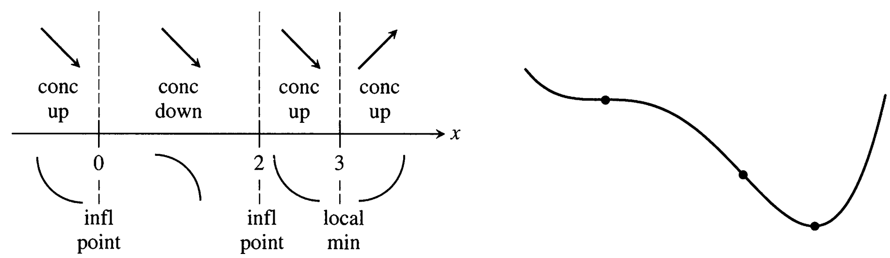
Step 2: *Rise and fall.* Sketch the sign pattern for y' and use it to describe the behavior of y.



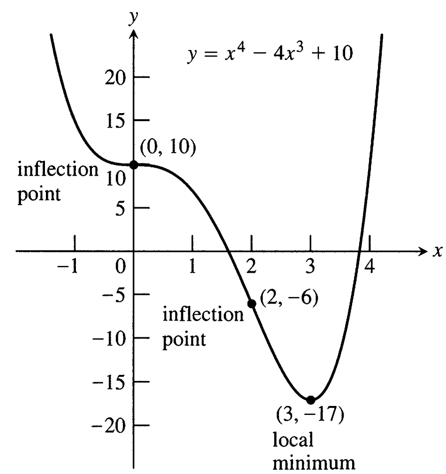
Step 3: *Concavity.* Sketch the sign pattern for y'' and use it to describe the way the graph bends.



Step 4: *Summary and general shape.* Summarize the information from steps 2 and 3. Show the shape over each interval. Then combine the shapes to show the curve's general form.



Step 5: *Specific points and curve.* Plot the curve's intercepts (if convenient) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape in step 4 as a guide to sketch the curve. (Plot additional points as needed.)



❑

The steps in Example 6 give a general procedure for graphing by hand.

Strategy for Graphing $y = f(x)$

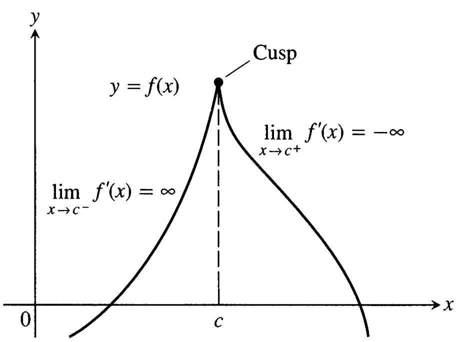
1. Find y' and y''.
2. Find the rise and fall of the curve.
3. Determine the concavity of the curve.
4. Make a summary and show the curve's general shape.
5. Plot specific points and sketch the curve.

EXAMPLE 7 Graph $y = x^{5/3} - 5x^{2/3}$.

Cusps

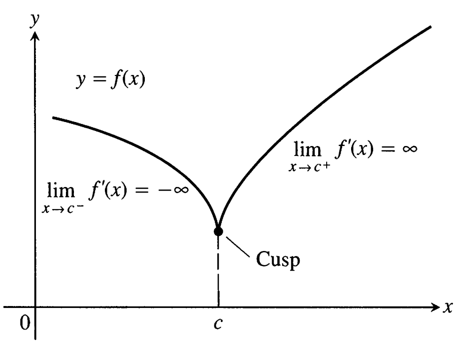
The graph of a continuous function $y = f(x)$ has a *cusp* at a point $x = c$ if the concavity is the same on both sides of c and either

1. $\lim_{x\to c^-} f'(x) = \infty$ and $\lim_{x\to c^+} f'(x) = -\infty$



or

2. $\lim_{x\to c^-} f'(x) = -\infty$ and $\lim_{x\to c^+} f'(x) = \infty$.



A cusp can be either a local maximum (1) or a local minimum (2).

Solution

Step 1: *Find y' and y''.*

$$y = x^{5/3} - 5x^{2/3} = x^{2/3}(x-5)$$

$$y' = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x-2)$$

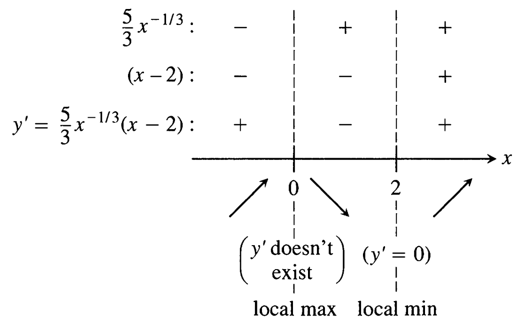
$$y'' = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x+1)$$

The x-intercepts are at $x = 0$ and $x = 5$.

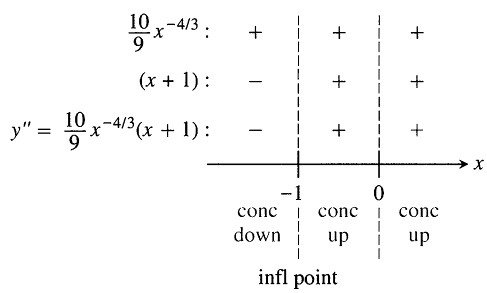
Critical points: $x = 0$, $x = 2$

Possible inflection points: $x = 0$, $x = -1$

Step 2: *Rise and fall.*



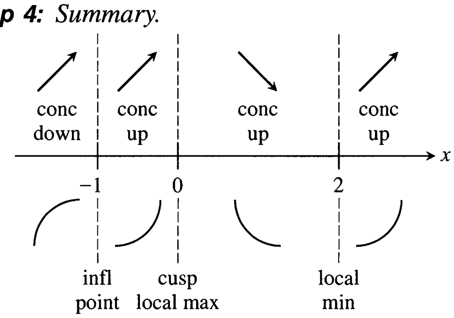
Step 3: *Concavity.*



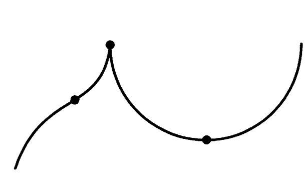
From the sign pattern for y'', we see that there is an inflection point at $x = -1$, but not at $x = 0$. However, knowing that

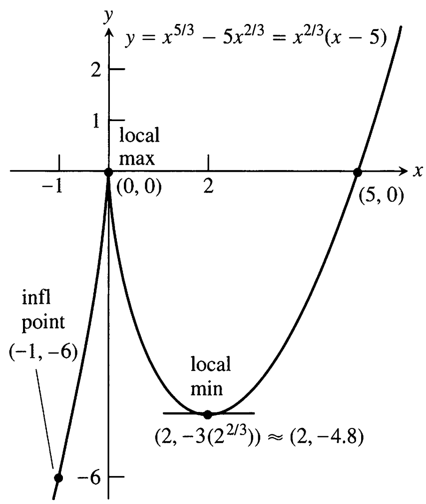
1. the function $y = x^{5/3} - 5x^{2/3}$ is continuous,
2. $y' \to \infty$ as $x \to 0^-$ and $y' \to -\infty$ as $x \to 0^+$ (see the formula for y' in step 2), and
3. the concavity does not change at $x = 0$ (step 3) tells us that the graph has a *cusp* at $x = 0$.

Step 4: *Summary.*



General shape.



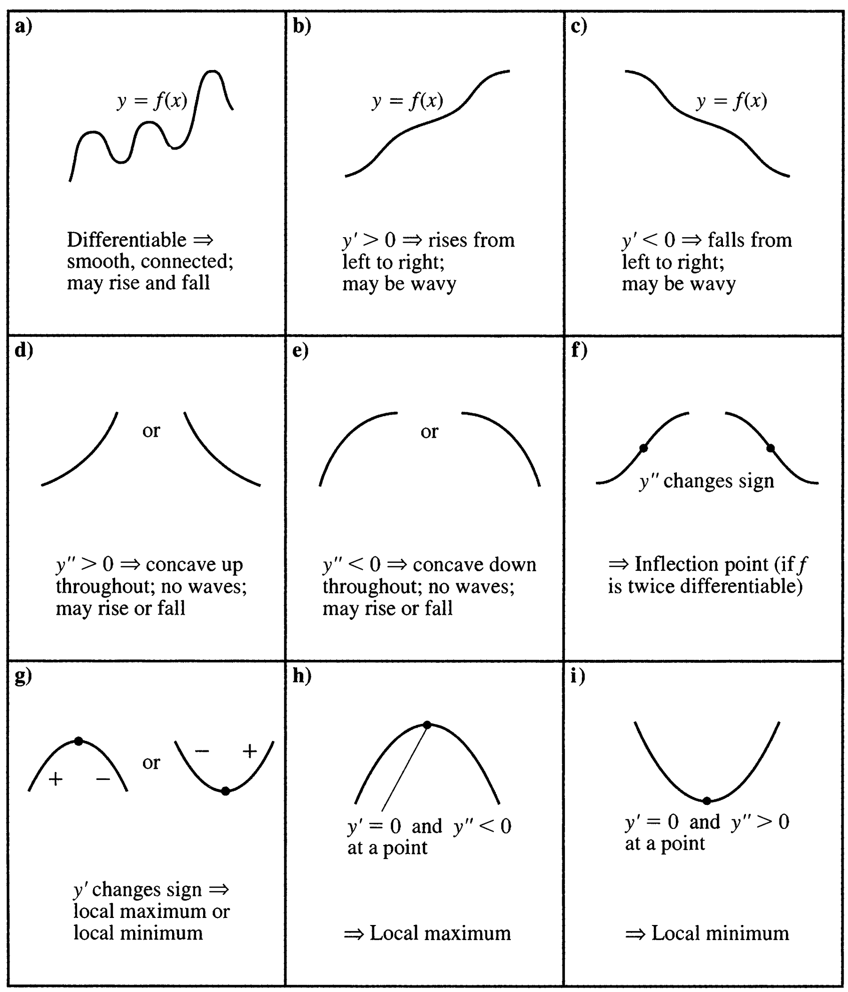


Step 5: *Specific points and curve.* See the figure to the left. ❑

Learning About Functions from Derivatives

Pause for a moment to see how remarkable the conclusions in Examples 6 and 7 really are. In each case, we have been able to recover almost everything we need to know about a differentiable function $y = f(x)$ by examining y'. We can find where the graph rises and falls and where the local extremes are assumed. We can differentiate y' to learn how the graph bends as it passes over the intervals of rise and fall. We can determine the shape of the function's graph. The only information we cannot get from the derivative is how to place the graph in the xy-plane. That requires evaluating the formula for f at various points. Or so it seems. But as we saw in Section 3.2, even *that* is nearly superfluous. All we really need, in addition to y', is the value of f at a single point.

What Derivatives Tell Us About Graphs



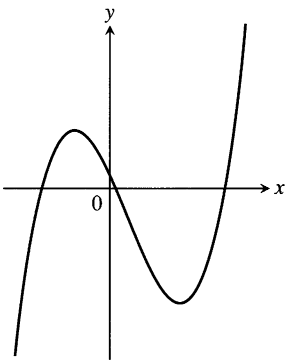
Exercises 3.4

Analyzing Graphed Functions

Identify the inflection points and local maxima and minima of the functions graphed in Exercises 1–8. Identify the intervals on which the functions are concave up and concave down.

1.

$y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$



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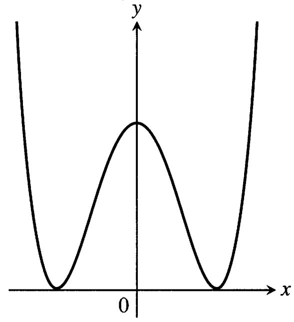
2.

$y = \frac{x^4}{4} - 2x^2 + 4$



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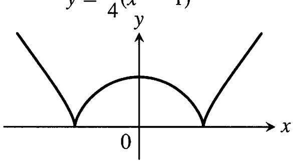
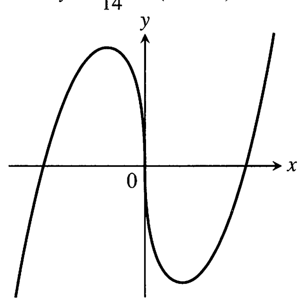
3.

$y = \frac{3}{4}(x^2 - 1)^{2/3}$



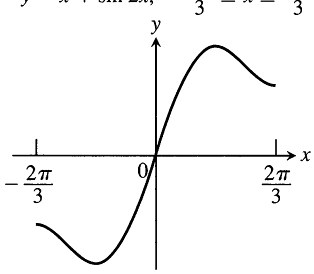
4.

$y = \frac{9}{14}x^{1/3}(x^2 - 7)$



5.

$y = x + \sin 2x, \ -\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$



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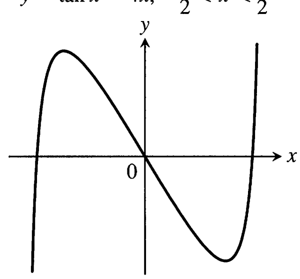
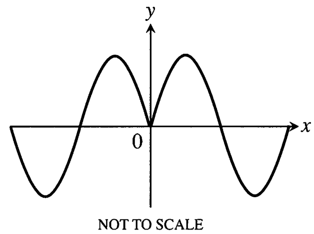
6.

$y = \tan x - 4x, \ -\frac{\pi}{2} < x < \frac{\pi}{2}$



7.

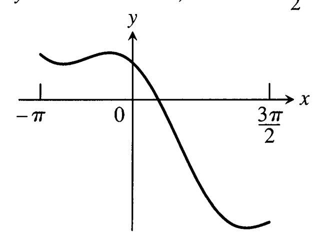
$y = \sin|x|, -2\pi \le x \le 2\pi$



(Generated by Mathematica)

8.

$y = 2\cos x - \sqrt{2}x, -\pi \le x \le \frac{3\pi}{2}$



Graphing Equations

Use the steps of the graphing procedure on page 214 to graph the equations in Exercises 9–40. Include the coordinates of any local extreme points and inflection points.

9. $y = x^2 - 4x + 3$

10. $y = 6 - 2x - x^2$

11. $y = x^3 - 3x + 3$

12. $y = x(6 - 2x)^2$

13. $y = -2x^3 + 6x^2 - 3$

14. $y = 1 - 9x - 6x^2 - x^3$

15. $y = (x - 2)^3 + 1$

16. $y = 1 - (x + 1)^3$

17. $y = x^4 - 2x^2 = x^2(x^2 - 2)$

18. $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$

19. $y = 4x^3 - x^4 = x^3(4 - x)$

20. $y = x^4 + 2x^3 = x^3(x + 2)$

21. $y = x^5 - 5x^4 = x^4(x - 5)$

22. $y = x\left(\frac{x}{2} - 5\right)^4$

23. $y = x + \sin x, \quad 0 \le x \le 2\pi$

24. $y = x - \sin x, \quad 0 \le x \le 2\pi$

25. $y = x^{1/5}$

26. $y = x^{3/5}$

27. $y = x^{2/5}$

28. $y = x^{4/5}$

29. $y = 2x - 3x^{2/3}$

30. $y = 5x^{2/5} - 2x$

31. $y = x^{2/3}\left(\frac{5}{2} - x\right)$

32. $y = x^{2/3}(x - 5)$

33. $y = x\sqrt{8 - x^2}$

34. $y = (2 - x^2)^{3/2}$

35. $y = \frac{x^2 - 3}{x - 2}, \quad x \ne 2$

36. $y = \frac{x^3}{3x^2 + 1}$

37. $y = |x^2 - 1|$

38. $y = |x^2 - 2x|$

39. $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x \le 0 \\ \sqrt{x}, & x > 0 \end{cases}$

40. $y = \sqrt{|x - 4|}$

Sketching the General Shape Knowing y'

Each of Exercises 41–62 gives the first derivative of a continuous function $y = f(x)$. Find y'' and then use steps 2–4 of the graphing procedure on page 214 to sketch the general shape of the graph of f.

41. $y' = 2 + x - x^2$

42. $y' = x^2 - x - 6$

43. $y' = x(x-3)^2$

44. $y' = x^2(2-x)$

45. $y' = x(x^2 - 12)$

46. $y' = (x-1)^2(2x+3)$

47. $y' = (8x - 5x^2)(4-x)^2$

48. $y' = (x^2 - 2x)(x-5)^2$

49. $y' = \sec^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

50. $y' = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

51. $y' = \cot \frac{\theta}{2}, \quad 0 < \theta < 2\pi$

52. $y' = \csc^2 \frac{\theta}{2}, \quad 0 < \theta < 2\pi$

53. $y' = \tan^2 \theta - 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

54. $y' = 1 - \cot^2 \theta, \quad 0 < \theta < \pi$

55. $y' = \cos t, \quad 0 \le t \le 2\pi$

56. $y' = \sin t, \quad 0 \le t \le 2\pi$

57. $y' = (x+1)^{-2/3}$

58. $y' = (x-2)^{-1/3}$

59. $y' = x^{-2/3}(x-1)$

60. $y' = x^{-4/5}(x+1)$

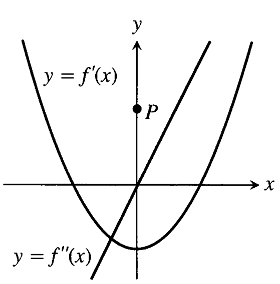
61. $y' = 2|x| = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$

62. $y' = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$

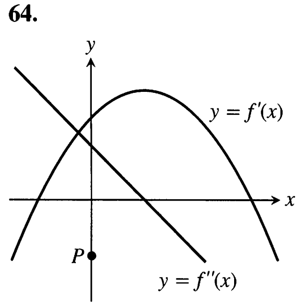
Sketching y from Graphs of y' and y''

Each of Exercises 63–66 shows the graphs of the first and second derivatives of a function $y = f(x)$. Copy the picture and add to it a sketch of the approximate graph of f, given that the graph passes through the point P.

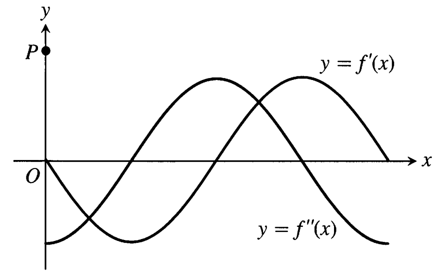
63.



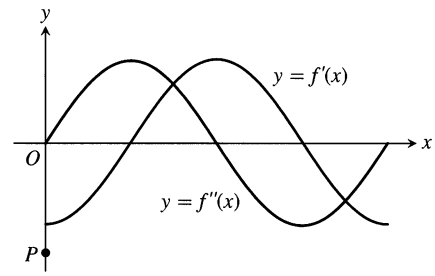
64.



65.

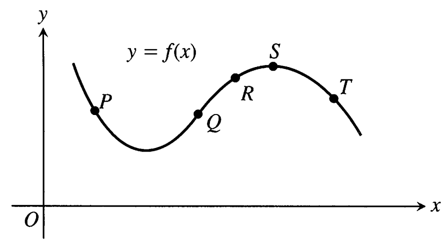


66.



Theory and Examples

67. The accompanying figure shows a portion of the graph of a twice-differentiable function $y = f(x)$. At each of the five labeled points, classify y' and y'' as positive, negative, or zero.



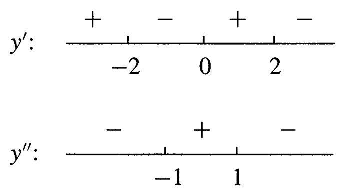
68. Sketch a smooth connected curve $y = f(x)$ with

$$f(-2) = 8, \quad f'(2) = f'(-2) = 0,$$
$$f(0) = 4, \quad f'(x) < 0 \text{ for } |x| < 2,$$
$$f(2) = 0, \quad f''(x) < 0 \text{ for } x < 0,$$
$$f'(x) > 0 \text{ for } |x| > 2, \quad f''(x) > 0 \text{ for } x > 0.$$

69. Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

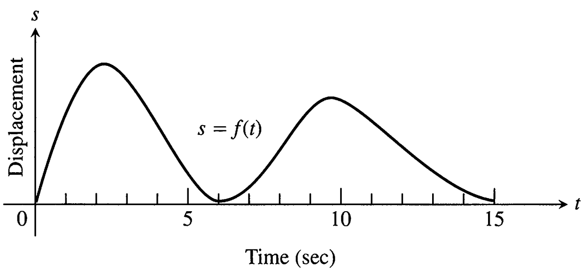
x	y	Derivatives
$x < 2$		$y' < 0, \quad y'' > 0$
2	1	$y' = 0, \quad y'' > 0$
$2 < x < 4$		$y' > 0, \quad y'' > 0$
4	4	$y' > 0, \quad y'' = 0$
$4 < x < 6$		$y' > 0, \quad y'' < 0$
6	7	$y' = 0, \quad y'' < 0$
$x > 6$		$y' < 0, \quad y'' < 0$

70. Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-2, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$ and $(2, 2)$ and whose first two derivatives have the following sign patterns:

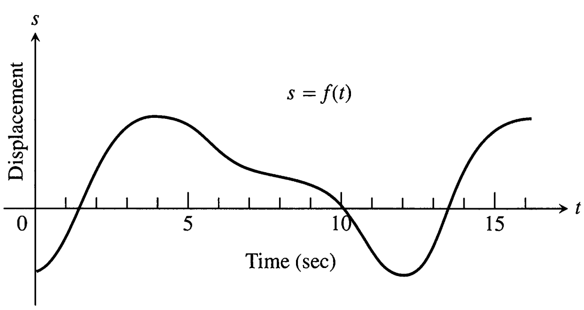


Velocity and acceleration. The graphs in Exercises 71 and 72 show the position $s = f(t)$ of a body moving back and forth on a coordinate line. (a) When is the body moving away from the origin? toward the origin? At approximately what times is the (b) velocity equal to zero? (c) acceleration equal to zero? (d) When is the acceleration positive? negative?

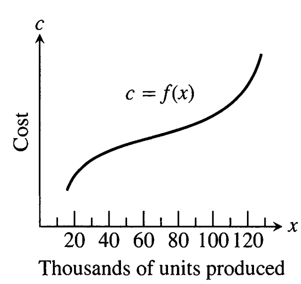
71.



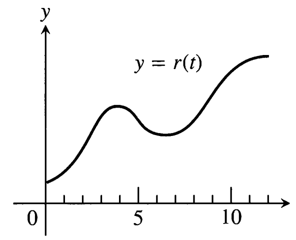
72.



