CHAPTER 4

Integration

OVERVIEW This chapter examines two processes and their relation to one another. One is the process by which we determine functions from their derivatives. The other is the process by which we arrive at exact formulas for such things as volume and area through successive approximations. Both processes are called integration.

Integration and differentiation are intimately connected. The nature of the connection is one of the most important ideas in all mathematics, and its independent discovery by Leibniz and Newton still constitutes one of the greatest technical advances of modern times.

4.1 Indefinite Integrals

One of the early accomplishments of calculus was predicting the future position of a moving body from one of its known locations and a formula for its velocity function. Today we view this as one of a number of occasions on which we determine a function from one of its known values and a formula for its rate of change. It is a routine process today, thanks to calculus, to calculate how fast a space vehicle needs to be going at a certain point to escape the earth's gravitational field or to predict the useful life of a sample of radioactive polonium-210 from its present level of activity and its rate of decay.

The process of determining a function from one of its known values and its derivative $f(x)$ has two steps. The first is to find a formula that gives us all the functions that could possibly have f as a derivative. These functions are the so-called antiderivatives of f, and the formula that gives them all is called the indefinite integral of f. The second step is to use the known function value to select the particular antiderivative we want from the indefinite integral. The first step is the subject of the present section; the second is the subject of the next.

Finding a formula that gives all of a function's antiderivatives might seem like an impossible task, or at least to require a little magic. But this is not the case at all. If we can find even one of a function's antiderivatives we can find them all, because of the first two corollaries of the Mean Value Theorem of Section 3.2.

Finding Antiderivatives—Indefinite Integrals

Definitions

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if

$$F'(x) = f(x)$$

for all x in the domain of f. The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign.** The function f is the **integrand** of the integral and x is the **variable of integration.**

According to Corollary 2 of the Mean Value Theorem (Section 3.2), once we have found one antiderivative F of a function f, the other antiderivatives of f differ from F by a constant. We indicate this in integral notation in the following way:

$$\boxed{\int f(x)\,dx = F(x) + C.} \qquad (1)$$

The constant C is the **constant of integration** or **arbitrary constant.** Equation (1) is read, "The indefinite integral of f with respect to x is $F(x) + C$." When we find $F(x) + C$, we say that we have **integrated** f and **evaluated** the integral.

EXAMPLE 1 Evaluate $\int 2x\,dx$.

Solution

$$\int 2x\,dx = \overbrace{x^2}^{\text{an antiderivative of } 2x} + \underbrace{C}_{\text{the arbitrary constant}}$$

The formula $x^2 + C$ generates all the antiderivatives of the function $2x$. The functions $x^2 + 1$, $x^2 - \pi$, and $x^2 + \sqrt{2}$ are all antiderivatives of the function $2x$, as you can check by differentiation. ❑

Many of the indefinite integrals needed in scientific work are found by reversing derivative formulas. You will see what we mean if you look at Table 4.1, which lists a number of standard integral forms side by side with their derivative-formula sources.

In case you are wondering why the integrals of the tangent, cotangent, secant, and cosecant do not appear in the table, the answer is that the usual formulas for them require logarithms. In Section 4.7, we will see that these functions do have antiderivatives, but we will have to wait until Chapters 6 and 7 to see what they are.

Table 4.1 Integral formulas

Indefinite integral	Reversed derivative formula
1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1,\ n$ rational	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$
$\int dx = \int 1\, dx = x + C$ (special case)	$\frac{d}{dx}(x) = 1$
2. $\int \sin kx\, dx = -\frac{\cos kx}{k} + C$	$\frac{d}{dx}\left(-\frac{\cos kx}{k}\right) = \sin kx$
3. $\int \cos kx\, dx = \frac{\sin kx}{k} + C$	$\frac{d}{dx}\left(\frac{\sin kx}{k}\right) = \cos kx$
4. $\int \sec^2 x\, dx = \tan x + C$	$\frac{d}{dx} \tan x = \sec^2 x$
5. $\int \csc^2 x\, dx = -\cot x + C$	$\frac{d}{dx}(-\cot x) = \csc^2 x$
6. $\int \sec x \tan x\, dx = \sec x + C$	$\frac{d}{dx} \sec x = \sec x \tan x$
7. $\int \csc x \cot x\, dx = -\csc x + C$	$\frac{d}{dx}(-\csc x) = \csc x \cot x$

EXAMPLE 2 *Selected integrals from Table 4.1*

a) $$\int x^5\, dx = \frac{x^6}{6} + C$$ Formula 1 with $n = 5$

b) $$\int \frac{1}{\sqrt{x}} dx = \int x^{-1/2}\, dx = 2x^{1/2} + C = 2\sqrt{x} + C$$ Formula 1 with $n = -1/2$

c) $$\int \sin 2x\, dx = -\frac{\cos 2x}{2} + C$$ Formula 2 with $k = 2$

d) $$\int \cos \frac{x}{2}\, dx = \int \cos \frac{1}{2}x\, dx = \frac{\sin (1/2)x}{1/2} + C = 2 \sin \frac{x}{2} + C$$ Formula 3 with $k = 1/2$

❑

Finding an integral formula can sometimes be difficult, but checking it, once found, is relatively easy: differentiate the right-hand side. The derivative should be the integrand.

EXAMPLE 3

Right: $$\int x \cos x\, dx = x \sin x + \cos x + C$$

Reason: The derivative of the right-hand side is the integrand:

$$\frac{d}{dx}(x \sin x + \cos x + C) = x \cos x + \sin x - \sin x + 0 = x \cos x.$$

Wrong: $\displaystyle\int x \cos x \, dx = x \sin x + C$

Reason: The derivative of the right-hand side is not the integrand:

$$\frac{d}{dx}(x \sin x + C) = x \cos x + \sin x + 0 \neq x \cos x.$$

❑

Do not worry about how to derive the correct integral formula in Example 3. We will present a technique for doing so in Chapter 7.

Rules of Algebra for Antiderivatives

Among the things we know about antiderivatives are these:

1. A function is an antiderivative of a constant multiple kf of a function f if and only if it is k times an antiderivative of f.
2. In particular, a function is an antiderivative of $-f$ if and only if it is the negative of an antiderivative of f.
3. A function is an antiderivative of a sum or difference $f \pm g$ if and only if it is the sum or difference of an antiderivative of f and an antiderivative of g.

When we express these observations in integral notation, we get the standard arithmetic rules for indefinite integration (Table 4.2).

Table 4.2 Rules for indefinite integration

1. *Constant Multiple Rule:*	$\displaystyle\int kf(x)\,dx = k\int f(x)\,dx$
(Does not work if k varies with x.)	
2. *Rule for Negatives:*	$\displaystyle\int -f(x)\,dx = -\int f(x)\,dx$
(Rule 1 with $k = -1$)	
3. *Sum and Difference Rule:*	$\displaystyle\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

EXAMPLE 4 *Rewriting the constant of integration*

$\displaystyle\int 5 \sec x \tan x \, dx = 5\int \sec x \tan x \, dx$	Table 4.2, Rule 1
$= 5(\sec x + C)$	Table 4.1, Formula 6
$= 5 \sec x + 5C$	First form
$= 5 \sec x + C'$	Shorter form, where C' is $5C$
$= 5 \sec x + C$	Usual form—no prime. Since 5 times an arbitrary constant is an arbitrary constant, we rename C'.

❑

What about all the different forms in Example 4? Each one gives all the antiderivatives of $f(x) = 5 \sec x \tan x$, so each answer is correct. But the least complicated of the three, and the usual choice, is

$$\int 5 \sec x \tan x \, dx = 5 \sec x + C.$$

Just as the Sum and Difference Rule for differentiation enables us to differentiate expressions term by term, the Sum and Difference Rule for integration enables us to integrate expressions term by term. When we do so, we combine the individual constants of integration into a single arbitrary constant at the end.

EXAMPLE 5 *Term-by-term integration*

Evaluate

$$\int (x^2 - 2x + 5)\, dx.$$

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)\, dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

If we do not recognize the antiderivative right away, we can generate it term by term with the Sum and Difference Rule:

$$\begin{aligned}\int (x^2 - 2x + 5)\, dx &= \int x^2\, dx - \int 2x\, dx + \int 5\, dx \\ &= \frac{x^3}{3} + C_1 - x^2 + C_2 + 5x + C_3.\end{aligned}$$

This formula is more complicated than it needs to be. If we combine C_1, C_2, and C_3 into a single constant $C = C_1 + C_2 + C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and *still* gives all the antiderivatives there are. For this reason we recommend that you go right to the final form even if you elect to integrate term by term. Write

$$\begin{aligned}\int (x^2 - 2x + 5)\, dx &= \int x^2\, dx - \int 2x\, dx + \int 5\, dx \\ &= \frac{x^3}{3} - x^2 + 5x + C.\end{aligned}$$

Find the simplest antiderivative you can for each part and add the constant at the end. ❑

The Integrals of $\sin^2 x$ and $\cos^2 x$

We can sometimes use trigonometric identities to transform integrals we do not know how to evaluate into integrals we do know how to evaluate. The integral formulas for $\sin^2 x$ and $\cos^2 x$ arise frequently in applications.

EXAMPLE 6

a) $$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$
$$= \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$
$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

b) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$
$$= \frac{x}{2} + \frac{\sin 2x}{4} + C$$
As in part (a), but with a sign change ❑

Exercises 4.1

Finding Antiderivatives

In Exercises 1–18, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. a) $2x$ **b)** x^2 **c)** $x^2 - 2x + 1$

2. a) $6x$ **b)** x^7 **c)** $x^7 - 6x + 8$

3. a) $-3x^{-4}$ **b)** x^{-4} **c)** $x^{-4} + 2x + 3$

4. a) $2x^{-3}$ **b)** $\dfrac{x^{-3}}{2} + x^2$ **c)** $-x^{-3} + x - 1$

5. a) $\dfrac{1}{x^2}$ **b)** $\dfrac{5}{x^2}$ **c)** $2 - \dfrac{5}{x^2}$

6. a) $-\dfrac{2}{x^3}$ **b)** $\dfrac{1}{2x^3}$ **c)** $x^3 - \dfrac{1}{x^3}$

7. a) $\dfrac{3}{2}\sqrt{x}$ **b)** $\dfrac{1}{2\sqrt{x}}$ **c)** $\sqrt{x} + \dfrac{1}{\sqrt{x}}$

8. a) $\dfrac{4}{3}\sqrt[3]{x}$ **b)** $\dfrac{1}{3\sqrt[3]{x}}$ **c)** $\sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x}}$

9. a) $\dfrac{2}{3}x^{-1/3}$ **b)** $\dfrac{1}{3}x^{-2/3}$ **c)** $-\dfrac{1}{3}x^{-4/3}$

10. a) $\dfrac{1}{2}x^{-1/2}$ **b)** $-\dfrac{1}{2}x^{-3/2}$ **c)** $-\dfrac{3}{2}x^{-5/2}$

11. a) $-\pi \sin \pi x$ **b)** $3 \sin x$ **c)** $\sin \pi x - 3 \sin 3x$

12. a) $\pi \cos \pi x$ **b)** $\dfrac{\pi}{2}\cos\dfrac{\pi x}{2}$ **c)** $\cos\dfrac{\pi x}{2} + \pi \cos x$

13. a) $\sec^2 x$ **b)** $\dfrac{2}{3}\sec^2\dfrac{x}{3}$ **c)** $-\sec^2\dfrac{3x}{2}$

14. a) $\csc^2 x$ **b)** $-\dfrac{3}{2}\csc^2\dfrac{3x}{2}$ **c)** $1 - 8\csc^2 2x$

15. a) $\csc x \cot x$ **b)** $-\csc 5x \cot 5x$
c) $-\pi \csc\dfrac{\pi x}{2}\cot\dfrac{\pi x}{2}$

16. a) $\sec x \tan x$ **b)** $4 \sec 3x \tan 3x$
c) $\sec\dfrac{\pi x}{2}\tan\dfrac{\pi x}{2}$

17. $(\sin x - \cos x)^2$ **18.** $(1 + 2\cos x)^2$

Evaluating Integrals

Evaluate the integrals in Exercises 19–58. Check your answers by differentiation.

19. $\displaystyle\int (x+1)\,dx$ **20.** $\displaystyle\int (5-6x)\,dx$

21. $\displaystyle\int \left(3t^2 + \frac{t}{2}\right)dt$ **22.** $\displaystyle\int \left(\frac{t^2}{2} + 4t^3\right)dt$

23. $\displaystyle\int (2x^3 - 5x + 7)\,dx$ **24.** $\displaystyle\int (1 - x^2 - 3x^5)\,dx$

25. $\displaystyle\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right)dx$ **26.** $\displaystyle\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right)dx$

27. $\displaystyle\int x^{-1/3}\,dx$ **28.** $\displaystyle\int x^{-5/4}\,dx$

29. $\displaystyle\int \left(\sqrt{x} + \sqrt[3]{x}\right)dx$ **30.** $\displaystyle\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right)dx$

31. $\displaystyle\int \left(8y - \frac{2}{y^{1/4}}\right)dy$ **32.** $\displaystyle\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right)dy$

33. $\int 2x\left(1-x^{-3}\right) dx$

34. $\int x^{-3}(x+1)\,dx$

35. $\int \frac{t\sqrt{t}+\sqrt{t}}{t^2}\,dt$

36. $\int \frac{4+\sqrt{t}}{t^3}\,dt$

37. $\int (-2\cos t)\,dt$

38. $\int (-5\sin t)\,dt$

39. $\int 7\sin\frac{\theta}{3}\,d\theta$

40. $\int 3\cos 5\theta\,d\theta$

41. $\int (-3\csc^2 x)\,dx$

42. $\int \left(-\frac{\sec^2 x}{3}\right) dx$

43. $\int \frac{\csc\theta\cot\theta}{2}\,d\theta$

44. $\int \frac{2}{5}\sec\theta\tan\theta\,d\theta$

45. $\int (4\sec x\tan x - 2\sec^2 x)\,dx$

46. $\int \frac{1}{2}(\csc^2 x - \csc x\cot x)\,dx$

47. $\int (\sin 2x - \csc^2 x)\,dx$

48. $\int (2\cos 2x - 3\sin 3x)\,dx$

49. $\int 4\sin^2 y\,dy$

50. $\int \frac{\cos^2 y}{7}\,dy$

51. $\int \frac{1+\cos 4t}{2}\,dt$

52. $\int \frac{1-\cos 6t}{2}\,dt$

53. $\int (1+\tan^2\theta)\,d\theta$
(*Hint:* $1+\tan^2\theta = \sec^2\theta$)

54. $\int (2+\tan^2\theta)\,d\theta$

55. $\int \cot^2 x\,dx$
(*Hint:* $1+\cot^2 x = \csc^2 x$)

56. $\int (1-\cot^2 x)\,dx$

57. $\int \cos\theta(\tan\theta + \sec\theta)\,d\theta$

58. $\int \frac{\csc\theta}{\csc\theta - \sin\theta}\,d\theta$

Checking Integration Formulas

Verify the integral formulas in Exercises 59–64 by differentiation. In Section 4.3, we will see where formulas like these come from.

59. $\int (7x-2)^3\,dx = \frac{(7x-2)^4}{28} + C$

60. $\int (3x+5)^{-2}\,dx = -\frac{(3x+5)^{-1}}{3} + C$

61. $\int \sec^2(5x-1)\,dx = \frac{1}{5}\tan(5x-1) + C$

62. $\int \csc^2\left(\frac{x-1}{3}\right) dx = -3\cot\left(\frac{x-1}{3}\right) + C$

63. $\int \frac{1}{(x+1)^2}\,dx = -\frac{1}{x+1} + C$

64. $\int \frac{1}{(x+1)^2}\,dx = \frac{x}{x+1} + C$

65. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a) $\int x\sin x\,dx = \frac{x^2}{2}\sin x + C$

b) $\int x\sin x\,dx = -x\cos x + C$

c) $\int x\sin x\,dx = -x\cos x + \sin x + C$

66. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a) $\int \tan\theta\sec^2\theta\,d\theta = \frac{\sec^3\theta}{3} + C$

b) $\int \tan\theta\sec^2\theta\,d\theta = \frac{1}{2}\tan^2\theta + C$

c) $\int \tan\theta\sec^2\theta\,d\theta = \frac{1}{2}\sec^2\theta + C$

67. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a) $\int (2x+1)^2\,dx = \frac{(2x+1)^3}{3} + C$

b) $\int 3(2x+1)^2\,dx = (2x+1)^3 + C$

c) $\int 6(2x+1)^2\,dx = (2x+1)^3 + C$

68. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a) $\int \sqrt{2x+1}\,dx = \sqrt{x^2+x+C}$

b) $\int \sqrt{2x+1}\,dx = \sqrt{x^2+x} + C$

c) $\int \sqrt{2x+1}\,dx = \frac{1}{3}\left(\sqrt{2x+1}\right)^3 + C$

Theory and Examples

69. Suppose that

$$f(x) = \frac{d}{dx}(1-\sqrt{x}) \quad \text{and} \quad g(x) = \frac{d}{dx}(x+2).$$

Find:

a) $\int f(x)\,dx$

b) $\int g(x)\,dx$

c) $\int [-f(x)]\,dx$

d) $\int [-g(x)]\,dx$

e) $\int [f(x)+g(x)]\,dx$

f) $\int [f(x)-g(x)]\,dx$

g) $\int [x + f(x)]\,dx$

h) $\int [g(x) - 4]\,dx$

70. Repeat Exercise 69, assuming that

$$f(x) = \frac{d}{dx} e^x \quad \text{and} \quad g(x) = \frac{d}{dx}(x \sin x).$$

4.2 Differential Equations, Initial Value Problems, and Mathematical Modeling

This section shows how to use a known value of a function to select a particular antiderivative from the functions in an indefinite integral. The ability to do this is important in mathematical modeling, the process by which we, as scientists, use mathematics to learn about reality.

Initial Value Problems

An equation like

$$\frac{dy}{dx} = f(x)$$

that has a derivative in it is called a **differential equation.** The problem of finding a function y of x when we know its derivative and its value y_0 at a particular point x_0 is called an **initial value problem.** We solve such a problem in two steps, as demonstrated in Example 1.

EXAMPLE 1 *Finding a body's velocity from its acceleration and initial velocity*

The acceleration of gravity near the surface of the earth is 9.8 m/sec^2. This means that the velocity v of a body falling freely in a vacuum changes at the rate of

$$\frac{dv}{dt} = 9.8 \text{ m/sec}^2.$$

If the body is dropped from rest, what will its velocity be t seconds after it is released?

Solution In mathematical terms, we want to solve the initial value problem that consists of

The differential equation: $\dfrac{dv}{dt} = 9.8$

The initial condition: $v = 0$ when $t = 0$ (abbreviated as $v(0) = 0$)

We first solve the differential equation by integrating both sides with respect to t:

$$\frac{dv}{dt} = 9.8 \qquad \text{The differential equation}$$

$$\int \frac{dv}{dt}\,dt = \int 9.8\,dt \qquad \text{Integrate with respect to } t.$$

$$v + C_1 = 9.8t + C_2 \qquad \text{Integrals evaluated}$$

$$v = 9.8t + C. \qquad \text{Constants combined as one}$$

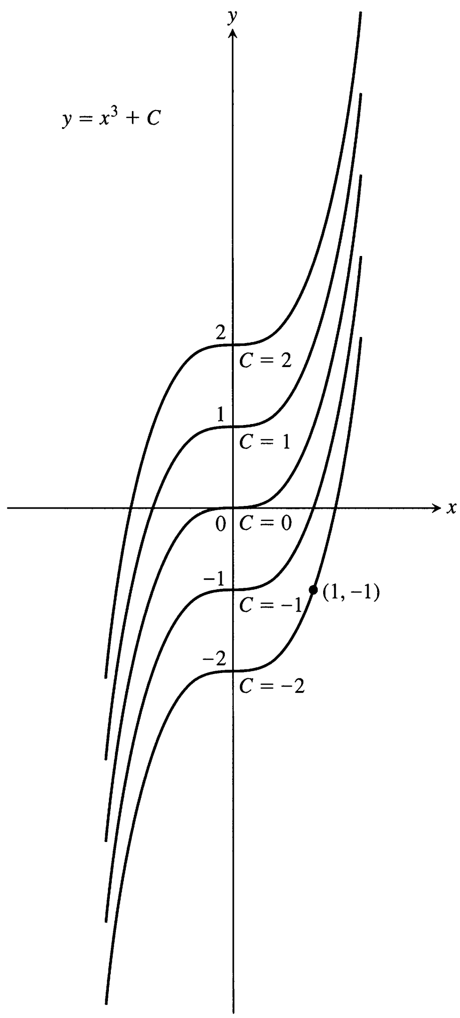
This last equation tells us that the body's velocity t seconds into the fall is $9.8t + C$ m/sec for some value of C. What value? We find out from the initial condition:

$$v = 9.8t + C$$
$$0 = 9.8(0) + C \qquad v(0) = 0$$
$$C = 0.$$

Conclusion: The body's velocity t seconds into the fall is

$$v = 9.8t + 0 = 9.8t \text{ m/sec.}$$ ❑

The indefinite integral $F(x) + C$ of the function $f(x)$ gives the **general solution** $y = F(x) + C$ of the differential equation $dy/dx = f(x)$. The general solution gives all the solutions of the equation (there are infinitely many, one for each value of C). We **solve** the differential equation by finding its general solution. We then solve the initial value problem by finding the **particular solution** that satisfies the initial condition $y(x_0) = y_0$ (y has the value y_0 when $x = x_0$).



4.1 The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 2 we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

EXAMPLE 2 *Finding a curve from its slope function and a point*

Find the curve whose slope at the point (x, y) is $3x^2$ if the curve is required to pass through the point $(1, -1)$.

Solution In mathematical language, we are asked to solve the initial value problem that consists of

The differential equation: $\dfrac{dy}{dx} = 3x^2$ The curve's slope is $3x^2$.

The initial condition: $y(1) = -1$.

To solve it we first solve the differential equation:

$$\frac{dy}{dx} = 3x^2$$
$$\int \frac{dy}{dx}\,dx = \int 3x^2\,dx$$
$$y = x^3 + C. \qquad \text{Constants of integration combined, giving the general solution}$$

This tells us that y equals $x^3 + C$ for some value of C. We find that value from the condition $y(1) = -1$:

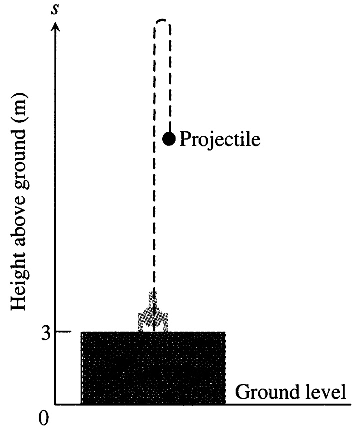
$$y = x^3 + C$$
$$-1 = (1)^3 + C$$
$$C = -2.$$ ❑

The curve we want is $y = x^3 - 2$ (Fig. 4.1).

In the next example, we have to integrate a second derivative twice to find the function we are looking for. The first integration,

$$\int \frac{d^2s}{dt^2}dt = \frac{ds}{dt} + C,$$

gives the function's first derivative. The second integration gives the function.



4.2 The sketch for modeling the projectile motion in Example 3.

EXAMPLE 3 *Finding a projectile's height from its acceleration, initial velocity, and initial position*

A heavy projectile is fired straight up from a platform 3 m above the ground, with an initial velocity of 160 m/sec. Assume that the only force affecting the projectile during its flight is from gravity, which produces a downward acceleration of 9.8 m/sec^2. Find an equation for the projectile's height above the ground as a function of time t if $t = 0$ when the projectile is fired. How high above the ground is the projectile 3 sec after firing?

Solution To model the problem, we draw a figure (Fig. 4.2) and let s denote the projectile's height above the ground at time t. We assume s to be a twice-differentiable function of t and represent the projectile's velocity and acceleration with the derivatives

$$v = \frac{ds}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Since gravity acts in the direction of *decreasing* s in our model, the initial value problem to solve is the following:

$$\text{The differential equation:} \quad \frac{d^2s}{dt^2} = -9.8$$

$$\text{The initial conditions:} \quad \frac{ds}{dt}(0) = 160 \quad \text{and} \quad s(0) = 3.$$

We integrate the differential equation with respect to t to find ds/dt:

$$\int \frac{d^2s}{dt^2}dt = \int (-9.8)\,dt$$

$$\frac{ds}{dt} = -9.8t + C_1.$$

We apply the first initial condition to find C_1:

$$160 = -9.8(0) + C_1 \qquad \frac{ds}{dt}(0) = 160$$

$$C_1 = 160.$$

This completes the formula for ds/dt:

$$\frac{ds}{dt} = -9.8t + 160.$$

We integrate ds/dt with respect to t to find s:

$$\int \frac{ds}{dt}dt = \int (-9.8t + 160)\,dt$$

$$s = -4.9t^2 + 160t + C_2.$$

We apply the second initial condition to find C_2:

$$3 = -4.9(0)^2 + 160(0) + C_2 \qquad s(0) = 3$$

$$C_2 = 3.$$

This completes the formula for s as a function of t:

$$s = -4.9t^2 + 160t + 3.$$

To find the projectile's height 3 sec into the flight, we set $t = 3$ in the formula for s. The height is

$$s = -4.9(3)^2 + 160(3) + 3 = 438.9 \text{ m}. \qquad \square$$

When we find a function from its first derivative, we have one arbitrary constant, as in Examples 1 and 2. When we find a function from its second derivative, we have to deal with two constants, one from each antidifferentiation, as in Example 3. To find a function from its third derivative would require us to find the values of three constants, and so on. In each case, the values of the constants are determined by the problem's initial conditions. Each time we find an antiderivative, we need an initial condition to tell us the value of C.

Sketching Solution Curves

The graph of a solution of a differential equation is called a **solution curve (integral curve).** The curves $y = x^3 + C$ in Fig. 4.1 are solution curves of the differential equation $dy/dx = 3x^2$. When we cannot find explicit formulas for the solution curves of an equation $dy/dx = f(x)$ (that is, we cannot find an antiderivative of f), we may still be able to find their general shape by examining derivatives.

EXAMPLE 4 Sketch the solutions of the differential equation

$$y' = \frac{1}{x^2 + 1}.$$

Solution

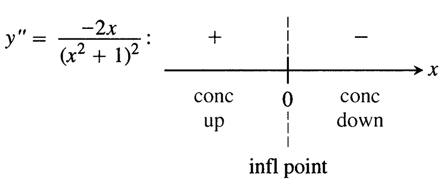
Step 1: y' and y''. As in Section 3.4, the curve's general shape is determined by y' and y''. We already know y':

$$y' = \frac{1}{x^2 + 1}.$$

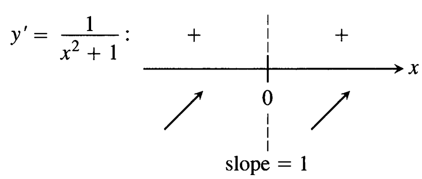
We find y'' by differentiation, in the usual way:

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{1}{x^2 + 1}\right) = \frac{-2x}{(x^2 + 1)^2}.$$

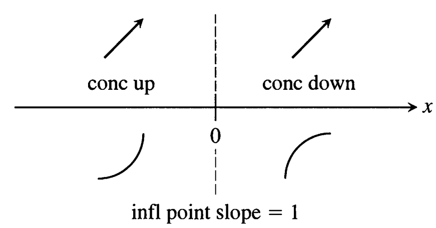
Step 3: *Concavity.* The second derivative changes from (+) to (−) at $x = 0$, so the curves all have an inflection point at $x = 0$.



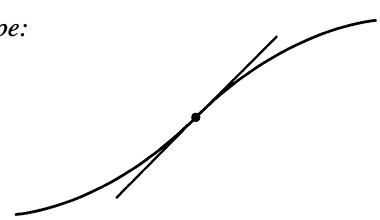
Step 2: *Rise and fall.* The domain of y' is $(-\infty, \infty)$. There are no critical points, so the solution curves have no cusps or extrema. The curves rise from left to right because $y' > 0$. At $x = 0$, the curves have slope 1.



Step 4: *Summary:*



General shape:

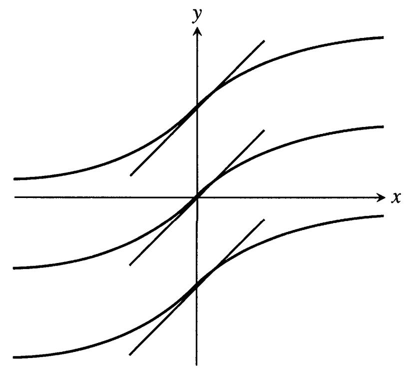


The first derivative tells us still more:

$$\lim_{x\to\pm\infty} y' = \lim_{x\to\pm\infty} \frac{1}{x^2+1} = 0,$$

so the curves level off as $x \to \pm\infty$.

Step 5: *Specific points and solution curves.* We plot an assortment of points on the y-axis where we know the curves' slope (it is 1 at $x = 0$), mark tangents with that slope for guidance, and sketch "parallel" curves of the right general shape.



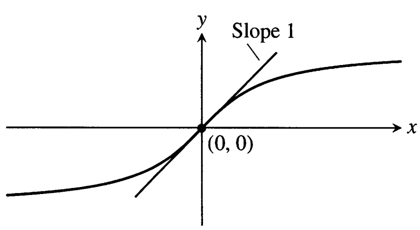
❑

EXAMPLE 5 Sketch the solution of the initial value problem

Differential equation: $\quad y' = \dfrac{1}{x^2+1}$

Initial condition: $\quad y = 0 \quad \text{when} \quad x = 0.$

Solution We find the solution's general shape (Example 4) and sketch the solution curve that passes through the point (0, 0) (Fig. 4.3). ❑



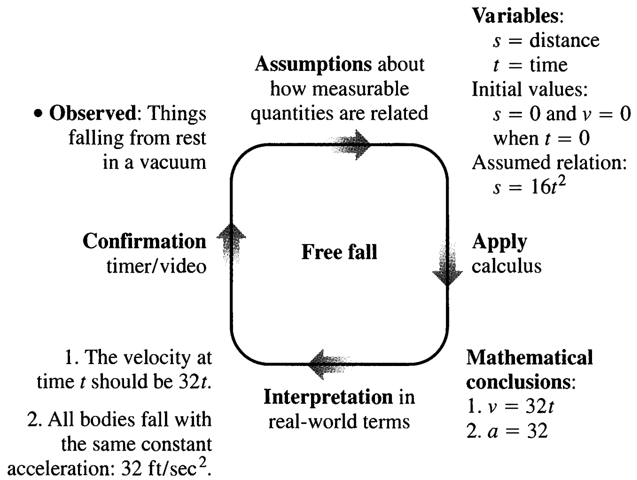
4.3 The solution curve in Example 5.

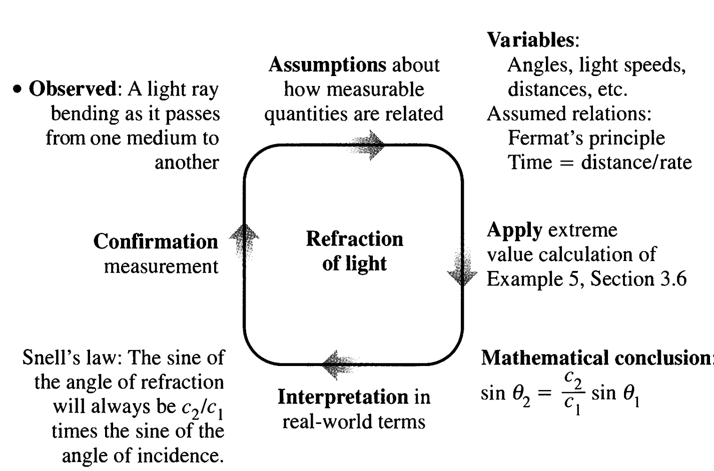
The technique we have learned for sketching solutions is particularly helpful when we are faced with an equation $dy/dx = f(x)$ that involves a function whose antiderivatives have no elementary formula. The antiderivatives of the function $f(x) = 1/(x^2+1)$ in Example 4 do have an elementary formula, as we will see in Chapter 6, but the antiderivatives of $g(x) = \sqrt{1+x^4}$ do not. To solve the equation $dy/dx = \sqrt{1+x^4}$, we must proceed either graphically or numerically.

Mathematical Modeling

The development of a mathematical model usually takes four steps: First we observe something in the real world (a ball bearing falling from rest or the trachea contracting during a cough, for example) and construct a system of mathematical variables and relationships that imitate some of its important features. We build a mathematical metaphor for what we see. Next we apply (usually) existing mathematics to the variables and relationships in the model to draw conclusions about them. After that we translate the mathematical conclusions into information about the system under study. Finally we check the information against observation to see if the model has predictive value. We also investigate the possibility that the model applies to other systems. The really good models are the ones that lead to conclusions that are consistent with observation, that have predictive value and broad application, and that are not too hard to use.

The natural cycle of mathematical imitation, deduction, interpretation, and confirmation is shown in the diagrams on the following page.



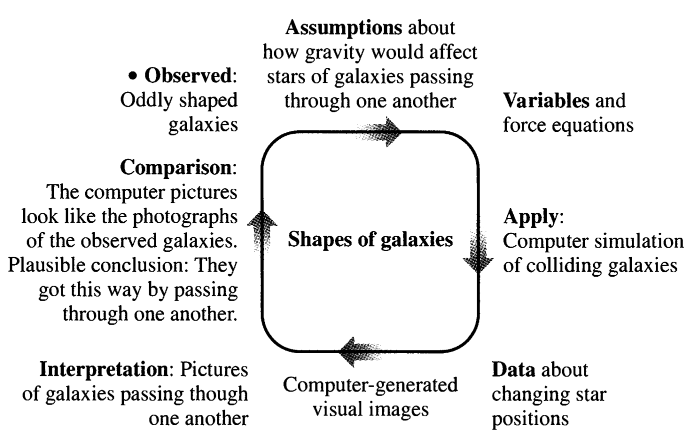


Computer Simulation

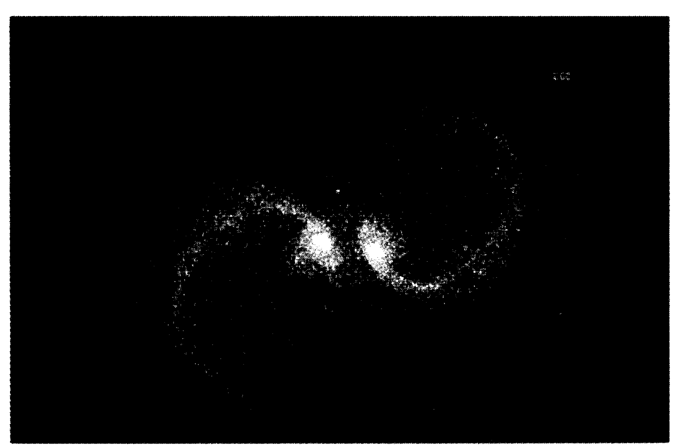
When a system we want to study is complicated, we can sometimes experiment first to see how the system behaves under different circumstances. But if this is not possible (the experiments might be expensive, time-consuming, or dangerous), we might run a series of simulated experiments on a computer—experiments that behave like the real thing, without the disadvantages. Thus we might model the effects of atomic war, the effect of waiting a year longer to harvest trees, the effect of crossing particular breeds of cattle, or the effect of reducing atmospheric ozone by 1%, all without having to pay the consequences or wait to see how things work out naturally.

