CHAPTER 5

Applications of Integrals

OVERVIEW Many things we want to know can be calculated with integrals: the areas between curves, the volumes and surface areas of solids, the lengths of curves, the amount of work it takes to pump liquids from belowground, the forces against floodgates, the coordinates of the points where solid objects will balance. We define all of these as limits of Riemann sums of continuous functions on closed intervals, that is, as integrals, and evaluate these limits with calculus.

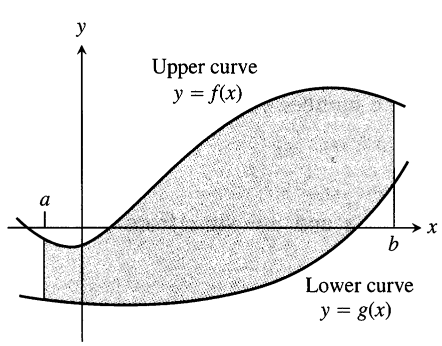
There is a pattern to how we define the integrals in applications, a pattern that, once learned, enables us to define new integrals when we need them. We look at specific applications first, then examine the pattern and show how it leads to integrals in new situations.

5.1 Areas Between Curves

This section shows how to find the areas of regions in the coordinate plane by integrating the functions that define the regions' boundaries.

The Basic Formula as a Limit of Riemann Sums

Suppose we want to find the area of a region that is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, and on the left and right by the lines $x = a$ and $x = b$ (Fig. 5.1). The region might accidentally have a shape whose area we could find with geometry, but if f and g are arbitrary continuous functions we usually have to find the area with an integral.



5.1 The region between $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

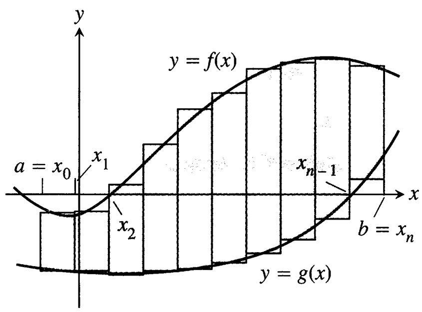
To see what the integral should be, we first approximate the region with n vertical rectangles based on a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ (Fig. 5.2, on the following page). The area of the kth rectangle (Fig. 5.3, on the following page) is

$$\Delta A_k = \text{height } \times \text{width } = [f(c_k) - g(c_k)]\,\Delta x_k.$$

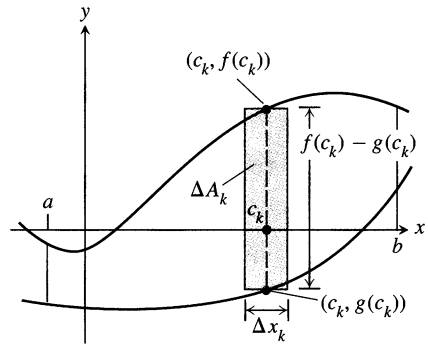
We then approximate the area of the region by adding the areas of the n rectangles:

$$A \approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} [f(c_k) - g(c_k)]\,\Delta x_k. \qquad \text{Riemann sum}$$

As $\| P \| \to 0$ the sums on the right approach the limit $\int_a^b [f(x) - g(x)]\,dx$ because



5.2 We approximate the region with rectangles perpendicular to the *x*-axis.



5.3 ΔA_k = area of kth rectangle, $f(c_k) - g(c_k)$ = height, Δx_k = width

f and g are continuous. We take the area of the region to be the value of this integral. That is,

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} [f(c_k) - g(c_k)]\, \Delta x_k = \int_a^b [f(x) - g(x)]\, dx.$$

Definition

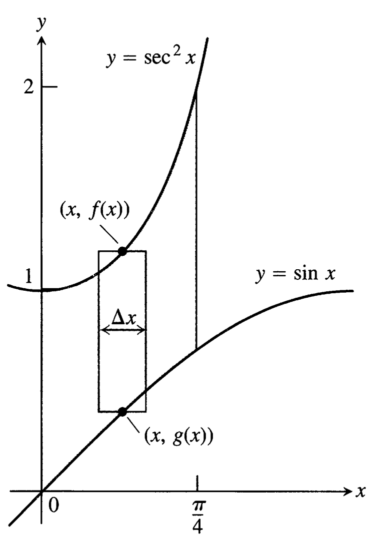
If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area** of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of $[f - g]$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx. \tag{1}$$

To apply Eq. (1) we take the following steps.

How to Find the Area Between Two Curves

1. *Graph the curves and draw a representative rectangle.* This reveals which curve is f (upper curve) and which is g (lower curve). It also helps find the limits of integration if you do not already know them.
2. *Find the limits of integration.*
3. *Write a formula for $f(x) - g(x)$.* Simplify it if you can.
4. *Integrate $[f(x) - g(x)]$ from a to b.* The number you get is the area.



5.4 The region in Example 1 with a typical approximating rectangle.

EXAMPLE 1 Find the area between $y = \sec^2 x$ and $y = \sin x$ from 0 to $\pi/4$.

Solution

Step 1: We sketch the curves and a vertical rectangle (Fig. 5.4). The upper curve is the graph of $f(x) = \sec^2 x$; the lower is the graph of $g(x) = \sin x$.

Step 2: The limits of integration are already given: $a = 0, b = \pi/4$.

Step 3: $f(x) - g(x) = \sec^2 x - \sin x$

Step 4:

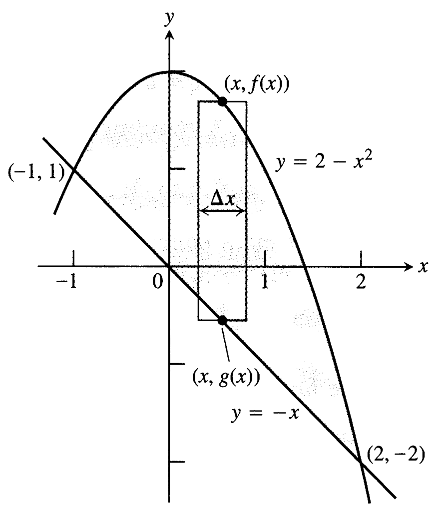
$$A = \int_0^{\pi/4} (\sec^2 x - \sin x)\,dx = \left[\tan x + \cos x\right]_0^{\pi/4}$$
$$= \left[1 + \frac{\sqrt{2}}{2}\right] - [0 + 1] = \frac{\sqrt{2}}{2}$$

❑

Curves That Intersect

When a region is determined by curves that intersect, the intersection points give the limits of integration.

EXAMPLE 2 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.



5.5 The region in Example 2 with a typical approximating rectangle.

Solution

Step 1: Sketch the curves and a vertical rectangle (Fig. 5.5). Identifying the upper and the lower curves, we take $f(x) = 2 - x^2$ and $g(x) = -x$. The x-coordinates of the intersection points are the limits of integration.

Step 2: We find the limits of integration by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x:

$$\begin{aligned} 2 - x^2 &= -x && \text{Equate } f(x) \text{ and } g(x). \\ x^2 - x - 2 &= 0 && \text{Rewrite.} \\ (x+1)(x-2) &= 0 && \text{Factor.} \\ x = -1, \quad x &= 2. && \text{Solve.} \end{aligned}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.

Step 3:

$$\begin{aligned} f(x) - g(x) &= (2 - x^2) - (-x) = 2 - x^2 + x && \text{Rearrangement} \\ &= 2 + x - x^2 && \text{a matter of taste} \end{aligned}$$

Step 4:

$$A = \int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$
$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)$$
$$= 6 + \frac{3}{2} - \frac{9}{3} = \frac{9}{2}$$

❑

Technology *The Intersection of Two Graphs* One of the difficult and sometimes frustrating parts of integration applications is finding the limits of integration. To do this you often have to find the zeroes of a function or the intersection points of two curves.

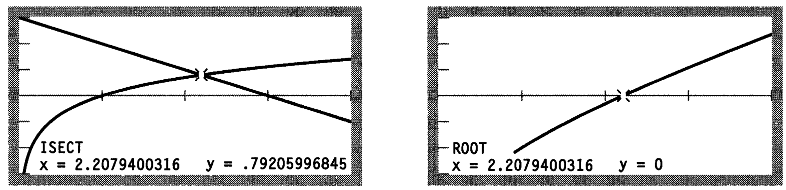
To solve the equation $f(x) = g(x)$ using a graphing utility, you enter

$$y_1 = f(x) \quad \text{and} \quad y_2 = g(x)$$

and use the grapher routine to find the points of intersection. Alternatively, you can solve the equation $f(x) - g(x) = 0$ with a root finder. Try both procedures with

$$f(x) = \ln x \quad \text{and} \quad g(x) = 3 - x.$$

When points of intersection are not clearly revealed or you suspect hidden behavior, additional work with the graphing utility or further use of calculus may be necessary.



a) The intersecting curves $y_1 = \ln x$ and $y_2 = 3 - x$, using a built-in function to find the intersection
b) Using a built-in root finder to find the zero of $f(x) = \ln x - 3 + x$

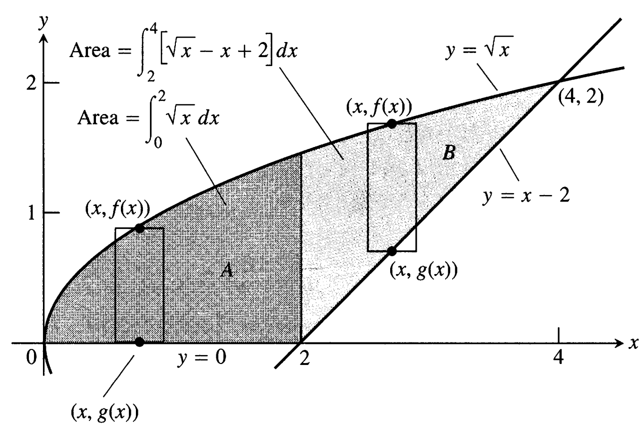
Boundaries with Changing Formulas

If the formula for a bounding curve changes at one or more points, we partition the region into subregions that correspond to the formula changes and apply Eq. (1) to each subregion.

EXAMPLE 3 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution

Step 1: The sketch (Fig. 5.6) shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (there is agreement at $x = 2$). We partition the region at $x = 2$ into subregions A and B and sketch a representative rectangle for each subregion.



5.6 When the formula for a bounding curve changes, the area integral changes to match (Example 3).

Step 2: The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations

$y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$
\begin{aligned}
\sqrt{x} &= x - 2 && \text{Equate } f(x) \text{ and } g(x).\\
x &= (x-2)^2 = x^2 - 4x + 4 && \text{Square both sides.}\\
x^2 - 5x + 4 &= 0 && \text{Rewrite.}\\
(x-1)(x-4) &= 0 && \text{Factor.}\\
x = 1, \qquad x &= 4. && \text{Solve.}
\end{aligned}
$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

Step 3: For $0 \le x \le 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \le x \le 4$: $\quad f(x) - g(x) = \sqrt{x} - (x-2) = \sqrt{x} - x + 2$

Step 4: We add the area of subregions A and B to find the total area:

$$
\begin{aligned}
\text{Total area} &= \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\,dx}_{\text{area of } B}\\
&= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4\\
&= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)\\
&= \frac{2}{3}(8) - 2 = \frac{10}{3}.
\end{aligned}
$$

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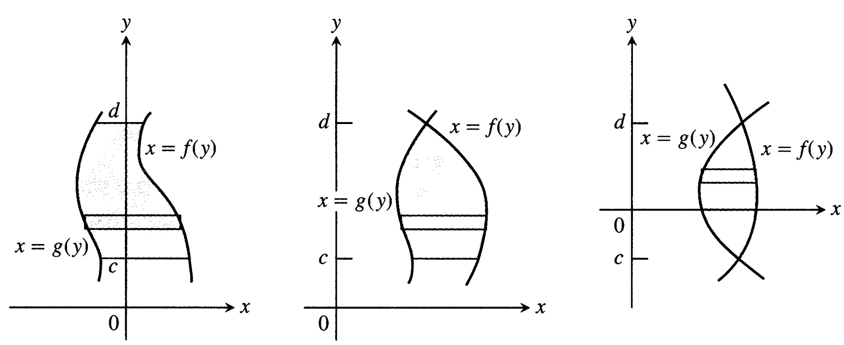
Integration with Respect to *y*

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

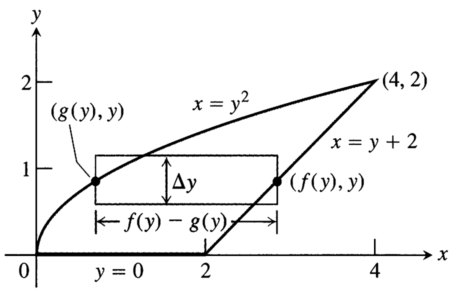
For regions like these



use the formula

$$A = \int_c^d [f(y) - g(y)]\,dy. \tag{2}$$

In Eq. (2), f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.



5.7 It takes two integrations to find the area of this region if we integrate with respect to *x*. It takes only one if we integrate with respect to *y* (Example 4).

EXAMPLE 4 Find the area of the region in Example 3 by integrating with respect to y.

Solution

Step 1: We sketch the region and a typical *horizontal* rectangle based on a partition of an interval of y-values (Fig. 5.7). The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$.

Step 2: The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y:

$$\begin{aligned} y + 2 &= y^2 && \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2. \\ y^2 - y - 2 &= 0 && \text{Rewrite.} \\ (y + 1)(y - 2) &= 0 && \text{Factor.} \\ y = -1, \qquad y &= 2 && \text{Solve.} \end{aligned}$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the x-axis.)

Step 3:

$$f(y) - g(y) = y + 2 - y^2 = 2 + y - y^2 \qquad \text{Rearrangement a matter of taste}$$

Step 4:

$$\begin{aligned} A = \int_a^b [f(y) - g(y)]\,dy &= \int_0^2 [2 + y - y^2]\,dy \\ &= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2 \\ &= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3} \end{aligned}$$

This is the result of Example 3, found with less work. ❑

Combining Integrals with Formulas from Geometry

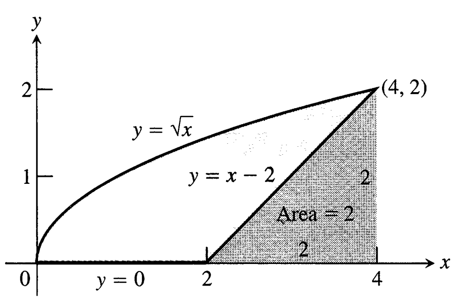
The fastest way to find an area may be to combine calculus and geometry.

EXAMPLE 5 *The Area of the Region in Example 3 Found the Fastest Way*

Find the area of the region in Example 3.

Solution The area we want is the area between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis, *minus* the area of a triangle with base 2 and height 2 (Fig. 5.8):

$$\begin{aligned} \text{Area} &= \int_0^4 \sqrt{x}\,dx - \frac{1}{2}(2)(2) \\ &= \frac{2}{3}x^{3/2}\Big]_0^4 - 2 \\ &= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}. \end{aligned}$$ ❑



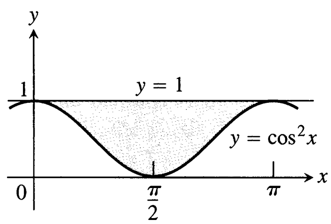
5.8 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle.

Moral of Examples 3–5 It is sometimes easier to find the area between two curves by integrating with respect to y instead of x. Also, it may help to combine geometry and calculus. After sketching the region, take a moment to determine the best way to proceed.

Exercises 5.1

Find the areas of the shaded regions in Exercises 1–8.

1.



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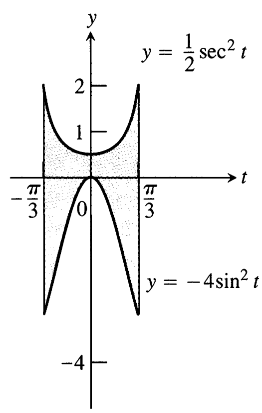

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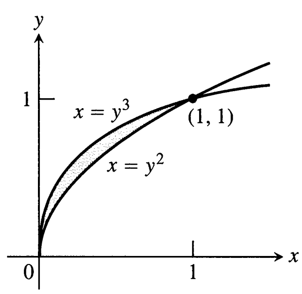

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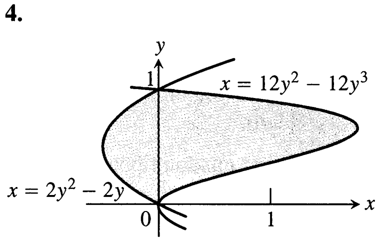

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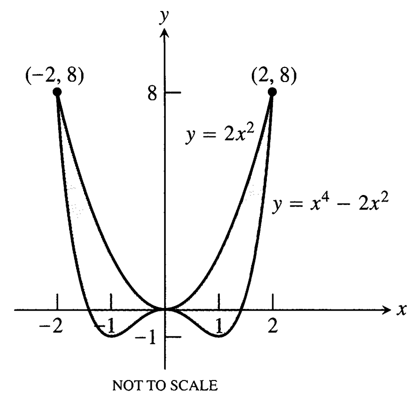

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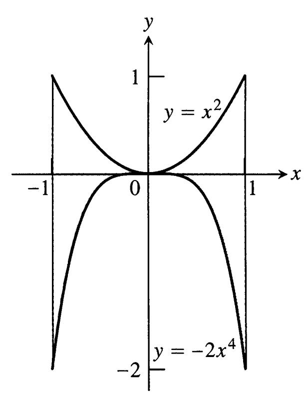

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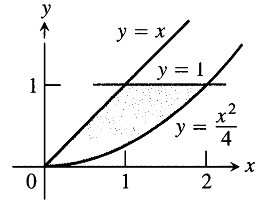

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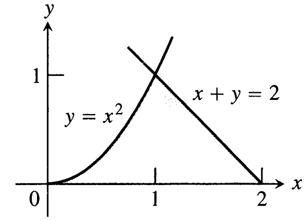
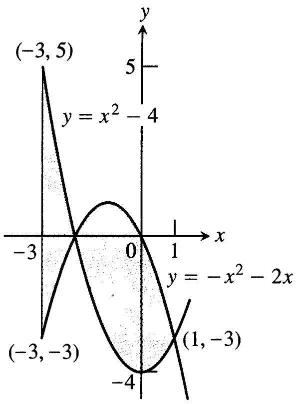


In Exercises 9–12, find the total shaded area.

9.



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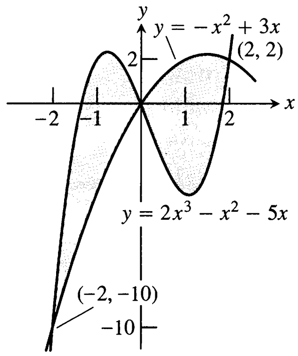

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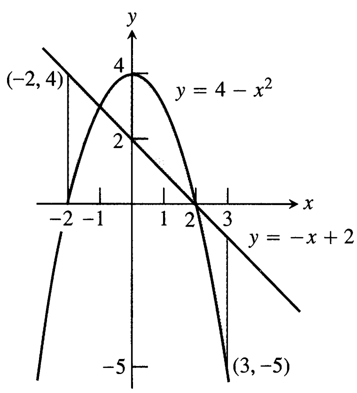

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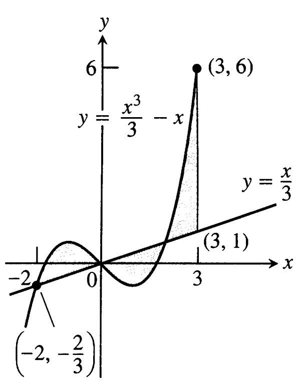


Find the areas of the regions enclosed by the lines and curves in Exercises 13–22.

13. $y = x^2 - 2$ and $y = 2$

14. $y = 2x - x^2$ and $y = -3$

15. $y = x^4$ and $y = 8x$

16. $y = x^2 - 2x$ and $y = x$

17. $y = x^2$ and $y = -x^2 + 4x$

18. $y = 7 - 2x^2$ and $y = x^2 + 4$

19. $y = x^4 - 4x^2 + 4$ and $y = x^2$

20. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

21. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

22. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

Find the areas of the regions enclosed by the lines and curves in Exercises 23–30.

23. $x = 2y^2$, $x = 0$, and $y = 3$

24. $x = y^2$ and $x = y + 2$

25. $y^2 - 4x = 4$ and $4x - y = 16$

26. $x - y^2 = 0$ and $x + 2y^2 = 3$

27. $x + y^2 = 0$ and $x + 3y^2 = 2$

28. $x - y^{2/3} = 0$ and $x + y^4 = 2$

29. $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$

30. $x = y^3 - y^2$ and $x = 2y$

Find the areas of the regions enclosed by the curves in Exercises 31–34.

31. $4x^2 + y = 4$ and $x^4 - y = 1$

32. $x^3 - y = 0$ and $3x^2 - y = 4$

33. $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

34. $x + y^2 = 3$ and $4x + y^2 = 0$

Find the areas of the regions enclosed by the lines and curves in Exercises 35–42.

35. $y = 2\sin x$ and $y = \sin 2x$, $0 \leq x \leq \pi$

36. $y = 8\cos x$ and $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$

37. $y = \cos(\pi x/2)$ and $y = 1 - x^2$

38. $y = \sin(\pi x/2)$ and $y = x$

39. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, and $x = \pi/4$

40. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \leq y \leq \pi/4$

41. $x = 3\sin y\sqrt{\cos y}$ and $x = 0$, $0 \leq y \leq \pi/2$

42. $y = \sec^2(\pi x/3)$ and $y = x^{1/3}$, $-1 \leq x \leq 1$

43. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.

44. Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.

45. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

46. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$.

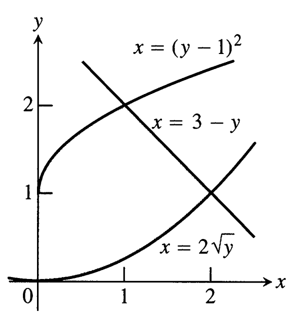
47. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

a) Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c, what are the coordinates of the points where the line and parabola intersect? Add them to your figure.
b) Find c by integrating with respect to y. (This puts c in the limits of integration.)
c) Find c by integrating with respect to x. (This puts c into the integrand as well.)

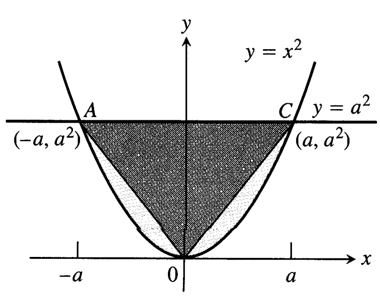
48. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to (a) x, (b) y.

49. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

50. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y-1)^2$, and above right by the line $x = 3 - y$.



51. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

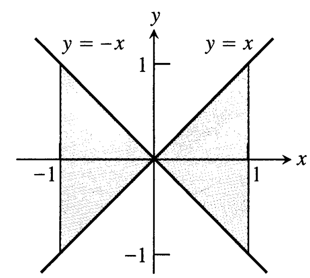


52. Suppose the area of the region between the graph of a positive continuous function f and the x-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.

53. Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

a) $\displaystyle\int_{-1}^{1} (x - (-x))\,dx = \int_{-1}^{1} 2x\,dx$

b) $\displaystyle\int_{-1}^{1} (-x - (x))\,dx = \int_{-1}^{1} -2x\,dx$



54. True, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ ($a < b$) is

$$\int_a^b [f(x) - g(x)]\,dx.$$

Give reasons for your answer.

CAS Explorations and Projects

In Exercises 55–58, you will find the area between curves in the plane when you cannot find their points of intersection using simple algebra. Use a CAS to perform the following steps:

a) Plot the curves together to see what they look like and how many points of intersection they have.
b) Use the numerical equation solver in your CAS to find all the points of intersection.
c) Integrate $|f(x) - g(x)|$ over consecutive pairs of intersection values.
d) Sum together the integrals found in part (c).

55. $f(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x + \dfrac{1}{3}, \quad g(x) = x - 1$

56. $f(x) = \dfrac{x^4}{2} - 3x^3 + 10, \quad g(x) = 8 - 12x$

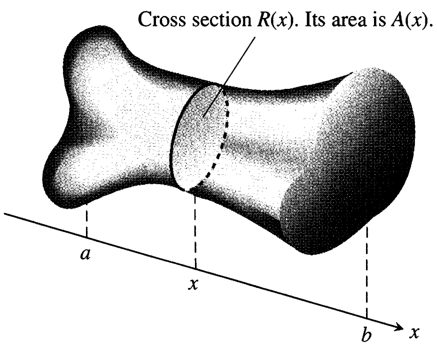
57. $f(x) = x + \sin(2x), \quad g(x) = x^3$

58. $f(x) = x^2 \cos x, \quad g(x) = x^3 - x$

5.2 Finding Volumes by Slicing

From the areas of regions with curved boundaries, we can calculate the volumes of cylinders with curved bases by multiplying base area by height. From the volumes of such cylinders, we can calculate the volumes of other solids.

Slicing



5.9 If the area $A(x)$ of the cross section $R(x)$ is a continuous function of x, we can find the volume of the solid by integrating $A(x)$ from a to b.

Suppose we want to find the volume of a solid like the one shown in Fig. 5.9. At each point x in the closed interval $[a, b]$ the cross section of the solid is a region $R(x)$ whose area is $A(x)$. This makes A a real-valued function of x. If it is also a continuous function of x, we can use it to define and calculate the volume of the solid as an integral in the following way.

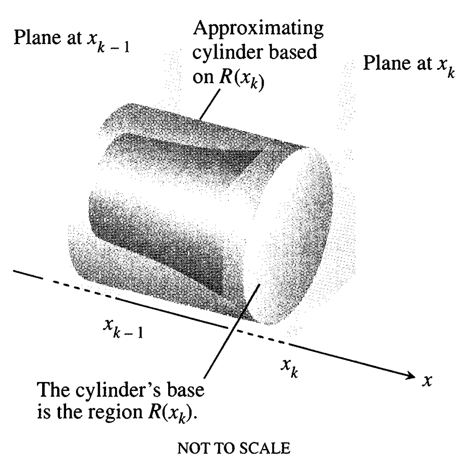
We partition the interval $[a, b]$ along the x-axis in the usual manner and slice the solid, as we would a loaf of bread, by planes perpendicular to the x-axis at the partition points. The kth slice, the one between the planes at x_{k-1} and x_k, has approximately the same volume as the cylinder between these two planes based on the region $R(x_k)$ (Fig. 5.10). The volume of this cylinder is

$$\begin{aligned} V_k &= \text{base area} \times \text{height} \\ &= A(x_k) \times (\text{distance between the planes at } x_{k-1} \text{ and } x_k) \\ &= A(x_k)\Delta x_k. \end{aligned}$$

The volume of the solid is therefore approximated by the cylinder volume sum

$$\sum_{k=1}^{n} A(x_k)\Delta x_k.$$

This is a Riemann sum for the function $A(x)$ on $[a, b]$. We expect the approximations from these sums to improve as the norm of the partition of $[a, b]$ goes to zero, so we define their limiting integral to be the volume of the solid.



5.10 Enlarged view of the slice of the solid between the planes at x_{k-1} and x_k and its approximating cylinder.

Definition

The **volume** of a solid of known integrable cross-section area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b:

$$V = \int_a^b A(x)\,dx. \tag{1}$$

To apply Eq. (1), we take the following steps.

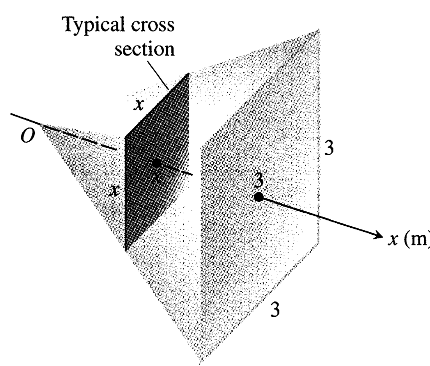
How to Find Volumes by the Method of Slicing

1. Sketch the solid and a typical cross section.
2. Find a formula for $A(x)$.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

EXAMPLE 1 A pyramid 3 m high has a square base that is 3 m on a side. The cross section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

Solution

Step 1: *A sketch.* We draw the pyramid with its altitude along the x-axis and its vertex at the origin and include a typical cross section (Fig. 5.11).



5.11 The cross sections of the pyramid in Example 1 are squares.

Step 2: *A formula for $A(x)$.* The cross section at x is a square x meters on a side, so its area is

$$A(x) = x^2.$$

Step 3: *The limits of integration.* The squares go from $x = 0$ to $x = 3$.

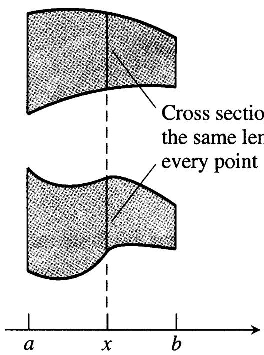
Step 4: *The volume.*

$$V = \int_a^b A(x)\,dx = \int_0^3 x^2\,dx = \left.\frac{x^3}{3}\right]_0^3 = 9.$$

The volume is 9 m^3. ❑

Bonaventura Cavalieri (1598–1647)

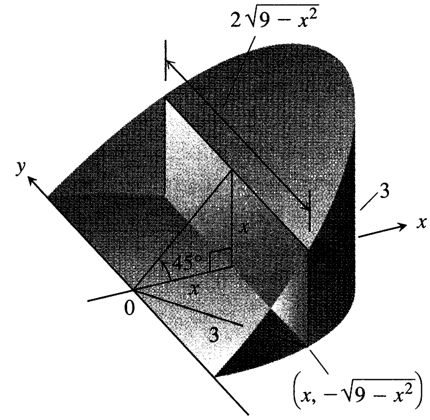
Cavalieri, a student of Galileo's, discovered that if two plane regions can be arranged to lie over the same interval of the x-axis in such a way that they have identical vertical cross sections at every point, then the regions have the same area. The theorem (and a letter of recommendation from Galileo) were enough to win Cavalieri a chair at the University of Bologna in 1629. The solid geometry version in Example 3, which Cavalieri never proved, was given his name by later geometers.



EXAMPLE 2 A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution

Step 1: *A sketch.* We draw the wedge and sketch a typical cross section perpendicular to the x-axis (Fig. 5.12).



5.12 The wedge of Example 2, sliced perpendicular to the x-axis. The cross sections are rectangles.

Step 2: *The formula for $A(x)$.* The cross section at x is a rectangle of area

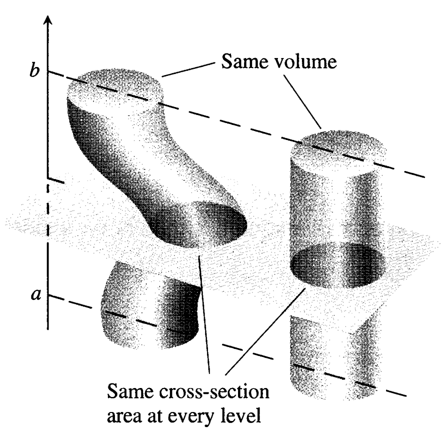
$$\begin{aligned} A(x) = (\text{height})(\text{width}) &= (x)\left(2\sqrt{9-x^2}\right) \\ &= 2x\sqrt{9-x^2}. \end{aligned}$$

Step 3: *The limits of integration.* The rectangles run from $x = 0$ to $x = 3$.

Step 4: *The volume.*

$$\begin{aligned} V &= \int_a^b A(x)\,dx = \int_0^3 2x\sqrt{9-x^2}\,dx \\ &= -\frac{2}{3}(9-x^2)^{3/2}\Big]_0^3 \\ &= 0 + \frac{2}{3}(9)^{3/2} \qquad \text{Let } u = 9-x^2,\ du = -2x\,dx, \text{ integrate, and substitute back.} \\ &= 18. \end{aligned}$$

❑



5.13 Cavalieri's theorem: These solids have the same volume. You can illustrate this yourself with stacks of coins.

EXAMPLE 3 *Cavalieri's Theorem*

Cavalieri's theorem says that solids with equal altitudes and identical parallel cross-section areas have the same volume (Fig. 5.13). We can see this immediately from Eq. (1) because the cross-section area function $A(x)$ is the same in each case. ❑

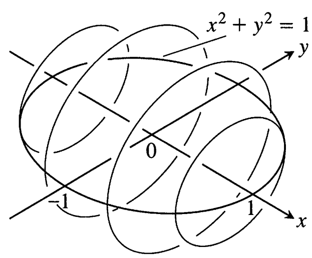
Exercises 5.2

Cross-Section Areas

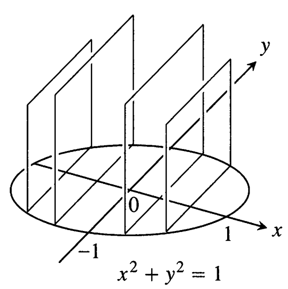
In Exercises 1 and 2, find a formula for the area $A(x)$ of the cross sections of the solid perpendicular to the x-axis.

1. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. In each case, the cross sections perpendicular to the x-axis between these planes run from the semicircle $y = -\sqrt{1-x^2}$ to the semicircle $y = \sqrt{1-x^2}$.

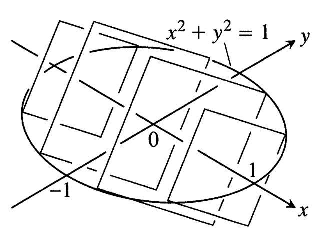
a) The cross sections are circular disks with diameters in the xy-plane.



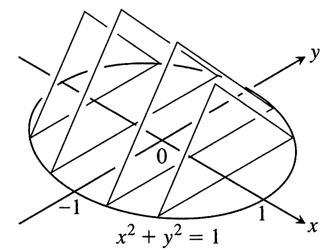
b) The cross sections are squares with bases in the xy-plane.



c) The cross sections are squares with diagonals in the xy-plane. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.)

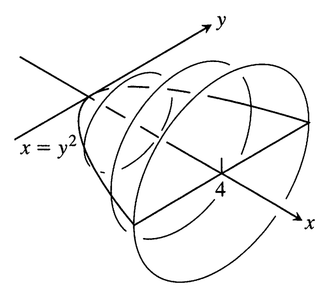


d) The cross sections are equilateral triangles with bases in the xy-plane.

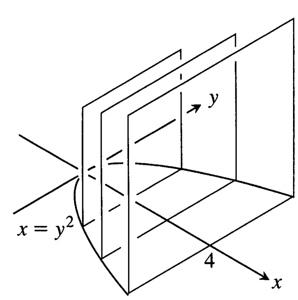


2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the x-axis between these planes run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.

a) The cross sections are circular disks with diameters in the xy-plane.



b) The cross sections are squares with bases in the xy-plane.



c) The cross sections are squares with diagonals in the xy-plane.

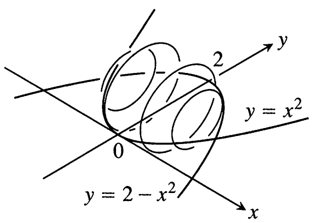
d) The cross sections are equilateral triangles with bases in the xy-plane.

Volumes by Slicing

Find the volumes of the solids in Exercises 3–12.

3. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.