73. *Marginal cost.* The accompanying graph shows the hypothetical cost $c = f(x)$ of manufacturing x items. At approximately what production level does the marginal cost change from decreasing to increasing?



74. The accompanying graph shows the monthly revenue of the Widget Corporation for the last twelve years. During approximately what time intervals was the marginal revenue increasing? decreasing?



75. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection? (*Hint:* Draw the sign pattern for y'.)

76. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2)(x - 4).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

77. For $x > 0$, sketch a curve $y = f(x)$ that has $f(1) = 0$ and $f'(x) = 1/x$. Can anything be said about the concavity of such a curve? Give reasons for your answer.

78. Can anything be said about the graph of a function $y = f(x)$ that has a continuous second derivative that is never zero? Give reasons for your answer.

79. If b, c, and d are constants, for what value of b will the curve $y = x^3 + bx^2 + cx + d$ have a point of inflection at $x = 1$? Give reasons for your answer.

80. *Horizontal tangents.* True, or false? Explain.

a) The graph of every polynomial of even degree (largest exponent even) has at least one horizontal tangent.

b) The graph of every polynomial of odd degree (largest exponent odd) has at least one horizontal tangent.

81. *Parabolas*

a) Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c$, $a \neq 0$.

b) When is the parabola concave up? concave down? Give reasons for your answers.

82. Is it true that the concavity of the graph of a twice-differentiable function $y = f(x)$ changes every time $f''(x) = 0$? Give reasons for your answer.

83. *Quadratic curves.* What can you say about the inflection points of a quadratic curve $y = ax^2 + bx + c$, $a \neq 0$? Give reasons for your answer.

84. *Cubic curves.* What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d$, $a \neq 0$? Give reasons for your answer.

Grapher Explorations

In Exercises 85–88, find the inflection points (if any) on the graph of the function and the coordinates of the points on the graph where the function has a local maximum or local minimum value. Then graph the function in a region large enough to show all these points simultaneously. Add to your picture the graphs of the function's first and second derivatives. How are the values at which these graphs intersect the x-axis related to the graph of the function? In what other ways are the graphs of the derivatives related to the graph of the function?

85. $y = x^5 - 5x^4 - 240$

86. $y = x^3 - 12x^2$

87. $y = \frac{4}{5}x^5 + 16x^2 - 25$

88. $y = \frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + 20$

89. Graph $f(x) = 2x^4 - 4x^2 + 1$ and its first two derivatives together. Comment on the behavior of f in relation to the signs and values of f' and f''.

90. Graph $f(x) = x \cos x$ and its second derivative together for $0 \le x \le 2\pi$. Comment on the behavior of the graph of f in relation to the signs and values of f''.

91. a) On a common screen, graph $f(x) = x^3 + kx$ for $k = 0$ and nearby positive and negative values of k. How does the value of k seem to affect the shape of the graph?

b) Find $f'(x)$. As you will see, $f'(x)$ is a quadratic function of x. Find the discriminant of the quadratic (the discriminant of $ax^2 + bx + c$ is $b^2 - 4ac$). For what values of k is the discriminant positive? zero? negative? For what values of k does f' have two zeros? one or no zeros? Now explain what the value of k has to do with the shape of the graph of f.

c) Experiment with other values of k. What appears to happen as $k \to -\infty$? as $k \to \infty$?

92. a) On a common screen, graph $f(x) = x^4 + kx^3 + 6x^2$, $-1 \le x \le 4$ for $k = -4$, and some nearby values of k. How does the value of k seem to affect the shape of the graph?

b) Find $f''(x)$. As you will see, $f''(x)$ is a quadratic function of x. What is the discriminant of this quadratic (see Exercise 91b)? For what values of k is the discriminant positive? zero? negative? For what values of k does $f''(x)$ have two zeros? one or no zeros? Now explain what the value of k has to do with the shape of the graph of f.

93. a) Graph $y = x^{2/3}(x^2 - 2)$ for $-3 \le x \le 3$. Then use calculus to confirm what the screen shows about concavity, rise, and fall. (Depending on your grapher, you may have to enter $x^{2/3}$ as $(x^2)^{1/3}$ to obtain a plot for negative values of x.)

b) Does the curve have a cusp at $x = 0$, or does it just have a corner with different right-hand and left-hand derivatives?

94. a) Graph $y = 9x^{2/3}(x - 1)$ for $-0.5 \le x \le 1.5$. Then use calculus to confirm what the screen shows about concavity, rise, and fall. What concavity does the curve have to the left of the origin? (Depending on your grapher, you may have to enter $x^{2/3}$ as $(x^2)^{1/3}$ to obtain a plot for negative values of x.)

b) Does the curve have a cusp at $x = 0$, or does it just have a corner with different right-hand and left-hand derivatives?

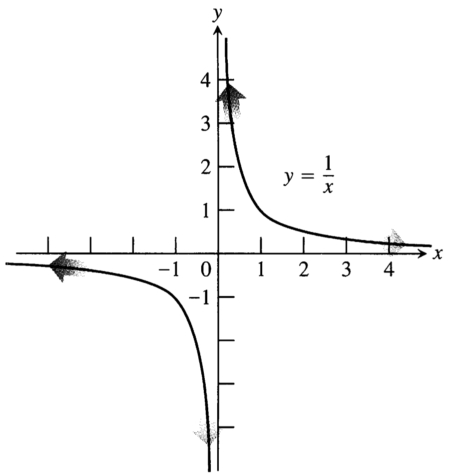
95. Does the curve $y = x^2 + 3 \sin 2x$ have a horizontal tangent near $x = -3$? Give reasons for your answer.

3.5 Limits as $x \to \pm\infty$, Asymptotes, and Dominant Terms

In this section, we analyze the graphs of rational functions (quotients of polynomial functions), as well as other functions with interesting limit behavior as $x \to \pm\infty$. Among the tools we use are asymptotes and dominant terms.

Limits as $x \to \pm\infty$

The function $f(x) = 1/x$ is defined for all $x \ne 0$ (Fig. 3.33). When x is positive and becomes increasingly large, $1/x$ becomes increasingly small. When x is negative and its magnitude becomes increasingly large, $1/x$ again becomes small. We summarize these observations by saying that $f(x) = 1/x$ has limit 0 as $x \to \pm\infty$.



3.33 The graph of $y = 1/x$.

Definitions

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x\to-\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

The symbol infinity (∞)

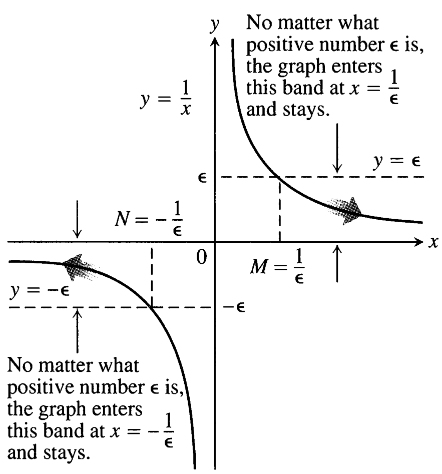
As always, the symbol ∞ does not represent a real number and we cannot use it in arithmetic in the usual way.

The strategy for calculating limits of functions as $x \to \pm\infty$ is similar to the one for finite limits in Section 1.2. There, we first found the limits of the constant and identity functions $y = k$ and $y = x$. We then extended these results to other functions by applying a theorem about limits of algebraic combinations. Here we do the same thing, except that the starting functions are $y = k$ and $y = 1/x$ instead of $y = k$ and $y = x$.

The basic facts to be verified by applying the formal definition are

$$\lim_{x\to\pm\infty} k = k \qquad \text{and} \qquad \lim_{x\to\pm\infty} \frac{1}{x} = 0. \tag{1}$$

We prove the latter and leave the former to Exercises 87 and 88.



3.34 The geometry behind the argument in Example 1.

EXAMPLE 1 Show that

a) $\displaystyle\lim_{x\to\infty} \frac{1}{x} = 0$ **b)** $\displaystyle\lim_{x\to-\infty} \frac{1}{x} = 0.$

Solution

a) Let $\epsilon > 0$ be given. We must find a number M such that for all x

$$x > M \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $M = 1/\epsilon$ or any larger positive number (Fig. 3.34). This proves $\lim_{x\to\infty} (1/x) = 0$.

b) Let $\epsilon > 0$ be given. We must find a number N such that for all x

$$x < N \quad \Rightarrow \quad \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon.$$

The implication will hold if $N = -1/\epsilon$ or any number less than $-1/\epsilon$ (Fig. 3.34). This proves $\lim_{x\to-\infty} (1/x) = 0$. ❑

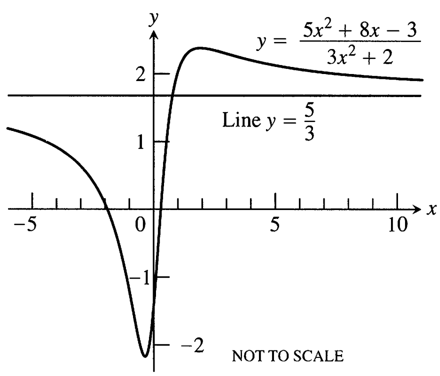
The following theorem enables us to build on Eqs. (1) to calculate other limits.

Theorem 6
Properties of Limits as $x \to \pm\infty$

The following rules hold if $\lim_{x\to\pm\infty} f(x) = L$ and $\lim_{x\to\pm\infty} g(x) = M$ (L and M real numbers).

1. *Sum Rule:* $\lim_{x\to\pm\infty} [f(x) + g(x)] = L + M$
2. *Difference Rule:* $\lim_{x\to\pm\infty} [f(x) - g(x)] = L - M$
3. *Product Rule:* $\lim_{x\to\pm\infty} f(x) \cdot g(x) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to\pm\infty} kf(x) = kL$ (any number k)
5. *Quotient Rule:* $\lim_{x\to\pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, if $M \neq 0$
6. *Power Rule:* If m and n are integers, then $\lim_{x\to\pm\infty} [f(x)]^{m/n} = L^{m/n}$ provided $L^{m/n}$ is a real number.

These properties are just like the properties in Theorem 1, Section 1.2, and we use them the same way.



3.35 The function in Example 3.

EXAMPLE 2

a) $$\lim_{x\to\infty} \left(5 + \frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{1}{x} \qquad \text{Sum Rule}$$
$$= 5 + 0 = 5 \qquad \text{Known values}$$

b) $$\lim_{x\to-\infty} \frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty} \pi\sqrt{3} \cdot \frac{1}{x} \cdot \frac{1}{x}$$
$$= \lim_{x\to-\infty} \pi\sqrt{3} \cdot \lim_{x\to-\infty} \frac{1}{x} \cdot \lim_{x\to-\infty} \frac{1}{x} \qquad \text{Product Rule}$$
$$= \pi\sqrt{3} \cdot 0 \cdot 0 = 0 \qquad \text{Known values}$$

❑

Limits of Rational Functions as $x \to \pm\infty$

To determine the limit of a rational function as $x \to \pm\infty$, we can divide the numerator and denominator by the highest power of x in the denominator. What happens then depends on the degrees of the polynomials involved.

The **degree** of the polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

$a_n \neq 0$, is n, the largest exponent.

EXAMPLE 3 *Numerator and denominator of same degree*

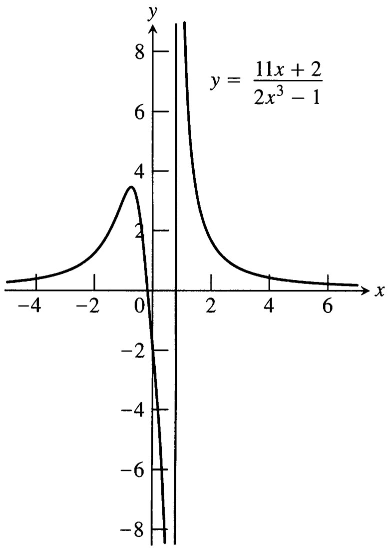
$$\lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x\to\infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)} \qquad \text{Divide numerator and denominator by } x^2.$$
$$= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3} \qquad \text{See Fig. 3.35.}$$

❑

EXAMPLE 4 *Degree of numerator less than degree of denominator*

$$\lim_{x\to-\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to-\infty} \frac{(11/x^2)+(2/x^3)}{2-(1/x^3)}$$ Divide numerator and denominator by x^3.

$$= \frac{0+0}{2-0} = 0$$ See Fig. 3.36. ❑



3.36 The graph of the function in Example 4. The graph approaches the x-axis as $|x|$ increases.

EXAMPLE 5 *Degree of numerator greater than degree of denominator*

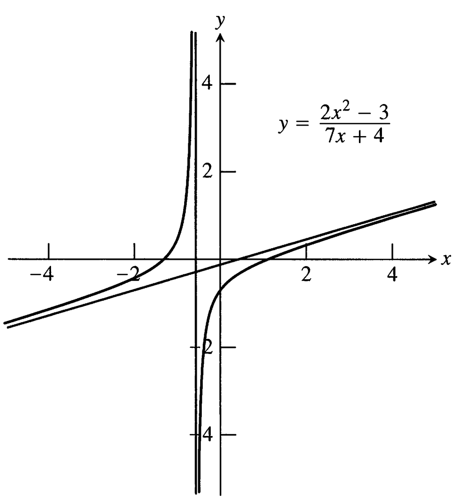
a) $$\lim_{x\to-\infty} \frac{2x^2-3}{7x+4} = \lim_{x\to-\infty} \frac{2x-(3/x)}{7+(4/x)}$$ Divide numerator and denominator by x.

$$= -\infty$$ The numerator now approaches $-\infty$ while the denominator approaches 7, so the ratio $\to -\infty$. See Fig. 3.37.

b) $$\lim_{x\to-\infty} \frac{-4x^3+7x}{2x^2-3x-10} = \lim_{x\to-\infty} \frac{-4x+(7/x)}{2-(3/x)-(10/x^2)}$$ Divide numerator and denominator by x^2.

$$= \infty$$ Numerator $\to \infty$. Denominator $\to 2$. Ratio $\to \infty$.

❑



3.37 The function in Example 5(a).

Examples 3–5 reveal a pattern for finding limits of rational functions as $x \to \pm\infty$.

1. If the numerator and the denominator have the same degree, the limit is the ratio of the polynomials' leading coefficients (Example 3).
2. If the degree of the numerator is less than the degree of the denominator, the limit is zero (Example 4).
3. If the degree of the numerator is greater than the degree of the denominator, the limit is $+\infty$ or $-\infty$, depending on the signs assumed by the numerator and denominator as $|x|$ becomes large (Example 5).

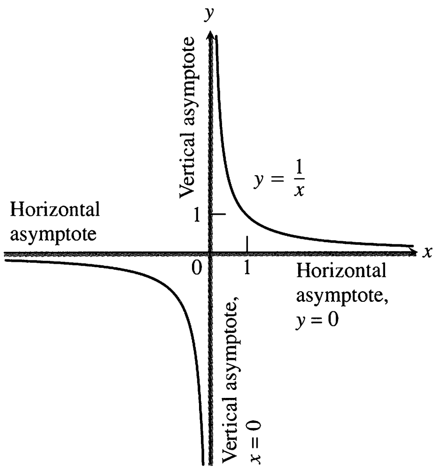
The **leading coefficient** of the polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$, is a_n, the coefficient of the highest-powered term.

Summary for Rational Functions

1. If deg $(f) =$ deg (g), $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{a_n}{b_n}$, the ratio of the leading coefficients of f and g.
2. If deg $(f) <$ deg (g), $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 0$.
3. If deg $(f) >$ deg (g), $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \pm\infty$, depending on the signs of numerator and denominator.

Horizontal and Vertical Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an *asymptote* of the graph.



3.38 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

EXAMPLE 6 The coordinate axes are asymptotes of the curve $y = 1/x$ (Fig. 3.38). The x-axis is an asymptote of the curve on the right because

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

and on the left because

$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

The y-axis is an asymptote of the curve both above and below because

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \qquad \text{and} \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Notice that the denominator is zero at $x = 0$ and the function is undefined. ❑

Definitions

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = b.$$

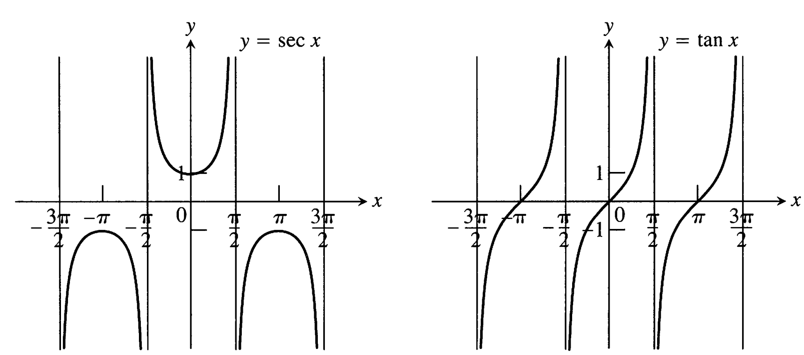
A line $x = a$ is a **vertical asymptote** of the graph if either

$$\lim_{x\to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x\to a^-} f(x) = \pm\infty.$$

EXAMPLE 7 The curves

$$y = \sec x = \frac{1}{\cos x} \qquad \text{and} \qquad y = \tan x = \frac{\sin x}{\cos x}$$

both have vertical asymptotes at odd-integer multiples of $\pi/2$, where $\cos x = 0$ (Fig. 3.39).

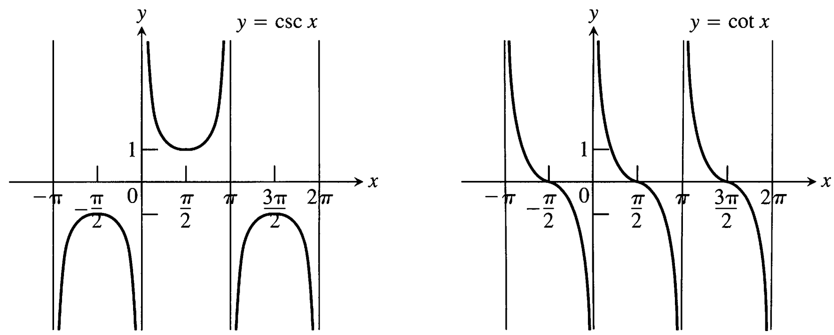


3.39 The graphs of sec x and tan x (Example 7).

The graphs of

$$y = \csc x = \frac{1}{\sin x} \qquad \text{and} \qquad y = \cot x = \frac{\cos x}{\sin x}$$

have vertical asymptotes at integer multiples of π, where $\sin x = 0$ (Fig. 3.40).



3.40 The graphs of csc x and cot x (Example 7). ❑

EXAMPLE 8 Find the asymptotes of the curve

$$y = \frac{x+3}{x+2}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to -2$, where the denominator is zero.

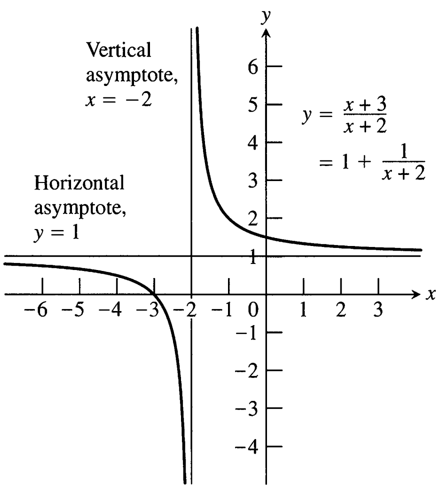
The asymptotes are quickly revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x+2)$ into $(x+3)$.

$$\begin{array}{r} 1 \\ x+2 \overline{)\, x+3} \\ \underline{x+2} \\ 1 \end{array}$$

This enables us to rewrite y:

$$y = 1 + \frac{1}{x+2}$$

From this we see that the curve in question is the graph of $y = 1/x$ shifted 1 unit up and 2 units left (Fig. 3.41). The asymptotes, instead of being the coordinate axes, are now the lines $y = 1$ and $x = -2$. ❑

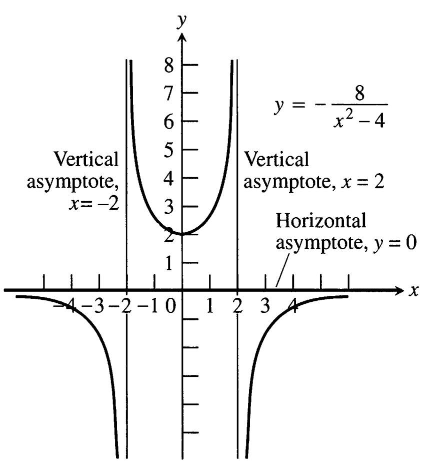


3.41 The lines $y = 1$ and $x = -2$ are asymptotes of the curve $y = (x+3)/(x+2)$ (Example 8).

EXAMPLE 9 Find the asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to \pm 2$, where the denominator is zero. Notice that f is an even function of x, so its graph is symmetric with respect to the y-axis.



3.42 The graph of $y = -8/(x^2 - 4)$ (Example 9). Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided.

The behavior as $x \to \pm\infty$. Since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is an asymptote of the graph to the right. By symmetry it is an asymptote to the left as well (Fig. 3.42).

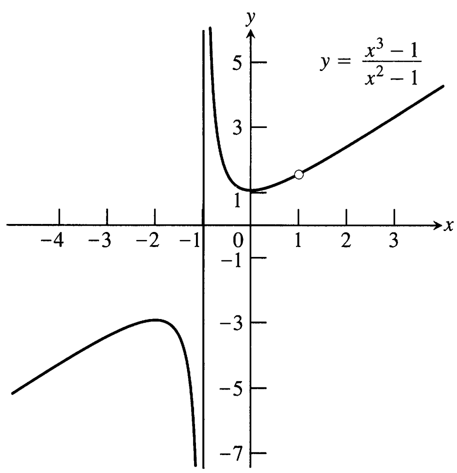
The behavior as $x \to \pm 2$. Since

$$\lim_{x \to 2^+} f(x) = -\infty \qquad \text{and} \qquad \lim_{x \to 2^-} f(x) = \infty,$$

the line $x = 2$ is an asymptote both from the right and from the left. By symmetry, the same holds for the line $x = -2$.

There are no other asymptotes because f has a finite limit at every other point. ❑

We might be tempted at this point to say that rational functions have vertical asymptotes where their denominators are zero. That is nearly true, but not quite. What is true is that rational functions *reduced to lowest terms* have vertical asymptotes where their denominators are zero.