We also bring computers in when the model we want to use has too many calculations to be practical any other way. NASA's space flight models are run on computers—they have to be to generate course corrections on time. If you want to model the behavior of galaxies that contain billions and billions of stars, a computer offers the only possible way. One of the most spectacular computer simulations in recent years, carried out by Alar Toomre at MIT, explained a peculiar galactic shape that was not consistent with our previous ideas about how galaxies are formed. The galaxies had acquired their odd shapes, Toomre concluded, by passing through one another (Fig. 4.4).

4.4 (a) The modeling cycle for the shapes of colliding galaxies. (b) The computer's image of how galaxies are reshaped by the collision.



(a)



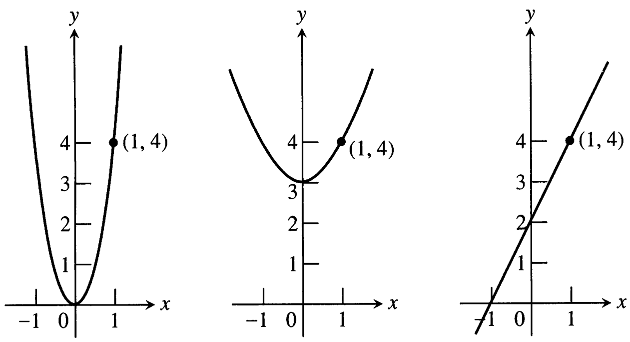
(b)

Exercises 4.2

Initial Value Problems

1. Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = 2x, \quad y = 4 \text{ when } x = 1?$$

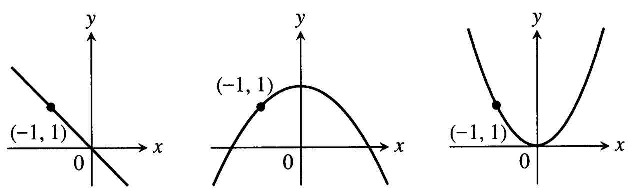


(a) (b) (c)

Give reasons for your answer.

2. Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = -x, \quad y = 1 \text{ when } x = -1?$$



(a) (b) (c)

Give reasons for your answer.

Solve the initial value problems in Exercises 3–22.

3. $\dfrac{dy}{dx} = 2x - 7, \quad y(2) = 0$

4. $\dfrac{dy}{dx} = 10 - x, \quad y(0) = -1$

5. $\dfrac{dy}{dx} = \dfrac{1}{x^2} + x, \quad x > 0; \quad y(2) = 1$

6. $\dfrac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0$

7. $\dfrac{dy}{dx} = 3x^{-2/3}, \quad y(-1) = -5$

8. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}, \quad y(4) = 0$

9. $\dfrac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

10. $\dfrac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

11. $\dfrac{dr}{d\theta} = -\pi \sin \pi\theta, \quad r(0) = 0$

12. $\dfrac{dr}{d\theta} = \cos \pi\theta, \quad r(0) = 1$

13. $\dfrac{dv}{dt} = \dfrac{1}{2} \sec t \tan t, \quad v(0) = 1$

14. $\dfrac{dv}{dt} = 8t + \csc^2 t, \quad v\left(\dfrac{\pi}{2}\right) = -7$

15. $\dfrac{d^2y}{dx^2} = 2 - 6x; \quad y'(0) = 4, \quad y(0) = 1$

16. $\dfrac{d^2y}{dx^2} = 0; \quad y'(0) = 2, \quad y(0) = 0$

17. $\dfrac{d^2r}{dt^2} = \dfrac{2}{t^3}; \quad \left.\dfrac{dr}{dt}\right|_{t=1} = 1, \quad r(1) = 1$

18. $\dfrac{d^2s}{dt^2} = \dfrac{3t}{8}; \quad \left.\dfrac{ds}{dt}\right|_{t=4} = 3, \quad s(4) = 4$

19. $\dfrac{d^3y}{dx^3} = 6; \quad y''(0) = -8, \quad y'(0) = 0, \quad y(0) = 5$

20. $\dfrac{d^3\theta}{dt^3} = 0; \quad \theta''(0) = -2, \quad \theta'(0) = -\dfrac{1}{2}, \quad \theta(0) = \sqrt{2}$

21. $y^{(4)} = -\sin t + \cos t;$
$y'''(0) = 7, \quad y''(0) = y'(0) = -1, \quad y(0) = 0$

22. $y^{(4)} = -\cos x + 8 \sin 2x;$
$y'''(0) = 0, \quad y''(0) = y'(0) = 1, \quad y(0) = 3$

Finding Position from Velocity

Exercises 23–26 give the velocity $v = ds/dt$ and initial position of a body moving along a coordinate line. Find the body's position at time t.

23. $v = 9.8t + 5, \quad s(0) = 10$

24. $v = 32t - 2, \quad s(1/2) = 4$

25. $v = \sin \pi t, \quad s(0) = 0$

26. $v = \dfrac{2}{\pi} \cos \dfrac{2t}{\pi}, \quad s(\pi^2) = 1$

Finding Position from Acceleration

Exercises 27–30 give the acceleration $a = d^2s/dt^2$, initial velocity, and initial position of a body moving on a coordinate line. Find the body's position at time t.

27. $a = 32; \quad v(0) = 20, \quad s(0) = 5$

28. $a = 9.8;\quad v(0) = -3,\quad s(0) = 0$

29. $a = -4\sin 2t;\quad v(0) = 2,\quad s(0) = -3$

30. $a = \dfrac{9}{\pi^2}\cos\dfrac{3t}{\pi};\quad v(0) = 0,\quad s(0) = -1$

Finding Curves

31. Find the curve $y = f(x)$ in the xy-plane that passes through the point (9, 4) and whose slope at each point is $3\sqrt{x}$.

32. **a)** Find a curve $y = f(x)$ with the following properties:

   i) $\dfrac{d^2y}{dx^2} = 6x$

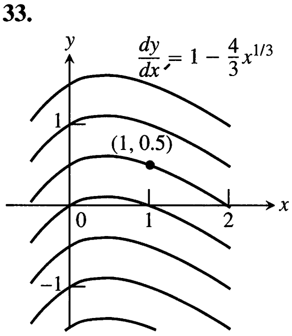
   ii) Its graph passes through the point (0, 1) and has a horizontal tangent there.

   b) How many curves like this are there? How do you know?

Solution (Integral) Curves

Exercises 33–36 show solution curves of differential equations. In each exercise, find an equation for the curve through the labeled point.

33.



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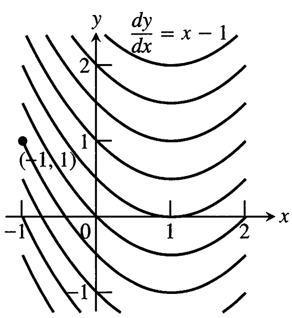

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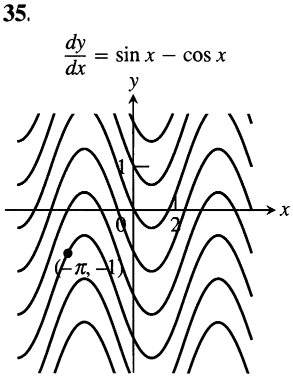

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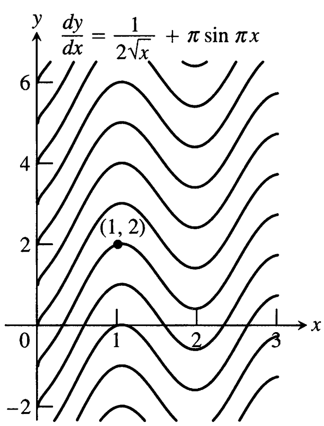


Use the technique described in Example 4 to sketch some of the solutions of the differential equations in Exercises 37–40. Then solve the equations to check on how well you did.

37. $\dfrac{dy}{dx} = 2x$

38. $\dfrac{dy}{dx} = -2x + 2$

39. $\dfrac{dy}{dx} = 1 - 3x^2$

40. $\dfrac{dy}{dx} = x^2$

Use the technique described in Examples 4 and 5 to sketch the solutions of the initial value problems in Exercises 41–44.

41. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}},\quad -1 < x < 1;\quad y(0) = 0$

42. $\dfrac{dy}{dx} = \sqrt{1+x^4},\quad y(0) = 1$

43. $\dfrac{dy}{dx} = \dfrac{1}{x^2+1} - 1,\quad y(0) = 1$

44. $\dfrac{dy}{dx} = \dfrac{x}{x^2+1},\quad y(0) = 0$

Applications

45. On the moon the acceleration of gravity is 1.6 m/sec^2. If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?

46. A rocket lifts off the surface of Earth with a constant acceleration of 20 m/sec^2. How fast will the rocket be going 1 min later?

47. With approximately what velocity do you enter the water if you dive from a 10-m platform? (Use $g = 9.8$ m/sec^2.)

48. CALCULATOR The acceleration of gravity near the surface of Mars is 3.72 m/sec^2. If a rock is blasted straight up from the surface with an initial velocity of 93 m/sec (about 208 mph), how high does it go? (*Hint:* When is the velocity zero?)

49. *Stopping a car in time.* You are driving along a highway at a steady 60 mph (88 ft/sec) when you see an accident ahead and slam on the brakes. What constant deceleration is required to stop your car in 242 ft? To find out, carry out the following steps.

   Step 1: Solve the initial value problem

   Differential equation: $\dfrac{d^2s}{dt^2} = -k \quad$ (k constant)

   Initial conditions: $\dfrac{ds}{dt} = 88$ and $s = 0$ when $t = 0$.

   Measuring time and distance from when the brakes are applied

   Step 2: Find the value of t that makes $ds/dt = 0$. (The answer will involve k.)

   Step 3: Find the value of k that makes $s = 242$ for the value of t you found in step 2.

50. *Stopping a motorcycle.* The State of Illinois Cycle Rider Safety Program requires riders to be able to brake from 30 mph (44 ft/sec) to 0 in 45 ft. What constant deceleration does it take to do that?

51. *Motion along a coordinate line.* A particle moves on a coordinate line with acceleration $a = d^2s/dt^2 = 15\sqrt{t} - (3/\sqrt{t})$, subject to the conditions that $ds/dt = 4$ and $s = 0$ when $t = 1$. Find

   a) the velocity $v = ds/dt$ in terms of t,

   b) the position s in terms of t.

52. *The hammer and the feather.* When *Apollo 15* astronaut David Scott dropped a hammer and a feather on the moon to demonstrate that in a vacuum all bodies fall with the same (constant) acceleration, he dropped them from about 4 ft above the ground. The television footage of the event shows the hammer and feather falling more slowly than on Earth, where, in a vacuum, they would have taken only half a second to fall the 4 ft. How long did it take the hammer and feather to fall 4 ft on the moon? To find out, solve the following initial value problem for s as a function of t. Then find the value of t that makes s equal to 0.

Differential equation: $\frac{d^2s}{dt^2} = -5.2 \text{ ft/sec}^2$

Initial conditions: $\frac{ds}{dt} = 0$ and $s = 4$ when $t = 0$

53. *Motion with constant acceleration.* The standard equation for the position s of a body moving with a constant acceleration a along a coordinate line is

$$s = \frac{a}{2}t^2 + v_0 t + s_0, \qquad (1)$$

where v_0 and s_0 are the body's velocity and position at time $t = 0$. Derive this equation by solving the initial value problem

Differential equation: $\frac{d^2s}{dt^2} = a$

Initial conditions: $\frac{ds}{dt} = v_0$ and $s = s_0$ when $t = 0$

54. (*Continuation of Exercise 53.*) *Free fall near the surface of a planet.* For free fall near the surface of a planet where the acceleration of gravity has a constant magnitude of g length-units/sec^2, Eq. (1) takes the form

$$s = -\frac{1}{2}gt^2 + v_0 t + s_0, \qquad (2)$$

where s is the body's height above the surface. The equation has a minus sign because the acceleration acts downward, in the direction of decreasing s. The velocity v_0 is positive if the object is rising at time $t = 0$, and negative if the object is falling.

Instead of using the result of Exercise 53, you can derive Eq. (2) directly by solving an appropriate initial value problem. What initial value problem? Solve it to be sure you have the right one, explaining the solution steps as you go along.

Theory and Examples

55. *Finding displacement from an antiderivative of velocity*

a) Suppose that the velocity of a body moving along the s-axis is

$$\frac{ds}{dt} = v = 9.8t - 3.$$

1) Find the body's displacement over the time interval from $t = 1$ to $t = 3$ given that $s = 5$ when $t = 0$.

2) Find the body's displacement from $t = 1$ to $t = 3$ given that $s = -2$ when $t = 0$.

3) Now find the body's displacement from $t = 1$ to $t = 3$ given that $s = s_0$ when $t = 0$.

b) Suppose the position s of a body moving along a coordinate line is a differentiable function of time t. Is it true that once you know an antiderivative of the velocity function ds/dt you can find the body's displacement from $t = a$ to $t = b$ even if you do not know the body's exact position at either of those times? Give reasons for your answer.

56. *Uniqueness of solutions.* If differentiable functions $y = F(x)$ and $y = G(x)$ both solve the initial value problem

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0,$$

on an interval I, must $F(x) = G(x)$ for every x in I? Give reasons for your answer.

4.3 Integration by Substitution—Running the Chain Rule Backward

A change of variable can often turn an unfamiliar integral into one we can evaluate. The method for doing this is called the substitution method of integration. It is one of the principal methods for evaluating integrals. This section shows how and why the method works.

The Generalized Power Rule in Integral Form

When u is a differentiable function of x and n is a rational number different from -1, the Chain Rule tells us

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

This same equation, from another point of view, says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n(du/dx)$. Therefore,

$$\int \left(u^n \frac{du}{dx}\right) dx = \frac{u^{n+1}}{n+1} + C.$$

The integral on the left-hand side of this equation is usually written in the simpler "differential" form,

$$\int u^n du,$$

obtained by treating the dx's as differentials that cancel. Combining the last two equations gives the following rule.

Equation (1) actually holds for any real exponent $n \neq -1$, as we will see in Chapter 6.

> If u is any differentiable function,
>
> $$\int u^n du = \frac{u^{n+1}}{n+1} + C. \qquad (n \neq -1,\ n \text{ rational}) \qquad (1)$$

In deriving Eq. (1) we assumed u to be a differentiable function of the variable x, but the name of the variable does not matter and does not appear in the final formula. We could have represented the variable with θ, t, y, or any other letter. Equation (1) says that whenever we can cast an integral in the form

$$\int u^n du, \qquad (n \neq -1)$$

with u a differentiable function and du its differential, we can evaluate the integral as $\left[u^{n+1}/(n+1)\right] + C$.

EXAMPLE 1 Evaluate $\int (x+2)^5\, dx$.

Solution We can put the integral in the form

$$\int u^n du$$

by substituting

$$u = x + 2, \qquad du = d(x+2) = \frac{d}{dx}(x+2) \cdot dx$$
$$= 1 \cdot dx = dx.$$

Then

$$\int (x+2)^5 dx = \int u^5 du \qquad u = x+2, \quad du = dx$$

$$= \frac{u^6}{6} + C \qquad \text{Integrate, using Eq. (1) with } n = 5.$$

$$= \frac{(x+2)^6}{6} + C. \qquad \text{Replace } u \text{ by } x+2.$$

❑

EXAMPLE 2

$$\int \sqrt{1+y^2} \cdot 2y\,dy = \int u^{1/2}\,du \qquad \text{Let } u = 1+y^2,\ du = 2y\,dy.$$

$$= \frac{u^{(1/2)+1}}{(1/2)+1} + C \qquad \text{Integrate, using Eq. (1) with } n = 1/2.$$

$$= \frac{2}{3}u^{3/2} + C \qquad \text{Simpler form}$$

$$= \frac{2}{3}(1+y^2)^{3/2} + C \qquad \text{Replace } u \text{ by } 1+y^2.$$

❑

EXAMPLE 3 *Adjusting the integrand by a constant*

$$\int \sqrt{4t-1}\,dt = \int u^{1/2} \cdot \frac{1}{4}du \qquad \text{Let } u = 4t-1,\ du = 4\,dt,\ (1/4)\,du = dt.$$

$$= \frac{1}{4}\int u^{1/2}du \qquad \text{With the 1/4 out front, the integral is now in standard form.}$$

$$= \frac{1}{4} \cdot \frac{u^{3/2}}{3/2} + C \qquad \text{Integrate, using Eq. (1) with } n = 1/2.$$

$$= \frac{1}{6}u^{3/2} + C \qquad \text{Simpler form}$$

$$= \frac{1}{6}(4t-1)^{3/2} + C \qquad \text{Replace } u \text{ by } 4t-1.$$

❑

Trigonometric Functions

If u is a differentiable function of x, then $\sin u$ is a differentiable function of x. The Chain Rule gives the derivative of $\sin u$ as

$$\frac{d}{dx}\sin u = \cos u\,\frac{du}{dx}.$$

From another point of view, however, this same equation says that $\sin u$ is one of the antiderivatives of the product $\cos u \cdot (du/dx)$. Therefore,

$$\int \left(\cos u\,\frac{du}{dx}\right) dx = \sin u + C.$$

A formal cancellation of the dx's in the integral on the left leads to the following rule.

> If u is a differentiable function, then
>
> $$\int \cos u\,du = \sin u + C. \qquad (2)$$

Equation (2) says that whenever we can cast an integral in the form

$$\int \cos u \, du,$$

we can integrate with respect to u to evaluate the integral as $\sin u + C$.

EXAMPLE 4

$$\int \cos(7\theta + 5)\, d\theta = \int \cos u \cdot \frac{1}{7} du \qquad \text{Let } u = 7\theta + 5,\ du = 7\,d\theta,\ (1/7)\,du = d\theta.$$

$$= \frac{1}{7}\int \cos u \, du \qquad \text{With (1/7) out front, the integral is now in standard form.}$$

$$= \frac{1}{7}\sin u + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{1}{7}\sin(7\theta + 5) + C \qquad \text{Replace } u \text{ by } 7\theta + 5.$$

❑

The companion formula for the integral of $\sin u$ when u is a differentiable function is

$$\int \sin u \, du = -\cos u + C. \tag{3}$$

EXAMPLE 5

$$\int x^2 \sin(x^3)\, dx = \int \sin(x^3) \cdot x^2 dx$$

$$= \int \sin u \cdot \frac{1}{3} du \qquad \text{Let } u = x^3,\ du = 3x^2\,dx,\ (1/3)du = x^2\,dx.$$

$$= \frac{1}{3}\int \sin u \, du$$

$$= \frac{1}{3}(-\cos u) + C \qquad \text{Integrate with respect to } u.$$

$$= -\frac{1}{3}\cos(x^3) + C \qquad \text{Replace } u \text{ by } x^3.$$

❑

The Chain Rule formulas for the derivatives of the tangent, cotangent, secant, and cosecant of a differentiable function u lead to the following integrals.

$$\int \sec^2 u \, du = \tan u + C \tag{4}$$

$$\int \csc^2 u \, du = -\cot u + C \tag{5}$$

$$\int \sec u \tan u \, du = \sec u + C \tag{6}$$

$$\int \csc u \cot u \, du = -\csc u + C \tag{7}$$

In each formula, u is a differentiable function of a real variable. Each formula can be checked by differentiating the right-hand side with respect to that variable. In each case, the Chain Rule applies to produce the integrand on the left.

EXAMPLE 6

$$\int \frac{1}{\cos^2 2\theta}\,d\theta = \int \sec^2 2\theta\,d\theta \qquad \sec 2\theta = \frac{1}{\cos 2\theta}$$

$$= \int \sec^2 u \cdot \frac{1}{2}\,du \qquad \text{Let } u = 2\theta,\ du = 2\,d\theta,\ d\theta = (1/2)\,du.$$

$$= \frac{1}{2}\int \sec^2 u\,du$$

$$= \frac{1}{2}\tan u + C \qquad \text{Integrate, using Eq. (4).}$$

$$= \frac{1}{2}\tan 2\theta + C \qquad \text{Replace } u \text{ by } 2\theta.$$

Check:

$$\frac{d}{d\theta}\left(\frac{1}{2}\tan 2\theta + C\right) = \frac{1}{2}\cdot\frac{d}{d\theta}(\tan 2\theta) + 0$$

$$= \frac{1}{2}\cdot\left(\sec^2 2\theta \cdot \frac{d}{d\theta}(2\theta)\right) \qquad \text{Chain Rule}$$

$$= \frac{1}{2}\cdot \sec^2 2\theta \cdot 2 = \frac{1}{\cos^2 2\theta}.$$

❑

The Substitution Method of Integration

The substitutions in the preceding examples are all instances of the following general rule.

$$\int f(g(x))\cdot g'(x)\,dx = \int f(u)\,du \qquad \text{1. Substitute } u = g(x),\ du = g'(x)\,dx.$$

$$= F(u) + C \qquad \text{2. Evaluate by finding an antiderivative } F(u) \text{ of } f(u). \text{ (Any one will do.)}$$

$$= F(g(x)) + C \qquad \text{3. Replace } u \text{ by } g(x).$$

These three steps are the steps of the substitution method of integration. The method works because $F(g(x))$ is an antiderivative of $f(g(x))\cdot g'(x)$ whenever F is an antiderivative of f:

$$\frac{d}{dx}F(g(x)) = F'(g(x))\cdot g'(x) \qquad \text{Chain Rule}$$

$$= f(g(x))\cdot g'(x) \qquad \text{Because } F' = f$$

Implicit in the substitution method is the assumption that we are replacing x by a function of u. Thus, the substitution $u = g(x)$ must be solvable for x to give x as a function $x = g^{-1}(u)$ ("g inverse of u"). The domains of u and x may need to be restricted on occasion to make this possible. You need not be concerned with this issue at the moment. We will discuss inverses in Section 6.1 and treat the theory of substitutions in greater detail in Sections 7.4 and 13.7.

The Substitution Method of Integration

Take these steps to evaluate the integral

$$\int f(g(x))g'(x)\,dx,$$

when f and g' are continuous functions:

Step 1: Substitute $u = g(x)$ and $du = g'(x)\,dx$ to obtain the integral

$$\int f(u)\,du.$$

Step 2: Integrate with respect to u.
Step 3: Replace u by $g(x)$ in the result.

EXAMPLE 7

$$\int (x^2+2x-3)^2(x+1)\,dx = \int u^2 \cdot \frac{1}{2}\,du \qquad \begin{aligned}&\text{Let } u = x^2+2x-3,\\ &du = 2x\,dx + 2\,dx\\ &\quad = 2(x+1)\,dx,\\ &(1/2)\,du = (x+1)\,dx.\end{aligned}$$

$$= \frac{1}{2}\int u^2\,du$$

$$= \frac{1}{2}\cdot\frac{u^3}{3} + C = \frac{1}{6}u^3 + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{1}{6}(x^2+2x-3)^3 + C \qquad \text{Replace } u.$$

❑

EXAMPLE 8

$$\int \sin^4 t \cos t\,dt = \int u^4\,du \qquad \begin{aligned}&\text{Let } u = \sin t,\\ &du = \cos t\,dt.\end{aligned}$$

$$= \frac{u^5}{5} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{\sin^5 t}{5} + C \qquad \text{Replace } u.$$

❑

The success of the substitution method depends on finding a substitution that will change an integral we cannot evaluate directly into one that we can. If the first substitution fails, we can try to simplify the integrand further with an additional substitution or two. (You will see what we mean if you do Exercises 47 and 48.) Alternatively, we can start afresh. There can be more than one good way to start, as in the next example.

EXAMPLE 9 Evaluate

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}.$$

Solution We can use the substitution method of integration as an exploratory tool: substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try $u = z^2 + 1$ or we might even press our luck and take u to be the entire cube root. Here is what happens in each case.

Solution 1 Substitute $u = z^2 + 1$.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}} \qquad \begin{aligned}&\text{Let } u = z^2+1,\\ &du = 2z\,dz.\end{aligned}$$

$$= \int u^{-1/3}\,du \qquad \text{In the form } \textstyle\int u^n\,du$$

$$= \frac{u^{2/3}}{2/3} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } z^2+1.$$

Solution 2 Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u} \qquad \text{Let } u = \sqrt[3]{z^2+1},\ u^3 = z^2+1,\ 3u^2\,du = 2z\,dz.$$

$$= 3\int u\,du$$

$$= 3\cdot\frac{u^2}{2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } (z^2+1)^{1/3}.$$

Exercises 4.3

Evaluating Integrals

Evaluate the indefinite integrals in Exercises 1–12 by using the given substitutions to reduce the integrals to standard form.

1. $\int \sin 3x\,dx,\quad u = 3x$

2. $\int x\sin(2x^2)\,dx,\quad u = 2x^2$

3. $\int \sec 2t\tan 2t\,dt,\quad u = 2t$

4. $\int \left(1-\cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt,\quad u = 1-\cos\frac{t}{2}$

5. $\int 28(7x-2)^{-5}\,dx,\quad u = 7x-2$

6. $\int x^3(x^4-1)^2\,dx,\quad u = x^4-1$

7. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}},\quad u = 1-r^3$

8. $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy,\quad u = y^4+4y^2+1$

9. $\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx,\quad u = x^{3/2}-1$

10. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx,\quad u = -\frac{1}{x}$

11. $\int \csc^2 2\theta\cot 2\theta\,d\theta$

a) Using $u = \cot 2\theta$ b) Using $u = \csc 2\theta$

12. $\int \frac{dx}{\sqrt{5x+8}}$

a) Using $u = 5x+8$ b) Using $u = \sqrt{5x+8}$

Evaluate the integrals in Exercises 13–46.

13. $\int \sqrt{3-2s}\,ds$

14. $\int (2x+1)^3\,dx$

15. $\int \frac{1}{\sqrt{5s+4}}\,ds$

16. $\int \frac{3\,dx}{(2-x)^2}$

17. $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

18. $\int 8\theta\sqrt[3]{\theta^2-1}\,d\theta$

19. $\int 3y\sqrt{7-3y^2}\,dy$

20. $\int \frac{4y\,dy}{\sqrt{2y^2+1}}$

21. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

22. $\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx$

23. $\int \cos(3z+4)\,dz$

24. $\int \sin(8z-5)\,dz$

25. $\int \sec^2(3x+2)\,dx$

26. $\int \tan^2 x\sec^2 x\,dx$

27. $\int \sin^5\frac{x}{3}\cos\frac{x}{3}\,dx$

28. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

29. $\int r^2\left(\frac{r^3}{18}-1\right)^5 dr$

30. $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

31. $\int x^{1/2}\sin(x^{3/2}+1)\,dx$

32. $\int x^{1/3}\sin(x^{4/3}-8)\,dx$

33. $\int \sec\left(v+\frac{\pi}{2}\right)\tan\left(v+\frac{\pi}{2}\right)\,dv$

34. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)\,dv$

35. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

36. $\int \frac{6\cos t}{(2+\sin t)^3}\,dt$

37. $\int \sqrt{\cot y}\,\csc^2 y\,dy$

38. $\int \frac{\sec z\tan z}{\sqrt{\sec z}}\,dz$

39. $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)\,dt$

40. $\int \frac{1}{\sqrt{t}}\cos(\sqrt{t}+3)\,dt$

41. $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta$

42. $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

43. $\int (s^3+2s^2-5s+5)(3s^2+4s-5)\,ds$

44. $\int (\theta^4-2\theta^2+8\theta-2)(\theta^3-\theta+2)\,d\theta$

45. $\int t^3(1+t^4)^3\,dt$

46. $\int \sqrt{\frac{x-1}{x^5}}\,dx$

Simplifying Integrals Step by Step

If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in Exercises 47 and 48.

47. $\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx$

a) $u=\tan x$, followed by $v=u^3$, then by $w=2+v$
b) $u=\tan^3 x$, followed by $v=2+u$
c) $u=2+\tan^3 x$

48. $\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx$

a) $u=x-1$, followed by $v=\sin u$, then by $w=1+v^2$
b) $u=\sin(x-1)$, followed by $v=1+u^2$
c) $u=1+\sin^2(x-1)$

Evaluate the integrals in Exercises 49 and 50.

49. $\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$

50. $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}}\,d\theta$

Initial Value Problems

Solve the initial value problems in Exercises 51–56.

51. $\frac{ds}{dt}=12t(3t^2-1)^3, \quad s(1)=3$

52. $\frac{dy}{dx}=4x(x^2+8)^{-1/3}, \quad y(0)=0$

53. $\frac{ds}{dt}=8\sin^2\left(t+\frac{\pi}{12}\right), \quad s(0)=8$

54. $\frac{dr}{d\theta}=3\cos^2\left(\frac{\pi}{4}-\theta\right), \quad r(0)=\frac{\pi}{8}$

55. $\frac{d^2s}{dt^2}=-4\sin\left(2t-\frac{\pi}{2}\right), \quad s'(0)=100,\ s(0)=0$

56. $\frac{d^2y}{dx^2}=4\sec^2 2x\tan 2x, \quad y'(0)=4,\ y(0)=-1$

57. The velocity of a particle moving back and forth on a line is $v=ds/dt=6\sin 2t$ m/sec for all t. If $s=0$ when $t=0$, find the value of s when $t=\pi/2$ sec.

58. The acceleration of a particle moving back and forth on a line is $a=d^2s/dt^2=\pi^2\cos\pi t$ m/sec^2 for all t. If $s=0$ and $v=8$ m/sec when $t=0$, find s when $t=1$ sec.

Theory and Examples

59. It looks as if we can integrate $2\sin x\cos x$ with respect to x in three different ways:

a) $\int 2\sin x\cos x\,dx=\int 2u\,du \qquad u=\sin x,$
$$=u^2+C_1=\sin^2 x+C_1$$

b) $\int 2\sin x\cos x\,dx=\int -2u\,du \qquad u=\cos x,$
$$=-u^2+C_2=-\cos^2 x+C_2$$

c) $\int 2\sin x\cos x\,dx=\int \sin 2x\,dx \qquad 2\sin x\cos x=\sin 2x$
$$=-\frac{\cos 2x}{2}+C_3.$$

Can all three integrations be correct? Give reasons for your answer.

60. The substitution $u=\tan x$ gives
$$\int \sec^2 x\tan x\,dx=\int u\,du=\frac{u^2}{2}+C=\frac{\tan^2 x}{2}+C.$$
The substitution $u=\sec x$ gives
$$\int \sec^2 x\tan x\,dx=\int u\,du=\frac{u^2}{2}+C=\frac{\sec^2 x}{2}+C.$$
Can both integrations be correct? Give reasons for your answer.

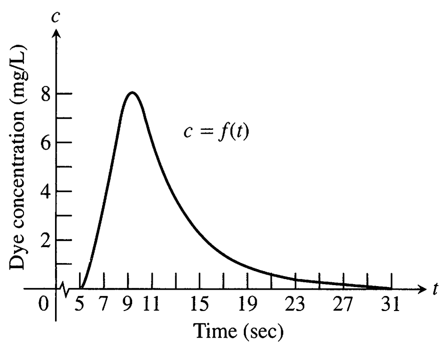
4.4 Estimating with Finite Sums

This section shows how practical questions can lead in natural ways to approximations by finite sums.

Area and Cardiac Output

The number of liters of blood your heart pumps in a minute is called your *cardiac output.* For a person at rest, the rate might be 5 or 6 liters per minute. During strenuous exercise the rate might be as high as 30 liters per minute. It might also be altered significantly by disease.

Instead of measuring a patient's cardiac output with exhaled carbon dioxide, as in Exercise 25 in Section 2.7, a doctor may prefer to use the dye-dilution technique described here. You inject 5 to 10 mg of dye in a main vein near the heart. The dye is drawn into the right side of the heart and pumped through the lungs and out the left side of the heart into the aorta, where its concentration can be measured every few seconds as the blood flows past. The data in Table 4.3 and the plot in Fig. 4.5 show the response of a healthy, resting patient to an injection of 5.6 mg of dye.



4.5 The dye concentrations from Table 4.3, plotted and fitted with a smooth curve. Time is measured with $t = 0$ at the time of injection. The dye concentrations are zero at the beginning, while the dye passes through the lungs. They then rise to a maximum at about $t = 9$ sec and taper to zero by $t = 31$ sec.

To calculate the patient's cardiac output, we divide the amount of dye by the area under the dye concentration curve and multiply the result by 60:

$$\text{Cardiac output} = \frac{\text{amount of dye}}{\text{area under curve}} \times 60. \tag{1}$$

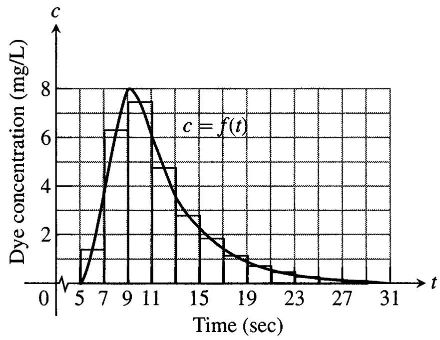
You can see why the formula works if you check the units in which the various quantities are measured. The amount of dye is in milligrams and the area is in (milligrams/liter) $\times$ seconds, which gives cardiac output in liters/minute:

$$\frac{\text{mg}}{\dfrac{\text{mg}}{\text{L}} \cdot \text{sec}} \cdot \frac{\text{sec}}{\text{min}} = \text{mg} \cdot \frac{\text{L}}{\text{mg} \cdot \text{sec}} \cdot \frac{\text{sec}}{\text{min}} = \frac{\text{L}}{\text{min}}.$$

In the example that follows, we estimate the area under the concentration curve in Fig. 4.5 and find the patient's cardiac output.

Table 4.3 Dye-dilution data

Seconds after injection t	Dye concentration (adjusted for recirculation) c	Seconds after injection t	Dye concentration (adjusted for recirculation) c
5	0	19	0.91
7	3.8	21	0.57
9	8.0	23	0.36
11	6.1	25	0.23
13	3.6	27	0.14
15	2.3	29	0.09
17	1.45	31	0



4.6 **The region under the concentration curve of Fig. 4.5 is approximated with rectangles. We ignore the portion from $t = 29$ to $t = 31$; its concentration is negligible.**

EXAMPLE 1 Find the cardiac output of the patient whose data appear in Table 4.3 and Fig. 4.5.

Solution We know the amount of dye to use in Eq. (1) (it is 5.6 mg), so all we need is the area under the concentration curve. None of the area formulas we know can be used for this irregularly shaped region. But we can get a good estimate of this area by approximating the region between the curve and the t-axis with rectangles and adding the areas of the rectangles (Fig. 4.6). Each rectangle omits some of the area under the curve but includes area from outside the curve, which compensates. In Fig. 4.6 each rectangle has a base 2 units long and a height that is equal to the height of the curve above the midpoint of the base. The rectangle's height acts as a sort of average value of the function over the time interval on which the rectangle stands. After reading rectangle heights from the curve, we multiply each rectangle's height and base to find its area, and then get the following estimate:

$$\begin{aligned}\text{Area under curve } &\approx \text{ sum of rectangle areas}\\ &\approx f(6)\cdot 2 + f(8)\cdot 2 + f(10)\cdot 2 + \cdots + f(28)\cdot 2\\ &\approx (1.4)(2) + (6.3)(2) + (7.5)(2) + \cdots + (0.1)(2)\\ &\approx (28.8)(2) = 57.6 \text{ mg}\cdot\text{sec/L}. \qquad (2)\end{aligned}$$

Dividing this figure into the amount of dye and multiplying by 60 gives a corresponding estimate of the cardiac output:

$$\text{Cardiac output} \approx \frac{\text{amount of dye}}{\text{area estimate}} \times 60 = \frac{5.6}{57.6} \times 60 \approx 5.8 \text{ L/min}.$$

The patient's cardiac output is about 5.8 L/min. ❑

Technology *Using a Grapher to Calculate Finite Sums* If your graphing utility has a method for evaluating sums, you might want to use it in this section. Later in the chapter, you will find it useful for approximating "definite" integrals. There will be other uses still later in your study of calculus.