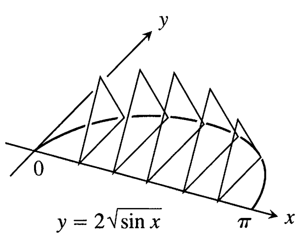
4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.



5. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the axis between these planes are vertical squares whose base edges run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

6. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.)

7. The base of the solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross sections perpendicular to the x-axis are



a) vertical equilateral triangles with bases running from the x-axis to the curve;

b) vertical squares with bases running from the x-axis to the curve.

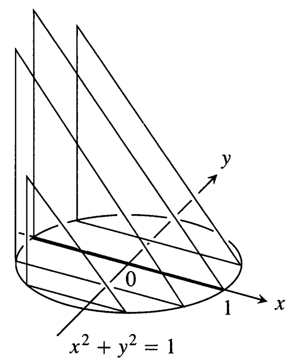
8. The solid lies between planes perpendicular to the x-axis at $x = -\pi/3$ and $x = \pi/3$. The cross sections perpendicular to the x-axis are

a) circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$;

b) vertical squares whose base edges run from the curve $y = \tan x$ to the curve $y = \sec x$.

9. The solid lies between planes perpendicular to the y-axis at $y = 0$ and $y = 2$. The cross sections perpendicular to the y-axis are circular disks with diameters running from the y-axis to the parabola $x = \sqrt{5}y^2$.

10. The base of the solid is the disk $x^2 + y^2 \leq 1$. The cross sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.



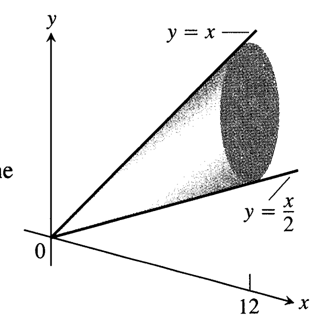
Cavalieri's Theorem

11. *A twisted solid.* A square of side length s lies in a plane perpendicular to a line L. One vertex of the square lies on L. As this square moves a distance h along L, the square turns one revolution about L to generate a corkscrew-like column with square cross sections.

a) Find the volume of the column.

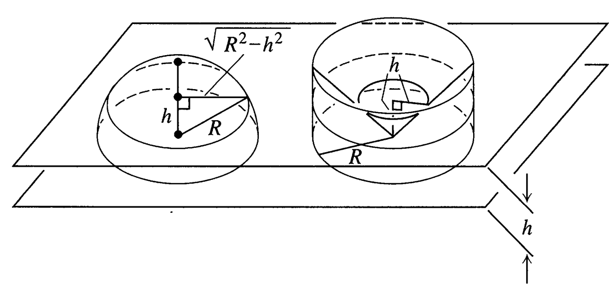
b) What will the volume be if the square turns twice instead of once? Give reasons for your answer.

12. A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 12$. The cross sections by planes perpendicular to the x-axis are circular disks whose diameters run from the line $y = x/2$ to the line $y = x$. Explain why the solid has the same volume as a right circular cone with base radius 3 and height 12.



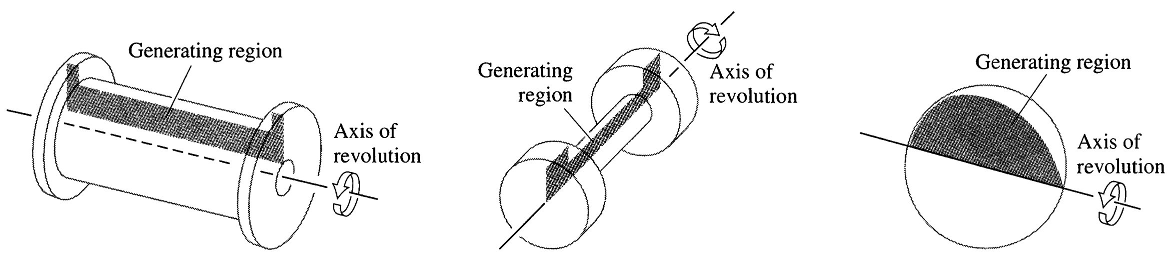
13. *Cavalieri's original theorem.* Prove Cavalieri's original theorem (marginal note, page 376), assuming that each region is bounded above and below by the graphs of continuous functions.

14. *The volume of a hemisphere (a classical application of Cavalieri's theorem).* Derive the formula $V = (2/3)\pi R^3$ for the volume of a hemisphere of radius R by comparing its cross sections with the cross sections of a solid right circular cylinder of radius R and height R from which a solid right circular cone of base radius R and height R has been removed.



5.3 Volumes of Solids of Revolution—Disks and Washers

The most common application of the method of slicing is to solids of revolution. **Solids of revolution** are solids whose shapes can be generated by revolving plane regions about axes. Thread spools are solids of revolution; so are hand weights and billiard balls. Solids of revolution sometimes have volumes we can find with formulas from geometry, as in the case of a billiard ball. But when we want to find the volume of a blimp or to predict the weight of a part we are going to have turned on a lathe, formulas from geometry are of little help and we turn to calculus for the answers.

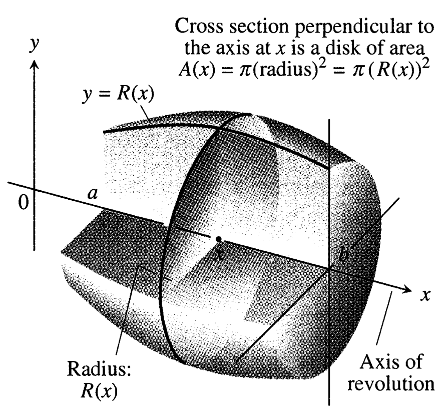


If we can arrange for the region to be the region between the graph of a continuous function $y = R(x)$, $a \le x \le b$, and the x-axis, and for the axis of revolution to be the x-axis (Fig. 5.14), we can find the solid's volume in the following way.

The typical cross section of the solid perpendicular to the axis of revolution is a disk of radius $R(x)$ and area

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

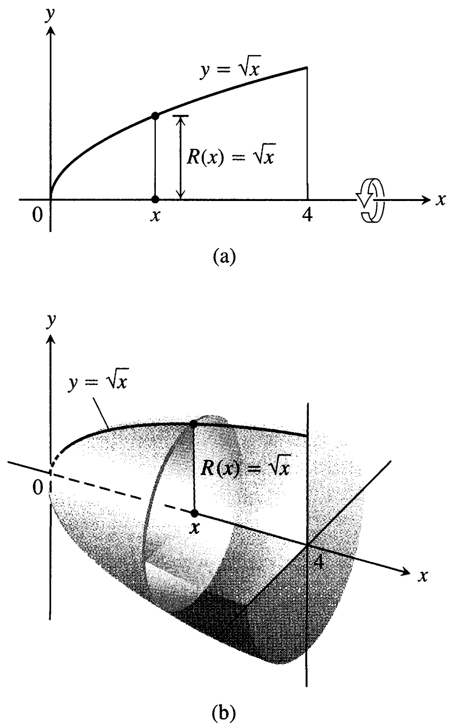
The solid's volume, being the integral of A from $x = a$ to $x = b$, is the integral of $\pi[R(x)]^2$ from a to b.



5.14 The solid generated by revolving the region between the curve $y = R(x)$ and the x-axis from a to b about the x-axis.

> **Volume of a Solid of Revolution (Rotation About the x-axis)**
>
> The volume of the solid generated by revolving about the x-axis the region between the x-axis and the graph of the continuous function $y = R(x)$, $a \le x \le b$, is
>
> $$V = \int_a^b \pi[\text{radius}]^2\,dx = \int_a^b \pi[R(x)]^2\,dx. \qquad (1)$$

EXAMPLE 1 The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.



5.15 The region (a) and solid (b) in Example 1.

Solution We draw figures showing the region, a typical radius, and the generated solid (Fig. 5.15). The volume is

$$\begin{aligned} V &= \int_a^b \pi[R(x)]^2\,dx && \text{Eq. (1)} \\ &= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx && R(x)=\sqrt{x} \\ &= \pi\int_0^4 x\,dx = \pi\frac{x^2}{2}\bigg]_0^4 = \pi\frac{(4)^2}{2} = 8\pi. \end{aligned}$$

❑

How to Find Volumes Using Eq. (1)

1. Draw the region and identify the radius function $R(x)$.
2. Square $R(x)$ and multiply by π.
3. Integrate to find the volume.

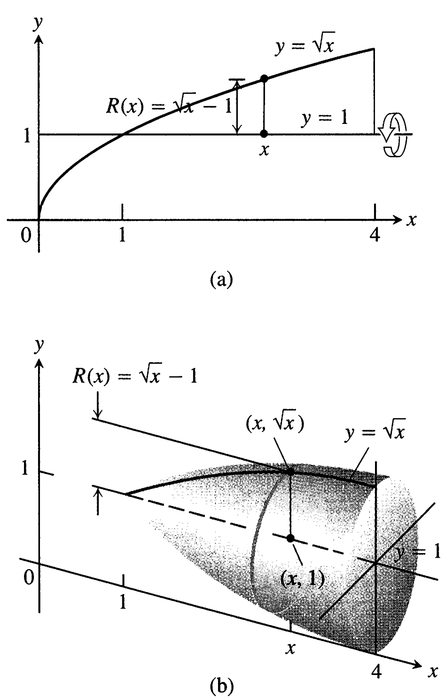
The axis of revolution in the next example is not the x-axis, but the rule for calculating the volume is the same: Integrate $\pi(\text{radius})^2$ between appropriate limits.

EXAMPLE 2 Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the lines $y=1$, $x=4$ about the line $y=1$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Fig. 5.16). The volume is

$$\begin{aligned} V &= \int_1^4 \pi[R(x)]^2\,dx && \text{Eq. (1)} \\ &= \int_1^4 \pi\left[\sqrt{x}-1\right]^2 dx && R(x)=\sqrt{x}-1 \\ &= \pi\int_1^4 \left[x-2\sqrt{x}+1\right]dx \\ &= \pi\left[\frac{x^2}{2}-2\cdot\frac{2}{3}x^{3/2}+x\right]_1^4 = \frac{7\pi}{6}. \end{aligned}$$

❑



5.16 The region (a) and solid (b) in Example 2.

To find the volume of a solid generated by revolving a region between the y-axis and a curve $x=R(y)$, $c\le y\le d$, about the y-axis, we use Eq. (1) with x replaced by y.

Volume of a Solid of Revolution (Rotation About the y-axis)

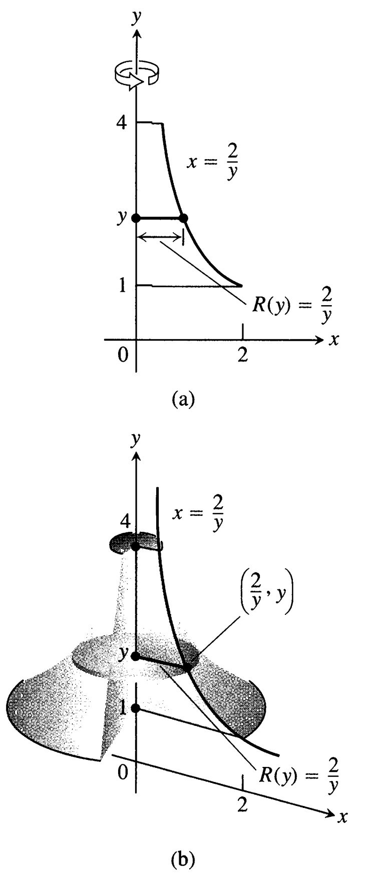
$$V=\int_c^d \pi(\text{radius})^2\,dy=\int_c^d \pi[R(y)]^2\,dy \qquad (2)$$

EXAMPLE 3 Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

Solution We draw figures showing the region, a typical radius, and the generated solid (Fig. 5.17). The volume is

$$\begin{aligned} V &= \int_1^4 \pi [R(y)]^2\, dy && \text{Eq. (2)} \\ &= \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy && R(y) = \frac{2}{y} \\ &= \pi \int_1^4 \frac{4}{y^2}\, dy = 4\pi \left[-\frac{1}{y}\right]_1^4 = 4\pi \left[\frac{3}{4}\right] \\ &= 3\pi. \end{aligned}$$

❑

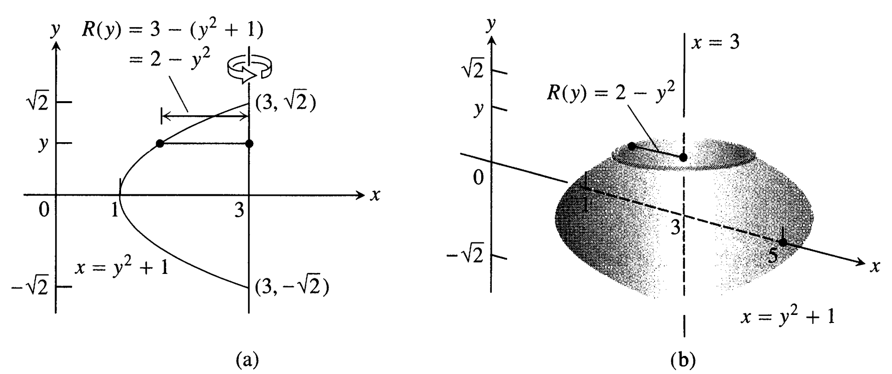


5.17 The region (a) and solid (b) in Example 3.

EXAMPLE 4 Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

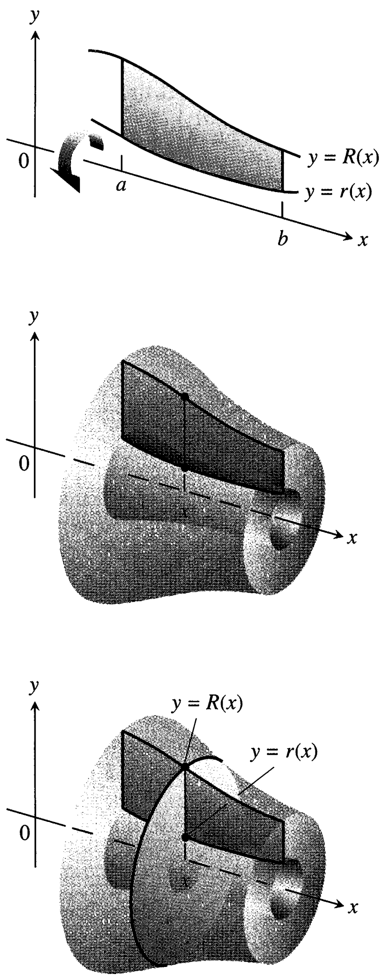
Solution We draw figures showing the region, a typical radius, and the generated solid (Fig. 5.18). The volume is

$$\begin{aligned} V &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [R(y)]^2\, dy && \text{Eq. (2)} \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [2 - y^2]^2\, dy && R(y) = 3 - (y^2 + 1) \\ & && \quad = 2 - y^2 \\ &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4]\, dy \\ &= \pi \left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}} \\ &= \frac{64\pi\sqrt{2}}{15}. \end{aligned}$$



5.18 The region (a) and solid (b) in Example 4.

❑



5.19 The cross sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it (Fig. 5.19). The cross sections perpendicular to the axis of revolution are washers instead of disks. The dimensions of a typical washer are

$$\text{Outer radius:} \quad R(x)$$

$$\text{Inner radius:} \quad r(x)$$

The washer's area is

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi\left([R(x)]^2 - [r(x)]^2\right).$$

The Washer Formula for Finding Volumes

$$V = \int_a^b \pi\left(\underbrace{[R(x)]^2}_{\substack{\text{outer}\\ \text{radius}\\ \text{squared}}} - \underbrace{[r(x)]^2}_{\substack{\text{inner}\\ \text{radius}\\ \text{squared}}}\right) dx \qquad (3)$$

Notice that the function integrated in Eq. (3) is $\pi(R^2 - r^2)$, not $\pi(R - r)^2$. Also notice that Eq. (3) gives the disk method formula if $r(x)$ is zero throughout $[a, b]$. Thus, the disk method is a special case of the washer method.

EXAMPLE 5 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution

Step 1: Draw the region and sketch a line segment across it perpendicular to the axis of revolution (the red segment in Fig. 5.20).

Step 2: Find the limits of integration by finding the x-coordinates of the intersection points.

$$x^2 + 1 = -x + 3$$

$$x^2 + x - 2 = 0$$

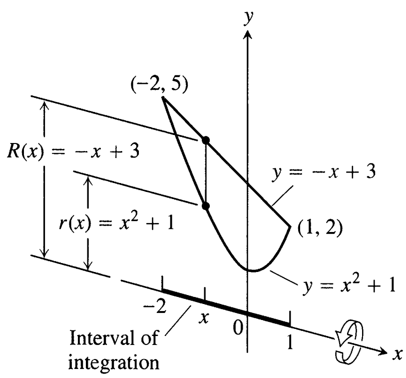
$$(x + 2)(x - 1) = 0$$

$$x = -2, \qquad x = 1$$

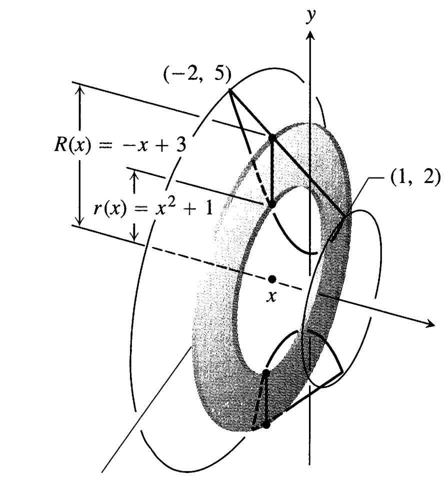
Step 3: Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x-axis along with the region. (We drew the washer in Fig. 5.21, but in your own work you need not do that.) These radii are the distances of the ends of the line segment from the axis of revolution.

$$\text{Outer radius:} \quad R(x) = -x + 3$$

$$\text{Inner radius:} \quad r(x) = x^2 + 1$$



5.20 The region in Example 5 spanned by a line segment perpendicular to the axis of revolution. When the region is revolved about the x-axis, the line segment will generate a washer.



Washer cross section
Outer radius: $R(x) = -x+3$
Inner radius: $r(x) = x^2 + 1$

5.21 The inner and outer radii of the washer swept out by the line segment in Fig. 5.20.

Step 4: Evaluate the volume integral.

$$V = \int_a^b \pi \left([R(x)]^2 - [r(x)]^2\right) dx \qquad \text{Eq. (3)}$$

$$= \int_{-2}^{1} \pi((-x+3)^2 - (x^2+1)^2)\, dx \qquad \text{Values from steps 2 and 3}$$

$$= \int_{-2}^{1} \pi(8 - 6x - x^2 - x^4)\, dx \qquad \text{Expressions squared and combined}$$

$$= \pi \left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5}$$

❑

How to Find Volumes by the Washer Method

1. *Draw the region and sketch a line segment across it perpendicular to the axis of revolution.* When the region is revolved, this segment will generate a typical washer cross section of the generated solid.
2. *Find the limits of integration.*
3. *Find the outer and inner radii* of the washer swept out by the line segment.
4. *Integrate* to find the volume.

To find the volume of a solid generated by revolving a region about the y-axis, we use the steps listed above but integrate with respect to y instead of x.

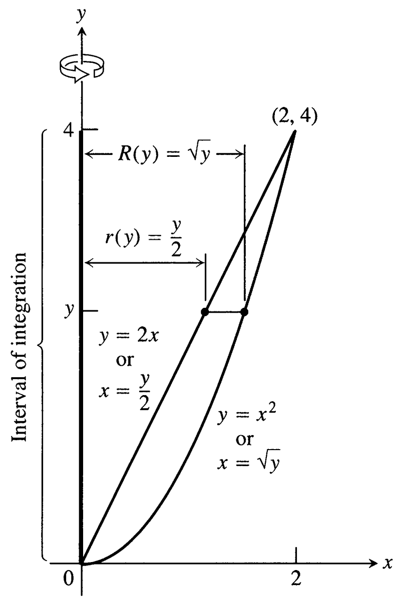
EXAMPLE 6 The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution

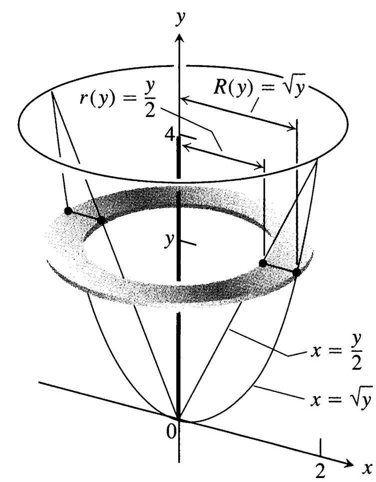
Step 1: Draw the region and sketch a line segment across it perpendicular to the axis of revolution, in this case the y-axis (Fig. 5.22).

Step 2: The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$.

Step 3: The radii of the washer swept out by the line segment are $R(y) = \sqrt{y}$, $r(y) = y/2$ (Figs. 5.22 and 5.23).



5.22 The region, limits of integration, and radii in Example 6.



5.23 The washer swept out by the line segment in Fig. 5.22.

Step 4:

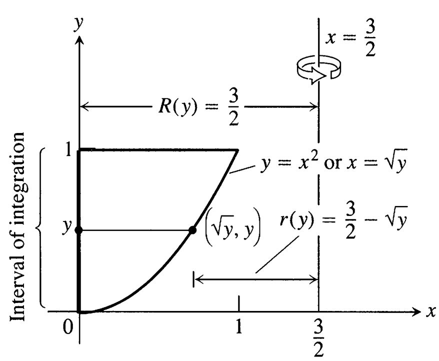
$$V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy \qquad \text{Eq. (3) with } y \text{ in place of } x$$

$$= \int_0^4 \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right) dy \qquad \text{Values from steps 2 and 3}$$

$$= \pi \int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi$$

❑

EXAMPLE 7 The region in the first quadrant enclosed by the parabola $y = x^2$, the y-axis, and the line $y = 1$ is revolved about the line $x = 3/2$ to generate a solid. Find the volume of the solid.



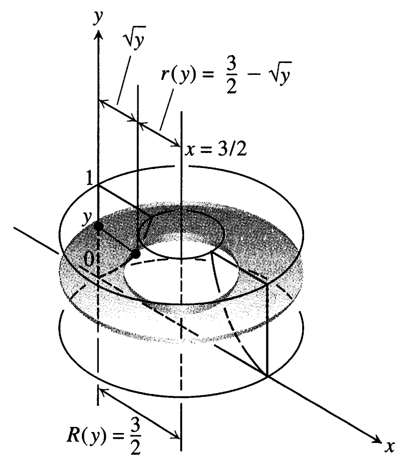
5.24 The region, limits of integration, and radii in Example 7.

Solution

Step 1: Draw the region and sketch a line segment across it perpendicular to the axis of revolution, in this case the line $x = 3/2$ (Fig. 5.24).

Step 2: The limits of integration are $y = 0$ to $y = 1$.

Step 3: The radii of the washer swept out by the line segment are $R(y) = 3/2$, $r(y) = (3/2) - \sqrt{y}$ (Figs. 5.24 and 5.25).



5.25 The washer swept out by the line segment in Fig. 5.24.

Step 4:

$$V = \int_c^d \pi\left([R(y)]^2 - [r(y)]^2\right) dy$$

Eq. (3) with y in place of x

$$= \int_0^1 \pi\left(\left[\frac{3}{2}\right]^2 - \left[\frac{3}{2} - \sqrt{y}\right]^2\right) dy$$

$$= \pi\int_0^1 (3\sqrt{y} - y)\, dy = \pi\left[2y^{3/2} - \frac{y^2}{2}\right]_0^1 = \frac{3\pi}{2}$$

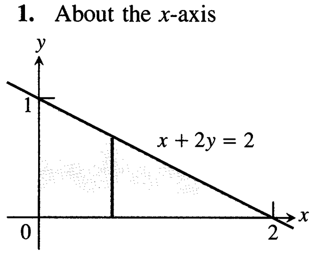
❑

Exercises 5.3

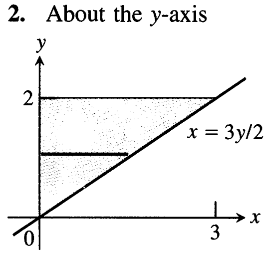
Volumes by the Disk Method

In Exercises 1–4, find the volume of the solid generated by revolving the shaded region about the given axis.

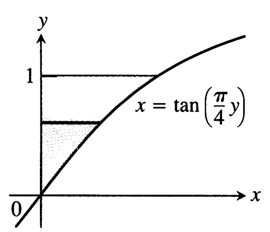
1. About the x-axis



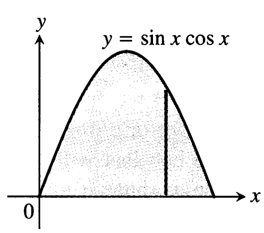
2. About the y-axis



3. About the y-axis



4. About the x-axis



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 5–10 about the x-axis.

5. $y = x^2, \quad y = 0, \quad x = 2$

6. $y = x^3, \quad y = 0, \quad x = 2$

7. $y = \sqrt{9 - x^2}, \quad y = 0$

8. $y = x - x^2, \quad y = 0$

9. $y = \sqrt{\cos x}, \quad 0 \le x \le \pi/2, \quad y = 0, \quad x = 0$

10. $y = \sec x, \quad y = 0, \quad x = -\pi/4, \quad x = \pi/4$

In Exercises 11 and 12, find the volume of the solid generated by revolving the region about the given line.

11. The region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y-axis, about the line $y = \sqrt{2}$

12. The region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2\sin x, 0 \le x \le \pi/2$, and on the left by the y-axis, about the line $y = 2$

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 13–18 about the y-axis.

13. $x = \sqrt{5}\, y^2, \quad x = 0, \quad y = -1, \quad y = 1$

14. $x = y^{3/2}, \quad x = 0, \quad y = 2$

15. $x = \sqrt{2 \sin 2y}, \quad 0 \le y \le \pi/2, \quad x = 0$

16. $x = \sqrt{\cos(\pi y/4)}, \quad -2 \le y \le 0, \quad x = 0$

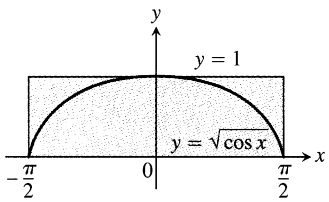
17. $x = 2/(y+1), \quad x = 0, \quad y = 0, \quad y = 3$

18. $x = \sqrt{2y}/(y^2+1), \quad x = 0, \quad y = 1$

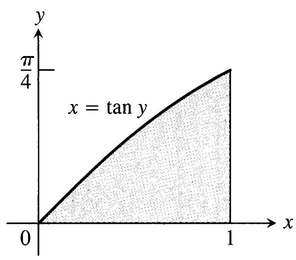
Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 19 and 20 about the indicated axes.

19. The x-axis



20. The y-axis



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 21–28 about the x-axis.

21. $y = x, \quad y = 1, \quad x = 0$

22. $y = 2x, \quad y = x, \quad x = 1$

23. $y = 2\sqrt{x}, \quad y = 2, \quad x = 0$

24. $y = -\sqrt{x}, \quad y = -2, \quad x = 0$

25. $y = x^2 + 1, \quad y = x + 3$

26. $y = 4 - x^2, \quad y = 2 - x$

27. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$

28. $y = \sec x, \quad y = \tan x, \quad x = 0, \quad x = 1$

In Exercises 29–34, find the volume of the solid generated by revolving each region about the y-axis.

29. The region enclosed by the triangle with vertices (1, 0), (2, 1), and (1, 1)

30. The region enclosed by the triangle with vertices (0, 1), (1, 0), and (1, 1)

31. The region in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis, and on the right by the line $x = 2$

32. The region bounded above by the curve $y = \sqrt{x}$ and below by the line $y = x$

33. The region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$

34. The region bounded on the left by the line $x = 4$ and on the right by the circle $x^2 + y^2 = 25$

In Exercises 35 and 36, find the volume of the solid generated by revolving each region about the given axis.

35. The region in the first quadrant bounded above by the curve $y = x^2$, below by the x-axis, and on the right by the line $x = 1$, about the line $x = -1$

36. The region in the second quadrant bounded above by the curve $y = -x^3$, below by the x-axis, and on the left by the line $x = -1$, about the line $x = -2$

Volumes of Solids of Revolution

37. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about

a) the x-axis;
b) the y-axis;
c) the line $y = 2$;
d) the line $x = 4$.

38. Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about

a) the line $x = 1$;
b) the line $x = 2$.

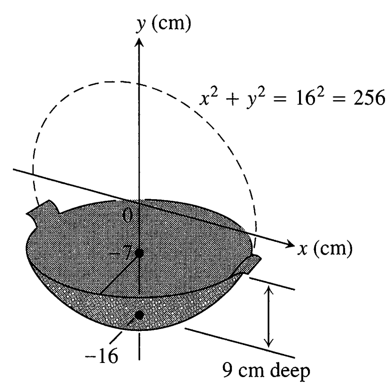
39. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about

a) the line $y = 1$;
b) the line $y = 2$;
c) the line $y = -1$.

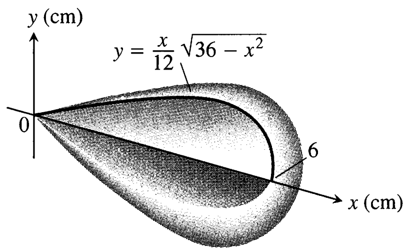
40. By integration, find the volume of the solid generated by revolving the triangular region with vertices (0, 0), $(b, 0)$, $(0, h)$ about

a) the x-axis;
b) the y-axis.

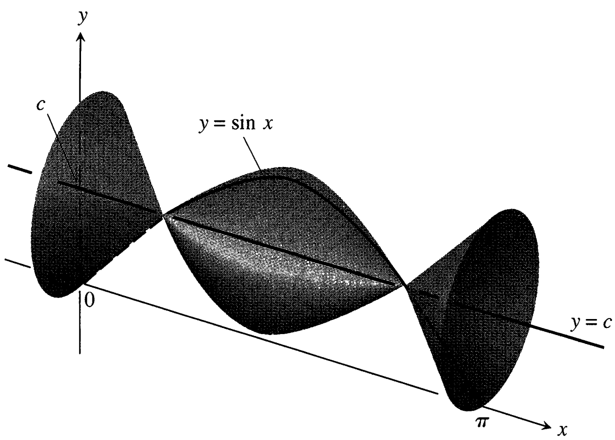
41. *Designing a wok.* You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that holds about 3 L if you make it 9 cm deep and give the sphere a radius of 16 cm. To be sure, you picture the wok as a solid of revolution, as shown here, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? (1 L = 1000 cm^3)



42. *Designing a plumb bob.* Having been asked to design a brass plumb bob that will weigh in the neighborhood of 190 g, you decide to shape it like the solid of revolution shown here. Find the plumb bob's volume. If you specify a brass that weighs 8.5 g/cm^3, how much will the plumb bob weigh (to the nearest gram)?



43. The arch $y = \sin x, 0 \le x \le \pi$, is revolved about the line $y = c$, $0 \le c \le 1$, to generate the solid in Fig. 5.26.



5.26 Exercise 43 asks for the value of c that minimizes the volume of this solid.

a) Find the value of c that minimizes the volume of the solid. What is the minimum value?

b) What value of c in [0, 1] maximizes the volume of the solid?

c) GRAPHER Graph the solid's volume as a function of c, first for $0 \le c \le 1$ and then on a larger domain. What happens to the volume of the solid as c moves away from [0, 1]? Does this make sense physically? Give reasons for your answer.

44. *An auxiliary fuel tank.* You are designing an auxiliary fuel tank that will fit under a helicopter's fuselage to extend its range. After some experimentation at your drawing board, you decide to shape the tank like the surface generated by revolving the curve $y = 1 - (x^2/16), -4 \le x \le 4$, about the x-axis (dimensions in feet).

a) How many cubic feet of fuel will the tank hold (to the nearest cubic foot)?

b) A cubic foot holds 7.481 gal. If the helicopter gets 2 mi to the gallon, how many additional miles will the helicopter be able to fly once the tank is installed (to the nearest mile)?

45. *The volume of a torus.* The disk $x^2 + y^2 \le a^2$ is revolved about the line $x = b$ $(b > a)$ to generate a solid shaped like a doughnut and called a *torus*. Find its volume. (*Hint:* $\int_{-a}^{a} \sqrt{a^2 - y^2}\, dy = \pi a^2/2$, since it is the area of a semicircle of radius a.)

46. a) A hemispherical bowl of radius a contains water to a depth h. Find the volume of water in the bowl.

b) (Related rates) Water runs into a sunken concrete hemispherical bowl of radius 5 m at the rate of 0.2 m^3/ sec. How fast is the water level in the bowl rising when the water is 4 m deep?

47. *Testing the consistency of the calculus definition of volume.* The volume formulas in this section are all consistent with the standard formulas from geometry.

a) As a case in point, show that if you revolve the region enclosed by the semicircle $y = \sqrt{a^2 - x^2}$ and the x-axis about the x-axis to generate a solid sphere, the disk formula for volume (Eq. 1) will give $(4/3)\pi a^3$ just as it should.

b) Use calculus to find the volume of a right circular cone of height h and base radius r.

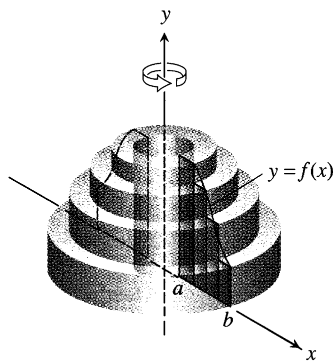
5.4

Cylindrical Shells

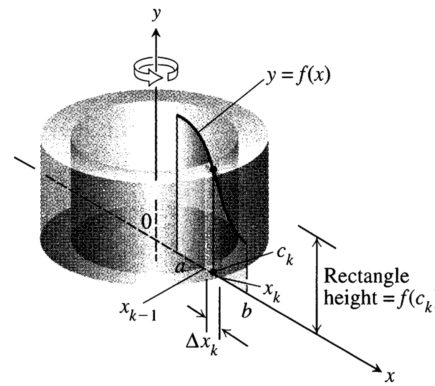
When we need to find the volume of a solid of revolution, cylindrical shells sometimes work better than washers (Fig. 5.27, on the following page). In part, the reason is that the formula they lead to does not require squaring.