3.43 The graph of $f(x) = (x^3 - 1)/(x^2 - 1)$ has one vertical asymptote, not two. The discontinuity at $x = 1$ is removable.

EXAMPLE 10 *A removable discontinuity at a zero of the denominator*

The graph of

$$f(x) = \frac{x^3 - 1}{x^2 - 1}$$

has a vertical asymptote at $x = -1$ but not at $x = 1$. Since

$$\frac{x^3 - 1}{x^2 - 1} = \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \frac{x^2+x+1}{x+1},$$

the function has a finite limit (3/2) as $x \to 1$ and the discontinuity is removable (Fig. 3.43). ❑

The Sandwich Theorem (Section 1.2, Theorem 4) also holds for limits as $x \to \pm\infty$. Here is a typical application.

EXAMPLE 11 Using the Sandwich Theorem, find the asymptotes of the curve

$$y = 2 + \frac{\sin x}{x}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to 0$, where the denominator is zero.

The behavior as $x \to 0$. We know that $\lim_{x \to 0} (\sin x)/x = 1$, so there is no asymptote at the origin.

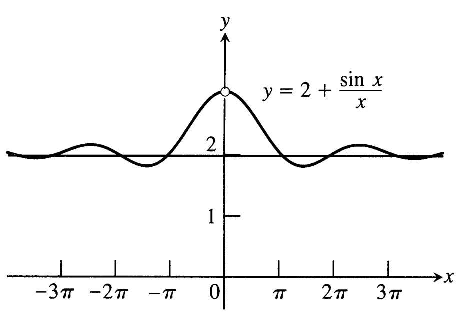
The behavior as $x \to \pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|,$$

and $\lim_{x \to \pm\infty} |1/x| = 0$, we have $\lim_{x \to \pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x \to \pm\infty} \left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2,$$

and the line $y = 2$ is an asymptote of the curve on both left and right (Fig. 3.44). ❑



3.44 A curve may cross one of its asymptotes infinitely often (Example 11).

Oblique Asymptotes

If the degree of the numerator of a rational function is one greater than the degree of the denominator, the graph has an **oblique asymptote,** that is, a linear asymptote that is neither vertical nor horizontal.

EXAMPLE 12 Find the asymptotes of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}.$$

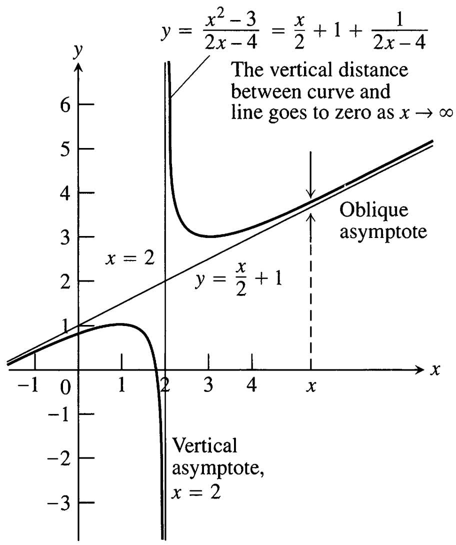
Solution We are interested in the behavior as $x \to \pm\infty$ and also as $x \to 2$, where the denominator is zero. We divide $(2x - 4)$ into $(x^2 - 3)$:

$$\begin{array}{r|l} & \frac{x}{2} + 1 \\ 2x - 4 & x^2 - 3 \\ & x^2 - 2x \\ & \quad 2x - 3 \\ & \quad 2x - 4 \\ & \qquad 1 \end{array}$$

This tells us that

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\frac{x}{2} + 1}_{\text{linear}} + \underbrace{\frac{1}{2x - 4}}_{\text{remainder}}. \tag{2}$$

Since $\lim_{x \to 2^+} f(x) = \infty$ and $\lim_{x \to 2^-} f(x) = -\infty$, the line $x = 2$ is a two-sided asymptote. As $x \to \pm\infty$, the remainder approaches 0 and $f(x) \to (x/2) + 1$. The line $y = (x/2) + 1$ is an asymptote both to the right and to the left (Fig. 3.45). ❑



3.45 The graph of $f(x) = (x^2 - 3)/(2x - 4)$ (Example 12).

Graphing with Asymptotes and Dominant Terms

Of all the observations we can make quickly about the function

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

in Example 12, probably the most useful is that

$$f(x) = \frac{x}{2} + 1 + \frac{1}{2x - 4}.$$

This tells us immediately that

$$f(x) \approx \frac{x}{2} + 1 \qquad \text{for } x \text{ numerically large}$$

$$f(x) \approx \frac{1}{2x - 4} \qquad \text{for } x \text{ near } 2$$

If we want to know how f behaves, this is the way to find out. It behaves like $y = (x/2) + 1$ when x is numerically large and the contribution of $1/(2x - 4)$ to the total value of f is insignificant. It behaves like $1/(2x - 4)$ when x is so close to 2 that $1/(2x - 4)$ makes the dominant contribution.

We say that $(x/2) + 1$ **dominates** when x is numerically large, and we say that $1/(2x - 4)$ dominates when x is near 2. **Dominant terms** like these are the key to predicting a function's behavior.

EXAMPLE 13 Graph the function

$$y = \frac{x^3 + 1}{x}.$$

Solution We investigate symmetry, dominant terms, asymptotes, rise, fall, extreme values, and concavity.

Step 1: *Symmetry.* There is none.

Step 2: *Find any dominant terms and asymptotes.* We write the rational function as a polynomial plus remainder:

$$y = x^2 + \frac{1}{x}. \tag{3}$$

For $|x|$ large, $y \approx x^2$. For x near zero, $y \approx 1/x$.

Equation (3) reveals a vertical asymptote at $x = 0$, where the denominator of the remainder is zero.

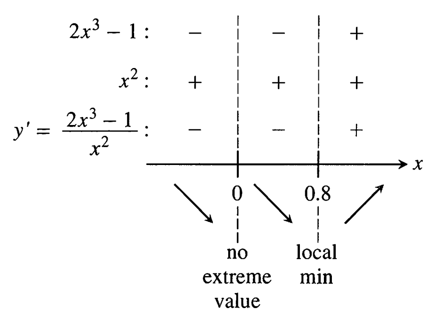
Step 3: *Find y′ and analyze the function's critical points. Where does the curve rise and fall?*

The first derivative

$$y' = 2x - \frac{1}{x^2} = \frac{2x^3 - 1}{x^2} \qquad \text{From Eq. (3)}$$

is undefined at $x = 0$ and zero when

$$2x - \frac{1}{x^2} = 0$$
$$2x^3 - 1 = 0$$
$$x^3 = \frac{1}{2}$$
$$x = \frac{1}{\sqrt[3]{2}} \approx 0.8.$$

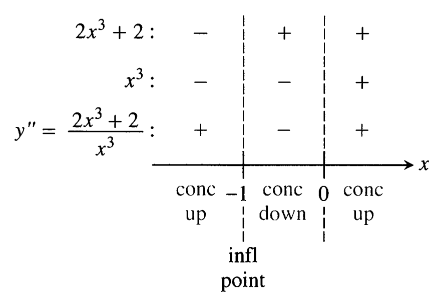


Step 4: *Find y″ and determine the curve's concavity.* The second derivative

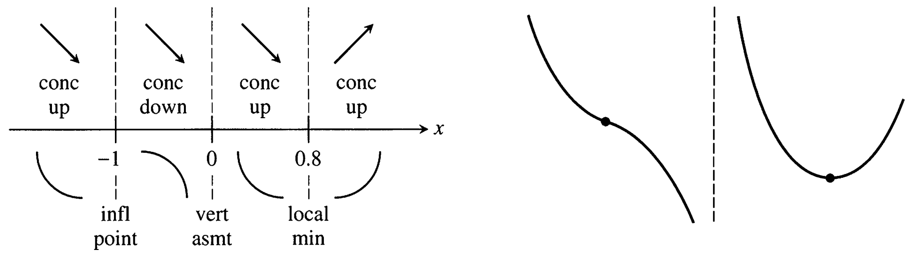
$$y'' = 2 + \frac{2}{x^3} = \frac{2x^3 + 2}{x^3}$$

is undefined at $x = 0$ and zero when

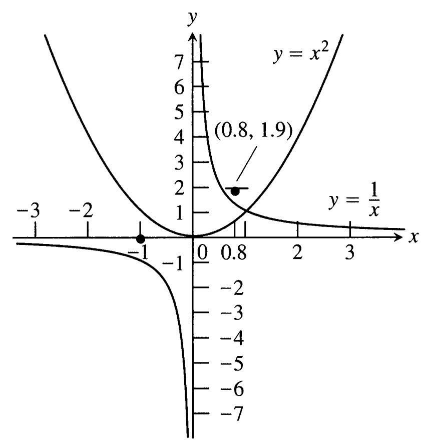
$$2 + \frac{2}{x^3} = 0$$
$$2x^3 + 2 = 0$$
$$x^3 = -1$$
$$x = -1.$$



Step 5: *Summarize the information from the preceding steps and sketch the curve's general shape.*

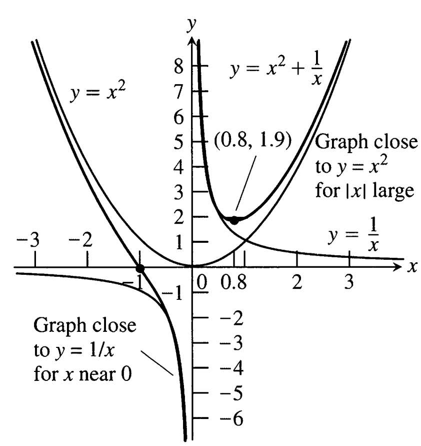


Step 6: *Plot the curve's intercepts, mark any horizontal tangents, and graph the dominant terms.* See Fig. 3.46. This provides a framework for graphing the curve.



3.46 The dominant terms and horizontal tangent provide a framework for graphing the function.

Step 7: *Now add the final curve to your figure, using the framework and the curve's general shape as guides.* See Fig. 3.47. ❑



3.47 The function, graphed with the aid of the framework in Fig. 3.46.

Hidden Behavior

Sometimes graphing f' or f'' will suggest where to zoom in on a computer generated graph of f to reveal behavior hidden in the grapher's original picture.

Checklist for Graphing a Function $y = f(x)$

1. Look for symmetry.
   Is the function even? odd?
2. Is the function a shift of a known function?
3. Analyze dominant terms.
   Divide rational functions into polynomial + remainder.
4. Check for asymptotes and removable discontinuities.
   Is there a zero denominator at any point?
   What happens as $x \to \pm\infty$?
5. Compute f' and solve $f' = 0$. Identify critical points and determine intervals of rise and fall.
6. Compute f'' to determine concavity and inflection points.
7. Sketch the graph's general shape.
8. Evaluate f at special values (endpoints, critical points, intercepts).
9. Graph f, using dominant terms, general shape, and special points for guidance.

Exercises 3.5

Calculating Limits as $x \to \pm\infty$

In Exercises 1–6, find the limit of each function (a) as $x \to \infty$ and (b) as $x \to -\infty$. (You may wish to visualize your answer with a grapher.)

1. $f(x) = \dfrac{2}{x} - 3$

2. $f(x) = \pi - \dfrac{2}{x^2}$

3. $g(x) = \dfrac{1}{2 + (1/x)}$

4. $g(x) = \dfrac{1}{8 - (5/x^2)}$

5. $h(x) = \dfrac{-5 + (7/x)}{3 - (1/x^2)}$

6. $h(x) = \dfrac{3 - (2/x)}{4 + \left(\sqrt{2}/x^2\right)}$

Find the limits in Exercises 7–10.

7. $\lim\limits_{x \to \infty} \dfrac{\sin 2x}{x}$

8. $\lim\limits_{\theta \to -\infty} \dfrac{\cos\theta}{3\theta}$

9. $\lim\limits_{t \to -\infty} \dfrac{2 - t + \sin t}{t + \cos t}$

10. $\lim\limits_{r \to \infty} \dfrac{r + \sin r}{2r + 7 - 5\sin r}$

Limits of Rational Functions

In Exercises 11–24, find the limit of each rational function (a) as $x \to \infty$ and (b) as $x \to -\infty$.

11. $f(x) = \dfrac{2x + 3}{5x + 7}$

12. $f(x) = \dfrac{2x^3 + 7}{x^3 - x^2 + x + 7}$

13. $f(x) = \dfrac{x + 1}{x^2 + 3}$

14. $f(x) = \dfrac{3x + 7}{x^2 - 2}$

15. $f(x) = \dfrac{1 - 12x^3}{4x^2 + 12}$

16. $g(x) = \dfrac{1}{x^3 - 4x + 1}$

17. $h(x) = \dfrac{7x^3}{x^3 - 3x^2 + 6x}$

18. $g(x) = \dfrac{3x^2 - 6x}{4x - 8}$

19. $f(x) = \dfrac{2x^5 + 3}{-x^2 + x}$

20. $g(x) = \dfrac{10x^5 + x^4 + 31}{x^6}$

21. $g(x) = \dfrac{x^4}{x^3 + 1}$

22. $h(x) = \dfrac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

23. $h(x) = \dfrac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$

24. $h(x) = \dfrac{-x^4}{x^4 - 7x^3 + 7x^2 + 9}$

Limits with Noninteger or Negative Powers

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x: divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 25–30.

25. $\lim\limits_{x \to \infty} \dfrac{2\sqrt{x} + x^{-1}}{3x - 7}$

26. $\lim\limits_{x \to \infty} \dfrac{2 + \sqrt{x}}{2 - \sqrt{x}}$

27. $\lim\limits_{x \to -\infty} \dfrac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

28. $\lim\limits_{x \to \infty} \dfrac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$

29. $\lim_{x\to\infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$

30. $\lim_{x\to-\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$

Inventing Graphs from Values and Limits

In Exercises 31–34, sketch the graph of a function $y = f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)

31. $f(0) = 0$, $f(1) = 2$, $f(-1) = -2$, $\lim_{x\to-\infty} f(x) = -1$, and $\lim_{x\to\infty} f(x) = 1$

32. $f(0) = 0$, $\lim_{x\to\pm\infty} f(x) = 0$, $\lim_{x\to 0^+} f(x) = 2$, and $\lim_{x\to 0^-} f(x) = -2$

33. $f(0) = 0$, $\lim_{x\to\pm\infty} f(x) = 0$, $\lim_{x\to 1^-} f(x) = \lim_{x\to -1^+} f(x) = \infty$, $\lim_{x\to 1^+} f(x) = -\infty$, and $\lim_{x\to -1^-} f(x) = -\infty$

34. $f(2) = 1$, $f(-1) = 0$, $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to 0^-} f(x) = -\infty$, and $\lim_{x\to-\infty} f(x) = 1$

Inventing Functions

In Exercises 35–38, find a function that satisfies the given conditions and sketch its graph. (The answers here are not unique. Any function that satisfies the conditions is acceptable. Feel free to use formulas defined in pieces if that will help.)

35. $\lim_{x\to\pm\infty} f(x) = 0$, $\lim_{x\to 2^-} f(x) = \infty$, and $\lim_{x\to 2^+} f(x) = \infty$

36. $\lim_{x\to\pm\infty} g(x) = 0$, $\lim_{x\to 3^-} g(x) = -\infty$, and $\lim_{x\to 3^+} g(x) = \infty$

37. $\lim_{x\to-\infty} h(x) = -1$, $\lim_{x\to\infty} h(x) = 1$, $\lim_{x\to 0^-} h(x) = -1$, and $\lim_{x\to 0^+} h(x) = 1$

38. $\lim_{x\to\pm\infty} k(x) = 1$, $\lim_{x\to 1^-} k(x) = \infty$, and $\lim_{x\to 1^+} k(x) = -\infty$

Graphing Rational Functions

Graph the rational functions in Exercises 39–66. Include the graphs and equations of the asymptotes and dominant terms.

39. $y = \frac{1}{x-1}$

40. $y = \frac{1}{x+1}$

41. $y = \frac{1}{2x+4}$

42. $y = \frac{-3}{x-3}$

43. $y = \frac{x+3}{x+2}$

44. $y = \frac{2x}{x+1}$

45. $y = \frac{2x^2+x-1}{x^2-1}$

46. $y = \frac{x^2-49}{x^2+5x-14}$

47. $y = \frac{x^2-1}{x}$

48. $y = \frac{x^2+4}{2x}$

49. $y = \frac{x^4+1}{x^2}$

50. $y = \frac{x^3+1}{x^2}$

51. $y = \frac{1}{x^2-1}$

52. $y = \frac{x^2}{x^2-1}$

53. $y = -\frac{x^2-2}{x^2-1}$

54. $y = \frac{x^2-4}{x^2-2}$

55. $y = \frac{x^2}{x-1}$

56. $y = -\frac{x^2}{x+1}$

57. $y = \frac{x^2-4}{x-1}$

58. $y = -\frac{x^2-4}{x+1}$

59. $y = \frac{x^2-x+1}{x-1}$

60. $y = -\frac{x^2-x+1}{x-1}$

61. $y = \frac{x^3-3x^2+3x-1}{x^2+x-2}$

62. $y = \frac{x^3+x-2}{x-x^2}$

63. $y = \frac{x}{x^2-1}$

64. $y = \frac{x-1}{x^2(x-2)}$

65. $y = \frac{8}{x^2+4}$ (Agnesi's witch)

66. $y = \frac{4x}{x^2+4}$ (Newton's serpentine)

Grapher Explorations

Graph the curves in Exercises 67–72 and explain the relation between the curve's formula and what you see.

67. $y = \frac{x}{\sqrt{4-x^2}}$

68. $y = \frac{-1}{\sqrt{4-x^2}}$

69. $y = x^{2/3} + \frac{1}{x^{1/3}}$

70. $y = 2\sqrt{x} + \frac{2}{\sqrt{x}} - 3$

71. $y = \sin\left(\frac{\pi}{x^2+1}\right)$

72. $y = -\cos\left(\frac{\pi}{x^2+1}\right)$

Graphing Terms

Each of the functions in Exercises 73–76 is given as the sum or difference of two terms. First graph the terms (with the same set of axes). Then, using these graphs as guides, sketch in the graph of the function.

73. $y = \sec x + \frac{1}{x}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

74. $y = \sec x - \frac{1}{x^2}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

75. $y = \tan x + \frac{1}{x^2}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

76. $y = \frac{1}{x} - \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Theory and Examples

77. Let $f(x) = (x^3 + x^2)/(x^2 + 1)$. Show that there is a value of c for which $f(c)$ equals

a) -2 **b)** $\cos 3$ **c)** $5{,}000{,}000$.

EXAMPLE 6

a) $d(\tan 2x) = \sec^2(2x)\,d(2x) = 2\sec^2 2x\,dx$

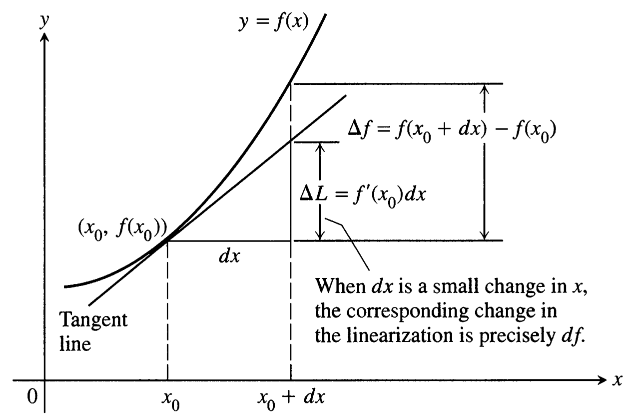
b) $d\left(\dfrac{x}{x+1}\right) = \dfrac{(x+1)\,dx - x\,d\,(x+1)}{(x+1)^2} = \dfrac{x\,dx + dx - x\,dx}{(x+1)^2} = \dfrac{dx}{(x+1)^2}$ ❑

Estimating Change with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point x_0 and we want to predict how much this value will change if we move to a nearby point $x_0 + dx$. If dx is small, f and its linearization L at x_0 will change by nearly the same amount. Since the values of L are simple to calculate, calculating the change in L offers a practical way to estimate the change in f.

In the notation of Fig. 3.65, the change in f is

$$\Delta f = f(x_0 + dx) - f(x_0).$$



3.65 If *dx* is small, the change in the linearization of *f* is nearly the same as the change in *f*.

The corresponding change in L is

$$\begin{aligned}\Delta L &= L(x_0 + dx) - L(x_0)\\ &= \underbrace{f(x_0) + f'(x_0)\Big[(x_0 + dx) - x_0\Big]}_{L(x_0+dx)} - \underbrace{f(x_0)}_{L(x_0)=f(x_0)}\\ &= f'(x_0)\,dx.\end{aligned}$$

Thus, the differential $df = f'(x)\,dx$ has a geometric interpretation: When df is evaluated at $x = x_0$, $df = \Delta L$, the change in the linearization of f corresponding to the change dx.

> **The Differential Estimate of Change**
>
> Let $f(x)$ be differentiable at $x = x_0$. The approximate change in the value of f when x changes from x_0 to $x_0 + dx$ is
>
> $$df = f'(x_0)\,dx.$$

Grapher Explorations—"Seeing" Limits at Infinity

Sometimes a change of variable can change an unfamiliar expression into one whose limit we know how to find. For example,

$$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{\theta\to 0^+} \sin\theta \qquad \text{Substitute } \theta = 1/x$$
$$= 0.$$

This suggests a creative way to "see" limits at infinity. Describe the procedure and use it to picture and determine limits in Exercises 103–108.

103. $\lim_{x\to\pm\infty} x\sin\frac{1}{x}$

104. $\lim_{x\to-\infty} \frac{\cos(1/x)}{1+(1/x)}$

105. $\lim_{x\to\pm\infty} \frac{3x+4}{2x-5}$

106. $\lim_{x\to\infty} \left(\frac{1}{x}\right)^{1/x}$

107. $\lim_{x\to\pm\infty} \left(3+\frac{2}{x}\right)\left(\cos\frac{1}{x}\right)$

108. $\lim_{x\to\infty} \left(\frac{3}{x^2}-\cos\frac{1}{x}\right)\left(1+\sin\frac{1}{x}\right)$

3.6 Optimization

To optimize something means to maximize or minimize some aspect of it. What is the size of the most profitable production run? What is the least expensive shape for an oil can? What is the stiffest beam we can cut from a 12-inch log? In the mathematical models in which we use functions to describe the things that interest us, we usually answer such questions by finding the greatest or smallest value of a differentiable function.