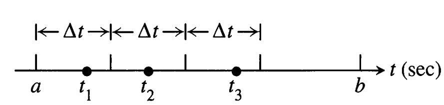
Distance Traveled

Suppose we know the velocity function $v = ds/dt = f(t)$ m/sec of a car moving down a highway and want to know how far the car will travel in the time interval $a \le t \le b$. If we know an antiderivative F of f, we can find the car's position function $s = F(t) + C$ and calculate the distance traveled as the difference between the car's positions at times $t = a$ and $t = b$ (as in Section 4.2, Exercise 55).

If we do not know an antiderivative of $v = f(t)$, we can approximate the answer with a sum in the following way. We partition $[a, b]$ into short time intervals *on each of which v is fairly constant.* Since velocity is the rate at which the car is traveling, we approximate the distance traveled on each time interval with the formula

$$\text{Distance} = \text{rate} \times \text{time} = f(t)\cdot\Delta t$$

and add the results across $[a, b]$. To be specific, suppose the partitioned interval looks like this



with the subintervals all of length Δt. Let t_1 be a point in the first subinterval. If the interval is short enough so the rate is almost constant, the car will move about $f(t_1)\Delta t$ m during that interval. If t_2 is a point in the second interval, the car will move an additional $f(t_2)\Delta t$ m during that interval, and so on. The sum of these products approximates the total distance D traveled from $t = a$ to $t = b$. If we use n subintervals, then

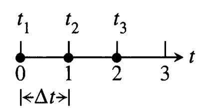
$$D \approx f(t_1)\,\Delta t + f(t_2)\,\Delta t + \cdots + f(t_n)\,\Delta t. \tag{3}$$

Let's try this on the projectile in Example 3, Section 4.2. The projectile was fired straight into the air. Its velocity t sec into the flight was $v = f(t) = 160 - 9.8t$ and it rose 435.9 m from a height of 3 m to a height of 438.9 m during the first 3 sec of flight.

EXAMPLE 2 The velocity function of a projectile fired straight into the air is $f(t) = 160 - 9.8t$. Use the summation technique just described to estimate how far the projectile rises during the first 3 sec. How close do the sums come to the exact figure of 435.9 m?

Solution We explore the results for different numbers of intervals and different choices of evaluation points.

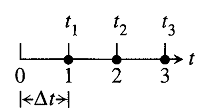
3 *subintervals of length* 1, *with f evaluated at left-hand endpoints:*



With f evaluated at $t = 0$, 1, and 2, we have

$$\begin{aligned} D &\approx f(t_1)\,\Delta t + f(t_2)\,\Delta t + f(t_3)\,\Delta t && \text{Eq. (1)} \\ &\approx [160 - 9.8(0)](1) + [160 - 9.8(1)](1) + [160 - 9.8(2)](1) \\ &\approx 450.6. \end{aligned}$$

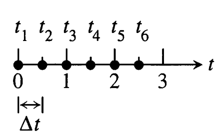
3 *subintervals of length* 1, *with f evaluated at right-hand endpoints:*

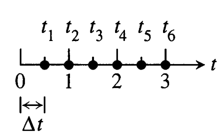


With f evaluated at $t = 1$, 2, and 3, we have

$$\begin{aligned} D &\approx f(t_1)\,\Delta t + f(t_2)\,\Delta t + f(t_3)\,\Delta t && \text{Eq. (1)} \\ &\approx [160 - 9.8(1)](1) + [160 - 9.8(2)](1) + [160 - 9.8(3)](1) \\ &\approx 421.2. \end{aligned}$$

With 6 *subintervals of length* 1/2, *we get*





Using left-hand endpoints: $D \approx 443.25$.

Using right-hand endpoints: $D \approx 428.55$.

These six-interval estimates are somewhat closer than the three-interval estimates. The results improve as the subintervals get shorter.

Table 4.4 Travel-distance estimates

Number of subintervals	Length of each subinterval	Left-endpoint sum	Right-endpoint sum
3	1	450.6	421.2
6	0.5	443.25	428.55
12	0.25	439.58	432.23
24	0.125	437.74	434.06
48	0.0625	436.82	434.98
96	0.03125	436.36	435.44
192	0.015625	436.13	435.67

Error magnitude $=$
|true value − calculated value|

As we can see in Table 4.4, the left-endpoint sums approach the true value 435.9 from above while the right-endpoint sums approach it from below. The true value lies between these upper and lower sums. The magnitude of the error in the closest entries is 0.23, a small percentage of the true value.

$$\text{Error percentage} = \frac{0.23}{435.9} \approx 0.05\%.$$

It would be safe to conclude from the table's last entries that the projectile rose about 436 m during its first 3 sec of flight. ❑

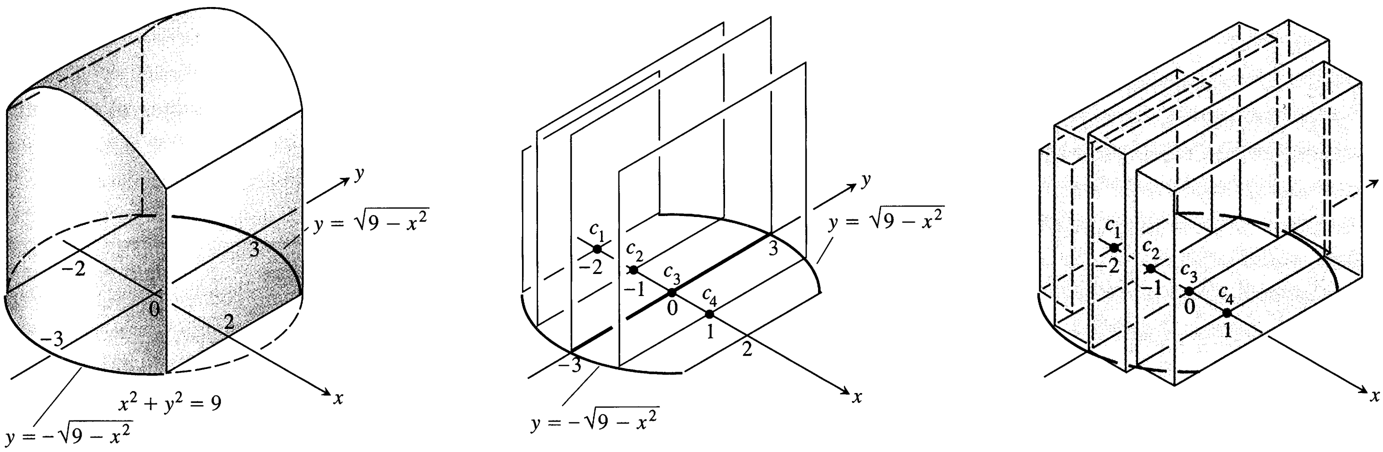
Notice the mathematical similarity between Examples 1 and 2. In each case, we have a function f defined on a closed interval and estimate what we want to know with a sum of function values multiplied by interval lengths. We can use similar sums to estimate volumes.

Volume

Here are two examples using finite sums to estimate volumes.

EXAMPLE 3 A solid lies between planes perpendicular to the x-axis at $x = -2$ and $x = 2$. The cross sections of the solid perpendicular to the axis between these planes are vertical squares whose base edges run from the semicircle $y = -\sqrt{9 - x^2}$ to the semicircle $y = \sqrt{9 - x^2}$ (Fig. 4.7a, on the following page). The height of the square at x is $2\sqrt{9 - x^2}$. Estimate the volume of the solid.

Solution We partition the interval $[-2,\ 2]$ on the x-axis into four subintervals of length $\Delta x = 1$. The solid's cross section at the left-hand endpoint of each subinterval is a square (Fig. 4.7b). On each of these squares we construct a right cylinder (square slab) of height 1 extending to the right (Fig. 4.7c). We add the cylinders' volumes to estimate the volume of the solid.



(a) (b) (c)

4.7 (a) The solid in Example 3. (b) Square cross sections of the solid at $x = -2, -1$, 0, and 1. (c) Rectangular cylinders (slabs) based on the cross sections to approximate the solid.

We calculate the volume of each cylinder with the formula $V = Ah$ (base area $\times$ height). The area of the solid's cross section at x is $A(x) = (\text{side})^2 = (2\sqrt{9-x^2})^2 = 4(9-x^2)$, so the sum of the volumes of the cylinders is

$$\begin{aligned}
S_4 &= A(c_1)\,\Delta x + A(c_2)\,\Delta x + A(c_3)\,\Delta x + A(c_4)\,\Delta x \\
&= 4(9 - c_1{}^2)(1) + 4(9 - c_2{}^2)(1) + 4(9 - c_3{}^2)(1) + 4(9 - c_4{}^2)(1) \\
&= 4\left[(9 - (-2)^2) + (9 - (-1)^2) + (9 - (0)^2) + (9 - (1)^2)\right] \\
&= 4\left[(9-4) + (9-1) + (9-0) + (9-1)\right] \\
&= 4(36 - 6) = 120.
\end{aligned}$$

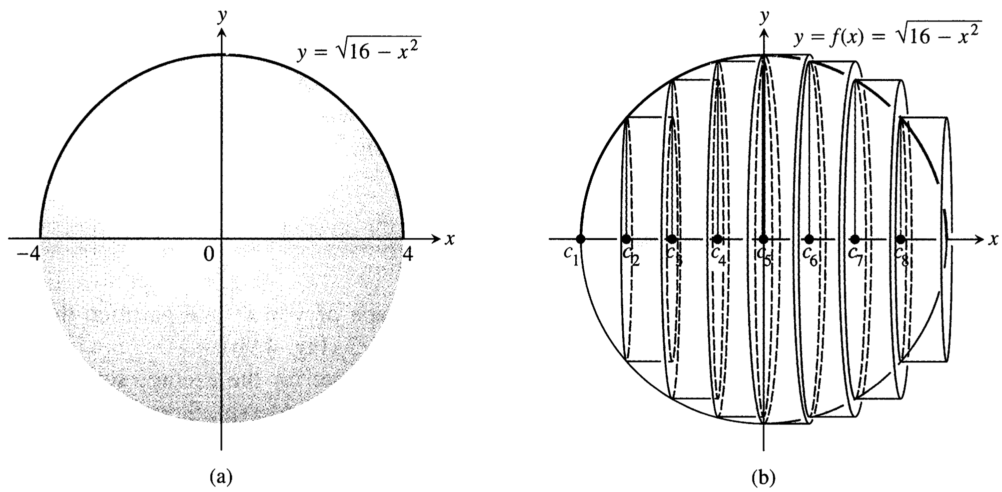
This compares favorably with the solid's true volume $V = 368/3 \approx 122.67$ (we will see how to calculate V in Section 4.7). The difference between S and V is a small percentage of V:

$$\begin{aligned}
\text{Error percentage} &= \frac{|V - S_4|}{V} = \frac{(368/3) - 120}{(368/3)} \\
&= \frac{8}{368} \approx 2.2\%.
\end{aligned}$$

With a finer partition (more subintervals) the approximation would be even better. ❑

EXAMPLE 4 Estimate the volume of a solid sphere of radius 4.

Solution We picture the sphere as if its surface were generated by revolving the graph of the function $f(x) = \sqrt{16 - x^2}$ about the x-axis (Fig. 4.8a). We partition the interval $-4 \le x \le 4$ into 8 subintervals of length $\Delta x = 1$. We then approximate the solid with right circular cylinders based on cross sections of the solid by planes perpendicular to the x-axis at the subintervals' left-hand endpoints (Fig. 4.8b). (The cylinder at $x = -4$ is degenerate because the cross section there is just a point.) We add the cylinders' volumes to estimate the volume of a sphere.



4.8 (a) The semicircle $y = \sqrt{16 - x^2}$ revolved about the x-axis to outline a sphere. (b) The solid sphere approximated with cross-section-based cylinders.

We calculate the volume of each cylinder with the formula $V = \pi r^2 h$. The sum of the eight cylinders' volumes is

$$\begin{aligned}
S_8 &= \pi\,[f(c_1)]^2\,\Delta x + \pi\,[f(c_2)]^2\,\Delta x + \pi\,[f(c_3)]^2\,\Delta x + \cdots + \pi\,[f(c_8)]^2\,\Delta x \\
&= \pi\left[\sqrt{16 - {c_1}^2}\right]^2 \Delta x + \pi\left[\sqrt{16 - {c_2}^2}\right]^2 \Delta x + \pi\left[\sqrt{16 - {c_3}^2}\right]^2 \Delta x \\
&\quad + \cdots + \pi\left[\sqrt{16 - {c_8}^2}\right]^2 \Delta x \\
&= \pi\left[(16 - (-4)^2) + (16 - (-3)^2) + (16 - (-2)^2) + \cdots + (16 - (3)^2)\right] \\
&= \pi\,[0 + 7 + 12 + 15 + 16 + 15 + 12 + 7] \\
&= 84\pi.
\end{aligned}$$

This compares favorably with the sphere's true volume,

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\,(4)^3 = \frac{256\pi}{3}.$$

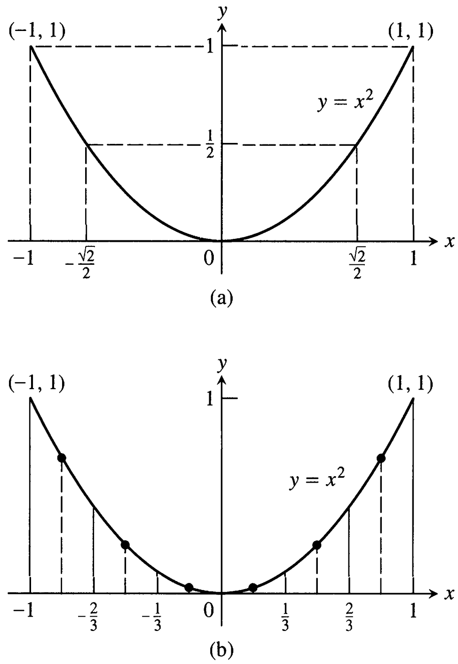
The difference between S_8 and V is a small percentage of V:

$$\begin{aligned}
\text{Error percentage} &= \frac{|V - S_8|}{V} = \frac{(256/3)\pi - 84\pi}{(256/3)\pi} \\
&= \frac{256 - 252}{256} = \frac{1}{64} \approx 1.6\%.
\end{aligned}$$

❑

The Average Value of a Nonnegative Function

To find the average of a finite set of values, we add them and divide by the number of values added. But what happens if we want to find the average of an infinite number of values? For example, what is the average value of the function $f(x) = x^2$ on the interval $[-1,\ 1]$? To see what this kind of "continuous" average might mean, imagine that we are pollsters sampling the function. We pick random x's between -1 and 1, square them, and average the squares. As we take larger samples, we expect this average to approach some number, which seems reasonable to call the *average of f over* $[-1,\ 1]$.



4.9 (a) The graph of $f(x) = x^2$, $-1 \le x \le 1$. (b) Values of f sampled at regular intervals.

The graph in Fig. 4.9(a) suggests that the average square should be less than 1/2, because numbers with squares less than 1/2 make up more than 70% of the interval $[-1,\ 1]$. If we had a computer to generate random numbers, we could carry out the sampling experiment described above, but it is much easier to estimate the average value with a finite sum.

EXAMPLE 5 Estimate the average value of the function $f(x) = x^2$ on the interval $[-1,\ 1]$.

Solution We look at the graph of $y = x^2$ and partition the interval [0, 1] into 6 subintervals of length $\Delta x = 1/3$ (Fig. 4.9b).

It appears that a good estimate for the average square on each subinterval is the square of the midpoint of the subinterval. Since the subintervals have the same length, we can average these six estimates to get a final estimate for the average value over $[-1,\ 1]$.

$$\text{Average value} \approx \frac{\left(-\frac{5}{6}\right)^2 + \left(-\frac{3}{6}\right)^2 + \left(-\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{3}{6}\right)^2 + \left(\frac{5}{6}\right)^2}{6}$$

$$\approx \frac{1}{6} \cdot \frac{25 + 9 + 1 + 1 + 9 + 25}{36} = \frac{70}{216} \approx 0.324$$

We will be able to show later that the average value is 1/3.

Notice that

$$\frac{\left(-\frac{5}{6}\right)^2 + \left(-\frac{3}{6}\right)^2 + \left(-\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{3}{6}\right)^2 + \left(\frac{5}{6}\right)^2}{6}$$

$$= \frac{1}{2}\left[\left(-\frac{5}{6}\right)^2 \cdot \frac{1}{3} + \left(-\frac{3}{6}\right)^2 \cdot \frac{1}{3} + \cdots + \left(\frac{5}{6}\right)^2 \cdot \frac{1}{3}\right]$$

$$= \frac{1}{\text{length of } [-1, 1]} \cdot \left[f\left(-\frac{5}{6}\right) \cdot \frac{1}{3} + f\left(-\frac{3}{6}\right) \cdot \frac{1}{3} + \cdots + f\left(\frac{5}{6}\right) \cdot \frac{1}{3}\right]$$

$$= \frac{1}{\text{length of } [-1, 1]} \cdot \left[\begin{array}{l}\text{a sum of function values} \\ \text{multiplied by interval lengths}\end{array}\right].$$

Once again our estimate has been achieved by multiplying function values by interval lengths and summing the results for all the intervals. ❑

Conclusion

The examples in this section describe instances in which sums of function values multiplied by interval lengths provide approximations that are good enough to answer practical questions. You will find additional examples in the exercises.

The distance approximations in Example 2 improved as the intervals involved became shorter and more numerous. We knew this because we had already found the exact answer with antiderivatives in Section 4.2. If we had made our partitions of the time interval still finer, would the sums have approached the exact answer as a limit? Is the connection between the sums and the antiderivative in this case just a coincidence? Could we have calculated the area in Example 1, the volumes in

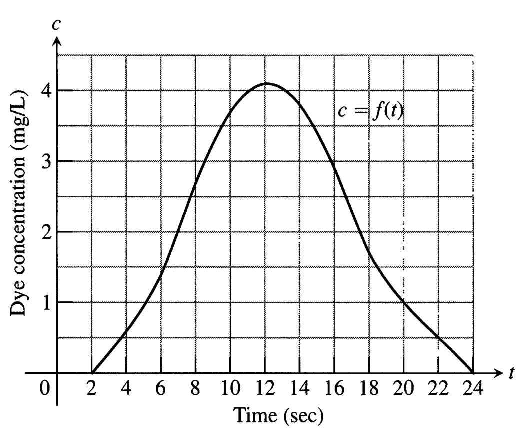
Examples 3 and 4, and the average value in Example 5 with antiderivatives as well? As we will see, the answers are "Yes, they would have," "No, it is not a coincidence," and "Yes, we could have."

Exercises 4.4

Cardiac Output

1. The table below gives dye concentrations for a dye-dilution cardiac-output determination like the one in Example 1. The amount of dye injected in this case was 5 mg instead of 5.6 mg. Use rectangles to estimate the area under the dye concentration curve and then go on to estimate the patient's cardiac output.

Seconds after injection t	Dye concentration (adjusted for recirculation) c
2	0
4	0.6
6	1.4
8	2.7
10	3.7
12	4.1
14	3.8
16	2.9
18	1.7
20	1.0
22	0.5
24	0



2. The accompanying table gives dye concentrations for a cardiac-output determination like the one in Example 1. The amount of dye injected in this case was 10 mg. Plot the data and connect the data points with a smooth curve. Estimate the area under the curve and calculate the cardiac output from this estimate.

Seconds after injection t	Dye concentration (adjusted for recirculation) c	Seconds after injection t	Dye concentration (adjusted for recirculation) c
0	0	16	7.9
2	0	18	7.8
4	0.1	20	6.1
6	0.6	22	4.7
8	2.0	24	3.5
10	4.2	26	2.1
12	6.3	28	0.7
14	7.5	30	0

Distance

3. The table below shows the velocity of a model train engine moving along a track for 10 sec. Estimate the distance traveled by the engine using 10 subintervals of length 1 with (a) left-endpoint values and (b) right-endpoint values.

Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

4. You are sitting on the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every five minutes for an hour, with the results shown in the table on the following page. About how far upstream did the bottle travel during that hour? Find an estimate using 12 subintervals of length 5 with (a) left-endpoint values and (b) right-endpoint values.

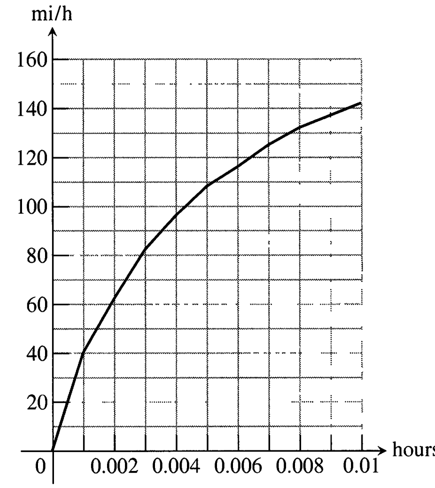
Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

5. You and a companion are about to drive a twisty stretch of dirt road in a car whose speedometer works but whose odometer (mileage counter) is broken. To find out how long this particular stretch of road is, you record the car's velocity at 10-sec intervals, with the results shown in the table below. Estimate the length of the road (a) using left-endpoint values and (b) using right-endpoint values.

Time (sec)	Velocity (converted to ft/sec) (30 mi/h = 44 ft/sec)	Time (sec)	Velocity (converted to ft/sec) (30 mi/h = 44 ft/sec)
0	0	70	15
10	44	80	22
20	15	90	35
30	35	100	44
40	30	110	30
50	44	120	35
60	35		

6. The table below gives data for the velocity of a vintage sports car accelerating from 0 to 142 mi/h in 36 sec (10 thousandths of an hour).

Time (h)	Velocity (mi/h)	Time (h)	Velocity (mi/h)
0.0	0	0.006	116
0.001	40	0.007	125
0.002	62	0.008	132
0.003	82	0.009	137
0.004	96	0.010	142
0.005	108		

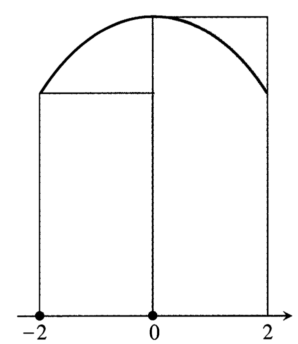


a) Use rectangles to estimate how far the car traveled during the 36 sec it took to reach 142 mi/h.

b) Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?

Volume

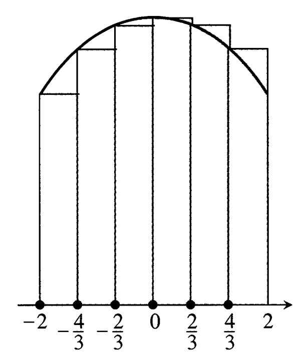
7. *(Continuation of Example 3.)* Suppose we use only two square cylinders to estimate the volume V of the solid in Example 3, as shown in profile in the figure here.



a) Find the sum S_2 of the volumes of the cylinders.

b) Express $|V - S_2|$ as a percentage of V to the nearest percent.

8. *(Continuation of Example 3.)* Suppose we use six square cylinders to estimate the volume V of the solid in Example 3, as shown in the accompanying profile view.



a) Find the sum S_6 of the volumes of the cylinders.

b) Express $|V - S_6|$ as a percentage of V to the nearest percent.

9. *(Continuation of Example 4.)* Suppose we approximate the volume V of the sphere in Example 4 by partitioning the interval $-4 \le x \le 4$ into four subintervals of length 2 and using cylinders based on the cross sections at the subintervals' left-hand endpoints. (As in Example 4, the leftmost cylinder will have a zero radius.)

a) Find the sum S_4 of the volumes of the cylinders.

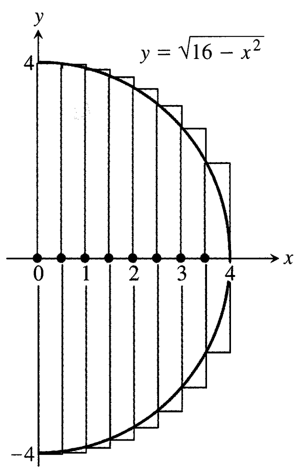
b) Express $|V - S_4|$ as a percentage of V to the nearest percent.

10. To estimate the volume V of a solid sphere of radius 5 you partition its diameter into five subintervals of length 2. You then slice the sphere with planes perpendicular to the diameter at the subintervals' left-hand endpoints and add the volumes of cylinders of height 2 based on the cross sections of the sphere determined by these planes.

a) Find the sum S_5 of the volumes of the cylinders.

b) Express $|V - S_5|$ as a percentage of V to the nearest percent.

11. To estimate the volume V of a solid hemisphere of radius 4, imagine its axis of symmetry to be the interval [0, 4] on the x-axis. Partition [0, 4] into eight subintervals of equal length and approximate the solid with cylinders based on the circular cross sections of the hemisphere perpendicular to the x-axis at the subintervals' left-hand endpoints. (See the accompanying profile view.)

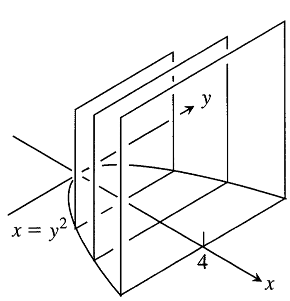


a) Find the sum S_8 of the volumes of the cylinders. Do you expect S_8 to overestimate V, or to underestimate V? Give reasons for your answer.

b) Express $|V - S_8|$ as a percentage of V to the nearest percent.

12. Repeat Exercise 11 using cylinders based on cross sections at the *right-hand* endpoints of the subintervals.

13. *Estimates with large error.* A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections of the solid perpendicular to the axis between these planes are vertical squares whose base edges run from the parabolic curve $y = -\sqrt{x}$ to the parabolic curve $y = \sqrt{x}$.

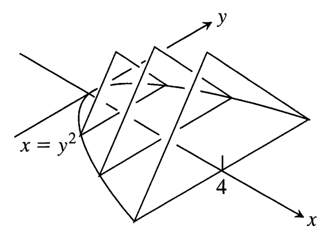


a) Find the sum S_4 of the volumes of the cylinders obtained by partitioning $0 \le x \le 4$ into four subintervals of length 1 based on the cross sections at the subinterval's right-hand endpoints.

b) The true volume is $V = 32$. Express $|V - S_4|$ as a percentage of V to the nearest percent.

c) Repeat parts (a) and (b) for the sum S_8.

14. *Estimates with large error.* A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections of the solid perpendicular to the axis between these planes are vertical equilateral triangles whose base edges run from the parabolic curve $y = -\sqrt{x}$ to the parabolic curve $y = \sqrt{x}$.



a) Find the sum S_4 of the volumes of the cylinders obtained by partitioning $0 \le x \le 4$ into four subintervals of length 1 based on the cross sections at the subinterval's left-hand endpoints.

b) The true volume is $V = 8\sqrt{3}$. Express $|V - S_4|$ as a percentage of V to the nearest percent.

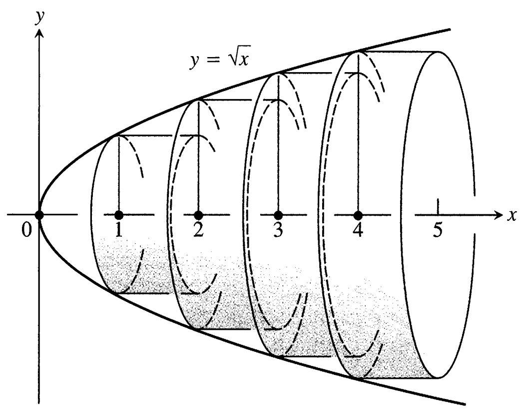
c) CALCULATOR Repeat parts (a) and (b) for the sum S_8.

15. A reservoir shaped like a hemispherical bowl of radius 8 m is filled with water to a depth of 4 m. (a) Find an estimate S of the water's volume by approximating the water with eight circumscribed solid cylinders. (b) As you will see in Section 4.7, Exercise 71, the water's volume is $V = 320\pi/3$ m^3. Find the error $|V - S|$ as a percentage of V to the nearest percent.

16. A rectangular swimming pool is 30 ft wide and 50 ft long. The table below shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using (a) left-endpoint values of h; (b) right-endpoint values of h.

Position x ft	Depth $h(x)$ ft	Position x ft	Depth $h(x)$ ft
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

17. The nose "cone" of a rocket is a paraboloid obtained by revolving the curve $y = \sqrt{x}$, $0 \le x \le 5$, about the x-axis, where x is measured in feet. To estimate the volume V of the nose cone, we partition [0, 5] into five subintervals of equal length, slice the cone with planes perpendicular to the x-axis at the subintervals' left-hand endpoints, and construct cylinders of height 1 based on cross sections at these points. (See the accompanying figure.)



a) Find the sum S_5 of the volumes of the cylinders. Do you expect S_5 to overestimate V, or to underestimate V? Give reasons for your answer.

b) As you will see in Section 4.7, Exercise 72, the volume of the nose cone is $V = 25\pi/2$ ft^3. Express $|V - S_5|$ as a percentage of V to the nearest percent.

18. Repeat Exercise 17 using cylinders based on cross sections at the *right-hand* endpoints of the subintervals.

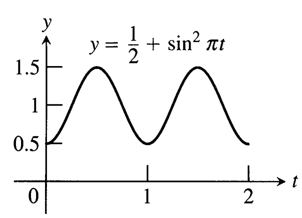
Average Value of a Function

In Exercises 19–22, use a finite sum to estimate the average value of f on the given interval by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

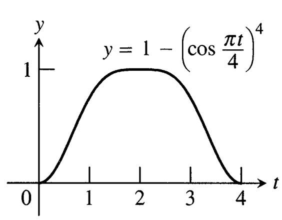
19. $f(x) = x^3$ on $[0, 2]$

20. $f(x) = 1/x$ on $[1, 9]$

21. $f(t) = (1/2) + \sin^2 \pi t$ on $[0, 2]$



22. $f(t) = 1 - \left(\cos \dfrac{\pi t}{4}\right)^4$ on $[0, 4]$



Velocity and Distance

23. An object is dropped straight down from an airplane. The object falls faster and faster but the acceleration is decreasing over time because of air resistance. The acceleration is measured in ft/sec^2 and recorded every second after the drop for 5 sec, as shown in the following table.

t	0	1	2	3	4	5
a	32.00	19.41	11.77	7.14	4.33	2.63

a) Find an upper estimate for the speed when $t = 5$.

b) Find a lower estimate for the speed when $t = 5$.

c) Find an upper estimate for the distance fallen when $t = 3$.

24. An object is shot straight upward from sea level with an initial velocity of 400 ft/sec. Assuming gravity is the only force acting on the object, give an upper estimate for its speed after 5 sec have elapsed. Use $g = 32$ ft/sec^2 for the gravitational constant. Find a lower estimate for the height attained after 5 sec.

Pollution Control

25. Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

Time (hours)	0	1	2	3	4
Leakage (gal/hr)	50	70	97	136	190
Time (hours)	5	6	7	8	
Leakage (gal/hr)	265	369	516	720	

a) Give an upper and a lower estimate of the total quantity of oil that has escaped after 5 hours.

b) Repeat part (a) for the quantity of oil that has escaped after 8 hours.

c) The tanker continues to leak 720 gal/h after the first 8 hours. If the tanker originally contained 25,000 gal of oil, approximately how many more hours will elapse in the worst case before all of the oil has spilled? in the best case?

26. A power plant generates electricity by burning oil. Pollutants produced as a result of the burning process are removed by scrubbers in the smoke stacks. Over time the scrubbers become less efficient and eventually they must be replaced when the amount of pollution released exceeds government standards. Measurements are taken at the end of each month determining the rate at which pollutants are released into the atmosphere, recorded as follows.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Pollutant release rate (tons/day)	0.2	0.25	0.27	0.34	0.45	0.52	0.63	0.70	0.81	0.85	0.89	0.95

a) Assuming a 30-day month and that new scrubbers allow only 0.05 tons/day released, give an upper estimate of the total tonnage of pollutant released by the end of June. What is a lower estimate?

b) In the best case, approximately when will a total of 125 tons of pollutant have been released into the atmosphere?

CAS Explorations and Projects

In Exercises 27–30, use a CAS to perform the following steps:

a) Plot the functions over the given interval.

b) Partition the interval into $n = 100$, 200, and 1000 subintervals of equal length, and evaluate the function at the midpoint of each subinterval.

c) Compute the average value of the function values generated in part (b).

d) Solve the equation $f(x) =$ (average value) for x using the average value calculated in (c) for the $n = 1000$ partitioning.

27. $f(x) = \sin x \quad \text{on} \quad [0, \pi]$

28. $f(x) = \sin^2 x \quad \text{on} \quad [0, \pi]$

29. $f(x) = x \sin \dfrac{1}{x} \quad \text{on} \quad \left[\dfrac{\pi}{4}, \pi\right]$

30. $f(x) = x \sin^2 \dfrac{1}{x} \quad \text{on} \quad \left[\dfrac{\pi}{4}, \pi\right]$

4.5 Riemann Sums and Definite Integrals

In the preceding section, we estimated distances, areas, volumes, and average values with finite sums. The terms in the sums were obtained by multiplying selected function values by the lengths of intervals. In this section, we say what it means for sums like these to approach a limit as the intervals involved become more numerous and shorter. We begin by introducing a compact notation for sums that contain large numbers of terms.

Sigma Notation for Finite Sums

We use the capital Greek letter Σ ("sigma") to write an abbreviation for the sum

$$f(t_1)\,\Delta t + f(t_2)\,\Delta t + \cdots + f(t_n)\,\Delta t$$

as $\sum_{k=1}^{n} f(t_k)\,\Delta t$, "the sum from k equals 1 to n of f of t_k times delta t." When we write a sum this way, we say that we have written it in sigma notation.

Definitions

Sigma Notation for Finite Sums

The symbol $\sum_{k=1}^{n} a_k$ denotes the sum $a_1 + a_2 + \cdots + a_n$. The a's are the **terms** of the sum: a_1 is the first term, a_2 is the second term, a_k is the ***k*th term,** and a_n is the nth and last term. The variable k is the **index of summation.** The values of k run through the integers from 1 to n. The number 1 is the **lower limit of summation;** the number n is the **upper limit of summation.**

EXAMPLE 1

The sum in sigma notation	The sum written out—one term for each value of k	The value of the sum
$\sum_{k=1}^{5} k$	$1+2+3+4+5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1)+(-1)^2(2)+(-1)^3(3)$	$-1+2-3=-2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1}+\frac{2}{2+1}$	$\frac{1}{2}+\frac{2}{3}=\frac{7}{6}$

❑

The lower limit of summation does not have to be 1; it can be any integer.

EXAMPLE 2 Express the sum $1+3+5+7+9$ in sigma notation.

Solution

Starting with $k=2$: $\quad 1+3+5+7+9=\sum_{k=2}^{6}(2k-3)$

Starting with $k=-3$: $\quad 1+3+5+7+9=\sum_{k=-3}^{1}(2k+7)$

The formula generating the terms changes with the lower limit of summation, but the terms generated remain the same. It is often simplest to start with $k=0$ or $k=1$.

Starting with $k=0$: $\quad 1+3+5+7+9=\sum_{k=0}^{4}(2k+1)$

Starting with $k=1$: $\quad 1+3+5+7+9=\sum_{k=1}^{5}(2k-1)$

❑

Algebra with Finite Sums

We can use the following rules whenever we work with finite sums.

Algebra Rules for Finite Sums

1. *Sum Rule:* $\quad \sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$
2. *Difference Rule:* $\quad \sum_{k=1}^{n}(a_k-b_k)=\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}b_k$
3. *Constant Multiple Rule:* $\quad \sum_{k=1}^{n}ca_k=c\cdot\sum_{k=1}^{n}a_k$ (Any number c)
4. *Constant Value Rule:* $\quad \sum_{k=1}^{n}c=n\cdot c$ (c is any constant value.)

There are no surprises in this list. The formal proofs can be done by mathematical induction (Appendix 1).

