CHAPTER 6

Transcendental Functions

OVERVIEW Many of the functions in mathematics and science are inverses of one another. The functions $\ln x$ and e^x are probably the best-known function–inverse pair, but others are nearly as important. The trigonometric functions, when suitably restricted, have important inverses, and there are other useful pairs of logarithmic and exponential functions. Less widely known are the hyperbolic functions and their inverses, functions that arise in the study of hanging cables, heat flow, and the friction encountered by objects falling through the air. We describe all of these functions in this chapter and look at the kinds of problems they solve.

6.1 Inverse Functions and Their Derivatives

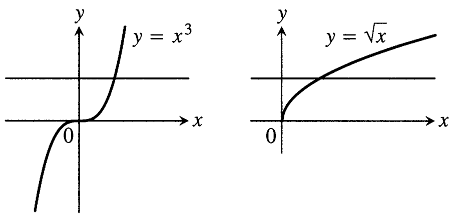
In this section, we define what it means for functions to be inverses of one another and look at what this says about the formulas, graphs, and derivatives of function–inverse pairs.

One-to-One Functions

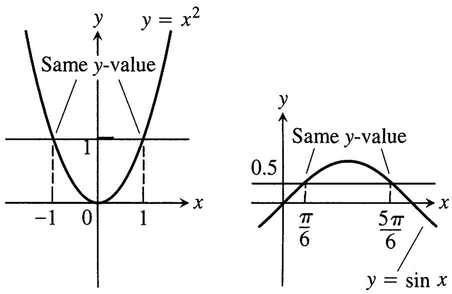
A function is a rule that assigns a value from its range to each point in its domain. Some functions assign the same value to more than one point. The squares of -1 and 1 are both 1; the sines of $\pi/3$ and $2\pi/3$ are both $\sqrt{3}/2$. Other functions never assume a given value more than once. The square roots and cubes of different numbers are always different. A function that has distinct values at distinct points is called one-to-one.

Definition
A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

EXAMPLE 1 $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$. ❑

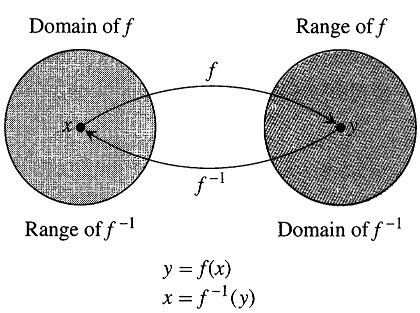


One-to-one: Graph meets each horizontal line at most once.



Not one-to-one: Graph meets one or more horizontal lines more than once.

6.1 Using the horizontal line test, we see that $y = x^3$ and $y = \sqrt{x}$ are one-to-one, but $y = x^2$ and $y = \sin x$ are not.



6.2 The inverse of a function f sends each output back to the input from which it came.

EXAMPLE 2 $g(x) = \sin x$ *is not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. The sine *is* one-to-one on $[0, \pi/2]$, however, because sines of angles in the first quadrant are distinct. ❑

The graph of a one-to-one function $y = f(x)$ can intersect a given horizontal line at most once. If it intersects the line more than once it assumes the same y-value more than once, and is therefore not one-to-one (Fig. 6.1).

The Horizontal Line Test

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

Inverses

Since each output of a one-to-one function comes from just one input, a one-to-one function can be reversed to send the outputs back to the inputs from which they came. The function defined by reversing a one-to-one function f is called the **inverse** of f. The symbol for the inverse of f is $\boldsymbol{f^{-1}}$, read "f inverse" (Fig. 6.2). The -1 in f^{-1} is *not* an exponent: $f^{-1}(x)$ does not mean $1/f(x)$.

As Fig. 6.2 suggests, the result of composing f and f^{-1} in either order is the **identity function,** the function that assigns each number to itself. This gives a way to test whether two functions f and g are inverses of one another. Compute $f \circ g$ and $g \circ f$. If $(f \circ g)(x) = (g \circ f)(x) = x$, then f and g are inverse of one another; otherwise they are not. If f cubes every number in its domain, g had better take cube roots or it isn't the inverse of f.

Functions f and g are an inverse pair if and only if

$$f(g(x)) = x \qquad \text{and} \qquad g(f(x)) = x.$$

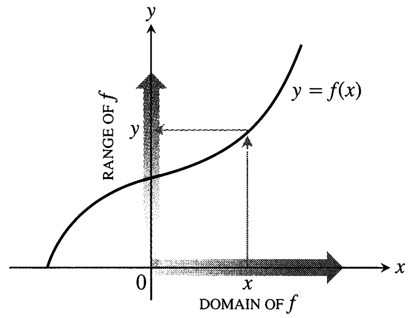
In this case, $g = f^{-1}$ and $f = g^{-1}$.

A function has an inverse if and only if it is one-to-one. This means, for example, that increasing functions have inverses and decreasing functions have inverses (Exercise 39). Functions with positive derivatives have inverses because they increase throughout their domains (Corollary 3 of the Mean Value Theorem, Section 3.2). Similarly, because they decrease throughout their domains, functions with negative derivatives have inverses.

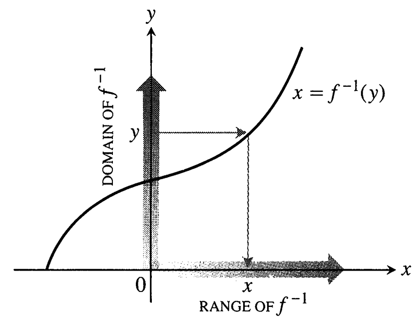
Finding Inverses

How is the graph of the inverse of a function related to the graph of the function? If the function is increasing, say, its graph rises from left to right, like the graph in Fig. 6.3(a). To read the graph, we start at the point x on the x-axis, go up to the graph, and then move over to the y-axis to read the value of y. If we start with y and want to find the x from which it came, we reverse the process (Fig. 6.3b).

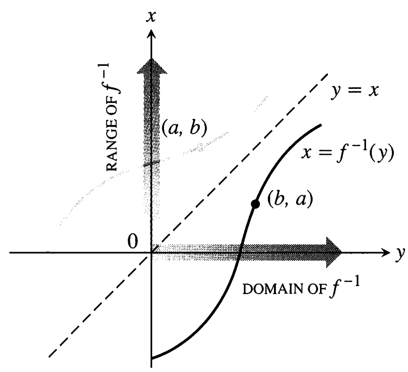
The graph of f is the graph of f^{-1} with the input–output pairs reversed. To display the graph in the usual way, we have to reverse the pairs by reflecting the



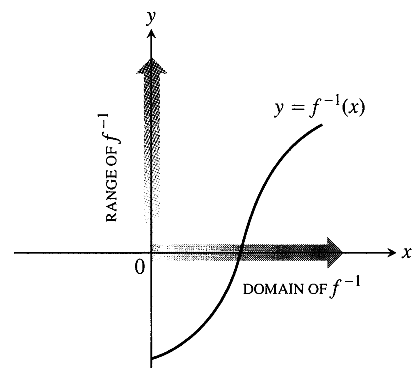
(a) To find the value of f at x, we start at x and go up to the curve and over to the y-axis.



(b) The graph of f can also serve as a graph of f^{-1}. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.



(c) To draw the graph of f^{-1} in the usual way, we reflect it in the line $y = x$.



(d) Then we interchange the letters x and y. We now have a graph of f^{-1} as a function of x.

6.3 The graph of $f^{-1}(x)$.

graph in the 45° line $y = x$ (Fig. 6.3c) and interchanging the letters x and y (Fig. 6.3d). This puts the independent variable, now called x, on the horizontal axis and the dependent variable, now called y, on the vertical axis. The graphs of $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y = x$.

The pictures in Fig. 6.3 tell us how to express f^{-1} as a function of x, which is stated at the left.

How to Express f^{-1} as a Function of x

Step 1: Solve the equation $y = f(x)$ for x in terms of y.

Step 2: Interchange x and y. The resulting formula will be $y = f^{-1}(x)$.

EXAMPLE 3 Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

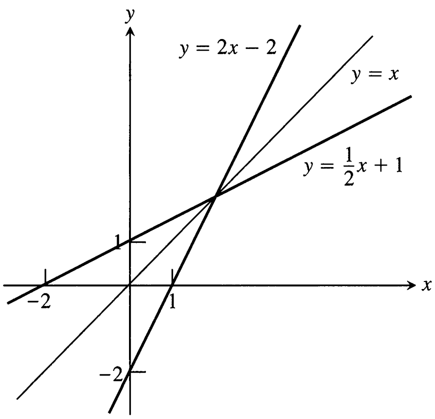
Solution

Step 1: *Solve for x in terms of y:*

$$y = \frac{1}{2}x + 1$$
$$2y = x + 2$$
$$x = 2y - 2.$$

Step 2: *Interchange x and y:* $y = 2x - 2$.

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$.



6.4 Graphing $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$.

To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

See Fig. 6.4. ❑

EXAMPLE 4 Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.

Solution

Step 1: *Solve for x in terms of y:*

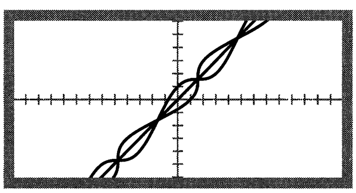
$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$

Step 2: *Interchange x and y:* $y = \sqrt{x}$.

The inverse of the function $y = x^2$, $x \geq 0$, is the function $y = \sqrt{x}$. See Fig. 6.5.

Notice that, unlike the restricted function $y = x^2$, $x \geq 0$, the unrestricted function $y = x^2$ is not one-to-one and therefore has no inverse. ❑



$$\begin{cases} x_1(t) = t \\ y_1(t) = t + \cos t \end{cases} \qquad \begin{cases} x_2(t) = t + \cos t \\ y_2(t) = t \end{cases}$$

$$\begin{cases} x_3(t) = t \\ y_3(t) = t \end{cases}$$

Technology *Using Parametric Equations to Graph Inverses* (See the Technology Notes in Section 2.3 for a discussion of parametric mode.) It is easy to graph the inverse of the function $y = f(x)$, using the parametric form

$$x(t) = f(t), \qquad y(t) = t.$$

You can graph the function and its inverse together, using

$$x_1(t) = t, \qquad y_1(t) = f(t) \qquad \text{(the function)}$$

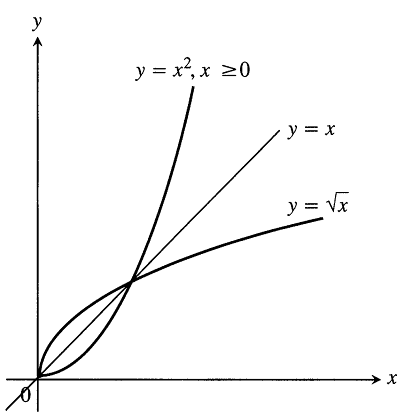
$$x_2(t) = f(t), \qquad y_2(t) = t \qquad \text{(its inverse)}$$

Even better, graph the function, its inverse, *and* the identity function $y = x$, expressed parametrically as

$$x_3(t) = t, \qquad y_3(t) = t \qquad \text{(the identity function)}$$

The graphing is particularly effective if done simultaneously.

Try it on the functions $y = x^5/(x^2 + 1)$ and $y = x + \cos x$. You will see the symmetry best if you use a square window (one in which the x- and y-axes are identically scaled).



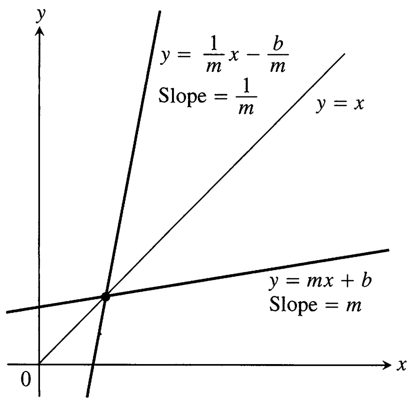
6.5 The functions $y = \sqrt{x}$ and $y = x^2$, $x \geq 0$, are inverses of one another.

Derivatives of Inverses of Differentiable Functions

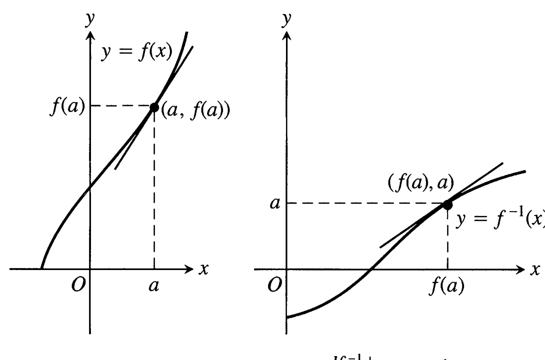
If we calculate the derivatives of $f(x) = (1/2)x + 1$ and its inverse $f^{-1}(x) = 2x - 2$ from Example 3, we see that

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2.$$



6.6 The slopes of nonvertical lines reflected across the line $y = x$ are reciprocals of one another.



6.7 The graphs of inverse functions have reciprocal slopes at corresponding points.

The derivatives are reciprocals of one another. The graph of f is the line $y = (1/2)x + 1$, and the graph of f^{-1} is the line $y = 2x - 2$ (Fig. 6.4). Their slopes are reciprocals of one another.

This is not a special case. Reflecting any nonhorizontal or nonvertical line across the line $y = x$ always inverts the line's slope. If the original line has slope $m \neq 0$ (Fig. 6.6), the reflected line has slope $1/m$ (Exercise 36).

The reciprocal relation between the slopes of graphs of inverses holds for other functions as well. If the slope of $y = f(x)$ at the point $(a, f(a))$ is $f'(a) \neq 0$, then the slope of $y = f^{-1}(x)$ at the corresponding point $(f(a), a)$ is $1/f'(a)$ (Fig. 6.7). Thus, the derivative of f^{-1} at $f(a)$ equals the reciprocal of the derivative of f at a. As you might imagine, we have to impose some mathematical conditions on f to be sure this conclusion holds. The usual conditions, from advanced calculus, are stated in Theorem 1.

Theorem 1

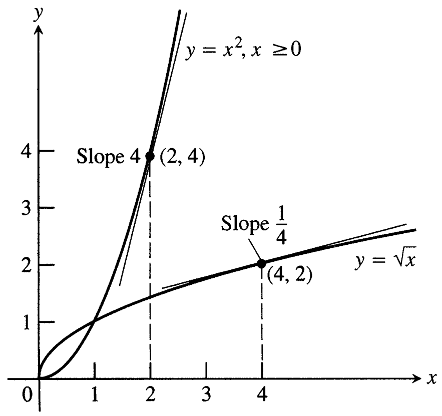
The Derivative Rule for Inverses

If f is differentiable at every point of an interval I and df/dx is never zero on I, then f^{-1} is differentiable at every point of the interval $f(I)$. The value of df^{-1}/dx at any particular point $f(a)$ is the reciprocal of the value of df/dx at a:

$$\left(\frac{df^{-1}}{dx}\right)_{x=f(a)} = \frac{1}{\left(\frac{df}{dx}\right)_{x=a}}. \tag{1}$$

In short,

$$(f^{-1})' = \frac{1}{f'}. \tag{2}$$



6.8 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4).

EXAMPLE 5 For $f(x) = x^2$, $x \geq 0$, and its inverse $f^{-1}(x) = \sqrt{x}$ (Fig. 6.8), we have

$$\frac{df}{dx} = \frac{d}{dx}(x^2) = 2x \qquad \text{and} \qquad \frac{df^{-1}}{dx} = \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0.$$

The point (4, 2) is the mirror image of the point (2, 4) across the line $y = x$.

$$\text{At the point } (2, 4): \qquad \frac{df}{dx} = 2x = 2(2) = 4.$$

$$\text{At the point } (4, 2): \qquad \frac{df^{-1}}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{4}} = \frac{1}{4} = \frac{1}{df/dx}.$$

❑

Equation (1) sometimes enables us to find specific values of df^{-1}/dx without knowing a formula for f^{-1}.

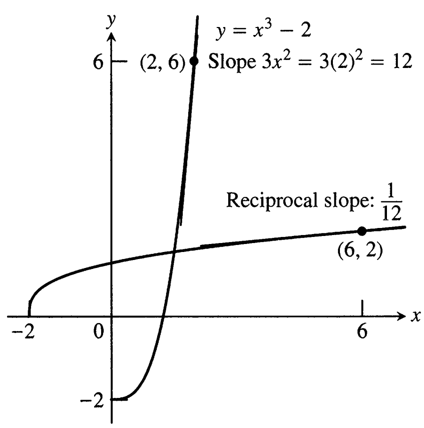
EXAMPLE 6 Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution

$$\left.\frac{df}{dx}\right|_{x=2} = \left.3x^2\right|_{x=2} = 12$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \frac{1}{12} \qquad \text{Eq. (1)}$$

See Fig. 6.9. ❑



6.9 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$.

Another Way to Look at Theorem 1

If $y = f(x)$ is differentiable at $x = a$ and we change x by a small amount dx, the corresponding change in y is approximately

$$dy = f'(a)\,dx.$$

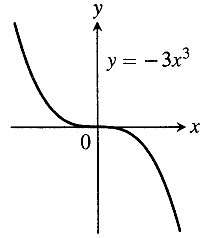
This means that y changes about $f'(a)$ times as fast as x and that x changes about $1/f'(a)$ times as fast as y.

Exercises 6.1

Identifying One-to-One Functions Graphically

Which of the functions graphed in Exercises 1–6 are one-to-one, and which are not?

1.



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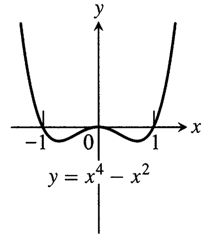

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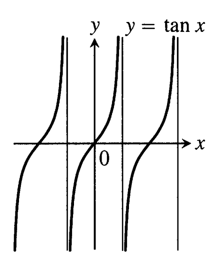

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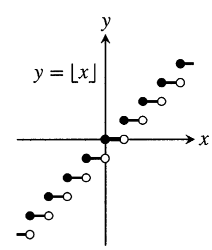

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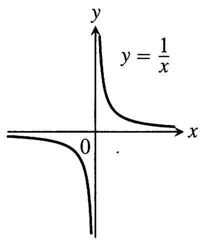
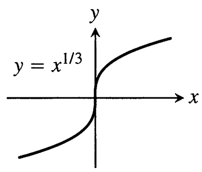


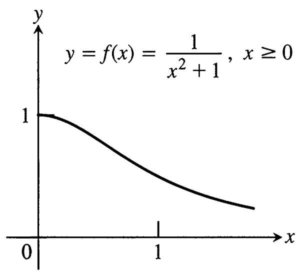
6.



Graphing Inverse Functions

Each of Exercises 7–10 shows the graph of a function $y = f(x)$. Copy the graph and draw in the line $y = x$. Then use symmetry with respect to the line $y = x$ to add the graph of f^{-1} to your sketch. (It is not necessary to find a formula for f^{-1}.) Identify the domain and range of f^{-1}.

7.



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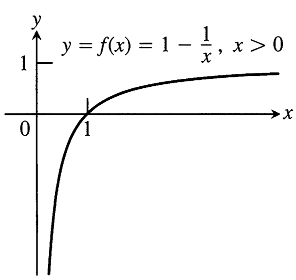

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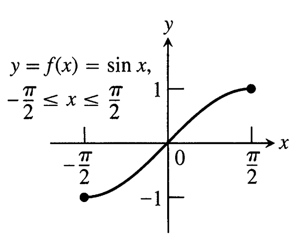
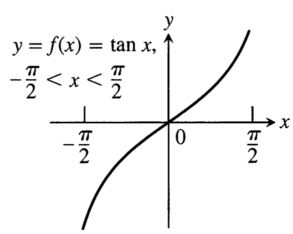


10.



11. a) Graph the function $f(x) = \sqrt{1 - x^2}$, $0 \leq x \leq 1$. What symmetry does the graph have?

b) Show that f is its own inverse. (Remember that $\sqrt{x^2} = x$ if $x \geq 0$.)

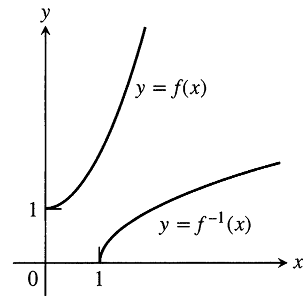
12. a) Graph the function $f(x) = 1/x$. What symmetry does the graph have?

b) Show that f is its own inverse.

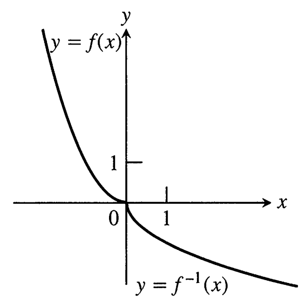
Formulas for Inverse Functions

Each of Exercises 13–18 gives a formula for a function $y = f(x)$ and shows the graphs of f and f^{-1}. Find a formula for f^{-1} in each case.

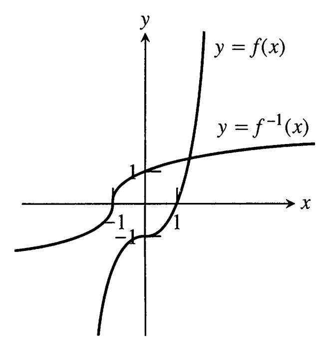
13. $f(x) = x^2 + 1, \quad x \geq 0$



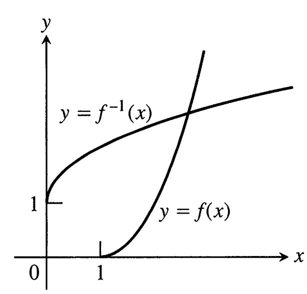
14. $f(x) = x^2, \quad x \leq 0$



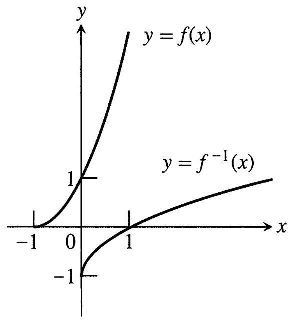
15. $f(x) = x^3 - 1$



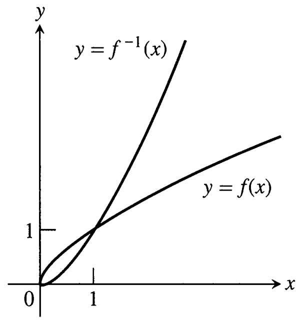
16. $f(x) = x^2 - 2x + 1, \quad x \geq 1$



17. $f(x) = (x+1)^2, \quad x \geq -1$



18. $f(x) = x^{2/3}, \quad x \geq 0$



Each of Exercises 19–24 gives a formula for a function $y = f(x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

19. $f(x) = x^5$

20. $f(x) = x^4, \quad x \geq 0$

21. $f(x) = x^3 + 1$

22. $f(x) = (1/2)x - 7/2$

23. $f(x) = 1/x^2, \quad x > 0$

24. $f(x) = 1/x^3, \quad x \neq 0$

Derivatives of Inverse Functions

In Exercises 25–28:

a) Find $f^{-1}(x)$.
b) Graph f and f^{-1} together.
c) Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that at these points $df^{-1}/dx = 1/(df/dx)$.

25. $f(x) = 2x + 3, \quad a = -1$

26. $f(x) = (1/5)x + 7, \quad a = -1$

27. $f(x) = 5 - 4x, \quad a = 1/2$

28. $f(x) = 2x^2, \quad x \geq 0, \quad a = 5$

29. a) Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.
b) Graph f and g over an x-interval large enough to show the graphs intersecting at (1, 1) and $(-1, -1)$. Be sure the picture shows the required symmetry in the line $y = x$.
c) Find the slopes of the tangents to the graphs of f and g at (1, 1) and $(-1, -1)$ (four tangents in all).
d) What lines are tangent to the curves at the origin?

30. a) Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.
b) Graph h and k over an x-interval large enough to show the graphs intersecting at (2, 2) and $(-2, -2)$. Be sure the picture shows the required symmetry about the line $y = x$.
c) Find the slopes of the tangents to the graphs at h and k at (2, 2) and $(-2, -2)$.
d) What lines are tangent to the curves at the origin?

31. Let $f(x) = x^3 - 3x^2 - 1$, $x \geq 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.

32. Let $f(x) = x^2 - 4x - 5$, $x > 2$. Find the value of df^{-1}/dx at the point $x = 0 = f(5)$.

33. Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point (2, 4) and has a slope of 1/3 there. Find the value of df^{-1}/dx at $x = 4$.

34. Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of g passes through the origin with slope 2. Find the slope of the graph of g^{-1} at the origin.

35. a) Find the inverse of the function $f(x) = mx$, where m is a constant different from zero.
b) What can you conclude about the inverse of a function $y = f(x)$ whose graph is a line through the origin with a nonzero slope m?

36. Show that the graph of the inverse of $f(x) = mx + b$, where m and b are constants and $m \neq 0$, is a line with slope $1/m$ and y-intercept $-b/m$.

37. a) Find the inverse of $f(x) = x + 1$. Graph f and its inverse together. Add the line $y = x$ to your sketch, drawing it with dashes or dots for contrast.
b) Find the inverse of $f(x) = x + b$ (b constant). How is the graph of f^{-1} related to the graph of f?
c) What can you conclude about the inverses of functions whose graphs are lines parallel to the line $y = x$?

38. a) Find the inverse of $f(x) = -x + 1$. Graph the line $y = -x + 1$ together with the line $y = x$. At what angle do the lines intersect?
b) Find the inverse of $f(x) = -x + b$ (b constant). What angle does the line $y = -x + b$ make with the line $y = x$?
c) What can you conclude about the inverses of functions whose graphs are lines perpendicular to the line $y = x$?

Increasing and Decreasing Functions

39. *Increasing functions and decreasing functions.* As in Section 3.2, a function $f(x)$ increases on an interval I if for any two points x_1 and x_2 in I,

$$x_2 > x_1 \quad \Rightarrow \quad f(x_2) > f(x_1).$$

Similarly, a function decreases on I if for any two points x_1 and x_2 in I,

$$x_2 > x_1 \quad \Rightarrow \quad f(x_2) < f(x_1).$$

Show that increasing functions and decreasing functions are one-to-one. That is, show that for any x_1 and x_2 in I, $x_2 \neq x_1$ implies $f(x_2) \neq f(x_1)$.

Use the results of Exercise 39 to show that the functions in Exercises 40–44 have inverses over their domains. Find a formula for df^{-1}/dx using Theorem 1.