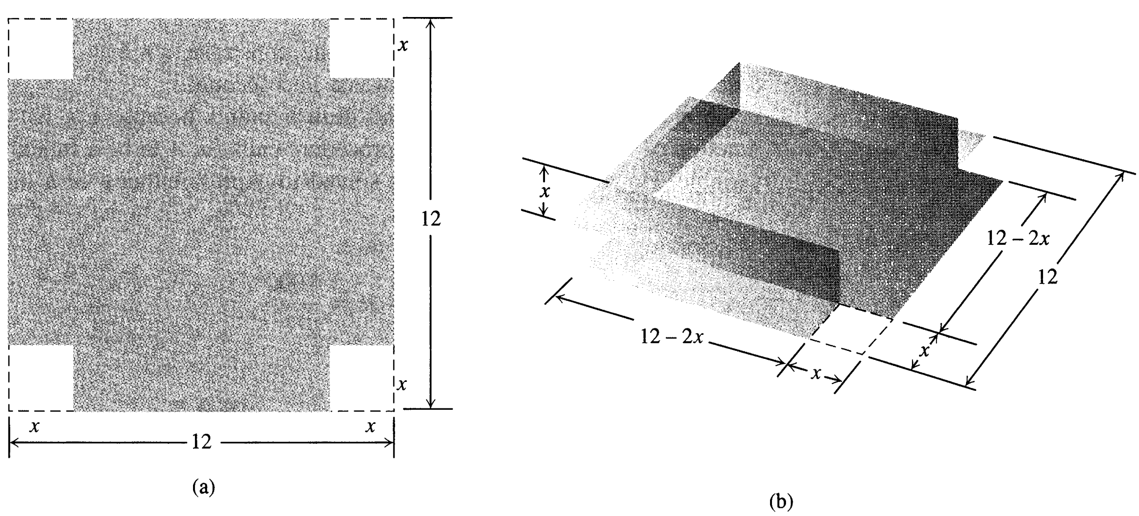
Examples from Business and Industry

EXAMPLE 1 *Metal fabrication*

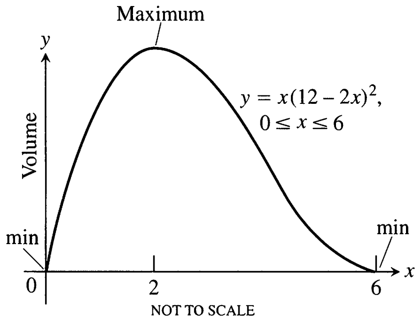
An open-top box is to be made by cutting small congruent squares from the corners of a 12-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution We start with a picture (Fig. 3.48). In the figure, the corner squares are x inches on a side. The volume of the box is a function of this variable:

$$V(x) = x(12-2x)^2 = 144x - 48x^2 + 4x^3. \qquad V = hlw$$



3.48 An open box made by cutting the corners from a square sheet of tin.



3.49 The volume of the box in Fig. 3.48 graphed as a function of x.

Since the sides of the sheet of tin are only 12 in. long, $x \le 6$ and the domain of V is the interval $0 \le x \le 6$.

A graph of V (Fig. 3.49) suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 2$. To learn more, we examine the first derivative of V with respect to x:

$$\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(2 - x)(6 - x).$$

Of the two zeros, $x = 2$ and $x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list. The values of V at this one critical point and two endpoints are

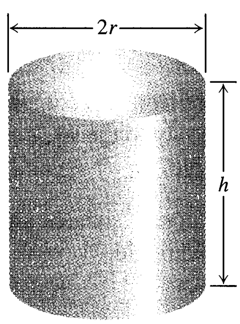
Critical-point value: $V(2) = 128$

Endpoint values: $V(0) = 0, \quad V(6) = 0.$

The maximum volume is 128 in^3. The cut-out squares should be 2 in. on a side. ❑

EXAMPLE 2 *Product design*

You have been asked to design a 1-L oil can shaped like a right circular cylinder. What dimensions will use the least material?



3.50 This 1-L can uses the least material when $h = 2r$ (Example 2).

Solution We picture the can as a right circular cylinder with height h and diameter $2r$ (Fig. 3.50). If r and h are measured in centimeters and the volume is expressed as 1000 cm^3, then r and h are related by the equation

$$\pi r^2 h = 1000. \qquad 1 \text{ L} = 1000 \text{ cm}^3 \tag{1}$$

How shall we interpret the phrase "least material"? One possibility is to ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions r and h that make the total surface area

$$A = \underbrace{2\pi r^2}_{\text{cylinder ends}} + \underbrace{2\pi r h}_{\text{cylinder wall}} \tag{2}$$

as small as possible while satisfying the constraint $\pi r^2 h = 1000$. (Exercise 18 describes one way we might take waste into account.)

We are not quite ready to find critical points because Eq. (2) gives A as a function of two variables and our procedure calls for A to be a function of a single variable. However, Eq. (1) can be solved to express either r or h in terms of the other.

Solving for h is easier, so we take

$$h = \frac{1000}{\pi r^2}.$$

This changes the formula for A to

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}.$$

For small r (a tall thin container, like a pipe), the term $2000/r$ dominates and A is large. For larger r (a short wide container, like a pizza pan), the term $2\pi r^2$

dominates and A is again large. If A has a minimum, it must be at a value of r that is neither too large nor too small.

Since A is differentiable throughout its domain $(0, \infty)$ and the domain has no endpoints, A can have a minimum only where $dA/dr = 0$.

$$A = 2\pi r^2 + \frac{2000}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2} \quad \text{Find } dA/dr.$$

$$4\pi r - \frac{2000}{r^2} = 0 \quad \text{Set it equal to 0.}$$

$$4\pi r^3 = 2000 \quad \text{Solve for } r.$$

$$r = \sqrt[3]{\frac{500}{\pi}} \quad \text{Critical point}$$

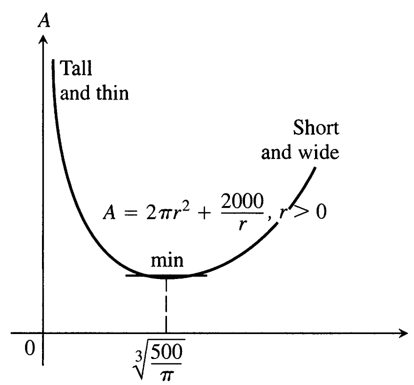
So something happens at $r = \sqrt[3]{500/\pi}$, but what?

If the domain of A were a closed interval, we could find out by evaluating A at this critical point and the endpoints and comparing the results. But the domain is not a closed interval, so we must learn what is happening at $r = \sqrt[3]{500/\pi}$ by determining the shape of A's graph. We can do this by investigating the second derivative, d^2A/dr^2:

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}.$$

The second derivative is positive throughout the domain of A. The value of A at $r = \sqrt[3]{500/\pi}$ is therefore an absolute minimum because the graph of A is concave up (Fig. 3.51).



3.51 The graph of $A = 2\pi r^2 + 2000/r$ is concave up.

When

$$r = \sqrt[3]{500/\pi},$$

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{500/\pi} = 2r. \quad \text{After some arithmetic} \quad (3)$$

Equation (3) tells us that the most efficient can has its height equal to its diameter. With a calculator we find

$$r \approx 5.42 \text{ cm}, \qquad h \approx 10.84 \text{ cm}.$$

❑

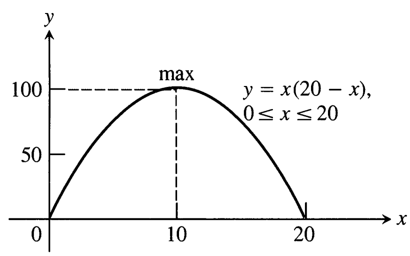
Strategy for Solving Max-Min Problems

1. *Read the problem.* Read the problem until you understand it. What is unknown? What is given? What is sought?
2. *Draw a picture.* Label any part that may be important to the problem.
3. *Introduce variables.* List every relation in the picture and in the problem as an equation or algebraic expression.
4. *Identify the unknown. Write an equation for it.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints.* Use what you know about the shape of the function's graph and the physics of the problem. Use the first and second derivatives to identify and classify critical points (where $f' = 0$ or does not exist).

Examples from Mathematics

EXAMPLE 3 *Products of numbers*

Find two positive numbers whose sum is 20 and whose product is as large as possible.



3.52 The product of x and $(20 - x)$ reaches a maximum value of 100 when $x = 10$ (Example 3).

Solution If one number is x, the other is $(20 - x)$. Their product is

$$f(x) = x(20 - x) = 20x - x^2.$$

We want the value or values of x that make $f(x)$ as large as possible. The domain of f is the closed interval $0 \le x \le 20$.

We evaluate f at the critical points and endpoints. The first derivative,

$$f'(x) = 20 - 2x,$$

is defined at every point of the interval $0 \le x \le 20$ and is zero only at $x = 10$. Listing the values of f at this one critical point and the endpoints gives

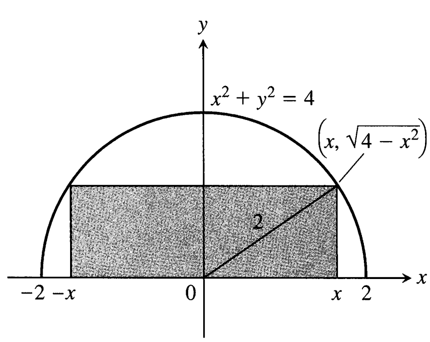
Critical-point value: $f(10) = 20(10) - (10)^2 = 100$

Endpoint values: $f(0) = 0, \quad f(20) = 0.$

We conclude that the maximum value is $f(10) = 100$. The corresponding numbers are $x = 10$ and $(20 - 10) = 10$ (Fig. 3.52). ❑

EXAMPLE 4 *Geometry*

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?



3.53 The rectangle and semicircle in Example 4.

Solution To describe the dimensions of the rectangle, we place the circle and rectangle in the coordinate plane (Fig. 3.53). The length, height, and area of the rectangle can then be expressed in terms of the position x of the lower right-hand corner:

$$\text{Length: } 2x \qquad \text{Height: } \sqrt{4 - x^2} \qquad \text{Area: } 2x \cdot \sqrt{4 - x^2}.$$

Notice that the values of x are to be found in the interval $0 \le x \le 2$, where the selected corner of the rectangle lies.

Our mathematical goal is now to find the absolute maximum value of the continuous function

$$A(x) = 2x\sqrt{4 - x^2}$$

on the domain [0, 2]. We do this by examining the values of A at the critical points and endpoints. The derivative

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2}$$

is not defined when $x = 2$ and is equal to zero when

$$\frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2} = 0$$

$$-2x^2 + 2(4 - x^2) = 0 \qquad \text{Multiply both sides by } \sqrt{4 - x^2}.$$

$$8 - 4x^2 = 0$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}.$$

Of the two zeros, $x = \sqrt{2}$ and $x = -\sqrt{2}$, only $x = \sqrt{2}$ lies in the interior of A's domain and makes the critical-point list. The values of A at the endpoints and at this one critical point are

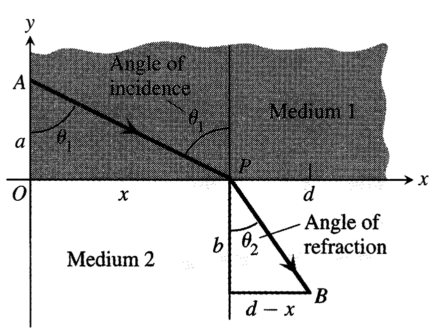
Critical-point value: $A(\sqrt{2}) = 2\sqrt{2}\sqrt{4 - 2} = 4$
Endpoint values: $A(0) = 0, \quad A(2) = 0.$

The area has a maximum value of 4 when the rectangle is $\sqrt{4 - x^2} = \sqrt{2}$ units high and $2x = 2\sqrt{2}$ units long. ❑

Fermat's Principle and Snell's Law

The speed of light depends on the medium through which it travels and tends to be slower in denser media. In a vacuum, it travels at the famous speed $c = 3 \times 10^8$ m/sec, but in the earth's atmosphere it travels slightly slower than that, and in glass slower still (about two-thirds as fast).

Fermat's principle in optics states that light always travels from one point to another along the quickest route. This observation enables us to predict the path light will take when it travels from a point in one medium (air, say) to a point in another medium (say, glass or water).



3.54 A light ray refracted (deflected from its path) as it passes from one medium to another. θ_1 is the angle of incidence and θ_2 is the angle of refraction.

EXAMPLE 5 Find the path that a ray of light will follow in going from a point A in a medium where the speed of light is c_1 across a straight boundary to a point B in a medium where the speed of light is c_2.

Solution Since light traveling from A to B will do so by the quickest route, we look for a path that will minimize the travel time.

We assume that A and B lie in the xy-plane and that the line separating the two media is the x-axis (Fig. 3.54).

In a uniform medium, where the speed of light remains constant, "shortest time" means "shortest path," and the ray of light will follow a straight line. Hence the path from A to B will consist of a line segment from A to a boundary point P, followed by another line segment from P to B. From the formula distance equals rate times time, we have

$$\text{time} = \frac{\text{distance}}{\text{rate}}.$$

The time required for light to travel from A to P is therefore

$$t_1 = \frac{AP}{c_1} = \frac{\sqrt{a^2 + x^2}}{c_1}.$$

From P to B the time is

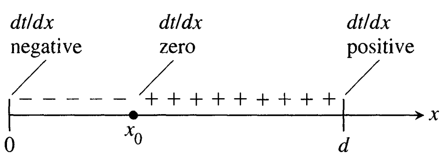
$$t_2 = \frac{PB}{c_2} = \frac{\sqrt{b^2 + (d - x)^2}}{c_2}.$$

The time from A to B is the sum of these:

$$t = t_1 + t_2 = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d - x)^2}}{c_2}. \tag{4}$$

Equation (4) expresses t as a differentiable function of x whose domain is $[0, d]$, and we want to find the absolute minimum value of t on this closed interval. We find

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{(d - x)}{c_2\sqrt{b^2 + (d - x)^2}}. \tag{5}$$



3.55 The sign pattern of *dt/dx* in Example 5.

In terms of the angles θ_1 and θ_2 in Fig. 3.54,

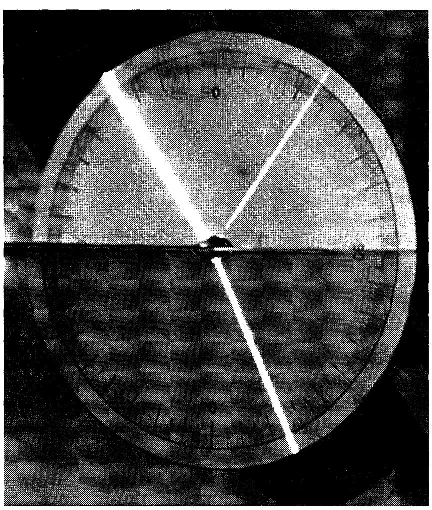
$$\frac{dt}{dx} = \frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2}. \tag{6}$$

We can see from Eq. (5) that $dt/dx < 0$ at $x = 0$ and $dt/dx > 0$ at $x = d$. Hence, $dt/dx = 0$ at some point x_0 in between (Fig. 3.55). There is only one such point because dt/dx is an increasing function of x (Exercise 52). At this point,

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}.$$

This equation is **Snell's law** or the **law of refraction.**

We conclude that the path the ray of light follows is the one described by Snell's law. Figure 3.56 shows how this works for air and water. ❑



3.56 For air and water at room temperature, the light velocity ratio is 1.33 and Snell's law becomes $\sin\theta_1 = 1.33\sin\theta_2$. In this laboratory photograph, $\theta_1 = 35.5°, \theta_2 = 26°$, and $(\sin 35.5°/\sin 26°) \approx 0.581/0.438 \approx 1.33$, as predicted.

This photograph also illustrates that angle of reflection = angle of incidence (Exercise 39).

Cost and Revenue in Economics

Here we want to point out two of the many places where calculus makes a contribution to economic theory. The first has to do with the relationship between profit, revenue (money received), and cost.

Suppose that

$r(x)$ = the revenue from selling x items

$c(x)$ = the cost of producing the x items

$p(x) = r(x) - c(x)$ = the profit from selling x items.

Developing a physical law

In developing a physical law, we typically observe an effect, measure values and list them in a table, and then try to find a rule by which one thing can be connected with another. The Alexandrian Greek Claudius Ptolemy (c. 100–c. 170 A.D.) tried to do this for the refraction of light by water. He made a table of angles of incidence and corresponding angles of refraction, with values very close to the ones we find for air and water today.

Angle in air (degrees)	Ptolemy's angle in water (degrees)	Modern angle in water (degrees)
10	8	7.5
20	15.5	15
30	22.5	22
40	28	29
50	35	35
60	40.5	40.5
70	45	45
80	50	47.6

The rule that connected these angles, however, eluded him, as it did everyone else for the next 1400 years. The Dutch mathematician Willebrord Snell (1580–1626) found it in 1621.

Finding a rule is nice, but the real glory of science is finding a way of thinking that makes the rule evident. Fermat discovered it around 1650. His idea was this: Of all the paths light might take to get from one point to another, it follows the path that takes the shortest time. In Example 5, you see how this principle leads to Snell's law. The derivation we give is Fermat's own.

For more on marginal revenue and cost, see the end of Section 2.3.

The marginal revenue and cost at this production level (x items) are

$$\frac{dr}{dx} = \text{marginal revenue}$$

$$\frac{dc}{dx} = \text{marginal cost.}$$

The first theorem is about the relationship of the profit p to these derivatives.

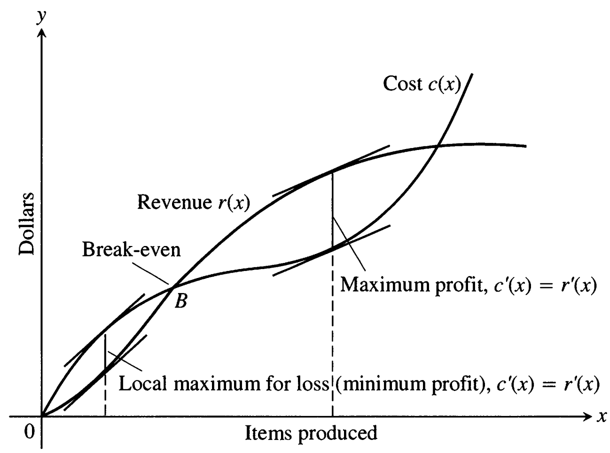
Theorem 7

Maximum profit (if any) occurs at a production level at which marginal revenue equals marginal cost.

Proof We assume that $r(x)$ and $c(x)$ are differentiable for all $x > 0$, so if $p(x) = r(x) - c(x)$ has a maximum value, it occurs at a production level at which $p'(x) = 0$. Since $p'(x) = r'(x) - c'(x)$, $p'(x) = 0$ implies

$$r'(x) - c'(x) = 0 \qquad \text{or} \qquad r'(x) = c'(x).$$

This concludes the proof (Fig. 3.57).



3.57 The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point *B*. To the left of *B*, the company operates at a loss. To the right, the company operates at a profit, with the maximum profit occurring where $c'(x) = r'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of market saturation and rising labor and material costs) and production levels become unprofitable again.

❑

What guidance do we get from Theorem 7? We know that a production level at which $p'(x) = 0$ need not be a level of maximum profit. It might be a level of minimum profit, for example. But if we are making financial projections for our company, we should look for production levels at which marginal cost seems to equal marginal revenue. If there is a most profitable production level, it will be one of these.

EXAMPLE 6 The cost and revenue functions at American Gadget are

$$r(x) = 9x \qquad \text{and} \qquad c(x) = x^3 - 6x^2 + 15x,$$

where x represents thousands of gadgets. Is there a production level that will maximize American Gadget's profit? If so, what is it?

Solution

$$r(x) = 9x, \qquad c(x) = x^3 - 6x^2 + 15x \qquad \text{Find } r'(x) \text{ and } c'(x).$$

$$r'(x) = 9, \qquad c'(x) = 3x^2 - 12x + 15$$

$$3x^2 - 12x + 15 = 9 \qquad \text{Set them equal.}$$

$$3x^2 - 12x + 6 = 0 \qquad \text{Rearrange.}$$

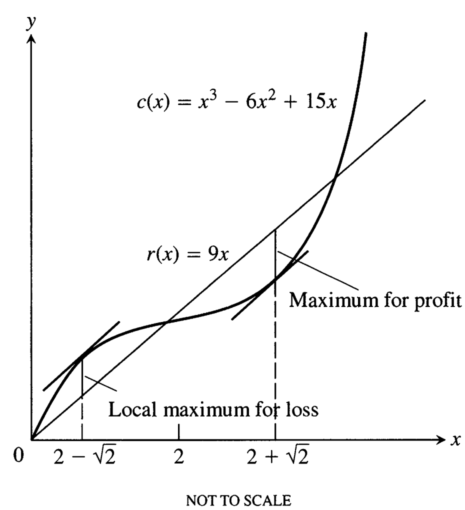
$$x^2 - 4x + 2 = 0$$

$$x = \frac{4 \pm \sqrt{16 - 4 \cdot 2}}{2} \qquad \text{Solve for } x \text{ with the quadratic formula.}$$

$$= \frac{4 \pm 2\sqrt{2}}{2}$$

$$= 2 \pm \sqrt{2}$$

The possible production levels for maximum profit are $x = 2 + \sqrt{2}$ thousand units and $x = 2 - \sqrt{2}$ thousand units. A quick glance at the graphs in Fig. 3.58 or at the corresponding values of r and c shows $x = 2 + \sqrt{2}$ to be a point of maximum profit and $x = 2 - \sqrt{2}$ to be a local maximum for loss.



3.58 The cost and revenue curves for Example 6. ❑

Another way to look for optimal production levels is to look for levels that minimize the average cost of the units produced. The next theorem helps us to find them.

Theorem 8

The production level (if any) at which average cost is smallest is a level at which the average cost equals the marginal cost.

Proof We start with

$$c(x) = \text{cost of producing } x \text{ items, } x > 0$$

$$\frac{c(x)}{x} = \text{average cost of producing } x \text{ items,}$$

assumed differentiable.