The Shell Formula

Suppose we revolve the tinted region in Fig. 5.28 (on the following page) about the y-axis to generate a solid. To estimate the volume of the solid, we can approximate the region with rectangles based on a partition P of the interval $[a, b]$ over which the region stands. The typical approximating rectangle is Δx_k units wide by $f(c_k)$ units high, where c_k is the midpoint of the rectangle's base. A formula from geometry tells



5.27 A solid of revolution approximated by cylindrical shells.



5.28 The shell swept out by the *k*th rectangle.

us that the volume of the shell swept out by the rectangle is

$$\Delta V_k = 2\pi \times \text{average shell radius } \times \text{shell height } \times \text{thickness},$$

which in our case is

$$\Delta V_k = 2\pi \times c_k \times f(c_k) \times \Delta x_k .$$

We approximate the volume of the solid by adding the volumes of the shells swept out by the n rectangles based on P:

$$V \approx \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi c_k \, f(c_k)\Delta x_k . \qquad \text{A Riemann sum}$$

The limit of this sum as $\|P\| \to 0$ gives the volume of the solid:

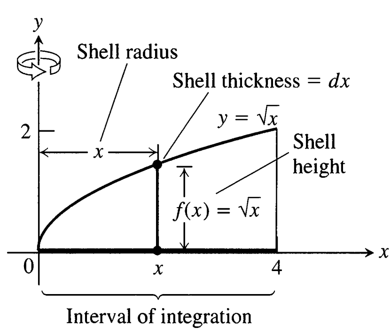
$$V = \lim_{\|P\|\to 0} \sum_{k=1}^{n} 2\pi c_k \, f(c_k)\Delta x_k = \int_a^b 2\pi x \, f(x)\, dx .$$

The Shell Formula for Revolution About the y-axis

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0, 0 \leq a \leq x \leq b$, about the y-axis is

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix} \begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx = \int_a^b 2\pi x \, f(x)\, dx . \qquad (1)$$

EXAMPLE 1 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

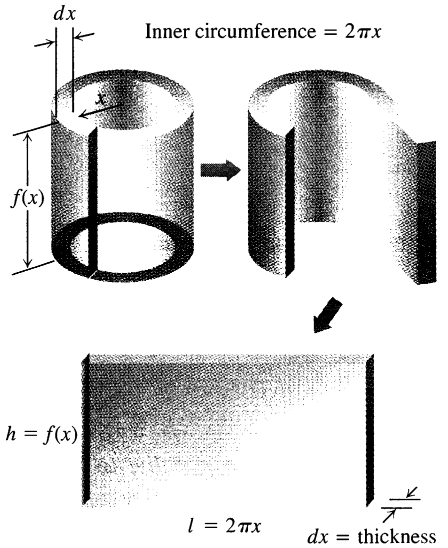


5.29 The region, shell dimensions, and interval of integration in Example 1.

Solution

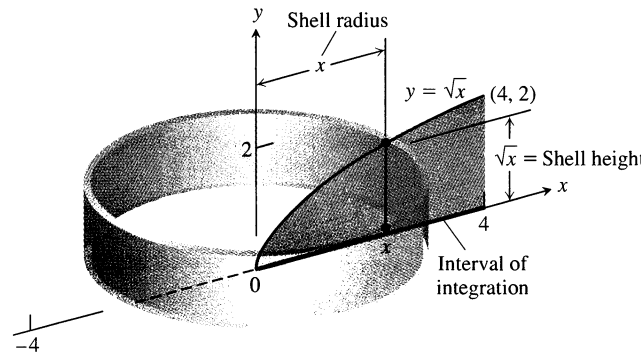
Step 1: Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Fig. 5.29). Label the segment's height (shell height) and distance from

One way to remember Eq. (1) is to imagine cutting and unrolling a cylindrical shell to get a (nearly) flat rectangular solid.



Almost a rectangular solid
$V \approx$ length $\times$ height $\times$ thickness
$\approx 2\pi x \cdot f(x) \cdot dx$

the axis of revolution (shell radius). The width of the segment is the shell thickness dx. (We drew the shell in Fig. 5.30, but you need not do that.)



5.30 The shell swept out by the line segment in Fig. 5.29.

Step 2: Find the limits of integration: x runs from $a = 0$ to $b = 4$.

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\\text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\\text{height}\end{pmatrix} dx \qquad \text{Eq. (1)}$$

$$= \int_0^4 2\pi(x)(\sqrt{x})\,dx \qquad \text{Values from steps 1 and 2}$$

$$= 2\pi \int_0^4 x^{3/2}\,dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}$$

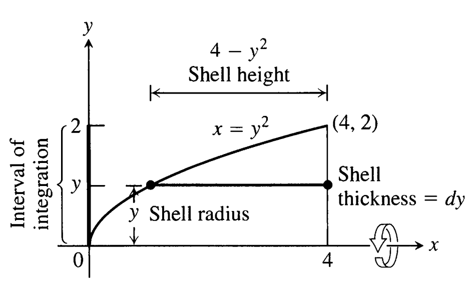
❑

Equation (1) is for vertical axes of revolution. For horizontal axes, we replace the x's with y's.

The Shell Formula for Revolution About the x-axis

$$V = \int_c^d 2\pi \begin{pmatrix}\text{shell}\\\text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\\text{height}\end{pmatrix} dy = \int_c^d 2\pi y f(y)\,dy \qquad (2)$$

(for $f(y) \geq 0$ and $0 \leq c \leq y \leq d$)

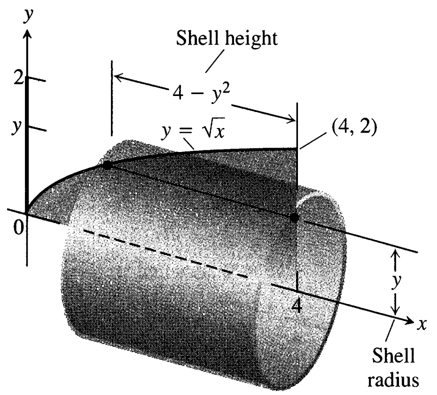


5.31 The region, shell dimensions, and interval of integration in Example 2.

EXAMPLE 2 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution

Step 1: Sketch the region and draw a line segment across it parallel to the axis of revolution (Fig. 5.31). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). The width of the segment is the shell thickness dy. (We drew the shell in Fig. 5.32, shown on the following page, but you need not do that.)



5.32 The shell swept out by the line segment in Fig. 5.31.

Step 2: Identify the limits of integration: y runs from $c = 0$ to $d = 2$.

Step 3: Integrate to find the volume.

$$V = \int_c^d 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dy \qquad \text{Eq. (2)}$$

$$= \int_0^2 2\pi(y)(4 - y^2)\,dy \qquad \text{Values from steps 1 and 2}$$

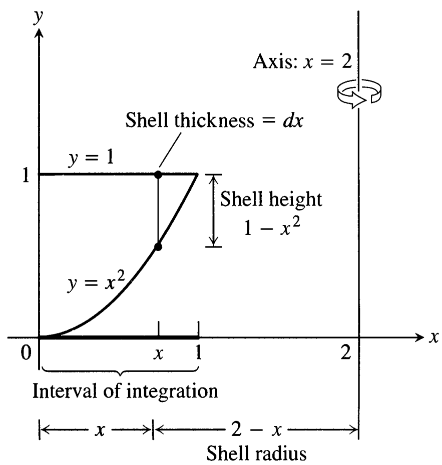
$$= 2\pi\left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi$$

This agrees with the disk method of calculation in Section 5.3, Example 1. ❑

How to Use the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these:

1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height), distance from the axis of revolution (shell radius), and width (shell thickness).
2. *Find* the limits of integration.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the appropriate variable (x or y) to find the volume.



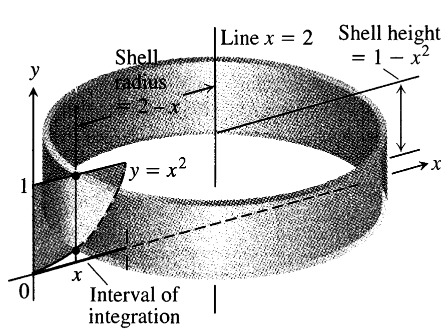
5.33 The region, shell dimensions, and interval of integration in Example 3.

In the next example, the axis of revolution is the vertical line $x = 2$.

EXAMPLE 3 The region in the first quadrant bounded by the parabola $y = x^2$, the y-axis, and the line $y = 1$ is revolved about the line $x = 2$ to generate a solid. Find the volume of the solid.

Solution

Step 1: Draw a line segment across the region parallel to the axis of revolution (the line $x = 2$) (Fig. 5.33). Label the segment's height (shell height), distance from the axis of revolution (shell radius), and width (in this case, dx). (We drew the shell in Fig. 5.34, but you need not do that.)



5.34 The shell swept out by the line segment in Fig. 5.33.

Step 2: The limits of integration: x runs from $a = 0$ to $b = 1$.

Step 3:

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx \qquad \text{Eq. (1)}$$

$$= \int_0^1 2\pi(2 - x)(1 - x^2)\,dx \qquad \text{Values from steps 1 and 2}$$

$$= 2\pi\int_0^1 (2 - x - 2x^2 + x^3)\,dx$$

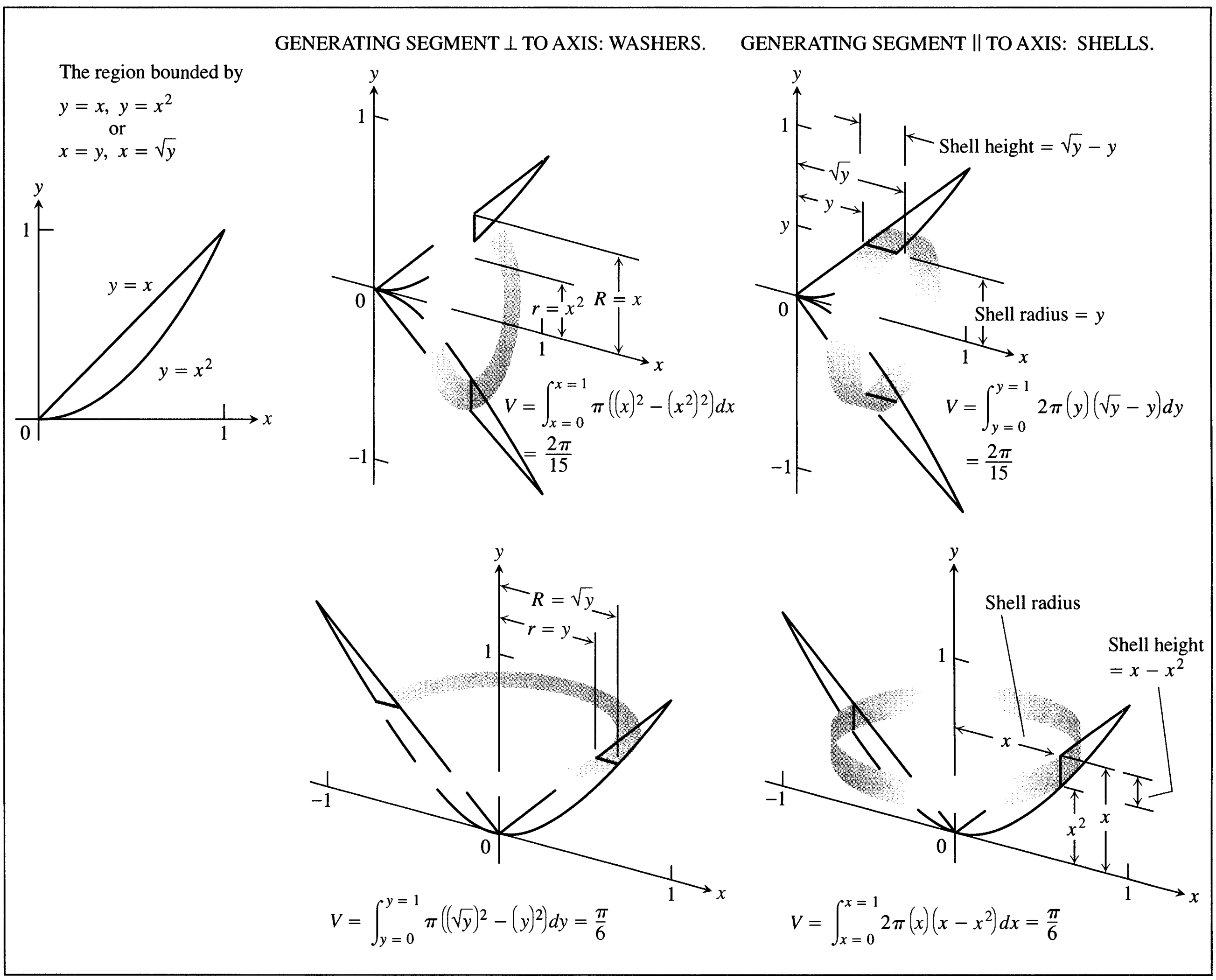
$$= \frac{13\pi}{6}$$

❑

Table 5.1 summarizes the washer and shell methods for the solid generated by revolving the region bounded by $y = x$ and $y = x^2$ about the coordinate axes. For this particular region, both methods work well for both axes of revolution. But this is not always the case. When a region is revolved about the y-axis, for example, and washers are used, we must integrate with respect to y. However, it may not be possible to express the integrand in terms of y. In such a case, the shell method allows us to integrate with respect to x instead.

The washer and shell methods for calculating volumes of solids of revolution always agree. In Section 6.1 (Exercise 52), we will be able to prove the equivalence for a broad class of solids.

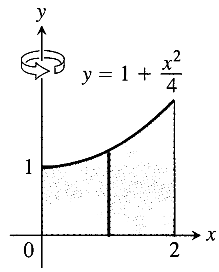
Table 5.1 Washers vs. shells



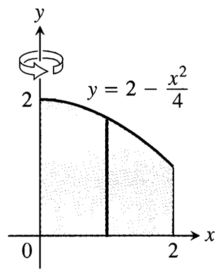
Exercises 5.4

In Exercises 1–6, use the shell method to find the volumes of the solids generated by revolving the shaded region about the indicated axis.

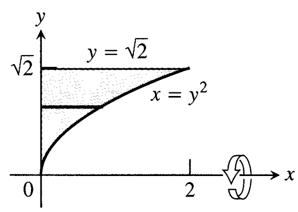
1.



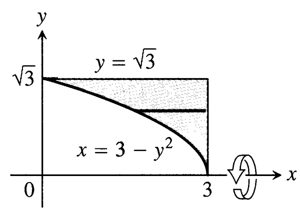
2.



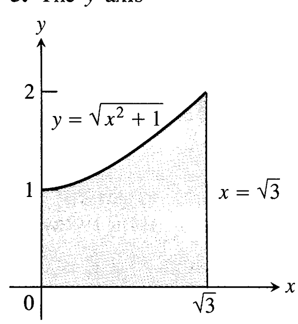
3.



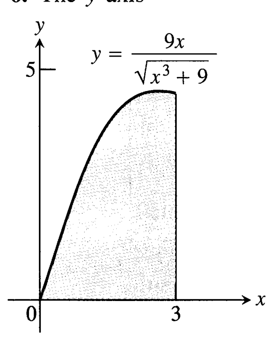
4.



5. The y-axis



6. The y-axis



Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–14 about the y-axis.

7. $y = x, \quad y = -x/2, \quad x = 2$

8. $y = 2x, \quad y = x/2, \quad x = 1$

9. $y = x^2, \quad y = 2 - x, \quad x = 0$, for $x \geq 0$

10. $y = 2 - x^2, \quad y = x^2, \quad x = 0$

11. $y = \sqrt{x}, \quad y = 0, \quad x = 4$

12. $y = 2x - 1, \quad y = \sqrt{x}, \quad x = 0$

13. $y = 1/x, \quad y = 0, \quad x = 1/2, \quad x = 2$

14. $y = 3/(2\sqrt{x}), \quad y = 0, \quad x = 1, \quad x = 4$

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 15–22 about the x-axis.

15. $x = \sqrt{y}, \quad x = -y, \quad y = 2$

16. $x = y^2, \quad x = -y, \quad y = 2$

17. $x = 2y - y^2, \quad x = 0$

18. $x = 2y - y^2, \quad x = y$

19. $y = |x|, \quad y = 1$

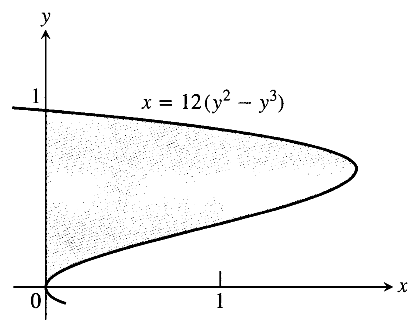
20. $y = x, \quad y = 2x, \quad y = 2$

21. $y = \sqrt{x}, \quad y = 0, \quad y = x - 2$

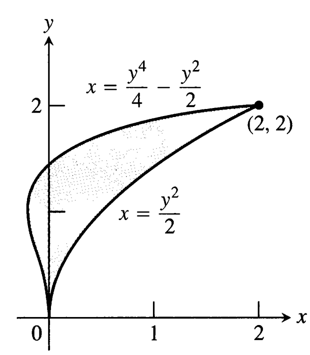
22. $y = \sqrt{x}, \quad y = 0, \quad y = 2 - x$

In Exercises 23 and 24, use the shell method to find the volumes of the solids generated by revolving the shaded regions about the indicated axes.

23. **a)** The x-axis
b) The line $y = 1$
c) The line $y = 8/5$
d) The line $y = -2/5$



24. **a)** The x-axis
b) The line $y = 2$
c) The line $y = 5$
d) The line $y = -5/8$



In Exercises 25–32, find the volumes of the solids generated by revolving the regions about the given axes. If you think it would be better to use disks or washers in any given instance, feel free to do so.

25. The triangle with vertices (1, 1), (1, 2), and (2, 2) about (a) the x-axis; (b) the y-axis; (c) the line $x = 10/3$; (d) the line $y = 1$

26. The region in the first quadrant bounded by the curve $x = y - y^3$ and the y-axis about (a) the x-axis; (b) the line $y = 1$

27. The region in the first quadrant bounded by $x = y - y^3$, $x = 1$, and $y = 1$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 1$; (d) the line $y = 1$

28. The triangular region bounded by the lines $2y = x + 4$, $y = x$, and $x = 0$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 4$; (d) the line $y = 8$

29. The region in the first quadrant bounded by $y = x^3$ and $y = 4x$ about (a) the x-axis; (b) the line $y = 8$

30. The region bounded by $y = \sqrt{x}$ and $y = x^2/8$ about (a) the x-axis; (b) the y-axis

31. The region bounded by $y = 2x - x^2$ and $y = x$ about (a) the y-axis; (b) the line $x = 1$

32. The region bounded by $y = \sqrt{x}$, $y = 2$, $x = 0$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 4$; (d) the line $y = 2$

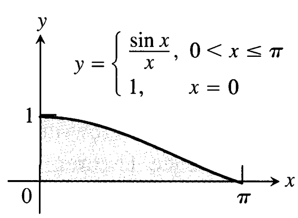
33. The region in the first quadrant that is bounded above by the curve $y = 1/x^{1/4}$, on the left by the line $x = 1/16$, and below by the line $y = 1$, is revolved about the x-axis to generate a solid. Find the volume of the solid by (a) the washer method; (b) the shell method.

34. The region in the first quadrant that is bounded above by the curve $y = 1/\sqrt{x}$, on the left by the line $y = 1/4$, and below by the line $y = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid by (a) the washer method; (b) the shell method.

35. Let $f(x) = \begin{cases} (\sin x)/x, & 0 < x \le \pi \\ 1, & x = 0. \end{cases}$

a) Show that $xf(x) = \sin x, 0 \le x \le \pi$.

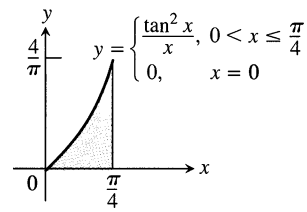
b) Find the volume of the solid generated by revolving the shaded region about the y-axis.



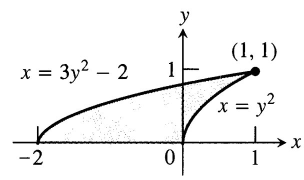
36. Let $g(x) = \begin{cases} (\tan x)^2/x, & 0 < x \le \pi/4 \\ 0, & x = 0. \end{cases}$

a) Show that $xg(x) = (\tan x)^2, 0 \le x \le \pi/4$.

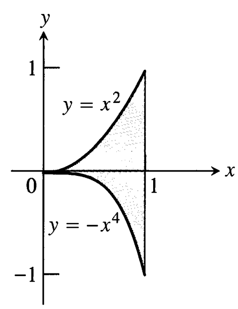
b) Find the volume of the solid generated by revolving the shaded region about the y-axis.



37. The region shown here is to be revolved about the x-axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Explain.



38. The region shown here is to be revolved about the y-axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Give reasons for your answers.



39. Suppose that the function $f(x)$ is nonnegative and continuous for $x \ge 0$. Suppose also that, for every positive number b, revolving the region enclosed by the graph of f, the coordinate axes, and the line $x = b$ about the y-axis generates a solid of volume $2\pi b^3$. Find $f(x)$.

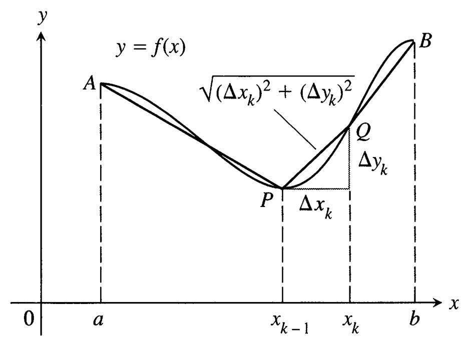
5.5 Lengths of Plane Curves

We approximate the length of a curved path in the plane the way we use a ruler to estimate the length of a curved road on a map, by measuring from point to point with straight-line segments and adding the results. There is a limit to the accuracy of such an estimate, however, imposed in part by how accurately we measure and in part by how many line segments we use.

With calculus we can usually do a better job because we can imagine using straight-line segments as short as we please, each set of segments making a polygonal path that fits the curve more tightly than before. When we proceed this way, with a smooth curve, the lengths of the polygonal paths approach a limit we can calculate with an integral.

The Basic Formula

Suppose we want to find the length of the curve $y = f(x)$ from $x = a$ to $x = b$. We partition $[a, b]$ in the usual way and connect the corresponding points on the curve with line segments to form a polygonal path that approximates the curve (Fig. 5.35). If we can find a formula for the length of the path, we will have a formula for approximating the length of the curve.



5.35 A typical segment *PQ* of a polygonal path approximating the curve *AB*.

The length of a typical line segment PQ (see the figure) is $\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$. The length of the curve is therefore approximated by the sum

$$\sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \tag{1}$$

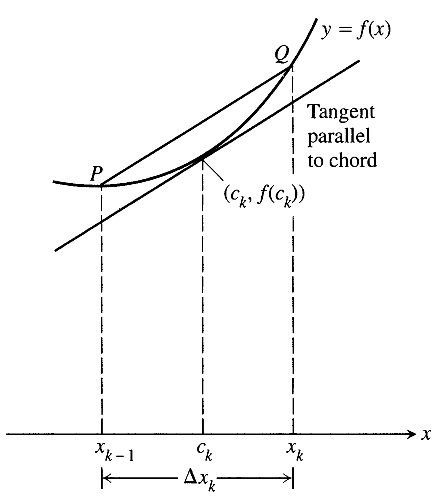
We expect the approximation to improve as the partition of $[a, b]$ becomes finer, and we would like to show that the sums in (1) approach a calculable limit as the norm of the partition goes to zero. To show this, we rewrite the sum in (1) in a form to which we can apply the Integral Existence Theorem from Chapter 4. Our starting point is the Mean Value Theorem for derivatives.

Definition

A function with a continuous first derivative is said to be **smooth** and its graph is called a **smooth curve.**

If f is smooth, by the Mean Value Theorem there is a point $(c_k, f(c_k))$ on the curve between P and Q where the tangent is parallel to the segment PQ (Fig. 5.36). At this point

$$f'(c_k) = \frac{\Delta y_k}{\Delta x_k}, \qquad \text{or} \qquad \Delta y_k = f'(c_k)\Delta x_k.$$



5.36 Enlargement of the arc *PQ* in Fig. 5.35.

With this substitution for Δy_k, the sums in (1) take the form

$$\sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (f'(c_k)\Delta x_k)^2} = \sum_{k=1}^{n} \sqrt{1 + (f'(c_k))^2}\, \Delta x_k.$$

A Riemann sum

Because $\sqrt{1 + (f'(x))^2}$ is continuous on $[a, b]$, the limit of the sums on the right as the norm of the partition goes to zero is $\int_a^b \sqrt{1 + (f'(x))^2}\, dx$. We define the length of the curve to be the value of this integral.

Definition

If f is smooth on $[a, b]$, the **length** of the curve $y = f(x)$ from a to b is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + (f'(x))^2}\, dy. \tag{2}$$

EXAMPLE 1 Find the length of the curve

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1, \qquad 0 \le x \le 1.$$

Solution We use Eq. (2) with $a = 0$, $b = 1$, and

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{1/2} = 2\sqrt{2} x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2} x^{1/2}\right)^2 = 8x.$$

The length of the curve from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$$

Eq. (2) with $a = 0$, $b = 1$

$$= \frac{2}{3} \cdot \frac{1}{8} (1 + 8x)^{3/2} \Big]_0^1 = \frac{13}{6}.$$

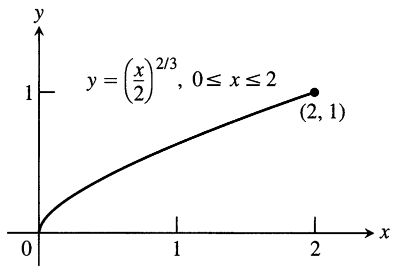
Let $u = 1 + 8x$, integrate, and replace u by $1 + 8x$. ❑

Dealing with Discontinuities in *dy/dx*

At a point on a curve where dy/dx fails to exist, dx/dy may exist and we may be able to find the curve's length by expressing x as a function of y and applying the following analogue of Eq. (2):

Formula for the Length of a Smooth Curve $x = g(y)$, $c \le y \le d$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + (g'(y))^2}\, dy. \tag{3}$$



5.37 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$.

EXAMPLE 2 Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Solution The derivative

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length with Eq. (2).

We therefore rewrite the equation to express x in terms of y:

$$y = \left(\frac{x}{2}\right)^{2/3}$$

$$y^{3/2} = \frac{x}{2} \qquad \text{Raise both sides to the power 3/2.}$$

$$x = 2y^{3/2}. \qquad \text{Solve for } x.$$

From this we see that the curve whose length we want is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Fig. 5.37).

The derivative

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2}$$

is continuous on $[0, 1]$. We may therefore use Eq. (3) to find the curve's length:

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_0^1 \sqrt{1+9y}\,dy \qquad \text{Eq. (3) with } c = 0,\ d = 1$$

$$= \frac{1}{9}\cdot\frac{2}{3}(1+9y)^{3/2}\Big]_0^1 \qquad \text{Let } u = 1+9y,\ du/9 = dy, \text{ integrate, and substitute back.}$$

$$= \frac{2}{27}(10\sqrt{10} - 1) \approx 2.27.$$

❑

The Short Differential Formula

The equations

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \quad \text{and} \quad L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \tag{4}$$

are often written with differentials instead of derivatives. This is done formally by thinking of the derivatives as quotients of differentials and bringing the dx and dy inside the radicals to cancel the denominators. In the first integral we have

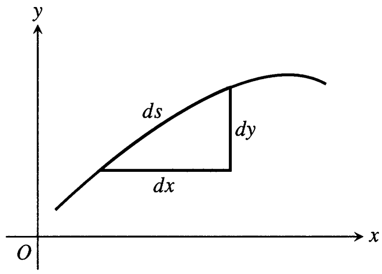
$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \frac{dy^2}{dx^2}}\,dx = \sqrt{dx^2 + \frac{dy^2}{dx^2}\,dx^2} = \sqrt{dx^2 + dy^2}.$$

In the second integral we have

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \sqrt{1 + \frac{dx^2}{dy^2}}\,dy = \sqrt{dy^2 + \frac{dx^2}{dy^2}\,dy^2} = \sqrt{dx^2 + dy^2}.$$

Thus the integrals in (4) reduce to the same differential formula:

$$L = \int_a^b \sqrt{dx^2 + dy^2}. \tag{5}$$



5.38 Diagram for remembering the equation $ds = \sqrt{dx^2 + dy^2}$.

Of course, dx and dy must be expressed in terms of a common variable, and appropriate limits of integration must be found before the integration in Eq. (5) is performed.

We can shorten Eq. (5) still further. Think of dx and dy as two sides of a small triangle whose "hypotenuse" is $ds = \sqrt{dx^2 + dy^2}$ (Fig. 5.38). The differential ds is then regarded as a differential of arc length that can be integrated between appropriate limits to give the length of the curve. With $\sqrt{dx^2 + dy^2}$ set equal to ds, the integral in Eq. (5) simply becomes the integral of ds.

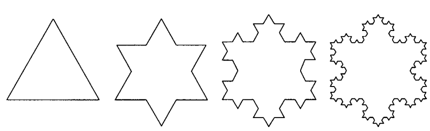
Definition

The Arc Length Differential and the Differential Formula for Arc Length

$$ds = \sqrt{dx^2 + dy^2} \qquad\qquad L = \int ds$$

arc length differential	differential formula for arc length

* Curves with Infinite Length



5.39 The first four polygonal approximations in the construction of Helga von Koch's snowflake.

As you may recall from Section 2.6, Helga von Koch's snowflake curve K is the limit curve of an infinite sequence $C_1, C_2, \ldots, C_n, \ldots$ of "triangular" polygonal curves. Figure 5.39 shows the first four curves in the sequence. Each time we introduce a new vertex in the construction process, it remains as a vertex in all subsequent curves and becomes a point on the limit curve K. This means that each of the C's is itself a polygonal approximation of K—the endpoints of its sides all belonging to K. The length of K should therefore be the limit of the lengths of the curves C_n. At least, that is what it should be if we apply the definition of length we developed for smooth curves.

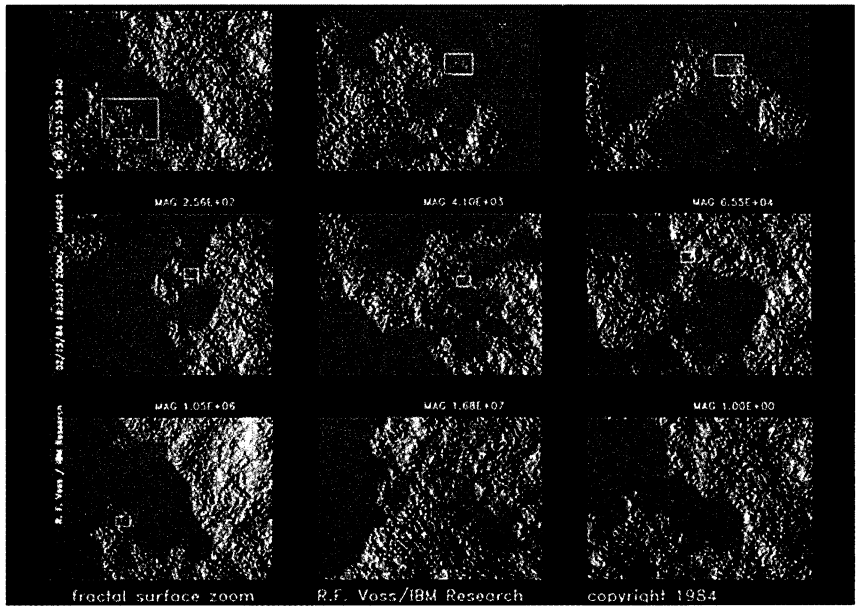
What, then, is the limit of the lengths of the curves C_n? If the original equilateral triangle C_1 has sides of length 1, the total length of C_1 is 3. To make C_2 from C_1, we replace each side of C_1 by four segments, each of which is one-third as long as the original side. The total length of C_2 is therefore 3(4/3). To get the length of C_3, we multiply by 4/3 again. We do so again to get the length of C_4. By the time we get out to C_n, we have a curve of length $3(4/3)^{n-1}$.

Curve Number	1	2	3	$\cdots$	n	$\cdots$
Length	3	$3\left(\frac{4}{3}\right)$	$3\left(\frac{4}{3}\right)^2$	$\cdots$	$3\left(\frac{4}{3}\right)^{n-1}$	$\cdots$

The length of C_{10} is nearly 40 and the length of C_{100} is greater than 7,000,000,000,000. The lengths grow too rapidly to have a finite limit. Therefore the snowflake curve has no length, or, if you prefer, infinite length.

What went wrong? Nothing. The formulas we derived for length are for the graphs of smooth functions, curves that are smooth enough to have a continuously turning tangent at every point. Helga von Koch's snowflake curve is too rough for that, and our derivative-based formulas do not apply.

Benoit Mandelbrot's theory of fractals has proved to be a rich source of curves with infinite length, curves that when magnified prove to be as rough and varied as they looked before magnification. Like coastlines on an ocean, such curves cannot be smoothed out by magnification (Fig. 5.40, on the following page).



5.40 Repeated magnifications of a fractal coastline. Like Helga Von Koch's snowflake curve, coasts like these are too rough to have a measurable length.

Exercises 5.5

Finding Integrals for Lengths of Curves

In Exercises 1–8:

a) Set up an integral for the length of the curve.

b) Graph the curve to see what it looks like.

c) Use your grapher's or computer's integral evaluator to find the curve's length numerically.

1. $y = x^2, \quad -1 \le x \le 2$

2. $y = \tan x, \quad -\pi/3 \le x \le 0$

3. $x = \sin y, \quad 0 \le y \le \pi$

4. $x = \sqrt{1 - y^2}, \quad -1/2 \le y \le 1/2$

5. $y^2 + 2y = 2x + 1$ from $(-1, -1)$ to $(7, 3)$

6. $y = \sin x - x \cos x, \quad 0 \le x \le \pi$

7. $y = \displaystyle\int_0^x \tan t \, dt, \quad 0 \le x \le \pi/6$

8. $x = \displaystyle\int_0^y \sqrt{\sec^2 t - 1} \, dt, \quad -\pi/3 \le y \le \pi/4$

Finding Lengths of Curves

Find the lengths of the curves in Exercises 9–18. If you have a grapher, you may want to graph these curves to see what they look like.

9. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$

10. $y = x^{3/2}$ from $x = 0$ to $x = 4$

11. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)

12. $x = (y^{3/2}/3) - y^{1/2}$ from $y = 1$ to $y = 9$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)

13. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)

14. $x = (y^3/6) + 1/(2y)$ from $y = 2$ to $y = 3$
(*Hint:* $1 + (dx/dy)^2$ is a perfect square.)

15. $y = (3/4)x^{4/3} - (3/8)x^{2/3} + 5, \quad 1 \le x \le 8$

16. $y = (x^3/3) + x^2 + x + 1/(4x + 4), \quad 0 \le x \le 2$

17. $x = \displaystyle\int_0^y \sqrt{\sec^4 t - 1} \, dt, \quad -\pi/4 \le y \le \pi/4$

18. $y = \displaystyle\int_{-2}^x \sqrt{3t^4 - 1} \, dt, \quad -2 \le x \le -1$

19. a) Find a curve through the point (1, 1) whose length integral (Eq. 2) is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}} \, dx.$$

b) How many such curves are there? Give reasons for your answer.

20. **a)** Find a curve through the point (0, 1) whose length integral (Eq. 3) is

$$L = \int_1^2 \sqrt{1 + \frac{1}{y^4}}\, dy.$$

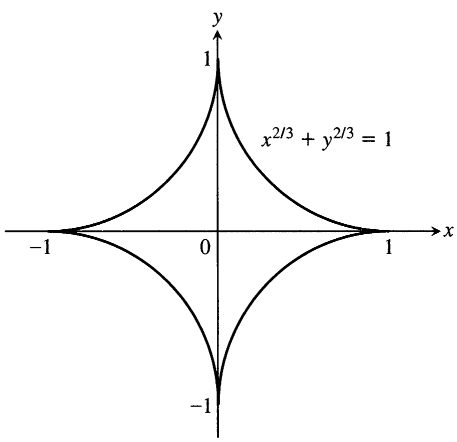
b) How many such curves are there? Give reasons for your answer.

21. Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt$$

from $x = 0$ to $x = \pi/4$.

22. *The length of an astroid.* The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not "asteroids") because of their starlike appearance (see the accompanying figure). Find the length of this particular astroid by finding the length of half the first-quadrant portion, $y = (1 - x^{2/3})^{3/2}$, $\sqrt{2}/4 \le x \le 1$, and multiplying by 8.



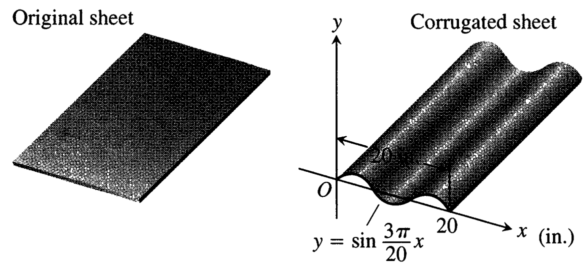
Numerical Integration

You may have wondered why so many of the curves we have been working with have unusual formulas. The reason is that the square root $\sqrt{1 + (dy/dx)^2}$ that appears in the integrals for length and surface area almost never leads to a function whose antiderivative we can find. In fact, the square root itself is a well-known source of nonelementary integrals. Most integrals for length and surface area have to be evaluated numerically, as in Exercises 23 and 24.

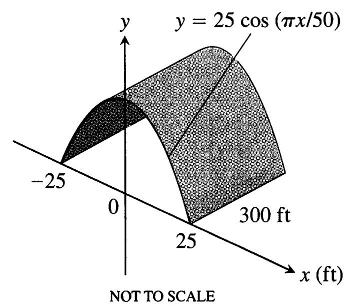
23. Your metal fabrication company is bidding for a contract to make sheets of corrugated iron roofing like the one shown here. The cross sections of the corrugated sheets are to conform to the curve

$$y = \sin \frac{3\pi}{20} x, \quad 0 \le x \le 20 \text{ in.}$$

If the roofing is to be stamped from flat sheets by a process that does not stretch the material, how wide should the original material be? To find out, use numerical integration to approximate the length of the sine curve to 2 decimal places.



24. Your engineering firm is bidding for the contract to construct the tunnel shown here. The tunnel is 300 ft long and 50 ft wide at the base. The cross section is shaped like one arch of the curve $y = 25 \cos(\pi x/50)$. Upon completion, the tunnel's inside surface (excluding the roadway) will be treated with a waterproof sealer that costs \$1.75 per square foot to apply. How much will it cost to apply the sealer? (*Hint:* Use numerical integration to find the length of the cosine curve.)



Theory and Examples

25. Is there a smooth curve $y = f(x)$ whose length over the interval $0 \le x \le a$ is always $\sqrt{2}a$? Give reasons for your answer.

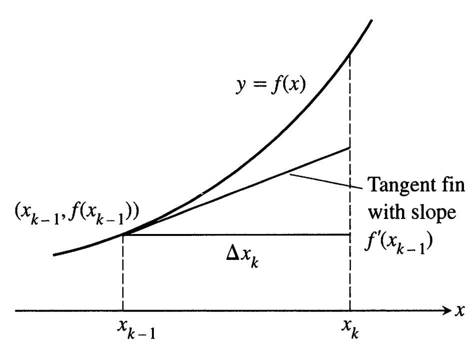
26. *Using tangent fins to derive the length formula for curves.* Assume f is smooth on $[a, b]$ and partition the interval $[a, b]$ in the usual way. In each subinterval $[x_{k-1}, x_k]$ construct the *tangent fin* at the point $(x_{k-1}, f(x_{k-1}))$, shown in the figure.

a) Show that the length of the kth tangent fin over the interval $[x_{k-1}, x_k]$ equals $\sqrt{(\Delta x_k)^2 + (f'(x_{k-1})\Delta x_k)^2}$.

b) Show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\text{length of } k\text{th tangent fin}) = \int_a^b \sqrt{1 + (f'(x))^2}\, dx,$$

which is the length L of the curve $y = f(x)$ from a to b.



CAS Explorations and Projects

In Exercises 27–32, use a CAS to perform the following steps for the given curve over the closed interval.

a) Plot the curve together with the polygonal path approximations for $n = 2,\ 4,\ 8$ partition points over the interval. (See Fig. 5.35.)
b) Find the corresponding approximation to the length of the curve by summing the lengths of the line segments.
c) Evaluate the length of the curve using an integral. Compare your approximations for $n = 2, 4, 8$ to the actual length given by the integral. How does the actual length compare with the approximations as n increases? Explain your answer.

27. $f(x) = \sqrt{1 - x^2}, \quad -1 \le x \le 1$

28. $f(x) = x^{1/3} + x^{2/3}, \quad 0 \le x \le 2$

29. $f(x) = \sin(\pi x^2), \quad 0 \le x \le \sqrt{2}$

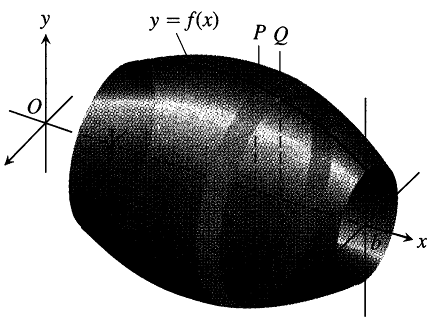
30. $f(x) = x^2 \cos x, \quad 0 \le x \le \pi$

31. $f(x) = \dfrac{x - 1}{4x^2 + 1}, \quad -\dfrac{1}{2} \le x \le 1$

32. $f(x) = x^3 - x^2, \quad -1 \le x \le 1$

5.6 Areas of Surfaces of Revolution

When you jump rope, the rope sweeps out a surface in the space around you, a surface called a surface of revolution. As you can imagine, the area of this surface depends on the rope's length and on how far away each segment of the rope swings. This section explores the relation between the area of a surface of revolution and the length and reach of the curve that generates it. The areas of more complicated surfaces will be treated in Chapter 14.



5.41 The surface generated by revolving the graph of a nonnegative function $y = f(x), a \le x \le b$, about the x-axis. The surface is a union of bands like the one swept out by the arc PQ.

The Basic Formula

Suppose we want to find the area of the surface swept out by revolving the graph of a nonnegative function $y = f(x), a \le x \le b$, about the x-axis. We partition $[a, b]$ in the usual way and use the points in the partition to partition the graph into short arcs. Figure 5.41 shows a typical arc PQ and the band it sweeps out as part of the graph of f.

As the arc PQ revolves about the x-axis, the line segment joining P and Q sweeps out part of a cone whose axis lies along the x-axis (magnified view in Fig. 5.42). A piece of a cone like this is called a *frustum* of the cone, *frustum* being Latin for "piece." The surface area of the frustum approximates the surface area of the band swept out by the arc PQ.

The surface area of the frustum of a cone (see Fig. 5.43) is 2π times the average of the base radii times the slant height:

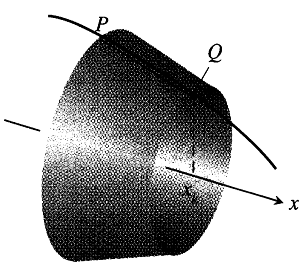
$$\text{Frustum surface area } = 2\pi \cdot \frac{r_1 + r_2}{2} \cdot L = \pi(r_1 + r_2)L.$$

For the frustum swept out by the segment PQ (Fig. 5.44), this works out to be

$$\text{Frustum surface area } = \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

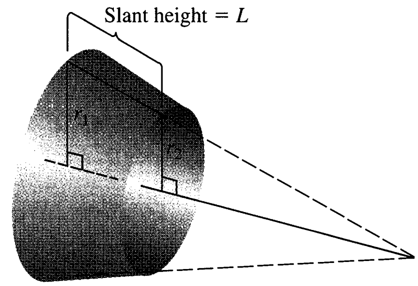
The area of the original surface, being the sum of the areas of the bands swept out by arcs like arc PQ, is approximated by the frustum area sum

$$\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \tag{1}$$

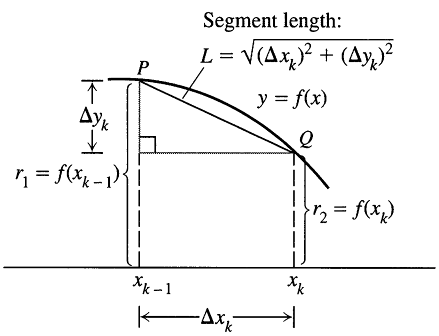


5.42 The line segment joining P and Q sweeps out a frustum of a cone.

We expect the approximation to improve as the partition of $[a, b]$ becomes finer, and we would like to show that the sums in (1) approach a calculable limit as the norm of the partition goes to zero.



5.43 The important dimensions of the frustum in Fig. 5.42.



5.44 Dimensions associated with the arc and segment *PQ*.

To show this, we try to rewrite the sum in (1) as the Riemann sum of some function over the interval from a to b. As in the calculation of arc length, we begin by appealing to the Mean Value Theorem for derivatives.

If f is smooth, then by the Mean Value Theorem, there is a point $(c_k, f(c_k))$ on the curve between P and Q where the tangent is parallel to the segment PQ (Fig. 5.45). At this point,

$$f'(c_k) = \frac{\Delta y_k}{\Delta x_k},$$

$$\Delta y_k = f'(c_k)\Delta x_k.$$

With this substitution for Δy_k, the sums in (1) take the form

$$\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (f'(c_k)\Delta x_k)^2}$$

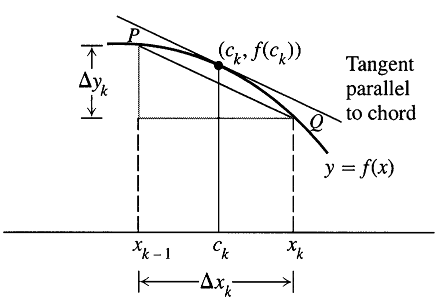
$$= \sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{1 + (f'(c_k))^2}\,\Delta x_k. \quad (2)$$

At this point there is both good news and bad news.

The bad news is that the sums in (2) are not the Riemann sums of any function because the points x_{k-1}, x_k, and c_k are not the same and there is no way to make them the same. The good news is that this does not matter. A theorem called Bliss's theorem, from advanced calculus, assures us that as the norm of the partition of $[a, b]$ goes to zero, the sums in Eq. (2) converge to

$$\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx$$

just the way we want them to. We therefore define this integral to be the area of the surface swept out by the graph of f from a to b.



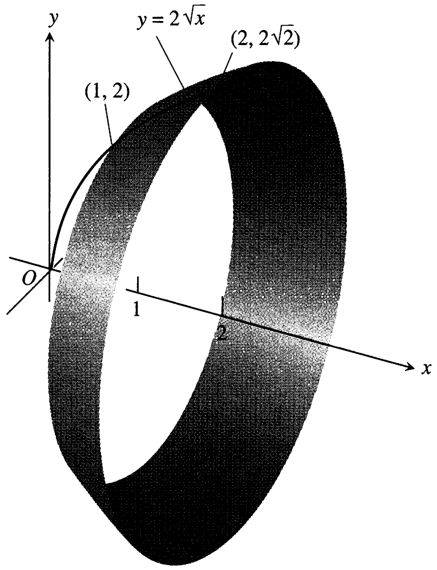
5.45 If f is smooth, the Mean Value Theorem guarantees the existence of a point on arc *PQ* where the tangent is parallel to segment *PQ*.

Definition

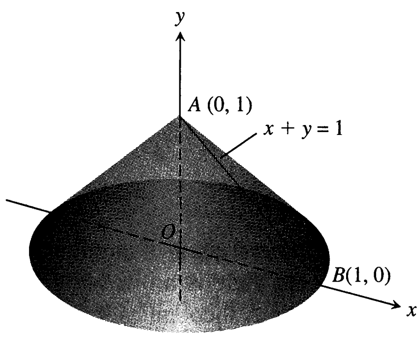
The Surface Area Formula for the Revolution About the *x*-axis

If the function $f(x) \geq 0$ is smooth on $[a, b]$, the **area** of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx. \quad (3)$$



5.46 Example 1 calculates the area of this surface.



5.47 Revolving line segment *AB* about the *y*-axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).

The square root in Eq. (3) is the same one that appears in the formula for the length of the generating curve.

EXAMPLE 1 Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$, about the x-axis (Fig. 5.46).

Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{Eq. (3)}$$

with

$$a = 1, \qquad b = 2, \qquad y = 2\sqrt{x}, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x}},$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}.$$

With these substitutions,

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\, \frac{\sqrt{x+1}}{\sqrt{x}}\, dx = 4\pi \int_1^2 \sqrt{x+1}\, dx$$

$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}(3\sqrt{3} - 2\sqrt{2}).$$

❑

Revolution About the *y*-axis

For revolution about the y-axis, we interchange x and y in Eq. (3).

Surface Area Formula for Revolution About the *y*-axis

If $x = g(y) \ge 0$ is smooth on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy. \qquad \textbf{(4)}$$

EXAMPLE 2 The line segment $x = 1 - y$, $0 \le y \le 1$, is revolved about the y-axis to generate the cone in Fig. 5.47. Find its lateral surface area.

Solution Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

To see how Eq. (4) gives the same result, we take

$$c = 0, \qquad d = 1, \qquad x = 1 - y, \qquad \frac{dx}{dy} = -1,$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

and calculate

$$\begin{aligned} S &= \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 2\pi (1 - y)\sqrt{2}\, dy \\ &= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right) \\ &= \pi\sqrt{2}. \end{aligned}$$

The results agree, as they should. ❑

The Short Differential Form

The equations

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{and} \qquad S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

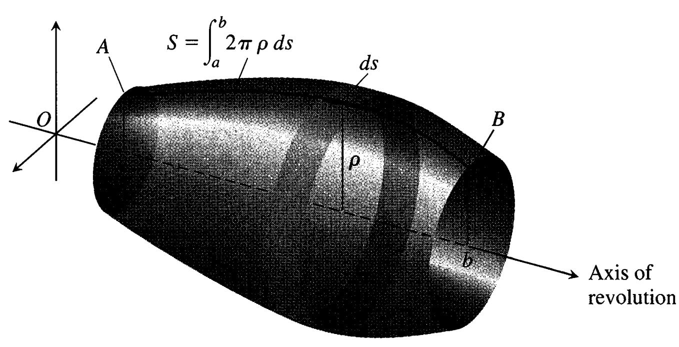
are often written in terms of the arc length differential $ds = \sqrt{dx^2 + dy^2}$ as

$$S = \int_a^b 2\pi y\, ds \qquad \text{and} \qquad S = \int_c^d 2\pi x\, ds.$$

In the first of these, y is the distance from the x-axis to an element of arc length ds. In the second, x is the distance from the y-axis to an element of arc length ds. Both integrals have the form

$$S = \int 2\pi(\text{radius})(\text{band width}) = \int 2\pi\rho\, ds,$$

where ρ is the radius from the axis of revolution to an element of arc length ds (Fig. 5.48).



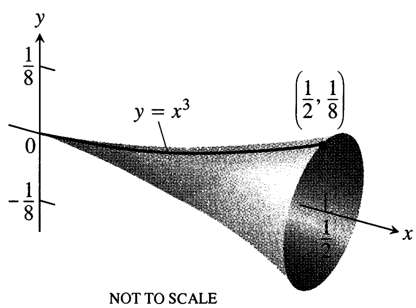
5.48 The area of the surface swept out by revolving arc AB about the axis shown here is $\int_a^b 2\pi\rho\, ds$. The exact expression depends on the formulas for ρ and ds.

If you wish to remember only one formula for surface area, you might make it the short differential form.

Short Differential Form

$$S = \int 2\pi\rho\, ds$$

In any particular problem, you would then express the radius function ρ and the arc length differential ds in terms of a common variable and supply limits of integration for that variable.



5.49 The surface generated by revolving the curve $y = x^3, 0 \leq x \leq 1/2$, about the x-axis could be the design for a champagne glass (Example 3).

EXAMPLE 3 Find the area of the surface generated by revolving the curve $y = x^3, 0 \leq x \leq 1/2$, about the x-axis (Fig. 5.49).

Solution We start with the short differential form:

$$\begin{aligned} S &= \int 2\pi\rho\, ds \\ &= \int 2\pi y\, ds && \text{For revolution about the } x\text{-axis, the radius function is } \rho = y. \\ &= \int 2\pi y \sqrt{dx^2 + dy^2}. && ds = \sqrt{dx^2 + dy^2} \end{aligned}$$

We then decide whether to express dy in terms of dx or dx in terms of dy. The original form of the equation, $y = x^3$, makes it easier to express dy in terms of dx, so we continue the calculation with

$$y = x^3, \qquad dy = 3x^2\, dx, \qquad \text{and} \qquad \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + (3x^2\, dx)^2} = \sqrt{1 + 9x^4}\, dx.$$

With these substitutions, x becomes the variable of integration and

$$\begin{aligned} S &= \int_{x=0}^{x=1/2} 2\pi y \sqrt{dx^2 + dy^2} \\ &= \int_0^{1/2} 2\pi x^3 \sqrt{1 + 9x^4}\, dx \\ &= 2\pi \left(\frac{1}{36}\right)\left(\frac{2}{3}\right)(1 + 9x^4)^{3/2}\Big]_0^{1/2} && \text{Substitute } u = 1 + 9x^4,\ du/36 = x^3 dx, \text{ integrate, and substitute back.} \\ &= \frac{\pi}{27}\left[\left(1 + \frac{9}{16}\right)^{3/2} - 1\right] \\ &= \frac{\pi}{27}\left[\left(\frac{25}{16}\right)^{3/2} - 1\right] = \frac{\pi}{27}\left(\frac{125}{64} - 1\right) \\ &= \frac{61\pi}{1728}. \end{aligned}$$

As with arc length calculations, even the simplest curves can provide a workout. ❑

Exercises 5.6

Finding Integrals for Surface Area

In Exercises 1–8:

a) Set up an integral for the area of the surface generated by revolving the given curve about the indicated axis.

b) Graph the curve to see what it looks like. If you can, graph the surface, too.

c) Use your grapher's or computer's integral evaluator to find the surface's area numerically.

1. $y = \tan x, \quad 0 \le x \le \pi/4; \quad x$-axis
2. $y = x^2, \quad 0 \le x \le 2; \quad x$-axis
3. $xy = 1, \quad 1 \le y \le 2; \quad y$-axis
4. $x = \sin y, \quad 0 \le y \le \pi; \quad y$-axis
5. $x^{1/2} + y^{1/2} = 3$ from $(4, 1)$ to $(1, 4)$; x-axis
6. $y + 2\sqrt{y} = x, \quad 1 \le y \le 2; \quad y$-axis
7. $x = \displaystyle\int_0^y \tan t\, dt, \quad 0 \le y \le \pi/3; \quad y$-axis
8. $y = \displaystyle\int_1^x \sqrt{t^2 - 1}\, dt, \quad 1 \le x \le \sqrt{5}; \quad x$-axis

Finding Surface Areas

9. Find the lateral (side) surface area of the cone generated by revolving the line segment $y = x/2, 0 \le x \le 4$, about the x-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

10. Find the lateral surface area of the cone generated by revolving the line segment $y = x/2, 0 \le x \le 4$ about the y-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

11. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \le x \le 3$, about the x-axis. Check your result with the geometry formula

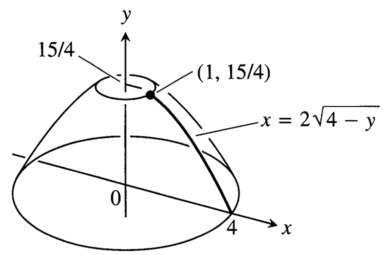
$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

12. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \le x \le 3$, about the y-axis. Check your result with the geometry formula

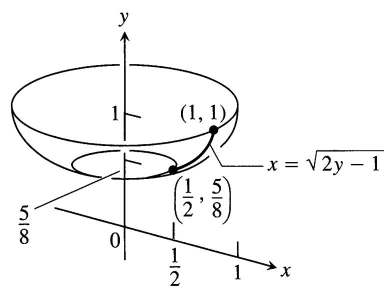
$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

Find the areas of the surfaces generated by revolving the curves in Exercises 13–22 about the indicated axes. If you have a grapher, you may want to graph these curves to see what they look like.

13. $y = x^3/9, \quad 0 \le x \le 2; \quad x$-axis
14. $y = \sqrt{x}, \quad 3/4 \le x \le 15/4; \quad x$-axis
15. $y = \sqrt{2x - x^2}, \quad 0.5 \le x \le 1.5; \quad x$-axis
16. $y = \sqrt{x + 1}, \quad 1 \le x \le 5; \quad x$-axis
17. $x = y^3/3, \quad 0 \le y \le 1; \quad y$-axis
18. $x = (1/3)y^{3/2} - y^{1/2}, \quad 1 \le y \le 3; \quad y$-axis
19. $x = 2\sqrt{4 - y}, \quad 0 \le y \le 15/4; \quad y$-axis



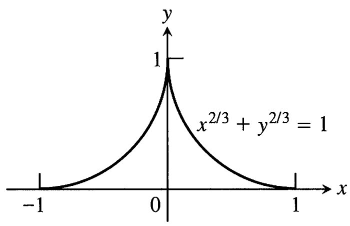
20. $x = \sqrt{2y - 1}, \quad 5/8 \le y \le 1; \quad y$-axis



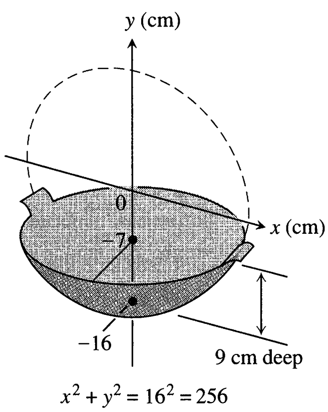
21. $x = (y^4/4) + 1/(8y^2), \quad 1 \le y \le 2; \quad x$-axis (*Hint:* Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dy, and evaluate the integral $S = \int 2\pi y\, ds$ with appropriate limits.)
22. $y = (1/3)(x^2 + 2)^{3/2}, \quad 0 \le x \le \sqrt{2}; \quad y$-axis (*Hint:* Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dx, and evaluate the integral $S = \int 2\pi x\, ds$ with appropriate limits.)
23. *Testing the new definition.* Show that the surface area of a sphere of radius a is still $4\pi a^2$ by using Eq. (3) to find the area of the surface generated by revolving the curve $y = \sqrt{a^2 - x^2}$, $-a \le x \le a$, about the x-axis.
24. *Testing the new definition.* The lateral (side) surface area of a cone of height h and base radius r should be $\pi r\sqrt{r^2 + h^2}$, the semiperimeter of the base times the slant height. Show that this is still the case by finding the area of the surface generated by revolving the line segment $y = (r/h)x$, $0 \le x \le h$, about the x-axis.
25. a) Write an integral for the area of the surface generated by revolving the curve $y = \cos x$, $-\pi/2 \le x \le \pi/2$, about the x-axis. In Section 7.4 we will see how to evaluate such integrals.

    b) CALCULATOR Find the surface area numerically.

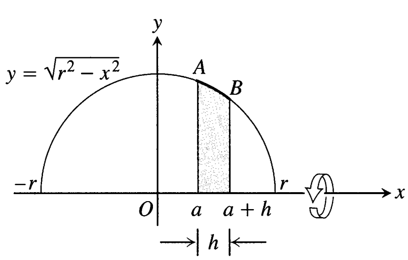
26. *The surface of an astroid.* Find the area of the surface generated by revolving about the x-axis the portion of the astroid $x^{2/3} + y^{2/3} = 1$ shown here. (*Hint:* Revolve the first-quadrant portion $y = (1 - x^{2/3})^{3/2}$, $0 \le x \le 1$, about the x-axis and double your result.)



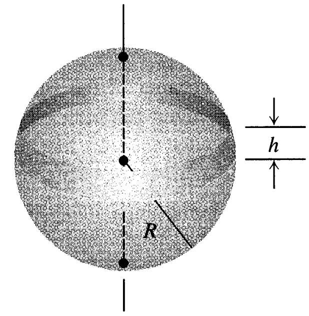
27. *Enameling woks.* Your company decided to put out a deluxe version of the successful wok you designed in Section 5.3, Exercise 41. The plan is to coat it inside with white enamel and outside with blue enamel. Each enamel will be sprayed on 0.5 mm thick before baking. (See diagram here.) Your manufacturing department wants to know how much enamel to have on hand for a production run of 5000 woks. What do you tell them? (Neglect waste and unused material and give your answer in liters. Remember that 1 cm^3 = 1 mL, so 1 L = 1000 cm^3.)



28. *Slicing bread.* Did you know that if you cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y = \sqrt{r^2 - x^2}$ shown here is revolved about the x-axis to generate a sphere. Let AB be an arc of the semicircle that lies above an interval of length h on the x-axis. Show that the area swept out by AB does not depend on the location of the interval. (It does depend on the length of the interval.)



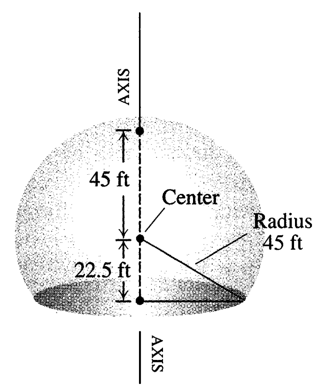
29. The shaded band shown here is cut from a sphere of radius R by parallel planes h units apart. Show that the surface area of the band is $2\pi Rh$.



30. Here is a schematic drawing of the 90-ft dome used by the U.S. National Weather Service to house radar in Bozeman, Mont.

a) How much outside surface is there to paint (not counting the bottom)?

b) CALCULATOR Express the answer to the nearest square foot.



31. *Surfaces generated by curves that cross the axis of revolution.* The surface area formula in Eq. (3) was developed under the assumption that the function f whose graph generated the surface was nonnegative over the interval $[a, b]$. For curves that cross the axis of revolution, we replace Eq. (3) with the absolute value formula

$$S = \int 2\pi\rho\, ds = \int 2\pi |f(x)|\, ds. \tag{5}$$

Use Eq. (5) to find the surface area of the double cone generated by revolving the line segment $y = x$, $-1 \le x \le 2$, about the x-axis.

32. *(Exercise 31, continued.)* Find the area of the surface generated by revolving the curve $y = x^3/9$, $-\sqrt{3} \le x \le \sqrt{3}$, about the x-axis. What do you think will happen if you drop the absolute value bars from Eq. (5) and attempt to find the surface area with the formula $S = \int 2\pi f(x)\, ds$ instead? Try it.

Numerical Integration

Find, to 2 decimal places, the areas of the surfaces generated by revolving the curves in Exercises 33–36 about the x-axis.

33. $y = \sin x, \quad 0 \le x \le \pi$

34. $y = x^2/4, \quad 0 \le x \le 2$

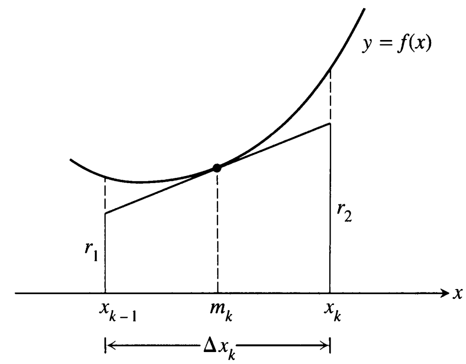
35. $y = x + \sin 2x, \quad -2\pi/3 \le x \le 2\pi/3$ (the curve in Section 3.4, Exercise 5)

36. $y = \frac{x}{12}\sqrt{36 - x^2}, 0 \le x \le 6$ (the surface of the plumb bob in Section 5.3, Exercise 44)

37. *An alternative derivation of the surface area formula.* Assume f is smooth on $[a, b]$ and partition $[a, b]$ in the usual way. In the kth subinterval $[x_{k-1}, x_k]$ construct the tangent line to the curve at the midpoint $m_k = (x_{k-1} + x_k)/2$, as in the figure here.

a) Show that $r_1 = f(m_k) - f'(m_k)\frac{\Delta x_k}{2}$ and $r_2 = f(m_k) + f'(m_k)\frac{\Delta x_k}{2}$.

b) Show that the length L_k of the tangent line segment in the kth subinterval is $L_k = \sqrt{(\Delta x_k)^2 + (f'(m_k)\Delta x_k)^2}$.



c) Show that the lateral surface area of the frustum of the cone swept out by the tangent line segment as it revolves about the x-axis is $2\pi f(m_k)\sqrt{1 + (f'(m_k))^2}\,\Delta x_k$.

d) Show that the area of the surface generated by revolving $y = f(x)$ about the x-axis over $[a, b]$ is

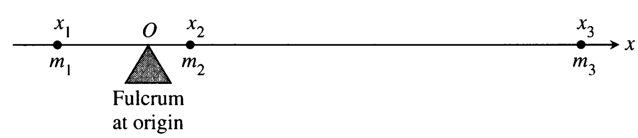
$$\lim_{n\to\infty} \sum_{k=1}^{n} \begin{pmatrix} \text{lateral surface area} \\ \text{of } k\text{th frustum} \end{pmatrix} = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

5.7 Moments and Centers of Mass

Many structures and mechanical systems behave as if their masses were concentrated at a single point, called the center of mass (Fig. 5.50, on the following page). It is important to know how to locate this point, and doing so is basically a mathematical enterprise. For the moment we deal with one- and two-dimensional objects. Three-dimensional objects are best done with the multiple integrals of Chapter 13.

Masses Along a Line

We develop our mathematical model in stages. The first stage is to imagine masses m_1, m_2, and m_3 on a rigid x-axis supported by a fulcrum at the origin.



The resulting system might balance, or it might not. It depends on how large the masses are and how they are arranged.

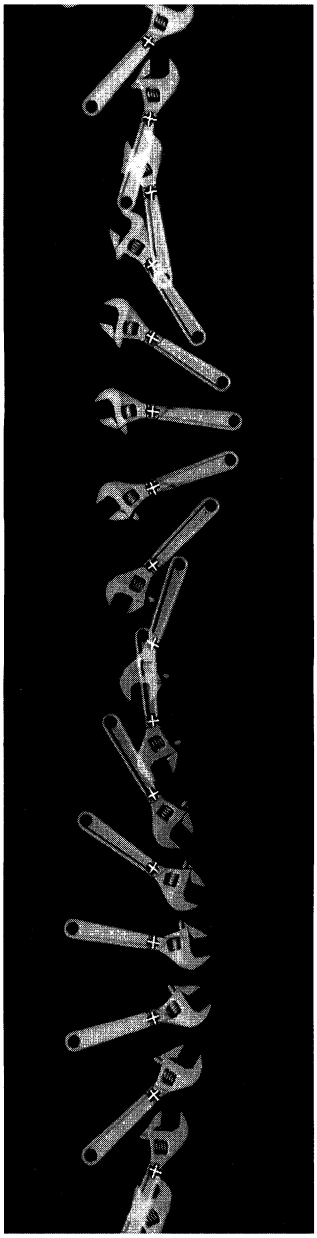
Each mass m_k exerts a downward force $m_k g$ equal to the magnitude of the mass times the acceleration of gravity. Each of these forces has a tendency to turn the axis about the origin, the way you turn a seesaw. This turning effect, called a **torque,** is measured by multiplying the force $m_k g$ by the signed distance x_k from the point of application to the origin. Masses to the left of the origin exert negative (counterclockwise) torque. Masses to the right of the origin exert positive (clockwise) torque.

Mass vs. weight

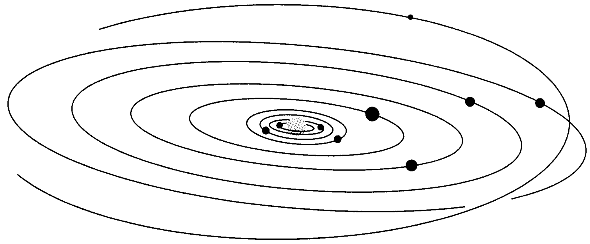
Weight is the force that results from gravity pulling on a mass. If an object of mass m is placed in a location where the acceleration of gravity is g, the object's weight there is

$$F = mg$$

(as in Newton's second law).



(a)



(b)

5.50 (a) The motion of this wrench gliding on ice seems haphazard until we notice that the wrench is simply turning about its center of mass as the center glides in a straight line. (b) The planets, asteroids, and comets of our solar system revolve about their collective center of mass. (It lies inside the sun.)

The sum of the torques measures the tendency of a system to rotate about the origin. This sum is called the **system torque.**

$$\text{System torque} = m_1gx_1 + m_2gx_2 + m_3gx_3 \tag{1}$$

The system will balance if and only if its torque is zero.

If we factor out the g in Eq. (1), we see that the system torque is

$$\underbrace{g}_{\substack{\text{a feature of the}\\ \text{environment}}}(\underbrace{m_1x_1 + m_2x_2 + m_3x_3}_{\substack{\text{a feature of}\\ \text{the system}}}).$$

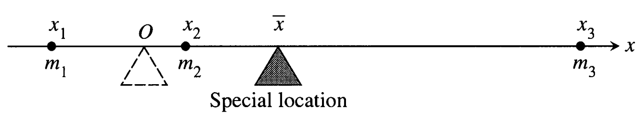
Thus the torque is the product of the gravitional acceleration g, which is a feature of the environment in which the system happens to reside, and the number $(m_1x_1 + m_2x_2 + m_3x_3)$, which is a feature of the system itself, a constant that stays the same no matter where the system is placed.

The number $(m_1x_1 + m_2x_2 + m_3x_3)$ is called the **moment of the system about the origin.** It is the sum of the **moments** m_1x_1, m_2x_2, m_3x_3 of the individual masses.

$$M_O = \text{Moment of system about origin} = \sum m_kx_k$$

(We shift to sigma notation here to allow for sums with more terms. For $\sum m_kx_k$, read "summation m_k times x_k.")

We usually want to know where to place the fulcrum to make the system balance, that is, at what point $\bar{x}$ to place it to make the torque zero.