40. $f(x) = (1/3)x + (5/6)$

41. $f(x) = 27x^3$

42. $f(x) = 1 - 8x^3$

43. $f(x) = (1-x)^3$

44. $f(x) = x^{5/3.}$

Theory and Applications

45. If $f(x)$ is one-to-one, can anything be said about $g(x) = -f(x)$? Give reasons for your answer.

46. If $f(x)$ is one-to-one and $f(x)$ is never 0, can anything be said about $h(x) = 1/f(x)$? Give reasons for your answer.

47. Suppose that the range of g lies in the domain of f so that the composite $f \circ g$ is defined. If f and g are one-to-one, can anything be said about $f \circ g$? Give reasons for your answer.

48. If a composite $f \circ g$ is one-to-one, must g be one-to-one? Give reasons for your answer.

49. Suppose $f(x)$ is positive, continuous, and increasing over the interval $[a, b]$. By interpreting the graph of f show that

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(y)\,dy = bf(b) - af(a).$$

50. Determine conditions on the constants a, b, c, and d so that the rational function

$$f(x) = \frac{ax+b}{cx+d}$$

has an inverse.

51. *Still another way to view Theorem 1.* If we write $g(x)$ for $f^{-1}(x)$, Eq. (1) can be written as

$$g'(f(a)) = \frac{1}{f'(a)}, \quad \text{or} \quad g'(f(a)) \cdot f'(a) = 1.$$

If we then write x for a, we get

$$g'(f(x)) \cdot f'(x) = 1.$$

The latter equation may remind you of the Chain Rule, and indeed there is a connection.

Assume that f and g are differentiable functions that are inverses of one another, so that $(g \circ f)(x) = x$. Differentiate both sides of this equation with respect to x, using the Chain Rule to express $(g \circ f)'(x)$ as a product of derivatives of g and f. What do you find? (This is not a proof of Theorem 1 because we assume here the theorem's conclusion that $g = f^{-1}$ is differentiable.)

52. *Equivalence of the washer and shell methods for finding volume.* Let f be differentiable on the interval $a \le x \le b$, with $a > 0$, and suppose that f has a differentiable inverse, f^{-1}. Revolve about the y-axis the region bounded by the graph of f and the lines $x = a$ and $y = f(b)$ to generate a solid. Then the values of the integrals given by the washer and shell methods for the volume have identical values:

$$\int_{f(a)}^{f(b)} \pi\Big((f^{-1}(y))^2 - a^2\Big)dy = \int_a^b 2\pi x(f(b) - f(x))\,dx.$$

To prove this equality, define

$$W(t) = \int_{f(a)}^{f(t)} \pi\Big((f^{-1}(y))^2 - a^2\Big)\,dy$$

$$S(t) = \int_a^t 2\pi x(f(t) - f(x))\,dx.$$

Then show that the functions W and S agree at a point of $[a, b]$ and have identical derivatives on $[a, b]$. As you saw in Section 4.2, Exercise 56, this will guarantee $W(t) = S(t)$ for all t in $[a, b]$. In particular, $W(b) = S(b)$. (Source: "Disks and Shells Revisited," by Walter Carlip, *American Mathematical Monthly,* Vol. 98, No. 2, February 1991, pp. 154–156.)

CAS Explorations and Projects

In Exercises 53–60, you will explore some functions and their inverses together with their derivatives and linear approximating functions at specified points. Perform the following steps using your CAS:

a) Plot the function $y = f(x)$ together with its derivative over the given interval. Explain why you know that f is one-to-one over the interval.
b) Solve the equation $y = f(x)$ for x as a function of y, and name the resulting inverse function g.
c) Find the equation for the tangent line to f at the specified point $(x_0, f(x_0))$.
d) Find the equation for the tangent line to g at the point $(f(x_0), x_0)$ located symmetrically across the 45° line $y = x$ (which is the graph of the identity function). Use Theorem 1 to find the slope of this tangent line.
e) Plot the functions f and g, the identity, the two tangent lines, and the line segment joining the points $(x_0, f(x_0))$ and $(f(x_0), x_0)$. Discuss the symmetries you see across the main diagonal.

53. $y = \sqrt{3x-2}, \quad \dfrac{2}{3} \le x \le 4, \quad x_0 = 3$

54. $y = \dfrac{3x+2}{2x-11}, \quad -2 \le x \le 2, \quad x_0 = 1/2$

55. $y = \dfrac{4x}{x^2+1}, \quad -1 \le x \le 1, \quad x_0 = 1/2$

56. $y = \dfrac{x^3}{x^2+1}, \quad -1 \le x \le 1, \quad x_0 = 1/2$

57. $y = x^3 - 3x^2 - 1, \quad 2 \le x \le 5, \quad x_0 = \dfrac{27}{10}$

58. $y = 2 - x - x^3, \quad -2 \le x \le 2, \quad x_0 = \dfrac{3}{2}$

59. $y = e^x, \quad -3 \le x \le 5, \quad x_0 = 1$

60. $y = \sin x, \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}, \quad x_0 = 1$

In Exercises 61 and 62, repeat the steps above to solve for the functions $y = f(x)$ and $x = f^{-1}(y)$ defined implicitly by the given equations over the interval.

61. $y^{1/3} - 1 = (x+2)^3, \quad -5 \le x \le 5, \quad x_0 = -3/2$

62. $\cos y = x^{1/5}, \quad 0 \le x \le 1, \quad x_0 = 1/2$

6.2

Natural Logarithms

The most important function–inverse pair in mathematics and science is the pair consisting of the natural logarithm function ln x and the exponential function e^x. The key to understanding e^x is ln x, so we introduce ln x first. The importance of logarithms came at first from the improvement they brought to arithmetic. The revolutionary properties of logarithms made possible the calculations of the great seventeenth-century advances in offshore navigation and celestial mechanics. Nowadays we do complicated arithmetic with calculators, but the properties of logarithms remain as important as ever.

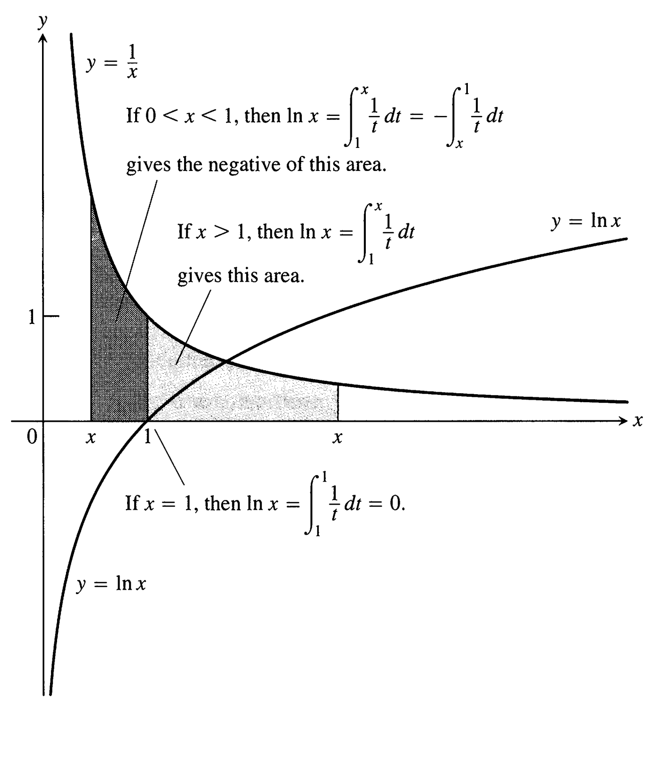
The Natural Logarithm Function

The natural logarithm of a positive number x, written as ln x, is the value of an integral.

Definition
The Natural Logarithm Function

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$

If $x > 1$, then $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$ (Fig. 6.10). For $0 < x < 1$, ln x gives the negative of the area under the curve from



6.10 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left.

Typical 2-place values of ln x

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

x to 1. The function is not defined for $x \leq 0$. We also have

$$\ln 1 = \int_1^1 \frac{1}{t}\, dt = 0. \qquad \text{Upper and lower limits equal}$$

Notice that we show the graph of $y = 1/x$ in Fig. 6.10 but use $y = 1/t$ in the integral. Using x for everything would have us writing

$$\ln x = \int_1^x \frac{1}{x}\, dx,$$

with x meaning two different things. So we change the variable of integration to t.

The Derivative of $y = \ln x$

By the first part of the Fundamental Theorem of Calculus (in Section 4.6),

$$\frac{d}{dx} \ln x = \frac{d}{dx} \int_1^x \frac{1}{t}\, dt = \frac{1}{x}.$$

For every positive value of x, therefore,

$$\boxed{\frac{d}{dx} \ln x = \frac{1}{x}.}$$

If u is a differentiable function of x whose values are positive, so that $\ln u$ is defined, then applying the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

to the function $y = \ln u$ gives

$$\frac{d}{dx} \ln u = \frac{d}{du} \ln u \cdot \frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

$$\boxed{\frac{d}{dx} \ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0 \qquad (1)}$$

EXAMPLE 1

$$\frac{d}{dx} \ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}$$ ❑

Notice the remarkable occurrence in Example 1. The function $y = \ln 2x$ has the same derivative as the function $y = \ln x$. This is true of $y = \ln ax$ for any number a:

$$\frac{d}{dx} \ln ax = \frac{1}{ax} \cdot \frac{d}{dx}(ax) = \frac{1}{ax}(a) = \frac{1}{x}. \qquad (2)$$

EXAMPLE 2 Equation (1) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}. \qquad \square$$

In the late 1500s, a Scottish baron, John Napier, invented a device called the *logarithm* that simplified arithmetic by replacing multiplication by addition. The equation that accomplished this was

$$\ln ax = \ln a + \ln x.$$

To multiply two positive numbers a and x, you looked up their logarithms in a table, added the logarithms, found the sum in the body of the table, and read the table backward to find the product ax.

Having the table was the key, of course, and Napier spent the last 20 years of his life working on a table he never finished (while the astronomer Tycho Brahe waited in vain for the information he needed to speed his calculations). The table was completed after Napier's death (and Brahe's) by Napier's friend Henry Briggs in London. Base 10 logarithms subsequently became known as Briggs's logarithms (what else?) and some books on navigation still refer to them this way.

Napier also invented an artillery piece that could hit a cow a mile away. Horrified by the weapon's accuracy, he stopped production and suppressed the cannon's design.

Properties of Logarithms

The properties that made logarithms the single most important improvement in arithmetic before the advent of modern computers are listed in Table 6.1. The properties made it possible to replace multiplication of positive numbers by addition, and division of positive numbers by subtraction. They also made it possible to replace exponentiation by multiplication. For the moment, we add the restriction that the exponent n in Rule 4 be a rational number. You will see why when we prove the rule.

EXAMPLE 3

a) $\ln 6 = \ln(2\cdot 3) = \ln 2 + \ln 3$ Product

b) $\ln 4 - \ln 5 = \ln\frac{4}{5} = \ln 0.8$ Quotient

c) $\ln\frac{1}{8} = -\ln 8$ Reciprocal

$= -\ln 2^3 = -3\ln 2$ Power $\square$

EXAMPLE 4

a) $\ln 4 + \ln\sin x = \ln(4\sin x)$ Product

b) $\ln\frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ Quotient

c) $\ln\sec x = \ln\frac{1}{\cos x} = -\ln\cos x$ Reciprocal

d) $\ln\sqrt[3]{x+1} = \ln(x+1)^{1/3} = \frac{1}{3}\ln(x+1)$ Power $\square$

***Proof that* $\ln ax = \ln a + \ln x$** The argument is unusual—and elegant. It starts by observing that $\ln ax$ and $\ln x$ have the same derivative (Eq. 2). According to Corollary 1 of the Mean Value Theorem, then, the functions must differ by a

Table 6.1 Properties of natural logarithms

For any numbers $a > 0$ and $x > 0$,

1. Product Rule:	$\ln ax = \ln a + \ln x$		
2. Quotient Rule:	$\ln\frac{a}{x} = \ln a - \ln x$		
3. Reciprocal Rule:	$\ln\frac{1}{x} = -\ln x$	Rule 2 with $a = 1$	
4. Power Rule:	$\ln x^n = n\ln x$		

constant, which means that

$$\ln ax = \ln x + C \tag{3}$$

for some C. With this much accomplished, it remains only to show that C equals $\ln a$.

Equation (3) holds for all positive values of x, so it must hold for $x = 1$. Hence,

$$\begin{aligned} \ln(a \cdot 1) &= \ln 1 + C \\ \ln a &= 0 + C && \ln 1 = 0 \\ C &= \ln a. && \text{Rearranged} \end{aligned}$$

Substituting $C = \ln a$ in Eq. (3) gives the equation we wanted to prove:

$$\ln ax = \ln a + \ln x. \tag{4}$$

❑

***Proof that* ln (a/x) = ln a − ln x** We get this from Eq. (4) in two stages. Equation (4) with a replaced by $1/x$ gives

$$\begin{aligned} \ln\frac{1}{x} + \ln x &= \ln\left(\frac{1}{x} \cdot x\right) \\ &= \ln 1 = 0, \end{aligned}$$

so that

$$\ln\frac{1}{x} = -\ln x.$$

Equation (4) with x replaced by $1/x$ then gives

$$\begin{aligned} \ln\frac{a}{x} &= \ln\left(a \cdot \frac{1}{x}\right) = \ln a + \ln\frac{1}{x} \\ &= \ln a - \ln x. \end{aligned}$$

❑

***Proof that* ln x^n = *n* ln *x* (assuming *n* rational)** We use the same-derivative argument again. For all positive values of x,

$$\begin{aligned} \frac{d}{dx}\ln x^n &= \frac{1}{x^n}\frac{d}{dx}(x^n) && \text{Eq. (1) with } u = x^n \\ &= \frac{1}{x^n} n x^{n-1} && \text{Here is where we need } n \text{ to be rational, at least for now. We have proved the Power Rule only for rational exponents.} \\ &= n \cdot \frac{1}{x} = \frac{d}{dx}(n \ln x). \end{aligned}$$

Since $\ln x^n$ and $n \ln x$ have the same derivative,

$$\ln x^n = n \ln x + C$$

for some constant C. Taking x to be 1 identifies C as zero, and we're done. ❑

As for using the rule $\ln x^n = n \ln x$ for irrational values of n, go right ahead and do so. It does hold for all n, and there is no need to pretend otherwise. From the point of view of mathematical development, however, we want you to be aware that the rule is far from proved.

The Graph and Range of $\ln x$

The derivative $d(\ln x)/dx = 1/x$ is positive for $x > 0$, so $\ln x$ is an increasing function of x. The second derivative, $-1/x^2$, is negative, so the graph of $\ln x$ is concave down.

We can estimate $\ln 2$ by numerical integration to be about 0.69. We therefore know that

$$\ln 2^n = n \ln 2 > n\left(\frac{1}{2}\right) = \frac{n}{2}$$

and

$$\ln 2^{-n} = -n \ln 2 < -n\left(\frac{1}{2}\right) = -\frac{n}{2}.$$

It follows that

$$\lim_{x\to\infty} \ln x = \infty \qquad \text{and} \qquad \lim_{x\to 0^+} \ln x = -\infty.$$

The domain of $\ln x$ is the set of positive real numbers; the range is the entire real line.

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the rules in Table 6.1 to simplify the formulas before differentiating. The process, called **logarithmic differentiation,** is illustrated in the next example.

EXAMPLE 5 Find dy/dx if $y = \dfrac{(x^2+1)(x+3)^{1/2}}{x-1}, \quad x > 1.$

Solution We take the natural logarithm of both sides and simplify the result with the rules in Table 6.1:

$$\begin{aligned} \ln y &= \ln \frac{(x^2+1)(x+3)^{1/2}}{x-1} \\ &= \ln\left((x^2+1)(x+3)^{1/2}\right) - \ln(x-1) && \text{Quotient Rule} \\ &= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1) && \text{Product Rule} \\ &= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1). && \text{Power Rule} \end{aligned}$$

We then take derivatives of both sides with respect to x, using Eq. (1) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right). \qquad \square$$

How to Differentiate $y = f(x) > 0$ by Logarithmic Differentiation

1. $\ln y = \ln f(x)$ — Take logs of both sides.
2. $\dfrac{d}{dx}\ln y = \dfrac{d}{dx}(\ln f(x))$ — Differentiate both sides . . .
3. $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{d}{dx}(\ln f(x))$ — . . . using Eq. (1) on the left.
4. $\dfrac{dy}{dx} = y\dfrac{d}{dx}(\ln f(x))$ — Solve for dy/dx.
5. $\dfrac{dy}{dx} = f(x)\dfrac{d}{dx}(\ln f(x))$ — Substitute $y = f(x)$.

The Integral $\int (1/u)\, du$

Equation (1) leads to the integral formula

$$\int \frac{1}{u}\, du = \ln u + C \tag{5}$$

when u is a positive differentiable function, but what if u is negative? If u is negative, then $-u$ is positive and

$$\begin{aligned} \int \frac{1}{u}\, du &= \int \frac{1}{(-u)}\, d(-u) \\ &= \ln(-u) + C. \qquad \text{Eq. (5) with } u \text{ replaced by } -u \end{aligned} \tag{6}$$

We can combine Eqs. (5) and (6) into a single formula by noticing that in each case the expression on the right is $\ln|u| + C$. In Eq. (5), $\ln u = \ln|u|$ because $u > 0$; in Eq. (6), $\ln(-u) = \ln|u|$ because $u < 0$. Whether u is positive or negative, the integral of $(1/u)\, du$ is $\ln|u| + C$.

If u is a nonzero differentiable function,

$$\int \frac{1}{u}\, du = \ln|u| + C. \tag{7}$$

We know that

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \qquad n \neq -1.$$

Equation (7) explains what to do when n equals -1.

Equation (7) says that integrals of a certain *form* lead to logarithms. That is,

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

whenever $f(x)$ is a differentiable function that maintains a constant sign on the domain given for it.

EXAMPLE 6

$$\int_0^2 \frac{2x}{x^2-5}\,dx = \int_{-5}^{-1}\frac{du}{u} = \ln|u|\Big]_{-5}^{-1} \qquad \begin{array}{l} u = x^2-5, \quad du = 2x\,dx, \\ u(0) = -5, \quad u(2) = -1 \end{array}$$

$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$$

❑

EXAMPLE 7

$$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta}\,d\theta = \int_1^5 \frac{2}{u}\,du \qquad \begin{array}{l} u = 3+2\sin\theta, \quad du = 2\cos\theta\,d\theta, \\ u(-\pi/2) = 1, \quad u(\pi/2) = 5 \end{array}$$

$$= 2\ln|u|\Big]_1^5$$

$$= 2\ln|5| - 2\ln|1| = 2\ln 5$$

❑

The Integrals of tan x and cot x

Equation (7) tells us at last how to integrate the tangent and cotangent functions. For the tangent,

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u} \qquad \begin{array}{l} u = \cos x, \\ du = -\sin x\,dx \end{array}$$

$$= -\int \frac{du}{u} = -\ln|u| + C \qquad \text{Eq. (7)}$$

$$= -\ln|\cos x| + C = \ln\frac{1}{|\cos x|} + C \qquad \text{Reciprocal Rule}$$

$$= \ln|\sec x| + C.$$

For the cotangent,

$$\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u} \qquad \begin{array}{l} u = \sin x, \\ du = \cos x\,dx \end{array}$$

$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

$$\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C = -\ln|\csc x| + C$$

EXAMPLE 8

$$\int_0^{\pi/6} \tan 2x\,dx = \int_0^{\pi/3} \tan u \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u\,du$$

$$= \frac{1}{2}\ln|\sec u|\Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Substitute $u = 2x$, $dx = du/2$, $u(0) = 0$, $u(\pi/6) = \pi/3$

❑

Exercises 6.2

Using the Properties of Logarithms

1. Express the following logarithms in terms of ln 2 and ln 3.

a) $\ln 0.75$ **b)** $\ln (4/9)$ **c)** $\ln (1/2)$

d) $\ln \sqrt[3]{9}$ **e)** $\ln 3\sqrt{2}$ **f)** $\ln \sqrt{13.5}$

2. Express the following logarithms in terms of ln 5 and ln 7.

a) $\ln (1/125)$ **b)** $\ln 9.8$ **c)** $\ln 7\sqrt{7}$

d) $\ln 1225$ **e)** $\ln 0.056$

f) $(\ln 35 + \ln (1/7))/(\ln 25)$

Use the properties of logarithms to simplify the expressions in Exercises 3 and 4.

3. a) $\ln \sin\theta - \ln\left(\dfrac{\sin\theta}{5}\right)$

b) $\ln (3x^2 - 9x) + \ln\left(\dfrac{1}{3x}\right)$

c) $\dfrac{1}{2}\ln (4t^4) - \ln 2$

4. a) $\ln \sec\theta + \ln \cos\theta$

b) $\ln (8x + 4) - 2\ln 2$

c) $3\ln \sqrt[3]{t^2 - 1} - \ln (t + 1)$

Derivatives of Logarithms

In Exercises 5–36, find the derivative of y with respect to x, t, or θ, as appropriate.

5. $y = \ln 3x$

6. $y = \ln kx$, k constant

7. $y = \ln (t^2)$

8. $y = \ln (t^{3/2})$

9. $y = \ln \dfrac{3}{x}$

10. $y = \ln \dfrac{10}{x}$

11. $y = \ln (\theta + 1)$

12. $y = \ln (2\theta + 2)$

13. $y = \ln x^3$

14. $y = (\ln x)^3$

15. $y = t(\ln t)^2$

16. $y = t\sqrt{\ln t}$

17. $y = \dfrac{x^4}{4}\ln x - \dfrac{x^4}{16}$

18. $y = \dfrac{x^3}{3}\ln x - \dfrac{x^3}{9}$

19. $y = \dfrac{\ln t}{t}$

20. $y = \dfrac{1 + \ln t}{t}$

21. $y = \dfrac{\ln x}{1 + \ln x}$

22. $y = \dfrac{x\ln x}{1 + \ln x}$

23. $y = \ln (\ln x)$

24. $y = \ln (\ln (\ln x))$

25. $y = \theta(\sin (\ln\theta) + \cos (\ln\theta))$

26. $y = \ln (\sec\theta + \tan\theta)$

27. $y = \ln \dfrac{1}{x\sqrt{x + 1}}$

28. $y = \dfrac{1}{2}\ln \dfrac{1 + x}{1 - x}$

29. $y = \dfrac{1 + \ln t}{1 - \ln t}$

30. $y = \sqrt{\ln \sqrt{t}}$

31. $y = \ln (\sec(\ln\theta))$

32. $y = \ln\left(\dfrac{\sqrt{\sin\theta\cos\theta}}{1 + 2\ln\theta}\right)$

33. $y = \ln\left(\dfrac{(x^2 + 1)^5}{\sqrt{1 - x}}\right)$

34. $y = \ln \sqrt{\dfrac{(x + 1)^5}{(x + 2)^{20}}}$

35. $y = \displaystyle\int_{x^2/2}^{x^2} \ln \sqrt{t}\,dt$

36. $y = \displaystyle\int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t\,dt$

Logarithmic Differentiation

In Exercises 37–50, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

37. $y = \sqrt{x(x + 1)}$

38. $y = \sqrt{(x^2 + 1)(x - 1)^2}$

39. $y = \sqrt{\dfrac{t}{t + 1}}$

40. $y = \sqrt{\dfrac{1}{t(t + 1)}}$

41. $y = \sqrt{\theta + 3}\sin\theta$

42. $y = (\tan\theta)\sqrt{2\theta + 1}$

43. $y = t(t + 1)(t + 2)$

44. $y = \dfrac{1}{t(t + 1)(t + 2)}$

45. $y = \dfrac{\theta + 5}{\theta\cos\theta}$

46. $y = \dfrac{\theta\sin\theta}{\sqrt{\sec\theta}}$

47. $y = \dfrac{x\sqrt{x^2 + 1}}{(x + 1)^{2/3}}$

48. $y = \sqrt{\dfrac{(x + 1)^{10}}{(2x + 1)^5}}$

49. $y = \sqrt[3]{\dfrac{x(x - 2)}{x^2 + 1}}$

50. $y = \sqrt[3]{\dfrac{x(x + 1)(x - 2)}{(x^2 + 1)(2x + 3)}}$

Integration

Evaluate the integrals in Exercises 51–68.

51. $\int_{-3}^{-2} \frac{dx}{x}$

52. $\int_{-1}^{0} \frac{3\,dx}{3x-2}$

53. $\int \frac{2y\,dy}{y^2-25}$

54. $\int \frac{8r\,dr}{4r^2-5}$

55. $\int_0^{\pi} \frac{\sin t}{2-\cos t}\,dt$

56. $\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta$

57. $\int_1^2 \frac{2\ln x}{x}\,dx$

58. $\int_2^4 \frac{dx}{x\ln x}$

59. $\int_2^4 \frac{dx}{x(\ln x)^2}$

60. $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

61. $\int \frac{3\sec^2 t}{6+3\tan t}\,dt$

62. $\int \frac{\sec y\tan y}{2+\sec y}\,dy$

63. $\int_0^{\pi/2} \tan\frac{x}{2}\,dx$

64. $\int_{\pi/4}^{\pi/2} \cot t\,dt$

65. $\int_{\pi/2}^{\pi} 2\cot\frac{\theta}{3}\,d\theta$

66. $\int_0^{\pi/12} 6\tan 3x\,dx$

67. $\int \frac{dx}{2\sqrt{x}+2x}$

68. $\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x+\tan x)}}$

Theory and Applications

69. Locate and identify the absolute extreme values of

a) $\ln(\cos x)$ on $[-\pi/4, \pi/3]$,

b) $\cos(\ln x)$ on $[1/2, 2]$.

70. a) Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.

b) Using part (a), show that $\ln x < x$ if $x > 1$.

71. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.

72. Find the area between the curve $y = \tan x$ and the x-axis from $x = -\pi/4$ to $x = \pi/3$.

73. The region in the first quadrant bounded by the coordinate axes, the line $y = 3$, and the curve $x = 2/\sqrt{y+1}$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

74. The region between the curve $y = \sqrt{\cot x}$ and the x-axis from $x = \pi/6$ to $x = \pi/2$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

75. The region between the curve $y = 1/x^2$ and the x-axis from $x = 1/2$ to $x = 2$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

76. In Section 5.4, Exercise 6, we revolved about the y-axis the region between the curve $y = 9x/\sqrt{x^3+9}$ and the x-axis from $x = 0$ to $x = 3$ to generate a solid of volume 36π. What volume do you get if you revolve the region about the x-axis instead? (See Section 5.4, Exercise 6, for a graph.)