If the average cost can be minimized, it will be at a production level at which

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 0$$

$$\frac{xc'(x) - c(x)}{x^2} = 0 \qquad \text{Quotient Rule}$$

$$xc'(x) - c(x) = 0 \qquad \text{Multiplied by } x^2$$

$$\underbrace{c'(x)}_{\substack{\text{marginal}\\ \text{cost}}} = \underbrace{\frac{c(x)}{x}}_{\substack{\text{average}\\ \text{cost}}}.$$

This completes the proof. ❑

Again we have to be careful about what Theorem 8 does and does not say. It does not say that there is a production level of minimum average cost—it says where to look to see if there is one. Look for production levels at which average cost and marginal cost are equal. Then check to see if any of them gives a minimum average cost.

EXAMPLE 7 The cost function at American Gadget is $c(x) = x^3 - 6x^2 + 15x$ (x in thousands of units). Is there a production level that minimizes average cost? If so, what is it?

Solution We look for levels at which average cost equals marginal cost.

Cost: $c(x) = x^3 - 6x^2 + 15x$

Marginal cost: $c'(x) = 3x^2 - 12x + 15$

Average cost: $\dfrac{c(x)}{x} = x^2 - 6x + 15$

$$3x^2 - 12x + 15 = x^2 - 6x + 15 \qquad \text{MC} = \text{AC}$$

$$2x^2 - 6x = 0$$

$$2x(x - 3) = 0$$

$$x = 0 \qquad \text{or} \qquad x = 3$$

Since $x > 0$, the only production level that might minimize average cost is $x = 3$ thousand units.

We check the derivatives:

$$\frac{c(x)}{x} = x^2 - 6x + 15 \qquad \text{Average cost}$$

$$\frac{d}{dx}\left(\frac{c(x)}{x}\right) = 2x - 6$$

$$\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = 2 > 0.$$

The second derivative is positive, so $x = 3$ gives an absolute minimum. ❑

Modeling Discrete Phenomena with Differentiable Functions

In case you are wondering how we can use differentiable functions $c(x)$ and $r(x)$ to describe the cost and revenue that come from producing a number of items x, which can only be an integer, here is the rationale.

When x is large, we can reasonably fit the cost and revenue data with smooth curves $c(x)$ and $r(x)$ that are defined not only at integer values of x but at the values in between. Once we have these differentiable functions, which are supposed to behave like the real cost and revenue when x is an integer, we can apply calculus to draw conclusions about their values. We then translate these mathematical conclusions into inferences about the real world that we hope will have predictive value. When they do, as is the case with the economic theory here, we say that the functions give a good model of reality.

What do we do when our calculus tells us that the best production level is a value of x that isn't an integer, as it did in Example 6 when it said that $x = 2 + \sqrt{2}$ thousand units would be the production level for maximum profit? The practical answer is to use the nearest convenient integer. For $x = 2 + \sqrt{2}$ thousand, we might use 3414, or perhaps 3410 or 3420 if we ship in boxes of 10.

Exercises 3.6

If you have a grapher, this is a good place to use it. We have included some specific grapher exercises but there is something to be learned from graphing in most of the other exercises as well.

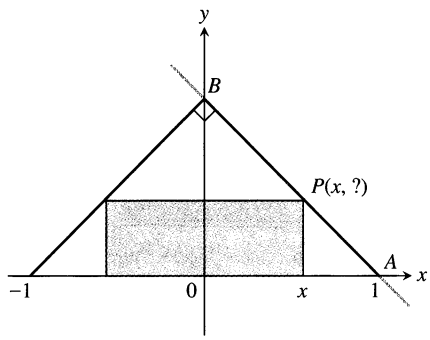
Whenever you are maximizing or minimizing a function of a single variable, we urge you to graph it over the domain that is appropriate to the problem you are solving. The graph will provide insight before you calculate and will furnish a visual context for understanding your answer.

Applications in Geometry

1. A sector shaped like a slice of pie is cut from a circle of radius r. The outer circular arc of the sector has length s. If the sector's total perimeter $(2r + s)$ is to be 100 m, what values of r and s will maximize the sector's area?
2. What is the largest possible area for a right triangle whose hypotenuse is 5 cm long?
3. What is the smallest perimeter possible for a rectangle whose area is 16 in^2?
4. Show that among all rectangles with a given perimeter, the one with the largest area is a square.
5. The figure shown here shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.
   a) Express the y-coordinate of P in terms of x. (You might start by writing an equation for the line AB.)

b) Express the area of the rectangle in terms of x.

c) What is the largest area the rectangle can have?



6. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

7. You are planning to make an open rectangular box from an 8-by-15-in. piece of cardboard by cutting squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way?

8. You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.

9. A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose?

10. A 216-m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

11. *The lightest steel holding tank.* Your iron works has contracted to design and build a 500-ft^3, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding $\frac{1}{2}$-in.-thick stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible. What dimensions do you tell the shop to use?

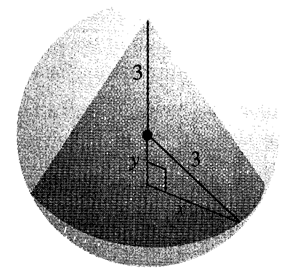
12. *Catching rainwater.* An 1125-ft^3 open-top rectangular tank with a square base x ft on a side and y ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product xy. If the cost is

$$c = 5(x^2 + 4xy) + 10xy,$$

what values of x and y will minimize it?

13. You are designing a poster to contain 50 in^2 of printing with margins of 4 in. each at top and bottom and 2 in. at each side. What overall dimensions will minimize the amount of paper used?

14. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



15. Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area? (*Hint:* $A = (1/2)ab \sin\theta$.)

16. Find the largest possible value of $s = 2x + y$ if x and y are side lengths in a right triangle whose hypotenuse is $\sqrt{5}$ units long.

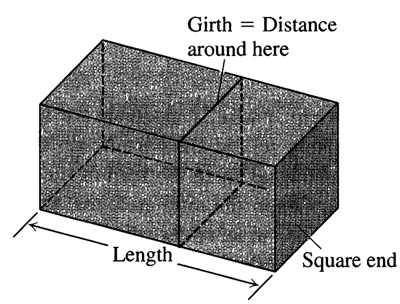
17. What are the dimensions of the lightest (least material) open-top right circular cylindrical can that will hold a volume of 1000 cm^3? Compare the result here with the result in Example 2.

18. You are designing 1000-cm^3 right circular cylindrical cans whose manufacture will take waste into account. There is no waste in cutting the aluminum for the sides, but the tops and bottoms of radius r will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used by each can will therefore be

$$A = 8r^2 + 2\pi rh$$

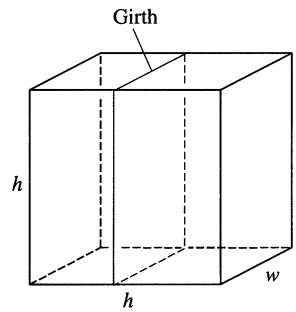
rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 2. In Example 2 the ratio of h to r for the most economical cans was 2 to 1. What is the ratio now?

19. a) The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 in. What dimensions will give a box with a square end the largest possible volume?



b) GRAPHER Graph the volume of a 108-in. box (length plus girth equals 108 in.) as a function of its length, and compare what you see with your answer in (a).

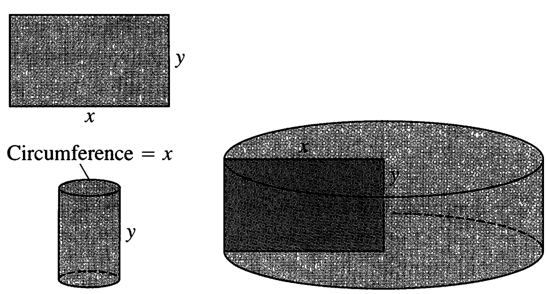
20. *(Continuation of Exercise 19.)* Suppose that instead of having a box with square ends you have a box with square sides so that its dimensions are h by h by w and the girth is $2h + 2w$. What dimensions will give the box its largest volume now?



21. Compare the answers to the following two construction problems.

a) A rectangular sheet of perimeter 36 cm and dimensions x cm by y cm is to be rolled into the cylinder shown here (a). What values of x and y give the largest volume?

b) The rectangular sheet of perimeter 36 cm and dimensions x by y is to be revolved about one of the sides of length y to sweep out the cyclinder shown here (b). What values of x and y give the largest volume?



(a) (b)

22. A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.

23. *Circle vs. square*

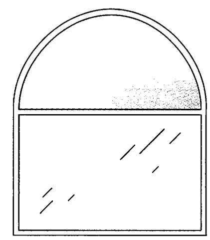
a) A 4-m length of wire is available for making a circle and a square. How should the wire be distributed between the two shapes to maximize the sum of the enclosed areas?

b) GRAPHER Graph the total area enclosed by the wire as a function of the circle's radius. Reconcile what you see with your answer in (a).

c) GRAPHER Now graph the total area enclosed by the wire as a function of the square's side length. Again, reconcile what you see with your answer in (a).

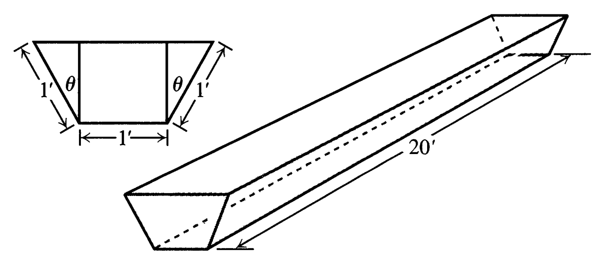
24. If the sum of the surface areas of a cube and a sphere is held constant, what ratio of an edge of the cube to the radius of the sphere will make the sum of the volumes (a) as small as possible, (b) as large as possible?

25. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass while the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.

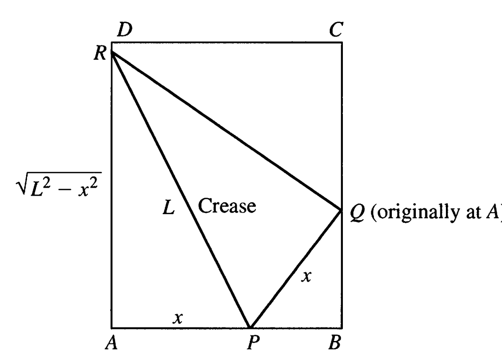


26. A silo (base not included) is to be constructed in the form of a cylinder surmounted by a hemisphere. The cost of construction per square unit of surface area is twice as great for the hemisphere as it is for the cylindrical sidewall. Determine the dimensions to be used if the volume is fixed and the cost of construction is to be kept to a minimum. Neglect the thickness of the silo and waste in construction.

27. The trough here is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?



28. A rectangular sheet of $8\frac{1}{2}$-by-11-in. paper shown here is placed on a flat surface, and one of the corners is placed on the opposite longer edge. The other corners are held in their original positions. With all four corners now held fixed, the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length L.



a) Try it with paper.
b) Show that $L^2 = 2x^3/(2x - 8.5)$.
c) What value of x minimizes L^2?
d) CALCULATOR Find the minimum value of L to the nearest tenth of an inch.
e) GRAPHER Graph L as a function of x and compare what you see with your answer in (d).

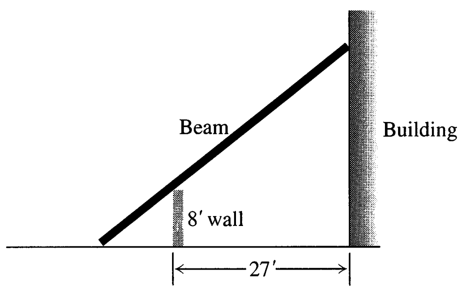
Physical Applications

29. The height of a body moving vertically is given by

$$s = -\frac{1}{2}gt^2 + v_0t + s_0, \quad g > 0,$$

with s in meters and t in seconds. Find the body's maximum height.

30. CALCULATOR The 8-ft wall shown here stands 27 ft from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.



31. *The strength of a beam.* The strength S of a rectangular wooden beam is proportional to its width w times the square of its depth d.

a) Find the dimensions of the strongest beam that can be cut from a 12-in.-diameter cylindrical log.
b) GRAPHER Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in (a).
c) GRAPHER On the same screen, or on a separate screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in (a). What would be the effect of changing to some other value of k? Try it.

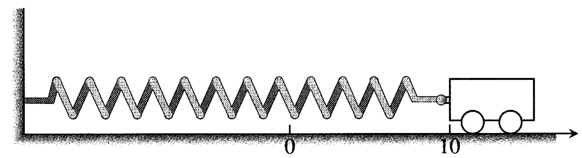
32. *The stiffness of a beam.* The stiffness S of a rectangular beam is proportional to its width times the cube of its depth.

a) Find the dimensions of the stiffest beam that can be cut from a 12-in.-diameter log.
b) GRAPHER Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in (a).
c) GRAPHER On the screen, or on a separate screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in (a). What would be the effect of changing to some other value of k? Try it.

33. Suppose that at any given time t (sec) the current i (amp) in an alternating current circuit is $i = 2\cos t + 2\sin t$. What is the peak current for this circuit (largest magnitude)?

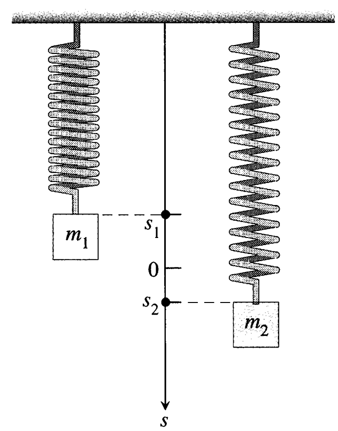
34. A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time t is $s = 10\cos \pi t$.

a) What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?
b) Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?



35. Two masses hanging side by side from springs have positions $s_1 = 2\sin t$ and $s_2 = \sin 2t$, respectively.

a) At what times in the interval $0 < t$ do the masses pass each other? (*Hint:* $\sin 2t = 2\sin t \cos t$.)
b) When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is this distance? (*Hint:* $\cos 2t = 2\cos^2 t - 1$.)



36. The positions of two particles on the s-axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$.

a) At what time(s) in the interval $0 \le t \le 2\pi$ do the particles meet?
b) What is the farthest apart the particles ever get?
c) When in the interval $0 \le t \le 2\pi$ is the distance between the particles changing the fastest?

37. Suppose that at time $t \ge 0$ the position of a particle moving on the x-axis is $x = (t - 1)(t - 4)^4$.

a) When is the particle at rest?
b) During what time interval does the particle move to the left?

c) What is the fastest the particle goes while moving to the left?

d) GRAPHER Graph x as a function of t for $0 \le t \le 6$. Graph dx/dt over the same interval, in another color if possible. Compare the graphs with one another and with your answers in (a)–(c).

38. At noon, ship A was 12 nautical miles due north of ship B. Ship A was sailing south at 12 knots (nautical miles per hour—a nautical mile is 2000 yd) and continued to do so all day. Ship B was sailing east at 8 knots and continued to do so all day.

a) Start counting time with $t = 0$ at noon and express the distance s between the ships as a function of t.

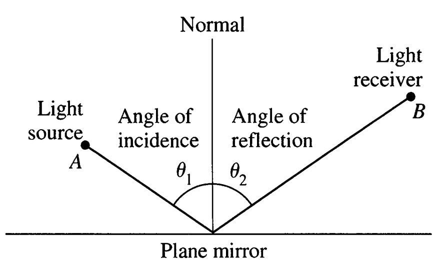
b) How rapidly was the distance between the ships changing at noon? One hour later?

c) CALCULATOR The visibility that day was 5 nautical miles. Did the ships ever sight each other?

d) GRAPHER Graph s and ds/dt together as functions of t for $-1 \le t \le 3$, using different colors if possible. Compare the graphs and reconcile what you see with your answers in (b) and (c).

e) The graph of ds/dt looks as if it might have a horizontal asymptote in the first quadrant. This in turn suggests that ds/dt approaches a limiting value at $t \to \infty$. What is this value? What is its relation to the ships' individual speeds?

39. Fermat's principle in optics states that light always travels from one point to another along a path that minimizes the travel time. Figure 3.59 shows light from a source A reflected by a plane mirror to a receiver at point B. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)



3.59 In studies of light reflection, the angles of incidence and reflection are measured from the line normal to the reflecting surface. Exercise 39 asks you to show that if light obeys Fermat's "least-time" principle, then $\theta_1 = \theta_2$.

40. *Tin pest.* Metallic tin, when kept below 13°C for a while, becomes brittle and crumbles to a gray powder. Tin objects eventually crumble to this gray powder spontaneously if kept in a cold climate for years. The Europeans who saw the tin organ pipes in their churches crumble away years ago called the change *tin pest* because it seemed to be contagious. And indeed it was, for the gray powder is a catalyst for its own formation.

A *catalyst* for a chemical reaction is a substance that controls the rate of the reaction without undergoing any permanent change in itself. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small and slowly again at the end, when most of the original substance is used up. But in between, when both the substance and its catalyst product are abundant, the reaction proceeds at a faster pace.

In some cases it is reasonable to assume that the rate $v = dx/dt$ of the reaction is proportional both to the amount of the original substance present and to the amount of product. That is, v may be considered to be a function of x alone, and

$$v = kx(a - x) = kax - kx^2,$$

where

$x =$ the amount of product

$a =$ the amount of substance at the beginning

$k =$ a positive constant.

At what value of x does the rate v have a maximum? What is the maximum value of v?

Mathematical Applications

41. Is the function $f(x) = x^2 - x + 1$ ever negative? Explain.

42. You have been asked to determine whether the function $f(x) = 3 + 4\cos x + \cos 2x$ is ever negative.

a) Explain why you need consider values of x only in the interval $[0, 2\pi]$.

b) Is f ever negative? Explain.

43. Find the points on the curve $y = \sqrt{x}$ nearest the point $(c, 0)$

a) if $c \ge 1/2$

b) if $c < 1/2$.

44. What value of a makes $f(x) = x^2 + (a/x)$ have (a) a local minimum of $x = 2$; (b) a point of inflection at $x = 1$?

45. What values of a and b make

$$f(x) = x^3 + ax^2 + bx$$

have (a) a local maximum at $x = -1$ and a local minimum at $x = 3$; (b) a local minimum at $x = 4$ and a point of inflection at $x = 1$?

46. Show that $f(x) = x^2 + (a/x)$ cannot have a local maximum for any value of a.

47. a) The function $y = \cot x - \sqrt{2}\csc x$ has an absolute maximum value on the interval $0 < x < \pi$. Find it.

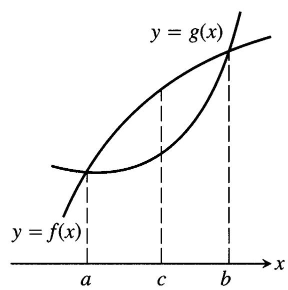
b) GRAPHER Graph the function and compare what you see with your answer in (a).

48. a) The function $y = \tan x + 3\cot x$ has an absolute minimum value on the interval $0 < x < \pi/2$. Find it.

b) GRAPHER Graph the function and compare what you see with your answer in (a).

49. How close does the curve $y = \sqrt{x}$ come to the point $(1/2, 16)$?

50. Let $f(x)$ and $g(x)$ be the differentiable functions graphed here. Point c is the point where the vertical distance between the curves is the greatest. Is there anything special about the tangents to the two curves at c? Give reasons for your answer.



51. Show that if a, b, c, and d are positive integers, then

$$\frac{(a^2+1)(b^2+1)(c^2+1)(d^2+1)}{abcd} \geq 16.$$

52. *The derivative dt/dx in Example 5*

a) Show that

$$f(x) = \frac{x}{\sqrt{a^2+x^2}}$$

is an increasing function of x.

b) Show that

$$g(x) = \frac{d-x}{\sqrt{b^2+(d-x)^2}}$$

is a decreasing function of x.

c) Show that

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2+x^2}} - \frac{d-x}{c_2\sqrt{b^2+(d-x)^2}}$$

is an increasing function of x.

Medicine

53. *Sensitivity to medicine (Continuation of Exercise 50, Section 2.2).* Find the amount of medicine to which the body is most sensitive by finding the value of M that maximizes the derivative dR/dM, where

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right)$$

and C is a constant.

54. *How we cough*

a) When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the questions of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity v can be modeled by the equation

$$v = c(r_0 - r)r^2 \text{ cm/sec}, \quad \frac{r_0}{2} \leq r \leq r_0,$$

where r_0 is the rest radius of the trachea in centimeters and c is a positive constant whose value depends in part on the length of the trachea.

Show that v is greatest when $r = (2/3)r_0$, that is, when the trachea is about 33% contracted. The remarkable fact is that x-ray photographs confirm that the trachea contracts about this much during a cough.

b) GRAPHER Take r_0 to be 0.5 and c to be 1, and graph v over the interval $0 \leq r \leq 0.5$. Compare what you see to the claim that v is at a maximum when $r = (2/3)r_0$.

Business and Economics

55. It costs you c dollars each to manufacture and distribute backpacks. If the backpacks sell at x dollars each, the number sold is given by $n = a/(x-c) + b(100-x)$, where a and b are certain positive constants. What selling price will bring a maximum profit?

56. You operate a tour service that offers the following rates:

a) \$200 per person if 50 people (the minimum number to book the tour) go on the tour.

b) For each additional person, up to a maximum of 80 people total, everyone's charge is reduced by \$2.

It costs \$6000 (a fixed cost) plus \$32 per person to conduct the tour. How many people does it take to maximize your profit?

57. *The best quantity to order.* One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), k is the cost of placing an order (the same, no matter how often you order), c is the cost of one item (a constant), m is the number of items sold each week (a constant), and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security). Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula.*)

58. (*Continuation of Exercise 57.*) Shipping costs sometimes depend on order size. When they do, it is more realistic to replace k by $k + bq$, the sum of k and a constant multiple of q. What is the most economical quantity to order now?

59. Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).

60. Suppose $c(x) = x^3 - 20x^2 + 20{,}000\,x$ is the cost of manufacturing x items. Find a production level that will minimize the average cost of making x items.