The torque of each mass about the fulcrum in this special location is

$$\text{Torque of } m_k \text{ about } \bar{x} = \begin{pmatrix}\text{signed distance}\\ \text{of } m_k \text{ from } \bar{x}\end{pmatrix}\begin{pmatrix}\text{downward}\\ \text{force}\end{pmatrix}$$

$$= (x_k - \bar{x})m_kg.$$

When we write the equation that says that the sum of these torques is zero, we get an equation we can solve for $\overline{x}$:

$$\sum (x_k - \overline{x}) m_k g = 0 \qquad \text{Sum of the torques equals zero}$$

$$g \sum (x_k - \overline{x}) m_k = 0 \qquad \text{Constant Multiple Rule for Sums}$$

$$\sum (m_k x_k - \overline{x} m_k) = 0 \qquad g \text{ divided out, } m_k \text{ distributed}$$

$$\sum m_k x_k - \sum \overline{x} m_k = 0 \qquad \text{Difference Rule for Sums}$$

$$\sum m_k x_k = \overline{x} \sum m_k \qquad \text{Rearranged, Constant Multiple Rule again}$$

$$\overline{x} = \frac{\sum m_k x_k}{\sum m_k}. \qquad \text{Solved for } \overline{x}$$

This last equation tells us to find $\overline{x}$ by dividing the system's moment about the origin by the system's total mass:

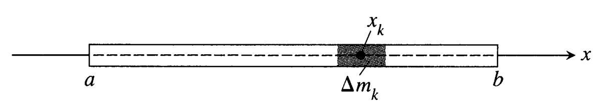
$$\overline{x} = \frac{\sum x_k m_k}{\sum m_k} = \frac{\text{system moment about origin}}{\text{system mass}}.$$

The point $\overline{x}$ is called the system's **center of mass.**

Wires and Thin Rods

In many applications, we want to know the center of mass of a rod or a thin strip of metal. In cases like these where we can model the distribution of mass with a continuous function, the summation signs in our formulas become integrals in a manner we now describe.

Imagine a long, thin strip lying along the x-axis from $x = a$ to $x = b$ and cut into small pieces of mass Δm_k by a partition of the interval $[a, b]$.



The kth piece is Δx_k units long and lies approximately x_k units from the origin. Now observe three things.

First, the strip's center of mass $\overline{x}$ is nearly the same as that of the system of point masses we would get by attaching each mass Δm_k to the point x_k:

$$\overline{x} \approx \frac{\text{system moment}}{\text{system mass}}.$$

Second, the moment of each piece of the strip about the origin is approximately $x_k \Delta m_k$, so the system moment is approximately the sum of the $x_k \Delta m_k$:

$$\text{System moment } \approx \sum x_k \Delta m_k.$$

Third, if the density of the strip at x_k is $\delta(x_k)$, expressed in terms of mass per unit length, and δ is continuous, then Δm_k is approximately equal to $\delta(x_k)\Delta x_k$ (mass per unit length times length):

$$\Delta m_k \approx \delta(x_k)\Delta x_k.$$

Combining these three observations gives

$$\overline{x} \approx \frac{\text{system moment}}{\text{system mass}} \approx \frac{\sum x_k \Delta m_k}{\sum \Delta m_k} \approx \frac{\sum x_k \delta(x_k) \Delta x_k}{\sum \delta(x_k) \Delta x_k}. \tag{2}$$

Density

A material's density is its mass per unit volume. In practice, however, we tend to use units we can conveniently measure. For wires, rods, and narrow strips we use mass per unit length. For flat sheets and plates we use mass per unit area.

The sum in the last numerator in Eq. (2) is a Riemann sum for the continuous function $x\delta(x)$ over the closed interval $[a, b]$. The sum in the denominator is a Riemann sum for the function $\delta(x)$ over this interval. We expect the approximations in (2) to improve as the strip is partitioned more finely, and we are led to the equation

$$\bar{x} = \frac{\int_a^b x\delta(x)\,dx}{\int_a^b \delta(x)\,dx}.$$

This is the formula we use to find $\bar{x}$.

To find a center of mass, divide moment by mass.

Moment, Mass, and Center of Mass of a Thin Rod or Strip Along the x-axis with Density Function $\delta(x)$

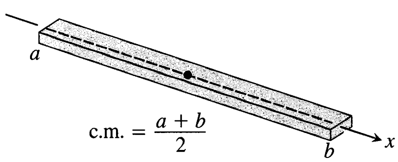
Moment about the origin: $$M_O = \int_a^b x\delta(x)\,dx \qquad \text{(3a)}$$

Mass: $$M = \int_a^b \delta(x)\,dx \qquad \text{(3b)}$$

Center of mass: $$\bar{x} = \frac{M_O}{M} \qquad \text{(3c)}$$

EXAMPLE 1 *Strips and rods of constant density*

Show that the center of mass of a straight, thin strip or rod of constant density lies halfway between its two ends.



5.51 The center of mass of a straight, thin rod or strip of constant density lies halfway between its ends.

Solution We model the strip as a portion of the x-axis from $x = a$ to $x = b$ (Fig. 5.51). Our goal is to show that $\bar{x} = (a + b)/2$, the point halfway between a and b.

The key is the density's having a constant value. This enables us to regard the function $\delta(x)$ in the integrals in Eqs. (3) as a constant (call it δ), with the result that

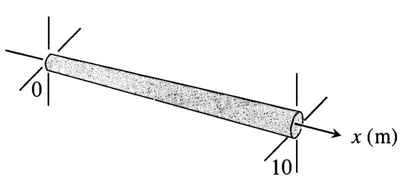
$$M_O = \int_a^b \delta x\,dx = \delta\int_a^b x\,dx = \delta\left[\frac{1}{2}x^2\right]_a^b = \frac{\delta}{2}(b^2 - a^2)$$

$$M = \int_a^b \delta\,dx = \delta\int_a^b dx = \delta\Big[x\Big]_a^b = \delta(b - a)$$

$$\bar{x} = \frac{M_O}{M} = \frac{\frac{\delta}{2}(b^2 - a^2)}{\delta(b - a)}$$

$$= \frac{a + b}{2}. \qquad \text{The } \delta\text{'s cancel in the formula for } \bar{x}.$$

❑



5.52 We can treat a rod of variable thickness as a rod of variable density. See Example 2.

EXAMPLE 2 *A variable density*

The 10-m-long rod in Fig. 5.52 thickens from left to right so that its density, instead of being constant, is $\delta(x) = 1 + (x/10)$ kg/m. Find the rod's center of mass.

Solution The rod's moment about the origin (Eq. 3a) is

$$M_O = \int_0^{10} x\delta(x)\,dx = \int_0^{10} x\left(1+\frac{x}{10}\right)dx = \int_0^{10}\left(x+\frac{x^2}{10}\right)dx$$

$$= \left[\frac{x^2}{2}+\frac{x^3}{30}\right]_0^{10} = 50+\frac{100}{3} = \frac{250}{3}\ \text{kg}\cdot\text{m}.$$

The units of a moment are mass × length.

The rod's mass (Eq. 3b) is

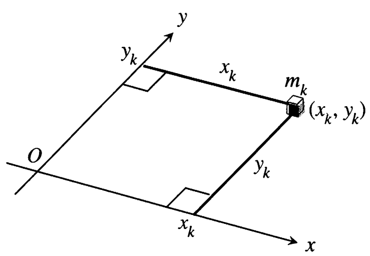
$$M = \int_0^{10}\delta(x)\,dx = \int_0^{10}\left(1+\frac{x}{10}\right)dx = \left[x+\frac{x^2}{20}\right]_0^{10} = 10+5 = 15\ \text{kg}.$$

The center of mass (Eq. 3c) is located at the point

$$\bar{x} = \frac{M_O}{M} = \frac{250}{3}\cdot\frac{1}{15} = \frac{50}{9} \approx 5.56\ \text{m}.$$

❑

Masses Distributed over a Plane Region



5.53 Each mass m_k has a moment about each axis.

Suppose we have a finite collection of masses located in the plane, with mass m_k at the point (x_k, y_k) (see Fig. 5.53). The mass of the system is

$$\text{System mass:} \qquad M = \sum m_k.$$

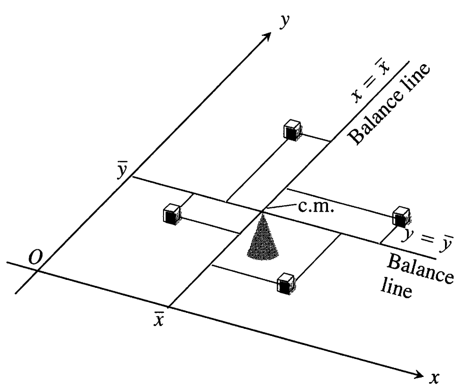
Each mass m_k has a moment about each axis. Its moment about the x-axis is $m_k y_k$, and its moment about the y-axis is $m_k x_k$. The moments of the entire system about the two axes are

$$\text{Moment about } x\text{-axis:} \qquad M_x = \sum m_k y_k,$$

$$\text{Moment about } y\text{-axis:} \qquad M_y = \sum m_k x_k.$$

The x-coordinate of the system's center of mass is defined to be

$$\bar{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k}. \tag{4}$$



5.54 A two-dimensional array of masses balances on its center of mass.

With this choice of $\bar{x}$, as in the one-dimensional case, the system balances about the line $x = \bar{x}$ (Fig. 5.54).

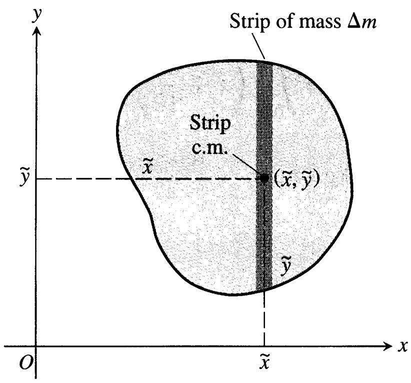
The y-coordinate of the system's center of mass is defined to be

$$\bar{y} = \frac{M_x}{M} = \frac{\sum m_k y_k}{\sum m_k}. \tag{5}$$

With this choice of $\bar{y}$, the system balances about the line $y = \bar{y}$ as well. The torques exerted by the masses about the line $y = \bar{y}$ cancel out. Thus, as far as balance is concerned, the system behaves as if all its mass were at the single point $(\bar{x}, \bar{y})$. We call this point the system's *center of mass.*

Thin, Flat Plates

In many applications, we need to find the center of mass of a thin, flat plate: a disk of aluminum, say, or a triangular sheet of steel. In such cases we assume the distribution of mass to be continuous, and the formulas we use to calculate $\bar{x}$ and $\bar{y}$ contain integrals instead of finite sums. The integrals arise in the following way.



5.55 A plate cut into thin strips parallel to the y-axis. The moment exerted by a typical strip about each axis is the moment its mass Δm would exert if concentrated at the strip's center of mass $(\tilde{x}, \tilde{y})$.

Imagine the plate occupying a region in the xy-plane, cut into thin strips parallel to one of the axes (in Fig. 5.55, the y-axis). The center of mass of a typical strip is $(\tilde{x}, \tilde{y})$. We treat the strip's mass Δm as if it were concentrated at $(\tilde{x}, \tilde{y})$. The moment of the strip about the y-axis is then $\tilde{x}\Delta m$. The moment of the strip about the x-axis is $\tilde{y}\Delta m$. Equations (4) and (5) then become

$$\bar{x} = \frac{M_y}{M} = \frac{\sum \tilde{x}\,\Delta m}{\sum \Delta m}, \qquad \bar{y} = \frac{M_x}{M} = \frac{\sum \tilde{y}\,\Delta m}{\sum \Delta m}.$$

As in the one-dimensional case, the sums are Riemann sums for integrals and approach these integrals as limiting values as the strips into which the plate is cut become narrower and narrower. We write these integrals symbolically as

$$\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm} \quad \text{and} \quad \bar{y} = \frac{\int \tilde{y}\,dm}{\int dm}.$$

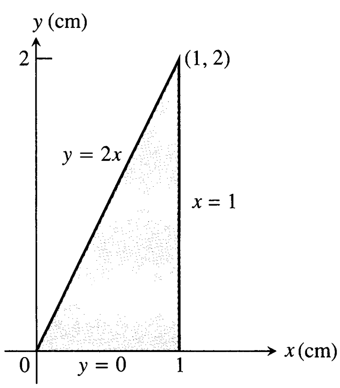
Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the xy-plane

Moment about the x-axis: $\quad M_x = \int \tilde{y}\,dm$

Moment about the y-axis: $\quad M_y = \int \tilde{x}\,dm \qquad$ **(6)**

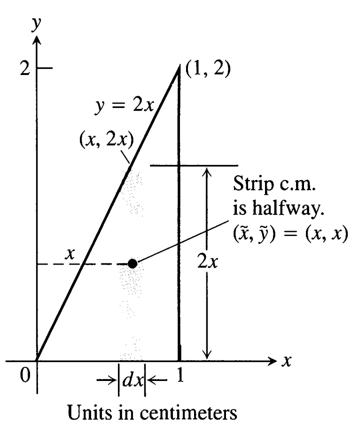
Mass: $\quad M = \int dm$

Center of mass: $\quad \bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$



5.56 The plate in Example 3.

To evaluate these integrals, we picture the plate in the coordinate plane and sketch a strip of mass parallel to one of the coordinates axes. We then express the strip's mass dm and the coordinates $(\tilde{x}, \tilde{y})$ of the strip's center of mass in terms of x or y. Finally, we integrate $\tilde{y}\,dm$, $\tilde{x}\,dm$, and dm between limits of integration determined by the plate's location in the plane.



5.57 Modeling the plate in Example 3 with vertical strips.

EXAMPLE 3 The triangular plate shown in Fig. 5.56 has a constant density of $\delta = 3\ \text{g/cm}^2$. Find (a) the plate's moment M_y about the y-axis, (b) the plate's mass M, and (c) the x-coordinate of the plate's center of mass (c.m.).

Solution

Method 1: *Vertical strips* (Fig. 5.57).

a) The moment M_y: The typical vertical strip has

center of mass (c.m.): $\quad (\tilde{x}, \tilde{y}) = (x, x)$,

length: $2x$, $\quad$ area: $dA = 2x\,dx$,

width: dx, $\quad$ mass: $dm = \delta\,dA = 3 \cdot 2x\,dx = 6x\,dx$,

distance of c.m. from y-axis: $\tilde{x} = x$.

The moment of the strip about the y-axis is

$$\tilde{x}\,dm = x \cdot 6x\,dx = 6x^2\,dx.$$

The moment of the plate about the y-axis is therefore

$$M_y = \int \tilde{x}\,dm = \int_0^1 6x^2\,dx = 2x^3\Big]_0^1 = 2\text{ g}\cdot\text{cm}.$$

b) The plate's mass:

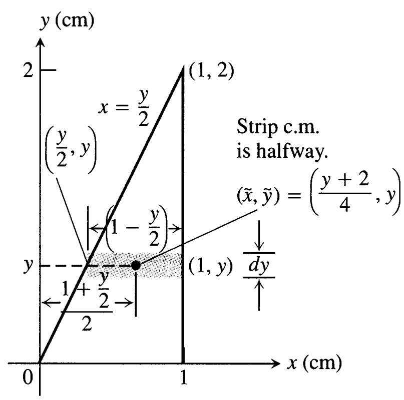
$$M = \int dm = \int_0^1 6x\,dx = 3x^2\Big]_0^1 = 3\text{ g}.$$

c) The x-coordinate of the plate's center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{2\text{ g}\cdot\text{cm}}{3\text{ g}} = \frac{2}{3}\text{ cm}.$$

By a similar computation we could find M_x and $\bar{y} = M_x/M$.

Method 2: *Horizontal strips* (Fig. 5.58).



5.58 Modeling the plate in Example 3 with horizontal strips.

a) The moment M_y: The y-coordinate of the center of mass of a typical horizontal strip is y (see the figure), so

$$\tilde{y} = y.$$

The x-coordinate is the x-coordinate of the point halfway across the triangle. This makes it the average of $y/2$ (the strip's left-hand x-value) and 1 (the strip's right-hand x-value):

$$\tilde{x} = \frac{(y/2) + 1}{2} = \frac{y}{4} + \frac{1}{2} = \frac{y+2}{4}.$$

We also have

length: $1 - \dfrac{y}{2} = \dfrac{2-y}{2}$,

width: dy,

area: $dA = \dfrac{2-y}{2}\,dy$,

mass: $dm = \delta\,dA = 3 \cdot \dfrac{2-y}{2}\,dy$,

distance of c.m. to y-axis: $\tilde{x} = \dfrac{y+2}{4}$.

The moment of the strip about the y-axis is

$$\tilde{x}\,dm = \frac{y+2}{4} \cdot 3 \cdot \frac{2-y}{2}\,dy = \frac{3}{8}(4 - y^2)\,dy.$$

The moment of the plate about the y-axis is

$$M_y = \int \tilde{x}\,dm = \int_0^2 \frac{3}{8}(4-y^2)\,dy = \frac{3}{8}\left[4y - \frac{y^3}{3}\right]_0^2 = \frac{3}{8}\left(\frac{16}{3}\right) = 2\text{ g}\cdot\text{cm}.$$

b) The plate's mass:

$$M = \int dm = \int_0^2 \frac{3}{2}(2-y)\,dy = \frac{3}{2}\left[2y - \frac{y^2}{2}\right]_0^2 = \frac{3}{2}(4-2) = 3\text{ g}.$$

How to Find a Plate's Center of Mass

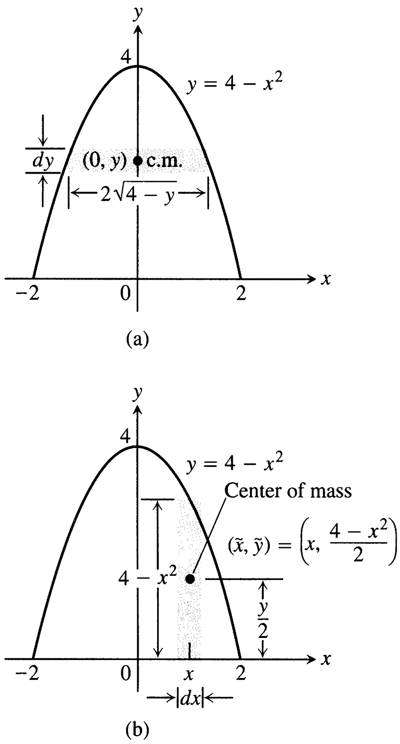
1. Picture the plate in the xy-plane.
2. Sketch a strip of mass parallel to one of the coordinate axes and find its dimensions.
3. Find the strip's mass dm and center of mass $(\tilde{x}, \tilde{y})$.
4. Integrate $\tilde{y}\,dm$, $\tilde{x}\,dm$, and dm to find M_x, M_y, and M.
5. Divide the moments by the mass to calculate $\bar{x}$ and $\bar{y}$.

c) The x-coordinate of the plate's center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{2 \text{ g} \cdot \text{cm}}{3 \text{ g}} = \frac{2}{3} \text{ cm}.$$

By a similar computation, we could find M_x and $\bar{y}$. ❑

If the distribution of mass in a thin, flat plate has an axis of symmetry, the center of mass will lie on this axis. If there are two axes of symmetry, the center of mass will lie at their intersection. These facts often help to simplify our work.



5.59 Modeling the plate in Example 4 with (a) horizontal strips leads to an inconvenient integration, so we model with (b) vertical strips instead.

EXAMPLE 4 Find the center of mass of a thin plate of constant density δ covering the region bounded above by the parabola $y = 4 - x^2$ and below by the x-axis (Fig. 5.59).

Solution Since the plate is symmetric about the y-axis and its density is constant, the distribution of mass is symmetric about the y-axis and the center of mass lies on the y-axis. This means that $\bar{x} = 0$. It remains to find $\bar{y} = M_x/M$.

A trial calculation with horizontal strips (Fig. 5.59a) leads to an inconvenient integration

$$M_x = \int_0^4 2\delta\, y\sqrt{4 - y}\, dy.$$

We therefore model the distribution of mass with vertical strips instead (Fig. 5.59b). The typical vertical strip has

center of mass (c.m): $(\tilde{x}, \tilde{y}) = \left(x, \dfrac{4 - x^2}{2}\right)$,

length: $4 - x^2$,

width: dx,

area: $dA = (4 - x^2)\, dx$,

mass: $dm = \delta\, dA = \delta(4 - x^2)\, dx$,

distance from c.m to x-axis: $\tilde{y} = \dfrac{4 - x^2}{2}$.

The moment of the strip about the x-axis is

$$\tilde{y}\, dm = \frac{4 - x^2}{2} \cdot \delta(4 - x^2)\, dx = \frac{\delta}{2}(4 - x^2)^2\, dx.$$

The moment of the plate about the x-axis is

$$M_x = \int \tilde{y}\, dm = \int_{-2}^{2} \frac{\delta}{2}(4 - x^2)^2\, dx$$

$$= \frac{\delta}{2}\int_{-2}^{2} (16 - 8x^2 + x^4)\, dx = \frac{256}{15}\delta. \tag{7}$$

The mass of the plate is

$$M = \int dm = \int_{-2}^{2} \delta(4 - x^2)\, dx = \frac{32}{3}\delta. \tag{8}$$

Therefore,

$$\bar{y} = \frac{M_x}{M} = \frac{(256/15)\,\delta}{(32/3)\,\delta} = \frac{8}{5}.$$

The plate's center of mass is the point

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{5}\right).$$

❑

EXAMPLE 5 *Variable density*

Find the center of mass of the plate in Example 4 if the density at the point (x, y) is $\delta = 2x^2$, twice the square of the distance from the point to the y-axis.

Solution The mass distribution is still symmetric about the y-axis, so $\bar{x} = 0$. With $\delta = 2x^2$, Eqs. (7) and (8) become

$$M_x = \int \tilde{y}\,dm = \int_{-2}^{2} \frac{\delta}{2}(4 - x^2)^2\,dx = \int_{-2}^{2} x^2(4 - x^2)^2\,dx$$

$$= \int_{-2}^{2} (16x^2 - 8x^4 + x^6)\,dx = \frac{2048}{105}, \qquad (7')$$

$$M = \int dm = \int_{-2}^{2} \delta(4 - x^2)\,dx = \int_{-2}^{2} 2x^2(4 - x^2)\,dx$$

$$= \int_{-2}^{2} (8x^2 - 2x^4)\,dx = \frac{256}{15}. \qquad (8')$$

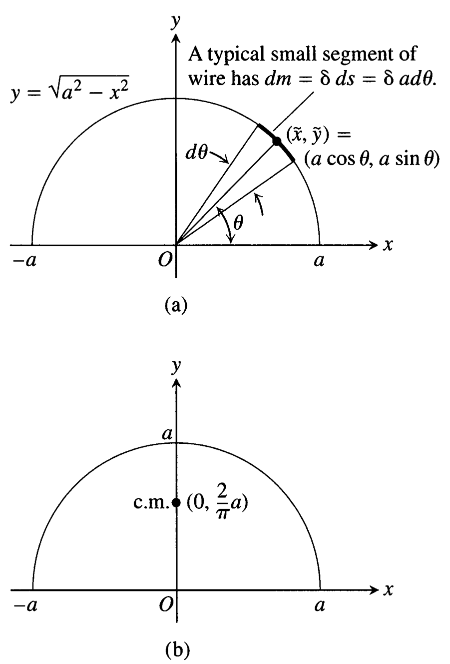
Therefore,

$$\bar{y} = \frac{M_x}{M} = \frac{2048}{105} \cdot \frac{15}{256} = \frac{8}{7}.$$

The plate's new center of mass is

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{7}\right).$$

❑



5.60 The semicircular wire in Example 6. (a) The dimensions and variables used in finding the center of mass. (b) The center of mass does not lie on the wire.

EXAMPLE 6 Find the center of mass of a wire of constant density δ shaped like a semicircle of radius a.

Solution We model the wire with the semicircle $y = \sqrt{a^2 - x^2}$ (Fig. 5.60). The distribution of mass is symmetric about the y-axis, so $\bar{x} = 0$. To find $\bar{y}$, we imagine the wire divided into short segments. The typical segment (Fig. 5.60a) has

length: $ds = a\,d\theta$,

mass: $dm = \delta\,ds = \delta\,a\,d\theta$, Mass per unit length times length

distance of c.m. to x-axis: $\tilde{y} = a\sin\theta$.

Hence,

$$\bar{y} = \frac{\int \tilde{y}\, dm}{\int dm} = \frac{\int_0^\pi a \sin\theta \cdot \delta a\, d\theta}{\int_0^\pi \delta a\, d\theta} = \frac{\delta a^2[-\cos\theta]_0^\pi}{\delta a\pi} = \frac{2}{\pi} a.$$

The center of mass lies on the axis of symmetry at the point $(0, 2a/\pi)$, about two-thirds of the way up from the origin (Fig. 5.60b). ❑

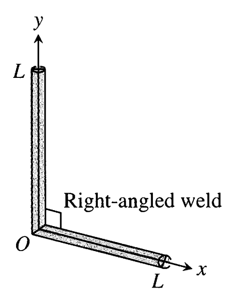
Centroids

When the density function is constant, it cancels out of the numerator and denominator of the formulas for $\bar{x}$ and $\bar{y}$. This happened in nearly every example in this section. As far as $\bar{x}$ and $\bar{y}$ were concerned, δ might as well have been 1. Thus, when the density is constant, the location of the center of mass is a feature of the geometry of the object and not of the material from which it is made. In such cases engineers may call the center of mass the **centroid** of the shape, as in "Find the centroid of a triangle or a solid cone." To do so, just set δ equal to 1 and proceed to find $\bar{x}$ and $\bar{y}$ as before, by dividing moments by masses.

Exercises 5.7

Thin Rods

1. An 80-lb child and a 100-lb child are balancing on a seesaw. The 80-lb child is 5 ft from the fulcrum. How far from the fulcrum is the 100-lb child?
2. The ends of a log are placed on two scales. One scale reads 100 kg and the other 200 kg. Where is the log's center of mass?
3. The ends of two thin steel rods of equal length are welded together to make a right-angled frame. Locate the frame's center of mass. (*Hint:* Where is the center of mass of each rod?)



4. You weld the ends of two steel rods into a right-angled frame. One rod is twice the length of the other. Where is the frame's center of mass? (*Hint:* Where is the center of mass of each rod?)

Exercises 5–12 give density functions of thin rods lying along various intervals of the x-axis. Use Eqs. (3a–c) to find each rod's moment about the origin, mass, and center of mass.

5. $\delta(x) = 4, \quad 0 \le x \le 2$
6. $\delta(x) = 4, \quad 1 \le x \le 3$
7. $\delta(x) = 1 + (x/3), \quad 0 \le x \le 3$
8. $\delta(x) = 2 - (x/4), \quad 0 \le x \le 4$
9. $\delta(x) = 1 + (1/\sqrt{x}), \quad 1 \le x \le 4$
10. $\delta(x) = 3(x^{-3/2} + x^{-5/2}), \quad 0.25 \le x \le 1$
11. $\delta(x) = \begin{cases} 2 - x, & 0 \le x < 1 \\ x, & 1 \le x \le 2 \end{cases}$
12. $\delta(x) = \begin{cases} x + 1, & 0 \le x < 1 \\ 2, & 1 \le x \le 2 \end{cases}$

Thin Plates with Constant Density

In Exercises 13–24, find the center of mass of a thin plate of constant density δ covering the given region.

13. The region bounded by the parabola $y = x^2$ and the line $y = 4$
14. The region bounded by the parabola $y = 25 - x^2$ and the x-axis
15. The region bounded by the parabola $y = x - x^2$ and the line $y = -x$
16. The region enclosed by the parabolas $y = x^2 - 3$ and $y = -2x^2$
17. The region bounded by the y-axis and the curve $x = y - y^3$, $0 \le y \le 1$
18. The region bounded by the parabola $x = y^2 - y$ and the line $y = x$
19. The region bounded by the x-axis and the curve $y = \cos x$, $-\pi/2 \le x \le \pi/2$

20. The region between the x-axis and the curve $y = \sec^2 x$, $-\pi/4 \le x \le \pi/4$

21. The region bounded by the parabolas $y = 2x^2 - 4x$ and $y = 2x - x^2$

22. a) The region cut from the first quadrant by the circle $x^2 + y^2 = 9$
    b) The region bounded by the x-axis and the semicircle $y = \sqrt{9 - x^2}$

    Compare your answer with the answer in (a).

23. The "triangular" region in the first quadrant between the circle $x^2 + y^2 = 9$ and the lines $x = 3$ and $y = 3$. (*Hint:* Use geometry to find the area.)

24. The region bounded above by the curve $y = 1/x^3$, below by the curve $y = -1/x^3$, and on the left and right by the lines $x = 1$ and $x = a > 1$. Also, find $\lim_{a \to \infty} \bar{x}$.

Thin Plates with Varying Density

25. Find the center of mass of a thin plate covering the region between the x-axis and the curve $y = 2/x^2$, $1 \le x \le 2$, if the plate's density at the point (x, y) is $\delta(x) = x^2$.

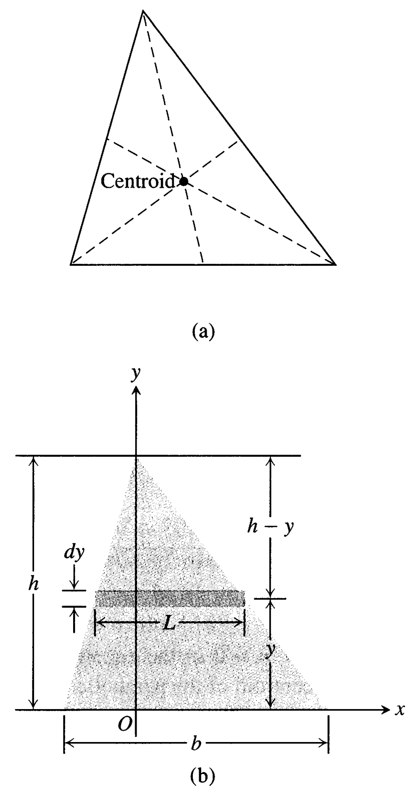
26. Find the center of mass of a thin plate covering the region bounded below by the parabola $y = x^2$ and above by the line $y = x$ if the plate's density at the point (x, y) is $\delta(x) = 12x$.

27. The region bounded by the curves $y = \pm 4/\sqrt{x}$ and the lines $x = 1$ and $x = 4$ is revolved about the y-axis to generate a solid.
    a) Find the volume of the solid.
    b) Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = 1/x$.
    c) Sketch the plate and show the center of mass in your sketch.

28. The region between the curve $y = 2/x$ and the x-axis from $x = 1$ to $x = 4$ is revolved about the x-axis to generate a solid.
    a) Find the volume of the solid.
    b) Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = \sqrt{x}$.
    c) Sketch the plate and show the center of mass in your sketch.

Centroids of Triangles

29. *The centroid of a triangle lies at the intersection of the triangle's medians (Fig. 5.61a).* You may recall that the point inside a triangle that lies one-third of the way from each side toward the opposite vertex is the point where the triangle's three medians intersect. Show that the centroid lies at the intersection of the medians by showing that it too lies one-third of the way from each side toward the opposite vertex. To do so, take the following steps.
    1. Stand one side of the triangle on the x-axis as in Fig. 5.61(b). Express dm in terms of L and dy.
    2. Use similar triangles to show that $L = (b/h)(h - y)$. Substitute this expression for L in your formula for dm.



5.61 The triangle in Exercise 29. (a) The centroid. (b) The dimensions and variables to use in locating the center of mass.

    3. Show that $\bar{y} = h/3$.
    4. Extend the argument to the other sides.

Use the result in Exercise 29 to find the centroids of the triangles whose vertices appear in Exercises 30–34. (*Hint:* Draw each triangle first.)

30. $(-1, 0), (1, 0), (0, 3)$
31. $(0, 0), (1, 0), (0, 1)$
32. $(0, 0), (a, 0), (0, a)$
33. $(0, 0), (a, 0), (0, b)$
34. $(0, 0), (a, 0), (a/2, b)$

Thin Wires

35. Find the moment about the x-axis of a wire of constant density that lies along the curve $y = \sqrt{x}$ from $x = 0$ to $x = 2$.

36. Find the moment about the x-axis of a wire of constant density that lies along the curve $y = x^3$ from $x = 0$ to $x = 1$.

37. Suppose the density of the wire in Example 6 is $\delta = k \sin\theta$ (k constant). Find the center of mass.

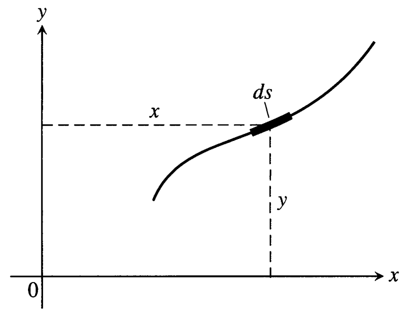
38. Suppose the density of the wire in Example 6 is $\delta = 1 + k\,|\cos\theta|$ (k constant). Find the center of mass.

Engineering Formulas

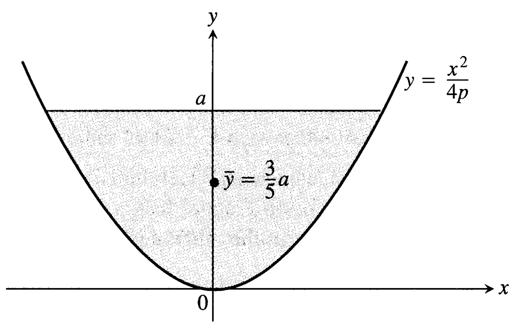
Verify the statements and formulas in Exercises 39–42.

39. The coordinates of the centroid of a differentiable plane curve are

$$\bar{x} = \frac{\int x\, ds}{\text{length}}, \qquad \bar{y} = \frac{\int y\, ds}{\text{length}}.$$

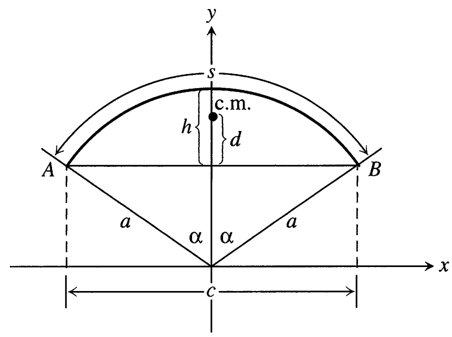


40. Whatever the value of $p > 0$ in the equation $y = x^2/(4p)$, the y-coordinate of the centroid of the parabolic segment shown here is $\bar{y} = (3/5)a$.



41. For wires and thin rods of constant density shaped like circular arcs centered at the origin and symmetric about the y-axis, the y-coordinate of the center of mass is

$$\bar{y} = \frac{a \sin\alpha}{\alpha} = \frac{ac}{s}.$$



42. (*Continuation of Exercise 41*)

a) Show that when α is small, the distance d from the centroid to chord AB is about $2h/3$ (in the notation of the figure here) by taking the following steps.

1. Show that

$$\frac{d}{h} = \frac{\sin\alpha - \alpha\cos\alpha}{\alpha - \alpha\cos\alpha}. \tag{9}$$

2. GRAPHER Graph

$$f(\alpha) = \frac{\sin\alpha - \alpha\cos\alpha}{\alpha - \alpha\cos\alpha}$$

and use TRACE to show that $\lim_{\alpha\to 0^+} f(\alpha) \approx 2/3$. (You will be able to confirm the suggested equality in Section 6.6, Exercise 74.)

b) CALCULATOR The error (difference between d and $2h/3$) is small even for angles greater than 45°. See for yourself by evaluating the right-hand side of Eq. (9) for $\alpha = 0.2$, 0.4, 0.6, 0.8, and 1.0 rad.

5.8 Work

In everyday life, *work* means an activity that requires muscular or mental effort. In science, the term refers specifically to a force acting on a body and the body's subsequent displacement. This section shows how to calculate work. The applications run from compressing railroad car springs and emptying subterranean tanks to forcing electrons together and lifting satellites into orbit.