77. Find the lengths of the following curves.

a) $y = (x^2/8) - \ln x, \quad 4 \le x \le 8$

b) $x = (y/4)^2 - 2\ln(y/4), \quad 4 \le y \le 12$

78. Find a curve through the point $(1, 0)$ whose length from $x = 1$ to $x = 2$ is

$$L = \int_1^2 \sqrt{1+\frac{1}{x^2}}\,dx.$$

79. CALCULATOR

a) Find the centroid of the region between the curve $y = 1/x$ and the x-axis from $x = 1$ to $x = 2$. Give the coordinates to 2 decimal places.

b) Sketch the region and show the centroid in your sketch.

80. a) Find the center of mass of a thin plate of constant density covering the region between the curve $y = 1/\sqrt{x}$ and the x-axis from $x = 1$ to $x = 16$.

b) Find the center of mass if, instead of being constant, the density function is $\delta(x) = 4/\sqrt{x}$.

Solve the initial value problems in Exercises 81 and 82.

81. $\frac{dy}{dx} = 1 + \frac{1}{x}, \quad y(1) = 3$

82. $\frac{d^2y}{dx^2} = \sec^2 x, \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1$

83. *The linearization of* $\ln(1+x)$ *at* $x = 0$. Instead of approximating x near $x = 1$, we approximate $\ln(1+x)$ near $x = 0$. We get a simpler formula this way.

a) Derive the linearization $\ln(1+x) \approx x$ at $x = 0$.

b) CALCULATOR Estimate to 5 decimal places the error involved in replacing $\ln(1+x)$ by x on the interval $[0, 0.1]$.

c) GRAPHER Graph $\ln(1+x)$ and x together for $0 \le x \le 0.5$. Use different colors, if available. At what points does the approximation of $\ln(1+x)$ seem best? least good? By reading coordinates from the graphs, find as good an upper bound for the error as your grapher will allow.

84. *Estimating values of* $\ln x$ *with Simpson's rule.* Although linearizations are good for replacing the logarithmic function over short intervals, Simpson's rule is better for estimating *particular* values of $\ln x$.

As a case in point, the values of $\ln(1.2)$ and $\ln(0.8)$ to 5 places are

$$\ln(1.2) = 0.18232, \quad \ln(0.8) = -0.22314.$$

Estimate $\ln(1.2)$ and $\ln(0.8)$ first with the formula $\ln(1+x) \approx x$ and then use Simpson's rule with $n = 2$. (Impressive, isn't it?)

85. Find

$$\lim_{x\to\infty} \frac{\ln(x^2)}{\ln x}.$$

Generalize this result.

86. *The derivative of* $\ln kx$. Could $y = \ln 2x$ and $y = \ln 3x$ possibly have the same derivative at each point? (Differentiate them to find out.) What about $y = \ln kx$, for other positive values of the constant k? Give reasons for your answer.

Grapher Explorations

87. Graph $\ln x$, $\ln 2x$, $\ln 4x$, $\ln 8x$, and $\ln 16x$ (as many as you can) together for $0 < x \leq 10$. What is going on? Explain.

88. Graph $y = \ln|\sin x|$ in the window $0 \leq x \leq 22$, $-2 \leq y \leq 0$. Explain what you see. How could you change the formula to turn the arches upside down?

89. **a)** Graph $y = \sin x$ and the curves $y = \ln(a + \sin x)$ for $a = 2, 4, 8, 20$, and 50 together for $0 \leq x \leq 23$.
b) Why do the curves flatten as a increases? (*Hint:* Find an a-dependent upper bound for $|y'|$.)

90. Does the graph of $y = \sqrt{x} - \ln x$, $x > 0$, have an inflection point? Try to answer the question (a) by graphing, (b) by using calculus.

6.3 The Exponential Function

Whenever we have a quantity y whose rate of change over time is proportional to the amount of y present, we have a function that satisfies the differential equation

$$\frac{dy}{dt} = ky.$$

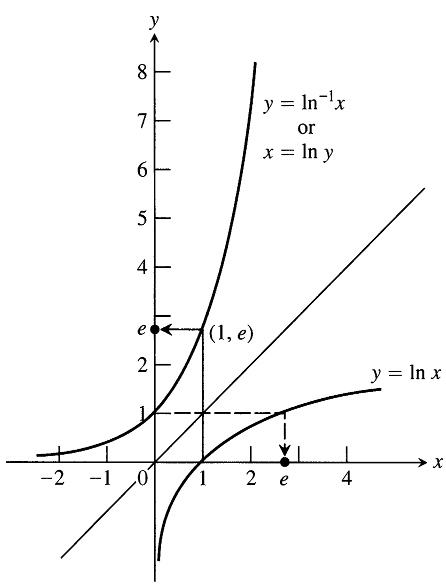
If, in addition, $y = y_0$ when $t = 0$, the function is the exponential function $y = y_0 e^{kt}$. This section defines the exponential function (it is the inverse of $\ln x$) and explores the properties that account for the amazing frequency with which the function appears in mathematics and its applications. We will look at some of these applications in Section 6.5.

The Inverse of ln *x* and the Number *e*

The function $\ln x$, being an increasing function of x with domain $(0, \infty)$ and range $(-\infty, \infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. The graph of $\ln^{-1} x$ is the graph of $\ln x$ reflected across the line $y = x$. As you can see,

$$\lim_{x\to\infty} \ln^{-1} x = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} \ln^{-1} x = 0.$$

The number $\ln^{-1} 1$ is denoted by the letter e (Fig. 6.11).



6.11 The graphs of $y = \ln x$ and $y = \ln^{-1} x$. The number e is $\ln^{-1} 1$.

Definition

$$e = \ln^{-1} 1$$

Although e is not a rational number, we will see in Chapter 8 that it is possible to find its value with a computer to as many places as we want with the formula

$$e = \lim_{n\to\infty} \left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n!}\right).$$

To 15 places,

$$e = 2.7\ 1828\ 1828\ 45\ 90\ 45.$$

The Function $y = e^x$

We can raise the number e to a rational power x in the usual way:

$$e^2 = e \cdot e, \qquad e^{-2} = \frac{1}{e^2}, \qquad e^{1/2} = \sqrt{e},$$

and so on. Since e is positive, e^x is positive too. This means that e^x has a logarithm. When we take the logarithm we find that

$$\ln e^x = x \ln e = x \cdot 1 = x. \tag{1}$$

Since $\ln x$ is one-to-one and $\ln(\ln^{-1} x) = x$, Eq. (1) tells us that

$$e^x = \ln^{-1} x \qquad \text{for } x \text{ rational.} \tag{2}$$

Equation (2) provides a way to extend the definition of e^x to irrational values of x. The function $\ln^{-1} x$ is defined for all x, so we can use it to assign a value to e^x at every point where e^x had no previous value.

Typical Values of e^x

x	e^x (rounded)
-1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

Definition

For every real number x, $e^x = \ln^{-1} x$.

Equations Involving $\ln x$ and e^x

Since $\ln x$ and e^x are inverses of one another, we have

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0) \tag{3}$$

$$\ln(e^x) = x \qquad (\text{all } x) \tag{4}$$

You might want to do parts of the next example on your calculator.

EXAMPLE 1

a) $\ln e^2 = 2$
b) $\ln e^{-1} = -1$
c) $\ln \sqrt{e} = \frac{1}{2}$
d) $\ln e^{\sin x} = \sin x$
e) $e^{\ln 2} = 2$
f) $e^{\ln(x^2+1)} = x^2 + 1$
g) $e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$ One way
h) $e^{3\ln 2} = (e^{\ln 2})^3 = 2^3 = 8$ Another way ❑

Useful Operating Rules

1. To remove logarithms from an equation, exponentiate both sides.
2. To remove exponentials, take the logarithm of both sides.

EXAMPLE 2 Find y if $\ln y = 3t + 5$.

Solution Exponentiate both sides:

$$e^{\ln y} = e^{3t+5}$$

$$y = e^{3t+5}. \qquad \text{Eq. (3)}$$

❑

EXAMPLE 3 Find k if $e^{2k} = 10$.

Solution Take the natural logarithm of both sides:

$$e^{2k} = 10$$

$$\ln e^{2k} = \ln 10$$

$$2k = \ln 10 \qquad \text{Eq. (4)}$$

$$k = \frac{1}{2} \ln 10.$$

❑

Laws of Exponents

Even though e^x is defined in a seemingly roundabout way as $\ln^{-1} x$, it obeys the familiar laws of exponents from algebra (Table 6.2).

Table 6.2 Laws of exponents for e^x

For all numbers x, x_1, and x_2,

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1x_2} = (e^{x_2})^{x_1}$

Proof of Law 1 Let

$$y_1 = e^{x_1} \quad \text{and} \quad y_2 = e^{x_2}. \tag{5}$$

Then

$$x_1 = \ln y_1 \quad \text{and} \quad x_2 = \ln y_2 \qquad \text{Take logs of both sides of Eqs. (5).}$$

$$x_1 + x_2 = \ln y_1 + \ln y_2$$

$$= \ln y_1 y_2 \qquad \text{Product Rule}$$

$$e^{x_1+x_2} = e^{\ln y_1 y_2} \qquad \text{Exponentiate.}$$

$$= y_1 y_2 \qquad e^{\ln u} = u$$

$$= e^{x_1} e^{x_2}.$$

❑

The proof of Law 4 is similar. Laws 2 and 3 follow from Law 1 (Exercise 78).

EXAMPLE 4

a) $e^{x+\ln 2} = e^x \cdot e^{\ln 2} = 2e^x$ Law 1

b) $e^{-\ln x} = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$ Law 2

c) $\dfrac{e^{2x}}{e} = e^{2x-1}$ Law 3

d) $(e^3)^x = e^{3x} = (e^x)^3$ Law 4

❑

The Derivative and Integral of e^x

The exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero. Starting with $y = e^x$, we have, in order,

$$y = e^x$$

$$\ln y = x \qquad \text{Logarithms of both sides}$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \qquad \text{Derivatives of both sides with respect to } x$$

$$\frac{dy}{dx} = y$$

$$\frac{dy}{dx} = e^x. \qquad y \text{ replaced by } e^x$$

The startling conclusion we draw from this sequence of equations is that e^x is its own derivative.

As we will see in Section 6.5, the only functions that behave this way are constant multiples of e^x.

$$\frac{d}{dx}e^x = e^x \tag{6}$$

EXAMPLE 5

$$\frac{d}{dx}(5e^x) = 5\frac{d}{dx}e^x = 5e^x$$

The Chain Rule extends Eq. (6) in the usual way to a more general form.

If u is any differentiable function of x, then

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}. \tag{7}$$

EXAMPLE 6

a) $\dfrac{d}{dx}e^{-x} = e^{-x}\dfrac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ Eq. (7) with $u = -x$

b) $\dfrac{d}{dx}e^{\sin x} = e^{\sin x}\dfrac{d}{dx}(\sin x) = e^{\sin x} \cdot \cos x$ Eq. (7) with $u = \sin x$

The integral equivalent of Eq. (7) is

Transcendental numbers and transcendental functions

Numbers that are solutions of polynomial equations with rational coefficients are called **algebraic:** -2 is algebraic because it satisfies the equation $x + 2 = 0$, and $\sqrt{3}$ is algebraic because it satisfies the equation $x^2 - 3 = 0$. Numbers that are not algebraic are called **transcendental,** a term coined by Euler to describe numbers, like e and π, that appeared to "transcend the power of algebraic methods." But it was not until a hundred years after Euler's death (1873) that Charles Hermite proved the transcendence of e in the sense that we describe. A few years later (1882), C. L. F. Lindemann proved the transcendence of π.

Today we call a function $y = f(x)$ algebraic if it satisfies an equation of the form

$$P_n y^n + \cdots + P_1 y + P_0 = 0$$

in which the P's are polynomials in x with rational coefficients. The function $y = 1/\sqrt{x+1}$ is algebraic because it satisfies the equation $(x+1)y^2 - 1 = 0$. Here the polynomials are $P_2 = x + 1$, $P_1 = 0$, and $P_0 = -1$. Polynomials and rational functions with rational coefficients are algebraic, as are all sums, products, quotients, rational powers, and rational roots of algebraic functions.

Functions that are not algebraic are called transcendental. The six basic trigonometric functions are transcendental, as are the inverses of the trigonometric functions and the exponential and logarithmic functions that are the main subject of the present chapter.

$$\int e^u \, du = e^u + C.$$

EXAMPLE 7

$$\int_0^{\ln 2} e^{3x} dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3} du \qquad u = 3x, \quad \frac{1}{3} du = dx, \ u(0) = 0,$$
$$u(\ln 2) = 3 \ln 2 = \ln 2^3 = \ln 8$$
$$= \frac{1}{3} \int_0^{\ln 8} e^u du$$
$$= \frac{1}{3} e^u \Big]_0^{\ln 8}$$
$$= \frac{1}{3}[8 - 1] = \frac{7}{3}$$

❑

EXAMPLE 8

$$\int_0^{\pi/2} e^{\sin x} \cos x \, dx = e^{\sin x} \Big]_0^{\pi/2} \qquad \text{Antiderivative from Example 6}$$
$$= e^1 - e^0 = e - 1$$

❑

EXAMPLE 9 *Solving an initial value problem*

Solve the initial value problem

$$e^y \frac{dy}{dx} = 2x, \qquad x > \sqrt{3}; \qquad y(2) = 0.$$

Solution We integrate both sides of the differential equation with respect to x to obtain

$$e^y = x^2 + C.$$

We use the initial condition to determine C:

$$C = e^0 - (2)^2$$
$$= 1 - 4 = -3.$$

This completes the formula for e^y:

$$e^y = x^2 - 3. \tag{8}$$

To find y, we take logarithms of both sides:

$$\ln e^y = \ln (x^2 - 3)$$
$$y = \ln (x^2 - 3). \tag{9}$$

Notice that the solution is valid for $x > \sqrt{3}$.

It is always a good idea to check a solution in the original equation. From Eqs. (8) and (9), we have

$$
\begin{aligned}
e^y \frac{dy}{dx} &= e^y \frac{d}{dx} \ln (x^2 - 3) && \text{Eq. (9)} \\
&= e^y \frac{2x}{x^2 - 3} \\
&= (x^2 - 3) \frac{2x}{x^2 - 3} && \text{Eq. (8)} \\
&= 2x.
\end{aligned}
$$

The solution checks. ❑

Exercises 6.3

Algebraic Calculations with the Exponential and Logarithm

Find simpler expressions for the quantities in Exercises 1–4.

1. a) $e^{\ln 7.2}$ **b)** $e^{-\ln x^2}$ **c)** $e^{\ln x - \ln y}$

2. a) $e^{\ln (x^2+y^2)}$ **b)** $e^{-\ln 0.3}$ **c)** $e^{\ln \pi x - \ln 2}$

3. a) $2 \ln \sqrt{e}$ **b)** $\ln (\ln e^e)$ **c)** $\ln (e^{-x^2-y^2})$

4. a) $\ln (e^{\sec \theta})$ **b)** $\ln (e^{(e^x)})$ **c)** $\ln (e^{2 \ln x})$

Solving Equations with Logarithmic or Exponential Terms

In Exercises 5–10, solve for y in terms of t or x, as appropriate.

5. $\ln y = 2t + 4$ **6.** $\ln y = -t + 5$

7. $\ln (y - 40) = 5t$ **8.** $\ln (1 - 2y) = t$

9. $\ln (y - 1) - \ln 2 = x + \ln x$

10. $\ln (y^2 - 1) - \ln (y + 1) = \ln (\sin x)$

In Exercises 11 and 12, solve for k.

11. a) $e^{2k} = 4$ **b)** $100e^{10k} = 200$ **c)** $e^{k/1000} = a$

12. a) $e^{5k} = \frac{1}{4}$ **b)** $80e^k = 1$ **c)** $e^{(\ln 0.8)k} = 0.8$

In Exercises 13–16, solve for t.

13. a) $e^{-0.3t} = 27$ **b)** $e^{kt} = \frac{1}{2}$ **c)** $e^{(\ln 0.2)t} = 0.4$

14. a) $e^{-0.01t} = 1000$ **b)** $e^{kt} = \frac{1}{10}$ **c)** $e^{(\ln 2)t} = \frac{1}{2}$

15. $e^{\sqrt{t}} = x^2$ **16.** $e^{(x^2)} e^{(2x+1)} = e^t$

Derivatives

In Exercises 17–36, find the derivative of y with respect to x, t, or θ, as appropriate.

17. $y = e^{-5x}$ **18.** $y = e^{2x/3}$

19. $y = e^{5-7x}$ **20.** $y = e^{(4\sqrt{x}+x^2)}$

21. $y = xe^x - e^x$ **22.** $y = (1 + 2x)e^{-2x}$

23. $y = (x^2 - 2x + 2)e^x$ **24.** $y = (9x^2 - 6x + 2)e^{3x}$

25. $y = e^\theta (\sin \theta + \cos \theta)$ **26.** $y = \ln (3\theta e^{-\theta})$

27. $y = \cos \left(e^{-\theta^2}\right)$ **28.** $y = \theta^3 e^{-2\theta} \cos 5\theta$

29. $y = \ln (3te^{-t})$ **30.** $y = \ln (2e^{-t} \sin t)$

31. $y = \ln \left(\frac{e^\theta}{1 + e^\theta}\right)$ **32.** $y = \ln \left(\frac{\sqrt{\theta}}{1 + \sqrt{\theta}}\right)$

33. $y = e^{(\cos t + \ln t)}$ **34.** $y = e^{\sin t}(\ln t^2 + 1)$

35. $y = \int_0^{\ln x} \sin e^t \, dt$ **36.** $y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t \, dt$

In Exercises 37–40, find dy/dx.

37. $\ln y = e^y \sin x$ **38.** $\ln xy = e^{x+y}$

39. $e^{2x} = \sin (x + 3y)$ **40.** $\tan y = e^x + \ln x$

Integrals

Evaluate the integrals in Exercises 41–62.

41. $\int (e^{3x} + 5e^{-x}) \, dx$ **42.** $\int (2e^x - 3e^{-2x}) \, dx$

43. $\int_{\ln 2}^{\ln 3} e^x \, dx$ **44.** $\int_{-\ln 2}^{0} e^{-x} \, dx$

45. $\int 8e^{(x+1)}\,dx$

46. $\int 2e^{(2x-1)}\,dx$

47. $\int_{\ln 4}^{\ln 9} e^{x/2}\,dx$

48. $\int_0^{\ln 16} e^{x/4}\,dx$

49. $\int \frac{e^{\sqrt{r}}}{\sqrt{r}}\,dr$

50. $\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}\,dr$

51. $\int 2t\,e^{-t^2}\,dt$

52. $\int t^3 e^{(t^4)}\,dt$

53. $\int \frac{e^{1/x}}{x^2}\,dx$

54. $\int \frac{e^{-1/x^2}}{x^3}\,dx$

55. $\int_0^{\pi/4} (1 + e^{\tan\theta})\sec^2\theta\,d\theta$

56. $\int_{\pi/4}^{\pi/2} (1 + e^{\cot\theta})\csc^2\theta\,d\theta$

57. $\int e^{\sec \pi t}\sec \pi t \tan \pi t\,dt$

58. $\int e^{\csc(\pi+t)}\csc(\pi+t)\cot(\pi+t)\,dt$

59. $\int_{\ln(\pi/6)}^{\ln(\pi/2)} 2e^v \cos e^v\,dv$

60. $\int_0^{\sqrt{\ln \pi}} 2x\,e^{x^2}\cos(e^{x^2})\,dx$

61. $\int \frac{e^r}{1+e^r}\,dr$

62. $\int \frac{dx}{1+e^x}$

Initial Value Problems

Solve the initial value problems in Exercises 63–66.

63. $\frac{dy}{dt} = e^t \sin(e^t - 2), \quad y(\ln 2) = 0$

64. $\frac{dy}{dt} = e^{-t}\sec^2(\pi e^{-t}), \quad y(\ln 4) = 2/\pi$

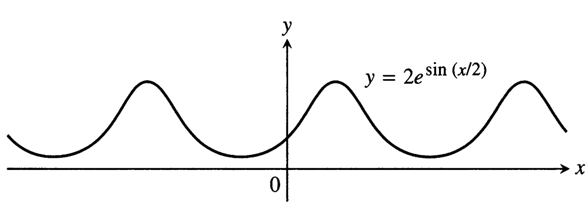
65. $\frac{d^2y}{dx^2} = 2e^{-x}, \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0$

66. $\frac{d^2y}{dt^2} = 1 - e^{2t}, \quad y(1) = -1 \quad \text{and} \quad y'(1) = 0$

Theory and Applications

67. Find the absolute maximum and minimum values of $f(x) = e^x - 2x$ on $[0, 1]$.

68. Where does the periodic function $f(x) = 2e^{\sin(x/2)}$ take on its extreme values and what are these values?



69. Find the absolute maximum value of $f(x) = x^2 \ln(1/x)$ and say where it is assumed.

70. GRAPHER Graph $f(x) = (x-3)^2 e^x$ and its first derivative together. Comment on the behavior of f in relation to the signs and values of f'. Identify significant points on the graphs with calculus, as necessary.

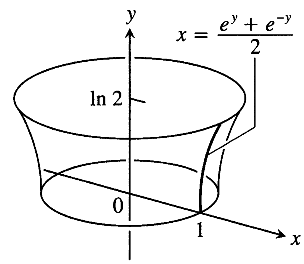
71. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{2x}$, below by the curve $y = e^x$, and on the right by the line $x = \ln 3$.

72. Find the area of the "triangular" region in the first quadrant that is bounded above by the curve $y = e^{x/2}$, below by the curve $y = e^{-x/2}$, and on the right by the line $x = 2\ln 2$.

73. Find a curve through the origin in the xy-plane whose length from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \frac{1}{4}e^x}\,dx.$$

74. Find the area of the surface generated by revolving the curve $x = (e^y + e^{-y})/2$, $0 \le y \le \ln 2$, about the y-axis.



75. a) Show that $\int \ln x\,dx = x\ln x - x + C$.
    b) Find the average value of $\ln x$ over $[1, e]$.

76. Find the average value of $f(x) = 1/x$ on $[1, 2]$.

77. *The linearization of* e^x *at* $x = 0$
    a) Derive the linear approximation $e^x \approx 1 + x$ at $x = 0$.
    b) CALCULATOR Estimate to 5 decimal places the magnitude of the error involved in replacing e^x by $1 + x$ on the interval $[0, 0.2]$.
    c) GRAPHER Graph e^x and $1 + x$ together for $-2 \le x \le 2$. Use different colors, if available. On what intervals does the approximation appear to overestimate e^x? underestimate e^x?

78. *Laws of Exponents.*
    a) Starting with the equation $e^{x_1}e^{x_2} = e^{x_1+x_2}$, derived in the text, show that $e^{-x} = 1/e^x$ for any real number x. Then show that $e^{x_1}/e^{x_2} = e^{x_1-x_2}$ for any numbers x_1 and x_2.
    b) Show that $(e^{x_1})^{x_2} = e^{x_1x_2} = (e^{x_2})^{x_1}$ for any numbers x_1 and x_2.

79. *A decimal representation of* e. Find e to as many decimal places as your calculator allows by solving the equation $\ln x = 1$.

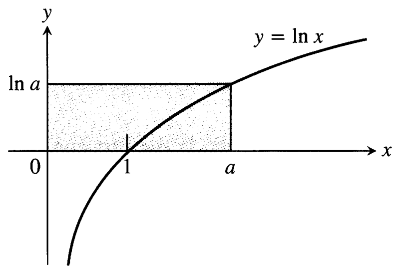
80. *The inverse relation between* e^x *and* $\ln x$. Find out how good your calculator is at evaluating the composites

$$e^{\ln x} \quad \text{and} \quad \ln(e^x).$$

81. Show that for any number $a > 1$

$$\int_1^a \ln x\,dx + \int_0^{\ln a} e^y dy = a \ln a.$$

(See accompanying figure.)



82. *The geometric, logarithmic, and arithmetic mean inequality*

a) Show that the graph of e^x is concave up over every interval of x-values.

b) Show, by reference to the accompanying figure, that if $0 < a < b$ then

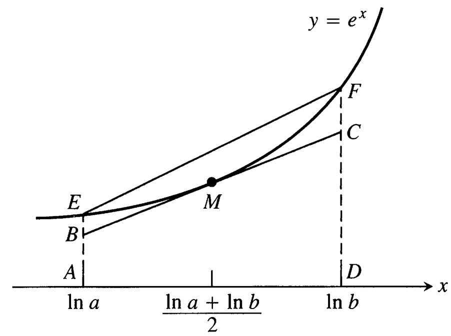
$$e^{(\ln a+\ln b)/2} \cdot (\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x\,dx < \frac{e^{\ln a}+e^{\ln b}}{2} \cdot (\ln b - \ln a).$$

c) Use the inequality in (b) to conclude that

$$\sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}.$$

This inequality says that the geometric mean of two positive numbers is less than their logarithmic mean, which in turn is less than their arithmetic mean.

(For more about this inequality, see "The Geometric, Logarithmic, and Arithmetic Mean Inequality" by Frank Burk, *American Mathematical Monthly,* Vol. 94, No. 6, June–July 1987, pp. 527–528.)



NOT TO SCALE

6.4 a^x and $\log_a x$

While we have not yet devised a way to raise positive numbers to any but rational powers, we have an exception in the number e. The definition $e^x = \ln^{-1} x$ defines e^x for every real value of x, irrational as well as rational. In this section, we show how this enables us to raise any other positive number to an arbitrary power and thus to define an exponential function $y = a^x$ for any positive number a. We also prove the Power Rule for differentiation in its final form (good for all exponents) and define functions like x^x and $(\sin x)^{\tan x}$ that involve raising the values of one function to powers given by another.

Just as e^x is but one of many exponential functions, $\ln x$ is one of many logarithmic functions, the others being the inverses of the function a^x. These logarithmic functions have important applications in science and engineering.

The Function a^x

Since $a = e^{\ln a}$ for any positive number a, we can think of a^x as $(e^{\ln a})^x = e^{x \ln a}$. We therefore make the following definition.