3.7

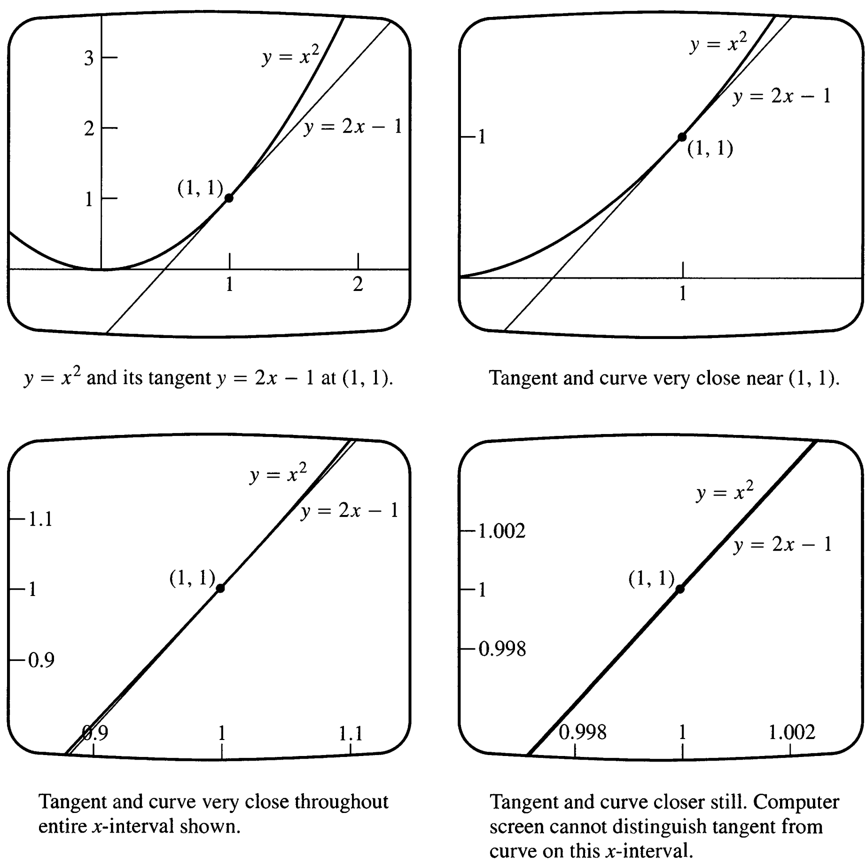
Linearization and Differentials

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications and are easier to work with. The approximating functions discussed in this section are called *linearizations.* They are based on tangent lines.

We introduce new variables dx and dy and define them in a way that gives new meaning to the Leibniz notation dy/dx. We will use dy to estimate error in measurement and sensitivity to change.

Linear Approximations

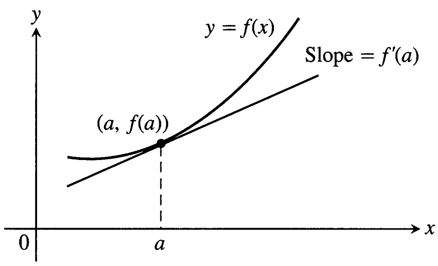
As you can see in Fig. 3.60, the tangent to a curve $y = f(x)$ lies close to the curve near the point of tangency. For a brief interval to either side, the y-values along the tangent line give a good approximation to the y-values on the curve.



3.60 The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.

In the notation of Fig. 3.61, the tangent passes through the point $(a, f(a))$, so its point–slope equation is

$$y = f(a) + f'(a)(x - a).$$



3.61 The equation of the tangent line is $y = f(a) + f'(a)(x - a)$.

Thus, the tangent is the graph of the function

$$L(x) = f(a) + f'(a)(x - a).$$

For as long as the line remains close to the graph of f, $L(x)$ gives a good approximation to $f(x)$.

Definitions

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a) \qquad (1)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

EXAMPLE 1 Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$.

Solution We evaluate Eq. (1) for f at $a = 0$. With

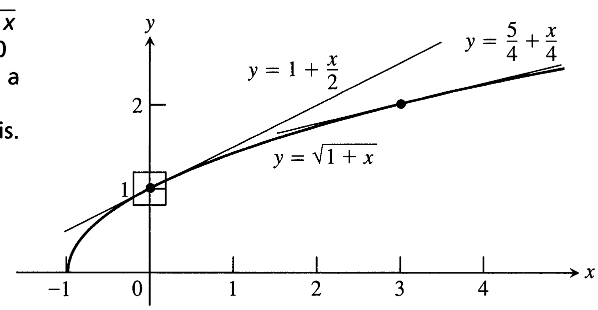
$$f'(x) = \frac{1}{2}(1 + x)^{-1/2},$$

we have $f(0) = 1$, $f'(0) = 1/2$, and

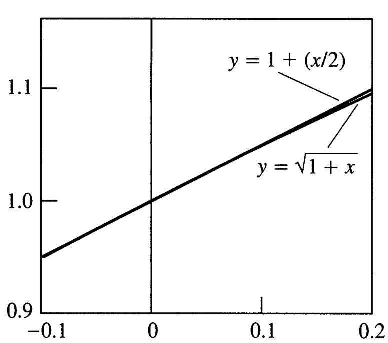
$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

See Fig. 3.62.

3.62 The graph of $y = \sqrt{1 + x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.63 shows a magnified view of the small window about 1 on the y-axis.



❑



3.63 Magnified view of the window in Fig. 3.62.

The approximation $\sqrt{1 + x} \approx 1 + (x/2)$ (Fig. 3.63) gives

$$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10, \qquad \text{Accurate to 2 decimals}$$

$$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025, \qquad \text{Accurate to 3 decimals}$$

$$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250. \qquad \text{Accurate to 5 decimals}$$

Do not be misled by these calculations into thinking that whatever we do with a linearization is better done with a calculator. In practice, we would never use a linearization to find a particular square root. The utility of a linearization is its ability to replace a complicated formula by a simpler one over an entire interval of values. If we have to work with $\sqrt{1+x}$ for x close to 0 and can tolerate the small amount of error involved, we can work with $1+(x/2)$ instead. Of course, we then need to know how much error there is. We will touch on this toward the end of the section but will not have the full story until Chapter 8.

A linear approximation normally loses accuracy away from its center. As Fig. 3.62 suggests, the approximation $\sqrt{1+x} \approx 1+(x/2)$ will probably be too crude to be useful near $x=3$. There, we need the linearization at $x=3$.

EXAMPLE 2 Find the linearization of $f(x)=\sqrt{1+x}$ at $x=3$.

Solution We evaluate Eq. (1) for f at $a=3$. With

$$f(3)=2, \qquad f'(3)=\frac{1}{2}(1+x)^{-1/2}\Big|_{x=3}=\frac{1}{4},$$

we have

$$L(x)=2+\frac{1}{4}(x-3)=\frac{5}{4}+\frac{x}{4}.$$

❑

At $x=3.2$, the linearization in Example 2 gives

$$\sqrt{1+x}=\sqrt{1+3.2}\approx\frac{5}{4}+\frac{3.2}{4}=1.250+0.800=2.050,$$

which differs from the true value $\sqrt{4.2}\approx 2.04939$ by less than one one-thousandth. The linearization in Example 1 gives

$$\sqrt{1+x}=\sqrt{1+3.2}\approx 1+\frac{3.2}{2}=1+1.6=2.6,$$

a result that is off by more than 25%.

EXAMPLE 3 The most important linear approximation for roots and powers is

$$(1+x)^k \approx 1+kx \qquad (x\approx 0;\ \text{any number } k) \tag{2}$$

(Exercise 20). This approximation, good for values of x sufficiently close to zero, has broad application.

Approximation ($x \approx 0$)	**Source: Eq. (2) with ...**
$\sqrt{1+x}\approx 1+\frac{x}{2}$	$k=1/2$
$\frac{1}{1-x}=(1-x)^{-1}\approx 1+(-1)(-x)=1+x$	$k=-1$; $-x$ in place of x
$\sqrt[3]{1+5x^4}=(1+5x^4)^{1/3}\approx 1+\frac{1}{3}(5x^4)=1+\frac{5}{3}x^4$	$k=1/3$; $5x^4$ in place of x
$\frac{1}{\sqrt{1-x^2}}=(1-x^2)^{-1/2}\approx 1+\left(-\frac{1}{2}\right)\left(-x^2\right)=1+\frac{x^2}{2}$	$k=-1/2$; $-x^2$ in place of x

❑

Common linear approximations, $x \approx 0$

$\sin x \approx x$

$\cos x \approx 1$

$\tan x \approx x$

$(1+x)^k \approx 1+kx$

(See the Exercises.)

EXAMPLE 4 Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ (Fig. 3.64).

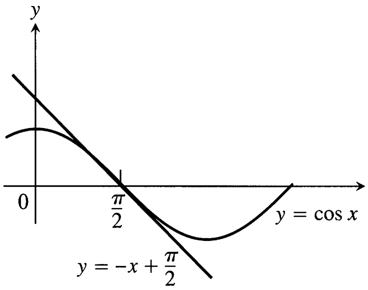
Solution With

$$f(\pi/2) = \cos(\pi/2) = 0 \quad \text{and} \quad f'(\pi/2) = -\sin(\pi/2) = -1,$$

we have

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned}$$

❑



3.64 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$.

Differentials

Definitions

Let $y = f(x)$ be a differentiable function. The **differential** $\boldsymbol{dx}$ is an independent variable. The **differential** $\boldsymbol{dy}$ is

$$dy = f'(x)\,dx.$$

The meaning of *dx* and *dy*

In most contexts, the differential dx of the independent variable is its change Δx, but we do not impose this restriction on the definition.

Unlike the independent variable dx, the variable dy is always a dependent variable. It depends on both x and dx.

EXAMPLE 5 Find dy if

a) $y = x^5 + 37x$ **b)** $y = \sin 3x$.

Solution

a) $dy = (5x^4 + 37)\,dx$ **b)** $dy = (3\cos 3x)\,dx$ ❑

If $dx \neq 0$ and we divide both sides of the equation $dy = f'(x)\,dx$ by dx, we obtain the familiar equation

$$\frac{dy}{dx} = f'(x).$$

This equation says that when $dx \neq 0$, we can regard the derivative dy/dx as a quotient of differentials.

We sometimes write

$$df = f'(x)\,dx$$

in place of $dy = f'(x)\,dx$, and call df the **differential** of f. For instance, if $f(x) = 3x^2 - 6$, then

$$df = d(3x^2 - 6) = 6x\,dx.$$

Every differentiation formula like

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

has a corresponding differential form like

$$d(u+v) = du + dv,$$

obtained by multiplying both sides by dx (Table 3.1).

Table 3.1 Formulas for differentials

$dc = 0$
$d(cu) = c\,du$
$d(u+v) = du + dv$
$d(uv) = u\,dv + v\,du$
$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$
$d(u^n) = nu^{n-1}\,du$
$d(\sin u) = \cos u\,du$
$d(\cos u) = -\sin u\,du$
$d(\tan u) = \sec^2 u\,du$
$d(\cot u) = -\csc^2 u\,du$
$d(\sec u) = \sec u \tan u\,du$
$d(\csc u) = -\csc u \cot u\,du$

EXAMPLE 6

a) $d(\tan 2x) = \sec^2(2x)\,d(2x) = 2\sec^2 2x\,dx$

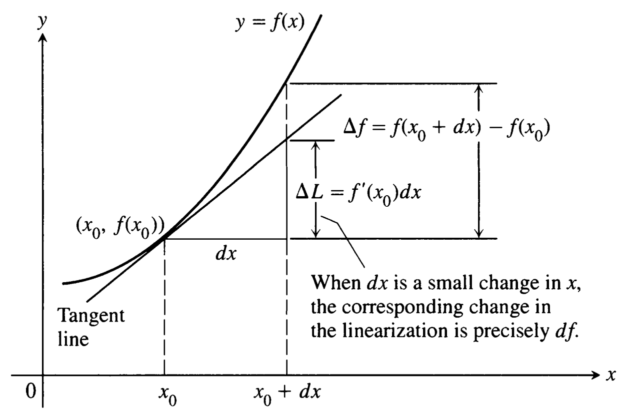
b) $d\left(\dfrac{x}{x+1}\right) = \dfrac{(x+1)\,dx - x\,d\,(x+1)}{(x+1)^2} = \dfrac{x\,dx + dx - x\,dx}{(x+1)^2} = \dfrac{dx}{(x+1)^2}$ ❑

Estimating Change with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point x_0 and we want to predict how much this value will change if we move to a nearby point $x_0 + dx$. If dx is small, f and its linearization L at x_0 will change by nearly the same amount. Since the values of L are simple to calculate, calculating the change in L offers a practical way to estimate the change in f.

In the notation of Fig. 3.65, the change in f is

$$\Delta f = f(x_0 + dx) - f(x_0).$$



3.65 If *dx* is small, the change in the linearization of *f* is nearly the same as the change in *f*.

The corresponding change in L is

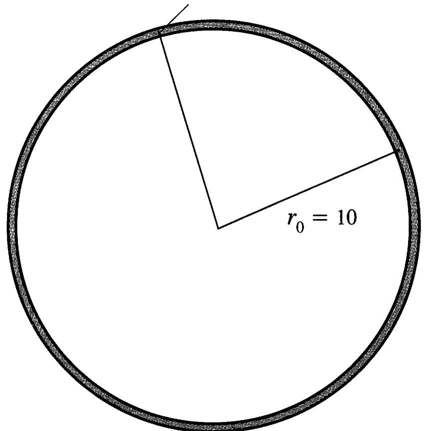
$$\begin{aligned}\Delta L &= L(x_0 + dx) - L(x_0)\\ &= \underbrace{f(x_0) + f'(x_0)\Big[(x_0 + dx) - x_0\Big]}_{L(x_0+dx)} - \underbrace{f(x_0)}_{L(x_0)=f(x_0)}\\ &= f'(x_0)\,dx.\end{aligned}$$

Thus, the differential $df = f'(x)\,dx$ has a geometric interpretation: When df is evaluated at $x = x_0$, $df = \Delta L$, the change in the linearization of f corresponding to the change dx.

The Differential Estimate of Change

Let $f(x)$ be differentiable at $x = x_0$. The approximate change in the value of f when x changes from x_0 to $x_0 + dx$ is

$$df = f'(x_0)\,dx.$$



3.66 When dr is small compared with r_0, as it is when $dr = 0.1$ and $r_0 = 10$, the differential $dA = 2\pi r_0 dr$ gives a good estimate of ΔA (Example 7).

EXAMPLE 7 The radius r of a circle increases from $r_0 = 10$ m to 10.1 m (Fig. 3.66). Estimate the increase in the circle's area A by calculating dA. Compare this with the true change ΔA.

Solution Since $A = \pi r^2$, the estimated increase is

$$dA = A'(r_0)\, dr = 2\pi r_0\, dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

The true change is

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = \underbrace{2\pi}_{dA} + \underbrace{0.01\pi}_{\text{error}}.$$

❑

Absolute, Relative, and Percentage Change

As we move from x_0 to a nearby point $x_0 + dx$, we can describe the change in f in three ways:

	True	Estimated
Absolute change	$\Delta f = f(x_0 + dx) - f(x_0)$	$df = f'(x_0)\, dx$
Relative change	$\dfrac{\Delta f}{f(x_0)}$	$\dfrac{df}{f(x_0)}$
Percentage change	$\dfrac{\Delta f}{f(x_0)} \times 100$	$\dfrac{df}{f(x_0)} \times 100$

EXAMPLE 8 The estimated percentage change in the area of the circle in Exercise 7 is

$$\frac{dA}{A(r_0)} \times 100 = \frac{2\pi}{100\pi} \times 100 = 2\%.$$

❑

EXAMPLE 9 *The earth's surface area*

Suppose the earth were a perfect sphere and we determined its radius to be 3959 $\pm$ 0.1 miles. What effect would the tolerance of $\pm$ 0.1 have on our estimate of the earth's surface area?

Solution The surface area of a sphere of radius r is $S = 4\pi r^2$. The uncertainty in the calculation of S that arises from measuring r with a tolerance of dr miles is about

$$dS = \left(\frac{dS}{dr}\right) dr = 8\pi r\, dr.$$

With $r = 3959$ and $dr = 0.1$, our estimate of S could be off by as much as

$$dS = 8\pi(3959)(0.1) \approx 9950 \text{ mi}^2,$$

to the nearest square mile, which is about the area of the state of Maryland. ❑

If we underestimated the radius of the earth by 528 ft during a calculation of the earth's surface area, we would leave out an area the size of the state of Maryland.

EXAMPLE 10 About how accurately should we measure the radius r of a sphere to calculate the surface area $S = 4\pi r^2$ within 1% of its true value?

Solution We want any inaccuracy in our measurement to be small enough to make the corresponding increment ΔS in the surface area satisfy the inequality

$$|\Delta S| \leq \frac{1}{100}S = \frac{4\pi r^2}{100}.$$

We replace ΔS in this inequality with

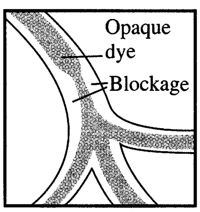
$$dS = \left(\frac{dS}{dr}\right)dr = 8\pi r\, dr.$$

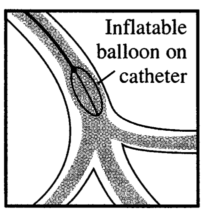
This gives

$$|8\pi r dr| \leq \frac{4\pi r^2}{100}, \qquad \text{or} \qquad |dr| \leq \frac{1}{8\pi r} \cdot \frac{4\pi r^2}{100} = \frac{1}{2}\frac{r}{100}.$$

We should measure r with an error dr that is no more than 0.5% of the true value. ❑

Angiography: An opaque dye is injected into a partially blocked artery to make the inside visible under x-rays. This reveals the location and severity of the blockage.





Angioplasty: A balloon-tipped catheter is inflated inside the artery to widen it at the blockage site.

EXAMPLE 11 *Unclogging arteries*

In the late 1830s, the French physiologist Jean Poiseuille ("pwa-*zoy*") discovered the formula we use today to predict how much the radius of a partially clogged artery has to be expanded to restore normal flow. His formula,

$$V = kr^4,$$

says that the volume V of fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube's radius r. How will a 10% increase in r affect V?

Solution The differentials of r and V are related by the equation

$$dV = \frac{dV}{dr}dr = 4kr^3 dr.$$

Hence,

$$\frac{dV}{V} = \frac{4kr^3 dr}{kr^4} = 4\frac{dr}{r}. \qquad \text{Dividing by } V = kr^4$$

The relative change in V is 4 times the relative change in r, so a 10% increase in r will produce a 40% increase in the flow. ❑

Sensitivity

The equation $df = f'(x)\,dx$ tells how sensitive the output of f is to a change in input at different values of x. The larger the value of f' at x, the greater is the effect of a given change dx.

EXAMPLE 12 You want to calculate the height of a bridge from the equation $s = 16t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculation be to a 0.1-sec error in measuring the time?

Solution The size of ds in the equation

$$ds = 32t\, dt$$

depends on how big t is. If $t = 2$ sec, the error caused by $dt = 0.1$ is only

$$ds = 32(2)(0.1) = 6.4 \text{ ft.}$$

Three seconds later, at $t = 5$ sec, the error caused by the same dt is

$$ds = 32(5)(0.1) = 16 \text{ ft.}$$

❑

The Error in the Approximation $\Delta f \approx df$

Let $f(x)$ be differentiable at $x = x_0$ and suppose that Δx is an increment of x. We have two ways to describe the change in f as x changes from x_0 to $x_0 + \Delta x$:

The true change: $\Delta f = f(x_0 + \Delta x) - f(x_0)$

The differential estimate: $df = f'(x_0)\Delta x.$

How well does df approximate Δf?

We measure the approximation error by subtracting df from Δf:

$$\begin{aligned}
\text{Approximation error } &= \Delta f - df \\
&= \Delta f - f'(x_0)\Delta x \\
&= \underbrace{f(x_0 + \Delta x) - f(x_0)}_{\Delta f} - f'(x_0)\Delta x \\
&= \underbrace{\left(\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - f'(x_0)\right)}_{\text{Call this part } \epsilon} \Delta x \\
&= \epsilon \cdot \Delta x.
\end{aligned}$$

As $\Delta x \to 0$, the difference quotient

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

approaches $f'(x_0)$ (remember the definition of $f'(x_0)$), so the quantity in parentheses becomes a very small number (which is why we called it ϵ). In fact, $\epsilon \to 0$ as $\Delta x \to 0$. When Δx is small, the approximation error $\epsilon\,\Delta x$ is smaller still.

$$\underbrace{\Delta f}_{\substack{\text{true}\\ \text{change}}} = \underbrace{f'(x_0)\Delta x}_{\substack{\text{estimated}\\ \text{change}}} + \underbrace{\epsilon\,\Delta x}_{\text{error}}$$

While we do not know exactly how small the error is and will not be able to make much progress on this front until Chapter 8, there is something worth noting here, namely the *form* taken by the equation.

If $y = f(x)$ is differentiable at $x = x_0$, and x changes from x_0 to $x_0 + \Delta x$, the change Δy in f is given by an equation of the form

$$\Delta y = f'(x_0)\Delta x + \epsilon\,\Delta x \qquad (3)$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

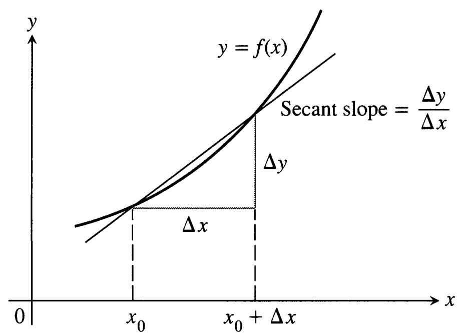
Surprising as it may seem, just knowing the form of Eq. (3) enables us to bring the proof of the Chain Rule to a successful conclusion.

Proof of the Chain Rule

You may recall our saying in Section 2.5 that the proof we wanted to give for the Chain Rule depended on ideas in Section 3.7, the present section. We were referring to Eq. (3), and here is the proof:

Our goal is to show that if $f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then the composite $y = f(g(x))$ is a differentiable function of x. More precisely, if g is differentiable at x_0 and f is differentiable at $g(x_0)$, then the composite is differentiable at x_0 and

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f'(g(x_0)) \cdot g'(x_0).$$



3.67 The graph of y as a function of x. The derivative of y with respect to x at $x = x_0$ is $\lim_{\Delta x \to 0} \Delta y/\Delta x$.

Let Δx be an increment in x and let Δu and Δy be the corresponding increments in u and y. As you can see in Fig. 3.67,

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

so our goal is to show that this limit is $f'(g(x_0)) \cdot g'(x_0)$.