EXAMPLE 3

a) $\sum_{k=1}^{n}(3k - k^2) = 3\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2$ — Difference Rule and Constant Multiple Rule

b) $\sum_{k=1}^{n}(-a_k) = \sum_{k=1}^{n}(-1)\cdot a_k = -1\cdot\sum_{k=1}^{n} a_k = -\sum_{k=1}^{n} a_k$ — Constant Multiple Rule

c) $\sum_{k=1}^{3}(k+4) = \sum_{k=1}^{3} k + \sum_{k=1}^{3} 4$ — Sum Rule

$$= (1+2+3) + (3\cdot 4)$$ Constant Value Rule

$$= 6 + 12 = 18$$ ❑

Sum Formulas for Positive Integers

Over the years people have discovered a variety of formulas for the values of finite sums. The most famous of these are the formula for the sum of the first n integers (Gauss discovered it at age 5) and the formulas for the sums of the squares and cubes of the first n integers.

The first n integers:	$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$	(1)
The first n squares:	$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$	(2)
The first n cubes:	$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$	(3)

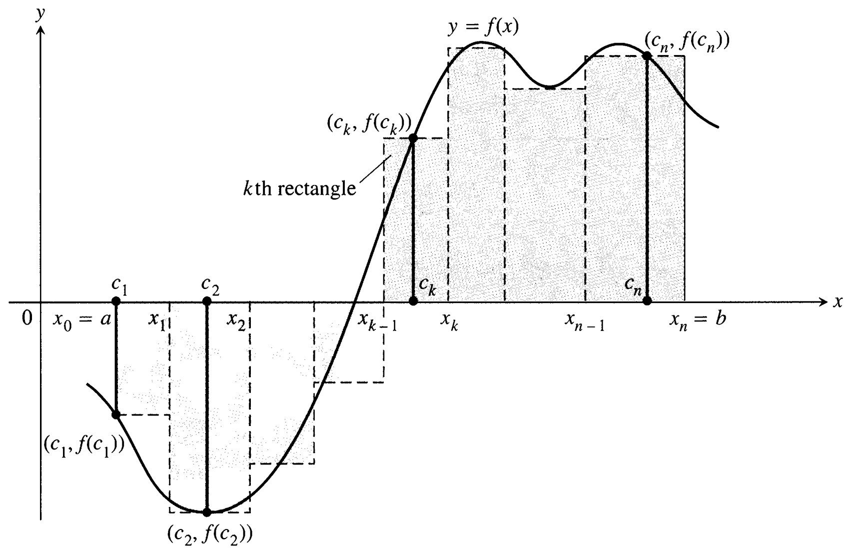
EXAMPLE 4 Evaluate $\sum_{k=1}^{4}(k^2 - 3k)$.

Solution We can use the algebra rules and known formulas to evaluate the sum without writing out the terms.

$$\sum_{k=1}^{4}(k^2 - 3k) = \sum_{k=1}^{4} k^2 - 3\sum_{k=1}^{4} k$$ Difference Rule and Constant Multiple Rule

$$= \frac{4(4+1)(8+1)}{6} - 3\left(\frac{4(4+1)}{2}\right)$$ Eqs. (2) and (1) with $n = 4$

$$= 30 - 30 = 0$$ ❑



4.10 The graph of a typical function $y = f(x)$ over a closed interval $[a, b]$. The rectangles approximate the region between the graph of the function and the x-axis.

Riemann Sums

The approximating sums in Section 4.4 are examples of a more general kind of sum called a *Riemann* ("*ree*-mahn") *sum.* The functions in the examples had nonnegative values, but the more general notion has no such restriction. Given an arbitrary continuous function $y = f(x)$ on an interval $[a, b]$ (Fig. 4.10), we partition the interval into n subintervals by choosing $n - 1$ points, say $x_1, x_2, \dots, x_{n-1}$, between a and b subject only to the condition that

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

To make the notation consistent, we usually denote a by x_0 and b by x_n. The set

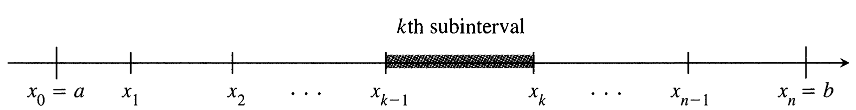
$$P = \{x_0, x_1, \dots, x_n\}$$

is called a **partition** of $[a, b]$.

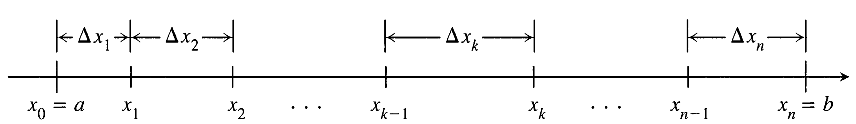
The partition P defines n closed **subintervals**

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n].$$

The typical closed subinterval $[x_{k-1}, x_k]$ is called the ***k*th subinterval** of P.

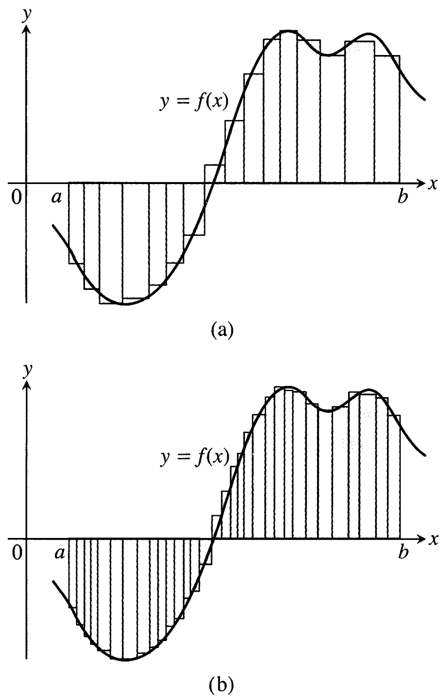


The length of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$.



In each subinterval $[x_{k-1}, x_k]$, we select a point c_k and construct a vertical rectangle from the subinterval to the point $(c_k, f(c_k))$ on the curve $y = f(x)$. The choice of c_k does not matter as long as it lies in $[x_{k-1}, x_k]$. See Fig. 4.10 again.

If $f(c_k)$ is positive, the number $f(c_k)\,\Delta x_k = \text{height} \times \text{base}$ is the area of the



4.11 The curve of Fig. 4.10 with rectangles from finer partitions of $[a, b]$. Finer partitions create more rectangles with shorter bases.

rectangle. If $f(c_k)$ is negative, then $f(c_k)\,\Delta x_k$ is the negative of the area. In any case, we add the n products $f(c_k)\,\Delta x_k$ to form the sum

$$S_P = \sum_{k=1}^{n} f(c_k)\,\Delta x_k.$$

This sum, which depends on P and the choice of the numbers c_k, is called a **Riemann sum for f on the interval $[a, b]$,** after German mathematician Georg Friedrich Bernhard Riemann (1826–1866), who studied the limits of such sums.

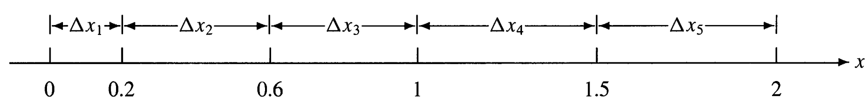
As the partitions of $[a, b]$ become finer, the rectangles defined by the partition approximate the region between the x-axis and the graph of f with increasing accuracy (Fig. 4.11). So we expect the associated Riemann sums to have a limiting value. To test this expectation, we need to develop a numerical way to say that partitions become finer and to determine whether the corresponding sums have a limit. We accomplish this with the following definitions.

The **norm** of a partition P is the partition's longest subinterval length. It is denoted by

$$\|P\| \qquad \text{(read "the norm of } P\text{")}.$$

The way to say that successive partitions of an interval become finer is to say that the norms of these partitions approach zero. As the norms go to zero, the subintervals become shorter and their number approaches infinity.

EXAMPLE 5 The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0, 2]$. There are five subintervals of P: $[0, 0.2]$, $[0.2, 0.6]$, $[0.6, 1]$, $[1, 1.5]$, and $[1.5, 2]$.



The lengths of the subintervals are $\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$, and $\Delta x_5 = 0.5$. The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$. In this example, there are two subintervals of this length. ❑

Definition

The Definite Integral as a Limit of Riemann Sums

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that the **limit** of the Riemann sums $\Sigma_{k=1}^{n} f(c_k)\,\Delta x_k$ on $[a, b]$ as $\|P\| \to 0$ is the number I if the following condition is satisfied:

Given any number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for every partition P of $[a, b]$

$$\|P\| < \delta \qquad \Rightarrow \qquad \left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - I \right| < \epsilon$$

for any choice of the numbers c_k in the subintervals $[x_{k-1}, x_k]$.

If the limit exists, we write

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = I.$$

We call I the **definite integral** of f over $[a, b]$, we say that f is **integrable** over $[a, b]$, and we say that the Riemann sums of f on $[a, b]$ **converge** to the number I.

We usually write I as $\int_a^b f(x)\,dx$, which is read "integral of f from a to b." Thus, if the limit exists,

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \int_a^b f(x)\,dx.$$

The amazing fact is that despite the variety in the Riemann sums $\Sigma f(c_k)\,\Delta x_k$ as the partitions change and the arbitrary choice of c_k's in the intervals of each new partition, the sums always have the same limit as $\|P\| \to 0$ as long as f is continuous. The need to establish the existence of this limit became clear as the nineteenth century progressed, and it was finally established when Riemann proved the following theorem in 1854. You can find a current version of Riemann's proof in most advanced calculus books.

Theorem 1
The Existence of Definite Integrals

All continuous functions are integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

Why should we expect such a theorem to hold? Imagine a typical partition P of the interval $[a, b]$. The function f, being continuous, has a minimum value $\min_k$ ("min kay") and a maximum value $\max_k$ ("max kay") on each subinterval. The products $\min_k \Delta x_k$ associated with the minimum values (Fig. 4.12a) add up to what we call the **lower sum** for f on P:

$$L = \min_1 \Delta x_1 + \min_2 \Delta x_2 + \cdots + \min_n \Delta x_n.$$

The products $\max_k \Delta x_k$ obtained from the maximum values (Fig. 4.12b) add up to the **upper sum** for f on P:

$$U = \max_1 \Delta x_1 + \max_2 \Delta x_2 + \cdots + \max_n \Delta x_n.$$

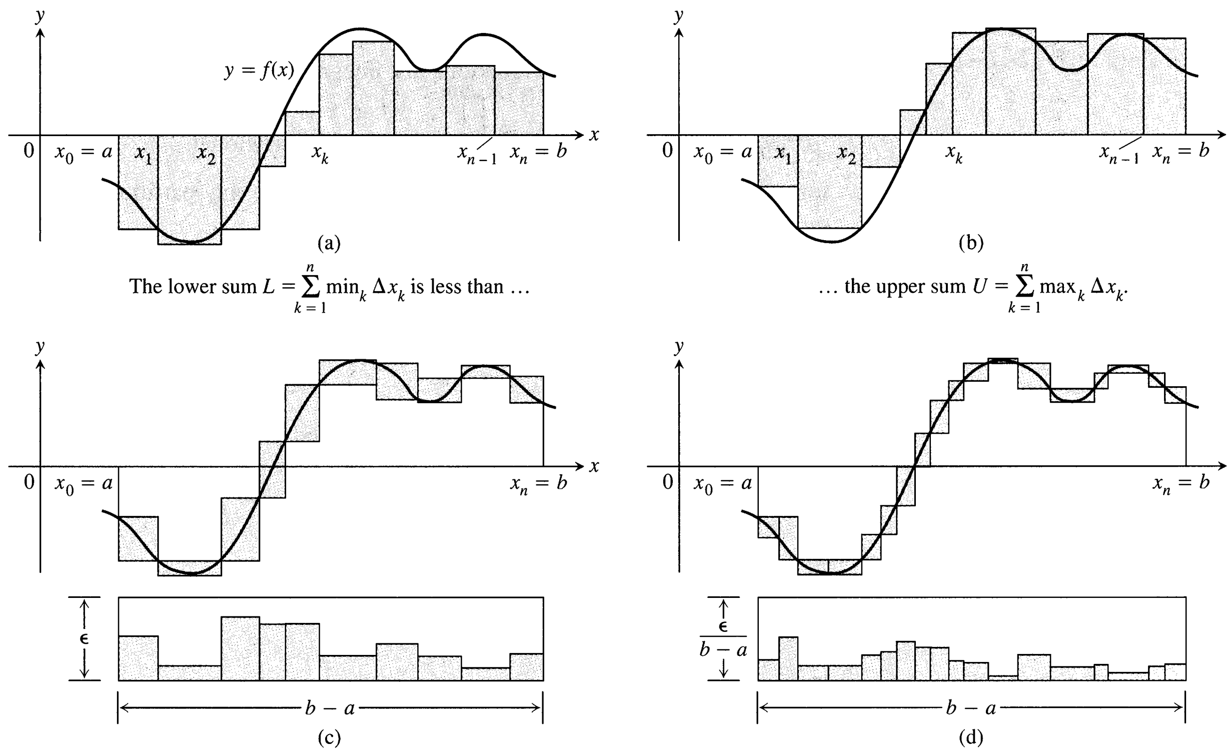
The difference $U - L$ between the upper and lower sums is the sum of the areas of the shaded blocks in Fig. 4.12(c). As $\|P\| \to 0$, the blocks in Fig. 4.12(c) become more numerous, narrower, and shorter. As Fig. 4.12(d) suggests, we can make the nonnegative number $U - L$ less than any prescribed positive ϵ by taking $\|P\|$ close enough to zero. In other words,

$$\lim_{\|P\|\to 0} (U - L) = 0, \tag{4}$$

and, as shown in more advanced texts,

$$\lim_{\|P\|\to 0} L = \lim_{\|P\|\to 0} U. \tag{5}$$

The fact that Eqs. (4) and (5) hold for any continuous function is a consequence of a special property, called *uniform continuity,* that continuous functions have on closed intervals. This property guarantees that as $\|P\| \to 0$ the blocks that make up the difference between U and L in Fig. 4.12(c) become less tall as they become less wide and that we can make them all as short as we please by making them narrow enough. Passing over the ϵ - δ arguments associated with uniform continuity



(a) The lower sum $L = \sum_{k=1}^{n} \min_k \Delta x_k$ is less than …

(b) … the upper sum $U = \sum_{k=1}^{n} \max_k \Delta x_k$.

(c) The difference $U - L$ can be made very small: less than $\epsilon \cdot (b - a)$.

(d) We can make $U - L$ smaller than any given positive ϵ by making $\|P\|$ small enough.

4.12 The difference between upper and lower sums.

keeps our derivation of Eq. (5) from being a proof. But the argument is right in spirit and gives a faithful portrait of the proof.

Assuming that Eq. (5) holds for any continuous function f on $[a, b]$, suppose we choose a point c_k from each subinterval $[x_{k-1}, x_k]$ of P and form the Riemann sum $\Sigma_{k=1}^{n} f(c_k)\,\Delta x_k$. Then $\min_k \le f(c_k) \le \max_k$ for each k, so

$$L \le \sum_{k=1}^{n} f(c_k)\,\Delta x_k \le U.$$

The Riemann sum for f is sandwiched between L and U. By a modified version of the Sandwich Theorem of Section 1.2, the limit of the Riemann sums as $\|P\| \to 0$ exists and equals the common limit of U and L:

$$\lim_{\|P\|\to 0} L = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \lim_{\|P\|\to 0} U.$$

Pause for a moment to see how remarkable this conclusion really is. It says that no matter how we choose the points c_k to form the Riemann sums as $\|P\| \to 0$, the limit is always the same. We can take every $f(c_k)$ to be the minimum value of f on $[x_{k-1}, x_k]$. The limit is the same. We can take every $f(c_k)$ to be the maximum value of f on $[x_{k-1}, x_k]$. The limit is the same. We can choose every c_k at random. The limit is the same.

Although we stated the integral existence theorem specifically for continuous functions, many discontinuous functions are integrable as well. We treat the integration of bounded piecewise continuous functions in Additional Exercises 11–18 at the end of this chapter. We explore the integration of unbounded functions in Section 7.6.

Functions with No Riemann Integral

While some discontinuous functions are integrable, others are not. The function

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ 0 & \text{when } x \text{ is irrational,} \end{cases}$$

for example, has no Riemann integral over [0, 1]. For any partition P of [0, 1], the upper and lower sums are

$$U = \sum \max_k \Delta x_k = \sum 1 \cdot \Delta x_k = \sum \Delta x_k = 1,$$

Every subinterval contains a rational number

$$L = \sum \min_k \Delta x_k = \sum 0 \cdot \Delta x_k = 0.$$

Every subinterval contains an irrational number.

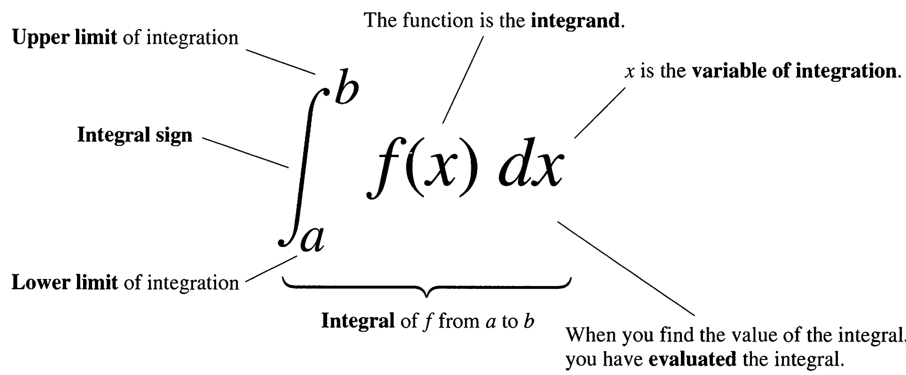
For the integral of f to exist over [0, 1], U and L would have to have the same limit as $\|P\| \to 0$. But they do not:

$$\lim_{\|P\|\to 0} L = 0 \qquad \text{while} \qquad \lim_{\|P\|\to 0} U = 1.$$

Therefore, f has no integral on [0, 1]. No constant multiple kf has an integral either, unless k is zero.

Terminology

There is a fair amount of terminology associated with the symbol $\int_a^b f(x)\,dx$.



The value of the definite integral of a function over any particular interval depends on the function and not on the letter we choose to represent its independent variable. If we decide to use t or u instead of x, we simply write the integral as

$$\int_a^b f(t)\,dt \qquad \text{or} \qquad \int_a^b f(u)\,du \qquad \text{instead of} \qquad \int_a^b f(x)\,dx.$$

No matter how we write the integral, it is still the same number, defined as a limit of Riemann sums. Since it does not matter what letter we use, the variable of integration is called a **dummy variable.**

EXAMPLE 6 Express the limit of Riemann sums

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} (3c_k{}^2 - 2c_k + 5)\,\Delta x_k$$

as an integral if P denotes a partition of the interval $[-1, 3]$.

Solution The function being evaluated at c_k in each term of the sum is $f(x) = 3x^2 - 2x + 5$. The interval being partitioned is $[-1, 3]$. The limit is therefore the integral of f from -1 to 3:

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} (3c_k^2 - 2c_k + 5)\,\Delta x_k = \int_{-1}^{3} (3x^2 - 2x + 5)\,dx.$$

❑

Constant Functions

Theorem 1 says nothing about how to *calculate* definite integrals. Except for a few special cases, that takes another theorem (Section 4.7). Among the exceptions are constant functions. Suppose that f has the constant value $f(x) = c$ over $[a, b]$. Then, no matter how the c_k's are chosen,

$$\begin{aligned} \sum_{k=1}^{n} f(c_k)\,\Delta x_k &= \sum_{k=1}^{n} c \cdot \Delta x_k && f(c_k) \text{ always equals } c. \\ &= c \cdot \sum_{k=1}^{n} \Delta x_k && \text{Constant Multiple Rule for Sums} \\ &= c(b - a). && \sum_{k=1}^{n} \Delta x_k = \text{length of interval } [a, b] = b - a \end{aligned}$$

Since the sums all have the value $c(b - a)$, their limit, the integral, does too.

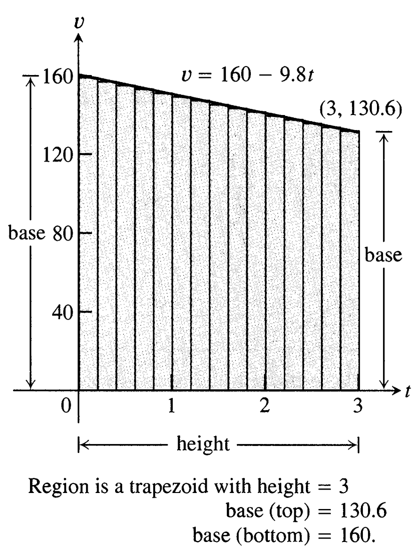
> If $f(x)$ has the constant value c on $[a, b]$, then
> $$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b - a).$$

EXAMPLE 7

a) $\displaystyle\int_{-1}^{4} 3\,dx = 3(4 - (-1)) = (3)(5) = 15$

b) $\displaystyle\int_{-1}^{4} (-3)\,dx = -3(4 - (-1)) = (-3)(5) = -15$

❑



4.13 Rectangles for a Riemann sum of the velocity function $f(t) = 160 - 9.8t$ over the interval $[0, 3]$.

The Area Under the Graph of a Nonnegative Function

The sums we used to estimate the height of the projectile in Section 4.4, Example 2, were Riemann sums for the projectile's velocity function

$$v = f(t) = 160 - 9.8t$$

on the interval $[0, 3]$. We can see from Fig. 4.13 how the associated rectangles approximate the trapezoid between the t-axis and the curve $v = 160 - 9.8t$. As the norm of the partition goes to zero, the rectangles fit the trapezoid with increasing accuracy and the sum of the areas they enclose approaches the trapezoid's area, which is

$$\text{Trapezoid area} = h \cdot \frac{b_1 + b_2}{2} = 3 \cdot \frac{160 + 130.6}{2} = 435.9.$$

This confirms our suspicion that the sums we were constructing in Section 4.4, Example 2, approached a limit of 435.9. Since the limit of these sums is also the integral of f from 0 to 3, we now know the value of the integral as well:

$$\int_0^3 (160 - 9.8t)\,dt = \text{trapezoid area} = 435.9.$$

We can exploit the connection between integrals and area in two ways. When we know a formula for the area of the region between the x-axis and the graph of a continuous nonnegative function $y = f(x)$, we can use it to evaluate the function's integral. When we do not know the region's area, we can use the function's integral to define and calculate the area.

Definition

Let $f(x) \geq 0$ be continuous on $[a, b]$. The **area** of the region between the graph of f and the x-axis is

$$A = \int_a^b f(x)\,dx.$$

Whenever we make a new definition, as we have here, consistency becomes an issue. Does the definition that we have just developed for nonstandard shapes give correct results for standard shapes? The answer is yes, but the proof is complicated and we will not go into it.

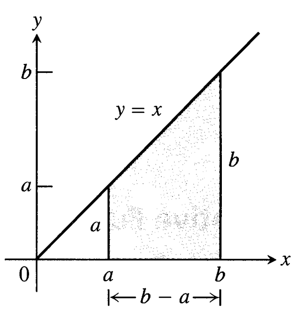
EXAMPLE 8 *Using an area to evaluate a definite integral*

Evaluate

$$\int_a^b x\,dx, \qquad 0 < a < b.$$

Solution We sketch the region under the curve $y = x$, $a \leq x \leq b$ (Fig. 4.14), and see that it is a trapezoid with height $(b - a)$ and bases a and b. The value of the integral is the area of this trapezoid:

$$\int_a^b x\,dx = (b - a) \cdot \frac{a + b}{2} = \frac{b^2}{2} - \frac{a^2}{2}.$$



4.14 The region in Example 8.

Thus,

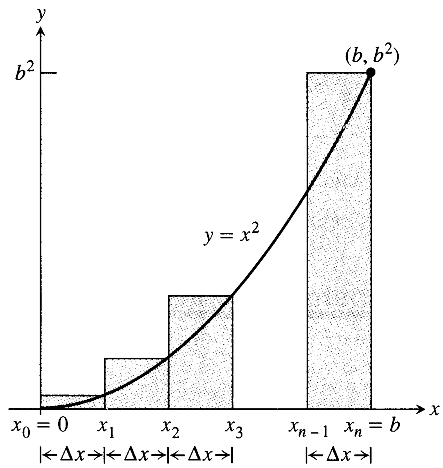
$$\int_1^{\sqrt{5}} x\,dx = \frac{(\sqrt{5})^2}{2} - \frac{(1)^2}{2} = 2$$

and so on.

Notice that $x^2/2$ is an antiderivative of x, further evidence of a connection between antiderivatives and summation. ❑

EXAMPLE 9 *Using a definite integral to find an area*

Find the area of the region between the parabola $y = x^2$ and the x-axis on the interval $[0, b]$.



4.15 The rectangles of the Riemann sums in Example 9.

Solution We evaluate the integral for the area as a limit of Riemann sums.

We sketch the region (a nonstandard shape) (Fig. 4.15) and partition $[0, b]$ into n subintervals of length $\Delta x = (b - 0)/n = b/n$. The points of the partition are

$$x_0 = 0, \quad x_1 = \Delta x, \quad x_2 = 2\Delta x, \quad \cdots, \quad x_{n-1} = (n-1)\,\Delta x, \quad x_n = n\Delta x = b.$$

We are free to choose the c_k's any way we please. We choose each c_k to be the right-hand endpoint of its subinterval, a choice that leads to manageable arithmetic. Thus, $c_1 = x_1, c_2 = x_2$, and so on. The rectangles defined by these choices have areas

$$f(c_1)\,\Delta x = f(\Delta x)\,\Delta x = (\Delta x)^2\,\Delta x = (1^2)(\Delta x)^3$$

$$f(c_2)\,\Delta x = f(2\Delta x)\,\Delta x = (2\Delta x)^2\,\Delta x = (2^2)(\Delta x)^3$$

$$\vdots$$

$$f(c_n)\,\Delta x = f(n\Delta x)\,\Delta x = (n\Delta x)^2\,\Delta x = (n^2)(\Delta x)^3.$$

The sum of these areas is

$$\begin{aligned} S_n &= \sum_{k=1}^{n} f(c_k)\,\Delta x \\ &= \sum_{k=1}^{n} k^2(\Delta x)^3 \\ &= (\Delta x)^3 \sum_{k=1}^{n} k^2 && (\Delta x)^3 \text{ is a constant.} \\ &= \frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} && \Delta x = b/n, \text{ and Eq. (2)} \\ &= \frac{b^3}{6} \cdot \frac{(n+1)(2n+1)}{n^2} \\ &= \frac{b^3}{6} \cdot \frac{2n^2+3n+1}{n^2} \\ &= \frac{b^3}{6} \cdot \left(2 + \frac{3}{n} + \frac{1}{n^2}\right). \end{aligned} \tag{6}$$

We can now use the definition of definite integral

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x$$

to find the area under the parabola from $x = 0$ to $x = b$ as

Notice that $x^3/3$ is an antiderivative of x^2.

$$\begin{aligned} \int_0^b x^2\,dx &= \lim_{n\to\infty} S_n && \text{In this example, } \|P\| \to 0 \text{ is equivalent to } n \to \infty. \\ &= \lim_{n\to\infty} \frac{b^3}{6} \cdot \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) && \text{Eq. (6)} \\ &= \frac{b^3}{6} \cdot (2+0+0) = \frac{b^3}{3}. \end{aligned}$$

With different values of b, we get

$$\int_0^1 x^2\,dx = \frac{1^3}{3} = \frac{1}{3}, \qquad \int_0^{1.5} x^2\,dx = \frac{(1.5)^3}{3} = \frac{3.375}{3} = 1.125,$$

and so on. ❑

Exercises 4.5

Sigma Notation

Write the sums in Exercises 1–6 without sigma notation. Then evaluate them.

1. $\sum_{k=1}^{2} \frac{6k}{k+1}$ **2.** $\sum_{k=1}^{3} \frac{k-1}{k}$

3. $\sum_{k=1}^{4} \cos k\pi$ **4.** $\sum_{k=1}^{5} \sin k\pi$

5. $\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k}$ **6.** $\sum_{k=1}^{4} (-1)^k \cos k\pi$

7. Which of the following express $1+2+4+8+16+32$ in sigma notation?

a) $\sum_{k=1}^{6} 2^{k-1}$ **b)** $\sum_{k=0}^{5} 2^k$ **c)** $\sum_{k=-1}^{4} 2^{k+1}$

8. Which of the following express $1-2+4-8+16-32$ in sigma notation?

a) $\sum_{k=1}^{6} (-2)^{k-1}$ **b)** $\sum_{k=0}^{5} (-1)^k 2^k$

c) $\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2}$

9. Which formula is not equivalent to the other two?

a) $\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1}$ **b)** $\sum_{k=0}^{2} \frac{(-1)^k}{k+1}$ **c)** $\sum_{k=-1}^{1} \frac{(-1)^k}{k+2}$

10. Which formula is not equivalent to the other two?

a) $\sum_{k=1}^{4} (k-1)^2$ **b)** $\sum_{k=-1}^{3} (k+1)^2$ **c)** $\sum_{k=-3}^{-1} k^2$

Express the sums in Exercises 11–16 in sigma notation. The form of your answer will depend on your choice of the lower limit of summation.

11. $1+2+3+4+5+6$ **12.** $1+4+9+16$

13. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ **14.** $2+4+6+8+10$

15. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$ **16.** $-\frac{1}{5}+\frac{2}{5}-\frac{3}{5}+\frac{4}{5}-\frac{5}{5}$

Values of Finite Sums

17. Suppose that $\sum_{k=1}^{n} a_k = -5$ and $\sum_{k=1}^{n} b_k = 6$. Find the values of

a) $\sum_{k=1}^{n} 3a_k$ **b)** $\sum_{k=1}^{n} \frac{b_k}{6}$ **c)** $\sum_{k=1}^{n} (a_k + b_k)$

d) $\sum_{k=1}^{n} (a_k - b_k)$ **e)** $\sum_{k=1}^{n} (b_k - 2a_k)$

18. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of

a) $\sum_{k=1}^{n} 8a_k$ **b)** $\sum_{k=1}^{n} 250b_k$

c) $\sum_{k=1}^{n} (a_k + 1)$ **d)** $\sum_{k=1}^{n} (b_k - 1)$

Use the algebra rules on p. 310 and the formulas in Eqs. (1)–(3) to evaluate the sums in Exercises 19–28.

19. a) $\sum_{k=1}^{10} k$ **b)** $\sum_{k=1}^{10} k^2$ **c)** $\sum_{k=1}^{10} k^3$

20. a) $\sum_{k=1}^{13} k$ **b)** $\sum_{k=1}^{13} k^2$ **c)** $\sum_{k=1}^{13} k^3$

21. $\sum_{k=1}^{7} (-2k)$ **22.** $\sum_{k=1}^{5} \frac{\pi k}{15}$

23. $\sum_{k=1}^{6} (3-k^2)$ **24.** $\sum_{k=1}^{6} (k^2-5)$

25. $\sum_{k=1}^{5} k(3k+5)$ **26.** $\sum_{k=1}^{7} k(2k+1)$

27. $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$ **28.** $\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$

Rectangles for Riemann Sums

In Exercises 29–32, graph each function $f(x)$ over the given interval. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\Sigma_{k=1}^{4} f(c_k)\,\Delta x_k$, given that c_k is the (a) left-hand endpoint, (b) right-hand endpoint, (c) midpoint of the kth subinterval. (Make a separate sketch for each set of rectangles.)

29. $f(x) = x^2 - 1, \quad [0, 2]$

30. $f(x) = -x^2, \quad [0, 1]$

31. $f(x) = \sin x, \quad [-\pi, \pi]$

32. $f(x) = \sin x + 1, \quad [-\pi, \pi]$

33. Find the norm of the partition $P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$.

34. Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.

Expressing Limits as Integrals

Express the limits in Exercises 35–42 as definite integrals.

35. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} c_k{}^2 \Delta x_k$, where P is a partition of $[0, 2]$

36. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} 2c_k{}^3 \Delta x_k$, where P is a partition of $[-1, 0]$

37. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (c_k{}^2 - 3c_k)\, \Delta x_k$, where P is a partition of $[-7, 5]$

38. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} \left(\frac{1}{c_k}\right) \Delta x_k$, where P is a partition of $[1, 4]$

39. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} \frac{1}{1 - c_k} \Delta x_k$, where P is a partition of $[2, 3]$

40. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} \sqrt{4 - c_k{}^2}\, \Delta x_k$, where P is a partition of $[0, 1]$

41. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (\sec c_k)\, \Delta x_k$, where P is a partition of $[-\pi/4, 0]$

42. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (\tan c_k)\, \Delta x_k$, where P is a partition of $[0, \pi/4]$

Constant Functions

Evaluate the integrals in Exercises 43–48.

43. $\int_{-2}^{1} 5\, dx$

44. $\int_{3}^{7} (-20)\, dx$

45. $\int_{0}^{3} (-160)\, dt$

46. $\int_{-4}^{-1} \frac{\pi}{2}\, d\theta$

47. $\int_{-2.1}^{3.4} 0.5\, ds$

48. $\int_{\sqrt{2}}^{\sqrt{18}} \sqrt{2}\, dr$

Using Area to Evaluate Integrals

In Exercises 49–56, graph the integrands and use areas to evaluate the integrals.

49. $\int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$

50. $\int_{1/2}^{3/2} (-2x + 4)\, dx$

51. $\int_{-3}^{3} \sqrt{9 - x^2}\, dx$

52. $\int_{-4}^{0} \sqrt{16 - x^2}\, dx$

53. $\int_{-2}^{1} |x|\, dx$

54. $\int_{-1}^{1} (1 - |x|)\, dx$

55. $\int_{-1}^{1} (2 - |x|)\, dx$

56. $\int_{-1}^{1} \left(1 + \sqrt{1 - x^2}\right) dx$

Use areas to evaluate the integrals in Exercises 57–60.

57. $\int_{0}^{b} x\, dx, \quad b > 0$

58. $\int_{0}^{b} 4x\, dx, \quad b > 0$

59. $\int_{a}^{b} 2s\, ds, \quad 0 < a < b$

60. $\int_{a}^{b} 3t\, dt, \quad 0 < a < b$

Evaluations

Use the results of Examples 8 and 9 to evaluate the integrals in Exercises 61–72.

61. $\int_{1}^{\sqrt{2}} x\, dx$

62. $\int_{0.5}^{2.5} x\, dx$

63. $\int_{\pi}^{2\pi} \theta\, d\theta$

64. $\int_{\sqrt{2}}^{5\sqrt{2}} r\, dr$

65. $\int_{0}^{\sqrt[3]{7}} x^2\, dx$

66. $\int_{0}^{0.3} s^2\, ds$

67. $\int_{0}^{1/2} t^2\, dt$

68. $\int_{0}^{\pi/2} \theta^2\, d\theta$

69. $\int_{a}^{2a} x\, dx$

70. $\int_{a}^{\sqrt{3}a} x\, dx$

71. $\int_{0}^{\sqrt[3]{b}} x^2\, dx$

72. $\int_{0}^{3b} x^2\, dx$

Finding Area

In Exercises 73–76, use a definite integral to find the area of the region between the given curve and the x-axis on the interval $[0, b]$, as in Example 9.

73. $y = 3x^2$

74. $y = \pi x^2$

75. $y = 2x$

76. $y = \frac{x}{2} + 1$

Theory and Examples

77. What values of a and b maximize the value of

$$\int_{a}^{b} (x - x^2)\, dx?$$

(*Hint:* Where is the integrand positive?)