Definition

For any numbers $a > 0$ and x,

$$a^x = e^{x \ln a}. \tag{1}$$

Table 6.3 Laws of exponents

For $a > 0$, and any x and y:

1. $a^x \cdot a^y = a^{x+y}$
2. $a^{-x} = \dfrac{1}{a^x}$
3. $\dfrac{a^x}{a^y} = a^{x-y}$
4. $(a^x)^y = a^{xy} = (a^y)^x$

EXAMPLE 1

a) $2^{\sqrt{3}} = e^{\sqrt{3}\ln 2}$

b) $2^{\pi} = e^{\pi \ln 2}$ ❑

The function a^x obeys the usual laws of exponents (Table 6.3). We omit the proofs.

The Power Rule (Final Form)

We can now define x^n for any $x > 0$ and any real number n as $x^n = e^{n \ln x}$. Therefore, the n in the equation $\ln x^n = n \ln x$ no longer needs to be rational—it can be any number as long as $x > 0$:

$$\begin{aligned} \ln x^n &= \ln (e^{n \ln x}) = n \ln x \cdot \ln e && \ln e^u = u, \text{ any } u \\ &= n \ln x. \end{aligned}$$

Together, the law $a^x/a^y = a^{x-y}$ and the definition $x^n = e^{n \ln x}$ enable us to establish the Power Rule for differentiation in its final form. Differentiating x^n with respect to x gives

$$\begin{aligned} \frac{d}{dx}x^n &= \frac{d}{dx}e^{n \ln x} && \text{Definition of } x^n,\ x > 0 \\ &= e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) && \text{Chain Rule for } e^u \\ &= x^n \cdot \frac{n}{x} && \text{The definition again} \\ &= n x^{n-1}. && \text{Table 6.3, Law 3} \end{aligned}$$

In short, as long as $x > 0$,

$$\frac{d}{dx}x^n = n x^{n-1}.$$

The Chain Rule extends this equation to the Power Rule's final form.

Power Rule (Final Form)

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

EXAMPLE 2

a) $\dfrac{d}{dx}x^{\sqrt{2}} = \sqrt{2}x^{\sqrt{2}-1} \qquad (x > 0)$

b) $\dfrac{d}{dx}(\sin x)^{\pi} = \pi (\sin x)^{\pi - 1} \cos x \qquad (\sin x > 0)$ ❑

The Derivative of a^x

We start with the definition $a^x = e^{x \ln a}$:

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) \qquad \text{Chain Rule}$$
$$= a^x \ln a.$$

If $a > 0$, then

$$\frac{d}{dx}a^x = a^x \ln a.$$

With the Chain Rule, we get a more general form.

> If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and
> $$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}. \qquad (2)$$

Equation (2) shows why e^x is the exponential function preferred in calculus. If $a = e$, then $\ln a = 1$ and Eq. (2) simplifies to

$$\frac{d}{dx}e^x = e^x \ln e = e^x.$$

EXAMPLE 3

a) $\dfrac{d}{dx}3^x = 3^x \ln 3$

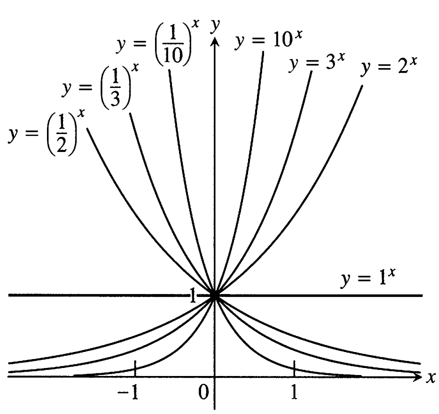
b) $\dfrac{d}{dx}3^{-x} = 3^{-x} \ln 3 \dfrac{d}{dx}(-x) = -3^{-x} \ln 3$

c) $\dfrac{d}{dx}3^{\sin x} = 3^{\sin x} \ln 3 \dfrac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$ ❑

From Eq. (2), we see that the derivative of a^x is positive if $\ln a > 0$, or $a > 1$, and negative if $\ln a < 0$, or $0 < a < 1$. Thus, a^x is an increasing function of x if $a > 1$ and a decreasing function of x if $0 < a < 1$. In each case, a^x is one-to-one. The second derivative

$$\frac{d^2}{dx^2}(a^x) = \frac{d}{dx}(a^x \ln a) = (\ln a)^2 a^x$$

is positive for all x, so the graph of a^x is concave up on every interval of the real line (Fig. 6.12).



6.12 Exponential functions decrease if $0 < a < 1$ and increase if $a > 1$. As $x \to \infty$, we have $a^x \to 0$ if $0 < a < 1$ and $a^x \to \infty$ if $a > 1$. As $x \to -\infty$, we have $a^x \to \infty$ if $0 < a < 1$ and $a^x \to 0$ if $a > 1$.

Other Power Functions

The ability to raise positive numbers to arbitrary real powers makes it possible to define functions like x^x and $x^{\ln x}$ for $x > 0$. We find the derivatives of such functions by rewriting the functions as powers of e.

EXAMPLE 4 Find dy/dx if $y = x^x, \quad x > 0$.

Solution Write x^x as a power of e:

$$y = x^x = e^{x \ln x}. \qquad \text{Eq. (1) with } a = x$$

Then differentiate as usual:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} e^{x \ln x} \\ &= e^{x \ln x} \frac{d}{dx}(x \ln x) \\ &= x^x \left(x \cdot \frac{1}{x} + \ln x \right) \\ &= x^x (1 + \ln x). \end{aligned}$$

❑

The Integral of a^u

If $a \neq 1$, so that $\ln a \neq 0$, we can divide both sides of Eq. (2) by $\ln a$ to obtain

$$a^u \frac{du}{dx} = \frac{1}{\ln a} \frac{d}{dx}(a^u).$$

Integrating with respect to x then gives

$$\int a^u \frac{du}{dx} dx = \int \frac{1}{\ln a} \frac{d}{dx}(a^u)\, dx = \frac{1}{\ln a} \int \frac{d}{dx}(a^u)\, dx = \frac{1}{\ln a} a^u + C.$$

Writing the first integral in differential form gives

$$\boxed{\int a^u du = \frac{a^u}{\ln a} + C. \qquad (3)}$$

EXAMPLE 5

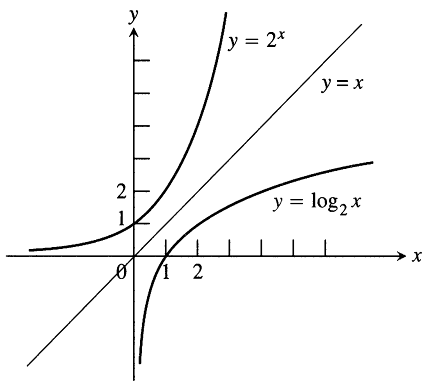
a) $$\int 2^x \, dx = \frac{2^x}{\ln 2} + C \qquad \text{Eq. (3) with } a = 2,\ u = x$$

b) $$\begin{aligned} &\int 2^{\sin x} \cos x \, dx \\ &= \int 2^u du = \frac{2^u}{\ln 2} + C \\ &= \frac{2^{\sin x}}{\ln 2} + C \qquad u = \sin x \text{ in Eq. (3)} \end{aligned}$$

❑

Logarithms with Base a

As we saw earlier, if a is any positive number other than 1, the function a^x is one-to-one and has a nonzero derivative at every point. It therefore has a differentiable inverse. We call the inverse the **logarithm of x with base a** and denote it by **$\log_a x$.**



6.13 The graph of 2^x and its inverse, $\log_2 x$.

Definition

For any positive number $a \neq 1$,

$$\log_a x = \text{ inverse of } a^x.$$

The graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the line $y = x$ (Fig. 6.13).

Since $\log_a x$ and a^x are inverses of one another, composing them in either order gives the identity function.

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0) \tag{4}$$

$$\log_a(a^x) = x \qquad (\text{all } x) \tag{5}$$

EXAMPLE 6

a) $\log_2(2^5) = 5$

b) $\log_{10}(10^{-7}) = -7$

c) $2^{\log_2(3)} = 3$

d) $10^{\log_{10}(4)} = 4$ ❑

The Evaluation of $\log_a x$

The evaluation of $\log_a x$ is simplified by the observation that $\log_a x$ is a numerical multiple of $\ln x$.

$$\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a} \tag{6}$$

We can derive Eq. (6) from Eq. (4):

$a^{\log_a(x)} = x$	Eq. (4)
$\ln a^{\log_a(x)} = \ln x$	Take the natural logarithm of both sides.
$\log_a(x) \cdot \ln a = \ln x$	The Power Rule in Table 6.1
$\log_a x = \dfrac{\ln x}{\ln a}$	Solve for $\log_a x$.

EXAMPLE 7

$$\log_{10} 2 = \frac{\ln 2}{\ln 10} \approx \frac{0.69315}{2.30259} \approx 0.30103$$ ❑

Table 6.4
Properties of base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:*
   $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule:*
   $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Reciprocal Rule:*
   $\log_a \frac{1}{y} = -\log_a y$
4. *Power Rule:*
   $\log_a x^y = y \log_a x$

The arithmetic properties of $\log_a x$ are the same as the ones for $\ln x$ (Table 6.4). These rules can be proved by dividing the corresponding rules for the natural logarithm function by $\ln a$. For example,

$$\ln xy = \ln x + \ln y \qquad \text{Rule 1 for natural logarithms ...}$$

$$\frac{\ln xy}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a} \qquad \text{... divided by } \ln a \text{ ...}$$

$$\log_a xy = \log_a x + \log_a y. \qquad \text{... gives Rule 1 for base } a \text{ logarithms.}$$

The Derivative of $\log_a u$

To find the derivative of a base a logarithm, we first convert it to a natural logarithm. If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}.$$

$$\boxed{\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}} \qquad (7)$$

EXAMPLE 8

$$\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10} \cdot \frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$$ ❑

Integrals Involving $\log_a x$

To evaluate integrals involving base a logarithms, we convert them to natural logarithms.

EXAMPLE 9

$$\int \frac{\log_2 x}{x}dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}dx \qquad \log_2 x = \frac{\ln x}{\ln 2}$$

$$= \frac{1}{\ln 2}\int u\,du \qquad u = \ln x, \quad du = \frac{1}{x}dx$$

$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$ ❑

*Base 10 Logarithms

Base 10 logarithms, often called **common logarithms,** appear in many scientific formulas. For example, earthquake intensity is often reported on the logarithmic **Richter scale.** Here the formula is

$$\text{Magnitude } R = \log_{10}\left(\frac{a}{T}\right) + B,$$

where a is the amplitude of the ground motion in microns at the receiving station, T is the period of the seismic wave in seconds, and B is an empirical factor that allows for the weakening of the seismic wave with increasing distance from the epicenter of the earthquake.

EXAMPLE 10 For an earthquake 10,000 km from the receiving station, $B = 6.8$. If the recorded vertical ground motion is $a = 10$ microns and the period is $T = 1$ sec, the earthquake's magnitude is

$$R = \log_{10}\left(\frac{10}{1}\right) + 6.8 = 1 + 6.8 = 7.8.$$

An earthquake of this magnitude does great damage near its epicenter. ❑

Most foods are acidic (pH < 7).

Food	pH Value
Bananas	4.5–4.7
Grapefruit	3.0–3.3
Oranges	3.0–4.0
Limes	1.8–2.0
Milk	6.3–6.6
Soft drinks	2.0–4.0
Spinach	5.1–5.7

The **pH scale** for measuring the acidity of a solution is a base 10 logarithmic scale. The pH value (hydrogen potential) of the solution is the common logarithm of the reciprocal of the solution's hydronium ion concentration, $[H_3O^+]$:

$$\text{pH} = \log_{10} \frac{1}{[H_3O^+]} = -\log_{10} [H_3O^+].$$

The hydronium ion concentration is measured in moles per liter. Vinegar has a pH of 3, distilled water a pH of 7, seawater a pH of 8.15, and household ammonia a pH of 12. The total scale ranges from about 0.1 for normal hydrochloric acid to 14 for a normal (1 N) solution of sodium hydroxide.

Another example of the use of common logarithms is the **decibel** or db ("dee bee") **scale** for measuring loudness. If I is the **intensity** of sound in watts per square meter, the decibel level of the sound is

$$\textbf{Sound level} = 10 \log_{10} (I \times 10^{12}) \text{ db}. \tag{8}$$

If you ever wondered why doubling the power of your audio amplifier increases the sound level by only a few decibels, Eq. (8) provides the answer. As the following example shows, doubling I adds only about 3 db.

Typical sound levels

Threshold of hearing	0 db
Rustle of leaves	10 db
Average whisper	20 db
Quiet automobile	50 db
Ordinary conversation	65 db
Pneumatic drill 10 feet away	90 db
Threshold of pain	120 db

EXAMPLE 11 Doubling I in Eq. (8) adds about 3 db. Writing log for $\log_{10}$ (a common practice), we have

$$\begin{aligned}
\text{Sound level with } I \text{ doubled} &= 10 \log (2I \times 10^{12}) && \text{Eq. (8) with } 2I \text{ for } I\\
&= 10 \log (2 \cdot I \times 10^{12})\\
&= 10 \log 2 + 10 \log (I \times 10^{12})\\
&= \text{original sound level} + 10 \log 2\\
&\approx \text{original sound level} + 3. && \log_{10} 2 \approx 0.30
\end{aligned}$$

❑

Exercises 6.4

Algebraic Calculations

Simplify the expressions in Exercises 1–4.

1. a) $5^{\log_5 7}$ b) $8^{\log_8 \sqrt{2}}$ c) $1.3^{\log_{1.3} 75}$ d) $\log_4 16$ e) $\log_3 \sqrt{3}$ f) $\log_4\left(\frac{1}{4}\right)$

2. a) $2^{\log_2 3}$ b) $10^{\log_{10}(1/2)}$ c) $\pi^{\log_\pi 7}$

d) $\log_{11} 121$ e) $\log_{121} 11$ f) $\log_3\left(\frac{1}{9}\right)$

3. a) $2^{\log_4 x}$ b) $9^{\log_3 x}$ c) $\log_2(e^{(\ln 2)(\sin x)})$

4. a) $25^{\log_5(3x^2)}$ b) $\log_e(e^x)$ c) $\log_4(2^{e^x \sin x})$

Express the ratios in Exercises 5 and 6 as ratios of natural logarithms and simplify.

5. a) $\dfrac{\log_2 x}{\log_3 x}$ b) $\dfrac{\log_2 x}{\log_8 x}$ c) $\dfrac{\log_x a}{\log_{x^2} a}$

6. a) $\dfrac{\log_9 x}{\log_3 x}$ b) $\dfrac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x}$ c) $\dfrac{\log_a b}{\log_b a}$

Solve the equations in Exercises 7–10 for x.

7. $3^{\log_3(7)} + 2^{\log_2(5)} = 5^{\log_5(x)}$
8. $8^{\log_8(3)} - e^{\ln 5} = x^2 - 7^{\log_7(3x)}$
9. $3^{\log_3(x^2)} = 5e^{\ln x} - 3 \cdot 10^{\log_{10}(2)}$
10. $\ln e + 4^{-2\log_4(x)} = \dfrac{1}{x}\log_{10}(100)$

Derivatives

In Exercises 11–38, find the derivative of y with respect to the given independent variable.

11. $y = 2^x$
12. $y = 3^{-x}$
13. $y = 5^{\sqrt{s}}$
14. $y = 2^{(s^2)}$
15. $y = x^{\pi}$
16. $y = t^{1-e}$
17. $y = (\cos\theta)^{\sqrt{2}}$
18. $y = (\ln\theta)^{\pi}$
19. $y = 7^{\sec\theta}\ln 7$
20. $y = 3^{\tan\theta}\ln 3$
21. $y = 2^{\sin 3t}$
22. $y = 5^{-\cos 2t}$
23. $y = \log_2 5\theta$
24. $y = \log_3(1 + \theta\ln 3)$
25. $y = \log_4 x + \log_4 x^2$
26. $y = \log_{25} e^x - \log_5\sqrt{x}$
27. $y = \log_2 r \cdot \log_4 r$
28. $y = \log_3 r \cdot \log_9 r$
29. $y = \log_3\left(\left(\dfrac{x+1}{x-1}\right)^{\ln 3}\right)$
30. $y = \log_5\sqrt{\left(\dfrac{7x}{3x+2}\right)^{\ln 5}}$
31. $y = \theta\sin(\log_7\theta)$
32. $y = \log_7\left(\dfrac{\sin\theta\cos\theta}{e^{\theta}2^{\theta}}\right)$
33. $y = \log_5 e^x$
34. $y = \log_2\left(\dfrac{x^2e^2}{2\sqrt{x+1}}\right)$
35. $y = 3^{\log_2 t}$
36. $y = 3\log_8(\log_2 t)$
37. $y = \log_2(8t^{\ln 2})$
38. $y = t\log_3\left(e^{(\sin t)(\ln 3)}\right)$

Logarithmic Differentiation

In Exercises 39–46, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

39. $y = (x+1)^x$
40. $y = x^{(x+1)}$
41. $y = (\sqrt{t})^t$
42. $y = t^{\sqrt{t}}$
43. $y = (\sin x)^x$
44. $y = x^{\sin x}$
45. $y = x^{\ln x}$
46. $y = (\ln x)^{\ln x}$

Integration

Evaluate the integrals in Exercises 47–56

47. $\displaystyle\int 5^x\,dx$
48. $\displaystyle\int (1.3)^x\,dx$
49. $\displaystyle\int_0^1 2^{-\theta}\,d\theta$
50. $\displaystyle\int_{-2}^0 5^{-\theta}\,d\theta$
51. $\displaystyle\int_1^{\sqrt{2}} x2^{(x^2)}\,dx$
52. $\displaystyle\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$
53. $\displaystyle\int_0^{\pi/2} 7^{\cos t}\sin t\,dt$
54. $\displaystyle\int_0^{\pi/4}\left(\frac{1}{3}\right)^{\tan t}\sec^2 t\,dt$
55. $\displaystyle\int_2^4 x^{2x}(1+\ln x)\,dx$
56. $\displaystyle\int_1^2 \frac{2^{\ln x}}{x}\,dx$

Evaluate the integrals in Exercises 57–60.

57. $\displaystyle\int 3x^{\sqrt{3}}dx$
58. $\displaystyle\int x^{\sqrt{2}-1}dx$
59. $\displaystyle\int_0^3 (\sqrt{2}+1)x^{\sqrt{2}}dx$
60. $\displaystyle\int_1^e x^{(\ln 2)-1}dx$

Evaluate the integrals in Exercises 61–70.

61. $\displaystyle\int \frac{\log_{10} x}{x}\,dx$
62. $\displaystyle\int_1^4 \frac{\log_2 x}{x}\,dx$
63. $\displaystyle\int_1^4 \frac{\ln 2\log_2 x}{x}\,dx$
64. $\displaystyle\int_1^e \frac{2\ln 10\log_{10} x}{x}\,dx$
65. $\displaystyle\int_0^2 \frac{\log_2(x+2)}{x+2}\,dx$
66. $\displaystyle\int_{1/10}^{10} \frac{\log_{10}(10x)}{x}\,dx$
67. $\displaystyle\int_0^9 \frac{2\log_{10}(x+1)}{x+1}\,dx$
68. $\displaystyle\int_2^3 \frac{2\log_2(x-1)}{x-1}\,dx$
69. $\displaystyle\int \frac{dx}{x\log_{10} x}$
70. $\displaystyle\int \frac{dx}{x(\log_8 x)^2}$

Evaluate the integrals in Exercises 71–74.

71. $\displaystyle\int_1^{\ln x}\frac{1}{t}dt,\quad x > 1$
72. $\displaystyle\int_1^{e^x}\frac{1}{t}\,dt$
73. $\displaystyle\int_1^{1/x}\frac{1}{t}\,dt,\quad x > 0$
74. $\displaystyle\frac{1}{\ln a}\int_1^x\frac{1}{t}\,dt,\quad x > 0$

Theory and Applications

75. Find the area of the region between the curve $y = 2x/(1+x^2)$ and the interval $-2 \le x \le 2$ of the x-axis.
76. Find the area of the region between the curve $y = 2^{1-x}$ and the interval $-1 \le x \le 1$ of the x-axis.

77. *Blood pH.* The pH of human blood normally falls between 7.37 and 7.44. Find the corresponding bounds for $[H_3O^+]$.

78. *Brain fluid pH.* The cerebrospinal fluid in the brain has a hydronium ion concentration of about $[H_3O^+] = 4.8 \times 10^{-8}$ moles per liter. What is the pH?

79. *Audio amplifiers.* By what factor k do you have to multiply the intensity of I of the sound from your audio amplifier to add 10 db to the sound level?

80. *Audio amplifiers.* You multiplied the intensity of the sound of your audio system by a factor of 10. By how many decibels did this increase the sound level?

81. In any solution, the product of the hydronium ion concentration $[H_3O^+]$ (moles/L) and the hydroxyl ion concentration $[OH^-]$ (moles/L) is about 10^{-14}.

a) What value of $[H_3O^+]$ minimizes the sum of the concentrations, $S = [H_3O^+] + [OH^-]$? (*Hint:* Change notation. Let $x = [H_3O^+]$.)

b) What is the pH of a solution in which S has this minimum value?

c) What ratio of $[H_3O^+]$ to $[OH^-]$ minimizes S?

82. Could $\log_a b$ possibly equal $1/\log_b a$? Give reasons for your answer.

Grapher Explorations

83. The equation $x^2 = 2^x$ has three solutions: $x = 2$, $x = 4$, and one other. Estimate the third solution as accurately as you can by graphing.

84. Could $x^{\ln 2}$ possibly be the same as $2^{\ln x}$ for $x > 0$? Graph the two functions and explain what you see.

85. *The linearization of 2^x*

a) Find the linearization of $f(x) = 2^x$ at $x = 0$. Then round its coefficients to 2 decimal places.

b) Graph the linearization and function together for $-3 \le x \le 3$ and $-1 \le x \le 1$.

86. *The linearization of $\log_3 x$*

a) Find the linearization of $f(x) = \log_3 x$ at $x = 3$. Then round its coefficients to 2 decimal places.

b) Graph the linearization and function together in the window $0 \le x \le 8$ and $2 \le x \le 4$.

Calculations with Other Bases

87. CALCULATOR Most scientific calculators have keys for $\log_{10} x$ and $\ln x$. To find logarithms to other bases, we use the equation $\log_a x = (\ln x)/(\ln a)$.

To find $\log_2 x$, find $\ln x$ and divide by $\ln 2$:	$\log_2 5 = \dfrac{\ln 5}{\ln 2} \approx 2.3219.$
To find $\ln x$ given $\log_2 x$, multiply by $\ln 2$:	$\ln 5 = \log_2 5 \cdot \ln 2 \approx 1.6094.$

Find the following logarithms to 5-decimal places.

a) $\log_3 8$ **b)** $\log_7 0.5$
c) $\log_{20} 17$ **d)** $\log_{0.5} 7$
e) $\ln x$, given that $\log_{10} x = 2.3$
f) $\ln x$, given that $\log_2 x = 1.4$
g) $\ln x$, given that $\log_2 x = -1.5$
h) $\ln x$, given that $\log_{10} x = -0.7$

88. *Conversion factors*

a) Show that the equation for converting base 10 logarithms to base 2 logarithms is

$$\log_2 x = \frac{\ln 10}{\ln 2} \log_{10} x.$$

b) Show that the equation for converting base a logarithms to base b logarithms is

$$\log_b x = \frac{\ln a}{\ln b} \log_a x.$$

6.5 Growth and Decay

In this section, we derive the law of exponential change and describe some of the applications that account for the importance of logarithmic and exponential functions.

The Law of Exponential Change

To set the stage once again, suppose we are interested in a quantity y (velocity, temperature, electric current, whatever) that increases or decreases at a rate that at any given time t is proportional to the amount present. If we also know the amount present at time $t = 0$, call it y_0, we can find y as a function of t by solving the following initial value problem:

Differential equation:	$\dfrac{dy}{dt} = ky$	(1)
Initial condition:	$y = y_0 \quad \text{when} \quad t = 0.$	

If y is positive and increasing, then k is positive, and we use Eq. (1) to say that the rate of growth is proportional to what has already been accumulated. If y is positive and decreasing, then k is negative, and we use Eq. (1) to say that the rate of decay is proportional to the amount still left.

We see right away that the constant function $y = 0$ is a solution of Eq. (1). To find the nonzero solutions, we divide Eq. (1) by y:

$$\frac{1}{y} \cdot \frac{dy}{dt} = k$$

$$\ln|y| = kt + C \qquad \text{Integrate with respect to } t;\ \int (1/u)\,du = \ln|u| + C.$$

$$|y| = e^{kt+C} \qquad \text{Exponentiate.}$$

$$|y| = e^{C} \cdot e^{kt} \qquad e^{a+b} = e^{a} \cdot e^{b}$$

$$y = \pm e^{C} e^{kt} \qquad \text{If } |y| = r, \text{ then } y = \pm r.$$

$$y = Ae^{kt}. \qquad A \text{ is a more convenient name for } \pm e^{C}.$$

By allowing A to take on the value 0 in addition to all possible values $\pm e^{C}$, we can include the solution $y = 0$ in the formula.

We find the right value of A for the initial value problem by solving for A when $y = y_0$ and $t = 0$:

$$y_0 = Ae^{k \cdot 0} = A.$$

The solution of the initial value problem is therefore $y = y_0 e^{kt}$.

The Law of Exponential Change

$$y = y_0 e^{kt} \tag{2}$$

Growth: $k > 0$ Decay: $k < 0$

The number k is the **rate constant** of the equation.

The derivation of Eq. (2) shows that the only functions that are their own derivatives are constant multiples of the exponential function.

Population Growth

Strictly speaking, the number of individuals in a population (of people, plants, foxes, or bacteria, for example) is a discontinuous function of time because it takes on discrete values. However, as soon as the number of individuals becomes large enough, it can safely be described with a continuous or even differentiable function.

If we assume that the proportion of reproducing individuals remains constant and assume a constant fertility, then at any instant t the birth rate is proportional to the number $y(t)$ of individuals present. If, further, we neglect departures, arrivals, and deaths, the growth rate dy/dt will be the same as the birth rate ky. In other words, $dy/dt = ky$, so that $y = y_0 e^{kt}$. As with all kinds of growth, there may be limitations imposed by the surrounding environment, but we will not go into these here.

EXAMPLE 1 One model for the way diseases spread assumes that the rate dy/dt at which the number of infected people changes is proportional to the number y. The more infected people there are, the faster the disease will spread. The fewer there are, the slower it will spread.