By Eq. (3),

$$\Delta u = g'(x_0)\Delta x + \epsilon_1 \Delta x = (g'(x_0) + \epsilon_1)\Delta x,$$

where $\epsilon_1 \to 0$ as $\Delta x \to 0$. Similarly,

$$\Delta y = f'(u_0)\Delta u + \epsilon_2 \Delta u = (f'(u_0) + \epsilon_2)\Delta u,$$

where $\epsilon_2 \to 0$ as $\Delta u \to 0$. Notice also that $\Delta u \to 0$ as $\Delta x \to 0$. Combining the equations for Δu and Δy gives

$$\Delta y = (f'(u_0) + \epsilon_2)(g'(x_0) + \epsilon_1)\Delta x,$$

so

$$\frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) + \epsilon_2 g'(x_0) + f'(u_0)\epsilon_1 + \epsilon_2\epsilon_1.$$

Since ϵ_1 and ϵ_2 go to zero as Δx goes to zero, three of the four terms on the right vanish in the limit, leaving

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) = f'(g(x_0)) \cdot g'(x_0).$$

This concludes the proof. ❑

⁂ The Conversion of Mass to Energy

Here is an example of how the approximation

$$\frac{1}{\sqrt{1 - x^2}} \approx 1 + \frac{1}{2}x^2 \qquad (4)$$

from Example 3 is used in an applied problem.

Newton's second law,

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} = ma,$$

is stated with the assumption that mass is constant, but we know this is not strictly true because the mass of a body increases with velocity. In Einstein's corrected formula, mass has the value

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \tag{5}$$

where the "rest mass" m_0 represents the mass of a body that is not moving and c is the speed of light, which is about 300,000 km/sec. When v is very small compared with c, v^2/c^2 is close to zero and it is safe to use the approximation

$$\frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right)$$

(Eq. 4 with $x = v/c$) to write

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0\left[1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right)\right] = m_0 + \frac{1}{2}m_0v^2\left(\frac{1}{c^2}\right),$$

or

$$m \approx m_0 + \frac{1}{2}m_0v^2\left(\frac{1}{c^2}\right). \tag{6}$$

Equation (6) expresses the increase in mass that results from the added velocity v.

In Newtonian physics, $(1/2)m_0v^2$ is the kinetic energy (KE) of the body, and if we rewrite Eq. (6) in the form

$$(m - m_0)c^2 \approx \frac{1}{2}m_0v^2,$$

we see that

$$(m - m_0)c^2 \approx \frac{1}{2}m_0v^2 = \frac{1}{2}m_0v^2 - \frac{1}{2}m_0(0)^2 = \Delta(\text{KE}),$$

or

$$(\Delta m)c^2 \approx \Delta(\text{KE}). \tag{7}$$

In other words, the change in kinetic energy $\Delta(\text{KE})$ in going from velocity 0 to velocity v is approximately equal to $(\Delta m)c^2$.

With c equal to 3×10^8 m/sec, Eq. (7) becomes

$$\Delta(\text{KE}) \approx 90{,}000{,}000{,}000{,}000{,}000\ \Delta m \text{ joules} \qquad \text{mass in kilograms}$$

and we see that a small change in mass can create a large change in energy. The energy released by exploding a 20-kiloton atomic bomb, for instance, is the result of converting only 1 gram of mass to energy. The products of the explosion weigh only 1 gram less than the material exploded. A U.S. penny weighs about 3 grams.

Exercises 3.7

Finding Linearizations

In Exercises 1–6, find the linearization $L(x)$ of $f(x)$ at $x = a$.

1. $f(x) = x^4$ at $x = 1$

2. $f(x) = x^{-1}$ at $x = 2$

3. $f(x) = x^3 - x$ at $x = 1$

4. $f(x) = x^3 - 2x + 3$ at $x = 2$

5. $f(x) = \sqrt{x}$ at $x = 4$

6. $f(x) = \sqrt{x^2 + 9}$ at $x = -4$

You want linearizations that will replace the functions in Exercises 7–12 over intervals that include the given points x_0. To make your subsequent work as simple as possible, you want to center each linearization not at x_0 but at a nearby integer $x = a$ at which the given function and its derivative are easy to evaluate. What linearization do you use in each case?

7. $f(x) = x^2 + 2x, \quad x_0 = 0.1$

8. $f(x) = x^{-1}, \quad x_0 = 0.6$

9. $f(x) = 2x^2 + 4x - 3, \quad x_0 = -0.9$

10. $f(x) = 1 + x, \quad x_0 = 8.1$

11. $f(x) = \sqrt[3]{x}, \quad x_0 = 8.5$

12. $f(x) = \dfrac{x}{x+1}, \quad x_0 = 1.3$

Linearizing Trigonometric Functions

In Exercises 13–16, find the linearization of f at $x = a$. Then graph the linearization and f together.

13. $f(x) = \sin x$ at (a) $x = 0$, (b) $x = \pi$

14. $f(x) = \cos x$ at (a) $x = 0$, (b) $x = -\pi/2$

15. $f(x) = \sec x$ at (a) $x = 0$, (b) $x = -\pi/3$

16. $f(x) = \tan x$ at (a) $x = 0$, (b) $x = \pi/4$

The Approximation $(1 + x)^k \approx 1 + kx$

17. Use the formula $(1 + x)^k \approx 1 + kx$ to find linear approximations of the following functions for values of x near zero.

a) $f(x) = (1 + x)^2$ b) $f(x) = \dfrac{1}{(1+x)^5}$

c) $g(x) = \dfrac{2}{1-x}$ d) $g(x) = (1 - x)^6$

e) $h(x) = 3(1 + x)^{1/3}$ f) $h(x) = \dfrac{1}{\sqrt{1+x}}$

18. *Faster than a calculator.* Use the approximation $(1 + x)^k \approx 1 + kx$ to estimate

a) $(1.0002)^{50}$ b) $\sqrt[3]{1.009}$.

19. Find the linearization of $f(x) = \sqrt{x+1} + \sin x$ at $x = 0$. How is it related to the individual linearizations for $\sqrt{x+1}$ and $\sin x$?

20. We know from the Power Rule that the equation

$$\frac{d}{dx}(1 + x)^k = k(1 + x)^{k-1}$$

holds for every rational number k. In Chapter 6, we will show that it holds for every irrational number as well. Assuming this result for now, show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$ for any number k.

Derivatives in Differential Form

In Exercises 21–32, find dy.

21. $y = x^3 - 3\sqrt{x}$

22. $y = x\sqrt{1 - x^2}$

23. $y = \dfrac{2x}{1 + x^2}$

24. $y = \dfrac{2\sqrt{x}}{3(1 + \sqrt{x})}$

25. $2y^{3/2} + xy - x = 0$

26. $xy^2 - 4x^{3/2} - y = 0$

27. $y = \sin(5\sqrt{x})$

28. $y = \cos(x^2)$

29. $y = 4\tan(x^3/3)$

30. $y = \sec(x^2 - 1)$

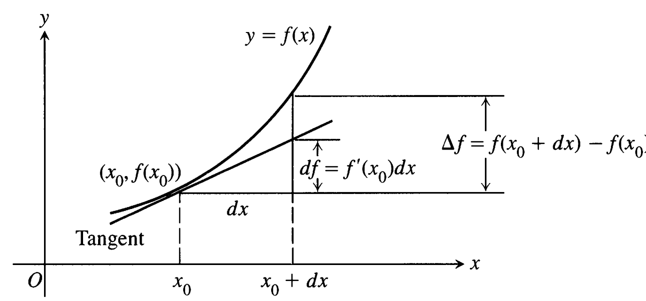
31. $y = 3\csc(1 - 2\sqrt{x})$

32. $y = 2\cot\left(\dfrac{1}{\sqrt{x}}\right)$

Approximation Error

In Exercises 33–38, each function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$. Find

a) the change $\Delta f = f(x_0 + dx) - f(x_0)$;
b) the value of the estimate $df = f'(x_0)\,dx$; and
c) the approximation error $|\Delta f - df|$.



33. $f(x) = x^2 + 2x, \quad x_0 = 0, \quad dx = 0.1$

34. $f(x) = 2x^2 + 4x - 3, \quad x_0 = -1, \quad dx = 0.1$

35. $f(x) = x^3 - x, \quad x_0 = 1, \quad dx = 0.1$

36. $f(x) = x^4, \quad x_0 = 1, \quad dx = 0.1$

37. $f(x) = x^{-1}, \quad x_0 = 0.5, \quad dx = 0.1$

38. $f(x) = x^3 - 2x + 3, \quad x_0 = 2, \quad dx = 0.1$

Differential Estimates of Change

In Exercises 39–44, write a differential formula that estimates the given change in volume or surface area.

39. The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from r_0 to $r_0 + dr$

40. The change in the volume $V = x^3$ of a cube when the edge lengths change from x_0 to $x_0 + dx$

41. The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from x_0 to $x_0 + dx$

42. The change in the lateral surface area $S = \pi r\sqrt{r^2+h^2}$ of a right circular cone when the radius changes from r_0 to $r_0 + dr$ and the height does not change

43. The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from r_0 to $r_0 + dr$ and the height does not change

44. The change in the lateral surface area $S = 2\pi rh$ of a right circular cylinder when the height changes from h_0 to $h_0 + dh$ and the radius does not change

Applications

45. The radius of a circle is increased from 2.00 to 2.02 m.

a) Estimate the resulting change in area.

b) Express the estimate in (a) as a percentage of the circle's original area.

46. The diameter of a tree was 10 in. During the following year, the circumference grew 2 in. About how much did the tree's diameter grow? the tree's cross-section area?

47. The edge of a cube is measured as 10 cm with an error of 1%. The cube's volume is to be calculated from this measurement. Estimate the percentage error in the volume calculation.

48. About how accurately should you measure the side of a square to be sure of calculating the area within 2% of its true value?

49. The diameter of a sphere is measured as 100 ± 1 cm and the volume is calculated from this measurement. Estimate the percentage error in the volume calculation.

50. Estimate the allowable percentage error in measuring the diameter D of a sphere if the volume is to be calculated correctly to within 3%.

51. The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated from a measurement of h and must be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of h, expressed as a percentage of h.

52. **a)** About how accurately must the interior diameter of a 10-m-high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?

b) About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank within 5% of the true amount?

53. A manufacturer contracts to mint coins for the federal government. How much variation dr in the radius of the coins can be tolerated if the coins are to weigh within 1/1000 of their ideal weight? Assume that the thickness does not vary.

54. *(Continuation of Example 11.)* By what percentage should r be increased to increase V by 50%?

55. *(Continuation of Example 12.)* Show that a 5% error in measuring t will cause about a 10% error in calculating s from the equation $s = 16t^2$.

56. *The effect of flight maneuvers on the heart.* The amount of work done in a unit of time by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where W is the work, P is the average blood pressure, V is the volume of blood pumped out during the unit of time, δ is the density of the blood, v is the average velocity of the exiting blood, and g is the acceleration of gravity.

When P, V, δ, and v remain constant, W becomes a function of g and the equation takes the simplified form

$$W = a + \frac{b}{g} \qquad (a,\ b \text{ constant}). \tag{8}$$

As a member of NASA's medical team, you want to know how sensitive W is to apparent changes in g caused by flight maneuvers, and this depends on the initial value of g. As part of your investigation, you decide to compare the effect on W of a given change dg on the moon, where $g = 5.2$ ft/sec^2, with the effect the same change dg would have on Earth, where $g = 32$ ft/sec^2. You use Eq. (8) to find the ratio of dW_{moon} to dW_{Earth}. What do you conclude?

57. *Sketching the change in a cube's volume.* The volume $V = x^3$ of a cube with edges of length x increases by an amount ΔV when x increases by an amount Δx. Show with a sketch how to represent ΔV geometrically as the sum of the volumes of

a) three slabs of dimensions x by x by Δx;

b) three bars of dimensions x by Δx by Δx;

c) one cube of dimensions Δx by Δx by Δx.

The differential formula $dV = 3x^2 dx$ estimates the change in V with the three slabs.

58. *Measuring the acceleration of gravity.* When the length L of a clock pendulum is held constant by controlling its temperature, the pendulum's period T depends on the acceleration of gravity g. The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in g. By keeping track of ΔT, we can estimate the variation in g from the equation $T = 2\pi(L/g)^{1/2}$ that relates T, g, and L.

a) With L held constant and g as the independent variable, calculate dT and use it to answer (b) and (c).

b) If g increases, will T increase, or decrease? Will a pendulum clock speed up, or slow down? Explain.

c) A clock with a 100-cm pendulum is moved from a location where $g = 980$ cm/sec^2 to a new location. This increases the period by $dT = 0.001$ sec. Find dg and estimate the value of g at the new location.

Theory and Examples

59. Show that the approximation of $\sqrt{1+x}$ by its linearization at the origin must improve as $x \to 0$ by showing that

$$\lim_{x\to 0} \frac{\sqrt{1+x}}{1+(x/2)} = 1.$$

60. Show that the approximation of tan x by its linearization at the origin must improve as $x \to 0$ by showing that

$$\lim_{x\to 0} \frac{\tan x}{x} = 1.$$

61. Suppose that the graph of a differentiable function $f(x)$ has a horizontal tangent at $x = a$. Can anything be said about the linearization of f at $x = a$? Give reasons for your answer.

62. *Reading derivatives from graphs.* The idea that differentiable curves flatten out when magnified can be used to estimate the values of the derivatives of functions at particular points. We magnify the curve until the portion we see looks like a straight line through the point in question, and then we use the screen's coordinate grid to read the slope of the curve as the slope of the line it resembles.

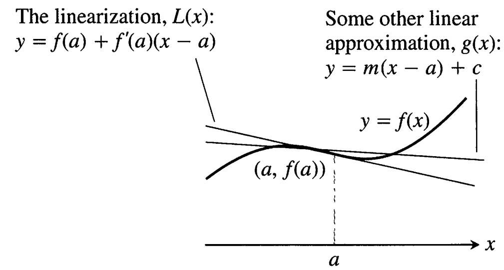
   a) To see how the process works, try it first with the function $y = x^2$ at $x = 1$. The slope you read should be 2.
   b) Then try it with the curve $y = e^x$ at $x = 1$, $x = 0$, and $x = -1$. In each case, compare your estimate of the derivative with the value of e^x at the point. What pattern do you see? Test it with other values of x. Chapter 6 will explain what is going on.

63. *Linearizations at inflection points.* As Fig. 3.64 suggests, linearizations fit particularly well at inflection points. You will understand why if you do Exercise 40 in Section 8.10 later in the book. As another example, graph *Newton's serpentine,* $f(x) = 4x/(x^2 + 1)$, together with its linearizations at $x = 0$ and $x = \sqrt{3}$.

64. *The linearization is the best linear approximation.* (This is why we use the linearization.) Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ is a linear function in which m and c are constants. If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using g as a linear approximation of f instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on g the conditions

   1. $E(a) = 0$ — The approximation error is zero at $x = a$.
   2. $\lim_{x\to a} \frac{E(x)}{x - a} = 0$ — The error is negligible when compared with $x - a$.

   then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization $L(x)$ gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $x - a$.



65. CALCULATOR Enter 2 in your calculator and take successive square roots by pressing the square root key repeatedly (or raising the displayed number repeatedly to the 0.5 power). What pattern do you see emerging? Explain what is going on. What happens if you take successive tenth roots instead?

66. CALCULATOR Repeat Exercise 65 with 0.5 in place of 2 as the original entry. What happens now? Can you use any positive number x in place of 2? Explain what is going on.

CAS Explorations and Projects

In Exercises 67–70, you will use a CAS to estimate the magnitude of the error in using the linearization in place of the function over a specified interval I. Perform the following steps:

a) Plot the function f over I.
b) Find the linearization L of the function at the point a.
c) Plot f and L together on a single graph.
d) Plot the absolute error $|f(x) - L(x)|$ over I and find its maximum value.
e) From your graph in part (d), estimate as large a $\delta > 0$ as you can, satisfying

$$|x - a| < \delta \Rightarrow |f(x) - L(x)| < \epsilon$$

for $\epsilon = 0.5$, 0.1, and 0.01. Then check graphically to see if your δ-estimate holds true.

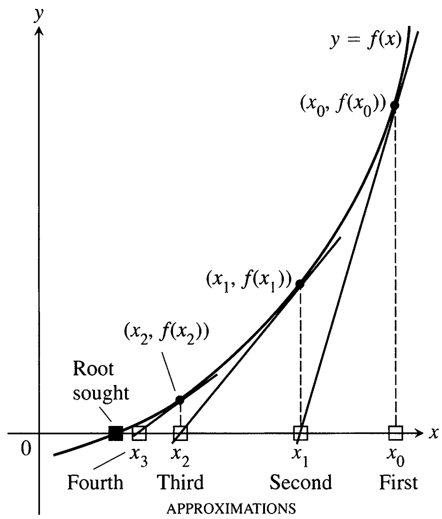
67. $f(x) = x^3 + x^2 - 2x, \quad [-1, 2], \quad a = 1$
68. $f(x) = \dfrac{x - 1}{4x^2 + 1}, \quad \left[-\dfrac{3}{4}, 1\right], \quad a = \dfrac{1}{2}$
69. $f(x) = x^{2/3}(x - 2), \quad [-2, 3], \quad a = 2$
70. $f(x) = \sqrt{x} - \sin x, \quad [0, 2\pi], \quad a = 2$

3.8 Newton's Method

We know simple formulas for solving linear and quadratic equations, and there are somewhat more complicated formulas for cubic and quartic equations (equations of degree three and four). At one time it was hoped that similar formulas might be found for quintic and higher degree equations, but the Norwegian mathematician

Neils Henrik Abel (1802–1829) showed that no formulas like these are possible for polynomial equations of degree greater than four.

When exact formulas for solving an equation $f(x) = 0$ are not available, we can turn to numerical techniques from calculus to approximate the solutions we seek. One of these techniques is *Newton's method* or, as it is more accurately called, the *Newton-Raphson method.* It is based on the idea of using tangent lines to replace the graph of $y = f(x)$ near the points where f is zero. Once again, linearization is the key to solving a practical problem.



3.68 Newton's method starts with an initial guess x_0 and (under favorable circumstances) improves the guess one step at a time.

The Theory

The goal of Newton's method for estimating a solution of an equation $f(x) = 0$ is to produce a sequence of approximations that approach the solution. We pick the first number x_0 of the sequence. Then, under favorable circumstances, the method does the rest by moving step by step toward a point where the graph of f crosses the x-axis (Fig. 3.68).

The initial estimate, x_0, may be found by graphing or just plain guessing. The method then uses the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$ to approximate the curve, calling the point where the tangent meets the x-axis x_1. The number x_1 is usually a better approximation to the solution than is x_0. The point x_2 where the tangent to the curve at $(x_1, f(x_1))$ crosses the x-axis is the next approximation in the sequence. We continue on, using each approximation to generate the next, until we are close enough to the root to stop.

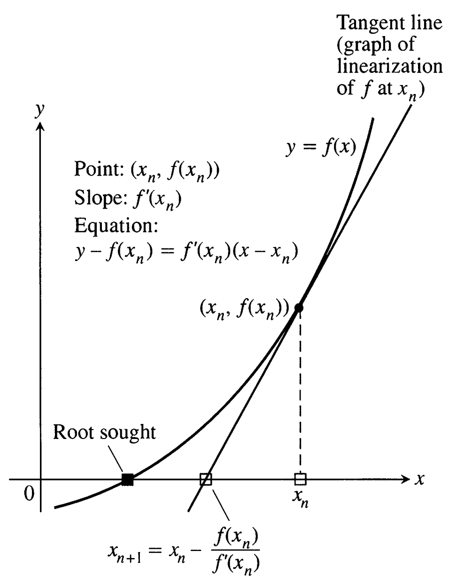
We can derive a formula for generating the successive approximations in the following way. Given the approximation x_n, the point–slope equation for the tangent to the curve at $(x_n, f(x_n))$ is

$$y - f(x_n) = f'(x_n)(x - x_n) \tag{1}$$

(Fig. 3.69). We find where the tangent crosses the x-axis by setting y equal to 0 in this equation and solving for x, giving, in turn,

$$\begin{aligned} 0 - f(x_n) &= f'(x_n)(x - x_n) && \text{Eq. (1) with } y = 0 \\ -f(x_n) &= f'(x_n)x - f'(x_n)x_n \\ f'(x_n)x &= f'(x_n)x_n - f(x_n) \\ x &= x_n - \frac{f(x_n)}{f'(x_n)}. && \text{Assuming } f'(x_n) \neq 0 \end{aligned}$$

This value of x is the next approximation, x_{n+1}.



3.69 The geometry of the successive steps of Newton's method. From x_n we go up to the curve and follow the tangent line down to find x_{n+1}.

The Strategy for Newton's Method

1. Guess a first approximation to a root of the equation $f(x) = 0$. A graph of $y = f(x)$ will help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad (f'(x_n) \neq 0) \tag{2}$$

where $f'(x_n)$ is the derivative of f at x_n.

The Practice

In our first example we find decimal approximations to $\sqrt{2}$ by estimating the positive root of the equation $f(x) = x^2 - 2 = 0$.

EXAMPLE 1 Find the positive root of the equation

$$f(x) = x^2 - 2 = 0.$$

Solution With $f(x) = x^2 - 2$ and $f'(x) = 2x$, Eq. (2) becomes

$$x_{n+1} = x_n - \frac{{x_n}^2 - 2}{2x_n}.$$

To use our calculator efficiently, we rewrite this equation in a form that uses fewer arithmetic operations:

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n}{2} + \frac{1}{x_n} \\ &= \frac{x_n}{2} + \frac{1}{x_n}. \end{aligned}$$

The equation

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

enables us to go from each approximation to the next with just a few keystrokes. With the starting value $x_0 = 1$, we get the results in the first column of the following table. (To 5 decimal places, $\sqrt{2} = 1.41421$.)

	Error	Number of correct figures
$x_0 = 1$	-0.41421	1
$x_1 = 1.5$	0.08579	1
$x_2 = 1.41667$	0.00246	3
$x_3 = 1.41422$	0.00001	5

❑

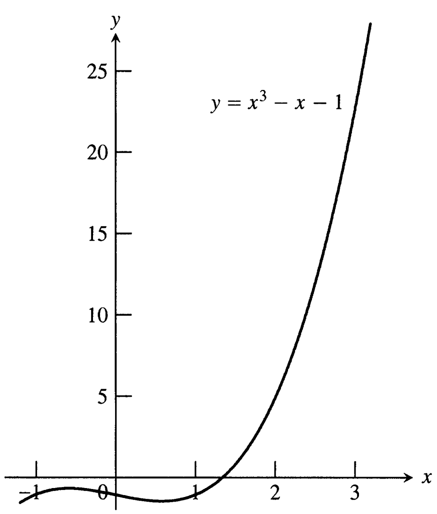
Algorithm and iteration

It is customary to call a specified sequence of computational steps like the one in Newton's method an *algorithm.* When an algorithm proceeds by repeating a given set of steps over and over, using the answer from the previous step as the input for the next, the algorithm is called *iterative* and each repetition is called an *iteration.* Newton's method is one of the really fast iterative techniques for finding roots.