78. What values of a and b minimize the value of

$$\int_{a}^{b} (x^4 - 2x^2)\, dx?$$

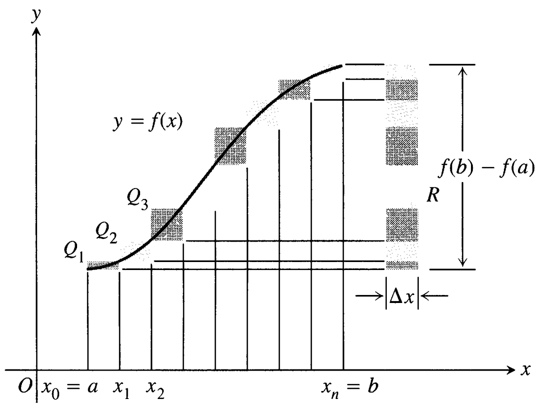
79. *Upper and lower sums for increasing functions*

a) Suppose the graph of a continuous function $f(x)$ rises steadily as x moves from left to right across an interval $[a, b]$. Let P be a partition of $[a, b]$ into n subintervals of length $\Delta x = (b - a)/n$. Show by referring to the accompanying figure that the difference between the upper and lower sums for f on this partition can be represented graphically as the area of a rectangle R whose dimensions are $[f(b) - f(a)]$ by Δx. (*Hint:* The difference $U - L$ is the sum of areas of rectangles whose diagonals $Q_0Q_1, Q_1Q_2, \ldots, Q_{n-1}Q_n$ lie along the curve. There is no overlapping when these rectangles are shifted horizontally onto R.)

b) Suppose that instead of being equal, the lengths Δx_k of the subintervals of the partition of $[a, b]$ vary in size. Show that

$$U - L \le |f(b) - f(a)|\Delta x_{\max},$$

where $\Delta x_{\max}$ is the norm of P, and hence that $\lim_{\|P\|\to 0} (U - L) = 0$.



80. *Upper and lower sums for decreasing functions (Continuation of Exercise 79)*

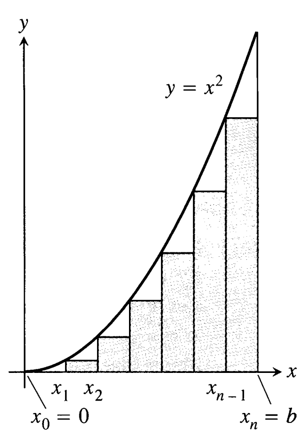
a) Draw a figure like the one in Exercise 79 for a continuous function $f(x)$ whose values decrease steadily as x moves from left to right across the interval $[a, b]$. Let P be a partition of $[a, b]$ into subintervals of equal length. Find an expression for $U - L$ that is analogous to the one you found for $U - L$ in Exercise 79(a).

b) Suppose that instead of being equal, the lengths Δx_k of the subintervals of P vary in size. Show that the inequality

$$U - L \le |f(b) - f(a)|\Delta x_{\max}$$

of Exercise 79(b) still holds and hence that $\lim_{\|P\|\to 0} (U - L) = 0$.

81. Evaluate $\int_0^b x^2\,dx,\ b > 0$, by carrying out the calculations of Example 9 with inscribed rectangles, as shown here, instead of circumscribed rectangles.



82. Let

$$S_n = \frac{1}{n}\left[\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \cdots + \frac{n-1}{n}\right].$$

Calculate $\lim_{n\to\infty} S_n$ by showing that S_n is an approximating sum of the integral

$$\int_0^1 x\,dx,$$

whose value we know from Example 8. (*Hint:* Partition [0, 1] into n intervals of equal length and write out the approximating sum for inscribed rectangles.)

83. Let

$$S_n = \frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{(n-1)^2}{n^3}.$$

To calculate $\lim_{n\to\infty} S_n$, show that

$$S_n = \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2\right]$$

and interpret S_n as an approximating sum of the integral

$$\int_0^1 x^2\,dx,$$

whose value we know from Example 9. (*Hint:* Partition [0, 1] into n intervals of equal length and write out the approximating sum for inscribed rectangles.)

84. Use the formula

$$\sin h + \sin 2h + \sin 3h + \cdots + \sin mh = \frac{\cos(h/2) - \cos((m + (1/2))h)}{2\sin(h/2)}$$

to find the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi/2$, in two steps:

a) Partition the interval $[0, \pi/2]$ into n subintervals of equal length and calculate the corresponding upper sum U; then

b) Find the limit of U as $n \to \infty$ and $\Delta x = (b - a)/n \to 0$.

CAS Explorations and Projects

If your CAS can draw rectangles associated with Riemann sums, use it to draw rectangles associated with Riemann sums that converge to the integrals in Exercises 85–90. Use $n = 4$, 10, 20, and 50 subintervals of equal length in each case.

85. $\displaystyle\int_0^1 (1 - x)\,dx = \frac{1}{2}$

86. $\displaystyle\int_0^1 (x^2 + 1)\,dx = \frac{4}{3}$

87. $\displaystyle\int_{-\pi}^{\pi} \cos x\,dx = 0$

88. $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx = 1$

89. $\int_{-1}^{1} |x|\,dx = 1$

90. $\int_{1}^{2} \frac{1}{x}\,dx$ (The integral's value is ln 2.)

91. a) Write the sum S_n in Exercise 82 in sigma notation and use your CAS to find $\lim_{n\to\infty} S_n$.

b) Do the same for the sum S_n in Exercise 83.

92. Write the sum $\sin h + \sin 2h + \cdots + \sin mh$ in Exercise 84 in sigma notation and use your CAS to find $\lim_{n\to\infty} S_n$.

93. *(Continuation of Section 4.4, Example 3.)* In sigma notation, the left-endpoint sum in Example 3, Section 4.4, is

$$S_4 = \sum_{k=1}^{4} 4\left[9 - (-2 + (k-1))^2\right].$$

a) Use sigma notation to write the analogous left-endpoint sums S_8 for eight subintervals of length 4/8 and S_{25} for 25 subintervals of length $4/25$.

b) Use sigma notation to write the left-endpoint sum S_n for n subintervals of length $4/n$.

c) Find $\lim_{n\to\infty} S_n$. How does this limit appear to be related to the volume of the solid?

94. *(Continuation of Section 4.4 Example 4.)* In sigma notation, the left-endpoint sum in Example 4, Section 4.4, is

$$S_8 = \sum_{k=1}^{8} \pi\left[16 - (-4 + (k-1))^2\right].$$

a) Use sigma notation to write the analogous left-endpoint sums S_{16} for 16 subintervals of length $1/2$ and S_{80} for 80 subintervals of length $1/10$.

b) Use sigma notation to write the left-endpoint sum S_n for n subintervals of length $8/n$.

c) Find $\lim_{n\to\infty} S_n$. How does this limit appear to be related to the volume of the sphere?

4.6 Properties, Area, and the Mean Value Theorem

This section describes working rules for integrals, examines the relationship between the integral of an arbitrary continuous function and area, and takes a fresh look at average value.

Properties of Definite Integrals

We often want to add and subtract definite integrals, multiply their integrands by constants, and compare them with other definite integrals. We do this with the rules in Table 4.5 (on the following page). All the rules except the first two follow from the way integrals are defined with Riemann sums. You might think that this would make them relatively easy to prove. After all, we might argue, sums have these properties so their limits should have them, too. But when we get down to the details we find that most of the proofs require complicated ϵ-δ arguments with norms of subdivisions and are not easy at all. We omit all but two of the proofs. The remaining proofs can be found in more advanced texts.

Notice that Rule 1 is a definition. We want every integral over an interval of zero length to be zero. Rule 1 extends the definition of definite integral to allow for the case $a = b$. Rule 2, also a definition, extends the definition of definite integral to allow for the case $b < a$. Rules 3 and 4 are like the analogous rules for limits and indefinite integrals. Once we know the integrals of two functions, we automatically know the integrals of all constant multiples of these functions and their sums and differences. We can also use Rules 3 and 4 repeatedly to evaluate integrals of arbitrary finite linear combinations of integrable functions term by term. For any

Table 4.5 Rules for definite integrals

1. *Zero:* $\displaystyle\int_a^a f(x)\,dx = 0$ (A definition)

2. *Order of Integration:* $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ (Also a definition)

3. *Constant Multiples:* $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ (Any number k)

   $\displaystyle\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ ($k=-1$)

4. *Sums and Differences:* $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If max f and min f are the maximum and minimum values of f on $[a,b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a,b]$ $\Rightarrow$ $\displaystyle\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

   $f(x) \ge 0$ on $[a,b]$ $\Rightarrow$ $\displaystyle\int_a^b f(x)\,dx \ge 0$

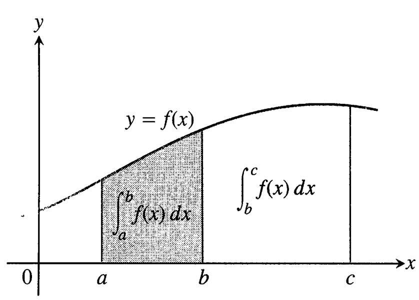
   (Special case)

constants $c_1, \dots, c_n$, regardless of sign, and functions $f_1(x), \dots, f_n(x)$, integrable on $[a,b]$,

$$\int_a^b (c_1 f_1(x) + \cdots + c_n f_n(x))\,dx = c_1\int_a^b f_1(x)\,dx + \cdots + c_n\int_a^b f_n(x)\,dx.$$

The proof, omitted, comes from mathematical induction.

Figure 4.16 illustrates Rule 5 with a positive function, but the rule applies to any integrable function.



4.16 Additivity for definite integrals:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\int_b^c f(x)\,dx = \int_a^c f(x)\,dx - \int_a^b f(x)\,dx.$$

Proof of Rule 3 Rule 3 says that the integral of k times a function is k times the integral of the function. This is true because

$$\begin{aligned}\int_a^b kf(x)\,dx &= \lim_{\|P\|\to 0}\sum_{i=1}^n kf(c_i)\Delta x_i\\ &= \lim_{\|P\|\to 0} k\sum_{i=1}^n f(c_i)\Delta x_i\\ &= k\lim_{\|P\|\to 0}\sum_{i=1}^n f(c_i)\Delta x_i = k\int_a^b f(x)\,dx.\end{aligned}$$

❑

Proof of Rule 6 Rule 6 says that the integral of f over $[a, b]$ is never smaller than the minimum value of f times the length of the interval and never larger than the maximum value of f times the length of the interval. The reason is that for every partition of $[a, b]$ and for every choice of the points c_k,

$$\begin{aligned}
\min f \cdot (b-a) &= \min f \cdot \sum_{k=1}^{n} \Delta x_k && \textstyle\sum_{k=1}^{n} \Delta x_k = b-a \\
&= \sum_{k=1}^{n} \min f \cdot \Delta x_k \\
&\le \sum_{k=1}^{n} f(c_k)\, \Delta x_k && \min f \le f(c_k) \\
&\le \sum_{k=1}^{n} \max f \cdot \Delta x_k && f(c_k) \le \max f \\
&= \max f \cdot \sum_{k=1}^{n} \Delta x_k \\
&= \max f \cdot (b-a).
\end{aligned}$$

In short, all Riemann sums for f on $[a, b]$ satisfy the inequality

$$\min f \cdot (b-a) \le \sum_{k=1}^{n} f(c_k)\, \Delta x_k \le \max f \cdot (b-a).$$

Hence their limit, the integral, does too. ❑

EXAMPLE 1 Suppose that

$$\int_{-1}^{1} f(x)\, dx = 5, \qquad \int_{1}^{4} f(x)\, dx = -2, \qquad \int_{-1}^{1} h(x)\, dx = 7.$$

Then

1. $\displaystyle\int_{4}^{1} f(x)\, dx = -\int_{1}^{4} f(x)\, dx = -(-2) = 2$ Rule 2

2. $\displaystyle\int_{-1}^{1} [2f(x) + 3h(x)]\, dx = 2\int_{-1}^{1} f(x)\, dx + 3\int_{-1}^{1} h(x)\, dx$
   $= 2(5) + 3(7) = 31$ Rules 3 and 4

3. $\displaystyle\int_{-1}^{4} f(x)\, dx = \int_{-1}^{1} f(x)\, dx + \int_{1}^{4} f(x)\, dx = 5 + (-2) = 3$ Rule 5 ❑

In Section 4.5 we learned to evaluate three general integrals:

$$\int_{a}^{b} c\, dx = c(b-a) \qquad \text{(Any constant } c\text{)} \tag{1}$$

$$\int_{a}^{b} x\, dx = \frac{b^2}{2} - \frac{a^2}{2} \qquad (0 < a < b) \tag{2}$$

$$\int_{0}^{b} x^2\, dx = \frac{b^3}{3} \qquad (b > 0). \tag{3}$$

The rules in Table 4.5 enable us to build on these results.

EXAMPLE 2 Evaluate $\int_0^2 \left(\frac{t^2}{4} - 7t + 5\right) dt$.

Solution

$$\begin{aligned} \int_0^2 \left(\frac{t^2}{4} - 7t + 5\right) dt &= \frac{1}{4}\int_0^2 t^2\,dt - 7\int_0^2 t\,dt + \int_0^2 5\,dt && \text{Rules 3 and 4} \\ &= \frac{1}{4}\left(\frac{(2)^3}{3}\right) - 7\left(\frac{(2)^2}{2} - \frac{(0)^2}{2}\right) + 5(2-0) && \text{Eqs. (1)–(3)} \\ &= \frac{2}{3} - 14 + 10 = -\frac{10}{3} \end{aligned}$$

❑

EXAMPLE 3 Evaluate $\int_2^3 x^2\,dx$.

Solution We cannot apply Eq. (3) directly because the lower limit of integration is different from 0. We can, however, use the Additivity Rule to express $\int_2^3 x^2\,dx$ as a difference of two integrals that *can* be evaluated with Eq. (3):

$$\begin{aligned} \int_0^2 x^2\,dx + \int_2^3 x^2\,dx &= \int_0^3 x^2\,dx && \text{Rule 5} \\ \int_2^3 x^2\,dx &= \int_0^3 x^2\,dx - \int_0^2 x^2\,dx && \text{Solve for } \textstyle\int_2^3 x^2 dx. \\ &= \frac{(3)^3}{3} - \frac{(2)^3}{3} && \text{Eq. (3) now applies.} \\ &= \frac{27}{3} - \frac{8}{3} = \frac{19}{3}. \end{aligned}$$

In Section 4.7, we will see how to evaluate $\int_2^3 x^2\,dx$ in a more direct way. ❑

The Max-Min Inequality for definite integrals (Rule 6) says that $\min f \cdot (b-a)$ is a **lower bound** for the value of $\int_a^b f(x)\,dx$ and that $\max f \cdot (b-a)$ is an **upper bound.**

EXAMPLE 4 Show that the value of

$$\int_0^1 \sqrt{1 + \cos x}\,dx$$

cannot possibly be 2.

Solution The maximum value of $\sqrt{1+\cos x}$ on $[0, 1]$ is $\sqrt{1+1} = \sqrt{2}$, so

$$\begin{aligned} \int_0^1 \sqrt{1+\cos x}\,dx &\le \max \sqrt{1+\cos x} \cdot (1-0) && \text{Table 4.5, Rule 6} \\ &\le \sqrt{2} \cdot 1 = \sqrt{2}. \end{aligned}$$

The integral cannot exceed $\sqrt{2}$, so it cannot possibly equal 2. ❑

EXAMPLE 5 Use the inequality $\cos x \ge (1 - x^2/2)$, which holds for all x, to find a lower bound for the value of $\int_0^1 \cos x\,dx$.

Solution

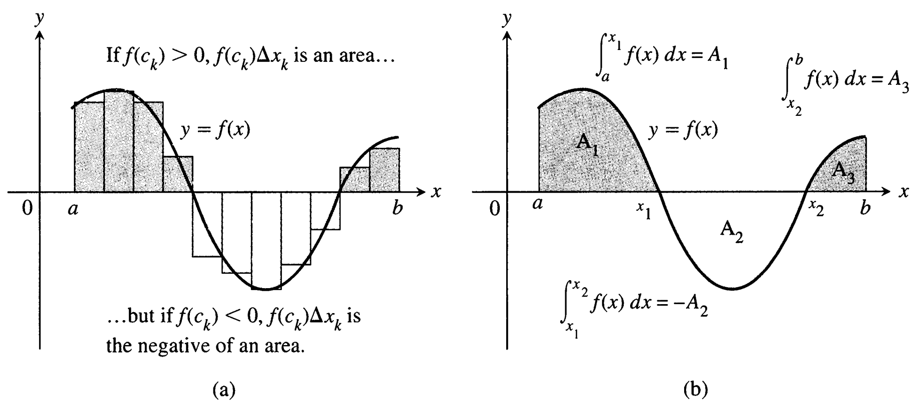
$$\begin{aligned}\int_0^1 \cos x\,dx &\geq \int_0^1 \left(1-\frac{x^2}{2}\right)dx && \text{Rule 7}\\ &\geq \int_0^1 1\,dx - \frac{1}{2}\int_0^1 x^2\,dx && \text{Rules 3 and 4}\\ &\geq 1\cdot(1-0) - \frac{1}{2}\cdot\frac{(1)^3}{3} = \frac{5}{6} \approx 0.83.\end{aligned}$$

The value of the integral is at least 5/6. ❑

Integrals and Total Area

If an integrable function $y = f(x)$ has both positive and negative values on an interval $[a, b]$, then the Riemann sums for f on $[a, b]$ add the areas of the rectangles that lie above the x-axis to the negatives of the areas of the rectangles that lie below it (Fig. 4.17). The resulting cancellation reduces the sums, so their limiting value is a number whose magnitude is less than the total area between the curve and the x-axis. The value of the integral is the area above the axis minus the area below the axis.

This means that we must take special care in finding areas by integration.

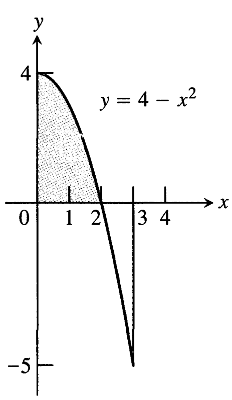


4.17 (a) The Riemann sums are algebraic sums of areas and so is the integral to which they converge. (b) The value of the integral of f from a to b is

$$\int_a^b f(x)\,dx = \int_a^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{b} f(x)\,dx = A_1 - A_2 + A_3.$$

EXAMPLE 6 Find the area of the region between the curve $y = 4 - x^2$, $0 \le x \le 3$, and the x-axis.

Solution The x-intercept of the curve partitions [0, 3] into subintervals on which $f(x) = 4 - x^2$ has the same sign (Fig. 4.18). To find the area of the region between the graph of f and the x-axis, we integrate f over each subinterval and add the absolute values of the results.



4.18 Part of the region in Example 6 lies below the x-axis.

Integral over [0, 2]:

$$\begin{aligned}\int_0^2 (4-x^2)\,dx &= \int_0^2 4\,dx - \int_0^2 x^2\,dx\\ &= 4(2-0) - \frac{(2)^3}{3} && \text{Eqs. (1) and (3)}\\ &= 8 - \frac{8}{3} = \frac{16}{3}\end{aligned}$$

How to Find the Area of the Region Between a Curve $y = f(x)$, $a \le x \le b$, and the x-axis

1. Partition $[a, b]$ with the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Integral over [2, 3]:

$$\int_2^3 (4 - x^2)\,dx = \int_2^3 4\,dx - \int_2^3 x^2\,dx$$

$$= 4(3-2) - \left(\frac{(3)^3}{3} - \frac{(2)^3}{3}\right) \qquad \text{Eq. (1) and Example 3}$$

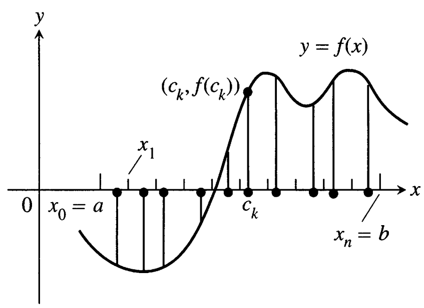
$$= 4 - \frac{19}{3} = -\frac{7}{3}$$

The region's area: $\quad \text{Area} = \dfrac{16}{3} + \left|-\dfrac{7}{3}\right| = \dfrac{23}{3}.$ ❑

The Average Value of an Arbitrary Continuous Function

In Section 4.4, Example 5, we discussed the average value of a nonnegative continuous function. We are now ready to define average value without requiring f to be nonnegative, and to show that every continuous function assumes its average value at least once.

We start once again with the idea from arithmetic that the average of n numbers is the sum of the numbers divided by n. For a continuous function f on a closed interval $[a, b]$ there may be infinitely many values to consider, but we can sample them in an orderly way. We partition $[a, b]$ into n subintervals of equal length (the length is $\Delta x = (b-a)/n$) and evaluate f at a point c_k in each subinterval (Fig. 4.19). The average of the n sampled values is



4.19 A sample of values of a function on an interval $[a, b]$.

$$\frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} = \frac{1}{n} \cdot \sum_{k=1}^{n} f(c_k) \qquad \text{The sum in sigma notation}$$

$$= \frac{\Delta x}{b-a} \cdot \sum_{k=1}^{n} f(c_k) \qquad \Delta x = \frac{b-a}{n}$$

$$= \frac{1}{b-a} \cdot \underbrace{\sum_{k=1}^{n} f(c_k)\,\Delta x}_{\text{a Riemann sum for } f \text{ on } [a,\,b]}$$

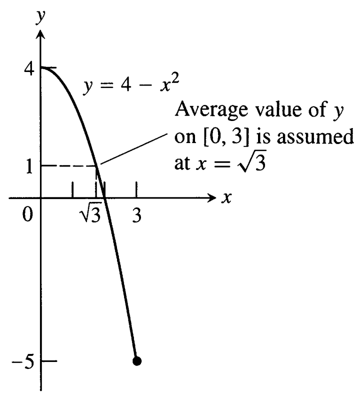
Thus, the average of the sampled values is always $1/(b-a)$ times a Riemann sum for f on $[a, b]$. As we increase the size of the sample and let the norm of the partition approach zero, the average must approach $(1/(b-a)) \int_a^b f(x)\,dx$. We are led by this remarkable fact to the following definition.

Definition

If f is integrable on $[a, b]$, its **average (mean) value** on $[a, b]$ is

$$\operatorname{av}(f) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

EXAMPLE 7 Find the average value of $f(x) = 4 - x^2$ on [0, 3]. Does f actually take on this value at some point in the given domain?



4.20 The average value of $f(x) = 4 - x^2$ on $[0, 3]$ occurs at $x = \sqrt{3}$ (Example 7).

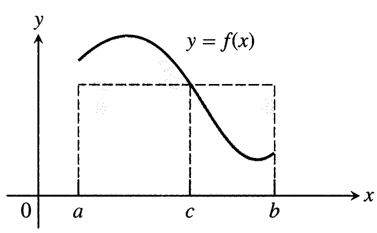
Solution

$$
\begin{aligned}
\operatorname{av}(f) &= \frac{1}{b-a}\int_a^b f(x)\,dx \\
&= \frac{1}{3-0}\int_0^3 (4-x^2)\,dx = \frac{1}{3}\left(\int_0^3 4\,dx - \int_0^3 x^2\,dx\right) \\
&= \frac{1}{3}\left(4(3-0) - \frac{(3)^3}{3}\right) = \frac{1}{3}(12-9) = 1
\end{aligned}
$$

The average value of $f(x) = 4 - x^2$ over the interval [0, 3] is 1. The function assumes this value when $4 - x^2 = 1$ or $x = \pm\sqrt{3}$. Since one of these points, $x = \sqrt{3}$, lies in [0, 3], the function does assume its average value in the given domain (Fig. 4.20). ❑

The Mean Value Theorem for Definite Integrals

The statement that a continuous function on a closed interval assumes its average value at least once in the interval is known as the Mean Value Theorem for Definite Integrals.



4.21 Theorem 2 for a positive function: At some point c in $[a, b]$,

$$f(c)\cdot(b-a) = \int_a^b f(x)\,dx.$$

Theorem 2

The Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

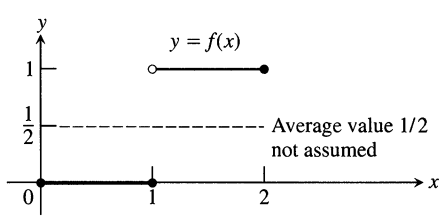
(Fig. 4.21).

In Example 7, we found a point where f assumed its average value by setting $f(x)$ equal to the calculated average value and solving for x. But this does not prove that such a point will always exist. It proves only that it existed in Example 7. To prove Theorem 2, we need a more general argument.

Proof of Theorem 2 If we divide both sides of the Max-Min Inequality (Rule 6) by $(b - a)$, we obtain

$$\min f \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \max f.$$

Since f is continuous, the Intermediate Value Theorem for Continuous Functions (Section 1.5) says that f must assume every value between $\min f$ and $\max f$. It must therefore assume the value $(1/(b-a))\int_a^b f(x)\,dx$ at some point c in $[a, b]$. ❑



4.22 A discontinuous function need not assume its average value.

The continuity of f is important here. A discontinuous function can step over its average value (Fig. 4.22).

What else can we learn from Theorem 2? Here is an example.

EXAMPLE 8 Show that if f is continuous on $[a, b]$, $a \neq b$, and if

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ at least once in $[a, b]$.

Solution The average value of f on $[a, b]$ is

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\cdot 0 = 0.$$

By Theorem 2, f assumes this value at some point c in $[a, b]$. ❑

Exercises 4.6

Using Properties and Known Values to Find Other Integrals

1. Suppose that f and g are continuous and that

$$\int_1^2 f(x)\,dx = -4, \quad \int_1^5 f(x)\,dx = 6, \quad \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 4.5 to find

a) $\int_2^2 g(x)\,dx$ **b)** $\int_5^1 g(x)\,dx$

c) $\int_1^2 3f(x)\,dx$ **d)** $\int_2^5 f(x)\,dx$

e) $\int_1^5 [f(x) - g(x)]\,dx$ **f)** $\int_1^5 [4f(x) - g(x)]\,dx$

2. Suppose that f and h are continuous and that

$$\int_1^9 f(x)\,dx = -1, \quad \int_7^9 f(x)\,dx = 5, \quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 4.5 to find

a) $\int_1^9 -2f(x)\,dx$ **b)** $\int_7^9 [f(x) + h(x)]\,dx$

c) $\int_7^9 [2f(x) - 3h(x)]\,dx$ **d)** $\int_9^1 f(x)\,dx$

e) $\int_1^7 f(x)\,dx$ **f)** $\int_9^7 [h(x) - f(x)]\,dx$

3. Suppose that $\int_1^2 f(x)\,dx = 5$. Find

a) $\int_1^2 f(u)\,du$ **b)** $\int_1^2 \sqrt{3}f(z)\,dz$

c) $\int_2^1 f(t)\,dt$ **d)** $\int_1^2 [-f(x)]\,dx$

4. Suppose that $\int_{-3}^0 g(t)\,dt = \sqrt{2}$. Find

a) $\int_0^{-3} g(t)\,dt$ **b)** $\int_{-3}^0 g(u)\,du$

c) $\int_{-3}^0 [-g(x)]\,dx$ **d)** $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr$

5. Suppose that f is continuous and that $\int_0^3 f(z)\,dz = 3$ and $\int_0^4 f(z)\,dz = 7$. Find

a) $\int_3^4 f(z)\,dz$ **b)** $\int_4^3 f(t)\,dt$

6. Suppose that h is continuous and that $\int_{-1}^1 h(r)\,dr = 0$ and $\int_{-1}^3 h(r)\,dr = 6$. Find

a) $\int_1^3 h(r)\,dr$ **b)** $-\int_3^1 h(u)\,du$

Evaluate the integrals in Exercises 7–18.

7. $\int_3^1 7\,dx$ **8.** $\int_0^{-2} \sqrt{2}\,dx$

9. $\int_0^2 5x\,dx$ **10.** $\int_3^5 \frac{x}{8}\,dx$

11. $\int_0^2 (2t - 3)\,dt$ **12.** $\int_0^{\sqrt{2}} \left(t - \sqrt{2}\right)\,dt$

13. $\int_2^1 \left(1 + \frac{z}{2}\right)\,dz$ **14.** $\int_3^0 (2z - 3)\,dz$

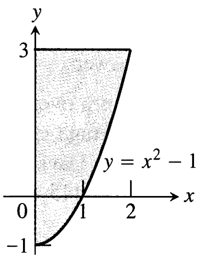
15. $\int_1^2 3u^2\,du$ **16.** $\int_{1/2}^1 24u^2\,du$

17. $\int_0^2 (3x^2 + x - 5)\,dx$ **18.** $\int_1^0 (3x^2 + x - 5)\,dx$

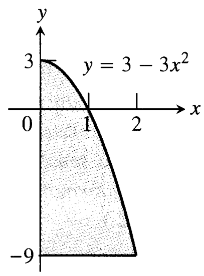
Area

In Exercises 19–22, find the total shaded area.

19.



20.



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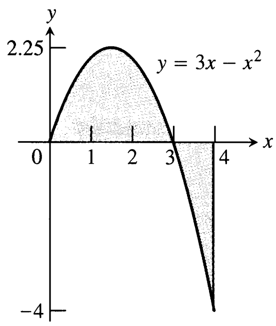
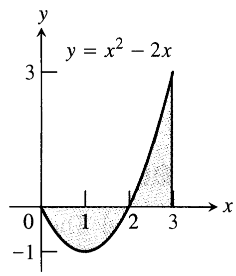


22.



In Exercises 23–26, graph the function over the given interval. Then (a) integrate the function over the interval and (b) find the area of the region between the graph and the x-axis.

23. $y = x^2 - 6x + 8, \quad [0, 3]$

24. $y = -x^2 + 5x - 4, \quad [0, 2]$

25. $y = 2x - x^2, \quad [0, 3]$

26. $y = x^2 - 4x, \quad [0, 5]$

Average Value

In Exercises 27–34, graph the function and find its average value over the given interval. At what point or points in the given interval does the function assume its average value?

27. $f(x) = x^2 - 1$ on $\left[0, \sqrt{3}\right]$

28. $f(x) = -\dfrac{x^2}{2}$ on $[0, 3]$

29. $f(x) = -3x^2 - 1$ on $[0, 1]$

30. $f(x) = 3x^2 - 3$ on $[0, 1]$

31. $f(t) = (t - 1)^2$ on $[0, 3]$

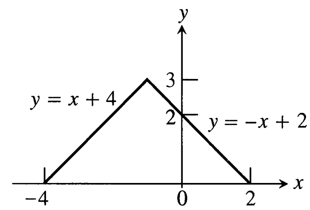
32. $f(t) = t^2 - t$ on $[-2, 1]$

33. $g(x) = |x| - 1$ on (a) $[-1, 1]$, (b) $[1, 3]$, and (c) $[-1, 3]$

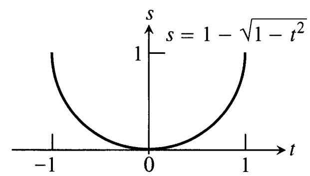
34. $h(x) = -|x|$ on (a) $[-1, 0]$, (b) $[0, 1]$, and (c) $[-1, 1]$

In Exercises 35–38, find the average value of the function over the given interval from the graph of f (without integrating).

35. $f(x) = \begin{cases} x + 4, & -4 \le x \le -1 \\ -x + 2, & -1 < x \le 2 \end{cases}$ on $[-4, 2]$



36. $f(t) = 1 - \sqrt{1 - t^2}$ on $[-1, 1]$



37. $f(t) = \sin t$ on $[0, 2\pi]$

38. $f(\theta) = \tan \theta$ on $\left[\dfrac{-\pi}{4}, \dfrac{\pi}{4}\right]$

Theory and Examples

39. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1 + x^2}\, dx.$$

40. (*Continuation of Exercise* 39.) Use the Max-Min Inequality to find upper and lower bounds for

$$\int_0^{0.5} \frac{1}{1 + x^2}\, dx \quad \text{and} \quad \int_{0.5}^1 \frac{1}{1 + x^2}\, dx.$$

Add these to arrive at an improved estimate of

$$\int_0^1 \frac{1}{1 + x^2}\, dx.$$

41. Show that the value of $\int_0^1 \sin(x^2)\, dx$ cannot possibly be 2.

42. Show that the value of $\int_0^1 \sqrt{x + 8}\, dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

43. Suppose that f is continuous and that $\int_1^2 f(x)\, dx = 4$. Show that $f(x) = 4$ at least once on $[1, 2]$.

44. Suppose that f and g are continuous on $[a, b]$, $a \ne b$, and that $\int_a^b (f(x) - g(x))\, dx = 0$. Show that $f(x) = g(x)$ at least once in $[a, b]$.

45. *Integrals of nonnegative functions.* Use the Max-Min Inequality to show that if f is integrable then

$$f(x) \ge 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\, dx \ge 0.$$

46. *Integrals of nonpositive functions.* Show that if f is integrable then

$$f(x) \leq 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \leq 0.$$

47. Use the inequality $\sin x \leq x$, which holds for $x \geq 0$, to find an upper bound for the value of $\int_0^1 \sin x\,dx$.

48. The inequality $\sec x \geq 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x\,dx$.

49. If $\mathrm{av}(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the number $\mathrm{av}(f)$ should have the same integral over $[a, b]$ that f does. Does it? That is, does

$$\int_a^b \mathrm{av}(f)\,dx = \int_a^b f(x)\,dx?$$

Give reasons for your answer.

50. It would be nice if average values of integrable functions obeyed the following rules on an interval $[a, b]$:

a) $\mathrm{av}(f+g) = \mathrm{av}(f) + \mathrm{av}(g)$

b) $\mathrm{av}(kf) = k\,\mathrm{av}(f)$ (any number k)

c) $\mathrm{av}(f) \leq \mathrm{av}(g)$ if $f(x) \leq g(x)$ on $[a, b]$.

Do these rules ever hold? Give reasons for your answers.

51. If you average 30 mi/h on a 150-mi trip and then return over the same 150 mi at the rate of 50 mi/h, what is your average speed for the trip? Give reasons for your answer. (Source: David H. Pleacher, *The Mathematics Teacher,* Vol. 85, No. 6, pp. 445–446, September 1992.)

52. A dam released 1000 m^3 of water at 10 m^3/min and then released another 1000 m^3 at 20 m^3/min. What was the average rate at which the water was released? Give reasons for your answer.

4.7 The Fundamental Theorem

This section presents the Fundamental Theorem of Integral Calculus. The independent discovery by Leibniz and Newton of this astonishing connection between integration and differentiation started the mathematical developments that fueled the scientific revolution for the next two hundred years and constitutes what is still regarded as the most important computational discovery in the history of the world.

The Fundamental Theorem, Part 1

If $f(t)$ is an integrable function, the integral from any fixed number a to another number x defines a function F whose value at x is

$$F(x) = \int_a^x f(t)\,dt. \tag{1}$$

For example, if f is nonnegative and x lies to the right of a, $F(x)$ is the area under the graph from a to x. The variable x is the upper limit of integration of an integral, but F is just like any other real-valued function of a real variable. For each value of the input x there is a well-defined numerical output, in this case the integral of f from a to x.

Equation (1) gives an important way to define new functions and to describe solutions of differential equations (more about this later). The reason for mentioning Eq. (1) now, however, is the connection it makes between integrals and derivatives. For if f is any continuous function whatever, then F is a differentiable function of x whose derivative is f itself. At every value of x,

$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_a^x f(t)\,dt = f(x).$$

This idea is so important that it is the first part of the Fundamental Theorem of Calculus.