Suppose that in the course of any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1000?

Solution We use the equation $y = y_0 e^{kt}$. There are three things to find:

1. the value of y_0,
2. the value of k,
3. the value of t that makes $y = 1000$.

Step 1: *The value of* y_0. We are free to count time beginning anywhere we want. If we count from today, then $y = 10{,}000$ when $t = 0$, so $y_0 = 10{,}000$. Our equation is now

$$y = 10{,}000\, e^{kt}. \tag{3}$$

Step 2: *The value of* k. When $t = 1$ year, the number of cases will be 80% of its present value, or 8000. Hence,

$$\begin{aligned} 8000 &= 10{,}000\, e^{k(1)} && \text{Eq. (3) with } t = 1 \text{ and } y = 8000 \\ e^k &= 0.8 \\ \ln(e^k) &= \ln 0.8 \\ k &= \ln 0.8. \end{aligned}$$

At any given time t,

$$y = 10{,}000\, e^{(\ln 0.8)t}. \tag{4}$$

Step 3: *The value of t that makes* $y = 1000$. We set y equal to 1000 in Eq. (4) and solve for t:

$$\begin{aligned} 1000 &= 10{,}000\, e^{(\ln 0.8)t} \\ e^{(\ln 0.8)t} &= 0.1 \\ (\ln 0.8)t &= \ln 0.1 && \text{Logs of both sides} \\ t &= \frac{\ln 0.1}{\ln 0.8} \approx 10.32 \text{ years.} \end{aligned}$$

It will take a little more than 10 years to reduce the number of cases to 1000. ❑

Continuously Compounded Interest

If you invest an amount A_0 of money at a fixed annual interest rate r (expressed as a decimal) and if interest is added to your account k times a year, it turns out that the amount of money you will have at the end of t years is

$$A_t = A_0\left(1+\frac{r}{k}\right)^{kt}. \tag{5}$$

The interest might be added ("compounded," bankers say) monthly ($k = 12$), weekly ($k = 52$), daily ($k = 365$), or even more frequently, say by the hour or by the minute. But there is still a limit to how much you will earn that way, and the limit is

$$\lim_{k\to\infty} A_t = \lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt}$$
$$= A_0 e^{rt}.$$

Evaluating

$$\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt}$$

involves what is called the indeterminate form 1^∞. We will see how to evaluate limits of this type in Section 6.6.

The resulting formula for the amount of money in your account after t years is

$$A(t) = A_0 e^{rt}. \tag{6}$$

Interest paid according to this formula is said to be **compounded continuously.** The number r is called the **continuous interest rate.**

EXAMPLE 2 Suppose you deposit \$621 in a bank account that pays 6% compounded continuously. How much money will you have 8 years later?

Solution We use Eq. (6) with $A_0 = 621$, $r = 0.06$, and $t = 8$:

$$A(8) = 621\,e^{(0.06)(8)} = 621\,e^{0.48} = 1003.58 \qquad \text{Nearest cent}$$

Had the bank paid interest quarterly ($k = 4$ in Eq. (5)), the amount in your account would have been \$1000.01. Thus the effect of continuous compounding, as compared with quarterly compounding, has been an addition of \$3.57. A bank might decide it would be worth this additional amount to be able to advertise, "We compound interest every second, night and day—better yet, we compound the interest continuously." ❑

Radioactivity

When an atom emits some of its mass as radiation, the remainder of the atom re-forms to make an atom of some new element. This process of radiation and change is called **radioactive decay,** and an element whose atoms go spontaneously through this process is called **radioactive.** Thus, radioactive carbon-14 decays into nitrogen; radium, through a number of intervening radioactive steps, decays into lead.

For radon-222 gas, t is measured in days and $k = 0.18$. For radium-226, which used to be painted on watch dials to make them glow at night (a dangerous practice), t is measured in years and $k = 4.3 \times 10^{-4}$. The decay of radium in the earth's crust is the source of the radon we sometimes find in our basements.

Experiments have shown that at any given time the rate at which a radioactive element decays (as measured by the number of nuclei that change per unit time) is approximately proportional to the number of radioactive nuclei present. Thus, the decay of a radioactive element is described by the equation $dy/dt = -ky$, $k > 0$. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

It is conventional to use $-k$ ($k > 0$) here instead of k ($k < 0$) to emphasize that y is decreasing.

$$y = y_0 e^{-kt}, \qquad k > 0.$$

EXAMPLE 3 *Half-life*

The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. It is a remarkable fact that the half-life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance.

To see why, let y_0 be the number of radioactive nuclei initially present in the sample. Then the number y present at any later time t will be $y = y_0 e^{-kt}$. We seek the value of t at which the number of radioactive nuclei present equals half the original number:

$$\begin{aligned} y_0 e^{-kt} &= \frac{1}{2} y_0 \\ e^{-kt} &= \frac{1}{2} \\ -kt &= \ln \frac{1}{2} = -\ln 2 && \text{Reciprocal Rule for logarithms} \\ t &= \frac{\ln 2}{k} \end{aligned}$$

This value of t is the half-life of the element. It depends only on the value of k; the number y_0 does not enter in. ❑

$$\boxed{\text{Half-life} = \frac{\ln 2}{k} \qquad (7)}$$

EXAMPLE 4 *Polonium-210*

The effective radioactive lifetime of polonium-210 is so short we measure it in days rather than years. The number of radioactive atoms remaining after t days in a sample that starts with y_0 radioactive atoms is

$$y = y_0 e^{-5 \times 10^{-3} t}.$$

Find the element's half-life.

Solution

$$\begin{aligned} \text{Half-life} &= \frac{\ln 2}{k} && \text{Eq. (7)} \\ &= \frac{\ln 2}{5 \times 10^{-3}} && \text{The } k \text{ from polonium's decay equation} \\ &\approx 139 \text{ days} \end{aligned}$$

❑

EXAMPLE 5 *Carbon-14*

People who do carbon-14 dating use a figure of 5700 years for its half-life (more about carbon-14 dating in the exercises). Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

Carbon-14 dating

The decay of radioactive elements can sometimes be used to date events from the Earth's past. The ages of rocks more than 2 billion years old have been measured by the extent of the radioactive decay of uranium (half-life 4.5 billion years!). In a living organism, the ratio of radioactive carbon, carbon-14, to ordinary carbon stays fairly constant during the lifetime of the organism, being approximately equal to the ratio in the organism's surroundings at the time. After the organism's death, however, no new carbon is ingested, and the proportion of carbon-14 in the organism's remains decreases as the carbon-14 decays. It is possible to estimate the ages of fairly old organic remains by comparing the proportion of carbon-14 they contain with the proportion assumed to have been in the organism's environment at the time it lived. Archaeologists have dated shells (which contain $CaCO_3$), seeds, and wooden artifacts this way. The estimate of 15,500 years for the age of the cave paintings at Lascaux, France, is based on carbon-14 dating. After generations of controversy, the Shroud of Turin, long believed by many to be the burial cloth of Christ, was shown by carbon-14 dating in 1988 to have been made after A.D. 1200.

Solution We use the decay equation $y = y_0 e^{-kt}$. There are two things to find:

1. the value of k,
2. the value of t when $y_0 e^{-kt} = 0.9 y_0$, or $e^{-kt} = 0.9$ — 90% of the radioactive nuclei still present

Step 1: *The value of k.* We use the half-life equation:

$$k = \frac{\ln 2}{\text{half-life}} = \frac{\ln 2}{5700} \qquad (\text{about } 1.2 \times 10^{-4})$$

Step 2: *The value of t that makes $e^{-kt} = 0.9$.*

$$\begin{aligned} e^{-kt} &= 0.9 \\ e^{-(\ln 2/5700)t} &= 0.9 \\ -\frac{\ln 2}{5700} t &= \ln 0.9 && \text{Logs of both sides} \\ t &= -\frac{5700 \ln 0.9}{\ln 2} \approx 866 \text{ years.} \end{aligned}$$

The sample is about 866 years old. ❑

Heat Transfer: Newton's Law of Cooling

Soup left in a tin cup cools to the temperature of the surrounding air. A hot silver ingot immersed in water cools to the temperature of the surrounding water. In situations like these, the rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium. This observation is called *Newton's law of cooling,* although it applies to warming as well, and there is an equation for it.

If T is the temperature of the object at time t, and T_S is the surrounding temperature, then

$$\frac{dT}{dt} = -k(T - T_S). \tag{8}$$

If we substitute y for $(T - T_S)$, then

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}(T - T_S) = \frac{dT}{dt} - \frac{d}{dt}(T_S) \\ &= \frac{dT}{dt} - 0 && T_S \text{ is a constant.} \\ &= \frac{dT}{dt} \end{aligned}$$

In terms of y, Eq. (8) therefore reads

$$\frac{dy}{dt} = -ky,$$

and we know that the solution to this differential equation is

$$y = y_0 e^{-kt}.$$

Thus, **Newton's law of cooling** is

$$T - T_S = (T_0 - T_S)e^{-kt}, \tag{9}$$

where T_0 is the value of T at time zero.

EXAMPLE 6 A hard-boiled egg at 98°C is put in a sink of 18°C water. After 5 minutes, the egg's temperature is 38°C. Assuming that the water has not warmed appreciably, how much longer will it take the egg to reach 20°C?

Solution We find how long it would take the egg to cool from 98°C to 20°C and subtract the 5 minutes that have already elapsed.

According to Eq. (9), the egg's temperature t minutes after it is put in the sink is

$$T = 18 + (98 - 18)e^{-kt} = 18 + 80e^{-kt}.$$

To find k, we use the information that $T = 38$ when $t = 5$:

$$\begin{aligned} 38 &= 18 + 80e^{-5k} \\ e^{-5k} &= \frac{1}{4} \\ -5k &= \ln\frac{1}{4} = -\ln 4 \\ k &= \frac{1}{5}\ln 4 = 0.2\ \ln 4 \qquad \text{(about 0.28).} \end{aligned}$$

The egg's temperature at time t is $T = 18 + 80e^{-(0.2\ln 4)t}$. Now find the time t when $T = 20$:

$$\begin{aligned} 20 &= 18 + 80e^{-(0.2\ln 4)t} \\ 80e^{-(0.2\ln 4)t} &= 2 \\ e^{-(0.2\ln 4)t} &= \frac{1}{40} \\ -(0.2\ln 4)t &= \ln\frac{1}{40} = -\ln 40 \\ t &= \frac{\ln 40}{0.2\ \ln 4} \approx 13\ \text{min.} \end{aligned}$$

The egg's temperature will reach 20°C about 13 min after it is put in water to cool. Since it took 5 min to reach 38°C, it will take about 8 min more to reach 20°C. ❑

Exercises 6.5

The answers to most of the following exercises are in terms of logarithms and exponentials. A calculator can be helpful, enabling you to express the answers in decimal form.

1. *Human evolution continues.* The analysis of tooth shrinkage by C. Loring Brace and colleagues at the University of Michigan's Museum of Anthropology indicates that human tooth size is continuing to decrease and that the evolutionary process did not come to a halt some 30,000 years ago as many scientists contend. In northern Europeans, for example, tooth size reduction now has a rate of 1% per 1000 years.

   a) If t represents time in years and y represents tooth size, use the condition that $y = 0.99y_0$ when $t = 1000$ to find the value of k in the equation $y = y_0 e^{kt}$. Then use this value of k to answer the following questions.

   b) In about how many years will human teeth be 90% of their present size?

   c) What will be our descendants' tooth size 20,000 years from now (as a percentage of our present tooth size)?

   (Source: *LSA Magazine,* Spring 1989, Vol. 12, No. 2, p. 19, Ann Arbor, MI.)

2. *Atmospheric pressure.* The earth's atmospheric pressure p is often modeled by assuming that the rate dp/dh at which p changes with the altitude h above sea level is proportional to p. Suppose that the pressure at sea level is 1013 millibars (about 14.7 pounds per square inch) and that the pressure at an altitude of 20 km is 90 millibars.

   a) Solve the initial value problem

   Differential equation: $dp/dh = kp$ (k a constant)

   Initial condition: $p = p_0$ when $h = 0$

   to express p in terms of h. Determine the values of p_0 and k from the given altitude-pressure data.

   b) What is the atmospheric pressure at $h = 50$ km?

   c) At what altitude does the pressure equal 900 millibars?

3. *First order chemical reactions.* In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount present. For the change of δ-glucono lactone into gluconic acid, for example,

$$\frac{dy}{dt} = -0.6y$$

   when t is measured in hours. If there are 100 grams of δ-glucono lactone present when $t = 0$, how many grams will be left after the first hour?

4. *The inversion of sugar.* The processing of raw sugar has a step called "inversion" that changes the sugar's molecular structure. Once the process has begun, the rate of change of the amount of raw sugar is proportional to the amount of raw sugar remaining. If 1000 kg of raw sugar reduces to 800 kg of raw sugar during the first 10 h, how much raw sugar will remain after another 14 h?

5. *Working underwater.* The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the differential equation

$$\frac{dL}{dx} = -kL.$$

   As a diver, you know from experience that diving to 18 ft in the Caribbean Sea cuts the intensity in half. You cannot work without artificial light when the intensity falls below one-tenth of the surface value. About how deep can you expect to work without artificial light?

6. *Voltage in a discharging capacitor.* Suppose that electricity is draining from a capacitor at a rate that is proportional to the voltage V across its terminals and that, if t is measured in seconds,

$$\frac{dV}{dt} = -\frac{1}{40}V.$$

   Solve this equation for V, using V_0 to denote the value of V when $t = 0$. How long will it take the voltage to drop to 10% of its original value?

7. *Cholera bacteria.* Suppose that the bacteria in a colony can grow unchecked, by the law of exponential change. The colony starts with 1 bacterium and doubles every half hour. How many bacteria will the colony contain at the end of 24 h? (Under favorable laboratory conditions, the number of cholera bacteria can double every 30 min. In an infected person, many bacteria are destroyed, but this example helps explain why a person who feels well in the morning may be dangerously ill by evening.)

8. *Growth of bacteria.* A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 h there are 10,000 bacteria. At the end of 5 h there are 40,000. How many bacteria were present initially?

9. *The incidence of a disease* (*Continuation of Example* 1). Suppose that in any given year the number of cases can be reduced by 25% instead of 20%.

   a) How long will it take to reduce the number of cases to 1000?

   b) How long will it take to eradicate the disease, that is, reduce the number of cases to less than 1?

10. *The U.S. population.* The Museum of Science in Boston displays a running total of the U.S. population. On May 11, 1993, the total was increasing at the rate of 1 person every 14 sec. The displayed population figure for 3:45 P.M. that day was 257,313,431.

    a) Assuming exponential growth at a constant rate, find the rate constant for the population's growth (people per 365-day year).

    b) At this rate, what will the U.S. population be at 3:45 P.M. Boston time on May 11, 2001?

11. *Oil depletion.* Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decreases at the continuous rate of 10% per year. When will the well's output fall to one-fifth of its present value?

12. *Continuous price discounting.* To encourage buyers to place 100-unit orders, your firm's sales department applies a continuous discount that makes the unit price a function $p(x)$ of the number of units x ordered. The discount decreases the price at the rate of \$0.01 per unit ordered. The price per unit for a 100-unit order is $p(100) = \$20.09$.

    a) Find $p(x)$ by solving the following initial value problem:

    Differential equation: $\dfrac{dp}{dx} = -\dfrac{1}{100}p$

    Initial condition: $p(100) = 20.09.$

    b) Find the unit price $p(10)$ for a 10-unit order and the unit price $p(90)$ for a 90-unit order.

    c) The sales department has asked you to find out if it is discounting so much that the firm's revenue, $r(x) = x \cdot p(x)$, will actually be less for a 100-unit order than, say, for a 90-unit order. Reassure them by showing that r has its maximum value at $x = 100$.

    d) GRAPHER Graph the revenue function $r(x) = xp(x)$ for $0 \le x \le 200$.

13. *Continuously compounded interest.* You have just placed A_0 dollars in a bank account that pays 4% interest, compounded continuously.

    a) How much money will you have in the account in 5 years?

    b) How long will it take your money to double? to triple?

14. *John Napier's question.* John Napier (1550–1617), the Scottish laird who invented logarithms, was the first person to answer the question What happens if you invest an amount of money at 100% interest, compounded continuously?

a) What does happen?
b) How long does it take to triple your money?
c) How much can you earn in a year?

Give reasons for your answers.

15. *Benjamin Franklin's will.* The Franklin Technical Institute of Boston owes its existence to a provision in a codicil to Benjamin Franklin's will. In part the codicil reads:

> I wish to be useful even after my Death, if possible, in forming and advancing other young men that may be serviceable to their Country in both Boston and Philadelphia. To this end I devote Two thousand Pounds Sterling, which I give, one thousand thereof to the Inhabitants of the Town of Boston in Massachusetts, and the other thousand to the inhabitants of the City of Philadelphia, in Trust and for the Uses, Interests and Purposes hereinafter mentioned and declared.

Franklin's plan was to lend money to young apprentices at 5% interest with the provision that each borrower should pay each year along

> . . . with the yearly Interest, one tenth part of the Principal, which sums of Principal and Interest shall be again let to fresh Borrowers. . . . If this plan is executed and succeeds as projected without interruption for one hundred Years, the Sum will then be one hundred and thirty-one thousand Pounds of which I would have the Managers of the Donation to the Inhabitants of the Town of Boston, then lay out at their discretion one hundred thousand Pounds in Public Works. . . . The remaining thirty-one thousand Pounds, I would have continued to be let out on Interest in the manner above directed for another hundred Years. . . . At the end of this second term if no unfortunate accident has prevented the operation the sum will be Four Millions and Sixty-one Thousand Pounds.

It was not always possible to find as many borrowers as Franklin had planned, but the managers of the trust did the best they could. At the end of 100 years from the reception of the Franklin gift, in January 1894, the fund had grown from 1000 pounds to almost exactly 90,000 pounds. In 100 years the original capital had multiplied about 90 times instead of the 131 times Franklin had imagined.

What rate of interest, compounded continuously for 100 years, would have multiplied Benjamin Franklin's original capital by 90?

16. (*Continuation of Exercise 15.*) In Benjamin Franklin's estimate that the original 1000 pounds would grow to 131,000 in 100 years, he was using an annual rate of 5% and compounding once each year. What rate of interest per year when compounded continuously for 100 years would multiply the original amount by 131?

17. *Radon-222.* The decay equation for radon-222 gas is known to be $y = y_0 e^{-0.18t}$, with t in days. About how long will it take the radon in a sealed sample of air to fall to 90% of its original value?

18. *Polonium-210.* The half-life of polonium is 139 days, but your sample will not be useful to you after 95% of the radioactive nuclei present on the day the sample arrives has disintegrated. For about how many days after the sample arrives will you be able to use the polonium?

19. *The mean life of a radioactive nucleus.* Physicists using the radioactivity equation $y = y_0 e^{-kt}$ call the number $1/k$ the *mean life* of a radioactive nucleus. The mean life of a radon nucleus is about $1/0.18 = 5.6$ days. The mean life of a carbon-14 nucleus is more than 8000 years. Show that 95% of the radioactive nuclei originally present in a sample will disintegrate within three mean lifetimes, i.e., by time $t = 3/k$. Thus, the mean life of a nucleus gives a quick way to estimate how long the radioactivity of a sample will last.

20. *Californium-252.* What costs \$27 million per gram and can be used to treat brain cancer, analyze coal for its sulfur content, and detect explosives in luggage? The answer is californium-252, a radioactive isotope so rare that only 8 g of it have been made in the western world since its discovery by Glenn Seaborg in 1950. The half-life of the isotope is 2.645 years—long enough for a useful service life and short enough to have a high radioactivity per unit mass. One microgram of the isotope releases 170 million neutrons per second.

a) What is the value of k in the decay equation for this isotope?
b) What is the isotope's mean life? (See Exercise 19.)
c) How long will it take 95% of a sample's radioactive nuclei to disintegrate?

21. *Cooling soup.* Suppose that a cup of soup cooled from 90°C to 60°C after 10 minutes in a room whose temperature was 20°C. Use Newton's law of cooling to answer the following questions.

a) How much longer would it take the soup to cool to 35°C?
b) Instead of being left to stand in the room, the cup of 90°C soup is put in a freezer whose temperature is -15°C. How long will it take the soup to cool from 90°C to 35°C?

22. *A beam of unknown temperature.* An aluminum beam was brought from the outside cold into a machine shop where the temperature was held at 65°. After 10 minutes, the beam warmed to 35°F and after another 10 minutes it was 50°F. Use Newton's law of cooling to estimate the beam's initial temperature.

23. *Surrounding medium of unknown temperature.* A pan of warm water (46°C) was put in a refrigerator. Ten minutes later, the water's temperature was 39°C; 10 minutes after that, it was 33°C. Use Newton's law of cooling to estimate how cold the refrigerator was.

24. *Silver cooling in air.* The temperature of an ingot of silver is 60°C above room temperature right now. Twenty minutes ago, it was 70°C above room temperature. How far above room temperature will the silver be

a) 15 minutes from now?

b) two hours from now?

c) When will the silver be 10°C above room temperature?

25. *The age of Crater Lake.* The charcoal from a tree killed in the volcanic eruption that formed Crater Lake in Oregon contained 44.5% of the carbon-14 found in living matter. About how old is Crater Lake?

26. *The sensitivity of carbon-14 dating to measurement.* To see the effect of a relatively small error in the estimate of the amount of carbon-14 in a sample being dated, consider this hypothetical situation:

a) A fossilized bone found in central Illinois in the year A.D. 2000 contains 17% of its original carbon-14 content. Estimate the year the animal died.

b) Repeat (a) assuming 18% instead of 17%.

c) Repeat (a) assuming 16% instead of 17%.

27. *Art forgery.* A painting attributed to Vermeer (1632–1675), which should contain no more than 96.2% of its original carbon-14, contains 99.5% instead. About how old is the forgery?

6.6 L'Hôpital's Rule

In the late seventeenth century, John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero. The rule is known today as **l'Hôpital's rule,** after Guillaume François Antoine de l'Hôpital (1661–1704), Marquis de St. Mesme, a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print.

Indeterminate Quotients

If functions $f(x)$ and $g(x)$ are both zero at $x = a$, then $\lim_{x\to a} f(x)/g(x)$ cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression known as an **indeterminate form.** Our experience so far has been that limits that lead to indeterminate forms may or may not be hard to find. It took a lot of work to find $\lim_{x\to 0} (\sin x)/x$ in Section 2.4. But we have had remarkable success with the limit

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

from which we calculate derivatives and which always produces $0/0$. L'Hôpital's rule enables us to draw on our success with derivatives to evaluate limits that lead to indeterminate forms.

Theorem 2

L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}. \tag{1}$$

Proof Working backward from $f'(a)$ and $g'(a)$, which are themselves limits, we have

$$\frac{f'(a)}{g'(a)} = \frac{\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a} \dfrac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)}$$

$$= \lim_{x\to a} \frac{f(x)-0}{g(x)-0}$$

$$= \lim_{x\to a} \frac{f(x)}{g(x)}.$$

❑

Caution

To apply l'Hôpital's rule to *f/g*, divide the derivative of *f* by the derivative of *g*. Do not fall into the trap of taking the derivative of *f/g*. The quotient to use is *f′/g′*, not *(f/g)′*.

EXAMPLE 1

a) $$\lim_{x\to 0} \frac{3x - \sin x}{x} = \left.\frac{3-\cos x}{1}\right|_{x=0} = 2$$

b) $$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \left.\frac{\dfrac{1}{2\sqrt{1+x}}}{1}\right|_{x=0} = \frac{1}{2}$$

c) $$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \left.\frac{1-\cos x}{3x^2}\right|_{x=0} = ? \qquad \text{Still } \frac{0}{0}$$

❑

What can we do about the limit in Example 1(c)? A stronger form of l'Hôpital's rule says that whenever the rule gives 0/0 we can apply it again, repeating the process until we get a different result. With this stronger rule we get

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} \qquad \text{Still } \frac{0}{0}\text{; apply the rule again.}$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} \qquad \text{Still } \frac{0}{0}\text{; apply the rule again.}$$

$$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}. \qquad \text{A different result. Stop.}$$

Theorem 3
L'Hôpital's Rule (Stronger Form)

Suppose that $f(a) = g(a) = 0$ and that f and g are differentiable on an open interval I containing a. Suppose also that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}, \tag{2}$$

if the limit on the right exists (or is ∞ or $-\infty$).

You will find a proof of the finite-limit case of Theorem 3 in Appendix 5.

A misnamed rule and the first differential calculus text

In 1694 John Bernoulli agreed to accept a retainer of 300 pounds per year from his former student l'Hôpital to solve problems for him and keep him up to date on calculus. One of the problems was the so-called 0/0 problem, which Bernoulli solved as agreed. When l'Hôpital published his notes on calculus in book form in 1696, the 0/0 rule appeared as a theorem. L'Hôpital acknowledged his debt to Bernoulli and, to avoid claiming authorship of the book's entire contents, had the book published anonymously. Bernoulli nevertheless accused l'Hôpital of plagiarism, an accusation inadvertently supported after l'Hôpital's death in 1704 by the publisher's promotion of the book as l'Hôpital's. By 1721, Bernoulli, a man so jealous he once threw his son Daniel out of the house for accepting a mathematics prize from the French Academy of Sciences, claimed to have been the author of the entire work. As puzzling and fickle as ever, history accepted Bernoulli's claim (until recently), but still named the rule after l'Hôpital.

EXAMPLE 2

$$\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - (x/2)}{x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2} - (1/2)}{2x} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8} \qquad \text{Not } \frac{0}{0}\text{; limit is found}$$

❑

When you apply l'Hôpital's rule, look for a change from $0/0$ to something else. This is where the limit is revealed.

EXAMPLE 3

$$\lim_{x\to 0} \frac{1 - \cos x}{x + x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0 \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

If we continue to differentiate in an attempt to apply l'Hôpital's rule once more, we get

$$\lim_{x\to 0} \frac{1 - \cos x}{x + x^2} = \lim_{x\to 0} \frac{\sin x}{1 + 2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2},$$

which is wrong. ❑

EXAMPLE 4

$$\lim_{x\to 0^+} \frac{\sin x}{x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\cos x}{2x} = \infty \qquad \text{Not } \frac{0}{0}\text{; answer is found.}$$

❑

L'Hôpital's rule also applies to quotients that lead to the indeterminate form ∞/∞. If $f(x)$ and $g(x)$ both approach infinity as $x \to a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the latter limit exists. The a here may itself be either finite or infinite.