Newton's method is the method used by most calculators to calculate roots because it converges so fast (more about this later). If the arithmetic in the table in Example 1 had been carried to 13 decimal places instead of 5, then going one step further would have given $\sqrt{2}$ correctly to more than 10 decimal places.

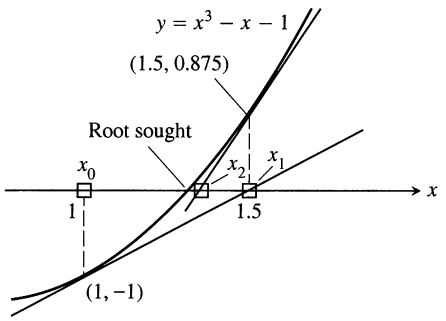
EXAMPLE 2 Find the x-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

Solution The curve crosses the line when $x^3 - x = 1$ or $x^3 - x - 1 = 0$. When does $f(x) = x^3 - x - 1$ equal zero? The graph of f (Fig. 3.70) shows a single root, located between $x = 1$ and $x = 2$. We apply Newton's method to f with the starting value $x_0 = 1$. The results are displayed in Table 3.2 and Fig. 3.71.



3.70 The graph of $f(x) = x^3 - x - 1$ crosses the x-axis between $x = 1$ and $x = 2$.

At $n = 5$ we come to the result $x_6 = x_5 = 1.3247\ 17957$. When $x_{n+1} = x_n$, Eq. (2) shows that $f(x_n) = 0$. We have found a solution of $f(x) = 0$ to 9 decimals.

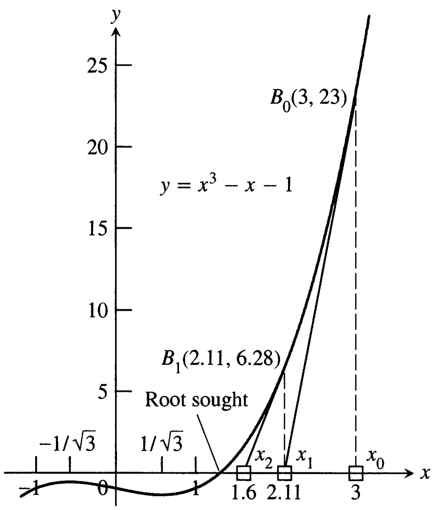


3.71 The first three x-values in Table 3.2.

Table 3.2 The result of applying Newton's method to $f(x) = x^3 - x - 1$ with $x_0 = 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
0	1	−1	2	1.5
1	1.5	0.875	5.75	1.3478 26087
2	1.3478 26087	0.1006 82173	4.4499 05482	1.3252 00399
3	1.3252 00399	0.0020 58362	4.2684 68293	1.3247 18174
4	1.3247 18174	0.0000 00924	4.2646 34722	1.3247 17957
5	1.3247 17957	−1.0437E-9	4.2646 32997	1.3247 17957

The equation $x^3 - x - 1 = 0$ is the equation we solved graphically in Section 1.5. Notice how much more rapidly and accurately we find the solution here. ❑



3.72 Any starting value x_0 to the right of $x = 1/\sqrt{3}$ will lead to the root.

In Fig. 3.72, we have indicated that the process in Example 2 might have started at the point $B_0(3, 23)$ on the curve, with $x_0 = 3$. Point B_0 is quite far from the x-axis, but the tangent at B_0 crosses the x-axis at about (2.11, 0), so x_1 is still an improvement over x_0. If we use Eq. (2) repeatedly as before, with $f(x) = x^3 - x - 1$ and $f'(x) = 3x^2 - 1$, we confirm the 9-place solution $x_6 = x_5 = 1.3247\ 17957$ in six steps.

The curve in Fig. 3.72 has a local maximum at $x = -1/\sqrt{3}$ and a local minimum at $x = +1/\sqrt{3}$. We would not expect good results from Newton's method if we were to start with x_0 between these points, but we can start any place to the right of $x = 1/\sqrt{3}$ and get the answer. It would not be very clever to do so, but we could even begin far to the right of B_0, for example with $x_0 = 10$. It takes a bit longer, but the process still converges to the same answer as before.

Convergence Is Usually Assured

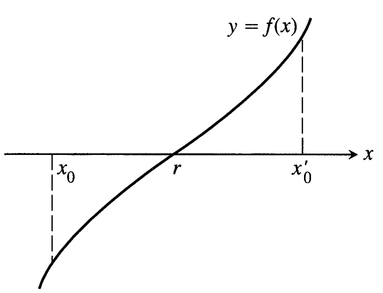
In practice, Newton's method usually converges with impressive speed, but since this is not guaranteed you must test that convergence is actually taking place. One way to do this would be to begin by graphing the function to find a good starting value for x_0. It is important to test that you are getting closer to a zero of the function, by evaluating $|f(x_n)|$, and to check that the method is converging, by evaluating $|x_n - x_{n+1}|$.

Theory does provide some help, however. A theorem from advanced calculus says that if

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1 \tag{3}$$

for all x in an interval about a root r, then the method will converge to r for any starting value x_0 in that interval. In practice, the theorem is somewhat hard to apply and convergence is evaluated by calculating $f(x_n)$ and $|x_n - x_{n+1}|$.

Inequality (3) is a *sufficient* but not a necessary condition. The method can and does converge in some cases where there is no interval about r on which the inequality holds. Newton's method always converges if the curve $y = f(x)$ is convex ("bulges") toward the x-axis in the interval between x_0 and the root sought. See Fig. 3.73.



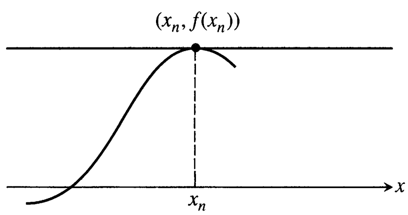
3.73 Newton's method will converge to r from either starting point.

Under favorable circumstances, the speed with which Newton's method converges to r is expressed by the advanced calculus formula

$$\underbrace{|x_{n+1} - r|}_{\text{error } e_{n+1}} \le \frac{\max |f''|}{2 \min |f'|} |x_n - r|^2 = \text{constant} \cdot \underbrace{|x_n - r|^2}_{\text{error } e_n}, \tag{4}$$

where max and min refer to the maximum and minimum values in an interval surrounding r. The formula says that the error in step $n+1$ is no greater than a constant times the square of the error in step n. This may not seem like much, but think of what it says. If the constant is less than or equal to 1, and $|x_n - r| < 10^{-3}$, then $|x_{n+1} - r| < 10^{-6}$. *In a single step* the method moves from three decimal places of accuracy to six!

The results in (3) and (4) both assume that f is "nice." In the case of (4), this means that f has only a single root at r, so that $f'(r) \neq 0$. If f has a multiple root at r, the convergence may be slower.



3.74 If $f'(x_n) = 0$, there is no intersection point to define x_{n+1}.

But Things Can Go Wrong

Newton's method stops if $f'(x_n) = 0$ (Fig. 3.74). In that case, try a new starting point. Of course, f and f' may have a common root. To detect whether this is so, you could first find the solutions of $f'(x) = 0$ and check f at those values. Or you could graph f and f' together.

Newton's method does not always converge. For instance, if

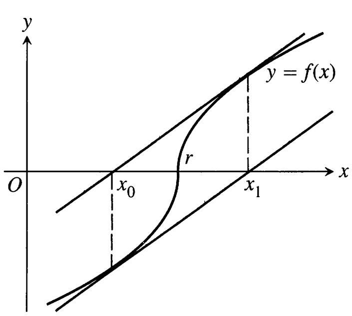
$$f(x) = \begin{cases} -\sqrt{r-x}, & x < r \\ \sqrt{x-r}, & x \ge r, \end{cases} \tag{5}$$

the graph will be like the one in Fig. 3.75. If we begin with $x_0 = r - h$, we get $x_1 = r + h$, and successive approximations go back and forth between these two values. No amount of iteration brings us closer to the root than our first guess.

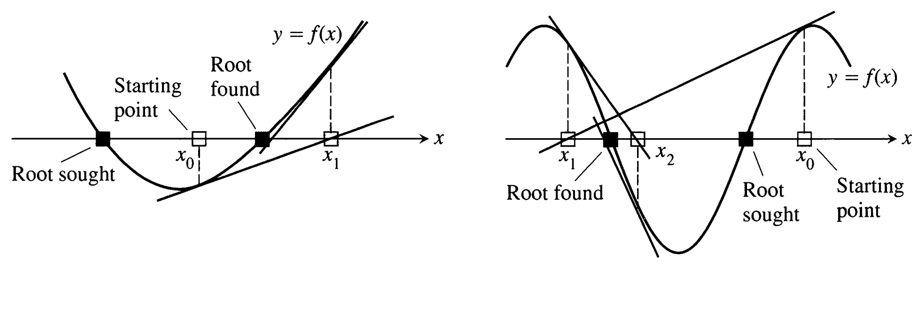
If Newton's method does converge, it converges to a root. In theory, that is. In practice, there are situations in which the method appears to converge but there is no root there. Fortunately, such situations are rare.

When Newton's method converges to a root, it may not be the root you have in mind. Figure 3.76 shows two ways this can happen.

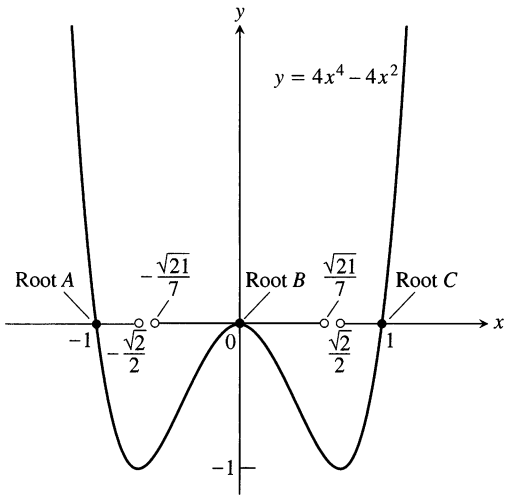
The solution then is to use everything you know about the curve—from graphs drawn by computer or from calculus-based analysis—to get a feeling for the shape of the curve near r and to choose an x_0 close to r. Use Newton's method and test its convergence as you go along. The chances are you will have no problems.



3.75 Newton's method fails to converge.

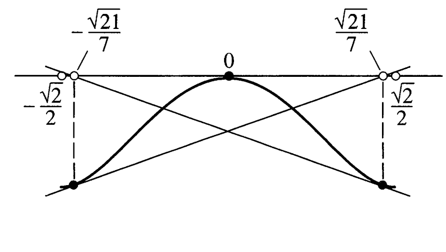


3.76 Newton's method may miss the root you want if you start too far away.

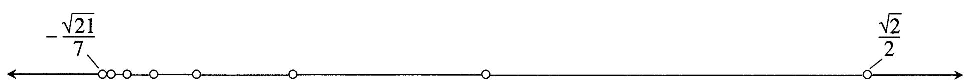


(a)

3.77 (a) Starting values in $(-\infty, -\sqrt{2}/2)$, $(-\sqrt{21}/7, \sqrt{21}/7)$, and $(\sqrt{2}/2, \infty)$ lead respectively to roots *A, B,* and *C.* (b) The values $x = \pm\sqrt{21}/7$ lead only to each other. (c) Between $\sqrt{21}/7$ and $\sqrt{2}/2$ there are infinitely many open intervals of points attracted to *A* alternating with open intervals of points attracted to *C.* This behavior is mirrored in the interval $(-\sqrt{2}/2, -\sqrt{21}/7)$.



(b)



(c)

⁜ Chaos in Newton's Method

The process of finding roots by Newton's method can be chaotic, meaning that for some equations the final outcome can be extremely sensitive to the starting value's location.

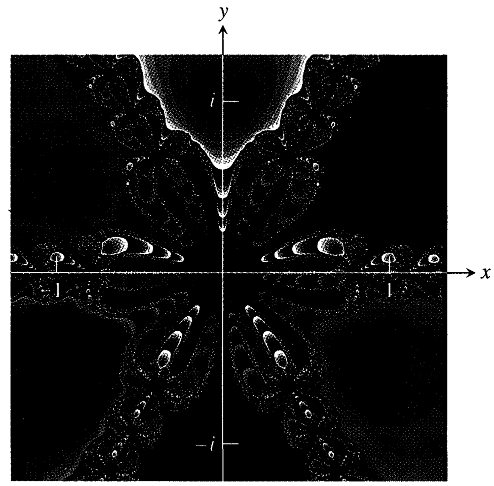
The equation $4x^4 - 4x^2 = 0$ is a case in point (Fig. 3.77a). Starting values in the blue zone on the x-axis lead to root *A*. Starting values in the black lead to root *B*, and starting values in the red zone lead to root *C*. The points $\pm\sqrt{2}/2$ give horizontal tangents. The points $\pm\sqrt{21}/7$ "cycle," each leading to the other, and back (Fig. 3.77b).

The interval between $\sqrt{21}/7$ and $\sqrt{2}/2$ contains infinitely many open intervals of points leading to root *A*, alternating with intervals of points leading to root *C* (Fig. 3.77c). The boundary points separating consecutive intervals (there are infinitely many) do not lead to roots, but cycle back and forth from one to another.

Here is where the "chaos" is truly manifested. As we select points that approach $\sqrt{21}/7$ from the right it becomes increasingly difficult to distinguish which lead to root *A* and which to root *C*. On the same side of $\sqrt{21}/7$, we find arbitrarily close together points whose ultimate destinations are far apart.

If we think of the roots as "attractors" of other points, the coloring in Fig. 3.77 shows the intervals of the points they attract (the "intervals of attraction"). You might think that points between roots *A* and *B* would be attracted to either *A* or *B*, but, as we see, that is not the case. Between *A* and *B* there are infinitely many intervals of points attracted to *C*. Similarly, between *B* and *C* lie infinitely many intervals of points attracted to *A*.

We encounter an even more dramatic example of chaotic behavior when we apply Newton's method to solve the complex-number equation $z^6 - 1 = 0$. It has six solutions: 1, -1, and the four numbers $\pm(1/2) \pm (\sqrt{3}/2)i$. As Fig. 3.78 (on the following page) suggests, each of the six roots has infinitely many "basins"



3.78 This computer-generated initial value portrait uses color to show where different points in the complex plane end up when they are used as starting values in applying Newton's method to solve the equation $z^6 - 1 = 0$. Red points go to 1, green points to $(1/2) + (\sqrt{3}/2)i$, dark blue points to $(-1/2) + (\sqrt{3}/2)i$, and so on. Starting values that generate sequences that do not arrive within 0.1 units of a root after 32 steps are colored black.

of attraction in the complex plane (Appendix 3). Starting points in red basins are attracted to the root 1, those in the green basin to the root $(1/2) + (\sqrt{3}/2)i$, and so on. Each basin has a boundary whose complicated pattern repeats without end under successive magnifications.

Exercises 3.8

Root Finding

1. Use Newton's method to estimate the solutions of the equation $x^2 + x - 1 = 0$. Start with $x_0 = -1$ for the left-hand solution and with $x_0 = 1$ for the solution on the right. Then, in each case, find x_2.

2. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2.

3. Use Newton's method to estimate the two zeros of the function $f(x) = x^4 + x - 3$. Start with $x_0 = -1$ for the left-hand zero and with $x_0 = 1$ for the zero on the right. Then, in each case, find x_2.

4. Use Newton's method to estimate the two zeros of the function $f(x) = 2x - x^2 + 1$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find x_2.

5. Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find x_2.

6. Use Newton's method to find the negative fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = -1$ and find x_2.

7. CALCULATOR At what value(s) of x does $\cos x = 2x$?

8. CALCULATOR At what value(s) of x does $\cos x = -x$?

9. CALCULATOR Use the Intermediate Value Theorem from Section 1.5 to show that $f(x) = x^3 + 2x - 4$ has a root between $x = 1$ and $x = 2$. Then find the root to 5 decimal places.

10. CALCULATOR Estimate π to as many decimal places as your calculator will display by using Newton's method to solve the equation $\tan x = 0$ with $x_0 = 3$.

Theory, Examples, and Applications

11. Suppose your first guess is lucky, in the sense that x_0 is a root of $f(x) = 0$. Assuming that $f'(x_0)$ is defined and not 0, what happens to x_1 and later approximations?

12. You plan to estimate $\pi/2$ to 5 decimal places by using Newton's method to solve the equation $\cos x = 0$. Does it matter what your starting value is? Give reasons for your answer.

13. *Oscillation.* Show that if $h > 0$, applying Newton's method to
$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$
leads to $x_1 = -h$ if $x_0 = h$ and to $x_1 = h$ if $x_0 = -h$. Draw a picture that shows what is going on.

14. *Approximations that get worse and worse.* Apply Newton's method to $f(x) = x^{1/3}$ with $x_0 = 1$, and calculate x_1, x_2, x_3, and x_4. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \to \infty$? Draw a picture that shows what is going on.

15. a) Explain why the following four statements ask for the same information:

i) Find the roots of $f(x) = x^3 - 3x - 1$.
ii) Find the x-coordinates of the intersections of the curve $y = x^3$ with the line $y = 3x + 1$.
iii) Find the x-coordinates of the points where the curve $y = x^3 - 3x$ crosses the horizontal line $y = 1$.
iv) Find the values of x where the derivative of $g(x) = (1/4)x^4 - (3/2)x^2 - x + 5$ equals zero.

a) CALCULATOR Use Newton's method to find the two negative zeros of $f(x) = x^3 - 3x - 1$ to 5 decimal places.

b) GRAPHER Graph $f(x) = x^3 - 3x - 1$ for $-2 \le x \le 2.5$. Use ZOOM and TRACE to estimate the zeros of f to 5 decimal places.

c) GRAPHER Graph $g(x) = 0.25x^4 - 1.5x^2 - x + 5$. Use ZOOM and TRACE with appropriate rescaling to find, to 5 decimal places, the values of x where the graph has horizontal tangents.

16. *Locating a planet.* To calculate a planet's space coordinates, we have to solve equations like $x = 1 + 0.5 \sin x$. Graphing the function $f(x) = x - 1 - 0.5 \sin x$ suggests that the function has a root near $x = 1.5$. Use one application of Newton's method to improve this estimate. That is, start with $x_0 = 1.5$ and find x_1. (The value of the root is 1.49870 to 5 decimal places.) Remember to use radians.

17. *Finding an ion concentration.* While trying to find the acidity of a saturated solution of magnesium hydroxide in hydrochloric acid, you derive the equation

$$\frac{3.64 \times 10^{-11}}{[H_3O^+]^2} = [H_3O^+] + 3.6 \times 10^{-4}$$

for the hydronium ion concentration $[H_3O^+]$. To find the value of $[H_3O^+]$, you set $x = 10^4[H_3O^+]$ and convert the equation to

$$x^3 + 3.6x^2 - 36.4 = 0.$$

You then solve this by Newton's method. What do you get for x? (Make it good to 2 decimal places.) For $[H_3O^+]$?

18. Show that Newton's method cannot converge to a point $x = c$ where the function's graph has an upward pointing cusp above the x-axis like the one in the margin on p. 215.

Computer or Programmable Calculator

Exercises 19–28 require a computer or programmable calculator.

19. The curve $y = \tan x$ crosses the line $y = 2x$ between $x = 0$ and $x = \pi/2$. Use Newton's method to find where.

20. Use Newton's method to find the two real solutions of the equation $x^4 - 2x^3 - x^2 - 2x + 2 = 0$.

21. a) How many solutions does the equation $\sin 3x = 0.99 - x^2$ have?

b) Use Newton's method to find them.

22. a) Does $\cos 3x$ ever equal x?

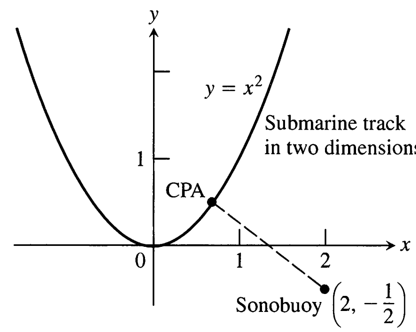
b) Use Newton's method to find where.

23. Find the four real zeros of the function $f(x) = 2x^4 - 4x^2 + 1$.

24. *The sonobuoy problem.* In submarine location problems it is often necessary to find a submarine's closest point of approach (CPA) to a sonobuoy (sound detector) in the water. Suppose that the submarine travels on a parabolic path $y = x^2$ and that the buoy is located at the point $(2, -1/2)$.

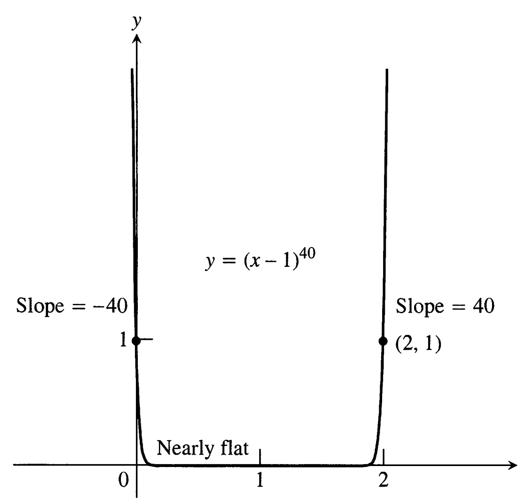
a) Show that the value of x that minimizes the distance between the submarine and the buoy is a solution of the equation $x = 1/(x^2 + 1)$.

b) Solve the equation $x = 1/(x^2 + 1)$ with Newton's method.



(Source: *The Contraction Mapping Principle,* by C. O. Wilde, UMAP Unit 326, Arlington, MA, COMAP, Inc.)

25. *Curves that are nearly flat at the root.* Some curves are so flat that, in practice, Newton's method stops too far from the root to give a useful estimate. Try Newton's method on $f(x) = (x - 1)^{40}$ with a starting value of $x_0 = 2$ to see how close your machine comes to the root $x = 1$.



26. *Finding a root different from the one sought.* All three roots of $f(x) = 4x^4 - 4x^2$ can be found by starting Newton's method near $x = \sqrt{21}/7$. Try it. See Fig. 3.77.