Theorem 3

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ has a derivative at every point of $[a, b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x), \qquad a \le x \le b. \tag{2}$$

This conclusion is beautiful, powerful, deep, and surprising, and Eq. (2) may well be the most important equation in mathematics. It says that the differential equation $dF/dx = f$ has a solution for every continuous function f. It says that every continuous function f is the derivative of some other function, namely $\int_a^x f(t)\,dt$. It says that every continuous function has an antiderivative. And it says that the processes of integration and differentiation are inverses of one another.

Proof of Theorem 3 We prove Theorem 3 by applying the definition of derivative directly to the function $F(x)$. This means writing out the difference quotient

$$\frac{F(x+h) - F(x)}{h} \tag{3}$$

and showing that its limit as $h \to 0$ is the number $f(x)$.

When we replace $F(x+h)$ and $F(x)$ by their defining integrals, the numerator in Eq. (3) becomes

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt.$$

The Additivity Rule for integrals (Table 4.5 in Section 4.6) simplifies the right-hand side to

$$\int_x^{x+h} f(t)\,dt,$$

so that Eq. (3) becomes

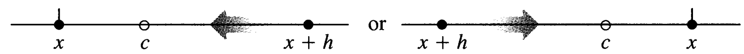
$$\begin{aligned} \frac{F(x+h) - F(x)}{h} &= \frac{1}{h}[F(x+h) - F(x)] \\ &= \frac{1}{h}\int_x^{x+h} f(t)\,dt. \end{aligned} \tag{4}$$

According to the Mean Value Theorem for Definite Integrals (Theorem 2 in the preceding section), the value of the last expression in Eq. (4) is one of the values taken on by f in the interval joining x and $x+h$. That is, for some number c in this interval,

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c). \tag{5}$$

We can therefore find out what happens to $(1/h)$ times the integral as $h \to 0$ by watching what happens to $f(c)$ as $h \to 0$.

What does happen to $f(c)$ as $h \to 0$? As $h \to 0$, the endpoint $x + h$ approaches x, pushing c ahead of it like a bead on a wire:



So c approaches x, and, since f is continuous at x, $f(c)$ approaches $f(x)$:

$$\lim_{h \to 0} f(c) = f(x). \tag{6}$$

Going back to the beginning, then, we have

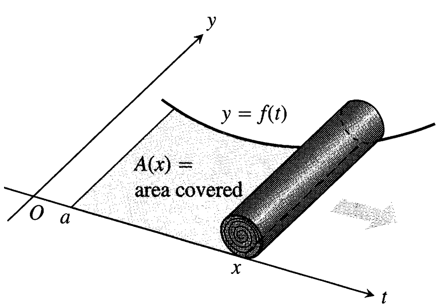
$$\begin{aligned}
\frac{dF}{dx} &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} && \text{Definition of derivative} \\
&= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt && \text{Eq. (4)} \\
&= \lim_{h \to 0} f(c) && \text{Eq. (5)} \\
&= f(x). && \text{Eq. (6)}
\end{aligned}$$

This concludes the proof. ❑

If the values of f are positive, the equation

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

has a nice geometric interpretation. For then the integral of f from a to x is the area $A(x)$ of the region between the graph of f and the x-axis from a to x. Imagine covering this region from left to right by unrolling a carpet of variable width $f(t)$ (Fig. 4.23). As the carpet rolls past x, the rate at which the floor is being covered is $f(x)$.



4.23 The rate at which the carpet covers the floor at the point x is the width of the carpet's leading edge as it rolls past x. In symbols, $dA/dx = f(x)$.

EXAMPLE 1

$$\frac{d}{dx} \int_{-\pi}^x \cos t\, dt = \cos x \qquad \text{Eq. (2) with } f(t) = \cos t$$

$$\frac{d}{dx} \int_0^x \frac{1}{1+t^2}\, dt = \frac{1}{1+x^2} \qquad \text{Eq. (2) with } f(t) = \frac{1}{1+t^2}$$

❑

EXAMPLE 2 Find dy/dx if

$$y = \int_1^{x^2} \cos t\, dt.$$

Solution Notice that the upper limit of integration is not x but x^2. To find dy/dx we must therefore treat y as the composite of

$$y = \int_1^u \cos t\, dt \qquad \text{and} \qquad u = x^2$$

and apply the Chain Rule:

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} && \text{Chain Rule} \\
&= \frac{d}{du}\int_1^u \cos t\,dt \cdot \frac{du}{dx} && \text{Substitute the formula for } y. \\
&= \cos u \cdot \frac{du}{dx} && \text{Eq. (2) with } f(t) = \cos t \\
&= \cos x^2 \cdot 2x && u = x^2 \\
&= 2x \cos x^2. && \text{Usual form}
\end{aligned}
$$

❑

EXAMPLE 3 Express the solution of the following initial value problem as an integral.

Differential equation: $\dfrac{dy}{dx} = \tan x$

Initial condition: $y(1) = 5$

Solution The function

$$F(x) = \int_1^x \tan t\,dt$$

is an antiderivative of $\tan x$. Hence the general solution of the equation is

$$y = \int_1^x \tan t\,dt + C.$$

As always, the initial condition determines the value of C:

$$
\begin{aligned}
5 &= \int_1^1 \tan t\,dt + C && y(1) = 5 \\
5 &= 0 + C && (7) \\
C &= 5.
\end{aligned}
$$

The solution of the initial value problem is

$$y = \int_1^x \tan t\,dt + 5.$$

How did we know where to start integrating when we constructed $F(x)$? We could have started anywhere, but the best value to start with is the initial value of x (in this case $x = 1$). Then the integral will be zero when we apply the initial condition (as it was in Eq. 7) and C will automatically be the initial value of y.

❑

The Evaluation of Definite Integrals

We now come to the second part of the Fundamental Theorem of Calculus, the part that describes how to evaluate definite integrals.

Theorem 4

The Fundamental Theorem of Calculus, Part 2

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{8}$$

How to Evaluate $\int_a^b f(x)\,dx$

1. Find an antiderivative F of f. Any antiderivative will do, so pick the simplest one you can.
2. Calculate the number $F(b) - F(a)$.

This number will be $\int_a^b f(x)\,dx$.

Notation

The usual notation for the number $F(b) - F(a)$ is $F(x)]_a^b$ when $F(x)$ has a single term, or $[F(x)]_a^b$ for $F(b) - F(a)$ when $F(x)$ has more than one term.

Theorem 4 says that to evaluate the definite integral of a continuous function f from a to b, all we need do is find an antiderivative F of f and calculate the number $F(b) - F(a)$. The existence of the antiderivative is assured by the first part of the Fundamental Theorem.

Proof of Theorem 4 To prove Theorem 4, we use the fact that functions with identical derivatives differ only by a constant. We already know one function whose derivative equals f, namely,

$$G(x) = \int_a^x f(t)\,dt.$$

Therefore, if F is any other such function, then

$$F(x) = G(x) + C \tag{9}$$

throughout $[a, b]$ for some constant C. When we use Eq. (9) to calculate $F(b) - F(a)$, we find that

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] \\ &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt - 0 = \int_a^b f(t)\,dt. \end{aligned}$$

This establishes Eq. (8) and concludes the proof. ❑

EXAMPLE 4

a) $$\int_0^\pi \cos x\,dx = \sin x\Big]_0^\pi = \sin \pi - \sin 0 = 0 - 0 = 0$$

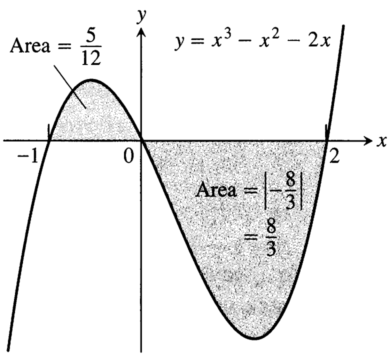
b) $$\int_{-\pi/4}^0 \sec x \tan x\,dx = \sec x\Big]_{-\pi/4}^0 = \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$$

c) $$\begin{aligned} \int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx &= \left[x^{3/2} + \frac{4}{x}\right]_1^4 \\ &= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right] \\ &= [8 + 1] - [5] = 4. \end{aligned}$$ ❑

Theorem 4 explains the formulas we derived for the integrals of x and x^2 in Section 4.5. We can now see that without any restriction on the signs of a and b,

$$\int_a^b x\,dx = \frac{x^2}{2}\bigg]_a^b = \frac{b^2}{2} - \frac{a^2}{2} \qquad \text{Because } x^2/2 \text{ is an antiderivative of } x$$

$$\int_a^b x^2\,dx = \frac{x^3}{3}\bigg]_a^b = \frac{b^3}{3} - \frac{a^3}{3} \qquad \text{Because } x^3/3 \text{ is an antiderivative of } x^2$$



4.24 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 5).

EXAMPLE 5 Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.

Solution First find the zeros of f. Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x+1)(x-2),$$

the zeros are $x = 0$, -1, and 2 (Fig. 4.24). The zeros partition $[-1, 2]$ into two subintervals: $[-1, 0]$, on which $f \ge 0$ and $[0, 2]$, on which $f \le 0$. We integrate f over each subinterval and add the absolute values of the calculated values.

Integral over $[-1, 0]$:
$$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

Integral over $[0, 2]$:
$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

Enclosed area:
$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$$ ❑

EXAMPLE 6 *Household electricity*

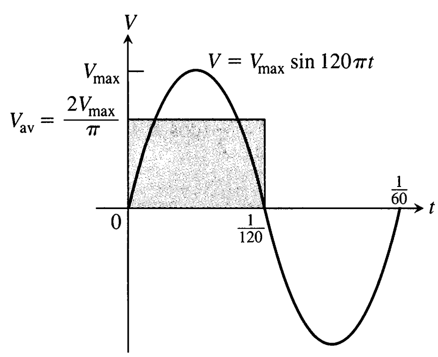
We model the voltage in our home wiring with the sine function

$$V = V_{\max} \sin 120\pi t,$$

which expresses the voltage V in volts as a function of time t in seconds. The function runs through 60 cycles each second (its frequency is 60 hertz, or 60 Hz). The positive constant $V_{\max}$ ("vee max") is the **peak voltage.**

The average value of V over a half-cycle (duration 1/120 sec; see Fig. 4.25) is

$$\begin{aligned} V_{av} &= \frac{1}{(1/120) - 0}\int_0^{1/120} V_{\max} \sin 120\pi t\,dt \\ &= 120V_{\max}\left[-\frac{1}{120\pi}\cos 120\pi t\right]_0^{1/120} \\ &= \frac{V_{\max}}{\pi}[-\cos\pi + \cos 0] \\ &= \frac{2V_{\max}}{\pi}. \end{aligned}$$



4.25 The graph of the household voltage $V = V_{\max} \sin 120\pi t$ over a full cycle. Its average value over a half-cycle is $2V_{\max}/\pi$. Its average value over a full cycle is zero.

The average value of the voltage over a full cycle, as we can see from Fig. 4.25, is zero. (Also see Exercise 64.) If we measured the voltage with a standard moving-coil galvanometer, the meter would read zero.

To measure the voltage effectively, we use an instrument that measures the square root of the average value of the square of the voltage, namely

$$V_{\text{rms}} = \sqrt{(V^2)_{\text{av}}}.$$

The subscript "rms" (read the letters separately) stands for "root mean square." Since the average value of $V^2 = (V_{\max})^2 \sin^2 120\pi t$ over a cycle is

$$(V^2)_{\text{av}} = \frac{1}{(1/60) - 0} \int_0^{1/60} (V_{\max})^2 \sin^2 120\pi t \, dt = \frac{(V_{\max})^2}{2} \tag{10}$$

(Exercise 64c), the rms voltage is

$$V_{\text{rms}} = \sqrt{\frac{(V_{\max})^2}{2}} = \frac{V_{\max}}{\sqrt{2}}. \tag{11}$$

The values given for household currents and voltages are always rms values. Thus, "115 volts ac" means that the rms voltage is 115. The peak voltage,

$$V_{\max} = \sqrt{2}\, V_{\text{rms}} = \sqrt{2} \cdot 115 \approx 163 \text{ volts},$$

obtained from Eq. (11), is considerably higher. ❑

Exercises 4.7

Evaluating Integrals

Evaluate the integrals in Exercises 1–26.

1. $\int_{-2}^{0} (2x + 5)\, dx$

2. $\int_{-3}^{4} \left(5 - \frac{x}{2}\right) dx$

3. $\int_{0}^{4} \left(3x - \frac{x^3}{4}\right) dx$

4. $\int_{-2}^{2} (x^3 - 2x + 3)\, dx$

5. $\int_{0}^{1} (x^2 + \sqrt{x})\, dx$

6. $\int_{0}^{5} x^{3/2}\, dx$

7. $\int_{1}^{32} x^{-6/5}\, dx$

8. $\int_{-2}^{-1} \frac{2}{x^2}\, dx$

9. $\int_{0}^{\pi} \sin x\, dx$

10. $\int_{0}^{\pi} (1 + \cos x)\, dx$

11. $\int_{0}^{\pi/3} 2 \sec^2 x\, dx$

12. $\int_{\pi/6}^{5\pi/6} \csc^2 x\, dx$

13. $\int_{\pi/4}^{3\pi/4} \csc \theta \cot \theta\, d\theta$

14. $\int_{0}^{\pi/3} 4 \sec u \tan u\, du$

15. $\int_{\pi/2}^{0} \frac{1 + \cos 2t}{2}\, dt$

16. $\int_{-\pi/3}^{\pi/3} \frac{1 - \cos 2t}{2}\, dt$

17. $\int_{-\pi/2}^{\pi/2} (8y^2 + \sin y)\, dy$

18. $\int_{-\pi/3}^{-\pi/4} \left(4 \sec^2 t + \frac{\pi}{t^2}\right) dt$

19. $\int_{1}^{-1} (r + 1)^2\, dr$

20. $\int_{-\sqrt{3}}^{\sqrt{3}} (t + 1)(t^2 + 4)\, dt$

21. $\int_{\sqrt{2}}^{1} \left(\frac{u^7}{2} - \frac{1}{u^5}\right) du$

22. $\int_{1/2}^{1} \left(\frac{1}{v^3} - \frac{1}{v^4}\right) dv$

23. $\int_{1}^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2}\, ds$

24. $\int_{9}^{4} \frac{1 - \sqrt{u}}{\sqrt{u}}\, du$

25. $\int_{-4}^{4} |x|\, dx$

26. $\int_{0}^{\pi} \frac{1}{2}(\cos x + |\cos x|)\, dx$

Evaluating Integrals Using Substitutions

In Exercises 27–34, use a substitution to find an antiderivative and then apply the Fundamental Theorem to evaluate the integral.

27. $\int_{0}^{1} (1 - 2x)^3\, dx$

28. $\int_{1}^{2} \sqrt{3x + 1}\, dx$

29. $\int_{0}^{1} t\sqrt{t^2 + 1}\, dt$

30. $\int_{-1}^{2} \frac{t\, dt}{\sqrt{2t^2 + 8}}$

31. $\displaystyle\int_0^{\pi} \sin^2\left(1+\frac{\theta}{2}\right) d\theta$

32. $\displaystyle\int_{3\pi/8}^{\pi/2} \sec^2(\pi - 2\theta)\, d\theta$

33. $\displaystyle\int_0^{\pi} \sin^2 \frac{x}{4} \cos \frac{x}{4}\, dx$

34. $\displaystyle\int_{2\pi/3}^{\pi} \tan^3 \frac{x}{4} \sec^2 \frac{x}{4}\, dx$

Area

In Exercises 35–40, find the total area between the region and the x-axis.

35. $y = -x^2 - 2x, \quad -3 \le x \le 2$

36. $y = 3x^2 - 3, \quad -2 \le x \le 2$

37. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

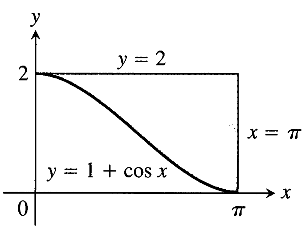
38. $y = x^3 - 4x, \quad -2 \le x \le 2$

39. $y = x^{1/3}, \quad -1 \le x \le 8$

40. $y = x^{1/3} - x, \quad -1 \le x \le 8$

Find the areas of the shaded regions in Exercises 41–44.

41.



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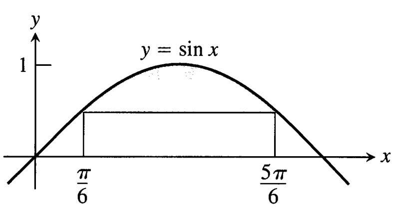

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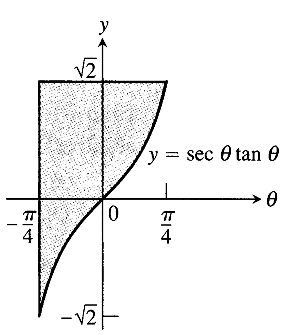

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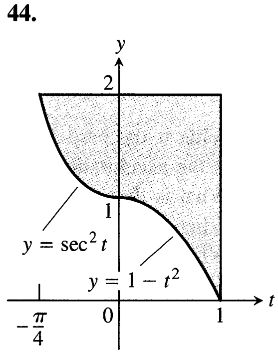


Derivatives of Integrals

Find the derivatives in Exercises 45–48 (a) by evaluating the integral and differentiating the result and (b) by differentiating the integral directly.

45. $\displaystyle\frac{d}{dx}\int_0^{\sqrt{x}} \cos t\, dt$

46. $\displaystyle\frac{d}{dx}\int_1^{\sin x} 3t^2\, dt$

47. $\displaystyle\frac{d}{dt}\int_0^{t^4} \sqrt{u}\, du$

48. $\displaystyle\frac{d}{d\theta}\int_0^{\tan\theta} \sec^2 y\, dy$

Find dy/dx in Exercises 49–54.

49. $\displaystyle y = \int_0^x \sqrt{1+t^2}\, dt$

50. $\displaystyle y = \int_1^x \frac{1}{t}\, dt, \quad x > 0$

51. $\displaystyle y = \int_0^{\sqrt{x}} \sin(t^2)\, dt$

52. $\displaystyle y = \int_0^{x^2} \cos\sqrt{t}\, dt$

53. $\displaystyle y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}, \quad |x| < \frac{\pi}{2}$

54. $\displaystyle y = \int_0^{\tan x} \frac{dt}{1+t^2}$

Initial Value Problems

Each of the following functions solves one of the initial value problems in Exercises 55–58. Which function solves which problem? Give brief reasons for your answers.

a) $\displaystyle y = \int_1^x \frac{1}{t}\, dt - 3$

b) $\displaystyle y = \int_0^x \sec t\, dt + 4$

c) $\displaystyle y = \int_{-1}^x \sec t\, dt + 4$

d) $\displaystyle y = \int_{\pi}^x \frac{1}{t}\, dt - 3$

55. $\displaystyle\frac{dy}{dx} = \frac{1}{x}, \quad y(\pi) = -3$

56. $y' = \sec x, \quad y(-1) = 4$

57. $y' = \sec x, \quad y(0) = 4$

58. $\displaystyle y' = \frac{1}{x}, \quad y(1) = -3$

Express the solutions of the initial value problems in Exercises 59–62 in terms of integrals.

59. $\displaystyle\frac{dy}{dx} = \sec x, \quad y(2) = 3$

60. $\displaystyle\frac{dy}{dx} = \sqrt{1+x^2}, \quad y(1) = -2$

61. $\displaystyle\frac{ds}{dt} = f(t), \quad s(t_0) = s_0$

62. $\displaystyle\frac{dv}{dt} = g(t), \quad v(t_0) = v_0$

Applications

63. *Archimedes' area formula for parabolas.* Archimedes (287–212 B.C.), inventor, military engineer, physicist, and the greatest mathematician of classical times in the western world, discovered

that the area under a parabolic arch is two-thirds the base times the height.

a) Use an integral to find the area under the arch

$$y = 6 - x - x^2, \quad -3 \le x \le 2.$$

b) Find the height of the arch.
c) Show that the area is two-thirds the base b times the height h.
d) Sketch the parabolic arch $y = h - (4h/b^2)x^2$, $-b/2 \le x \le b/2$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x-axis.

64. (*Continuation of Example 6.*)

a) Show by evaluating the integral in the expression

$$\frac{1}{(1/60) - 0} \int_0^{1/60} V_{\max} \sin 120\pi t \, dt$$

that the average value of $V = V_{\max} \sin 120\pi t$ over a full cycle is zero.

b) The circuit that runs your electric stove is rated 240 volts rms. What is the peak value of the allowable voltage?

c) Show that

$$\int_0^{1/60} (V_{\max})^2 \sin^2 120\pi t \, dt = \frac{(V_{\max})^2}{120}.$$

65. *Cost from marginal cost.* The marginal cost of printing a poster when x posters have been printed is

$$\frac{dc}{dx} = \frac{1}{2\sqrt{x}}$$

dollars. Find (a) $c(100) - c(1)$, the cost of printing posters 2–100; (b) $c(400) - c(100)$, the cost of printing posters 101–400.

66. *Revenue from marginal revenue.* Suppose that a company's marginal revenue from the manufacture and sale of egg beaters is

$$\frac{dr}{dx} = 2 - 2/(x + 1)^2,$$

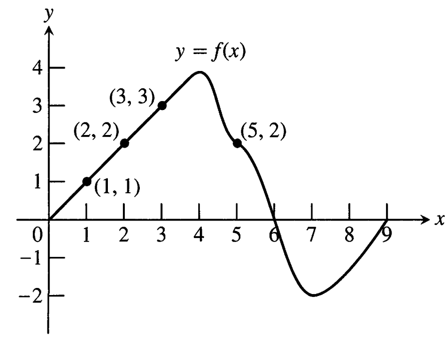
where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand egg beaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

Drawing Conclusions about Motion from Graphs

67. Suppose that f is the differentiable function shown in the accompanying graph and that the position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x)\, dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.

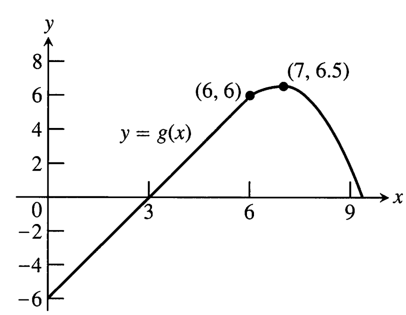


a) What is the particle's velocity at time $t = 5$?
b) Is the acceleration of the particle at time $t = 5$ positive, or negative?
c) What is the particle's position at time $t = 3$?
d) At what time during the first 9 sec does s have its largest value?
e) Approximately when is the acceleration zero?
f) When is the particle moving toward the origin? away from the origin?
g) On which side of the origin does the particle lie at time $t = 9$?

68. Suppose that g is the differentiable function graphed here and that the position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t g(x)\, dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.



a) What is the particle's velocity at $t = 3$?
b) Is the acceleration at time $t = 3$ positive, or negative?
c) What is the particle's position at time $t = 3$?
d) When does the particle pass through the origin?
e) When is the acceleration zero?
f) When is the particle moving away from the origin? toward the origin?
g) On which side of the origin does the particle lie at $t = 9$?

Volumes from Section 4.4

69. (*Continuation of Section 4.4, Example 3.*) The approximating sum for the volume of the solid in Example 3, Section 4.4, was a Riemann sum for an integral. What integral? Evaluate it to find the volume.

70. (*Continuation of Section 4.4, Example 4.*) The approximating sum for the volume of the sphere in Example 4, Section 4.4, was a Riemann sum for an integral. What integral? Evaluate it to find the volume.

71. (*Continuation of Section 4.4, Exercise 15.*) The approximating sums for the volume of water in Exercise 15, Section 4.4, are Riemann sums for an integral. What integral? Evaluate it to find the volume.

72. (*Continuation of Section 4.4, Exercise 17.*) The approximating sums for the volume of the rocket nose cone in Exercise 17, Section 4.4, is a Riemann sum for an integral. What integral? Evaluate it to find the volume.

Theory and Examples

73. Show that if k is a positive constant, then the area between the x-axis and one arch of the curve $y = \sin kx$ is $2/k$.

74. Find

$$\lim_{x\to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{t^4+1}\, dt.$$

75. Suppose $\int_1^x f(t)\, dt = x^2 - 2x + 1$. Find $f(x)$.

76. Find $f(4)$ if $\int_0^x f(t)\, dt = x \cos \pi x$.

77. Find the linearization of

$$f(x) = 2 - \int_2^{x+1} \frac{9}{1+t}\, dt$$

at $x = 1$.

78. Find the linearization of

$$g(x) = 3 + \int_1^{x^2} \sec(t-1)\, dt$$

at $x = -1$.

79. Suppose that f has a positive derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t)\, dt?$$

Give reasons for your answers.

a) g is a differentiable function of x.
b) g is a continuous function of x.
c) The graph of g has a horizontal tangent at $x = 1$.
d) g has a local maximum at $x = 1$.
e) g has a local minimum at $x = 1$.
f) The graph of g has an inflection point at $x = 1$.
g) The graph of dg/dx crosses the x-axis at $x = 1$.

80. Suppose that f has a negative derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$h(x) = \int_0^x f(t)\, dt?$$

Give reasons for your answers.

a) h is a twice-differentiable function of x.
b) h and dh/dx are both continuous.
c) The graph of h has a horizontal tangent at $x = 1$.
d) h has a local maximum at $x = 1$.
e) h has a local minimum at $x = 1$.
f) The graph of h has an inflection point at $x = 1$.
g) The graph of dh/dx crosses the x-axis at $x = 1$.

Grapher Explorations

81. *The Fundamental Theorem.* If f is continuous, we expect

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt$$

to equal $f(x)$, as in the proof of Part 1 of the Fundamental Theorem. For instance, if $f(t) = \cos t$, then

$$\frac{1}{h} \int_x^{x+h} \cos t\, dt = \frac{\sin(x+h) - \sin x}{h}. \qquad (12)$$

The right-hand side of Eq. (12) is the difference quotient for the derivative of the sine, and we expect its limit as $h \to 0$ to be $\cos x$.

Graph $\cos x$ for $-\pi \le x \le 2\pi$. Then, in a different color if possible, graph the right-hand side of Eq. (12) as a function of x for $h = 2$, 1, 0.5, and 0.1. Watch how the latter curves converge to the graph of the cosine as $h \to 0$.

82. Repeat Exercise 81 for $f(t) = 3t^2$. What is

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} 3t^2\, dt = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}?$$

Graph $f(x) = 3x^2$ for $-1 \le x \le 1$. Then graph the quotient $((x+h)^3 - x^3)/h$ as a function of x for $h = 1$, 0.5, 0.2, and 0.1. Watch how the latter curves converge to the graph of $3x^2$ as $h \to 0$.

CAS Explorations and Projects

In Exercises 83–86, let $F(x) = \int_a^x f(t)\, dt$ for the specified function f and interval $[a, b]$. Use a CAS to perform the following steps and answer the questions posed.

a) Plot the functions f and F together over $[a, b]$.
b) Solve the equation $F'(x) = 0$. What can you see to be true about the graphs of f and F at points where $F'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem coupled with information provided by the first derivative? Explain your answer.
c) Over what intervals (approximately) is the function F increasing and decreasing? What is true about f over those intervals?

d) Calculate the derivative f' and plot it together with F. What can you see to be true about the graph of F at points where $f'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem? Explain your answer.

83. $f(x) = x^3 - 4x^2 + 3x, \quad [0, 4]$

84. $f(x) = 2x^4 - 17x^3 + 46x^2 - 43x + 12, \quad \left[0, \dfrac{9}{2}\right]$

85. $f(x) = \sin 2x \cos \dfrac{x}{3}, \quad [0, 2\pi]$

86. $f(x) = x \cos \pi x, \quad [0, 2\pi]$

In Exercises 87–90, let $F(x) = \int_a^{u(x)} f(t)\,dt$ for the specified a, u, and f. Use a CAS to perform the following steps and answer the questions posed.

a) Find the domain of F.

b) Calculate $F'(x)$ and determine its zeros. For what points in its domain is F increasing? decreasing?

c) Calculate $F''(x)$ and determine its zero. Identify the local extrema and the points of inflection of F.

d) Using the information from parts (a)–(c), draw a rough hand-sketch of $y = F(x)$ over its domain. Then graph $F(x)$ on your CAS to support your sketch.

87. $a = 1, \quad u(x) = x^2, \quad f(x) = \sqrt{1 - x^2}$

88. $a = 0, \quad u(x) = x^2, \quad f(x) = \sqrt{1 - x^2}$

89. $a = 0, \quad u(x) = 1 - x, \quad f(x) = x^2 - 2x - 3$

90. $a = 0, \quad u(x) = 1 - x^2, \quad f(x) = x^2 - 2x - 3$

91. Calculate $\dfrac{d}{dx} \displaystyle\int_a^{u(x)} f(t)\,dt$ and check your answer using a CAS.

92. Calculate $\dfrac{d^2}{dx^2} \displaystyle\int_a^{u(x)} f(t)\,dt$ and check your answer using a CAS.

4.8 Substitution in Definite Integrals

There are two methods for evaluating a definite integral by substitution, and they both work well. One is to find the corresponding indefinite integral by substitution and use one of the resulting antiderivatives to evaluate the definite integral by the Fundamental Theorem. The other is to use the following formula.

This formula first appeared in a book written by Isaac Barrow (1630–1677), Newton's teacher and predecessor at Cambridge University.

Substitution in Definite Integrals

THE FORMULA

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \tag{1}$$

HOW TO USE IT

Substitute $u = g(x)$, $du = g'(x)\,dx$, and integrate from $g(a)$ to $g(b)$.

To use the formula, make the same u-substitution you would use to evaluate the corresponding indefinite integral. Then integrate with respect to u from the value u has at $x = a$ to the value u has at $x = b$.

EXAMPLE 1 Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3 + 1}\,dx$.

Solution We have two choices.

Method 1: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du \qquad \text{Let } u = x^3+1,\ du = 3x^2\,dx.$$

$$= \frac{2}{3}u^{3/2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{2}{3}(x^3+1)^{3/2} + C \qquad \text{Replace } u \text{ by } x^3+1.$$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1} \qquad \text{Use the integral just found, with limits of integration for } x.$$

$$= \frac{2}{3}\left[((1)^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral and evaluate the transformed integral with the transformed limits given by Eq. (1).

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

$$= \int_0^2 \sqrt{u}\,du \qquad \begin{aligned}&\text{Let } u = x^3+1,\ du = 3x^2\,dx.\\ &\text{When } x=-1,\ u=(-1)^3+1=0.\\ &\text{When } x=1,\ u=(1)^3+1=2.\end{aligned}$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2 \qquad \text{Evaluate the new definite integral.}$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$ ❑

Which method is better—transforming the integral, integrating, and transforming back to use the original limits of integration, or evaluating the transformed integral with transformed limits? In Example 1, the second method seems easier, but that is not always the case. As a rule, it is best to know both methods and to use whichever one seems better at the time.

Here is another example of evaluating a transformed integral with transformed limits.

EXAMPLE 2

$$\int_{\pi/4}^{\pi/2} \cot\theta\,\csc^2\theta\,d\theta = \int_1^0 u\cdot(-du) \qquad \begin{aligned}&\text{Let } u=\cot\theta,\ du = -\csc^2\theta\,d\theta\\ &\qquad -du = \csc^2\theta\,d\theta.\\ &\text{When } \theta=\pi/4,\ u=\cot(\pi/4)=1.\\ &\text{When } \theta=\pi/2,\ u=\cot(\pi/2)=0.\end{aligned}$$

$$= -\int_1^0 u\,du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$ ❑

Technology *Visualizing Integrals with Elusive Antiderivatives* Many integrable functions, such as the important

$$f(x) = e^{-x^2}$$

from probability theory, *do not* have antiderivatives that can be expressed in terms of elementary functions. Nevertheless, we know the antiderivative of f exists by Part 1 of the Fundamental Theorem of Calculus. Use your graphing utility to visualize the integral function

$$F(x) = \int_0^x e^{-t^2}\,dt.$$

What can you say about $F(x)$? Where is it increasing and decreasing? Where are its extreme values, if any? What can you say about the concavity of its graph?

Exercises 4.8

Evaluating Definite Integrals

Evaluate the integrals in Exercises 1–24.

1. a) $\int_0^3 \sqrt{y+1}\,dy$ **b)** $\int_{-1}^0 \sqrt{y+1}\,dy$

2. a) $\int_0^1 r\sqrt{1-r^2}\,dr$ **b)** $\int_{-1}^1 r\sqrt{1-r^2}\,dr$

3. a) $\int_0^{\pi/4} \tan x \sec^2 x\,dx$ **b)** $\int_{-\pi/4}^0 \tan x \sec^2 x\,dx$

4. a) $\int_0^{\pi} 3\cos^2 x \sin x\,dx$ **b)** $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx$

5. a) $\int_0^1 t^3(1+t^4)^3\,dt$ **b)** $\int_{-1}^1 t^3(1+t^4)^3\,dt$

6. a) $\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\,dt$ **b)** $\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\,dt$

7. a) $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr$ **b)** $\int_0^1 \frac{5r}{(4+r^2)^2}\,dr$

8. a) $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$ **b)** $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\,dv$

9. a) $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$ **b)** $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

10. a) $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx$ **b)** $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\,dx$

11. a) $\int_0^{\pi/6} (1-\cos 3t)\sin 3t\,dt$

b) $\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\,dt$

12. a) $\int_{-\pi/2}^0 \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

b) $\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

13. a) $\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$ **b)** $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\,dz$

14. a) $\int_{-\pi/2}^0 \frac{\sin w}{(3+2\cos w)^2}\,dw$

b) $\int_0^{\pi/2} \frac{\sin w}{(3+2\cos w)^2}\,dw$

15. $\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\,dt$ **16.** $\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$

17. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\,d\theta$

18. $\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)d\theta$

19. $\int_0^{\pi} 5(5-4\cos t)^{1/4}\sin t\,dt$

20. $\int_0^{\pi/4} (1-\sin 2t)^{3/2}\cos 2t\,dt$

21. $\int_0^1 (4y-y^2+4y^3+1)^{-2/3}(12y^2-2y+4)\,dy$

22. $\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\,dy$

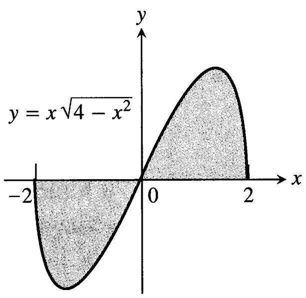
23. $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\,d\theta$

24. $\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt$

Area

Find the total areas of the shaded regions in Exercises 25–28.

25.



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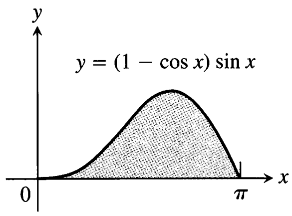

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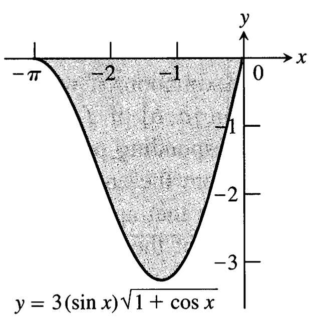

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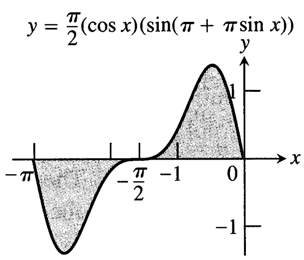


Theory and Examples

29. Suppose that $F(x)$ is an antiderivative of $f(x) = (\sin x)/x$, $x > 0$. Express

$$\int_1^3 \frac{\sin 2x}{x}\,dx$$

in terms of F.

30. Show that if f is continuous, then

$$\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx.$$

31. Suppose that

$$\int_0^1 f(x)\,dx = 3.$$

Find

$$\int_{-1}^0 f(x)\,dx$$

if (a) f is odd, (b) f is even.