EXAMPLE 5

a) $$\lim_{x\to(\pi/2)^-} \frac{\sec x}{1 + \tan x} \qquad \frac{\infty}{\infty}$$

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

b) $$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

❑

Indeterminate Products and Differences

We can sometimes handle the indeterminate forms $0 \cdot \infty$ and $\infty - \infty$ by using algebra to get $0/0$ or ∞/∞ instead. Here again, we do not mean to suggest that there is a number $0 \cdot \infty$ or $\infty - \infty$ any more than we mean to suggest that there is a number $0/0$ or ∞/∞. These forms are not numbers but descriptions of function behavior.

EXAMPLE 6

$$\lim_{x \to 0^+} x \cot x \qquad 0 \cdot \infty;\ \text{rewrite } x \cot x.$$

$$= \lim_{x \to 0^+} x \cdot \frac{1}{\tan x} \qquad \cot x = \frac{1}{\tan x}$$

$$= \lim_{x \to 0^+} \frac{x}{\tan x} \qquad \text{Now } \frac{0}{0}$$

$$= \lim_{x \to 0^+} \frac{1}{\sec^2 x} = \frac{1}{1} = 1$$

❑

EXAMPLE 7 Find $\displaystyle\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$.

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions.

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}, \qquad \text{Common denominator is } x \sin x.$$

and then apply l'Hôpital's rule to the result:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \qquad \frac{0}{0}$$

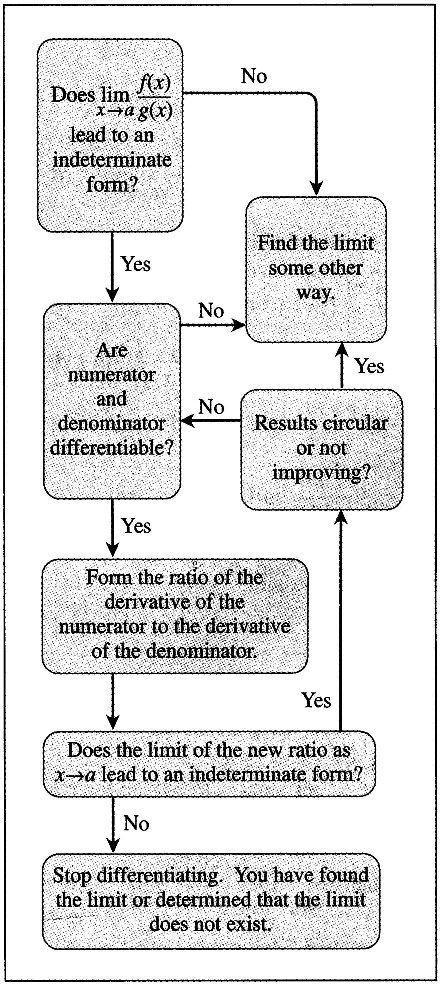
$$= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0$$

❑

Indeterminate Powers

Limits that lead to the indeterminate forms 1^∞, 0^0, and ∞^0 can sometimes be handled by taking logarithms first. We use l'Hôpital's rule to find the limit of the logarithm and then exponentiate to find the original function behavior.



Flowchart 6.1 L'Hôpital's rule

If $\lim_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

EXAMPLE 8 Show that $\lim_{x\to 0^+} (1+x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^∞. We let $f(x) = (1+x)^{1/x}$ and find $\lim_{x\to 0^+} \ln f(x)$. Since

$$\begin{aligned}\ln f(x) &= \ln (1+x)^{1/x}\\ &= \frac{1}{x}\ln(1+x),\end{aligned}$$

l'Hôpital's rule now applies to give

$$\begin{aligned}\lim_{x\to 0^+} \ln f(x) &= \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \qquad \frac{0}{0}\\ &= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1}\\ &= \frac{1}{1} = 1.\end{aligned}$$

Therefore,

$$\lim_{x\to 0^+} (1+x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = e. \qquad \square$$

EXAMPLE 9 Find $\lim_{x\to\infty} x^{1/x}$.

Solution The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{1/x}$ and find $\lim_{x\to\infty} \ln f(x)$. Since

$$\begin{aligned}\ln f(x) &= \ln x^{1/x}\\ &= \frac{\ln x}{x},\end{aligned}$$

l'Hôpital's rule gives

$$\begin{aligned}\lim_{x\to\infty} \ln f(x) &= \lim_{x\to\infty} \frac{\ln x}{x} \qquad \frac{\infty}{\infty}\\ &= \lim_{x\to\infty} \frac{1/x}{1}\\ &= \frac{0}{1} = 0.\end{aligned}$$

Therefore,

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1. \qquad \square$$

Exercises 6.6

Applying l'Hôpital's Rule

Use l'Hôpital's rule to find the limits in Exercises 1–42.

1. $\lim_{x \to 2} \dfrac{x-2}{x^2-4}$

2. $\lim_{x \to -5} \dfrac{x^2-25}{x+5}$

3. $\lim_{t \to -3} \dfrac{t^3-4t+15}{t^2-t-12}$

4. $\lim_{t \to 1} \dfrac{t^3-1}{4t^3-t-3}$

5. $\lim_{x \to \infty} \dfrac{5x^2-3x}{7x^2+1}$

6. $\lim_{x \to \infty} \dfrac{x-8x^2}{12x^2+5x}$

7. $\lim_{t \to 0} \dfrac{\sin t^2}{t}$

8. $\lim_{t \to 0} \dfrac{\sin 5t}{t}$

9. $\lim_{x \to 0} \dfrac{8x^2}{\cos x - 1}$

10. $\lim_{x \to 0} \dfrac{\sin x - x}{x^3}$

11. $\lim_{\theta \to \pi/2} \dfrac{2\theta - \pi}{\cos(2\pi - \theta)}$

12. $\lim_{\theta \to -\pi/3} \dfrac{3\theta + \pi}{\sin(\theta + (\pi/3))}$

13. $\lim_{\theta \to \pi/2} \dfrac{1-\sin\theta}{1+\cos 2\theta}$

14. $\lim_{x \to 1} \dfrac{x-1}{\ln x - \sin \pi x}$

15. $\lim_{x \to 0} \dfrac{x^2}{\ln(\sec x)}$

16. $\lim_{x \to \pi/2} \dfrac{\ln(\csc x)}{(x-(\pi/2))^2}$

17. $\lim_{t \to 0} \dfrac{t(1-\cos t)}{t - \sin t}$

18. $\lim_{t \to 0} \dfrac{t \sin t}{1-\cos t}$

19. $\lim_{x \to (\pi/2)^-} \left(x - \dfrac{\pi}{2}\right) \sec x$

20. $\lim_{x \to (\pi/2)^-} \left(\dfrac{\pi}{2} - x\right) \tan x$

21. $\lim_{\theta \to 0} \dfrac{3^{\sin\theta} - 1}{\theta}$

22. $\lim_{\theta \to 0} \dfrac{(1/2)^{\theta} - 1}{\theta}$

23. $\lim_{x \to 0} \dfrac{x2^x}{2^x - 1}$

24. $\lim_{x \to 0} \dfrac{3^x - 1}{2^x - 1}$

25. $\lim_{x \to \infty} \dfrac{\ln(x+1)}{\log_2 x}$

26. $\lim_{x \to \infty} \dfrac{\log_2 x}{\log_3(x+3)}$

27. $\lim_{x \to 0^+} \dfrac{\ln(x^2+2x)}{\ln x}$

28. $\lim_{x \to 0^+} \dfrac{\ln(e^x-1)}{\ln x}$

29. $\lim_{y \to 0} \dfrac{\sqrt{5y+25} - 5}{y}$

30. $\lim_{y \to 0} \dfrac{\sqrt{ay+a^2} - a}{y}, \quad a > 0$

31. $\lim_{x \to \infty} (\ln 2x - \ln(x+1))$

32. $\lim_{x \to 0^+} (\ln x - \ln \sin x)$

33. $\lim_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$

34. $\lim_{x \to 0^+} \left(\dfrac{3x+1}{x} - \dfrac{1}{\sin x}\right)$

35. $\lim_{x \to 1^+} \left(\dfrac{1}{x-1} - \dfrac{1}{\ln x}\right)$

36. $\lim_{x \to 0^+} (\csc x - \cot x + \cos x)$

37. $\lim_{x \to \infty} \displaystyle\int_x^{2x} \dfrac{1}{t}\, dt$

38. $\lim_{x \to \infty} \dfrac{1}{x \ln x} \displaystyle\int_1^x \ln t\, dt$

39. $\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{e^{\theta} - \theta - 1}$

40. $\lim_{h \to 0} \dfrac{e^h - (1+h)}{h^2}$

41. $\lim_{t \to \infty} \dfrac{e^t + t^2}{e^t - t}$

42. $\lim_{x \to \infty} x^2 e^{-x}$

Limits Involving Bases and Exponents

Find the limits in Exercises 43–52.

43. $\lim_{x \to 1^+} x^{1/(1-x)}$

44. $\lim_{x \to 1^+} x^{1/(x-1)}$

45. $\lim_{x \to \infty} (\ln x)^{1/x}$

46. $\lim_{x \to e^+} (\ln x)^{1/(x-e)}$

47. $\lim_{x \to 0^+} x^{-1/\ln x}$

48. $\lim_{x \to \infty} x^{1/\ln x}$

49. $\lim_{x \to \infty} (1+2x)^{1/(2\ln x)}$

50. $\lim_{x \to 0} (e^x + x)^{1/x}$

51. $\lim_{x \to 0^+} x^x$

52. $\lim_{x \to 0^+} \left(1 + \dfrac{1}{x}\right)^x$

Theory and Applications

L'Hôpital's rule does not help with the limits in Exercises 53–56. Try it—you just keep on cycling. Find the limits some other way.

53. $\lim_{x \to \infty} \dfrac{\sqrt{9x+1}}{\sqrt{x+1}}$

54. $\lim_{x \to 0^+} \dfrac{\sqrt{x}}{\sqrt{\sin x}}$

55. $\lim_{x \to (\pi/2)^-} \dfrac{\sec x}{\tan x}$

56. $\lim_{x \to 0^+} \dfrac{\cot x}{\csc x}$

57. Which one is correct, and which one is wrong? Give reasons for your answers.

   a) $\lim_{x \to 3} \dfrac{x-3}{x^2-3} = \lim_{x \to 3} \dfrac{1}{2x} = \dfrac{1}{6}$

   b) $\lim_{x \to 3} \dfrac{x-3}{x^2-3} = \dfrac{0}{6} = 0$

58. Which one is correct, and which one is wrong? Give reasons for your answers.

a) $\lim_{x\to 0} \frac{x^2-2x}{x^2-\sin x} = \lim_{x\to 0} \frac{2x-2}{2x-\cos x}$

$= \lim_{x\to 0} \frac{2}{2+\sin x} = \frac{2}{2+0} = 1$

b) $\lim_{x\to 0} \frac{x^2-2x}{x^2-\sin x} = \lim_{x\to 0} \frac{2x-2}{2x-\cos x} = \frac{-2}{0-1} = 2$

59. Only one of these calculations is correct. Which one? Why are the others wrong? Give reasons for your answers.

a) $\lim_{x\to 0^+} x\ln x = 0 \cdot (-\infty) = 0$

b) $\lim_{x\to 0^+} x\ln x = 0 \cdot (-\infty) = -\infty$

c) $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{(1/x)} = \frac{-\infty}{\infty} = -1$

d) $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{(1/x)}$

$= \lim_{x\to 0^+} \frac{(1/x)}{(-1/x^2)} = \lim_{x\to 0^+} (-x) = 0$

60. Let

$$f(x) = \begin{cases} x+2, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$g(x) = \begin{cases} x+1, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Show that

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 1 \quad \text{but that} \quad \lim_{x\to 0} \frac{f(x)}{g(x)} = 2.$$

Doesn't this contradict l'Hôpital's rule? Give reasons for your answers.

61. Find a value of c that makes the function

$$f(x) = \begin{cases} \dfrac{9x - 3\sin 3x}{5x^3}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$. Explain why your value of c works.

62. Find a value of c that makes the function

$$g(\theta) = \begin{cases} \dfrac{(\tan\theta)^2}{\sin(4\theta^2/\pi)}, & \theta \neq 0 \\ c, & \theta = 0 \end{cases}$$

continuous from the right at $\theta = 0$. Explain why your value of c works.

63. *The continuous compound interest formula.* In deriving the formula $A(t) = A_0 e^{rt}$ in Section 6.5, we claimed that

$$\lim_{k\to\infty} A_0 \left(1 + \frac{r}{k}\right)^{kt} = A_0 e^{rt}.$$

This equation will hold if

$$\lim_{k\to\infty} \left(1 + \frac{r}{k}\right)^{kt} = e^{rt},$$

and this, in turn, will hold if

$$\lim_{k\to\infty} \left(1 + \frac{r}{k}\right)^{k} = e^{r}.$$

As you can see, the limit leads to the indeterminate form 1^∞. Verify the limit using l'Hôpital's rule.

64. Given that $x > 0$, find the maximum value, if any, of

a) $x^{1/x}$

b) x^{1/x^2}

c) x^{1/x^n} (n a positive integer)

d) Show that $\lim_{x\to\infty} x^{1/x^n} = 1$ for every positive integer n.

Grapher Explorations

65. *Determining the value of e.*

a) Use l'Hôpital's rule to show that

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

b) CALCULATOR See how close you can come to

$$e = 2.7\ 1828\ 1828\ 45\ 90\ 45$$

by evaluating $f(x) = (1 + (1/x))^x$ for $x = 10$, 10^2, 10^3, ...and so on. You can expect the approximations to approach e at first, but on some calculators they will move away again as round-off errors take their toll.

c) If you have a grapher, you may prefer to do part (b) by graphing $f(x) = (1 + (1/x))^x$ for large values of x, using TRACE to display the coordinates along the graph. Again, you may expect to find decreasing accuracy as x increases and, beyond $x = 10^{10}$ or so, erratic behavior.

66. This exercise explores the difference between the limit

$$\lim_{x\to\infty} \left(1 + \frac{1}{x^2}\right)^x$$

and the limit

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e,$$

studied in Exercise 65.

a) Graph

$$f(x) = \left(1 + \frac{1}{x^2}\right)^x \quad \text{and} \quad g(x) = \left(1 + \frac{1}{x}\right)^x$$

together for $x \geq 0$. How does the behavior of f compare with that of g? Estimate the value of $\lim_{x\to\infty} f(x)$.

b) Confirm your estimate of $\lim_{x\to\infty} f(x)$ by calculating it with l'Hôpital's rule.

67. a) Estimate the value of

$$\lim_{x\to\infty} (x - \sqrt{x^2 + x})$$

by graphing $f(x) = x - \sqrt{x^2 + x}$ over a suitably large interval of x-values.

b) Now confirm your estimate by finding the limit with l'Hôpital's rule. As the first step, multiply $f(x)$ by the fraction $(x+\sqrt{x^2+x})/(x+\sqrt{x^2+x})$ and simplify the new numerator.

68. Estimate the value of

$$\lim_{x\to 2}\frac{x^2-4}{\sqrt{x^2+5}-3}$$

by graphing. Then confirm your estimate with l'Hôpital's rule.

69. Estimate the value of

$$\lim_{x\to 1}\frac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$$

by graphing. Then confirm your estimate with l'Hôpital's rule.

70. a) Estimate the value of

$$\lim_{x\to 1}\frac{(x-1)^2}{x\ln x-x-\cos\pi x}$$

by graphing $f(x)=(x-1)^2/(x\ln x-x-\cos\pi x)$ near $x=1$. Then confirm your estimate with l'Hôpital's rule.

b) Graph f for $0<x\le 11$.

71. *The continuous extension of* $(\sin x)^x$ *to* $[0, \pi]$

a) Graph $f(x)=(\sin x)^x$ on the interval $0\le x\le\pi$. What value would you assign to f to make it continuous at $x=0$?

b) Verify your conclusion in (a) by finding $\lim_{x\to 0^+} f(x)$ with l'Hôpital's rule.

c) Returning to the graph, estimate the maximum value of f on $[0, \pi]$. About where is max f taken on?

d) Sharpen your estimate in (c) by graphing f' in the same window to see where its graph crosses the x-axis. To simplify your work, you might want to delete the exponential factor from the expression for f' and graph just the factor that has a zero.

e) Sharpen your estimate of the location of max f further still by solving the equation $f'=0$ numerically.

f) CALCULATOR Estimate max f by evaluating f at the locations you found in (c), (d), and (e). What is your best value for max f?

72. *The function* $(\sin x)^{\tan x}$. (*Continuation of Exercise* 71.)

a) Graph $f(x)=(\sin x)^{\tan x}$ on the interval $-7\le x\le 7$. How do you account for the gaps in the graph? How wide are the gaps?

b) Now graph f on the interval $0\le x\le\pi$. The function is not defined at $x=\pi/2$, but the graph has no break at this point. What is going on? What value does the graph appear to give for f at $x=\pi/2$? (*Hint:* Use l'Hôpital's rule to find lim f as $x\to(\pi/2)^-$ and $x\to(\pi/2)^+$.)

c) Continuing with the graphs in (b), find max f and min f as accurately as you can and estimate the values of x at which they are taken on.

73. *The place of* $\ln x$ *among the powers of* x. The natural logarithm

$$\ln x=\int_1^x\frac{1}{t}\,dt$$

fills the gap in the set of formulas

$$\int t^{k-1}dt=\frac{t^k}{k}+C,\quad k\ne 0, \tag{3}$$

but the formulas themselves do not reveal how well the logarithm fits in. We can see the nice fit graphically if we select from Eq. (3) the specific antiderivatives

$$\int_1^x t^{k-1}dt=\frac{x^k-1}{k},\quad x>0,$$

and compare their graphs with the graph of $\ln x$.

a) Graph the functions $f(x)=(x^k-1)/k$ together with $\ln x$ on the interval $0\le x\le 50$ for $k=\pm1$, ±0.5, ±0.1, and ±0.05.

b) Show that

$$\lim_{k\to 0}\frac{x^k-1}{k}=\ln x.$$

(Based on "The Place of $\ln x$ Among the Powers of x" by Henry C. Finlayson, *American Mathematical Monthly,* Vol. 94, No. 5, May 1987, p. 450.)

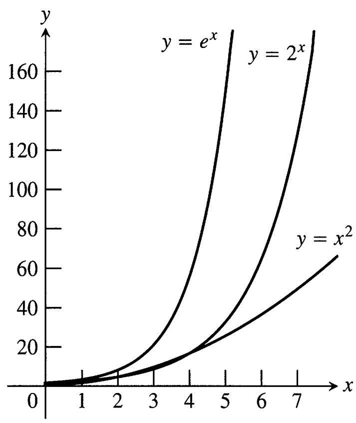
74. *Confirmation of the limit in Section 5.7, Exercise 42.* Estimate the value of

$$\lim_{\alpha\to 0^+}\frac{\sin\alpha-\alpha\cos\alpha}{\alpha-\alpha\cos\alpha}$$

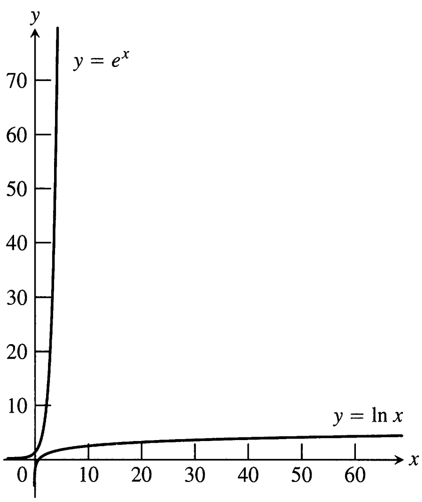
as closely as you can by graphing. Then confirm your estimate with l'Hôpital's rule.

6.7 Relative Rates of Growth

This section shows how to compare the rates at which functions of x grow as x becomes large and introduces the so-called little-oh and big-oh notation sometimes used to describe the results of these comparisons. *We restrict our attention to functions whose values eventually become and remain positive as* $x\to\infty$.



6.14 The graphs of e^x, 2^x, and x^2.



6.15 Scale drawings of the graphs of e^x and $\ln x$.

Relatives Rates of Growth

You may have noticed that exponential functions like 2^x and e^x seem to grow more rapidly as x gets large than the polynomials and rational functions we graphed in Chapter 3. These exponentials certainly grow more rapidly than x itself, and you can see 2^x outgrowing x^2 as x increases in Fig. 6.14. In fact, as $x \to \infty$, the functions 2^x and e^x grow faster than any power of x, even $x^{1,000,000}$ (Exercise 19).

To get a feeling for how rapidly the values of $y = e^x$ grow with increasing x, think of graphing the function on a large blackboard, with the axes scaled in centimeters. At $x = 1$ cm, the graph is $e^1 \approx 3$ cm above the x-axis. At $x = 6$ cm, the graph is $e^6 \approx 403$ cm ≈ 4 m high (it is about to go through the ceiling if it hasn't done so already). At $x = 10$ cm, the graph is $e^{10} \approx 22{,}026$ cm ≈ 220 m high, higher than most buildings. At $x = 24$ cm, the graph is more than halfway to the moon, and at $x = 43$ cm from the origin, the graph is high enough to reach past the sun's closest stellar neighbor, the red dwarf star Proxima Centauri:

$$\begin{aligned} e^{43} &\approx 4.73 \times 10^{18} \text{ cm} \\ &= 4.73 \times 10^{13} \text{ km} \\ &\approx 1.58 \times 10^{8} \text{ light-seconds} \\ &\approx 5.0 \text{ light-years} \end{aligned}$$

In a vacuum, light travels at 300,000 km/sec.

The distance to Proxima Centauri is about 4.22 light-years. Yet with $x = 43$ cm from the origin, the graph is still less than 2 feet to the right of the y-axis.

In contrast, logarithmic functions like $y = \log_2 x$ and $y = \ln x$ grow more slowly as $x \to \infty$ than any positive power of x (Exercise 21). With axes scaled in centimeters, you have to go nearly 5 light-years out on the x-axis to find a point where the graph of $y = \ln x$ is even $y = 43$ cm high. See Fig. 6.15.

These important comparisons of exponential, polynomial, and logarithmic functions can be made precise by defining what it means for a function $f(x)$ to grow faster than a function $g(x)$ as $x \to \infty$.

Definition

Rates of Growth as $x \to \infty$

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f **grows faster than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

or, equivalently, if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

We also say that g **grows slower than** f as $x \to \infty$.

2. f and g **grow at the same rate** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0. \qquad L \text{ finite and not zero}$$

According to these definitions, $y = 2x$ does not grow faster than $y = x$. The two functions grow at the same rate because

$$\lim_{x\to\infty} \frac{2x}{x} = \lim_{x\to\infty} 2 = 2,$$

which is a finite, nonzero limit. The reason for this apparent disregard of common sense is that we want "f grows faster than g" to mean that for large x-values g is negligible when compared with f.

EXAMPLE 1 e^x grows faster than x^2 as $x \to \infty$ because

$$\underbrace{\lim_{x\to\infty} \frac{e^x}{x^2}}_{\infty/\infty} = \underbrace{\lim_{x\to\infty} \frac{e^x}{2x}}_{\infty/\infty} = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Using l'Hôpital's rule twice

❑

EXAMPLE 2

a) 3^x grows faster than 2^x as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{3^x}{2^x} = \lim_{x\to\infty} \left(\frac{3}{2}\right)^x = \infty.$$

b) As part (a) suggests, exponential functions with different bases never grow at the same rate as $x \to \infty$. If $a > b > 0$, then a^x grows faster than b^x. Since $(a/b) > 1$,

$$\lim_{x\to\infty} \frac{a^x}{b^x} = \lim_{x\to\infty} \left(\frac{a}{b}\right)^x = \infty.$$

❑

EXAMPLE 3 x^2 grows faster than $\ln x$ as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{x^2}{\ln x} = \lim_{x\to\infty} \frac{2x}{1/x} = \lim_{x\to\infty} 2x^2 = \infty.$$

l'Hôpital's rule

❑

EXAMPLE 4 $\ln x$ grows slower than x as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1}$$

l'Hôpital's rule

$$= \lim_{x\to\infty} \frac{1}{x} = 0.$$

❑

EXAMPLE 5 In contrast to exponential functions, logarithmic functions with different bases a and b always grow at the same rate as $x \to \infty$:

$$\lim_{x\to\infty} \frac{\log_a x}{\log_b x} = \lim_{x\to\infty} \frac{\ln x/\ln a}{\ln x/\ln b} = \frac{\ln b}{\ln a}.$$

The limiting ratio is always finite and never zero. ❑

If f grows at the same rate as g as $x \to \infty$, and g grows at the same rate as h as $x \to \infty$, then f grows at the same rate as h as $x \to \infty$. The reason is that

$$\lim_{x\to\infty} \frac{f}{g} = L_1 \qquad \text{and} \qquad \lim_{x\to\infty} \frac{g}{h} = L_2$$

together imply

$$\lim_{x\to\infty} \frac{f}{h} = \lim_{x\to\infty} \frac{f}{g} \cdot \frac{g}{h} = L_1 L_2.$$

If L_1 and L_2 are finite and nonzero, then so is $L_1 L_2$.

EXAMPLE 6 Show that $\sqrt{x^2+5}$ and $(2\sqrt{x}-1)^2$ grow at the same rate as $x \to \infty$.

Solution We show that the functions grow at the same rate by showing that they both grow at the same rate as the function x:

$$\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{x} = \lim_{x\to\infty} \sqrt{1+\frac{5}{x^2}} = 1,$$

$$\lim_{x\to\infty} \frac{(2\sqrt{x}-1)^2}{x} = \lim_{x\to\infty} \left(\frac{2\sqrt{x}-1}{\sqrt{x}}\right)^2 = \lim_{x\to\infty} \left(2-\frac{1}{\sqrt{x}}\right)^2 = 4.$$ ❑

Order and Oh-Notation

Here we introduce the "little-oh" and "big-oh" notation invented by number theorists a hundred years ago and now commonplace in mathematical analysis and computer science.

Definition

A function f is **of smaller order than** g as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing $\boldsymbol{f = o(g)}$ ("f is little-oh of g").

Notice that saying $f = o(g)$ as $x \to \infty$ is another way to say that f grows slower than g as $x \to \infty$.