Work Done by a Constant Force

When a body moves a distance d along a straight line as a result of being acted on by a force of constant magnitude F in the direction of motion, we calculate the

Joules

The joule, abbreviated J and pronounced "jewel," is named after the English physicist James Prescott Joule (1818–1889). The defining equation is

$$1 \text{ joule} = (1 \text{ newton})(1 \text{ meter}).$$

In symbols, $1 \text{ J} = 1 \text{ N} \cdot \text{m}$.

It takes a force of about 1 N to lift an apple from a table. If you lift it 1 m you have done about 1 J of work on the apple. If you then eat the apple you will have consumed about 80 food calories, the heat equivalent of nearly 335,000 joules. If this energy were directly useful for mechanical work, it would enable you to lift 335,000 more apples up 1 m.

work W done by the force on the body with the formula

$$W = Fd \qquad \text{(Constant-force formula for work).} \tag{1}$$

Right away we can see a considerable difference between what we are used to calling work and what this formula says work is. If you push a car down the street, you will be doing work on the car, both by your own reckoning and by Eq. (1). But if you push against the car and the car does not move, Eq. (1) says you will do no work on the car, even if you push for an hour.

From Eq. (1) we see that the unit of work in any system is the unit of force multiplied by the unit of distance. In SI units (SI stands for *Système International,* or International System), the unit of force is a newton, the unit of distance is a meter, and the unit of work is a newton-meter (N · m). This combination appears so often it has a special name, the **joule.** In the British system, the unit of work is the foot-pound, a unit frequently used by engineers.

EXAMPLE 1 If you jack up the side of a 2000-lb car 1.25 ft to change a tire (you have to apply a constant vertical force of about 1000 lb) you will perform $1000 \times 1.25 = 1250$ ft-lb of work on the car. In SI units, you have applied a force of 4448 N through a distance of 0.381 m to do $4448 \times 0.381 \approx 1695$ J of work. ❑

Work Done by a Variable Force

If the force you apply varies along the way, as it will if you are lifting a leaking bucket or compressing a spring, the formula $W = Fd$ has to be replaced by an integral formula that takes the variation in F into account.

Suppose that the force performing the work acts along a line that we can model with the x-axis and that its magnitude F is a continuous function of the position. We want to find the work done over the interval from $x = a$ to $x = b$. We partition $[a, b]$ in the usual way and choose an arbitrary point c_k in each subinterval $[x_{k-1}, x_k]$. If the subinterval is short enough, F, being continuous, will not vary much from x_{k-1} to x_k. The amount of work done across the interval will be about $F(c_k)$ times the distance Δx_k, the same as it would be if F were constant and we could apply Eq. (1). The total work done from a to b is therefore approximated by the Riemann sum

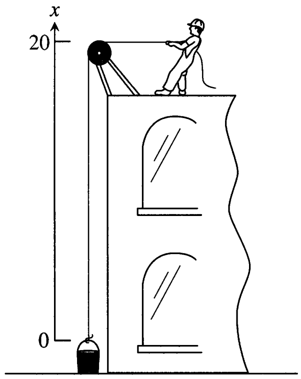
$$\sum_{k=1}^{n} F(c_k)\Delta x_k. \tag{2}$$

We expect the approximation to improve as the norm of the partition goes to zero, so we define the work done by the force from a to b to be the integral of F from a to b.

Definition

The **work** done by a variable force $F(x)$ directed along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\, dx. \tag{3}$$



5.62 The leaky bucket in Example 3.

The units of the integral are joules if F is in newtons and x is in meters, and foot-pounds if F is in pounds and x in feet.

EXAMPLE 2 The work done by a force of $F(x) = 1/x^2$ N along the x-axis from $x = 1$ m to $x = 10$ m is

$$W = \int_1^{10} \frac{1}{x^2}\,dx = -\frac{1}{x}\Bigg]_1^{10} = -\frac{1}{10} + 1 = 0.9 \text{ J.}$$ ❑

EXAMPLE 3 A leaky 5-lb bucket is lifted from the ground into the air by pulling in 20 ft of rope at a constant speed (Fig. 5.62). The rope weighs 0.08 lb/ft. The bucket starts with 2 gal of water (16 lb) and leaks at a constant rate. It finishes draining just as it reaches the top. How much work was spent

a) lifting the water alone;
b) lifting the water and bucket together;
c) lifting the water, bucket, and rope?

Solution

a) *The water alone.* The force required to lift the water is equal to the water's weight, which varies steadily from 16 to 0 lb over the 20-ft lift. When the bucket is x ft off the ground, the water weighs

$$F(x) = \underbrace{16}_{\substack{\text{original weight}\\ \text{of water}}}\underbrace{\left(\frac{20 - x}{20}\right)}_{\substack{\text{proportion left}\\ \text{at elevation } x}} = 16\left(1 - \frac{x}{20}\right) = 16 - \frac{4x}{5} \text{ lb.}$$

The work done is

$$\begin{aligned} W &= \int_a^b F(x)\,dx \qquad \text{Use Eq. (3) for variable forces.} \\ &= \int_0^{20} \left(16 - \frac{4x}{5}\right) dx = \left[16x - \frac{2x^2}{5}\right]_0^{20} = 320 - 160 = 160 \text{ ft} \cdot \text{lb.} \end{aligned}$$

b) *The water and bucket together.* According to Eq. (1), it takes $5 \times 20 = 100$ ft · lb to lift a 5-lb weight 20 ft. Therefore

$$160 + 100 = 260 \text{ ft} \cdot \text{lb}$$

of work were spent lifting the water and bucket together.

c) *The water, bucket, and rope.* Now the total weight at level x is

$$F(x) = \underbrace{\left(16 - \frac{4x}{5}\right)}_{\substack{\text{variable}\\ \text{weight}\\ \text{of water}}} + \underbrace{5}_{\substack{\text{constant}\\ \text{weight}\\ \text{of bucket}}} + \underbrace{(\overset{\text{lb/ft}}{0.08})(\overset{\text{ft}}{20 - x})}_{\substack{\text{weight of rope}\\ \text{paid out at}\\ \text{elevation } x}}.$$

The work lifting the rope is

$$\text{Work on rope} = \int_0^{20} (0.08)(20 - x)\,dx = \int_0^{20} (1.6 - 0.08x)\,dx$$

$$= \Big[1.6x - 0.04x^2\Big]_0^{20} = 32 - 16 = 16 \text{ ft} \cdot \text{lb}.$$

The total work for the water, bucket, and rope combined is

$$160 + 100 + 16 = 276 \text{ ft} \cdot \text{lb}.$$

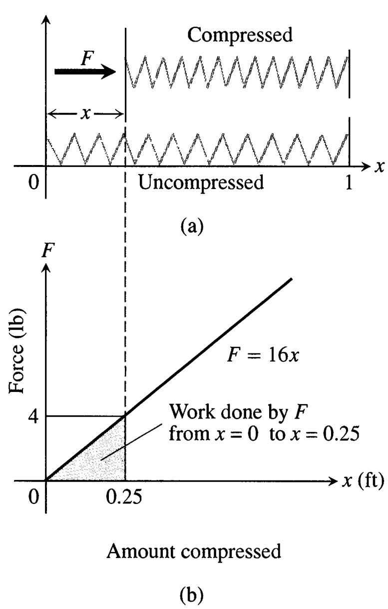
❑

Hooke's Law for Springs: $F = kx$

Hooke's law says that the force it takes to stretch or compress a spring x length units from its natural (unstressed) length is proportional to x. In symbols,

$$F = kx. \tag{4}$$

The constant k, measured in force units per unit length, is a characteristic of the spring, called the **force constant** (or spring constant) of the spring. Hooke's law (Eq. 4) gives good results as long as the force doesn't distort the metal in the spring. We assume that the forces in this section are too small to do that.



5.63 The force F needed to hold a spring under compression increases linearly as the spring is compressed.

EXAMPLE 4 Find the work required to compress a spring from its natural length of 1 ft to a length of 0.75 ft if the force constant is $k = 16$ lb/ft.

Solution We picture the uncompressed spring laid out along the x-axis with its movable end at the origin and its fixed end at $x = 1$ ft (Fig. 5.63). This enables us to describe the force required to compress the spring from 0 to x with the formula $F = 16x$. To compress the spring from 0 to 0.25 ft, the force must increase from

$$F(0) = 16 \cdot 0 = 0 \text{ lb} \qquad \text{to} \qquad F(0.25) = 16 \cdot 0.25 = 4 \text{ lb}.$$

The work done by F over this interval is

$$W = \int_0^{0.25} 16x\,dx = 8x^2\Big]_0^{0.25} = 0.5 \text{ ft} \cdot \text{lb}.$$

Eq. (3) with $a = 0$, $b = 0.25$, $F(x) = 16x$

❑

EXAMPLE 5 A spring has a natural length of 1 m. A force of 24 N stretches the spring to a length of 1.8 m.

a) Find the force constant k.
b) How much work will it take to stretch the spring 2 m beyond its natural length?
c) How far will a 45-N force stretch the spring?

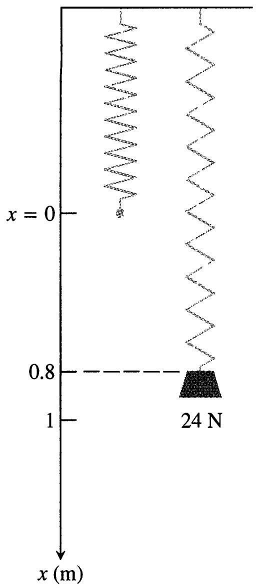
Solution

a) *The force constant.* We find the force constant from Eq. (4). A force of 24 N stretches the spring 0.8 m, so

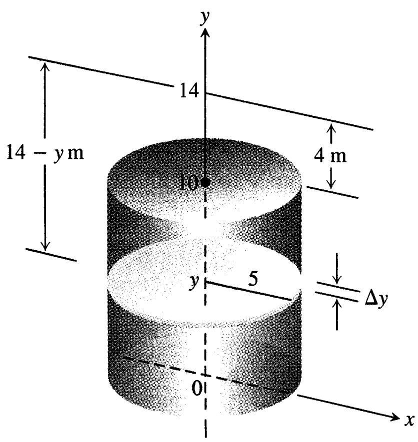
$$24 = k(0.8)$$

Eq. (4) with $F = 24$, $x = 0.8$

$$k = 24/0.8 = 30 \text{ N/m}.$$



5.64 A 24-N weight stretches this spring 0.8 m beyond its unstressed length.



5.65 To find the work it takes to pump the water from a tank, think of lifting the water one thin slab at a time.

How to Find Work Done During Pumping

1. Draw a figure with a coordinate system.
2. Find the weight F of a thin horizontal slab of liquid.
3. Find the work ΔW it takes to lift the slab to its destination.
4. Integrate the work expression from the base to the surface of the liquid.

b) *The work to stretch the spring* 2 m. We imagine the unstressed spring hanging along the x-axis with its free end at $x = 0$ (Fig. 5.64). The force required to stretch the spring x m beyond its natural length is the force required to pull the free end of the spring x units from the origin. Hooke's law with $k = 30$ says that this force is

$$F(x) = 30x.$$

The work done by F on the spring from $x = 0$ m to $x = 2$ m is

$$W = \int_0^2 30x\,dx = 15x^2\Big]_0^2 = 60 \text{ J}.$$

c) *How far will a* 45-N *force stretch the spring?* We substitute $F = 45$ in the equation $F = 30x$ to find

$$45 = 30x, \qquad \text{or} \qquad x = 1.5 \text{ m}.$$

A 45-N force will stretch the spring 1.5 m. No calculus is required to find this. ❑

Pumping Liquids from Containers

How much work does it take to pump all or part of the liquid from a container? To find out, we imagine lifting the liquid out one thin horizontal slab at a time and applying the equation $W = Fd$ to each slab. We then evaluate the integral this leads to as the slabs become thinner and more numerous. The integral we get each time depends on the weight of the liquid and the dimensions of the container, but the way we find the integral is always the same. The next examples show what to do.

EXAMPLE 6 How much work does it take to pump the water from a full upright circular cylindrical tank of radius 5 m and height 10 m to a level of 4 m above the top of the tank?

Solution We draw the tank (Fig. 5.65), add coordinate axes, and imagine the water divided into thin horizontal slabs by planes perpendicular to the y-axis at the points of a partition P of the interval [0, 10].

The typical slab between the planes at y and $y + \Delta y$ has a volume of

$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi(5)^2\Delta y = 25\pi\,\Delta y \text{ m}^3.$$

The force F required to lift the slab is equal to its weight,

$$F = 9800\Delta V$$

$$= 9800(25\pi\,\Delta y) = 245{,}000\pi\,\Delta y \text{ N}.$$

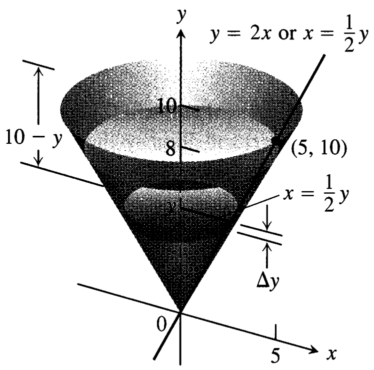
Water weighs 9800 N/m^3.

The distance through which F must act is about $(14 - y)$ m, so the work done lifting the slab is about

$$\Delta W = \text{force} \times \text{distance} = 245{,}000\pi(14 - y)\Delta y \text{ J}.$$

The work it takes to lift all the water is approximately

$$W \approx \sum_0^{10} \Delta W = \sum_0^{10} 245{,}000\pi(14 - y)\Delta y \text{ J}.$$



5.66 The olive oil in Example 7.

This is a Riemann sum for the function $245{,}000\pi(14-y)$ over the interval $0 \leq y \leq 10$. The work of pumping the tank dry is the limit of these sums as $\| P \| \to 0$:

$$W = \int_0^{10} 245{,}000\pi(14-y)\,dy = 245{,}000\pi \int_0^{10} (14-y)\,dy$$

$$= 245{,}000\pi \left[14y - \frac{y^2}{2} \right]_0^{10} = 245{,}000\pi[90]$$

$$\approx 69{,}272{,}118 \approx 69.3 \times 10^6 \text{ J.}$$

A 1-horsepower output motor rated at 746 J/sec could empty the tank in a little less than 26 h. ❑

EXAMPLE 7 The conical tank in Fig. 5.66 is filled to within 2 ft of the top with olive oil weighing 57 lb/ft^3. How much work does it take to pump the oil to the rim of the tank?

Solution We imagine the oil divided into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval [0, 8].

The typical slab between the planes at y and $y + \Delta y$ has a volume of about

$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi \left(\frac{1}{2}y\right)^2 \Delta y = \frac{\pi}{4} y^2 \Delta y \text{ ft}^3.$$

The force $F(y)$ required to lift this slab is equal to its weight,

$$F(y) = 57\,\Delta V = \frac{57\pi}{4} y^2 \Delta y \text{ lb.}$$

Weight = weight per unit volume × volume

The distance through which $F(y)$ must act to lift this slab to the level of the rim of the cone is about $(10 - y)$ ft, so the work done lifting the slab is about

$$\Delta W = \frac{57\pi}{4}(10-y)y^2 \Delta y \text{ ft} \cdot \text{lb.}$$

The work done lifting all the slabs from $y = 0$ to $y = 8$ to the rim is approximately

$$W \approx \sum_0^8 \frac{57\pi}{4}(10-y)y^2 \Delta y \text{ ft} \cdot \text{lb.}$$

This is a Riemann sum for the function $(57\pi/4)(10-y)y^2$ on the interval from $y = 0$ to $y = 8$. The work of pumping the oil to the rim is the limit of these sums as the norm of the partition goes to zero.

$$W = \int_0^8 \frac{57\pi}{4}(10-y)y^2\,dy$$

$$= \frac{57\pi}{4} \int_0^8 (10y^2 - y^3)\,dy$$

$$= \frac{57\pi}{4} \left[\frac{10y^3}{3} - \frac{y^4}{4} \right]_0^8 \approx 30{,}561 \text{ ft} \cdot \text{lb.}$$

❑

Exercises 5.8

Work Done by a Variable Force

1. The workers in Example 3 changed to a larger bucket that held 5 gal (40 lb) of water, but the new bucket had an even larger leak so that it, too, was empty by the time it reached the top. Assuming that the water leaked out at a steady rate, how much work was done lifting the water? (Do not include the rope and bucket.)

2. The bucket in Example 3 is hauled up twice as fast so that there is still 1 gal (8 lb) of water left when the bucket reaches the top. How much work is done lifting the water this time? (Do not include the rope and bucket.)

3. A mountain climber is about to haul up a 50-m length of hanging rope. How much work will it take if the rope weighs 0.624 N/m?

4. A bag of sand originally weighing 144 lb was lifted at a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been lifted 18 ft. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)

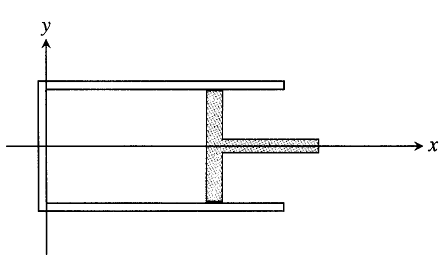
5. An electric elevator with a motor at the top has a multistrand cable weighing 4.5 lb/ft. When the car is at the first floor, 180 ft of cable are paid out, and effectively 0 ft are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?

6. When a particle of mass m is at $(x, 0)$, it is attracted toward the origin with a force whose magnitude is k/x^2. If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time it reaches $x = a$, $0 < a < b$.

7. Suppose that the gas in a circular cylinder of cross-section area A is being compressed by a piston. If p is the pressure of the gas in pounds per square inch and V is the volume in cubic inches, show that the work done in compressing the gas from state (p_1, V_1) to state (p_2, V_2) is given by the equation

$$\text{Work} = \int_{(p_1, V_1)}^{(p_2, V_2)} p\,dV.$$

(*Hint:* In the coordinates suggested in the figure here, $dV = A\,dx$. The force against the piston is pA.)



8. *(Continuation of Exercise 7.)* Use the integral in Exercise 7 to find the work done in compressing the gas from $V_1 = 243$ in^3 to $V_2 = 32$ in^3 if $p_1 = 50$ lb/in^3 and p and V obey the gas law $pV^{1.4} = \text{constant}$ (for adiabatic processes).

Springs

9. It took 1800 J of work to stretch a spring from its natural length of 2 m to a length of 5 m. Find the spring's force constant.

10. A spring has a natural length of 10 in. An 800-lb force stretches the spring to 14 in. (a) Find the force constant. (b) How much work is done in stretching the spring from 10 in. to 12 in.? (c) How far beyond its natural length will a 1600-lb force stretch the spring?

11. A force of 2 N will stretch a rubber band 2 cm (0.02 m). Assuming Hooke's law applies, how far will a 4-N force stretch the rubber band? How much work does it take to stretch the rubber band this far?

12. If a force of 90 N stretches a spring 1 m beyond its natural length, how much work does it take to stretch the spring 5 m beyond its natural length?

13. *Subway car springs.* It takes a force of 21,714 lb to compress a coil spring assembly on a New York City Transit Authority subway car from its free height of 8 in. to its fully compressed height of 5 in.

a) What is the assembly's force constant?

b) How much work does it take to compress the assembly the first half inch? the second half inch? Answer to the nearest in · lb.

(Data courtesy of Bombardier, Inc., Mass Transit Division, for spring assemblies in subway cars delivered to the New York City Transit Authority from 1985 to 1987.)

14. A bathroom scale is compressed 1/16 in. when a 150-lb person stands on it. Assuming the scale behaves like a spring that obeys Hooke's law, how much does someone who compresses the scale 1/8 in. weigh? How much work is done compressing the scale 1/8 in.?

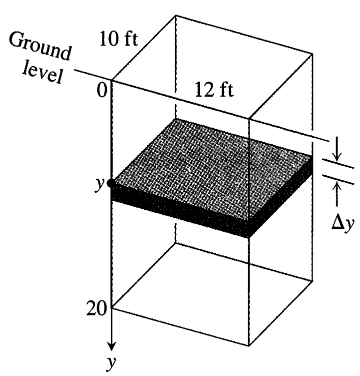
Pumping Liquids from Containers

The Weight of Water

Because of variations in the earth's gravitational field, the weight of a cubic foot of water at sea level can vary from about 62.26 lb at the equator to as much as 62.59 lb near the poles, a variation of about 0.5%. A cubic foot that weighs about 62.4 lb in Melbourne and New York City will weigh 62.5 lb in Juneau and Stockholm. While 62.4 is a typical figure and a common textbook value, there is considerable variation.

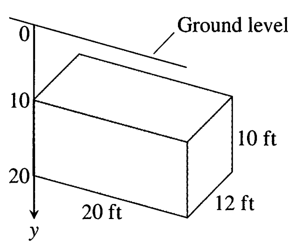
15. The rectangular tank shown here, with its top at ground level, is used to catch runoff water. Assume that the water weighs 62.4 lb/ft^3.

a) How much work does it take to empty the tank by pumping the water back to ground level once the tank is full?
b) If the water is pumped to ground level with a (5/11)-hp motor (work output 250 ft • lb/sec), how long will it take to empty the full tank (to the nearest minute)?
c) Show that the pump in part (b) will lower the water level 10 ft (halfway) during the first 25 min of pumping.
d) What are the answers to parts (a) and (b) in a location where water weighs 62.26 lb/ft^3? 62.59 lb/ft^3?



16. The rectangular cistern (storage tank for rainwater) shown here has its top 10 ft below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level.

a) How much work will it take to empty the cistern?
b) How long will it take a (1/2)-hp pump, rated at 275 ft • lb/sec, to pump the tank dry?
c) How long will it take the pump in part (b) to empty the tank halfway? (It will be less than half the time required to empty the tank completely.)
d) What are the answers to parts (a)–(c) in a location where water weighs 62.26 lb/ft^3? 62.59 lb/ft^3?

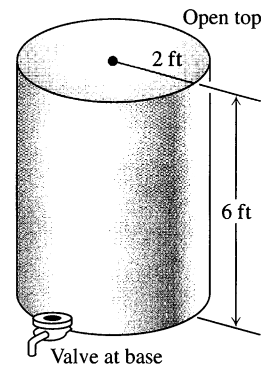


17. How much work would it take to pump the water from the tank in Example 6 to the level of the top of the tank (instead of 4 m higher)?

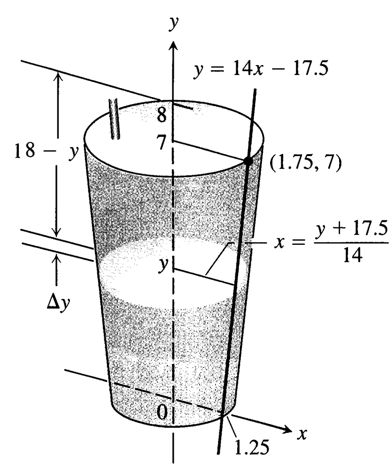
18. Suppose that, instead of being full, the tank in Example 6 is only half full. How much work does it take to pump the remaining water to a level 4 m above the top of the tank?

19. A vertical right circular cylindrical tank measures 30 ft high and 20 ft in diameter. It is full of kerosene weighing 51.2 lb/ft^3. How much work does it take to pump the kerosene to the level of the top of the tank?

20. The cylindrical tank shown here can be filled by pumping water from a lake 15 ft below the bottom of the tank. There are two ways to go about it. One is to pump the water through a hose attached to a valve in the bottom of the tank. The other is to attach the hose to the rim of the tank and let the water pour in. Which way will be faster? Give reasons for your answer.



21. CALCULATOR The truncated conical container shown here is full of strawberry milkshake that weighs 4/9 oz/in^3. As you can see, the container is 7 in. deep, 2.5 in. across at the base, and 3.5 in. across at the top (a standard size at Brigham's in Boston). The straw sticks up an inch above the top. About how much work does it take to suck up the milkshake through the straw (neglecting friction)? Answer in inch-ounces.

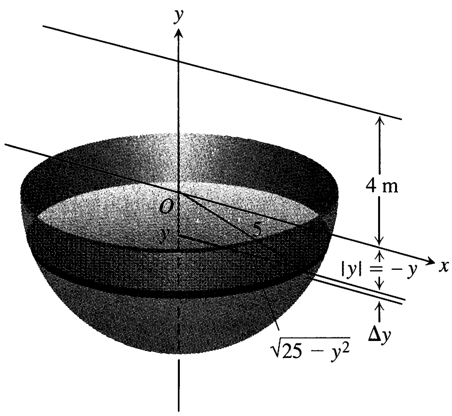


Dimensions in inches

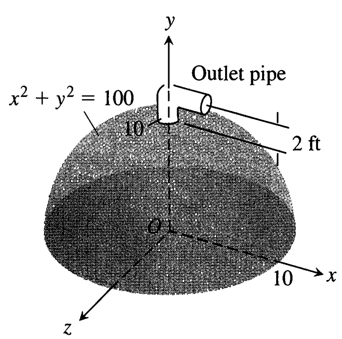
22. a) Suppose the conical container in Example 7 contains milk (weighing 64.5 lb/ft^3) instead of olive oil. How much work will it take to pump the contents to the rim?
b) How much work will it take to pump the oil in Example 7 to a level 3 ft above the cone's rim?

23. To design the interior surface of a huge stainless steel tank, you revolve the curve $y = x^2, 0 \leq x \leq 4$, about the y-axis. The container, with dimensions in meters, is to be filled with seawater, which weighs 10,000 N/m^3. How much work will it take to empty the tank by pumping the water to the tank's top?

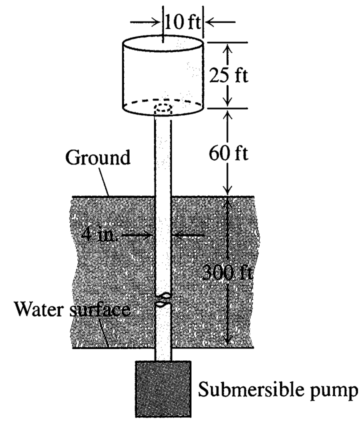
24. We model pumping from spherical containers the way we do from other containers, with the axis of integration along the vertical axis of the sphere. Use the figure here to find how much work it takes to empty a full hemispherical water reservoir of radius 5 m by pumping the water to a height of 4 m above the top of the reservoir. Water weighs 9800 N/m^3.



25. You are in charge of the evacuation and repair of the storage tank shown here. The tank is a hemisphere of radius 10 ft and is full of benzene weighing 56 lb/ft^3. A firm you contacted says it can empty the tank for 1/2¢ per foot-pound of work. Find the work required to empty the tank by pumping the benzene to an outlet 2 ft above the top of the tank. If you have $5000 budgeted for the job, can you afford to hire the firm?



26. Your town has decided to drill a well to increase its water supply. As the town engineer, you have determined that a water tower will be necessary to provide the pressure needed for distribution, and you have designed the system shown here. The water is to be pumped from a 300-ft well through a vertical 4-in. pipe into the base of a cylindrical tank 20 ft in diameter and 25 ft high. The base of the tank will be 60 ft aboveground. The pump is a 3-hp pump, rated at 1650 ft • lb/sec. To the nearest hour, how long will it take to fill the tank the first time? (Include the time it takes to fill the pipe.) Assume water weighs 62.4 lb/ft^3.



NOT TO SCALE

Other Applications

27. *Putting a satellite in orbit.* The strength of the earth's gravitational field varies with the distance r from the earth's center, and the magnitude of the gravitational force experienced by a satellite of mass m during and after launch is

$$F(r) = \frac{mMG}{r^2}.$$

Here, $M = 5.975 \times 10^{24}$ kg is the earth's mass, $G = 6.6720 \times 10^{-11}$ N • m^{2}kg^{-2} is the universal gravitational constant, and r is measured in meters. The work it takes to lift a 1000-kg satellite from the earth's surface to a circular orbit 35,780 km above the earth's center is therefore given by the integral

$$\text{Work} = \int_{6,370,000}^{35,780,000} \frac{1000MG}{r^2}\, dr \text{ joules}.$$

Evaluate the integral. The lower limit of integration is the earth's radius in meters at the launch site. (This calculation does not take into account energy spent lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

28. *Forcing electrons together.* Two electrons r meters apart repel each other with a force of

$$F = \frac{23 \times 10^{-29}}{r^2} \text{ newtons}.$$

a) Suppose one electron is held fixed at the point (1, 0) on the x-axis (units in meters). How much work does it take to move a second electron along the x-axis from the point $(-1, 0)$ to the origin?

b) Suppose an electron is held fixed at each of the points $(-1, 0)$ and $(1, 0)$. How much work does it take to move a third electron along the x-axis from (5, 0) to (3, 0)?

Work and Kinetic Energy

29. If a variable force of magnitude $F(x)$ moves a body of mass

m along the x-axis from x_1 to x_2, the body's velocity v can be written as dx/dt (where t represents time). Use Newton's Second Law of Motion $F = m(dv/dt)$ and the Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

to show that the net work done by the force in moving the body from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F(x)\,dx = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2,$$

where v_1 and v_2 are the body's velocities at x_1 and x_2. In physics the expression $(1/2)mv^2$ is called the *kinetic energy* of the body moving with velocity v. Therefore, *the work done by the force equals the change in the body's kinetic energy,* and we can find the work by calculating this change.

In Exercises 30–36, use the result of Exercise 29.

30. *Tennis.* A 2-oz tennis ball was served at 160 ft/sec (about 109 mph). How much work was done on the ball to make it go this fast? (To find the ball's mass from its weight, express the weight in pounds and divide by 32 ft/sec^2, the acceleration of gravity.)

31. *Baseball.* How many foot-pounds of work does it take to throw a baseball 90 mph? A baseball weighs 5 oz = 0.3125 lb.

32. *Golf.* A 1.6-oz golf ball is driven off the tee at a speed of 280 ft/sec (about 191 mph). How many foot-pounds of work are done getting the ball into the air?

33. *Tennis.* During the match in which Pete Sampras won the 1990 U.S. Open men's tennis championship, Sampras hit a serve that was clocked at a phenomenal 124 mph. How much work did Sampras have to do on the 2-oz ball to get it to that speed?

34. *Football.* A quarterback threw a 14.5-oz football 88 ft/sec (60 mph). How many foot-pounds of work were done on the ball to get it to this speed?

35. *Softball.* How much work has to be performed on a 6.5-oz softball to pitch it 132 ft/sec (90 mph)?

36. *A ball bearing.* A 2-oz steel ball bearing is placed on a vertical spring whose force constant is $k = 18$ lb/ft. The spring is compressed 3 inches and released. About how high does the ball bearing go?

Weight vs. Mass

Weight is the force that results from gravity pulling on a mass. The two are related by the equation in Newton's second law,

$$\text{Weight} = \text{mass} \times \text{acceleration}.$$

Thus,

$$\text{Newtons} = \text{kilograms} \times \text{m/sec}^2,$$
$$\text{Pounds} = \text{slugs} \times \text{ft/sec}^2.$$

To convert mass to weight, multiply by the acceleration of gravity. To convert weight to mass, divide by the acceleration of gravity.

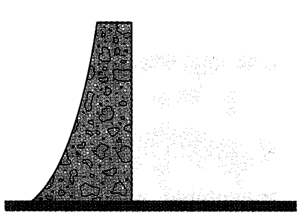
5.9 Fluid Pressures and Forces

We make dams thicker at the bottom than at the top (Fig. 5.67) because the pressure against them increases with depth. It is a remarkable fact that the pressure at any point on a dam depends only on how far below the surface the point is and not on how much the surface of the dam happens to be tilted at that point. The pressure, in pounds per square foot at a point h feet below the surface, is always $62.4h$. The number 62.4 is the weight-density of water in pounds per cubic foot.

The formula, pressure $= 62.4h$, makes sense when you think of the units involved:

$$\frac{\text{lb}}{\text{ft}^2} = \frac{\text{lb}}{\text{ft}^3} \times \text{ft}.$$

As you can see, this equation depends only on units and not on the fluid involved. The pressure h feet below the surface of any fluid is the fluid's weight-density times h.



5.67 To withstand the increasing pressure, dams are built thicker as they go down.

Weight-density

A fluid's weight-density is its weight per unit volume. Typical values (lb/ft^3) are

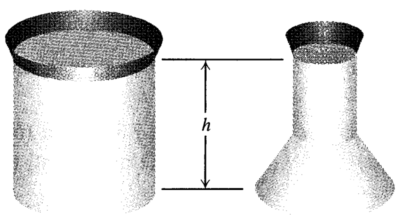
Gasoline	42
Mercury	849
Milk	64.5
Molasses	100
Olive oil	57
Seawater	64
Water	62.4

The Pressure-Depth Equation

In a fluid that is standing still, the pressure p at depth h is the fluid's weight-density w times h:

$$p = wh. \tag{1}$$

In this section we use the equation $p = wh$ to derive a formula for the total force exerted by a fluid against all or part of a vertical or horizontal containing wall.



5.68 These containers are filled with water to the same depth and have the same base area. The total force is therefore the same on the bottom of each container. The containers' shapes do not matter here.

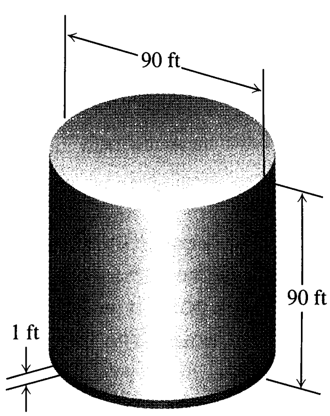
The Constant-Depth Formula for Fluid Force

In a container of fluid with a flat horizontal base, the total force exerted by the fluid against the base can be calculated by multiplying the area of the base by the pressure at the base. We can do this because total force equals force per unit area (pressure) times area. (See Fig. 5.68.) If F, p, and A are the total force, pressure, and area, then

$$\begin{aligned} F &= \text{total force} = \text{force per unit area} \times \text{area} \\ &= \text{pressure} \times \text{area} = pA \\ &= whA. \qquad \text{$p = wh$ from Eq. (1)} \end{aligned}$$

Fluid Force on a Constant-Depth Surface

$$F = pA = whA \tag{2}$$



5.69 Schematic drawing of the molasses tank in Example 1. How much force did the lowest foot of the vertical wall have to withstand when the tank was full? It takes an integral to find out. Notice that the proportions of the tank were ideal.

EXAMPLE 1 *The Great Molasses Flood*

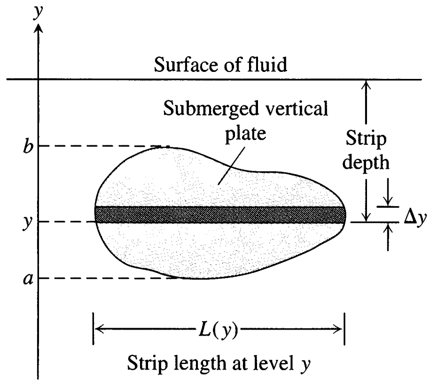
At 1:00 P.M. on January 15, 1919, an unusually warm day, a 90-ft-high, 90-ft-diameter cylindrical metal tank in which the Puritan Distilling Company was storing molasses at the corner of Foster and Commercial streets in Boston's North End exploded. The molasses flooded into the streets, 30 ft deep, trapping pedestrians and horses, knocking down buildings, and oozing into homes. It was eventually tracked all over town and even made its way into the suburbs (on trolley cars and people's shoes). It took weeks to clean up.