32. a) Show that

$$\int_{-a}^{a} h(x)\,dx = \begin{cases} 0 & \text{if } h \text{ is odd} \\ 2\int_0^a h(x)\,dx & \text{if } h \text{ is even.} \end{cases}$$

b) Test the result in part (a) with $h(x) = \sin x$ and with $h(x) = \cos x$, taking $a = \pi/2$ in each case.

33. If f is a continuous function, find the value of the integral

$$I = \int_0^a \frac{f(x)\,dx}{f(x)+f(a-x)}$$

by making the substitution $u = a - x$ and adding the resulting integral to I.

34. By using a substitution, prove that for all positive numbers x and y,

$$\int_x^{xy} \frac{1}{t}\,dt = \int_1^y \frac{1}{t}\,dt.$$

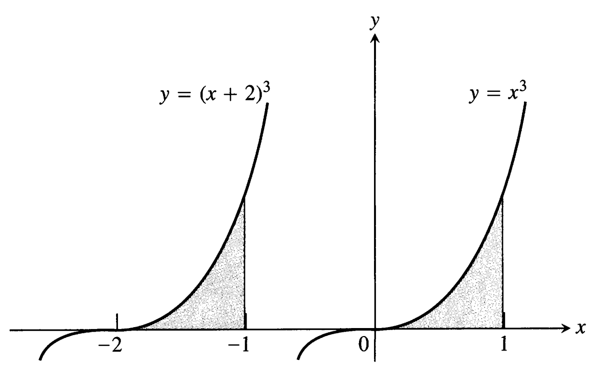
The Shift Property for Definite Integrals

A basic property of definite integrals is their invariance under translation, as expressed by the equation.

$$\int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x+c)\,dx. \tag{2}$$

The equation holds whenever f is integrable and defined for the necessary values of x. For example (Fig. 4.26),

$$\int_{-2}^{-1} (x+2)^3\,dx = \int_0^1 x^3\,dx. \tag{3}$$



4.26 The integrations in Eq. (3). The shaded regions, being congruent, have equal areas.

35. Use a substitution to verify Eq. (2).

36. For each of the following functions, graph $f(x)$ over $[a, b]$ and $f(x+c)$ over $[a-c, b-c]$ to convince yourself that Eq. (2) is reasonable.

a) $f(x) = x^2,\quad a = 0,\quad b = 1,\quad c = 1$
b) $f(x) = \sin x,\quad a = 0,\quad b = \pi,\quad c = \pi/2$
c) $f(x) = \sqrt{x-4},\quad a = 4,\quad b = 8,\quad c = 5$

4.9

Numerical Integration

As we have seen, the ideal way to evaluate a definite integral $\int_a^b f(x)\,dx$ is to find a formula $F(x)$ for one of the antiderivatives of $f(x)$ and calculate the number $F(b) - F(a)$. But some antiderivatives are hard to find, and still others, like the antiderivatives of $(\sin x)/x$ and $\sqrt{1+x^4}$, have no elementary formulas. We do not mean merely that no one has yet succeeded in finding elementary formulas for the antiderivatives of $(\sin x)/x$ and $\sqrt{1+x^4}$. We mean it has been proved that no such formulas exist.

Whatever the reason, when we cannot evaluate a definite integral with an antiderivative, we turn to numerical methods such as the trapezoidal rule and Simpson's rule, described in this section.

The Trapezoidal Rule

When we cannot find a workable antiderivative for a function f that we have to integrate, we partition the interval of integration, replace f by a closely fitting polynomial on each subinterval, integrate the polynomials, and add the results to approximate the integral of f. The higher the degrees of the polynomials for a given partition, the better the results. For a given degree, the finer the partition, the better the results, until we reach limits imposed by round-off and truncation errors.

The length $h = (b - a)/n$ is called the **step size.** It is conventional to use h in this context instead of Δx.

The polynomials do not need to be of high degree to be effective. Even line segments (graphs of polynomials of degree 1) give good approximations if we use enough of them. To see why, suppose we partition the domain $[a, b]$ of f into n subintervals of length $\Delta x = h = (b - a)/n$ and join the corresponding points on the curve with line segments (Fig. 4.27). The vertical lines from the ends of the segments to the partition points create a collection of trapezoids that approximate the region between the curve and the x-axis. We add the areas of the trapezoids, counting area above the x-axis as positive and area below the axis as negative:

$$\begin{aligned} T &= \frac{1}{2}(y_0 + y_1)h + \frac{1}{2}(y_1 + y_2)h + \cdots + \frac{1}{2}(y_{n-2} + y_{n-1})h + \frac{1}{2}(y_{n-1} + y_n)h \\ &= h\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right) \\ &= \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n), \end{aligned}$$

where

$$y_0 = f(a), \quad y_1 = f(x_1), \quad \ldots, \quad y_{n-1} = f(x_{n-1}), \quad y_n = f(b).$$

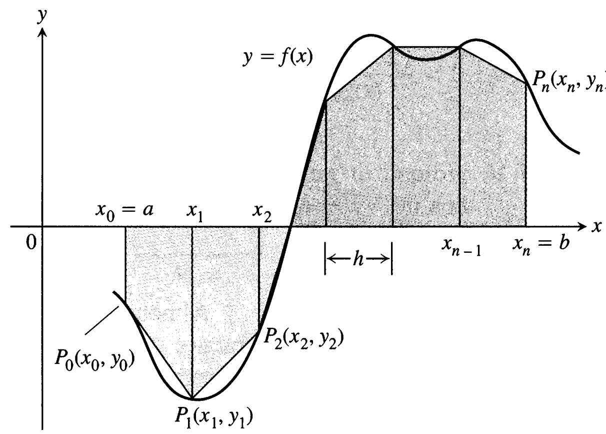
The trapezoidal rule says: Use T to estimate the integral of f from a to b.

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n) \tag{1}$$

(for n subintervals of length $h = (b - a)/n$ and $y_k = f(x_k)$).



4.27 The trapezoidal rule approximates short stretches of the curve $y = f(x)$ with line segments. To estimate the integral of f from a to b, we add the "signed" areas of the trapezoids made by joining the ends of the segments to the x-axis.

Table 4.6

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

EXAMPLE 1 Use the trapezoidal rule with $n = 4$ to estimate

$$\int_1^2 x^2\,dx.$$

Compare the estimate with the exact value of the integral.

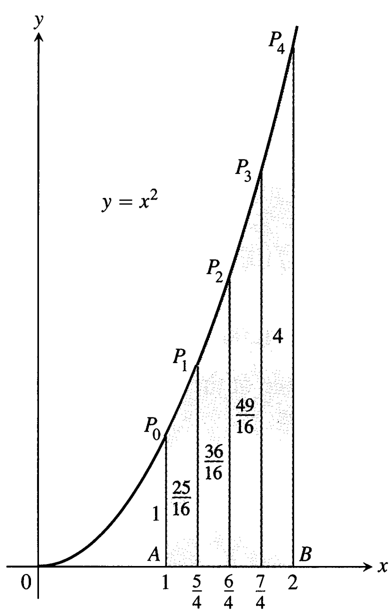
Solution To find the trapezoidal approximation, we divide the interval of integration into four subintervals of equal length and list the values of $y = x^2$ at the endpoints and partition points (see Table 4.6). We then evaluate Eq. (1) with $n = 4$ and $h = 1/4$:

$$\begin{aligned} T &= \frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) \\ &= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right) = \frac{75}{32} \\ &= 2.34375. \end{aligned}$$

The exact value of the integral is

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\bigg]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} = 2.\overline{3}.$$

The approximation is a slight overestimate. Each trapezoid contains slightly more than the corresponding strip under the curve (Fig. 4.28). ❑



4.28 The trapezoidal approximation of the area under the graph of $y = x^2$ from $x = 1$ to $x = 2$ is a slight overestimate.

Controlling the Error in the Trapezoidal Approximation

Pictures suggest that the magnitude of the error

$$E_T = \int_a^b f(x)\,dx - T \tag{2}$$

in the trapezoidal approximation will decrease as the **step size** h decreases, because the trapezoids fit the curve better as their number increases. A theorem from advanced calculus assures us that this will be the case if f has a continuous second derivative.

The Error Estimate for the Trapezoidal Rule

If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then

$$|E_T| \leq \frac{b-a}{12} h^2 M. \tag{3}$$

Although theory tells us there will always be a smallest safe value of M, in practice we can hardly ever find it. Instead, we find the best value we can and go on from there to estimate $|E_T|$. This may seem sloppy, but it works. To make $|E_T|$ small for a given M, we make h small.

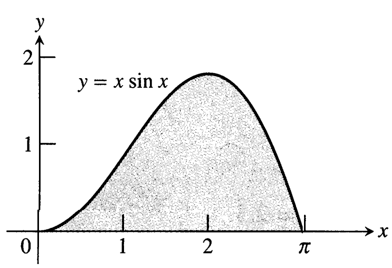
EXAMPLE 2 Find an upper bound for error in the approximation found in Example 1 for the value of

$$\int_1^2 x^2\, dx.$$

Solution We first find an upper bound M for the magnitude of the second derivative of $f(x) = x^2$ on the interval $1 \leq x \leq 2$. Since $f''(x) = 2$ for all x, we may safely take $M = 2$. With $b - a = 1$ and $h = 1/4$, Eq. (3) gives

$$|E_T| \leq \frac{b-a}{12} h^2 M = \frac{1}{12}\left(\frac{1}{4}\right)^2 (2) = \frac{1}{96}.$$

This is precisely what we find when we subtract $T = 75/32$ from $\int_1^2 x^2\, dx = 7/3$, since $|7/3 - 75/32| = |-1/96|$. Here our estimate gave the error's magnitude *exactly,* but this is exceptional. ❑



4.29 Graph of the integrand in Example 3.

EXAMPLE 3 Find an upper bound for the error incurred in estimating

$$\int_0^\pi x \sin x\, dx$$

with the trapezoidal rule with $n = 10$ steps (Fig. 4.29).

Solution With $a = 0$, $b = \pi$, and $h = (b - a)/n = \pi/10$, Eq. (3) gives

$$|E_T| \leq \frac{b-a}{12} h^2 M = \frac{\pi}{12}\left(\frac{\pi}{10}\right)^2 M = \frac{\pi^3}{1200} M.$$

The number M can be any upper bound for the magnitude of the second derivative of $f(x) = x \sin x$ on $[0, \pi]$. A routine calculation gives

$$f''(x) = 2 \cos x - x \sin x,$$

so

$$\begin{aligned} |f''(x)| &= |2 \cos x - x \sin x| \\ &\leq 2|\cos x| + |x||\sin x| && \text{Triangle inequality: } |a+b| \leq |a| + |b| \\ &\leq 2 \cdot 1 + \pi \cdot 1 = 2 + \pi. && |\cos x| \text{ and } |\sin x| \text{ never exceed 1, and } 0 \leq x \leq \pi. \end{aligned}$$

We can safely take $M = 2 + \pi$. Therefore,

$$|E_T| \le \frac{\pi^3}{1200} M = \frac{\pi^3(2+\pi)}{1200} < 0.133. \qquad \text{Rounded up to be safe}$$

The absolute error is no greater than 0.133.

For greater accuracy, we would not try to improve M but would take more steps. With $n = 100$ steps, for example, $h = \pi/100$ and

$$|E_T| \le \frac{\pi}{12}\left(\frac{\pi}{100}\right)^2 M = \frac{\pi^3(2+\pi)}{120{,}000} < 0.00133 = 1.33 \times 10^{-3}.$$ ❑

EXAMPLE 4 As we will see in Chapter 6, the value of ln 2 can be calculated from the integral

$$\ln 2 = \int_1^2 \frac{1}{x}\, dx.$$

How many subintervals (steps) should be used in the trapezoidal rule to approximate the integral with an error of magnitude less than 10^{-4}?

Solution To determine n, the number of subintervals, we use Eq. (3) with

$$b - a = 2 - 1 = 1, \qquad h = \frac{b-a}{n} = \frac{1}{n},$$

$$f''(x) = \frac{d^2}{dx^2}(x^{-1}) = 2x^{-3} = \frac{2}{x^3}.$$

Then

$$\left|E_T\right| \le \frac{b-a}{12} h^2 \max\left|f''(x)\right| = \frac{1}{12}\left(\frac{1}{n}\right)^2 \max\left|\frac{2}{x^3}\right|,$$

where max refers to the interval [1, 2].

This is one of the rare cases where we can find the exact value of $\max|f''|$. On [1, 2], $y = 2/x^3$ decreases steadily from a maximum of $y = 2$ to a minimum of $y = 1/4$. Therefore,

$$|E_T| \le \frac{1}{12}\left(\frac{1}{n}\right)^2 \cdot 2 = \frac{1}{6n^2}.$$

The error's absolute value will therefore be less than 10^{-4} if

$$\frac{1}{6n^2} < 10^{-4},$$

$$\frac{10^4}{6} < n^2, \qquad \text{Multiply both sides by } 10^4 n^2.$$

$$\frac{100}{\sqrt{6}} < |n|, \qquad \text{Square roots of both sides}$$

$$\frac{100}{\sqrt{6}} < n, \qquad n \text{ is positive.}$$

$$40.83 < n. \qquad \text{Rounded up, to be safe}$$

Simpson's one-third rule

The idea of using the formula

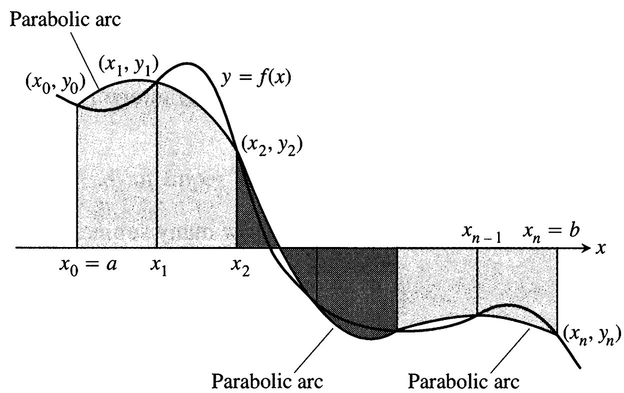
$$A = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

to estimate the area under a curve is known as Simpson's one-third rule. But the rule was in use long before Thomas Simpson (1720–1761) was born. It is another of history's beautiful quirks that one of the ablest mathematicians of eighteenth-century England is remembered not for his successful texts and his contributions to mathematical analysis but for a rule that was never his, that he never laid claim to, and that bears his name only because he happened to mention it in a book he wrote.

The first integer beyond 40.83 is $n = 41$. With $n = 41$ subintervals we can guarantee calculating $\ln 2$ with an error of magnitude less than 10^{-4}. Any larger n will work, too. ❑

Simpson's Rule

Simpson's rule for approximating $\int_a^b f(x)\,dx$ is based on approximating f with quadratic polynomials instead of linear polynomials. We approximate the graph with parabolic arcs instead of line segments (Fig. 4.30).

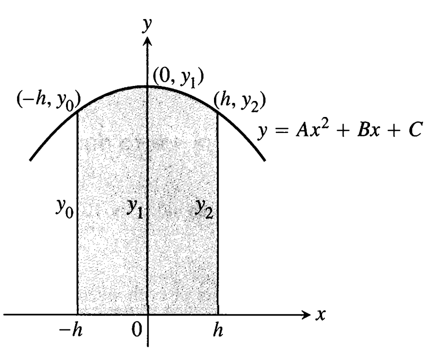


4.30 Simpson's rule approximates short stretches of curve with parabolic arcs.

The integral of the quadratic polynomial $y = Ax^2 + Bx + C$ in Fig. 4.31 from $x = -h$ to $x = h$ is

$$\int_{-h}^{h} (Ax^2 + Bx + C)\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2) \tag{4}$$

(Appendix 4). Simpson's rule follows from partitioning $[a, b]$ into an even number of subintervals of equal length h, applying Eq. (4) to successive interval pairs, and adding the results.



4.31 By integrating from $-h$ to h, we find the shaded area to be

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

Simpson's Rule

To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n). \tag{5}$$

The y's are the values of f at the partition points

$$x_0 = a,\ x_1 = a + h,\ x_2 = a + 2h,\ \ldots,\ x_{n-1} = a + (n-1)h,\ x_n = b.$$

The number n is even, and $h = (b - a)/n$.

Error Control for Simpson's Rule

The magnitude of the Simpson's rule error,

$$E_S = \int_a^b f(x)\,dx - S, \tag{6}$$

decreases with the step size, as we would expect from our experience with the trapezoidal rule. The inequality for controlling the Simpson's rule error, however, assumes f to have a continuous fourth derivative instead of merely a continuous second derivative. The formula, once again from advanced calculus, is this:

The Error Estimate for Simpson's Rule

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then

$$|E_S| \leq \frac{b-a}{180} h^4 M. \tag{7}$$

As with the trapezoidal rule, we can almost never find the smallest possible value of M. We just find the best value we can and go on from there to estimate $|E_S|$.

EXAMPLE 5 Use Simpson's rule with $n = 4$ to approximate

$$\int_0^1 5x^4 \, dx.$$

What estimate does Eq. (7) give for the error in the approximation?

Solution Again we have chosen an integral whose exact value we can calculate directly:

$$\int_0^1 5x^4 \, dx = x^5 \Big]_0^1 = 1.$$

To find the Simpson approximation, we partition the interval of integration into four subintervals and evaluate $f(x) = 5x^4$ at the partition points (Table 4.7).

Table 4.7

x	$y = 5x^4$
0	0
$\frac{1}{4}$	$\frac{5}{256}$
$\frac{2}{4}$	$\frac{80}{256}$
$\frac{3}{4}$	$\frac{405}{256}$
1	5

We then evaluate Eq. (5) with $n = 4$ and $h = 1/4$:

$$\begin{aligned} S &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \\ &= \frac{1}{12}\left(0 + 4\left(\frac{5}{256}\right) + 2\left(\frac{80}{256}\right) + 4\left(\frac{405}{256}\right) + 5\right) \approx 1.00260. \end{aligned}$$

To estimate the error, we first find an upper bound M for the magnitude of the fourth derivative of $f(x) = 5x^4$ on the interval $0 \leq x \leq 1$. Since the fourth derivative has the constant value $f^{(4)}(x) = 120$, we may safely take $M = 120$. With $b - a = 1$ and $h = 1/4$, Eq. (7) gives

$$|E_S| \leq \frac{b-a}{180} h^4 M = \frac{1}{180}\left(\frac{1}{4}\right)^4 (120) = \frac{1}{384} < 0.00261.$$ ❑

Which Rule Gives Better Results?

The answer lies in the error-control formulas

$$|E_T| \leq \frac{b-a}{12} h^2 M, \qquad |E_S| \leq \frac{b-a}{180} h^4 M.$$

Trapezoidal vs. Simpson

If Simpson's rule is more accurate, why bother with the trapezoidal rule? There are two reasons. First, the trapezoidal rule is useful in a number of specific applications because it leads to much simpler expressions. Second, the trapezoidal rule is the basis for *Rhomberg integration,* one of the most satisfactory machine methods when high precision is required.

The M's of course mean different things, the first being an upper bound on $|f''|$ and the second an upper bound on $|f^{(4)}|$. But there is more. The factor $(b-a)/180$ in the Simpson formula is one-fifteenth of the factor $(b-a)/12$ in the trapezoidal formula. More important still, the Simpson formula has an h^4 while the trapezoidal formula has only an h^2. If h is one-tenth, then h^2 is one-hundredth but h^4 is only one ten-thousandth. If both M's are 1, for example, and $b-a=1$, then, with $h=1/10$,

$$|E_T| \le \frac{1}{12}\left(\frac{1}{10}\right)^2 \cdot 1 = \frac{1}{1200},$$

while

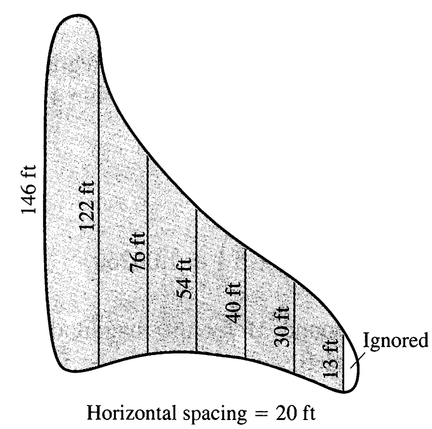
$$|E_S| \le \frac{1}{180}\left(\frac{1}{10}\right)^4 \cdot 1 = \frac{1}{1{,}800{,}000} = \frac{1}{1500} \cdot \frac{1}{1200}.$$

For roughly the same amount of computational effort, we get better accuracy with Simpson's rule—at least in this case.

The h^2 versus h^4 is the key. If h is less than 1, then h^4 can be significantly smaller than h^2. On the other hand, if h equals 1, there is no difference between h^2 and h^4. If h is greater than 1, the value of h^4 may be significantly larger than the value of h^2. In the latter two cases, the error-control formulas offer little help. We have to go back to the geometry of the curve $y=f(x)$ to see whether trapezoids or parabolas, if either, are going to give the results we want.

Working with Numerical Data

The next example shows how we can use Simpson's rule to estimate the integral of a function from values measured in the laboratory or in the field even when we have no formula for the function. We can use the trapezoidal rule the same way.



4.32 The swamp in Example 6.

EXAMPLE 6 A town wants to drain and fill a small polluted swamp (Fig. 4.32). The swamp averages 5 ft deep. About how many cubic yards of dirt will it take to fill the area after the swamp is drained?

Solution To calculate the volume of the swamp, we estimate the surface area and multiply by 5. To estimate the area, we use Simpson's rule with $h=20$ ft and the y's equal to the distances measured across the swamp, as shown in Fig. 4.32.

$$\begin{aligned} S &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6) \\ &= \frac{20}{3}(146 + 488 + 152 + 216 + 80 + 120 + 13) = 8100. \end{aligned}$$

The volume is about $(8100)(5) = 40{,}500$ ft^3 or 1500 yd^3. ❑

Round-off Errors

Although decreasing the step size h reduces the error in the Simpson and trapezoidal approximations in theory, it may fail to do so in practice. When h is very small, say $h=10^{-5}$, the round-off errors in the arithmetic required to evaluate S and T may accumulate to such an extent that the error formulas no longer describe what is going on. Shrinking h below a certain size can actually make things worse. While this will not be an issue in the present book, you should consult a text on numerical analysis for alternative methods if you are having problems with round-off.

Exercises 4.9

Estimating Integrals

The instructions for the integrals in Exercises 1–10 have two parts, one for the trapezoidal rule and one for Simpson's rule.

I. *Using the trapezoidal rule*

- **a)** Estimate the integral with $n = 4$ steps and use Eq. (3) to find an upper bound for $|E_T|$.
- **b)** Evaluate the integral directly, and use Eq. (2) to find $|E_T|$.
- **c)** CALCULATOR Use the formula $(|E_T|/\text{true value}) \times 100$ to express $|E_T|$ as a percentage of the integral's true value.

II. *Using Simpson's rule*

- **a)** Estimate the integral with $n = 4$ steps and use Eq. (7) to find an upper bound for $|E_S|$.
- **b)** Evaluate the integral directly, and use Eq. (6) to find $|E_S|$.
- **c)** CALCULATOR Use the formula $(|E_S|/\text{true value}) \times 100$ to express $|E_S|$ as a percentage of the integral's true value.

1. $\int_1^2 x\,dx$

2. $\int_1^3 (2x-1)\,dx$

3. $\int_{-1}^1 (x^2+1)\,dx$

4. $\int_{-2}^0 (x^2-1)\,dx$

5. $\int_0^2 (t^3+t)\,dt$

6. $\int_{-1}^1 (t^3+1)\,dt$

7. $\int_1^2 \frac{1}{s^2}\,ds$

8. $\int_2^4 \frac{1}{(s-1)^2}\,ds$

9. $\int_0^{\pi} \sin t\,dt$

10. $\int_0^1 \sin \pi t\,dt$

In Exercises 11–14, use the tabulated values of the integrand to estimate the integral with (a) the trapezoidal rule and (b) Simpson's rule with $n = 8$ steps. Round your answers to 5 decimal places. Then (c) find the integral's exact value and the approximation error E_T or E_S, as appropriate, from Eqs. (2) and (6).

11. $\int_0^1 x\sqrt{1-x^2}\,dx$

x	$x\sqrt{1-x^2}$
0	0.0
0.125	0.12402
0.25	0.24206
0.375	0.34763
0.5	0.43301
0.625	0.48789
0 75	0.49608
0.875	0.42361
1.0	0

12. $\int_0^3 \frac{\theta}{\sqrt{16+\theta^2}}\,d\theta$

θ	$\theta/\sqrt{16+\theta^2}$
0	0.0
0.375	0.09334
0.75	0.18429
1.125	0.27075
1.5	0.35112
1.875	0.42443
2.25	0.49026
2.625	0.58466
3.0	0.6

13. $\int_{-\pi/2}^{\pi/2} \frac{3\cos t}{(2+\sin t)^2}\,dt$

t	$(3\cos t)/(2+\sin t)^2$
−1.57080	0.0
−1.17810	0.99138
−0.78540	1.26906
−0.39270	1.05961
0	0.75
0.39270	0.48821
0.78540	0.28946
1.17810	0.13429
1.57080	0

14. $\int_{\pi/4}^{\pi/2} (\csc^2 y)\sqrt{\cot y}\,dy$

y	$(\csc^2 y)\sqrt{\cot y}$
0.78540	2.0
0.88357	1.51606
0.98175	1.18237
1.07992	0.93998
1.17810	0.75402
1.27627	0.60145
1.37445	0.46364
1.47262	0.31688
1.57080	0

The Minimum Number of Subintervals

In Exercises 15–26, use Eqs. (3) and (7), as appropriate, to estimate the minimum number of subintervals needed to approximate the integrals with an error of magnitude less than 10^{-4} by (a) the trapezoidal rule and (b) Simpson's rule. (The integrals in Exercises 15–22 are the integrals from Exercises 1–8.)

15. $\int_1^2 x\,dx$

16. $\int_1^3 (2x-1)\,dx$

17. $\int_{-1}^1 (x^2+1)\,dx$

18. $\int_{-2}^0 (x^2-1)\,dx$

19. $\int_0^2 (t^3 + t)\,dt$

20. $\int_{-1}^1 (t^3 + 1)\,dt$

21. $\int_1^2 \frac{1}{s^2}\,ds$

22. $\int_2^4 \frac{1}{(s-1)^2}\,ds$

23. $\int_0^3 \sqrt{x+1}\,dx$

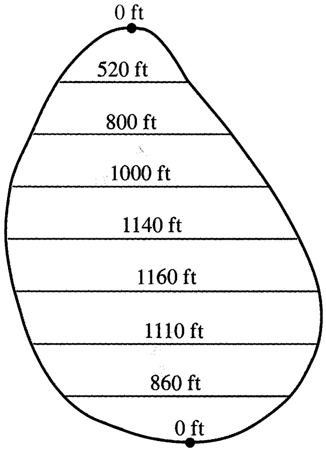
24. $\int_0^3 \frac{1}{\sqrt{x+1}}\,dx$

25. $\int_0^2 \sin(x+1)\,dx$

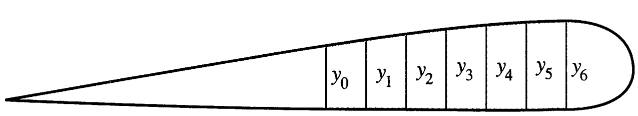
26. $\int_{-1}^1 \cos(x+\pi)\,dx$

Applications

27. As the fish-and-game warden of your township, you are responsible for stocking the town pond with fish before fishing season. The average depth of the pond is 20 ft. You plan to start the season with one fish per 1000 ft^3. You intend to have at least 25% of the opening day's fish population left at the end of the season. What is the maximum number of licenses the town can sell if the average seasonal catch is 20 fish per license?

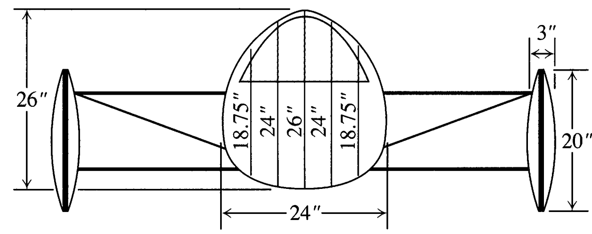


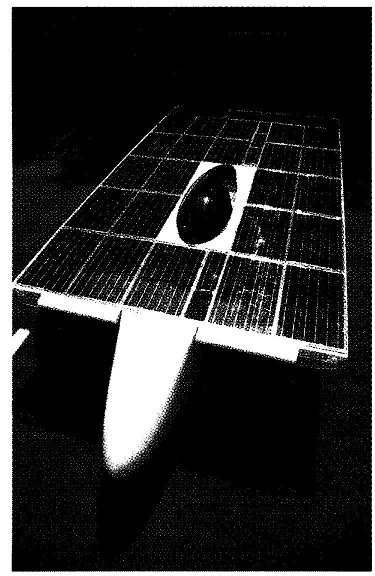
28. CALCULATOR The design of a new airplane requires a gasoline tank of constant cross-section area in each wing. A scale drawing of a cross section is shown here. The tank must hold 5000 lb of gasoline, which has a density of 42 lb/ft^3. Estimate the length of the tank.



$y_0 = 1.5$ ft, $y_1 = 1.6$ ft, $y_2 = 1.8$ ft, $y_3 = 1.9$ ft,
$y_4 = 2.0$ ft, $y_5 = y_6 = 2.1$ ft Horizontal spacing = 1 ft

29. CALCULATOR A vehicle's aerodynamic drag is determined in part by its cross-section area and, all other things being equal, engineers try to make this area as small as possible. Use Simpson's rule to estimate the cross-section area of James Worden's solar-powered Solectria car at MIT (Fig. 4.33).





4.33 Solectria cars are produced by Selectron Corp., Arlington, MA (Exercise 29).

30. The accompanying table shows time-to-speed data for a 1994 Ford Mustang Cobra accelerating from rest to 130 mph. How far had the Mustang traveled by the time it reached this speed?

Speed change	Seconds
Zero to 30 mph	2.2
40 mph	3.2
50 mph	4.5
60 mph	5.9
70 mph	7.8
80 mph	10.2
90 mph	12.7
100 mph	16.0
110 mph	20.6
120 mph	26.2
130 mph	37.1

Source: *Car and Driver,* April 1994.

Theory and Examples

31. *Polynomials of low degree.* The magnitude of the error in the trapezoidal approximation of $\int_a^b f(x)\,dx$ is

$$|E_T| = \frac{b-a}{12} h^2 \,|f''(c)|,$$

where c is some point (usually unidentified) in (a, b). If f is a linear function of x, then $f''(c) = 0$, so $E_T = 0$ and T gives the exact value of the integral for any value of h. This is no surprise, really, for if f is linear, the line segments approximating the graph of f fit the graph exactly. The surprise comes with Simpson's rule. The magnitude of the error in Simpson's rule is

$$|E_S| = \frac{b-a}{180} h^4 \,|f^{(4)}(c)|,$$

where once again c lies in (a, b). If f is a polynomial of degree less than 4, then $f^{(4)} = 0$ no matter what c is, so $E_S = 0$ and S gives the integral's exact value—even if we use only two steps. As a case in point, use Simpson's rule with $n = 2$ to estimate

$$\int_0^2 x^3\,dx.$$

Compare your answer with the integral's exact value.

32. *Usable values of the sine-integral function.* The sine-integral function,

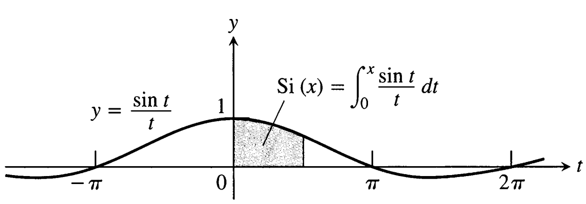
$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt, \qquad \text{“Sine integral of } x\text{”}$$

is one of the many functions in engineering whose formulas cannot be simplified. There is no elementary formula for the antiderivative of $(\sin t)/t$. The values of $\mathrm{Si}(x)$, however, are readily estimated by numerical integration.

Although the notation does not show it explicitly, the function being integrated is

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0, \end{cases}$$

the continuous extension of $(\sin t)/t$ to the interval $[0, x]$. The function has derivatives of all orders at every point of its domain. Its graph is smooth (Fig. 4.34) and you can expect good results from Simpson's rule.



4.34 The continuous extension of $y = (\sin t)/t$. The sine-integral function Si(x) is the subject of Exercise 32.

a) Use the fact that $|f^{(4)}| \le 1$ on $[0, \pi/2]$ to give an upper bound for the error that will occur if

$$\mathrm{Si}\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{\sin t}{t}\,dt$$

is estimated by Simpson's rule with $n = 4$.

b) Estimate $\mathrm{Si}(\pi/2)$ by Simpson's rule with $n = 4$.

c) Express the error bound you found in (a) as a percentage of the value you found in (b).

33. (*Continuation of Example 3.*) The error bounds in Eqs. (3) and (7) are “worst case” estimates, and the trapezoidal and Simpson rules are often more accurate than the bounds suggest. The trapezoidal rule estimate of

$$\int_0^\pi x \sin x\,dx$$

in Example 3 is a case in point.

a) Use the trapezoidal rule with $n = 10$ to approximate the value of the integral. The table to the right gives the necessary y-values.

x	$x \sin x$
0	0
$(0.1)\pi$	0.09708
$(0.2)\pi$	0.36932
$(0.3)\pi$	0.76248
$(0.4)\pi$	1.19513
$(0.5)\pi$	1.57080
$(0.6)\pi$	1.79270
$(0.7)\pi$	1.77912
$(0.8)\pi$	1.47727
$(0.9)\pi$	0.87372
π	0

b) Find the magnitude of the difference between π, the integral's value, and your approximation in (a). You will find the difference to be considerably less than the upper bound of 0.133 calculated with $n = 10$ in Example 3.

c) GRAPHER The upper bound of 0.133 for $|E_T|$ in Example 3 could have been improved somewhat by having a better bound for

$$|f''(x)| = |2 \cos x - x \sin x|$$

on $[0, \pi]$. The upper bound we used was $2 + \pi$. Graph f'' over $[0, \pi]$ and use TRACE or ZOOM to improve this upper bound.

Use the improved upper bound as M in Eq. (3) to make an improved estimate of $|E_T|$. Notice that the trapezoidal rule approximation in (a) is also better than this improved estimate would suggest.

34. CALCULATOR (*Continuation of Exercise 33*)

a) GRAPHER Show that the fourth derivative of $f(x) = x \sin x$ is

$$f^{(4)}(x) = -4 \cos x + x \sin x.$$

Use TRACE or ZOOM to find an upper bound M for the values of $|f^{(4)}|$ on $[0, \pi]$.

b) Use the value of M from (a) together with Eq. (7) to obtain an upper bound for the magnitude of the error in estimating the value of

$$\int_0^\pi x \sin x\,dx$$

with Simpson's rule with $n = 10$ steps.

c) Use the data in the table in Exercise 33 to estimate $\int_0^{\pi} x \sin x \, dx$ with Simpson's rule with $n = 10$ steps.

d) To 6 decimal places, find the magnitude of the difference between your estimate in (c) and the integral's true value, π. You will find the error estimate obtained in (b) to be quite good.

You are planning to use Simpson's rule to estimate the values of the integrals in Exercises 35 and 36. Before proceeding, you turn to Eq. (7) to determine the step size h needed to assure the accuracy you want. What happens? Can this be avoided by using the trapezoidal rule and Eq. (3) instead? Give reasons for your answers.

35. $\displaystyle\int_0^4 x^{3/2}\, dx$

36. $\displaystyle\int_0^1 x^{5/2}\, dx$

Numerical Integrator

As we mentioned at the beginning of the section, the definite integrals of many continuous functions cannot be evaluated with the Fundamental Theorem of Calculus because their antiderivatives lack elementary formulas. Numerical integration offers a practical way to estimate the values of these so-called *nonelementary integrals.* If your calculator or computer has a numerical integration routine, try it on the integrals in Exercises 37–40.