EXAMPLE 7

$$\ln x = o(x) \text{ as } x \to \infty \quad \text{because} \quad \lim_{x\to\infty} \frac{\ln x}{x} = 0$$

$$x^2 = o(x^3+1) \text{ as } x \to \infty \quad \text{because} \quad \lim_{x\to\infty} \frac{x^2}{x^3+1} = 0$$ ❑

Definition

Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \le M,$$

for x sufficiently large. We indicate this by writing $\boldsymbol{f = O(g)}$ ("f is big-oh of g").

EXAMPLE 8

$$x + \sin x = O(x) \text{ as } x \to \infty \quad \text{because} \quad \frac{x + \sin x}{x} \le 2 \text{ for } x \text{ sufficiently large.}$$

EXAMPLE 9

$$e^x + x^2 = O(e^x) \text{ as } x \to \infty \quad \text{because} \quad \frac{e^x + x^2}{e^x} \to 1 \text{ as } x \to \infty,$$

$$x = O(e^x) \text{ as } x \to \infty \quad \text{because} \quad \frac{x}{e^x} \to 0 \text{ as } x \to \infty.$$

If you look at the definitions again, you will see that $f = o(g)$ implies $f = O(g)$ for functions that are positive for x sufficiently large. Also, if f and g grow at the same rate, then $f = O(g)$ and $g = O(f)$ (Exercise 11).

Sequential vs. Binary Search

Computer scientists sometimes measure the efficiency of an algorithm by counting the number of steps a computer must take to make the algorithm do something. There can be significant differences in how efficiently algorithms perform, even if they are designed to accomplish the same task. These differences are often described in big-oh notation. Here is an example.

Webster's Third New International Dictionary lists about 26,000 words that begin with the letter a. One way to look up a word, or to learn if it is not there, is to read through the list one word at a time until you either find the word or determine that it is not there. This method, called sequential search, makes no particular use of the words' alphabetical arrangement. You are sure to get an answer, but it might take 26,000 steps.

Another way to find the word or to learn it is not there is to go straight to the middle of the list (give or take a few words). If you do not find the word, then go to the middle of the half that contains it and forget about the half that does not. (You know which half contains it because you know the list is ordered alphabetically.) This method eliminates roughly 13,000 words in a single step. If you do not find the word on the second try, then jump to the middle of the half that contains it. Continue this way until you have either found the word or divided the list in half so many times there are no words left. How many times do you have to divide the list to find the word or learn that it is not there? At most 15, because

$$(26{,}000/2^{15}) < 1.$$

That certainly beats a possible 26,000 steps.

For a list of length n, a sequential search algorithm takes on the order of n steps to find a word or determine that it is not in the list. A binary search, as the second algorithm is called, takes on the order of $\log_2 n$ steps. The reason is that if $2^{m-1} < n \le 2^m$, then $m - 1 < \log_2 n \le m$, and the number of bisections required to narrow the list to one word will be at most $m = \lceil \log_2 n \rceil$, the integer ceiling for $\log_2 n$.

Big-oh notation provides a compact way to say all this. The number of steps in a sequential search of an ordered list is $O(n)$; the number of steps in a binary search is $O(\log_2 n)$. In our example, there is a big difference between the two (26,000 vs. 15), and the difference can only increase with n because n grows faster than $\log_2 n$ as $n \to \infty$.

To find an item in a list of length n:

A sequential search takes $O(n)$ steps.

A binary search takes $O(\log_2 n)$ steps.

Exercises 6.7

Comparisons with the Exponential e^x

1. Which of the following functions grow faster than e^x as $x \to \infty$? Which grow at the same rate as e^x? Which grow slower?

   a) $x+3$ b) $x^3+\sin^2 x$ c) $\sqrt{x}$ d) 4^x e) $(3/2)^x$ f) $e^{x/2}$ g) $e^x/2$ h) $\log_{10} x$

2. Which of the following functions grow faster than e^x as $x \to \infty$? Which grow at the same rate as e^x? Which grow slower?

   a) $10x^4+30x+1$ b) $x\ln x - x$ c) $\sqrt{1+x^4}$ d) $(5/2)^x$ e) e^{-x} f) xe^x g) $e^{\cos x}$ h) e^{x-1}

Comparisons with the Power x^2

3. Which of the following functions grow faster than x^2 as $x \to \infty$? Which grow at the same rate as x^2? Which grow slower?

   a) x^2+4x b) x^5-x^2 c) $\sqrt{x^4+x^3}$ d) $(x+3)^2$ e) $x\ln x$ f) 2^x g) x^3e^{-x} h) $8x^2$

4. Which of the following functions grow faster than x^2 as $x \to \infty$? Which grow at the same rate as x^2? Which grow slower?

   a) $x^2+\sqrt{x}$ b) $10x^2$ c) x^2e^{-x} d) $\log_{10}(x^2)$ e) x^3-x^2 f) $(1/10)^x$ g) $(1.1)^x$ h) x^2+100x

Comparisons with the Logarithm ln x

5. Which of the following functions grow faster than $\ln x$ as $x \to \infty$? Which grow at the same rate as $\ln x$? Which grow slower?

   a) $\log_3 x$ b) $\ln 2x$ c) $\ln\sqrt{x}$ d) $\sqrt{x}$ e) x f) $5\ln x$ g) $1/x$ h) e^x

6. Which of the following functions grow faster than $\ln x$ as $x \to \infty$? Which grow at the same rate as $\ln x$? Which grow slower?

   a) $\log_2(x^2)$ b) $\log_{10} 10x$ c) $1/\sqrt{x}$ d) $1/x^2$ e) $x-2\ln x$ f) e^{-x} g) $\ln(\ln x)$ h) $\ln(2x+5)$

Ordering Functions by Growth Rates

7. Order the following functions from slowest growing to fastest growing as $x \to \infty$.

   a) e^x b) x^x c) $(\ln x)^x$ d) $e^{x/2}$

8. Order the following functions from slowest growing to fastest growing as $x \to \infty$.

   a) 2^x b) x^2 c) $(\ln 2)^x$ d) e^x

Big-oh and Little-oh; Order

9. True, or false? As $x \to \infty$,

   a) $x=o(x)$ b) $x=o(x+5)$ c) $x=O(x+5)$ d) $x=O(2x)$ e) $e^x=o(e^{2x})$ f) $x+\ln x=O(x)$ g) $\ln x=o(\ln 2x)$ h) $\sqrt{x^2+5}=O(x)$

10. True, or false? As $x \to \infty$,

   a) $\frac{1}{x+3}=O\left(\frac{1}{x}\right)$ b) $\frac{1}{x}+\frac{1}{x^2}=O\left(\frac{1}{x}\right)$ c) $\frac{1}{x}-\frac{1}{x^2}=o\left(\frac{1}{x}\right)$ d) $2+\cos x=O(2)$ e) $e^x+x=O(e^x)$ f) $x\ln x=o(x^2)$ g) $\ln(\ln x)=O(\ln x)$ h) $\ln(x)=o(\ln(x^2+1))$

11. Show that if positive functions $f(x)$ and $g(x)$ grow at the same rate as $x \to \infty$, then $f=O(g)$ and $g=O(f)$.

12. When is a polynomial $f(x)$ of smaller order than a polynomial $g(x)$ as $x \to \infty$? Give reasons for your answer.

13. When is a polynomial $f(x)$ of at most the order of a polynomial $g(x)$ as $x \to \infty$? Give reasons for your anwer.

14. *Simpson's rule and the trapedzoidal rule.* The definitions in the present section can be made more general by lifting the restriction that $x \to \infty$ and considering limits as $x \to a$ for any

real number a. Show that the error E_S in the Simpson's rule approximation of a definite integral is $O(h^4)$ as $h \to 0$ while the error E_T in the trapezoidal rule approximation is $O(h^2)$. This gives another way to explain the relative accuracies of the two approximation methods.

Other Comparisons

15. What do the conclusions we drew in Section 3.5 about the limits of rational functions tell us about the relative growth of polynomials as $x \to \infty$?

16. GRAPHER

a) Investigate

$$\lim_{x\to\infty} \frac{\ln (x+1)}{\ln x} \quad \text{and} \quad \lim_{x\to\infty} \frac{\ln (x+999)}{\ln x}.$$

Then use l'Hôpital's rule to explain what you find.

b) Show that the value of

$$\lim_{x\to\infty} \frac{\ln (x+a)}{\ln x}$$

is the same no matter what value you assign to the constant a. What does this say about the relative rates at which the functions $f(x) = \ln (x+a)$ and $g(x) = \ln x$ grow?

17. Show that $\sqrt{10x+1}$ and $\sqrt{x+1}$ grow at the same rate as $x \to \infty$ by showing that they both grow at the same rate as $\sqrt{x}$ as $x \to \infty$.

18. Show that $\sqrt{x^4+x}$ and $\sqrt{x^4-x^3}$ grow at the same rate as $x \to \infty$ by showing that they both grow at the same rate as x^2 as $x \to \infty$.

19. Show that e^x grows faster as $x \to \infty$ than x^n for any positive integer n, even $x^{1{,}000{,}000}$. (*Hint:* What is the nth derivative of x^n?)

20. *The function e^x outgrows any polynomial.* Show that e^x grows faster as $x \to \infty$ than any polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

21. a) Show that $\ln x$ grows slower as $x \to \infty$ than $x^{1/n}$ for any positive integer n, even $x^{1/1{,}000{,}000}$.

b) CALCULATOR Although the values of $x^{1/1{,}000{,}000}$ eventually overtake the values of $\ln x$, you have to go way out on the x-axis before this happens. Find a value of x greater than 1 for which $x^{1/1{,}000{,}000} > \ln x$. You might start by observing that when $x > 1$ the equation $\ln x = x^{1/1{,}000{,}000}$ is equivalent to the equation $\ln (\ln x) = (\ln x)/1{,}000{,}000$.

c) CALCULATOR Even $x^{1/10}$ takes a long time to overtake $\ln x$. Experiment with a calculator to find the value of x at which the graphs of $x^{1/10}$ and $\ln x$ cross, or, equivalently, at which $\ln x = 10 \ln (\ln x)$. Bracket the crossing point between powers of 10 and then close in by successive halving.

d) GRAPHER (*Continuation of part c.*) The value of x at which $\ln x = 10 \ln (\ln x)$ is too far out for some graphers and root finders to identify. Try it on the equipment available to you and see what happens.

22. *The function $\ln x$ grows slower than any polynomial.* Show that $\ln x$ grows slower as $x \to \infty$ than any nonconstant polynomial.

Algorithms and Searches

23. a) Suppose you have three different algorithms for solving the same problem and each algorithm takes a number of steps that is of the order of one of the functions listed here:

$$n \log_2 n, \quad n^{3/2}, \quad n(\log_2 n)^2.$$

Which of the algorithms is the most efficient in the long run? Give reasons for your answer.

b) GRAPHER Graph the functions in part (a) together to get a sense of how rapidly each one grows.

24. Repeat Exercise 23 for the functions

$$n, \quad \sqrt{n}\log_2 n, \quad (\log_2 n)^2.$$

25. CALCULATOR Suppose you are looking for an item in an ordered list one million items long. How many steps might it take to find that item with a sequential search? a binary search?

26. CALCULATOR You are looking for an item in an ordered list 450,000 items long (the length of *Webster's Third New International Dictionary*). How many steps might it take to find the item with a sequential search? a binary search?

6.8 Inverse Trigonometric Functions

Inverse trigonometric functions arise when we want to calculate angles from side measurements in triangles. They also provide useful antiderivatives and appear frequently in the solutions of differential equations. This section shows how the functions are defined, graphed, and evaluated.

Defining the Inverses

The six basic trigonometric functions are not one-to-one (their values repeat), but we can restrict their domains to intervals on which they are one-to-one.

Domain Restrictions That Make the Trigonometric Functions One-to-One

Function	Domain	Range
$\sin x$	$[-\pi/2\ , \pi/2]$	$[-1, 1]$
$\cos x$	$[0, \pi]$	$[-1, 1]$
$\tan x$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$
$\cot x$	$(0, \pi)$	$(-\infty, \infty)$
$\sec x$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$
$\csc x$	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$
\begin{aligned}
y &= \sin^{-1} x \quad &\text{or} \quad y &= \text{arc } \sin x\\
y &= \cos^{-1} x \quad &\text{or} \quad y &= \text{arc } \cos x\\
y &= \tan^{-1} x \quad &\text{or} \quad y &= \text{arc } \tan x\\
y &= \cot^{-1} x \quad &\text{or} \quad y &= \text{arc } \cot x\\
y &= \sec^{-1} x \quad &\text{or} \quad y &= \text{arc } \sec x\\
y &= \csc^{-1} x \quad &\text{or} \quad y &= \text{arc } \csc x
\end{aligned}
$$

These equations are read "y equals the arc sine of x" or "y equals arc sin x" and so on.

Caution The -1 in the expressions for the inverse means "inverse." It does *not* mean reciprocal. For example, the *reciprocal* of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$.

The domains of the inverses are chosen to satisfy the following relationships.

$$\sec^{-1} x = \cos^{-1}(1/x) \tag{1}$$

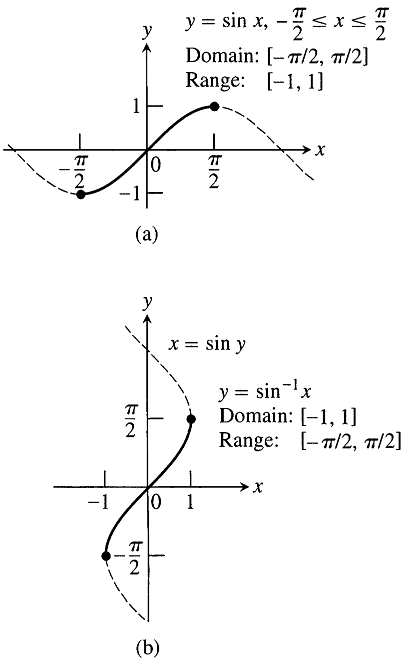
$$\csc^{-1} x = \sin^{-1}(1/x) \tag{2}$$

$$\cot^{-1} x = \pi/2 - \tan^{-1} x \tag{3}$$

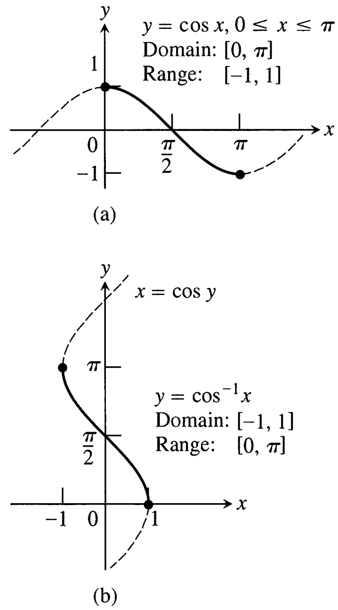
We can use these relationships to find values of $\sec^{-1} x$, $\csc^{-1} x$, and $\cot^{-1} x$ on calculators that give only $\cos^{-1} x$, $\sin^{-1} x$, and $\tan^{-1} x$. As in some of the examples that follow, we can also find a few of the more common values of $\sec^{-1} x$, $\csc^{-1} x$, and $\cot^{-1} x$ using reference right triangles.

The Arc Sine and Arc Cosine

The arc sine of x is an angle whose sine is x. The arc cosine is an angle whose cosine is x.



6.16 The graphs of (a) $y = \sin x$, $-\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.



6.17 The graphs of (a) $y = \cos x$, $0 \le x \le \pi$, and (b) its inverse, $y = \cos^{-1} x$. The graph of $\cos^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

Definition

$\boldsymbol{y = \sin^{-1} x}$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$\boldsymbol{y = \cos^{-1} x}$ is the number in $[0, \pi]$ for which $\cos y = x$.

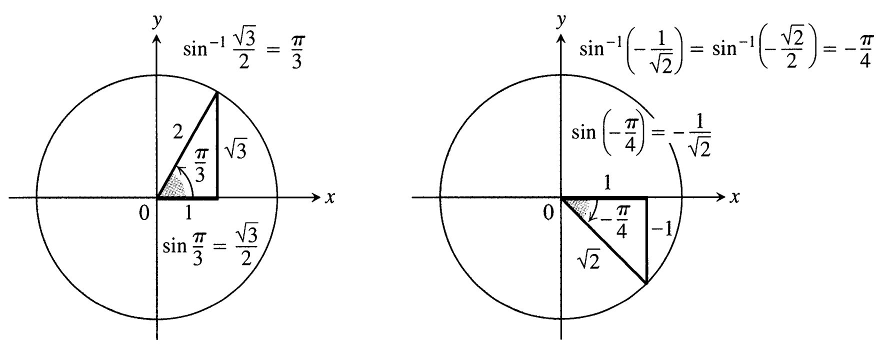
The graph of $y = \sin^{-1} x$ (Fig. 6.16) is symmetric about the origin (it lies along the graph of $x = \sin y$). The arc sine is therefore an odd function:

$$\sin^{-1}(-x) = -\sin^{-1} x. \tag{4}$$

The graph of $y = \cos^{-1} x$ (Fig. 6.17) has no such symmetry.

EXAMPLE 1 *Common values of* $\sin^{-1} x$

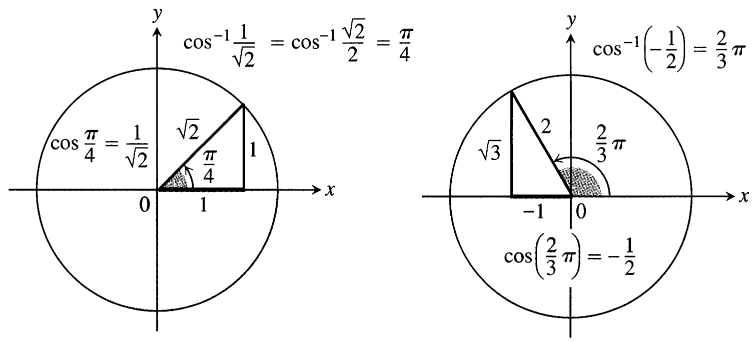
x	$\sin^{-1} x$
$\sqrt{3}/2$	$\pi/3$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/6$
$-1/2$	$-\pi/6$
$-\sqrt{2}/2$	$-\pi/4$
$-\sqrt{3}/2$	$-\pi/3$



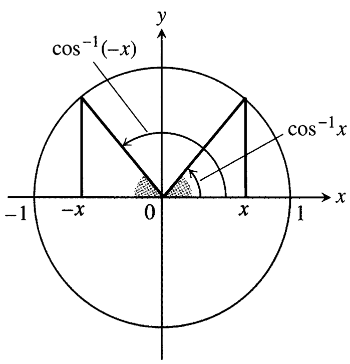
The angles come from the first and fourth quadrants because the range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$. ❑

EXAMPLE 2 *Common values of* $\cos^{-1} x$

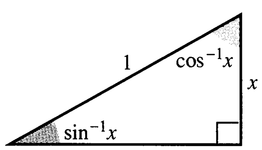
x	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/3$
$-1/2$	$2\pi/3$
$-\sqrt{2}/2$	$3\pi/4$
$-\sqrt{3}/2$	$5\pi/6$



The angles come from the first and second quadrants because the range of $\cos^{-1} x$ is $[0, \pi]$. ❑



6.18 $\cos^{-1} x + \cos^{-1}(-x) = \pi$



6.19 In this figure,
$\sin^{-1} x + \cos^{-1} x = \pi/2$.

Identities Involving Arc Sine and Arc Cosine

As we can see from Fig. 6.18, the arc cosine of x satisfies the identity

$$\cos^{-1} x + \cos^{-1}(-x) = \pi, \tag{5}$$

or

$$\cos^{-1}(-x) = \pi - \cos^{-1} x. \tag{6}$$

And we can see from the triangle in Fig. 6.19 that for $x > 0$,

$$\sin^{-1} x + \cos^{-1} x = \pi/2. \tag{7}$$

Equation (7) holds for the other values of x in $[-1, 1]$ as well, but we cannot conclude this from the triangle in Fig. 6.19. It is, however, a consequence of Eqs. (4) and (6) (Exercise 55).

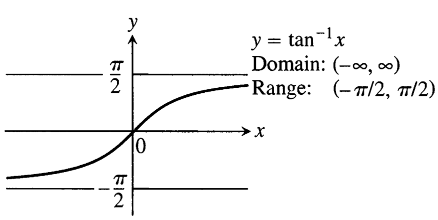
Inverses of tan x, cot x, sec x, and csc x

The arc tangent of x is an angle whose tangent is x. The arc cotangent of x is an angle whose cotangent is x.

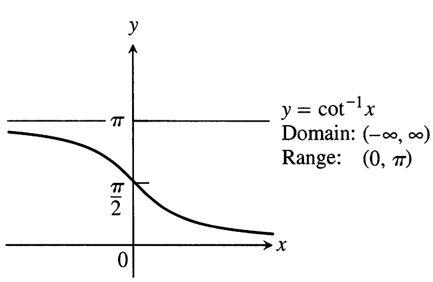
Definition

$\mathbf{y = tan^{-1} x}$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$\mathbf{y = cot^{-1} x}$ is the number in $(0, \pi)$ for which $\cot y = x$.



6.20 The graph of $y = \tan^{-1} x$.



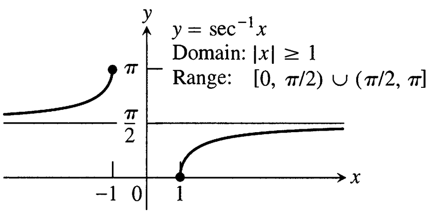
6.21 The graph of $y = \cot^{-1} x$.

We use open intervals to avoid values where the tangent and cotangent are undefined.

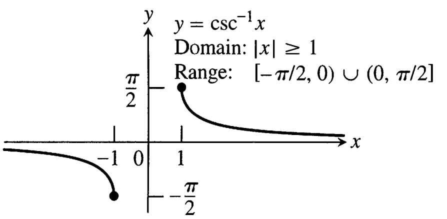
The graph of $y = \tan^{-1} x$ is symmetric about the origin because it is a branch of the graph $x = \tan y$ that is symmetric about the origin (Fig. 6.20). Algebraically this means that

$$\tan^{-1}(-x) = -\tan^{-1} x; \tag{8}$$

the arc tangent is an odd function. The graph of $y = \cot^{-1} x$ has no such symmetry (Fig. 6.21).



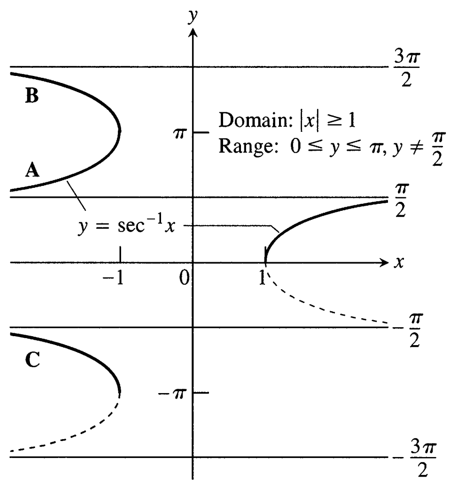
6.22 The graph of $y = \sec^{-1} x$.



6.23 The graph of $y = \csc^{-1} x$.

The inverses of the restricted forms of $\sec x$ and $\csc x$ are chosen to be the functions graphed in Figs. 6.22 and 6.23.

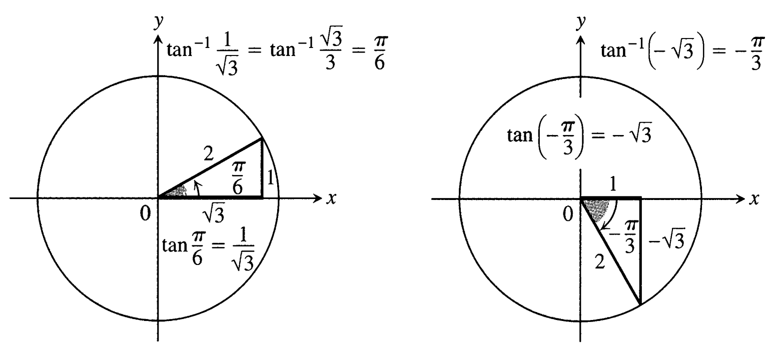
Caution There is no general agreement about how to define $\sec^{-1} x$ for negative values of x. We chose angles in the second quadrant between $\pi/2$ and π. This choice makes $\sec^{-1} x = \cos^{-1}(1/x)$. It also makes $\sec^{-1} x$ an increasing function on each interval of its domain. Some tables choose $\sec^{-1} x$ to lie in $[-\pi, -\pi/2)$ for $x < 0$ and some texts choose it to lie in $[\pi, 3\pi/2)$ (Fig. 6.24). These choices simplify the formula for the derivative (our formula needs absolute value signs) but fail to satisfy the computational equation $\sec^{-1} x = \cos^{-1}(1/x)$.



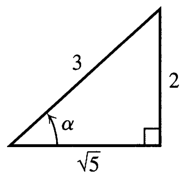
6.24 There are several logical choices for the left-hand branch of $y = \sec^{-1} x$. With choice **A**, Eq. (1) holds, but the formula for the derivative of the arc secant is complicated by absolute value bars. Choices **B** and **C** lead to a simpler derivative formula, but Eq. (1) no longer holds. Most calculators use Eq. (1), so we chose **A**.

EXAMPLE 3 *Common values of* $\tan^{-1} x$

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$



The angles come from the first and fourth quadrants because the range of $\tan^{-1} x$ is $[-\pi/2, \pi/2]$. ❑



6.25 If $\alpha = \sin^{-1}(2/3)$, then the values of the other basic trigonometric functions of α can be read from this triangle (Example 4).