Given that the molasses weighed $100\ \text{lb/ft}^3$, what was the total force exerted by the molasses against the bottom of the tank at the time it blew? Assuming the tank was full, we can find out from Eq. (2):

$$\text{Total force} = whA = (100)(90)(\pi(45)^2) \approx 57{,}255{,}526 \text{ lb.} \qquad \square$$

How about the force against the walls of the tank? For example, what was the total force against the bottom foot-wide band of tank wall (Fig. 5.69)? The area of the band was

$$A = 2\pi rh = 2\pi(45)(1) = 90\pi \text{ ft}^2.$$



5.70 The force exerted by a fluid against one side of a thin horizontal strip is about ΔF = pressure $\times$ area = $w \times$ (strip depth) $\times$ $L(y)\,\Delta y$. The plate here is flat, but it might have been curved instead, like the vertical wall of a cylindrical tank. Whatever the case, the strip length is measured along the surface of the plate.

The tank was 90 ft deep, so the pressure near the bottom was approximately

$$p = wh = (100)(90) = 9000 \text{ lb/ft}^2.$$

Therefore the total force against the band was approximately

$$F = whA = (9000)(90\pi) \approx 2,544,690 \text{ lb}.$$

But this is not exactly right. The top of the band was 89 ft below the surface, not 90, and the pressure there was less. To find out exactly what the force on the band was, we need to take into account the variation of the pressure across the band.

The Variable-Depth Formula

Suppose we want to know the force exerted by a fluid against one side of a vertical plate submerged in a fluid of weight-density w. To find it, we model the plate as a region extending from $y = a$ to $y = b$ in the xy-plane (Fig. 5.70). We partition $[a, b]$ in the usual way and imagine the region to be cut into thin horizontal strips by planes perpendicular to the y-axis at the partition points. The typical strip from y to $y + \Delta y$ is Δy units wide by $L(y)$ units long. We assume $L(y)$ to be a continuous function of y.

The pressure varies across the strip from top to bottom, just as it did in the molasses tank. But if the strip is narrow enough, the pressure will remain close to its bottom-edge value of $w \times$ (strip depth). The force exerted by the fluid against one side of the strip will be about

$$\begin{aligned} \Delta F &= (\text{pressure along bottom edge}) \times (\text{area}) \\ &= w \times (\text{strip depth}) \times L(y)\Delta y. \end{aligned}$$

The force against the entire plate will be about

$$\sum_a^b \Delta F = \sum_a^b (w \times (\text{strip depth}) \times L(y)\Delta y). \tag{3}$$

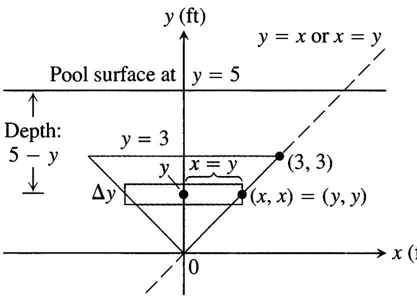
The sum in (3) is a Riemann sum for a continuous function on $[a, b]$, and we expect the approximations to improve as the norm of the partition goes to zero. We define the force against the plate to be the limit of these sums.

Definition
The Integral for Fluid Force

Suppose that a plate submerged vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y-axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level y. Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (\text{strip depth}) \cdot L(y)\,dy. \tag{4}$$

EXAMPLE 2 A flat isosceles right triangular plate with base 6 ft and height 3 ft is submerged vertically, base up, 2 ft below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.



5.71 To find the force on one side of the submerged plate in Example 2, we can use a coordinate system like the one here.

Solution We establish a coordinate system to work in by placing the origin at the plate's bottom vertex and running the y-axis upward along the plate's axis of symmetry (Fig. 5.71). (We will look at other coordinate systems in Exercises 3 and 4.) The surface of the pool lies along the line $y = 5$ and the plate's top edge along the line $y = 3$. The plate's right-hand edge lies along the line $y = x$, with the upper right vertex at (3, 3). The length of a thin strip at level y is

$$L(y) = 2x = 2y.$$

The depth of the strip beneath the surface is $(5 - y)$. The force exerted by the water against one side of the plate is therefore

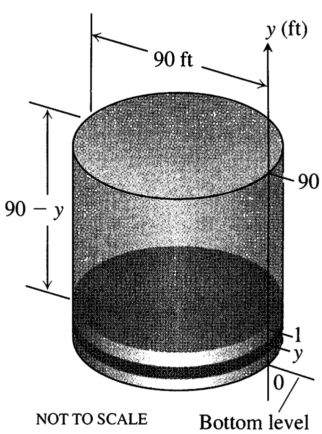
$$\begin{aligned} F &= \int_a^b w \times \begin{pmatrix}\text{strip}\\ \text{depth}\end{pmatrix} \times L(y)\,dy \qquad \text{Eq. (4)} \\ &= \int_0^3 62.4(5-y)\,2y\,dy \\ &= 124.8\int_0^3 (5y - y^2)\,dy \\ &= 124.8\left[\frac{5}{2}y^2 - \frac{y^3}{3}\right]_0^3 = 1684.8 \text{ lb.} \end{aligned}$$

❑

How to Find Fluid Force

Whatever coordinate system you use, you can find the fluid force against one side of a submerged vertical plate or wall by taking these steps.

1. *Find expressions* for the length and depth of a typical thin horizontal strip.
2. *Multiply their product by the fluid's weight-density w and integrate* over the interval of depths occupied by the plate or wall.



5.72 The molasses tank with the coordinate origin at the bottom (Example 3).

EXAMPLE 3 We can now calculate exactly the force exerted by the molasses against the bottom 1-ft band of the Puritan Distilling Company's storage tank when the tank was full.

The tank was a right circular cylindrical tank 90 ft high and 90 ft in diameter. Using a coordinate system with the origin at the bottom of the tank and the y-axis pointing up (Fig. 5.72), we find that the typical horizontal strip at level y has

Strip depth: $90 - y$,

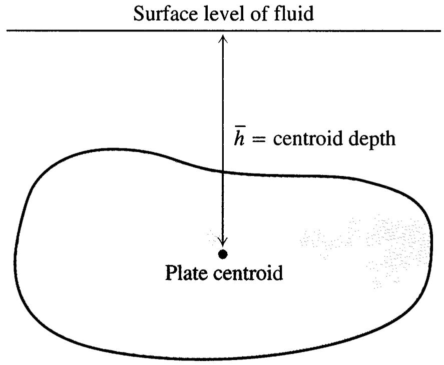
Strip length: $\pi \times$ tank diameter $= 90\pi$.

The force against the band is therefore

$$\text{Force} = \int_0^1 w(\text{depth})(\text{length})\,dy = \int_0^1 100(90-y)(90\pi)\,dy \qquad \text{For molasses, } w = 100$$

$$= 9000\pi\int_0^1 (90-y)\,dy \approx 2{,}530{,}553 \text{ lb.}$$

❑



5.73 The force against one side of the plate is $w \cdot \bar{h} \cdot$ plate area.

As expected, the force is slightly less than the constant-depth estimate following Example 1.

Fluid Forces and Centroids

If we know the location of the centroid of a submerged flat vertical plate (Fig. 5.73), we can take a shortcut to find the force against one side of the plate. From Eq. (4),

$$\begin{aligned} F &= \int_a^b w \times (\text{strip depth}) \times L(y)\, dy \\ &= w \int_a^b (\text{strip depth}) \times L(y)\, dy \\ &= w \times (\text{moment about surface level line of region occupied by plate}) \\ &= w \times (\text{depth of plate's centroid}) \times (\text{area of plate}). \end{aligned}$$

Fluid Forces and Centroids

The force of a fluid of weight-density w against one side of a submerged flat vertical plate is the product of w, the distance $\bar{h}$ from the plate's centroid to the fluid surface, and the plate's area:

$$F = w\bar{h}A. \tag{5}$$

EXAMPLE 4 Use Eq. (5) to find the force in Example 2.

Solution The centroid of the triangle (Fig. 5.71) lies on the y-axis, one-third of the way from the base to the vertex, so $\bar{h} = 3$. The triangle's area is

$$\begin{aligned} A &= \frac{1}{2}(\text{base})(\text{height}) \\ &= \frac{1}{2}(6)(3) = 9. \end{aligned}$$

Hence,

$$\begin{aligned} F &= w\bar{h}A = (62.4)(3)(9) \\ &= 1684.8 \text{ lb.} \end{aligned}$$

❑

Equation (5) says that the fluid force on one side of a submerged flat vertical plate is the same as it would be if the plate's entire area lay $\bar{h}$ units beneath the surface. For many shapes, the location of the centroid can be found in a table, and Eq. (5) gives a practical way to find F. Of course, the centroid's location was found by someone who performed an integration equivalent to evaluating the integral in Eq. (4). We recommend for now that you practice your mathematical modeling by drawing pictures and thinking things through the way we did when we developed Eq. (4). Then check your results, when you conveniently can, with Eq. (5).

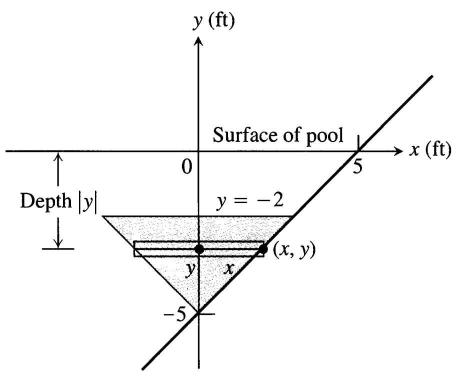
Exercises 5.9

The weight-densities of the fluids in the following exercises can be found in the table on page 428.

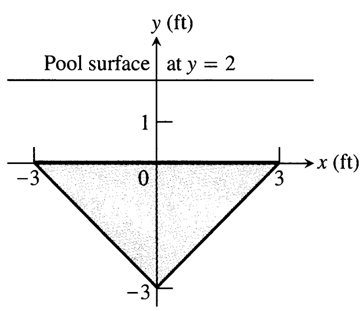
1. What was the total fluid force against the cylindrical inside wall of the molasses tank in Example 1 when the tank was full? half full?

2. What was the total fluid force against the bottom 1-ft band of the inside wall of the molasses tank in Example 1 when the tank was half full?

3. Calculate the fluid force on one side of the plate in Example 2 using the coordinate system shown here.



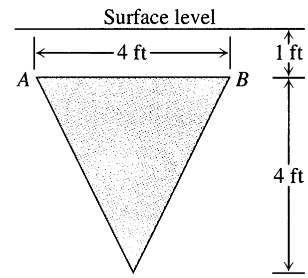
4. Calculate the fluid force on one side of the plate in Example 2 using the coordinate system shown here.



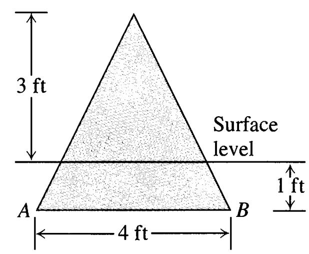
5. The plate in Example 2 is lowered another 2 ft into the water. What is the fluid force on one side of the plate now?

6. The plate in Example 2 is raised to put its top edge at the surface of the pool. What is the fluid force on one side of the plate now?

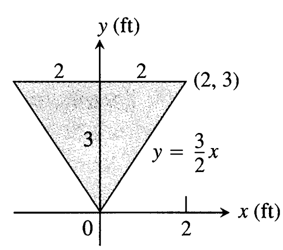
7. The isosceles triangular plate shown here is submerged vertically 1 ft below the surface of a freshwater lake.
   a) Find the fluid force against one face of the plate.
   b) What would be the fluid force on one side of the plate if the water were seawater instead of freshwater?



8. The plate in Exercise 7 is revolved 180° about line AB so that part of the plate sticks out of the lake, as shown here. What force does the water exert on one face of the plate now?



9. The vertical ends of a watering trough are isosceles triangles like the one shown here (dimensions in feet).
   a) Find the fluid force against the ends when the trough is full.
   b) CALCULATOR How many inches do you have to lower the water level in the trough to cut the fluid force on the ends in half? (Answer to the nearest half inch.)
   c) Does it matter how long the trough is? Give reasons for your answer.



10. The vertical ends of a watering trough are squares 3 ft on a side.
    a) Find the fluid force against the ends when the trough is full.
    b) CALCULATOR How many inches do you have to lower the water level in the trough to reduce the fluid force by 25%?
    c) Does it matter how long the trough is? Give reasons for your answer.

11. The viewing portion of the rectangular glass window in a typical fish tank at the New England Aquarium in Boston is 63 in. wide and runs from 0.5 in. below the water's surface to 33.5 in. below the surface. Find the fluid force against this portion of the window. The weight-density of seawater is 64 lb/ft^3. (In case you were wondering, the glass is 3/4 in. thick and the tank walls extend 4 in. above the water to keep the fish from jumping out.)

12. A horizontal rectangular freshwater fish tank with base 2×4 ft and height 2 ft (interior dimensions) is filled to within 2 in. of the top.
   a) Find the fluid force against each side and end of the tank.
   b) If the tank is sealed and stood on end (without spilling), so that one of the square ends is the base, what does that do to the fluid forces on the rectangular sides?

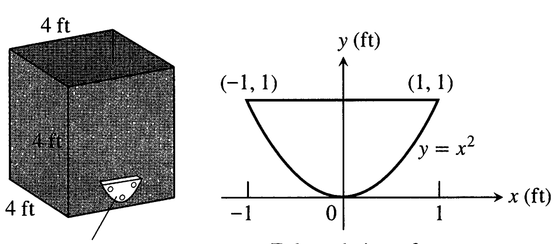
13. CALCULATOR A rectangular milk carton measures 3.75×3.75 in. at the base and is 7.75 in. tall. Find the force of the milk on one side when the carton is full.

14. CALCULATOR A standard olive oil can measures 5.75 by 3.5 in. at the base and is 10 in. tall. Find the fluid force against the base and each side when the can is full.

15. A semicircular plate 2 ft in diameter sticks straight down into fresh water with the diameter along the surface. Find the force exerted by the water on one side of the plate.

16. A tank truck hauls milk in a 6-ft-diameter horizontal right circular cylindrical tank. How much force does the milk exert on each end of the tank when the tank is half full?

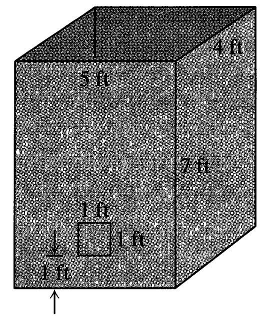
17. The cubical metal tank shown here has a parabolic gate, held in place by bolts and designed to withstand a fluid force of 160 lb without rupturing. The liquid you plan to store has a weight-density of 50 lb/ft^3.
   a) What is the fluid force on the gate when the liquid is 2 ft deep?
   b) What is the maximum height to which the container can be filled without exceeding its design limitation?



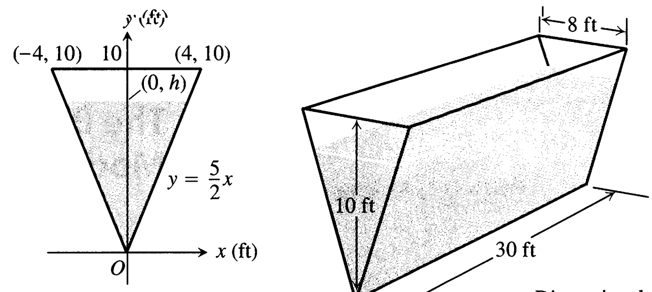
Parabolic gate

Enlarged view of parabolic gate

18. The rectangular tank shown here has a 1 ft × 1 ft square window 1 ft above the base. The window is designed to withstand a fluid force of 312 lb without cracking.
   a) What fluid force will the window have to withstand if the tank is filled with water to a depth of 3 ft?
   b) To what level can the tank be filled with water without exceeding the window's design limitation?



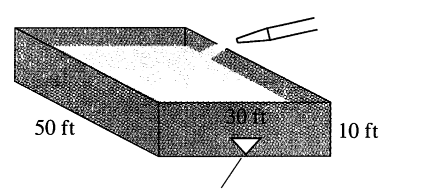
19. CALCULATOR The end plates of the trough shown here were designed to withstand a fluid force of 6667 lb. How many cubic feet of water can the tank hold without exceeding this limitation? Round down to the nearest cubic foot.



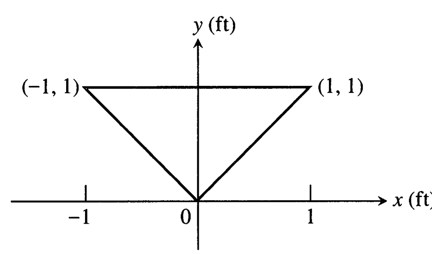
End view of trough

Dimensional view of trough

20. Water is running into the rectangular swimming pool shown here at the rate of 1000 ft^3/h.
   a) Find the fluid force against the triangular drain plate after 9 h of filling.
   b) The drain plate is designed to withstand a fluid force of 520 lb. How high can you fill the pool without exceeding this limitation?



Triangular drain plate

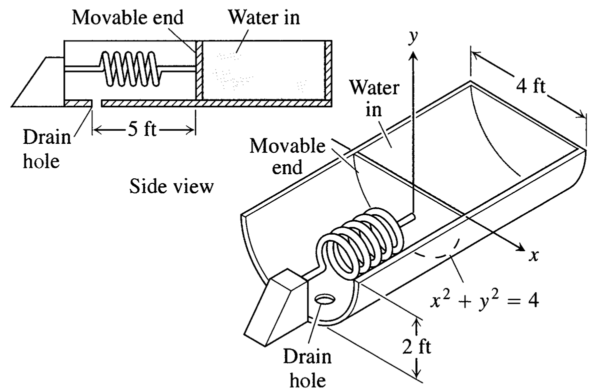


Enlarged view of drain plate

21. A vertical rectangular plate a units long by b units wide is submerged in a fluid of weight density w with its long edges parallel to the fluid's surface. Find the average value of the pressure along the vertical dimension of the plate. Explain your answer.

22. *(Continuation of Exercise 21.)* Show that the force exerted by the fluid on one side of the plate is the average value of the pressure (found in Exercise 21) times the area of the plate.

23. Water pours into the tank here at the rate of 4 ft^3/min. The tank's cross sections are 4-ft-diameter semicircles. One end of the tank is movable, but moving it to increase the volume compresses a spring. The spring constant is $k = 100$ lb/ft. If the end of the tank moves 5 ft against the spring, the water will drain out of a safety hole in the bottom at the rate of 5 ft^3/min. Will the movable end reach the hole before the tank overflows?



5.10 The Basic Pattern and Other Modeling Applications

There is a pattern to what we did in the preceding sections. In each section we wanted to measure something that was modeled or described by one or more continuous functions. In Section 5.1 it was the area between the graphs of two continuous functions. In Section 5.2 it was the volume of a solid. In Section 5.8 it was the work done by a force whose magnitude was a continuous function, and so on. In each case we responded by partitioning the interval on which the function or functions were defined and approximating what we wanted to measure with Riemann sums over the interval. We used the integral defined by the limit of the Riemann sums to define and calculate what we wanted to measure. Table 5.2 shows the pattern.

Literally thousands of things in biology, chemistry, economics, engineering, finance, geology, medicine, and other fields (the list would fill pages) are modeled and calculated by exactly this process.

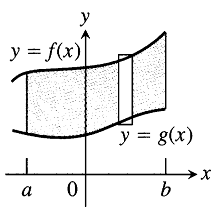
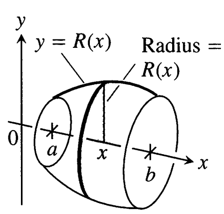
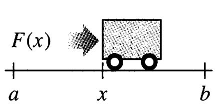
This section reviews the process and looks at a few more of the integrals it leads to.

Displacement vs. Distance Traveled

If a body with position function $s(t)$ moves along a coordinate line without changing direction, we can calculate the total distance it travels from $t = a$ to $t = b$ by integrating its velocity function $v(t)$ from $t = a$ to $t = b$, as we did in Chapter 4. If the body changes direction one or more times during the trip, we need to integrate the body's *speed* $|v(t)|$ to find the total distance traveled. Integrating the velocity will give only the body's **displacement,** $s(b) - s(a)$, the difference between its initial and final positions.

To see why, partition the time interval $a \leq t \leq b$ into subintervals in the usual way and let Δt_k denote the length of the kth interval. If Δt_k is small enough, the body's velocity $v(t)$ will not change much from t_{k-1} to t_k and the right-hand

Table 5.2 The phases of developing an integral to calculate something

Phase 1	**Phase 2**	**Phase 3**
We describe or model something we want to measure in terms of one or more continuous functions defined on a closed interval $[a, b]$.	We partition $[a, b]$ into subintervals of length Δx_k and choose a point c_k in each subinterval. We approximate what we want to measure with a finite sum. We identify the sum as a Riemann sum of a continuous function over $[a, b]$.	The approximations improve as the norm of the partition goes to zero. The Riemann sums approach a limiting integral. We use the integral to define and calculate what we originally wanted to measure.
The area between the curves $y = f(x)$, $y = g(x)$ on $[a, b]$ when $f(x) \geq g(x)$ 	$\sum [f(c_k) - g(c_k)]\,\Delta x_k$	$A = \lim_{\lVert P \rVert \to 0} \sum [f(c_k) - g(c_k)]\,\Delta x_k$ $= \int_a^b [f(x) - g(x)]\,dx$
The volume of the solid defined by revolving the curve $y = R(x)$, $a \leq x \leq b$, about the x-axis. 	$\sum \pi [R(c_k)]^2\,\Delta x_k$	$V = \lim_{\lVert P \rVert \to 0} \sum \pi [R(c_k)]^2\,\Delta x_k$ $= \int_a^b \pi [R(x)]^2\,dx$
The work done by a continuous variable force of magnitude $F(x)$ directed along the x-axis from a to b 	$\sum F(c_k)\,\Delta x_k$	$W = \lim_{\lVert P \rVert \to 0} \sum F(c_k)\,\Delta x_k$ $= \int_a^b F(x)\,dx$

endpoint value $v(t_k)$ will give a good approximation of the velocity throughout the interval. Accordingly, the change in the body's position coordinate during the kth time interval will be about

$$v(t_k)\Delta t_k.$$

The change will be positive if $v(t_k)$ is positive and negative if $v(t_k)$ is negative.

In either case, the distance traveled during the kth interval will be about

$$|v(t_k)|\Delta t_k.$$

The total trip distance will be approximately

$$\sum_{k=1}^{n} |v(t_k)|\Delta t_k. \tag{1}$$

The sum in Eq. (1) is a Riemann sum for the speed $|v(t)|$ on the interval $[a, b]$. We expect the approximations to improve as the norm of the partition of $[a, b]$ goes to zero. It therefore looks as if we should be able to calculate the total distance traveled by the body by integrating the body's speed from a to b. In practice, this turns out to be the right thing to do. The mathematical model predicts the distance correctly every time.

$$\boxed{\text{Distance traveled} = \int_a^b |v(t)|\,dt}$$

If we wish to predict how far up or down the line from its initial position a body will end up when a trip is over, we integrate v instead of its absolute value.

To see why, let $s(t)$ be the body's position at time t and let F be an antiderivative of v. Then

$$s(t) = F(t) + C$$

for some constant C. The displacement caused by the trip from $t = a$ to $t = b$ is

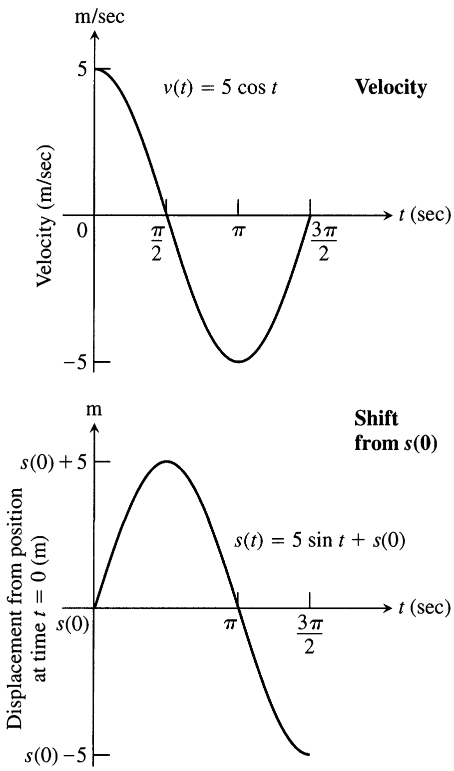
$$\begin{aligned} s(b) - s(a) &= (F(b) + C) - (F(a) + C) \\ &= F(b) - F(a) = \int_a^b v(t)\,dt. \end{aligned}$$

$$\boxed{\text{Displacement} = \int_a^b v(t)\,dt}$$

EXAMPLE 1 The velocity of a body moving along a line from $t = 0$ to $t = 3\pi/2$ sec was

$$v(t) = 5\cos t \text{ m/sec.}$$

Find the total distance traveled and the body's displacement.



5.74 The velocity and displacement of the body in Example 1.

Solution

$$\begin{aligned}
\text{Distance traveled} &= \int_0^{3\pi/2} |5\cos t|\, dt \qquad \text{Distance is the integral of speed.}\\
&= \int_0^{\pi/2} 5\cos t\, dt + \int_{\pi/2}^{3\pi/2} (-5\cos t)\, dt\\
&= 5\sin t\Big]_0^{\pi/2} - 5\sin t\Big]_{\pi/2}^{3\pi/2}\\
&= 5(1-0) - 5(-1-1) = 5 + 10 = 15\text{ m}\\
\text{Displacement} &= \int_0^{3\pi/2} 5\cos t\, dt \qquad \text{Displacement is the integral of velocity.}\\
&= 5\sin t\Big]_0^{3\pi/2} = 5(-1) - 5(0) = -5\text{ m}
\end{aligned}$$

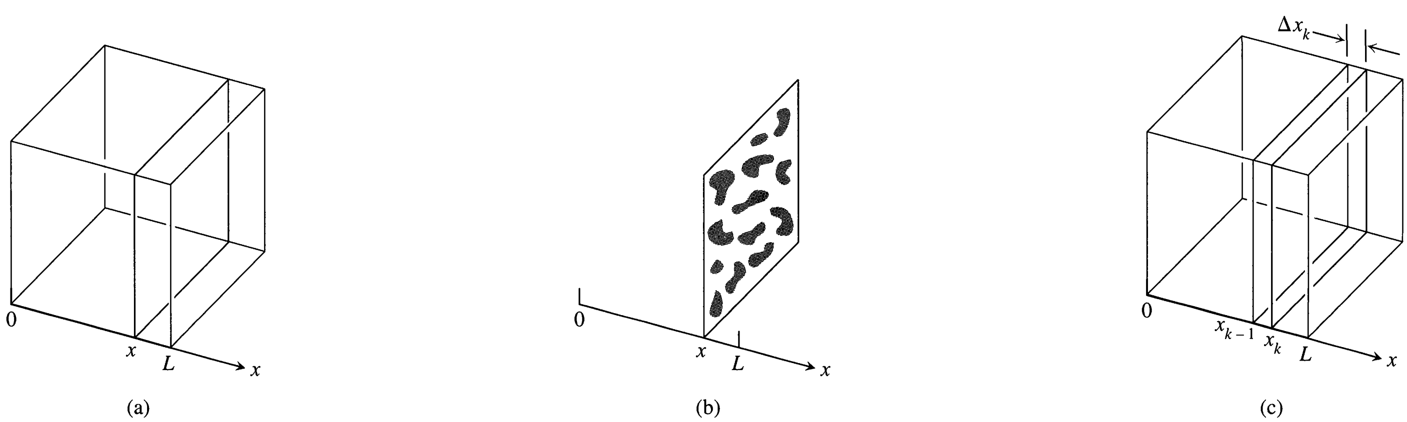
During the trip, the body traveled 5 m forward and 10 m backward for a total distance of 15 m. This displaced the body 5 m to the left (Fig. 5.74). ❑

Delesse's Rule

As you may know, the sugar in an apple starts turning into starch as soon as the apple is picked, and the longer the apple sits around, the starchier it becomes. You can tell fresh apples from stale by both flavor and consistency.

To find out how much starch is in a given apple, we can look at a thin slice under a microscope. The cross sections of the starch granules will show up clearly, and it is easy to estimate the proportion of the viewing area they occupy. This two-dimensional proportion will be the same as the three-dimensional proportion of uncut starch granules in the apple itself. The apparently magical equality of these proportions was first discovered by a French geologist, Achille Ernest Delesse, in the 1840s. Its explanation lies in the notion of average value.

Suppose we want to find the proportion of some granular material in a solid and that the sample we have chosen to analyze is a cube whose edges have length L. We picture the cube with an x-axis along one edge and imagine slicing the cube with planes perpendicular to points of the interval $[0, L]$ (Fig. 5.75). Call the proportion of the area of the slice at x occupied by the granular material of interest (starch, in our apple example) $r(x)$ and assume r is a continuous function of x.



5.75 The steps leading to Delesse's rule: (a) a slice through a sample cube; (b) the granular material in the slice; (c) the slab between consecutive slices determined by a partition of $[0, L]$.

Now partition the interval $[0, L]$ into subintervals in the usual way. Imagine the cube sliced into thin slices by planes at the subdivision points. The length Δx_k of the kth subinterval is the distance between the planes at x_{k-1} and x_k. If the planes are close enough together, the sections cut from the grains by the planes will resemble cylinders with bases in the plane at x_k. The proportion of granular material between the planes will be about the same as the proportion of cylinder base area in the plane at x_k, which in turn will be about $r(x_k)$. Thus the amount of granular material in the slab between the two planes will be about

$$\text{(Proportion)} \times \text{(slab volume)} = r(x_k)L^2\,\Delta x_k.$$

The amount of granular material in the entire sample cube will be about

$$\sum_{k=1}^{n} r(x_k)L^2\;\Delta x_k.$$

This sum is a Riemann sum for the function $r(x)L^2$ over the interval $[0, L]$. We expect the approximations by sums like these to improve as the norm of the subdivision of $[0, L]$ goes to zero and therefore expect the integral

$$\int_0^L r(x)L^2\,dx$$

to give the amount of granular material in the sample cube.

We can obtain the proportion of granular material in the sample by dividing this amount by the cube's volume, L^3. If we have chosen our sample well, this will also be the proportion of granular material in the solid from which the sample was taken. Putting it all together, we get

$$\begin{aligned}
\begin{matrix}\text{Proportion of granular}\\ \text{material in solid}\end{matrix} &= \begin{matrix}\text{Proportion of granular}\\ \text{material in the sample cube}\end{matrix}\\
&= \frac{\int_0^L r(x)L^2\,dx}{L^3} = \frac{L^2\int_0^L r(x)\,dx}{L^3} = \frac{1}{L}\int_0^L r(x)\,dx\\
&= \text{average value of } r(x) \text{ over } [0, L]\\
&= \text{proportion of area occupied by granular material in a typical cross section.}
\end{aligned}$$

This is Delesse's rule. Once we have found $\bar{r}$, the average of $r(x)$ over $[0, L]$, we have found the proportions of granular material in the solid.

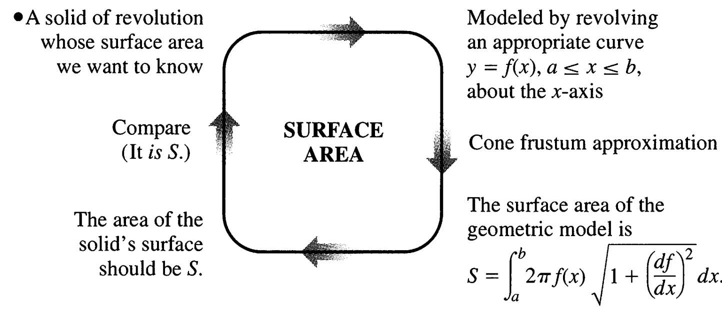
In practice, $\bar{r}$ is found by averaging over a number of cross sections. There are several things to watch out for in the process. In addition to the possibility that the granules cluster in ways that make representative samples difficult to find, there is the possibility that we might not recognize a granule's trace for what it is. Some cross sections of normal red blood cells look like disks and ovals, while others look surprisingly like dumbbells. We do not want to dismiss the dumbbells as experimental error the way one research group did a few years ago.

Delesse's rule

Achille Ernest Delesse was a mid-nineteenth-century mining engineer interested in determining the composition of rocks. To find out how much of a particular mineral a rock contained, he cut it through, polished an exposed face, and covered the face with transparent waxed paper, trimmed to size. He then traced on the paper the exposed portions of the mineral that interested him. After weighing the paper, he cut out the mineral traces and weighed them. The ratio of the weights gave not only the proportion of the surface occupied by the mineral but, more important, the proportion of the entire rock occupied by the mineral. This rule is still used by petroleum geologists today. A two-dimensional analogue of it is used to determine the porosities of the ceramic filters that extract organic molecules in chemistry laboratories and screen out microbes in water purifiers.

Useless Integrals—Bad Models

Some of the integrals we get from forming Riemann sums do what we want, but others do not. It all depends on how we choose to model the problems we want to solve. Some choices are good; others are not. Here is an example.



5.76 The modeling cycle for surface area.

We use the surface area formula

$$S = \int_a^b 2\pi f(x) \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx \tag{2}$$

because it has predictive value and always gives results consistent with information from other sources. In other words, the model we used to derive the formula (Fig. 5.76) was a good one.

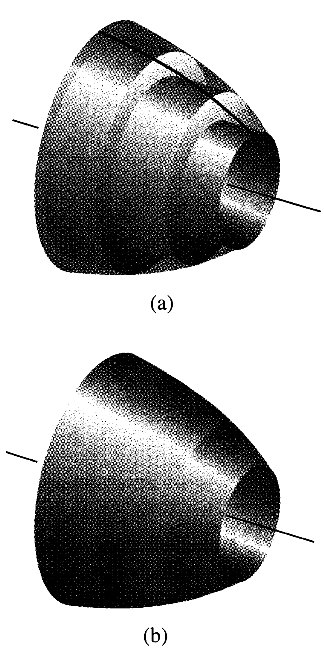
Why not find the surface area by approximating with cylindrical bands instead of conical bands, as suggested in Fig. 5.77? The Riemann sums we get this way converge just as nicely as the ones based on conical bands, and the resulting integral is simpler. Instead of Eq. (2), we get

$$S = \int_a^b 2\pi f(x)\, dx. \tag{3}$$

After all, we might argue, we used cylinders to derive good volume formulas, so why not use them again to derive surface area formulas?

The answer is that the formula in Eq. (3) has no predictive value and almost never gives results consistent with other calculations. The comparison step in the modeling process fails for this formula.

There is a moral here: Just because we end up with a nice-looking integral does not mean it will do what we want. Constructing an integral is not enough—we have to test it too (Exercises 15 and 16).