37. $\displaystyle\int_0^1 \sqrt{1+x^4}\, dx$ — A nonelementary integral that came up in Newton's research

38. $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\, dx$ — The integral from Exercise 32. To avoid division by zero, you may have to start the integration at a small positive number like 10^{-6} instead of 0.

39. $\displaystyle\int_0^{\pi/2} \sin(x^2)\, dx$ — An integral associated with the diffraction of light

40. $\displaystyle\int_0^{\pi/2} 40\sqrt{1-0.64\cos^2 t}\, dt$ — The length of the ellipse $(x^2/25) + (y^2/9) = 1$

Chapter 4 Questions to Guide Your Review

1. Can a function have more than one antiderivative? If so, how are the antiderivatives related? Explain.
2. What is an indefinite integral? How do you evaluate one? What general formulas do you know for evaluating indefinite integrals?
3. How can you sometimes use a trigonometric identity to transform an unfamiliar intregal into one you know how to evaluate?
4. How can you sometimes solve a differential equation of the form $dy/dx = f(x)$?
5. What is an initial value problem? How do you solve one? Give an example.
6. If you know the acceleration of a body moving along a coordinate line as a function of time, what more do you need to know to find the body's position function? Give an example.
7. How do you sketch the solutions of a differential equation $dy/dx = f(x)$ when you do not know an antiderivative of f? How would you sketch the solution of an initial value problem $dy/dx = f(x)$, $y(x_0) = y_0$ under these circumstances?
8. How can you sometimes evaluate indefinite integrals by substitution? Give examples.
9. How can you sometimes estimate quantities like distance traveled, area, volume, and average value with finite sums? Why might you want to do so?
10. What is sigma notation? What advantage does it offer? Give examples.
11. What rules are available for calculating with sigma notation?
12. What is a Riemann sum? Why might you want to consider such a sum?
13. What is the norm of a partition of a closed interval?
14. What is the definite integral of a function f over a closed interval $[a, b]$? When can you be sure it exists?
15. What is the relation between definite integrals and area? Describe some other interpretations of definite integrals.
16. Describe the rules for working with definite integrals (Table 4.5). Give examples.
17. What is the average value of an integrable function over a closed interval? Must the function assume its average value? Explain.
18. What does a function's average value have to do with sampling a function's values?
19. What is the Fundamental Theorem of Calculus? Why is it so important? Illustrate each part of the theorem with an example.
20. How does the Fundamental Theorem provide a solution to the initial value problem $dy/dx = f(x)$, $y(x_0) = y_0$, when f is continuous?
21. How does the method of substitution work for definite integrals? Give examples.
22. How is integration by substitution related to the Chain Rule?
23. You are collaborating to produce a short "how-to" manual for

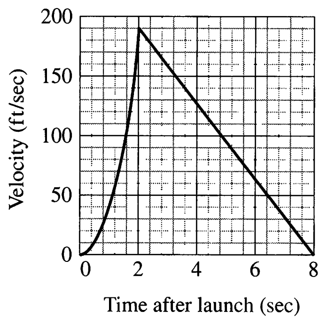
numerical integration, and you are writing about the trapezoidal rule. (a) What would you say about the rule itself and how to use it? how to achieve accuracy? (b) What would you say if you were writing about Simpson's rule instead?

24. How would you compare the relative merits of Simpson's rule and the trapezoidal rule?

CHAPTER 4 PRACTICE EXERCISES

Finite Sums and Estimates

1. The accompanying figure shows the graph of the velocity (ft/sec) of a model rocket for the first 8 sec after launch. The rocket accelerated straight up for the first 2 sec and then coasted to reach its maximum height at $t = 8$ sec.

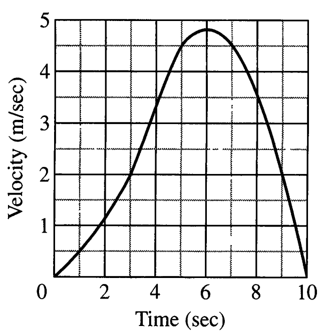


a) Assuming that the rocket was launched from ground level, about how high did it go? (This is the rocket in Section 2.3, Exercise 19, but you do not need to do Exercise 19 to do the exercise here.)

b) Sketch a graph of the rocket's height aboveground as a function of time for $0 \le t \le 8$.

2. a) The accompanying figure shows the velocity (m/sec) of a body moving along the s-axis during the time interval from $t = 0$ to $t = 10$ sec. About how far did the body travel during those 10 sec?

b) Sketch a graph of s as a function of t for $0 \le t \le 10$ assuming $s(0) = 0$.



3. Suppose that $\sum_{k=1}^{10} a_k = -2$ and $\sum_{k=1}^{10} b_k = 25$. Find the value of

a) $\displaystyle\sum_{k=1}^{10} \frac{a_k}{4}$ **b)** $\displaystyle\sum_{k=1}^{10} (b_k - 3a_k)$

c) $\displaystyle\sum_{k=1}^{10} (a_k + b_k - 1)$ **d)** $\displaystyle\sum_{k=1}^{10} \left(\frac{5}{2} - b_k\right)$

4. Suppose that $\sum_{k=1}^{20} a_k = 0$ and $\sum_{k=1}^{20} b_k = 7$. Find the values of

a) $\displaystyle\sum_{k=1}^{20} 3a_k$ **b)** $\displaystyle\sum_{k=1}^{20} (a_k + b_k)$

c) $\displaystyle\sum_{k=1}^{20} \left(\frac{1}{2} - \frac{2b_k}{7}\right)$ **d)** $\displaystyle\sum_{k=1}^{20} (a_k - 2)$

Definite Integrals

In Exercises 5–8, express each limit as a definite integral. Then evaluate the integral to find the value of the limit. In each case, P is a partition of the given interval and the numbers c_k are chosen from the subintervals of P.

5. $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} (2c_k - 1)^{-1/2}\, \Delta x_k$, where P is a partition of $[1, 5]$

6. $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k(c_k{}^2 - 1)^{1/3}\, \Delta x_k$, where P is a partition of $[1, 3]$

7. $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} \left(\cos\left(\frac{c_k}{2}\right)\right) \Delta x_k$, where P is a partition of $[-\pi, 0]$

8. $\displaystyle\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\sin c_k)(\cos c_k)\, \Delta x_k$, where P is a partition of $[0, \pi/2]$

9. If $\int_{-2}^{2} 3f(x)\,dx = 12$, $\int_{-2}^{5} f(x)\,dx = 6$, and $\int_{-2}^{5} g(x)\,dx = 2$, find the values of the following.

a) $\displaystyle\int_{-2}^{2} f(x)\,dx$ **b)** $\displaystyle\int_{2}^{5} f(x)\,dx$

c) $\displaystyle\int_{5}^{-2} g(x)\,dx$ **d)** $\displaystyle\int_{-2}^{5} (-\pi g(x))\,dx$

e) $\displaystyle\int_{-2}^{5} \left(\frac{f(x) + g(x)}{5}\right) dx$

10. If $\int_0^2 f(x)\,dx = \pi$, $\int_0^2 7g(x)\,dx = 7$, and $\int_0^1 g(x)\,dx = 2$, find the values of the following.

a) $\displaystyle\int_0^2 g(x)\,dx$

b) $\displaystyle\int_1^2 g(x)\,dx$

c) $\displaystyle\int_2^0 f(x)\,dx$

d) $\displaystyle\int_0^2 \sqrt{2}\,f(x)\,dx$

e) $\displaystyle\int_0^2 (g(x) - 3f(x))\,dx$

Area

In Exercises 11–14, find the total area of the region between the graph of f and the x-axis.

11. $f(x) = x^2 - 4x + 3, \quad 0 \le x \le 3$

12. $f(x) = 1 - (x^2/4), \quad -2 \le x \le 3$

13. $f(x) = 5 - 5x^{2/3}, \quad -1 \le x \le 8$

14. $f(x) = 1 - \sqrt{x}, \quad 0 \le x \le 4$

Initial Value Problems

Solve the initial value problems in Exercises 15–18.

15. $\dfrac{dy}{dx} = \dfrac{x^2+1}{x^2}, \quad y(1) = -1$

16. $\dfrac{dy}{dx} = \left(x + \dfrac{1}{x}\right)^2, \quad y(1) = 1$

17. $\dfrac{d^2r}{dt^2} = 15\sqrt{t} + \dfrac{3}{\sqrt{t}}; \quad r'(1) = 8, \quad r(1) = 0$

18. $\dfrac{d^3r}{dt^3} = -\cos t; \quad r''(0) = r'(0) = 0, \quad r(0) = -1$

19. Show that $y = x^2 + \displaystyle\int_1^x \frac{1}{t}\,dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = 2 - \frac{1}{x^2}; \quad y'(1) = 3, \quad y(1) = 1.$$

20. Show that $y = \int_0^x (1 + 2\sqrt{\sec t}\,)\,dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = \sqrt{\sec x}\tan x; \quad y'(0) = 3, \quad y(0) = 0.$$

Express the solutions of the initial value problems in Exercises 21 and 22 in terms of integrals.

21. $\dfrac{dy}{dx} = \dfrac{\sin x}{x}, \quad y(5) = -3$

22. $\dfrac{dy}{dx} = \sqrt{2 - \sin^2 x}, \quad y(-1) = 2$

Evaluating Indefinite Integrals

Evaluate the integrals in Exercises 23–44.

23. $\displaystyle\int (x^3 + 5x - 7)\,dx$

24. $\displaystyle\int \left(8t^3 - \frac{t^2}{2} + t\right) dt$

25. $\displaystyle\int \left(3\sqrt{t} + \frac{4}{t^2}\right) dt$

26. $\displaystyle\int \left(\frac{1}{2\sqrt{t}} - \frac{3}{t^4}\right) dt$

27. $\displaystyle\int \frac{r\,dr}{(r^2+5)^2}$

28. $\displaystyle\int \frac{6r^2 dr}{(r^3 - \sqrt{2}\,)^3}$

29. $\displaystyle\int 3\theta\sqrt{2 - \theta^2}\,d\theta$

30. $\displaystyle\int \frac{\theta^2}{9\sqrt{73 + \theta^3}}\,d\theta$

31. $\displaystyle\int x^3(1 + x^4)^{-1/4}\,dx$

32. $\displaystyle\int (2 - x)^{3/5}\,dx$

33. $\displaystyle\int \sec^2 \frac{s}{10}\,ds$

34. $\displaystyle\int \csc^2 \pi s\,ds$

35. $\displaystyle\int \csc\sqrt{2}\,\theta\cot\sqrt{2}\,\theta\,d\theta$

36. $\displaystyle\int \sec\frac{\theta}{3}\tan\frac{\theta}{3}\,d\theta$

37. $\displaystyle\int \sin^2 \frac{x}{4}\,dx$

38. $\displaystyle\int \cos^2 \frac{x}{2}\,dx$

39. $\displaystyle\int 2(\cos x)^{-1/2}\sin x\,dx$

40. $\displaystyle\int (\tan x)^{-3/2}\sec^2 x\,dx$

41. $\displaystyle\int (2\theta + 1 + 2\cos(2\theta + 1))\,d\theta$

42. $\displaystyle\int \left(\frac{1}{\sqrt{2\theta - \pi}} + 2\sec^2(2\theta - \pi)\right) d\theta$

43. $\displaystyle\int \left(t - \frac{2}{t}\right)\left(t + \frac{2}{t}\right) dt$

44. $\displaystyle\int \frac{(t+1)^2 - 1}{t^4}\,dt$

Evaluating Definite Integrals

Evaluate the integrals in Exercises 45–70.

45. $\displaystyle\int_{-1}^1 (3x^2 - 4x + 7)\,dx$

46. $\displaystyle\int_0^1 (8s^3 - 12s^2 + 5)\,ds$

47. $\displaystyle\int_1^2 \frac{4}{v^2}\,dv$

48. $\displaystyle\int_1^{27} x^{-4/3}\,dx$

49. $\displaystyle\int_1^4 \frac{dt}{t\sqrt{t}}$

50. $\displaystyle\int_1^4 \frac{(1 + \sqrt{u})^{1/2}}{\sqrt{u}}\,du$

51. $\displaystyle\int_0^1 \frac{36\,dx}{(2x+1)^3}$

52. $\displaystyle\int_0^1 \frac{dr}{\sqrt[3]{(7 - 5r)^2}}$

53. $\displaystyle\int_{1/8}^1 x^{-1/3}(1 - x^{2/3})^{3/2}\,dx$

54. $\displaystyle\int_0^{1/2} x^3(1 + 9x^4)^{-3/2}\,dx$

55. $\displaystyle\int_0^\pi \sin^2 5r\,dr$

56. $\displaystyle\int_0^{\pi/4} \cos^2\left(4t - \frac{\pi}{4}\right) dt$

57. $\displaystyle\int_0^{\pi/3} \sec^2\theta\,d\theta$

58. $\displaystyle\int_{\pi/4}^{3\pi/4} \csc^2 x\,dx$

59. $\displaystyle\int_{\pi}^{3\pi} \cot^2 \frac{x}{6}\, dx$

60. $\displaystyle\int_{0}^{\pi} \tan^2 \frac{\theta}{3}\, d\theta$

61. $\displaystyle\int_{-\pi/3}^{0} \sec x \tan x\, dx$

62. $\displaystyle\int_{\pi/4}^{3\pi/4} \csc z \cot z\, dz$

63. $\displaystyle\int_{0}^{\pi/2} 5(\sin x)^{3/2} \cos x\, dx$

64. $\displaystyle\int_{-1}^{1} 2x \sin(1 - x^2)\, dx$

65. $\displaystyle\int_{-\pi/2}^{\pi/2} 15 \sin^4 3x \cos 3x\, dx$

66. $\displaystyle\int_{0}^{2\pi/3} \cos^{-4}\left(\frac{x}{2}\right) \sin\left(\frac{x}{2}\right) dx$

67. $\displaystyle\int_{0}^{\pi/2} \frac{3 \sin x \cos x}{\sqrt{1 + 3 \sin^2 x}}\, dx$

68. $\displaystyle\int_{0}^{\pi/4} \frac{\sec^2 x}{(1 + 7 \tan x)^{2/3}}\, dx$

69. $\displaystyle\int_{0}^{\pi/3} \frac{\tan \theta}{\sqrt{2 \sec \theta}}\, d\theta$

70. $\displaystyle\int_{\pi^2/36}^{\pi^2/4} \frac{\cos \sqrt{t}}{\sqrt{t} \sin \sqrt{t}}\, dt$

Average Values

71. Find the average value of $f(x) = mx + b$

- **a)** over $[-1, 1]$
- **b)** over $[-k, k]$

72. Find the average value of

- **a)** $y = \sqrt{3x}$ over $[0, 3]$
- **b)** $y = \sqrt{ax}$ over $[0, a]$

73. Let f be a function that is differentiable on $[a, b]$. In Chapter 1 we defined the average rate of change of f over $[a, b]$ to be

$$\frac{f(b) - f(a)}{b - a}$$

and the instantaneous rate of change of f at x to be $f'(x)$. In this chapter we defined the average value of a function. For the new definition of average to be consistent with the old one, we should have

$$\frac{f(b) - f(a)}{b - a} = \text{average value of } f' \text{ on } [a, b].$$

Is this the case? Give reasons for your answer.

74. Is it true that the average value of an integrable function over an interval of length 2 is half the function's integral over the interval? Give reasons for your answer.

Numerical Integration

75. CALCULATOR According to the error-bound formula for Simpson's rule, how many subintervals should you use to be sure of estimating the value of

$$\ln 3 = \int_1^3 \frac{1}{x}\, dx$$

by Simpson's rule with an error of no more than 10^{-4} in absolute value? (Remember that for Simpson's rule, the number of subintervals has to be even.)

76. A brief calculation shows that if $0 \le x \le 1$, then the second derivative of $f(x) = \sqrt{1 + x^4}$ lies between 0 and 8. Based on this, about how many subdivisions would you need to estimate the integral of f from 0 to 1 with an error no greater than 10^{-3} in absolute value using the trapezoidal rule?

77. A direct calculation shows that

$$\int_0^{\pi} 2 \sin^2 x\, dx = \pi.$$

How close do you come to this value by using the trapezoidal rule with $n = 6$? Simpson's rule with $n = 6$? Try them and find out.

78. You are planning to use Simpson's rule to estimate the value of the integral

$$\int_1^2 f(x)\, dx$$

with an error magnitude less than 10^{-5}. You have determined that $|f^{(4)}(x)| \le 3$ throughout the interval of integration. How many subintervals should you use to assure the required accuracy? (Remember that for Simpson's rule the number has to be even.)

79. CALCULATOR Compute the average value of the temperature function

$$f(x) = 37 \sin\left(\frac{2\pi}{365}(x - 101)\right) + 25$$

for a 365-day year. This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is 25.7°F, which is slightly higher than the average value of $f(x)$. Figure 2.42 shows why.

80. *Specific heat of a gas.* Specific heat C_v is the amount of heat required to raise the temperature of a given mass of gas with constant volume by 1°C, measured in units of cal/deg-mole (calories per degree gram molecule). The specific heat of oxygen depends on its temperature T and satisfies the formula

$$C_v = 8.27 + 10^{-5}(26T - 1.87T^2).$$

Find the average value of C_v for $20° \le T \le 675°\text{C}$ and the temperature at which it is attained.

Theory and Examples

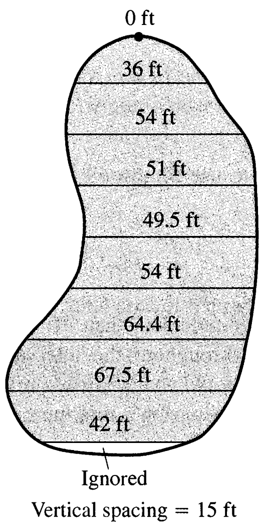
81. Is it true that every function $y = f(x)$ that is differentiable on $[a, b]$ is itself the derivative of some function on $[a, b]$? Give reasons for your answer.

82. Suppose that $F(x)$ is an antiderivative of $f(x) = \sqrt{1 + x^4}$. Express $\int_0^1 \sqrt{1 + x^4}\, dx$ in terms of F and give a reason for your answer.

83. Find dy/dx if $y = \int_x^1 \sqrt{1 + t^2}\, dt$. Explain the main steps in your calculation.

84. Find dy/dx if $y = \int_{\cos x}^0 (1/(1 - t^2))\, dt$. Explain the main steps in your calculation.

85. *A new parking lot.* To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for \$11,000. The cost to clear the land will be \$0.10 a square foot, and the lot will cost \$2.00 a square foot to pave. Can the job be done for \$11,000?



86. Skydivers A and B are in a helicopter hovering at 6400 ft. Skydiver A jumps and descends for 4 sec before opening her parachute. The helicopter then climbs to 7000 ft and hovers there. Forty-five seconds after A leaves the aircraft, B jumps and descends for 13 sec before opening her parachute. Both skydivers descend at 16 ft/sec with parachute open. Assume that the skydivers fall freely (no effective air resistance) before their parachutes open.

a) At what altitude does A's parachute open?
b) At what altitude does B's parachute open?
c) Which skydiver lands first?

Average Daily Inventory

Average value is used in economics to study such things as average daily inventory. If $I(t)$ is the number of radios, tires, shoes, or whatever product a firm has on hand on day t (we call I an **inventory function**), the average value of I over a time period $[0, T]$ is called the firm's average daily inventory for the period.

$$\textbf{Average daily inventory} = \operatorname{av}(I) = \frac{1}{T}\int_0^T I(t)\,dt.$$

If h is the dollar cost of holding one item per day, the product $\operatorname{av}(I)\cdot h$ is the **average daily holding cost** for the period.

87. As a wholesaler, Tracey Burr Distributors receives a shipment of 1200 cases of chocolate bars every 30 days. TBD sells the chocolate to retailers at a steady rate, and t days after a shipment arrives, its inventory of cases on hand is $I(t) = 1200 - 40t$, $0 \le t \le 30$. What is TBD's average daily inventory for the 30-day period? What is its average daily holding cost if the cost of holding one case is 3¢ a day?

88. Rich Wholesale Foods, a manufacturer of cookies, stores its cases of cookies in an air-conditioned warehouse for shipment every 14 days. Rich tries to keep 600 cases on reserve to meet occasional peaks in demand, so a typical 14-day inventory function is $I(t) = 600 + 600t$, $0 \le t \le 14$. The daily holding cost for each case is 4¢ per day. Find Rich's average daily inventory and average daily holding cost.

89. Solon Container receives 450 drums of plastic pellets every 30 days. The inventory function (drums on hand as a function of days) is $I(t) = 450 - t^2/2$. Find the average daily inventory. If the holding cost for one drum is 2¢ per day, find the average daily holding cost.

90. Mitchell Mailorder receives a shipment of 600 cases of athletic socks every 60 days. The number of cases on hand t days after the shipment arrives is $I(t) = 600 - 20\sqrt{15t}$. Find the average daily inventory. If the holding cost for one case is 1/2¢ per day, find the average daily holding cost.

CHAPTER 4 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Theory and Examples

1. a) If $\int_0^1 7f(x)\,dx = 7$, does $\int_0^1 f(x)\,dx = 1$?

b) If $\int_0^1 f(x)\,dx = 4$ and $f(x) \ge 0$, does $\int_0^1 \sqrt{f(x)}\,dx = \sqrt{4} = 2$?

Give reasons for your answers.

2. Suppose $\int_{-2}^2 f(x)\,dx = 4$, $\int_2^5 f(x)\,dx = 3$, $\int_{-2}^5 g(x)\,dx = 2$.

Which, if any, of the following statements are true?

a) $\int_5^2 f(x)\,dx = -3$

b) $\int_{-2}^5 (f(x) + g(x)) = 9$

c) $f(x) \le g(x)$ on the interval $-2 \le x \le 5$

3. Show that

$$y = \frac{1}{a} \int_0^x f(t) \sin a(x - t)\, dt$$

solves the initial value problem

$$\frac{d^2y}{dx^2} + a^2 y = f(x), \quad \frac{dy}{dx} = 0 \quad \text{and} \quad y = 0 \text{ when } x = 0.$$

(*Hint:* $\sin(ax - at) = \sin ax \cos at - \cos ax \sin at$.)

4. Suppose x and y are related by the equation

$$x = \int_0^y \frac{1}{\sqrt{1 + 4t^2}}\, dt.$$

Show that d^2y/dx^2 is proportional to y and find the constant of proportionality.

5. Find $f(4)$ if

a) $\displaystyle\int_0^{x^2} f(t)\, dt = x \cos \pi x,$

b) $\displaystyle\int_0^{f(x)} t^2\, dt = x \cos \pi x.$

6. Find $f(\pi/2)$ from the following information.

i) f is positive and continuous.

ii) The area under the curve $y = f(x)$ from $x = 0$ to $x = a$ is

$$\frac{a^2}{2} + \frac{a}{2} \sin a + \frac{\pi}{2} \cos a.$$

7. The area of the region in the xy-plane enclosed by the x-axis, the curve $y = f(x)$, $f(x) \ge 0$, and the lines $x = 1$ and $x = b$ is equal to $\sqrt{b^2 + 1} - \sqrt{2}$ for all $b > 1$. Find $f(x)$.

8. Prove that

$$\int_0^x \left(\int_0^u f(t)\, dt \right) du = \int_0^x f(u)(x - u)\, du.$$

(*Hint:* Express the integral on the right-hand side as the difference of two integrals. Then show that both sides of the equation have the same derivative with respect to x.)

9. Find the equation for the curve in the xy-plane that passes through the point $(1, -1)$ if its slope at x is always $3x^2 + 2$.

10. You sling a shovelful of dirt up from the bottom of a hole with an initial velocity of 32 ft/sec. The dirt must rise 17 ft above the release point to clear the edge of the hole. Is that enough speed to get the dirt out, or had you better duck?

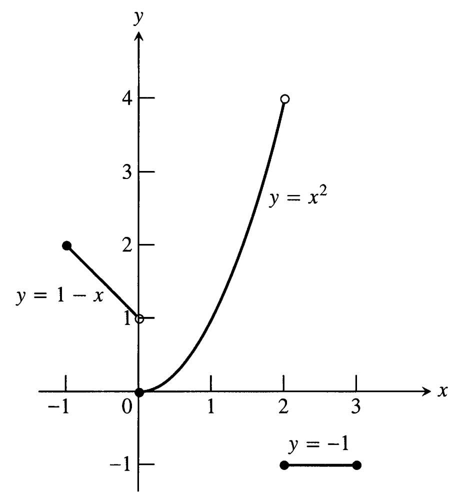
Bounded Piecewise Continuous Functions

Although we are mainly interested in continous functions, many functions in applications are piecewise continuous. All bounded piecewise continuous functions are integrable (as are many unbounded functions, as we will see in Chapter 7). **Bounded** on an interval I means that for some finite constant M, $|f(x)| \le M$ for all x in I. **Piecewise continuous** on I means that I can be partitioned into open or half open subintervals on which f is continuous. To integrate a bounded piecewise continuous function that has a continuous extension to each closed subinterval of the partition, we integrate the individual extensions and add the results. The integral of the function

$$f(x) = \begin{cases} 1 - x, & -1 \le x < 0 \\ x^2, & 0 \le x < 2 \\ -1, & 2 \le x \le 3, \end{cases}$$

(Fig. 4.35) over $[-1, 3]$ is

$$\begin{aligned} \int_{-1}^3 f(x)\, dx &= \int_{-1}^0 (1 - x)\, dx + \int_0^2 x^2\, dx + \int_2^3 (-1)\, dx \\ &= \left[x - \frac{x^2}{2} \right]_{-1}^0 + \left[\frac{x^3}{3} \right]_0^2 + \Big[-x \Big]_2^3 \\ &= \frac{3}{2} + \frac{8}{3} - 1 = \frac{19}{6}. \end{aligned}$$



4.35 Piecewise continuous functions like this are integrated piece by piece.

The Fundamental Theorem applies to bounded piecewise continuous functions with the restriction that $(d/dx) \int_a^x f(t)\, dt$ is expected to equal $f(x)$ only at values of x at which f is continuous. There is a similar restriction on Leibniz's rule below.

Graph the functions in Exercises 11–16 and integrate them over their domains.

11. $f(x) = \begin{cases} x^{2/3}, & -8 \le x < 0 \\ -4, & 0 \le x \le 3, \end{cases}$

12. $f(x) = \begin{cases} \sqrt{-x}, & -4 \le x < 0 \\ x^2 - 4, & 0 \le x \le 3 \end{cases}$

13. $g(t) = \begin{cases} t, & 0 \le t < 1 \\ \sin \pi t, & 1 \le t \le 2 \end{cases}$

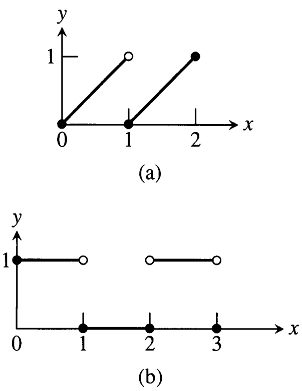
14. $h(z) = \begin{cases} \sqrt{1 - z}, & 0 \le z < 1 \\ (7z - 6)^{-1/3}, & 1 \le z \le 2 \end{cases}$

15. $f(x) = \begin{cases} 1, & -2 \le x < -1 \\ 1 - x^2, & -1 \le x < 1 \\ 2, & 1 \le x \le 2 \end{cases}$

16. $h(r) = \begin{cases} r, & -1 \le r < 0 \\ 1 - r^2, & 0 \le r < 1 \\ 1, & 1 \le r \le 2 \end{cases}$

17. Find the average value of the function graphed in Fig. 4.36(a).

18. Find the average value of the function graphed in Fig. 4.36(b).



4.36 The graphs for Exercises 17 and 18.

Leibniz's Rule

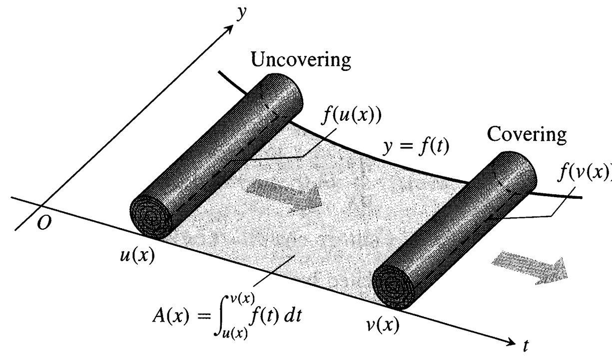
In applications, we sometimes encounter functions like

$$f(x) = \int_{\sin x}^{x^2} (1 + t)\, dt \qquad \text{and} \qquad g(x) = \int_{\sqrt{x}}^{2\sqrt{x}} \sin t^2\, dt,$$

defined by integrals that have variable upper limits of integration and variable lower limits of integration at the same time. The first integral can be evaluated directly but the second cannot. We may find the derivative of either integral, however, by a formula called **Leibniz's rule:**

> **Leibniz's Rule**
>
> If f is continuous on $[a, b]$, and $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then
>
> $$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t)\, dt = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}.$$

Figure 4.37 gives a geometric interpretation of Leibniz's rule. It shows a carpet of variable width $f(t)$ that is being rolled up at the left at the same time x as it is being unrolled at the right. (In this interpretation time is x, not t.) At time x, the floor is covered from $u(x)$ to $v(x)$. The rate du/dx at which the carpet is being rolled up need not be the same as the rate dv/dx at which the carpet is being



4.37 Rolling and unrolling a carpet: a geometric interpretation of Leibniz's rule:

$$\frac{dA}{dx} = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.$$

laid down. At any given time x, the area covered by carpet is

$$A(x) = \int_{u(x)}^{v(x)} f(t)\, dt.$$

At what rate is the covered area changing? At the instant x, $A(x)$ is increasing by the width $f(v(x))$ of the unrolling carpet times the rate dv/dx at which the carpet is being unrolled. That is, $A(x)$ is being increased at the rate

$$f(v(x)) \frac{dv}{dx}.$$

At the same time, A is being decreased at the rate

$$f(u(x)) \frac{du}{dx},$$

the width at the end that is being rolled up times the rate du/dx. The net rate of change in A is

$$\frac{dA}{dx} = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx},$$

which is precisely Leibniz's rule.

To prove the rule, let F be an antiderivative of f on $[a, b]$. Then

$$\int_{u(x)}^{v(x)} f(t)\, dt = F(v(x)) - F(u(x)). \tag{1}$$

Differentiating both sides of this equation with respect to x gives the equation we want:

$$\begin{aligned} \frac{d}{dx} \int_{u(x)}^{v(x)} f(t)\, dt &= \frac{d}{dx}\Big[F(v(x)) - F(u(x))\Big] \\ &= F'(v(x)) \frac{dv}{dx} - F'(u(x)) \frac{du}{dx} \qquad \text{Chain Rule} \\ &= f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}. \end{aligned}$$

You will see another way to derive the rule in Chapter 12, Additional Exercise 3.

Use Leibniz's rule to find the derivatives of the functions in Exercises 19–21.

19. $f(x) = \int_{1/x}^{x} \frac{1}{t}\, dt$ **20.** $f(x) = \int_{\cos x}^{\sin x} \frac{1}{1-t^2}\, dt$

21. $g(y) = \int_{\sqrt{y}}^{2\sqrt{y}} \sin t^2\, dt$

22. Use Leibniz's rule to find the value of x that maximizes the value of the integral

$$\int_{x}^{x+3} t(5-t)\, dt.$$

Problems like this arise in the mathematical theory of political elections. See "The Entry Problem in a Political Race," by Steven J. Brams and Philip D. Straffin, Jr., in *Political Equilibrium,* Peter Ordeshook and Kenneth Shepfle, Editors, Kluwer-Nijhoff, Boston, 1982, pp. 181–195.

Approximating Finite Sums with Integrals

In many applications of calculus, integrals are used to approximate finite sums—the reverse of the usual procedure of using finite sums to approximate integrals. Here is an example.

EXAMPLE 7 Estimate the sum of the square roots of the first n positive integers, $\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}$.

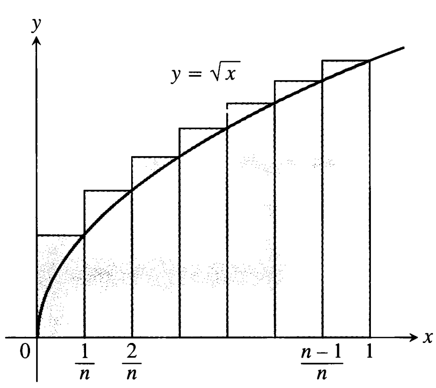
Solution See Fig. 4.38. The integral

$$\int_0^1 \sqrt{x}\, dx = \frac{2}{3}x^{3/2}\Big]_0^1 = \frac{2}{3}$$

is the limit of the sums

$$S_n = \sqrt{\frac{1}{n}}\cdot\frac{1}{n} + \sqrt{\frac{2}{n}}\cdot\frac{1}{n} + \cdots + \sqrt{\frac{n}{n}}\cdot\frac{1}{n}$$

$$= \frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}{n^{3/2}}.$$



4.38 The relation of the circumscribed rectangles to the integral $\int_0^1 \sqrt{x}\, dx$ leads to an estimate of the sum $\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}$.

Therefore, when n is large, S_n will be close to 2/3 and we will have

$$\text{Root sum} = \sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} = S_n \cdot n^{3/2} \approx \frac{2}{3}n^{3/2}.$$

The following table shows how good the approximation can be.

n	Root sum	$(2/3)n^{3/2}$	Relative error
10	22.468	21.082	1.386/22.468≈ 6%
50	239.04	235.70	1.4%
100	671.46	666.67	0.7%
1000	21,097	21,082	0.07%

❑

23. Evaluate

$$\lim_{n\to\infty} \frac{1^5+2^5+3^5+\cdots+n^5}{n^6}$$

by showing that the limit is

$$\int_0^1 x^5\, dx$$

and evaluating the integral.

24. See Exercise 23. Evaluate

$$\lim_{n\to\infty} \frac{1}{n^4}(1^3+2^3+3^3+\cdots+n^3).$$

25. Let $f(x)$ be a continuous function. Express

$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n}{n}\right)\right]$$

as a definite integral.

26. Use the result of Exercise 25 to evaluate

a) $\lim_{n\to\infty} \frac{1}{n^2}(2+4+6+\cdots+2n)$,

b) $\lim_{n\to\infty} \frac{1}{n^{16}}(1^{15}+2^{15}+3^{15}+\cdots+n^{15})$,

c) $\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\cdots+\sin\frac{n\pi}{n}\right)$.

What can be said about the following limits?

d) $\lim_{n\to\infty} \frac{1}{n^{17}}(1^{15}+2^{15}+3^{15}+\cdots+n^{15})$

e) $\lim_{n\to\infty} \frac{1}{n^{15}}(1^{15}+2^{15}+3^{15}+\cdots+n^{15})$

27. a) Show that the area A_n of an n-sided regular polygon in a circle of radius r is

$$A_n = \frac{nr^2}{2}\sin\frac{2\pi}{n}.$$

b) Find the limit of A_n as $n\to\infty$. Is this answer consistent with what you know about the area of a circle?

28. *The error function.* The error function,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

important in probability and in the theories of heat flow and signal transmission, must be evaluated numerically because there is no elementary expression for the antiderivative of e^{-t^2}.

a) Use Simpson's rule with $n = 10$ to estimate erf(1).

b) In [0, 1],

$$\left| \frac{d^4}{dt^4} \left(e^{-t^2} \right) \right| \leq 12.$$

Give an upper bound for the magnitude of the error of the estimate in (a).