5.77 Why not use (a) cylindrical bands instead of (b) conical bands to approximate surface area?

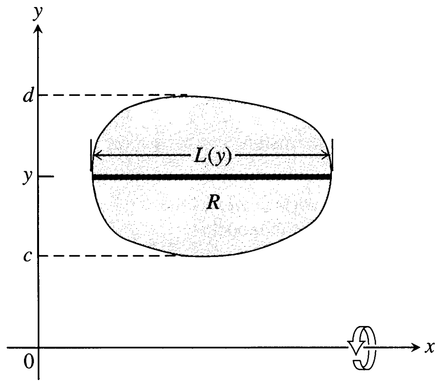
The Theorems of Pappus

In the third century, an Alexandrian Greek named Pappus discovered two formulas that relate centroids to surfaces and solids of revolution. The formulas provide shortcuts to a number of otherwise lengthy calculations.

Theorem 1
Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$V = 2\pi \rho A. \tag{4}$$



5.78 The region R is to be revolved (once) about the x-axis to generate a solid. A 1700-year-old theorem says that the solid's volume can be calculated by multiplying the region's area by the distance traveled by its centroid during the revolution.

Proof We draw the axis of revolution as the x-axis with the region R in the first quadrant (Fig. 5.78). We let $L(y)$ denote the length of the cross section of R perpendicular to the y-axis at y. We assume $L(y)$ to be continuous.

By the method of cylindrical shells, the volume of the solid generated by revolving the region about the x-axis is

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\,dy = 2\pi \int_c^d y\,L(y)\,dy. \tag{5}$$

The y-coordinate of R's centroid is

$$\bar{y} = \frac{\int_c^d \tilde{y}\,dA}{A} = \frac{\int_c^d y\,L(y)\,dy}{A},$$

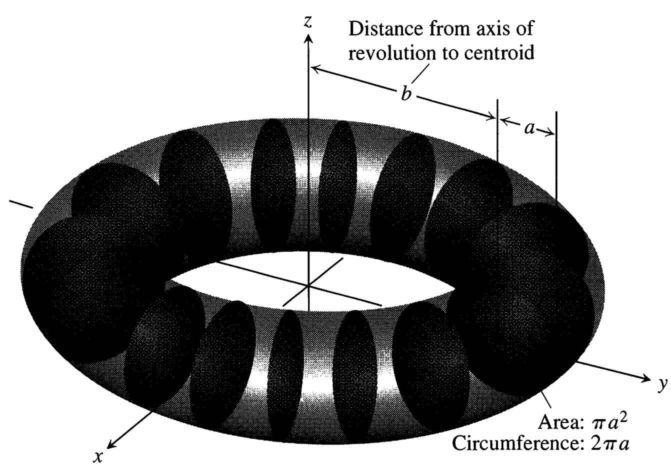
so that

$$\int_c^d y\,L(y)\,dy = A\bar{y}.$$

Substituting $A\bar{y}$ for the last integral in Eq. (5) gives $V = 2\pi\bar{y}A$. With ρ equal to $\bar{y}$, we have $V = 2\pi\rho A$. ❑

EXAMPLE 2 The volume of the torus (doughnut) generated by revolving a circular disk of radius a about an axis in its plane at a distance $b \geq a$ from its center (Fig. 5.79) is

$$V = 2\pi(b)(\pi a^2) = 2\pi^2 ba^2.$$ ❑

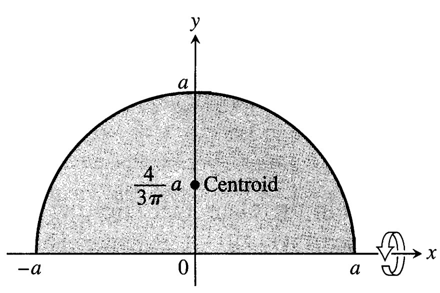


5.79 With Pappus's first theorem, we can find the volume of a torus without having to integrate (Example 2).

EXAMPLE 3 Locate the centroid of a semicircular region.

Solution We model the region as the region between the semicircle $y = \sqrt{a^2 - x^2}$ (Fig. 5.80) and the x-axis and imagine revolving the region about the x-axis to generate a solid sphere. By symmetry, the x-coordinate of the centroid is $\bar{x} = 0$. With $\bar{y} = \rho$ in Eq. (4), we have

$$\bar{y} = \frac{V}{2\pi A} = \frac{(4/3)\pi a^3}{2\pi(1/2)\pi a^2} = \frac{4}{3\pi}a.$$ ❑



5.80 With Pappus's first theorem, we can locate the centroid of a semicircular region without having to integrate (Example 3).

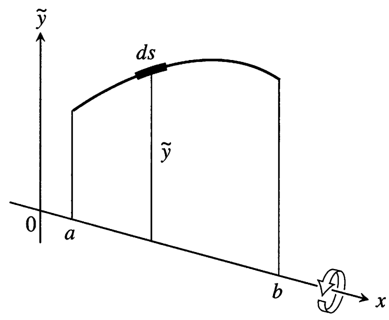
Theorem 2

Pappus's Theorem for Surface Areas

If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length of the arc times the distance traveled by the arc's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$S = 2\pi\rho L. \tag{6}$$

The proof we give assumes that we can model the axis of revolution as the x-axis and the arc as the graph of a smooth function of x.



5.81 Figure for Pappus's area theorem.

Proof We draw the axis of revolution as the x-axis with the arc extending from $x = a$ to $x = b$ in the first quadrant (Fig. 5.81). The area of the surface generated by the arc is

$$S = \int_{x=a}^{x=b} 2\pi y\, ds = 2\pi \int_{x=a}^{x=b} y\, ds. \tag{7}$$

The y-coordinate of the arc's centroid is

$$\bar{y} = \frac{\displaystyle\int_{x=a}^{x=b} \tilde{y}\, ds}{\displaystyle\int_{x=a}^{x=b} ds} = \frac{\displaystyle\int_{x=a}^{x=b} y\, ds}{L}.$$

$L = \int ds$ is the arc's length and $\tilde{y} = y$.

Hence

$$\int_{x=a}^{x=b} y\, ds = \bar{y}L.$$

Substituting $\bar{y}L$ for the last integral in Eq. (7) gives $S = 2\pi\bar{y}L$. With ρ equal to $\bar{y}$, we have $S = 2\pi\rho L$.

EXAMPLE 4 The surface area of the torus in Example 2 is

$$S = 2\pi(b)(2\pi a) = 4\pi^2 ba.$$

❑

Exercises 5.10

Distance and Displacement

In Exercises 1–8, the function $v(t)$ is the velocity in meters per second of a body moving along a coordinate line. (a) Graph v to see where it is positive and negative. Then find (b) the total distance traveled by the body during the given time interval and (c) the body's displacement.

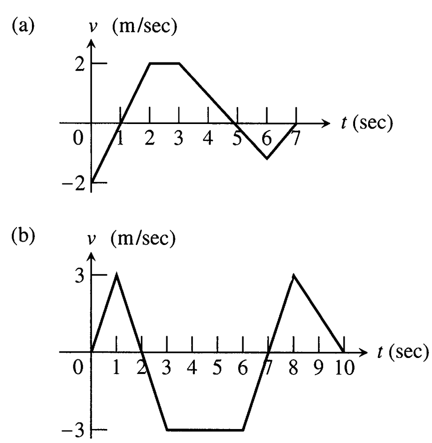
1. $v(t) = 5\cos t, \quad 0 \le t \le 2\pi$

2. $v(t) = \sin \pi t, \quad 0 \le t \le 2$

3. $v(t) = 6\sin 3t, \quad 0 \le t \le \pi/2$

4. $v(t) = 4\cos 2t, \quad 0 \le t \le \pi$

5. $v(t) = 49 - 9.8t, \quad 0 \le t \le 10$
6. $v(t) = 8 - 1.6t, \quad 0 \le t \le 10$
7. $v(t) = 6t^2 - 18t + 12 = 6(t-1)(t-2), \quad 0 \le t \le 2$
8. $v(t) = 6t^2 - 18t + 12 = 6(t-1)(t-2), \quad 0 \le t \le 3$
9. The function $s = (1/3)t^3 - 3t^2 + 8t$ gives the position of a body moving on the horizontal s-axis at time $t \ge 0$ (s in meters, t in seconds).
   a) Show that the body is moving to the right at time $t = 0$.
   b) When does the body move to the left?
   c) What is the body's position at time $t = 3$?
   d) When $t = 3$, what is the total distance the body has traveled?
   e) GRAPHER Graph s as a function of t and comment on the relationship of the graph to the body's motion.
10. The function $s = -t^3 + 6t^2 - 9t$ gives the position of a body moving on the horizontal s-axis at time $t \ge 0$ (s in meters, t in seconds).
   a) Show that the body is moving to the left at $t = 0$.
   b) When does the body move to the right?
   c) Does the body ever move to the right of the origin? Give reasons for your answer.
   d) What is the body's position at time $t = 3$?
   e) What is the total distance the particle has traveled by the time $t = 3$?
   f) GRAPHER Graph s as a function of t and comment on the relationship of the graph to the body's motion.
11. Here are the velocity graphs of two bodies moving on a coordinate line. Find the total distance traveled and the body's displacement for the given time interval.

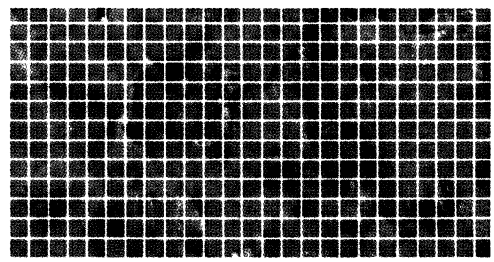


12. CALCULATOR The table at the top of the next column shows the velocity of a model train engine moving back and forth on a track for 10 sec. Use Simpson's rule to find the resulting displacement and total distance traveled.

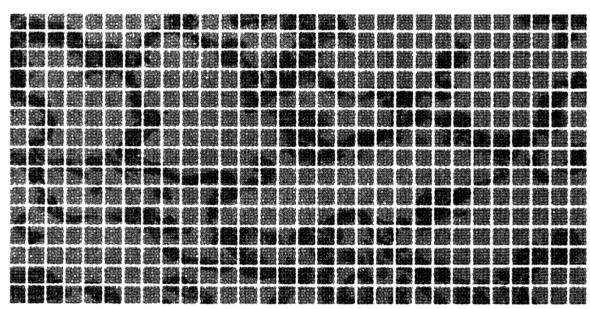
Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	−11
1	12	7	− 6
2	22	8	2
3	10	9	6
4	− 5	10	0
5	−13		

Delesse's Rule

13. The photograph here shows a grid superimposed on the polished face of a piece of granite. Use the grid and Delesse's rule to estimate the proportion of shrimp-colored granular material in the rock.

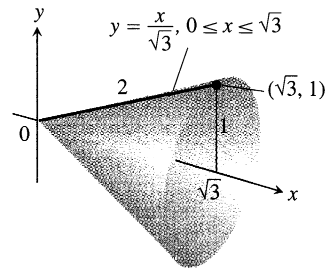


14. The photograph here shows a grid superimposed on a microscopic view of a stained section of human lung tissue. The clear spaces between the cells are cross sections of the lung's air sacks (called *alveoli,* accent on the second syllable). Use the grid and Delesse's rule to estimate the proportion of air space in the lung.



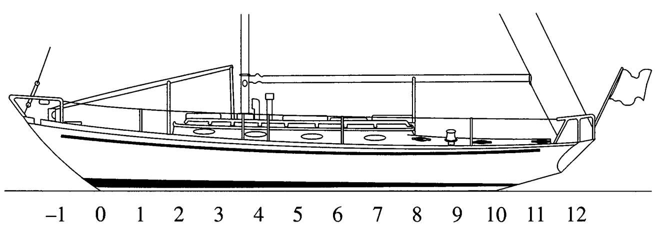
Modeling Surface Area

15. *Modeling surface area.* The lateral surface area of the cone swept out by revolving the line segment $y = x/\sqrt{3}, 0 \le x \le \sqrt{3}$, about the x-axis should be (1/2)(base circumference)(slant height) $= (1/2)(2\pi)(2) = 2\pi$. What do you get if you use Eq. (3) with $f(x) = x/\sqrt{3}$?



16. *Modeling surface area.* The only surface for which Eq. (3) gives the area we want is a cylinder. Show that Eq. (3) gives $S = 2\pi rh$ for the cylinder swept out by revolving the line segment $y = r, 0 \le x \le h$, about the x-axis.

17. *A sailboat's displacement.* To find the volume of water displaced by a sailboat, the common practice is to partition the waterline into 10 subintervals of equal length, measure the cross section area $A(x)$ of the submerged portion of the hull at each partition point, and then use Simpson's rule to estimate the integral of $A(x)$ from one end of the waterline to the other. The table here lists the area measurements at "Stations" 0 through 10, as the partition points are called, for the cruising sloop *Pipedream,* shown here. The common subinterval length (distance between consecutive stations) is $h = 2.54$ ft (about 2′ 6 1/2″, chosen for the convenience of the builder).



a) Estimate *Pipedream*'s displacement volume to the nearest cubic foot.

Station	Submerged area (ft^2)
0	0
1	1.07
2	3.84
3	7.82
4	12.20
5	15.18
6	16.14
7	14.00
8	9.21
9	3.24
10	0

b) The figures in the table are for seawater, which weighs 64 lb/ft^3. How many pounds of water does *Pipedream* displace? (Displacement is given in pounds for small craft, and long tons [1 long ton = 2240 lb] for larger vessels.)

(Data from *Skene's Elements of Yacht Design,* Francis S. Kinney, Dodd, Mead & Company, Inc., 1962)

18. *Prismatic coefficients (Continuation of Exercise 17).* A boat's prismatic coefficient is the ratio of the displacement volume to the volume of a prism whose height equals the boat's waterline length and whose base equals the area of the boat's largest submerged cross section. The best sailboats have prismatic coefficients between 0.51 and 0.54. Find *Pipedream*'s prismatic coefficient, given a waterline length of 25.4 ft and a largest submerged cross section area of 16.14 ft^2 (at Station 6).

The Theorems of Pappus

19. The square region with vertices (0, 2), (2, 0), (4, 2), and (2, 4) is revolved about the x-axis to generate a solid. Find the volume and surface area of the solid.

20. Use a theorem of Pappus to find the volume generated by revolving about the line $x = 5$ the triangular region bounded by the coordinate axes and the line $2x + y = 6$. (As you saw in Exercise 31 of Section 5.7, the centroid of a triangle lies at the intersection of the medians, one-third of the way from the midpoint of each side toward the opposite vertex.)

21. Find the volume of the torus generated by revolving the circle $(x - 2)^2 + y^2 = 1$ about the y-axis.

22. Use the theorems of Pappus to find the lateral surface area and the volume of a right circular cone.

23. Use the second theorem of Pappus and the fact that the surface area of a sphere of radius a is $4\pi a^2$ to find the centroid of the semicircle $y = \sqrt{a^2 - x^2}$.

24. As found in Exercise 23, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface swept out by revolving the semicircle about the line $y = a$.

25. The area of the region R enclosed by the semiellipse $y = (b/a)\sqrt{a^2 - x^2}$ and the x-axis is $(1/2)\pi ab$ and the volume of the ellipsoid generated by revolving R about the x-axis is $(4/3)\pi ab^2$. Find the centroid of R. Notice the remarkable fact that the location is independent of a.

26. As found in Example 3, the centroid of the region enclosed by the x-axis and the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 4a/3\pi)$. Find the volume of the solid generated by revolving this region about the line $y = -a$.

27. The region of Exercise 26 is revolved about the line $y = x - a$ to generate a solid. Find the volume of the solid.

28. As found in Exercise 23, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface generated by revolving the semicircle about the line $y = x - a$.

29. Find the moment about the x-axis of the semicircular region in Example 3. If you use results already known, you will not need to integrate.

CHAPTER 5 QUESTIONS TO GUIDE YOUR REVIEW

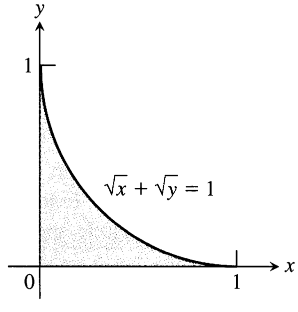
1. How do you define and calculate the area of the region between the graphs of two continuous functions? Give an example.
2. How do you define and calculate the volumes of solids by the method of slicing? Give an example.
3. How are the disk and washer methods for calculating volumes derived from the method of slicing? Give examples of volume calculations by these methods.
4. Describe the method of cylindrical shells. Give an example.
5. How do you define and calculate the length of the graph of a smooth function over a closed interval? Give an example. What about functions that do not have continuous first derivatives?
6. How do you define and calculate the area of the surface swept out by revolving the graph of a smooth function $y = f(x)$, $a \le x \le b$, about the x-axis? Give an example.
7. What is a center of mass?
8. How do you locate the center of mass of a straight, narrow rod or strip of material? Give an example. If the density of the material is constant, you can tell right away where the center of mass is. Where is it?
9. How do you locate the center of mass of a thin flat plate of material? Give an example.
10. How do you define and calculate the work done by a variable force directed along a portion of the x-axis? How do you calculate the work it takes to pump a liquid from a tank? Give examples.
11. How do you calculate the force exerted by a liquid against a portion of a vertical wall? Give an example.
12. Suppose you know the velocity function $v(t)$ of a body that will be moving back and forth along a coordinate line from time $t = a$ to time $t = b$. How can you predict how much the motion will shift the body's position? How can you predict the total distance the body will travel?
13. What does Delesse's rule say? Give an example.
14. What do Pappus's two theorems say? Give examples of how they are used to calculate surface areas and volumes and to locate centroids.
15. There is a basic pattern to the way we constructed integrals in this chapter. What is it? Give examples.

CHAPTER 5 PRACTICE EXERCISES

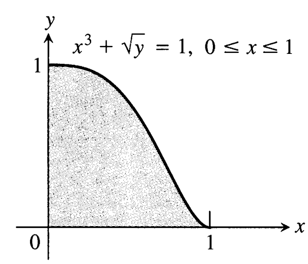
Areas

Find the areas of the regions enclosed by the curves and lines in Exercises 1–12.

1. $y = x, \quad y = 1/x^2, \quad x = 2$
2. $y = x, \quad y = 1/\sqrt{x}, \quad x = 2$
3. $\sqrt{x} + \sqrt{y} = 1, \quad x = 0, \quad y = 0$



4. $x^3 + \sqrt{y} = 1, \quad x = 0, \quad y = 0, \quad \text{for} \quad 0 \le x \le 1$



5. $x = 2y^2, \quad x = 0, \quad y = 3$
6. $x = 4 - y^2, \quad x = 0$
7. $y^2 = 4x, \quad y = 4x - 2$
8. $y^2 = 4x + 4, \quad y = 4x - 16$
9. $y = \sin x, \quad y = x, \quad 0 \le x \le \pi/4$

10. $y = |\sin x|, \quad y = 1, \quad -\pi/2 \le x \le \pi/2$

11. $y = 2\sin x, \quad y = \sin 2x, \quad 0 \le x \le \pi$

12. $y = 8\cos x, \quad y = \sec^2 x, \quad -\pi/3 \le x \le \pi/3$

13. Find the area of the "triangular" region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$.

14. Find the area of the "triangular" region bounded on the left by $y = \sqrt{x}$, on the right by $y = 6 - x$, and below by $y = 1$.

15. Find the extreme values of $f(x) = x^3 - 3x^2$ and find the area of the region enclosed by the graph of f and the x-axis.

16. Find the area of the region cut from the first quadrant by the curve $x^{1/2} + y^{1/2} = a^{1/2}$.

17. Find the total area of the region enclosed by the curve $x = y^{2/3}$ and the lines $x = y$ and $y = -1$.

18. Find the total area of the region between the curves $y = \sin x$ and $y = \cos x$ for $0 \le x \le 3\pi/2$.

Volumes

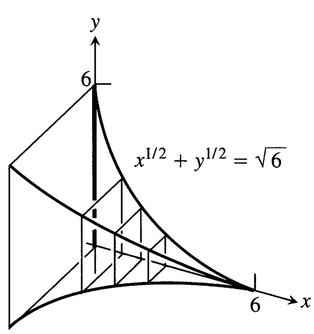
Find the volumes of the solids in Exercises 19–24.

19. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 1$. The cross sections perpendicular to the x-axis between these planes are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = \sqrt{x}$.

20. The base of the solid is the region in the first quadrant between the line $y = x$ and the parabola $y = 2\sqrt{x}$. The cross sections of the solid perpendicular to the x-axis are equilateral triangles whose bases stretch from the line to the curve.

21. The solid lies between planes perpendicular to the x-axis at $x = \pi/4$ and $x = 5\pi/4$. The cross sections between these planes are circular disks whose diameters run from the curve $y = 2\cos x$ to the curve $y = 2\sin x$.

22. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 6$. The cross sections between these planes are squares whose bases run from the x-axis up to the curve $x^{1/2} + y^{1/2} = \sqrt{6}$.



23. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross sections of the solid perpendicular to the x-axis between these planes are circular disks whose diameters run from the curve $x^2 = 4y$ to the curve $y^2 = 4x$.

24. The base of the solid is the region bounded by the parabola $y^2 = 4x$ and the line $x = 1$ in the xy-plane. Each cross section perpendicular to the x-axis is an equilateral triangle with one edge in the plane. (The triangles all lie on the same side of the plane.)

25. Find the volume of the solid generated by revolving the region bounded by the x-axis, the curve $y = 3x^4$, and the lines $x = 1$ and $x = -1$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 1$; (d) the line $y = 3$.

26. Find the volume of the solid generated by revolving the "triangular" region bounded by the curve $y = 4/x^3$ and the lines $x = 1$ and $y = 1/2$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 2$; (d) the line $y = 4$.

27. Find the volume of the solid generated by revolving the region bounded on the left by the parabola $x = y^2 + 1$ and on the right by the line $x = 5$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 5$.

28. Find the volume of the solid generated by revolving the region bounded by the parabola $y^2 = 4x$ and the line $y = x$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 4$; (d) the line $y = 4$.

29. Find the volume of the solid generated by revolving the "triangular" region bounded by the x-axis, the line $x = \pi/3$, and the curve $y = \tan x$ in the first quadrant about the x-axis.

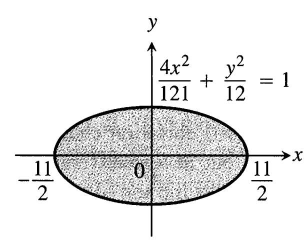
30. Find the volume of the solid generated by revolving the region bounded by the curve $y = \sin x$ and the lines $x = 0$, $x = \pi$, and $y = 2$ about the line $y = 2$.

31. Find the volume of the solid generated by revolving the region between the x-axis and the curve $y = x^2 - 2x$ about (a) the x-axis; (b) the line $y = -1$; (c) the line $x = 2$; (d) the line $y = 2$.

32. Find the volume of the solid generated by revolving about the x-axis the region bounded by $y = 2\tan x$, $y = 0$, $x = -\pi/4$, and $x = \pi/4$. (The region lies in the first and third quadrants and resembles a skewed bow tie.)

33. A round hole of radius $\sqrt{3}$ ft is bored through the center of a solid sphere of radius 2 ft. Find the volume of material removed from the sphere.

34. CALCULATOR The profile of a football resembles the ellipse shown here. Find the football's volume to the nearest cubic inch.



Lengths of Curves

Find the lengths of the curves in Exercises 35–38.

35. $y = x^{1/2} - (1/3)x^{3/2}, \quad 1 \le x \le 4$

36. $x = y^{2/3}, \quad 1 \le y \le 8$

37. $y = (5/12)x^{6/5} - (5/8)x^{4/5}, \quad 1 \le x \le 32$

38. $x = (y^3/12) + (1/y), \quad 1 \le y \le 2$

Areas of Surfaces of Revolution

In Exercises 39–42, find the areas of the surfaces generated by revolving the curves about the given axes.

39. $y = \sqrt{2x+1}, \quad 0 \le x \le 3, \quad x\text{-axis}$

40. $y = x^3/3, \quad 0 \le x \le 1, \quad x\text{-axis}$

41. $x = \sqrt{4y - y^2}, \quad 1 \le y \le 2, \quad y\text{-axis}$

42. $x = \sqrt{y}, \quad 2 \le y \le 6, \quad y\text{-axis}$

Centroids and Centers of Mass

43. Find the centroid of a thin, flat plate covering the region enclosed by the parabolas $y = 2x^2$ and $y = 3 - x^2$.

44. Find the centroid of a thin, flat plate covering the region enclosed by the x-axis, the lines $x = 2$ and $x = -2$, and the parabola $y = x^2$.

45. Find the centroid of a thin, flat plate covering the "triangular" region in the first quadrant bounded by the y-axis, the parabola $y = x^2/4$, and the line $y = 4$.

46. Find the centroid of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$.

47. Find the center of mass of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$ if the density function is $\delta(y) = 1 + y$. (Use horizontal strips.)

48. **a)** Find the center of mass of a thin plate of constant density covering the region between the curve $y = 3/x^{3/2}$ and the x-axis from $x = 1$ to $x = 9$.

b) Find the plate's center of mass if, instead of being constant, the density is $\delta(x) = x$. (Use vertical strips.)

Work

49. A rock climber is about to haul up 100 N (about 22.5 lb) of equipment that has been hanging beneath her on 40 m of rope that weighs 0.8 newton per meter. How much work will it take? (*Hint:* Solve for the rope and equipment separately; then add.)

50. You drove an 800-gal tank truck from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-ft elevation change in 50 min. Assuming that the water leaked out at a steady rate, how much work was spent in carrying water to the top? Do not count the work done in getting yourself and the truck there. Water weighs 8 lb/U.S. gal.

51. If a force of 20 lb is required to hold a spring 1 ft beyond its unstressed length, how much work does it take to stretch the spring this far? an additional foot?

52. A force of 200 N will stretch a garage door spring 0.8 m beyond its unstressed length. How far will a 300-N force stretch the spring? How much work does it take to stretch the spring this far?

53. A reservoir shaped like a right circular cone, point down, 20 ft across the top and 8 ft deep, is full of water. How much work does it take to pump the water to a level 6 ft above the top?

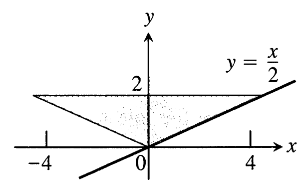
54. *(Continuation of Exercise 53.)* The reservoir is filled to a depth of 5 ft, and the water is to be pumped to the same level as the top. How much work does it take?

55. A right circular conical tank, point down, with top radius 5 ft and height 10 ft is filled with a liquid whose weight-density is 60 lb/ft^3. How much work does it take to pump the liquid to a point 2 ft above the tank? If the pump is driven by a motor rated at 275 ft • lb/sec (1/2-hp), how long will it take to empty the tank?

56. A storage tank is a right circular cylinder 20 ft long and 8 ft in diameter with its axis horizontal. If the tank is half full of olive oil weighing 57 lb/ft^3, find the work done in emptying it through a pipe that runs from the bottom of the tank to an outlet that is 6 ft above the top of the tank.

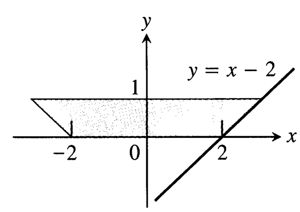
Fluid Force

57. The vertical triangular plate shown here is the end plate of a trough full of water ($w = 62.4$). What is the fluid force against the plate?



UNITS IN FEET

58. The vertical trapezoidal plate shown here is the end plate of a trough full of maple syrup weighing 75 lb/ft^3. What is the force exerted by the syrup against the end plate of the trough when the syrup is 10 in. deep?

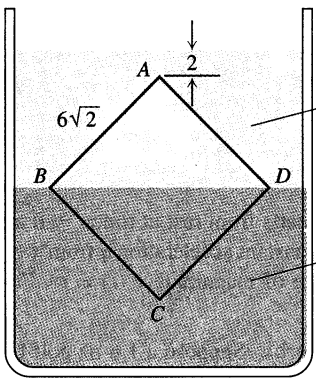


UNITS IN FEET

59. A flat vertical gate in the face of a dam is shaped like the parabolic region between the curve $y = 4x^2$ and the line $y = 4$, with measurements in feet. The top of the gate lies 5 ft below the surface of the water. Find the force exerted by the water against the gate ($w = 62.4$).

60. CALCULATOR You plan to store mercury ($w = 849$ lb/ft^3) in a vertical right circular cylindrical tank of radius 1 ft whose interior side wall can withstand a total fluid force of 40,000 lb. About how many cubic feet of mercury can you store in the tank at any one time?

61. The container profiled in Fig. 5.82 is filled with two nonmixing liquids of weight density w_1 and w_2. Find the fluid force on one side of the vertical square plate $ABCD$. The points B and D lie in the boundary layer and the square is $6\sqrt{2}$ ft on a side.

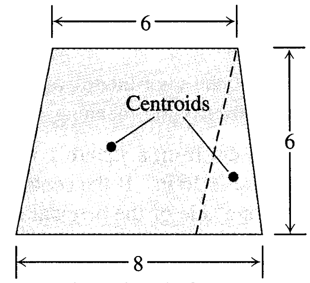


5.82 Profile of the container in Exercise 61.

62. The isosceles trapezoidal plate shown here is submerged vertically in water ($w = 62.4$) with its upper edge 4 ft below the surface. Find the fluid force on one side of the plate in two different ways:

a) By evaluating an integral.

b) By dividing the plate into a parallelogram and an isosceles triangle, locating their centroids, and using the equation $F = w\bar{h}A$ from Section 5.9.



Distance and Displacement

In Exercises 63–66, the function $v = f(t)$ is the velocity (m/sec) of a body moving along a coordinate line. Find (a) the total distance the body travels during the given time interval and (b) the body's displacement.

63. $v = t^2 - 8t + 12, \quad 0 \le t \le 6$

64. $v = t^3 - 3t^2 + 2t, \quad 0 \le t \le 2$

65. $v = 5\cos t, \quad 0 \le t \le 3\pi/2$

66. $v = -\pi \sin \pi t, \quad 0 \le t \le 3/2$

CHAPTER 5 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Volume and Length

1. A solid is generated by revolving about the x-axis the region bounded by the graph of the continuous function $y = f(x)$, the x-axis, and the fixed line $x = a$ and the variable line $x = b$, $b > a$. Its volume, for all b, is $b^2 - ab$. Find $f(x)$.

2. A solid is generated by revolving about the x-axis the region bounded by the graph of the continuous function $y = f(x)$, the x-axis, and the lines $x = 0$ and $x = a$. Its volume, for all $a > 0$, is $a^2 + a$. Find $f(x)$.

3. Suppose that the increasing function $f(x)$ is smooth for $x \ge 0$ and that $f(0) = a$. Let $s(x)$ denote the length of the graph of f from $(0, a)$ to $(x, f(x))$, $x > 0$. Find $f(x)$ if $s(x) = Cx$ for some constant C. What are the allowable values for C?

4. a) Show that for $0 < \alpha \le \pi/2$,

$$\int_0^\alpha \sqrt{1 + \cos^2 \theta}\, d\theta > \sqrt{\alpha^2 + \sin^2 \alpha}.$$

b) Generalize the result in (a).

Moments and Centers of Mass

5. Find the centroid of the region bounded below by the x-axis and above by the curve $y = 1 - x^n$, n an even positive integer. What is the limiting position of the centroid as $n \to \infty$?

6. CALCULATOR If you haul a telephone pole on a two-wheeled carriage behind a truck, you want the wheels to be three feet or so behind the pole's center of mass to provide an adequate "tongue" weight. NYNEX's class 1 40-ft wooden poles have a 27-in. circumference at the top and a 43.5-in. circumference at the base. About how far from the top is the center of mass?

7. Suppose that a thin metal plate of area A and constant density δ occupies a region R in the xy-plane, and let M_y be the plate's moment about the y-axis. Show that the plate's moment about the line $x = b$ is

a) $M_y - b\,\delta\, A$ if the plate lies to the right of the line, and

b) $b\,\delta\, A - M_y$ if the plate lies to the left of the line.

8. Find the center of mass of a thin plate covering the region bounded by the curve $y^2 = 4ax$ and the line $x = a$, $a =$ positive constant, if the density at (x, y) is directly proportional to (a) x, (b) $|y|$.

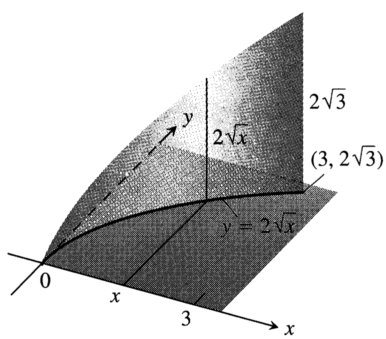
9. a) Find the centroid of the region in the first quadrant bounded by two concentric circles and the coordinate axes, if the circles have radii a and b, $0 < a < b$, and their centers are at the origin.

b) Find the limits of the coordinates of the centroid as a approaches b and discuss the meaning of the result.

10. A triangular corner is cut from a square 1 ft on a side. The area of the triangle removed is 36 in^2. If the centroid of the remaining region is 7 in. from one side of the original square, how far is it from the remaining sides?

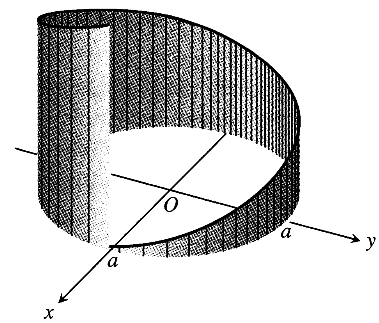
Surface Area

11. At points on the curve $y = 2\sqrt{x}$, line segments of length $h = y$ are drawn perpendicular to the xy-plane (Fig. 5.83). Find the area of the surface formed by these perpendiculars from (0, 0) to $(3, 2\sqrt{3})$.



5.83 The surface in Exercise 11.

12. At points on a circle of radius a, line segments are drawn perpendicular to the plane of the circle, the perpendicular at each point P being of length ks, where s is the length of the arc of the circle measured counterclockwise from $(a, 0)$ to P and k is a positive constant, as shown here. Find the area of the surface formed by the perpendiculars along the arc beginning at $(a, 0)$ and extending once around the circle.



Work

13. A particle of mass m starts from rest at time $t = 0$ and is moved along the x-axis with constant acceleration a from $x = 0$ to $x = h$ against a variable force of magnitude $F(t) = t^2$. Find the work done.

14. *Work and kinetic energy.* Suppose a 1.6-oz golf ball is placed on a vertical spring with force constant $k = 2$ lb/in. The spring is compressed 6 in. and released. About how high does the ball go (measured from the spring's rest position)?

Fluid Force

15. A triangular plate ABC is submerged in water with its plane vertical. The side AB, 4 ft long, is 6 ft below the surface of the water, while the vertex C is 2 ft below the surface. Find the force exerted by the water on one side of the plate.

16. A vertical rectangular plate is submerged in a fluid with its top edge parallel to the fluid's surface. Show that the force exerted by the fluid on one side of the plate equals the average value of the pressure up and down the plate times the area of the plate.

17. The *center of pressure* on one side of a plane region submerged in a fluid is defined to be the point at which the total force exerted by the fluid can be applied without changing its total moment about any axis in the plane. Find the depth to the center of pressure (a) on a vertical rectangle of height h and width b if its upper edge is in the surface of the fluid; (b) on a vertical triangle of height h and base b if the vertex opposite b is a ft and the base b is $(a + h)$ ft below the surface of the fluid.