EXAMPLE 4 Find $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$ if

$$\alpha = \sin^{-1}\frac{2}{3}. \tag{12}$$

Solution Equation (12) says that $\sin\alpha = 2/3$. We picture α as an angle in a right triangle with opposite side 2 and hypotenuse 3 (Fig. 6.25). The length of the remaining side is

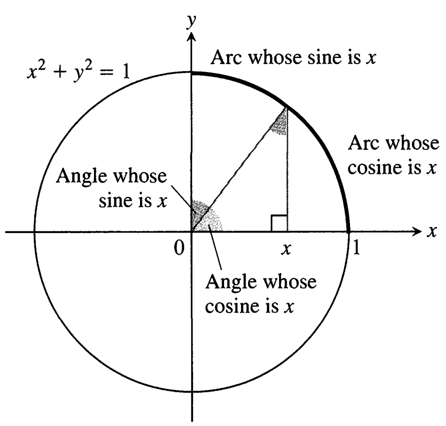
$$\sqrt{(3)^2 - (2)^2} = \sqrt{9-4} = \sqrt{5}. \qquad \text{Pythagorean theorem}$$

We add this information to the figure and then read the values we want from the completed triangle:

$$\cos\alpha = \frac{\sqrt{5}}{3}, \quad \tan\alpha = \frac{2}{\sqrt{5}}, \quad \sec\alpha = \frac{3}{\sqrt{5}}, \quad \csc\alpha = \frac{3}{2}, \quad \cot\alpha = \frac{\sqrt{5}}{2}.$$ ❑

The "arc" in arc sine and arc cosine

In case you are wondering about the "arc," look at the accompanying figure. It gives a geometric interpretation of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ for angles in the first quadrant. For a unit circle, the equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x." When angles were measured by intercepted arc lengths, as they once were, this was a natural way to speak. Today it can sound a bit strange, but the language has stayed with us. The arc cosine has a similar interpretation.

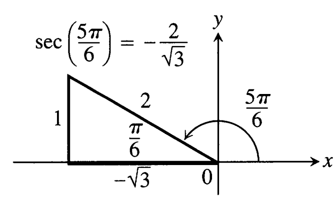


EXAMPLE 5 Find $\cot\left(\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) + \csc^{-1}(-2)\right)$.

Solution We work from inside out, using reference triangles to exhibit ratios and angles.

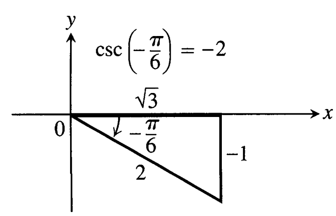
Step 1: Negative values of the secant come from second-quadrant angles:

$$\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \sec^{-1}\left(\frac{2}{-\sqrt{3}}\right) = \frac{5\pi}{6}.$$



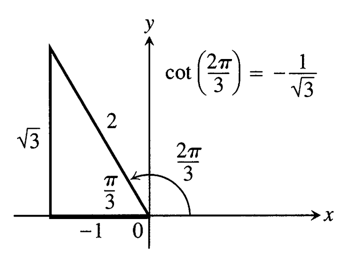
Step 2: Negative values of the cosecant come from fourth-quadrant angles:

$$\csc^{-1}(-2) = \csc^{-1}\left(\frac{2}{-1}\right) = -\frac{\pi}{6}.$$



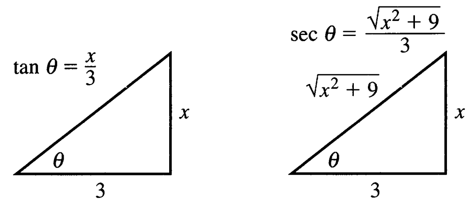
Step 3:

$$\begin{aligned}\cot\left(\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) + \csc^{-1}(-2)\right) &= \cot\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) \\ &= \cot\left(\frac{2\pi}{3}\right) \\ &= -\frac{1}{\sqrt{3}}\end{aligned}$$



❑

EXAMPLE 6 Find $\sec\left(\tan^{-1}\dfrac{x}{3}\right)$.



Solution We let $\theta = \tan^{-1}(x/3)$ (to give the angle a name) and picture θ in a right triangle with

$$\tan\theta = \text{opposite}/\text{adjacent} = x/3.$$

The length of the triangle's hypotenuse is

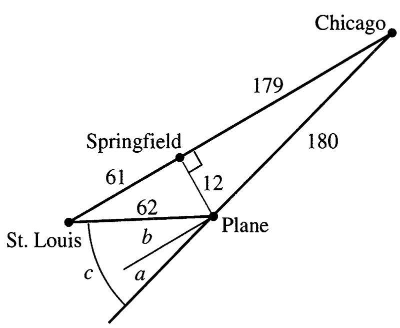
$$\sqrt{x^2+3^2} = \sqrt{x^2+9}.$$

Thus,

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \sec\theta$$

$$= \frac{\sqrt{x^2+9}}{3}. \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

❑



6.26 Diagram for drift correction (Example 7), with distances rounded to the nearest mile (drawing not to scale).

EXAMPLE 7 *Drift correction*

During an airplane flight from Chicago to St. Louis the navigator determines that the plane is 12 mi off course, as shown in Fig. 6.26. Find the angle a for a course parallel to the original, correct course, the angle b, and the correction angle $c = a + b$.

Solution

$$a = \sin^{-1}\frac{12}{180} \approx 0.067 \text{ radian} \approx 3.8^\circ$$

$$b = \sin^{-1}\frac{12}{62} \approx 0.195 \text{ radian} \approx 11.2^\circ$$

$$c = a + b \approx 15^\circ.$$

❑

Exercises 6.8

Common Values of Inverse Trignonometric Functions

Use reference triangles like those in Examples 1–3 to find the angles in Exercises 1–12.

1. a) $\tan^{-1} 1$ **b)** $\tan^{-1}(-\sqrt{3})$ **c)** $\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

2. a) $\tan^{-1}(-1)$ **b)** $\tan^{-1}\sqrt{3}$ **c)** $\tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$

3. a) $\sin^{-1}\left(\dfrac{-1}{2}\right)$ **b)** $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ **c)** $\sin^{-1}\left(\dfrac{-\sqrt{3}}{2}\right)$

4. a) $\sin^{-1}\left(\frac{1}{2}\right)$ b) $\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ c) $\sin^{-1}\frac{\sqrt{3}}{2}\Big)$

5. a) $\cos^{-1}\left(\frac{1}{2}\right)$ b) $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ c) $\cos^{-1}\frac{\sqrt{3}}{2}\Big)$

6. a) $\cos^{-1}\left(\frac{-1}{2}\right)$ b) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ c) $\cos^{-1}\frac{-\sqrt{3}}{2}\Big)$

7. a) $\sec^{-1}(-\sqrt{2})$ b) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ c) $\sec^{-1}(-2)$

8. a) $\sec^{-1}\sqrt{2}$ b) $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ c) $\sec^{-1} 2$

9. a) $\csc^{-1}\sqrt{2}$ b) $\csc^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ c) $\csc^{-1} 2$

10. a) $\csc^{-1}(-\sqrt{2})$ b) $\csc^{-1}\left(\frac{2}{\sqrt{3}}\right)$ c) $\csc^{-1}(-2)$

11. a) $\cot^{-1}(-1)$ b) $\cot^{-1}\sqrt{3}$ c) $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

12. a) $\cot^{-1} 1$ b) $\cot^{-1}(-\sqrt{3})$ c) $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Trigonometric Function Values

13. Given that $\alpha = \sin^{-1}(5/13)$, find $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

14. Given that $\alpha = \tan^{-1}(4/3)$, find $\sin\alpha$, $\cos\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

15. Given that $\alpha = \sec^{-1}(-\sqrt{5})$, find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\csc\alpha$, and $\cot\alpha$.

16. Given that $\alpha = \sec^{-1}(-\sqrt{13}/2)$, find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\csc\alpha$, and $\cot\alpha$.

Evaluating Trigonometric and Inverse Trigonometric Terms

Find the values in Exercises 17–28.

17. $\sin\ \cos^{-1}\frac{\sqrt{2}}{2}\Big)$

18. $\sec\left(\cos^{-1}\frac{1}{2}\right)$

19. $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

20. $\cot\ \sin^{-1}\ -\frac{\sqrt{3}}{2}\Big)\Big)$

21. $\csc(\sec^{-1} 2) + \cos(\tan^{-1}(-\sqrt{3}))$

22. $\tan(\sec^{-1} 1) + \sin(\csc^{-1}(-2))$

23. $\sin\left(\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)\right)$

24. $\cot\left(\sin^{-1}\left(-\frac{1}{2}\right) - \sec^{-1} 2\right)$

25. $\sec(\tan^{-1} 1 + \csc^{-1} 1)$

26. $\sec(\cot^{-1}\sqrt{3} + \csc^{-1}(-1))$

27. $\sec^{-1}\left(\sec\left(-\frac{\pi}{6}\right)\right)$ (The answer is *not* $-\pi/6$.)

28. $\cot^{-1}\left(\cot\left(-\frac{\pi}{4}\right)\right)$ (The answer is *not* $-\pi/4$.)

Finding Trigonometric Expressions

Evaluate the expressions in Exercises 29–40.

29. $\sec\left(\tan^{-1}\frac{x}{2}\right)$

30. $\sec(\tan^{-1} 2x)$

31. $\tan(\sec^{-1} 3y)$

32. $\tan\left(\sec^{-1}\frac{y}{5}\right)$

33. $\cos(\sin^{-1} x)$

34. $\tan(\cos^{-1} x)$

35. $\sin(\tan^{-1}\sqrt{x^2 - 2x})$, $x \geq 2$

36. $\sin\left(\tan^{-1}\frac{x}{\sqrt{x^2+1}}\right)$

37. $\cos\left(\sin^{-1}\frac{2y}{3}\right)$

38. $\cos\left(\sin^{-1}\frac{y}{5}\right)$

39. $\sin\left(\sec^{-1}\frac{x}{4}\right)$

40. $\sin\ \sec^{-1}\frac{\sqrt{x^2+4}}{x}\Big)$

Limits

Find the limits in Exercises 41–48. (If in doubt, look at the function's graph.)

41. $\lim_{x\to 1^-} \sin^{-1} x$

42. $\lim_{x\to -1^+} \cos^{-1} x$

43. $\lim_{x\to\infty} \tan^{-1} x$

44. $\lim_{x\to -\infty} \tan^{-1} x$

45. $\lim_{x\to\infty} \sec^{-1} x$

46. $\lim_{x\to -\infty} \sec^{-1} x$

47. $\lim_{x\to\infty} \csc^{-1} x$

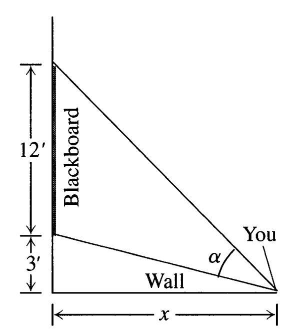
48. $\lim_{x\to -\infty} \csc^{-1} x$

Applications and Theory

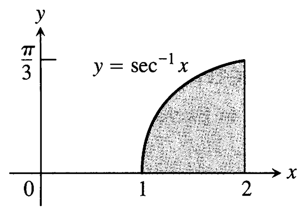
49. You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 ft long and starts 3 ft from the wall you are sitting next to. Show that your viewing angle is

$$\alpha = \cot^{-1}\frac{x}{15} - \cot^{-1}\frac{x}{3}$$

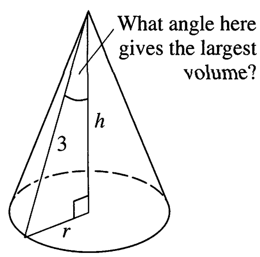
if you are x ft from the front wall.



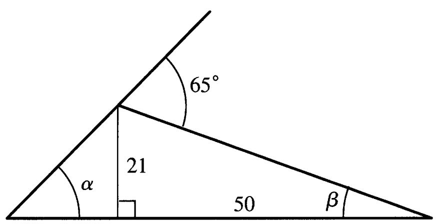
50. The region between the curve $y = \sec^{-1} x$ and the x-axis from $x = 1$ to $x = 2$ (shown here) is revolved about the y-axis to generate a solid. Find the volume of the solid.



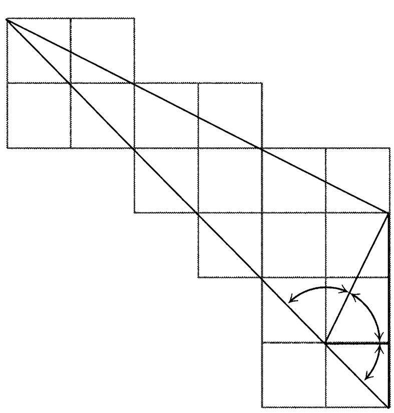
51. The slant height of the cone shown here is 3 m. How large should the indicated angle be to maximize the cone's volume?



52. Find the angle α.

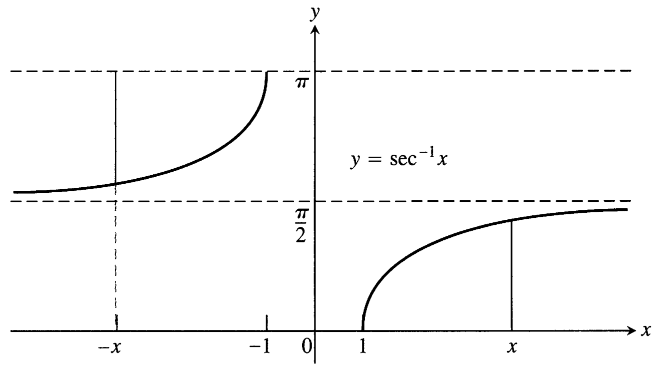


53. Here is an informal proof that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$. Explain what is going on.



54. *Two derivations of the identity* $\sec^{-1}(-x) = \pi - \sec^{-1} x$.

a) (*Geometric*) Here is a pictorial proof that $\sec^{-1}(-x) = \pi - \sec^{-1} x$. See if you can tell what is going on.



b) (*Algebraic*) Derive the identity $\sec^{-1}(-x) = \pi - \sec^{-1} x$ by combining the following two equations from the text:

$$\cos^{-1}(-x) = \pi - \cos^{-1} x \qquad \text{Eq. (6)}$$
$$\sec^{-1} x = \cos^{-1}(1/x) \qquad \text{Eq. (1)}$$

55. *The identity* $\sin^{-1} x + \cos^{-1} x = \pi/2$. Figure 6.19 establishes the identity for $0 < x < 1$. To establish it for the rest of $[-1, 1]$, verify by direct calculation that it holds for $x = 1$, 0, and -1. Then, for values of x in $(-1, 0)$, let $x = -a$, $a > 0$, and apply Eqs. (4) and (6) to the sum $\sin^{-1}(-a) + \cos^{-1}(-a)$.

56. Show that the sum $\tan^{-1} x + \tan^{-1}(1/x)$ is constant.

Which of the expressions in Exercises 57–60 are defined, and which are not? Give reasons for your answers.

57. a) $\tan^{-1} 2$ **b)** $\cos^{-1} 2$

58. a) $\csc^{-1} \frac{1}{2}$ **b)** $\csc^{-1} 2$

59. a) $\sec^{-1} 0$ **b)** $\sin^{-1} \sqrt{2}$

60. a) $\cot^{-1}\left(-\frac{1}{2}\right)$ **b)** $\cos^{-1}(-5)$

Calculator Explorations

61. Find the values of

a) $\sec^{-1} 1.5$ **b)** $\csc^{-1}(-1.5)$ **c)** $\cot^{-1} 2$

62. Find the values of

a) $\sec^{-1}(-3)$ **b)** $\csc^{-1} 1.7$ **c)** $\cot^{-1}(-2)$

Grapher Explorations

In Exercises 63–65, find the domain and range of each composite function. Then graph the composites on separate screens. Do the graphs make sense in each case? Give reasons for your answers. Comment on any differences you see.

63. a) $y = \tan^{-1}(\tan x)$ **b)** $y = \tan(\tan^{-1} x)$

64. a) $y = \sin^{-1}(\sin x)$ **b)** $y = \sin(\sin^{-1} x)$

65. a) $y = \cos^{-1}(\cos x)$ **b)** $y = \cos(\cos^{-1} x)$

66. Graph $y = \sec(\sec^{-1} x) = \sec(\cos^{-1}(1/x))$. Explain what you see.

67. *Newton's serpentine.* Graph Newton's serpentine, $y = 4x/(x^2+1)$. Then graph $y = 2\sin(2\tan^{-1} x)$ in the same graphing window. What do you see? Explain.

68. Graph the rational function $y = (2-x^2)/x^2$. Then graph $y = \cos(2\sec^{-1} x)$ in the same graphing window. What do you see? Explain.

6.9 Derivatives of Inverse Trigonometric Functions; Integrals

Inverse trigonometric functions provide antiderivatives for a variety of functions that arise in mathematics, engineering, and physics. In this section we find the derivatives of the inverse trigonometric functions (Table 6.5) and discuss related integrals.

Table 6.5 Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$$
2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$$
3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{du/dx}{1+u^2}$$
4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{du/dx}{1+u^2}$$
5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$$
6. $$\frac{d(\csc^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$$

EXAMPLE 1

a) $$\frac{d}{dx}\sin^{-1}(x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}$$

b) $$\frac{d}{dx}\tan^{-1}\sqrt{x+1} = \frac{1}{1+(\sqrt{x+1})^2} \cdot \frac{d}{dx}(\sqrt{x+1})$$
$$= \frac{1}{x+2} \cdot \frac{1}{2\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}(x+2)}$$

c) $$\frac{d}{dx}\sec^{-1}(-3x) = \frac{1}{|-3x|\sqrt{(-3x)^2-1}} \cdot \frac{d}{dx}(-3x)$$
$$= \frac{-3}{|3x|\sqrt{9x^2-1}} = \frac{-1}{|x|\sqrt{9x^2-1}}.$$ ❑

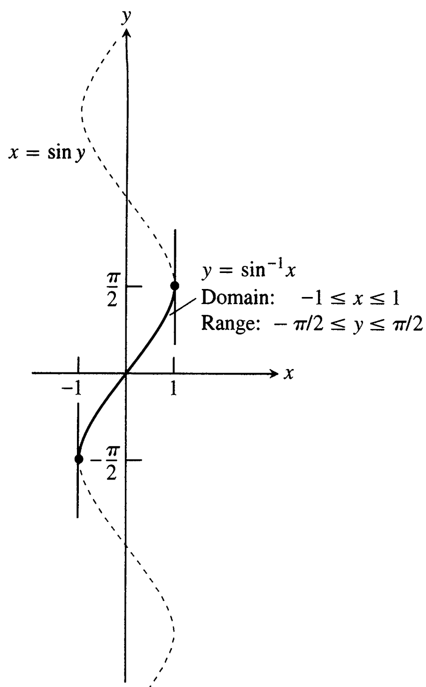
EXAMPLE 2

$$\int_0^1 \frac{e^{\tan^{-1} x}}{1+x^2}dx = \int_0^{\pi/4} e^u du \qquad u = \tan^{-1} x, \quad du = \frac{dx}{1+x^2}, \quad u(0) = 0, \quad u(1) = \pi/4$$
$$= e^u\Big]_0^{\pi/4} = e^{\pi/4} - 1$$ ❑

We derive Formulas 1 and 5 from Table 6.5. The derivation of Formula 3 is similar. Formulas 2, 4, and 6 can be derived from Formulas 1, 3, and 5 by differentiating appropriate identities (Exercises 81–83).

The Derivative of $y \sin^{-1} u$

We know that the function $x = \sin y$ is differentiable in the interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Theorem 1 in Section 6.1 therefore assures us that the inverse function $y = \sin^{-1} x$ is differentiable throughout



6.27 The graph of $y = \sin^{-1} x$ has vertical tangents at $x = -1$ and $x = 1$.

the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = 1$ or $x = -1$ because the tangents to the graph are vertical at these points (see Fig. 6.27).

We find the derivative of $y = \sin^{-1} x$ as follows:

$$\sin y = x \qquad y = \sin^{-1} x \quad \Leftrightarrow \quad \sin y = x$$

$$\frac{d}{dx}(\sin y) = 1 \qquad \text{Derivative of both sides with respect to } x$$

$$\cos y \frac{dy}{dx} = 1 \qquad \text{Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \qquad \text{We can divide because } \cos y > 0 \text{ for } -\pi/2 < y < \pi/2.$$

$$= \frac{1}{\sqrt{1 - x^2}} \qquad \text{Fig. 6.28}$$

The derivative of $y = \sin^{-1} x$ with respect to x is

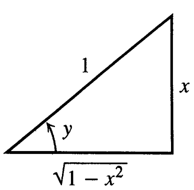
$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

to $y = \sin^{-1} u$ to obtain

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}, \qquad |u| < 1.$$



6.28 In the reference right triangle above,

$$\sin y = \frac{x}{1} = x,$$

$$\cos y = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}.$$

The Derivative of $y = \sec^{-1} u$

We find the derivative of $y = \sec^{-1} x$, $|x| > 1$, in a similar way.

$$\sec y = x \qquad y = \sec^{-1} x \quad \Leftrightarrow \quad \sec y = x$$

$$\frac{d}{dx}(\sec y) = 1 \qquad \text{Derivative of both sides with respect to } x$$

$$\sec y \tan y \frac{dy}{dx} = 1 \qquad \text{Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} \qquad \text{Since } |x| > 1,\ y \text{ lies in } (0, \pi/2) \cup (\pi/2, \pi) \text{ and } \sec y \tan y \ne 0.$$

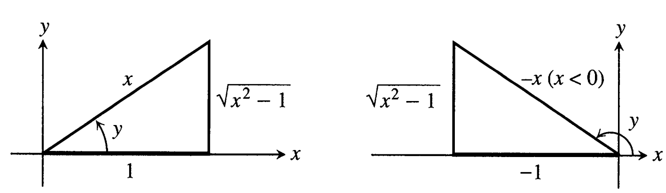
$$= \pm \frac{1}{x\sqrt{x^2 - 1}} \qquad \text{Fig. 6.29}$$

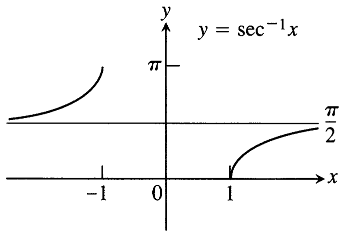
6.29 In both quadrants, $\sec y = x$. In the first quadrant,

$$\tan y = \sqrt{x^2 - 1}/1 = \sqrt{x^2 - 1}.$$

In the second quadrant,

$$\tan y = \sqrt{x^2 - 1}/(-1) = -\sqrt{x^2 - 1}.$$





6.30 The slope of the curve $y = \sec^{-1} x$ is positive for both $x < -1$ and $x > 1$.

What do we do about the sign? A glance at Fig. 6.30 shows that for $|x| > 1$ the slope of the graph of $y = \sec^{-1} x$ is always positive. Therefore,

$$\frac{d}{dx}(\sec^{-1} x) = \begin{cases} \dfrac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2-1}} & \text{if } x < -1. \end{cases} \tag{1}$$

With absolute values, we can write Eq. (1) as a single formula:

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \qquad |x| > 1.$$

If u is a differentiable function of x with $|u| > 1$, we can then apply the Chain Rule to obtain

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \qquad |u| > 1.$$

Integration Formulas

The derivative formulas in Table 6.5 yield three useful integration formulas in Table 6.6.

Table 6.6 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$) (2)

2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u) (3)

3. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $u^2 > a^2$) (4)

The derivative formulas in Table 6.5 have $a = 1$, but in most integrations $a \neq 1$, and the formulas in Table 6.6 are more useful. They are readily verified by differentiating the functions on the right-hand sides.

EXAMPLE 3

a) $$\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x)\Big]_{\sqrt{2}/2}^{\sqrt{3}/2}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

b) $$\int_0^1 \frac{dx}{1+x^2} = \tan^{-1}(x)\Big]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

c) $$\int_{2/\sqrt{3}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}(x)\Big]_{2/\sqrt{3}}^{\sqrt{2}} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$ ❑

EXAMPLE 4

a) $$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{(3)^2 - x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$$ Eq. (2) with $a = 3$, $u = x$

b) $$\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2 - u^2}}$$ $a = \sqrt{3}$, $u = 2x$, and $du/2 = dx$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$ Eq. (2)

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

EXAMPLE 5 Evaluate $\displaystyle\int \frac{dx}{\sqrt{4x - x^2}}$.

For more about completing the square, see the end papers of this book.

Solution The expression $\sqrt{4x - x^2}$ does not match any of the formulas in Table 6.6, so we first rewrite $4x - x^2$ by completing the square:

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = 4 - (x-2)^2.$$

Then we substitute $a = 2$, $u = x - 2$, and $du = dx$ to get

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x-2)^2}}$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}}$$ $a = 2$, $u = x - 2$, and $du = dx$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C$$ Eq. (2)

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

EXAMPLE 6

a) $$\int \frac{dx}{10 + x^2} = \frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{x}{\sqrt{10}}\right) + C$$ Eq. (3) with $a = \sqrt{10}$, $u = x$

b) $$\int \frac{dx}{7 + 3x^2} = \frac{1}{\sqrt{3}}\int \frac{du}{a^2 + u^2}$$ $a = \sqrt{7}$, $u = \sqrt{3}x$, and $du/\sqrt{3} = dx$

$$= \frac{1}{\sqrt{3}} \cdot \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$ Eq. (3)

$$= \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{\sqrt{3}x}{\sqrt{7}}\right) + C$$

$$= \frac{1}{\sqrt{21}}\tan^{-1}\left(\frac{\sqrt{3}x}{\sqrt{7}}\right) + C$$

EXAMPLE 7 Evaluate $\displaystyle\int \frac{dx}{4x^2 + 4x + 2}$.

Solution We complete the square on the binomial $4x^2 + 4x$:

$$4x^2 + 4x + 2 = 4(x^2 + x) + 2 = 4\left(x^2 + x + \frac{1}{4}\right) + 2 - \frac{4}{4}$$

$$= 4\left(x + \frac{1}{2}\right)^2 + 1 = (2x + 1)^2 + 1.$$

Then we substitute $a = 1$, $u = 2x + 1$, and $du/2 = dx$ to get

$$\int \frac{dx}{4x^2 + 4x + 2} = \int \frac{dx}{(2x+1)^2 + 1} = \frac{1}{2}\int \frac{du}{u^2 + a^2} \qquad a = 1,\ u = 2x + 1, \text{ and } du/2 = dx$$

$$= \frac{1}{2} \cdot \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) \qquad \text{Eq. (3)}$$

$$= \frac{1}{2} \tan^{-1}(2x + 1) + C \qquad a = 1,\ u = 2x + 1$$

❑

EXAMPLE 8 Evaluate $\displaystyle\int \frac{dx}{x\sqrt{4x^2 - 5}}$.

Solution

$$\int \frac{dx}{x\sqrt{4x^2 - 5}} = \int \frac{\frac{du}{2}}{\frac{u}{2}\sqrt{u^2 - a^2}} \qquad u = 2x,\ x = u/2,\ dx = du/2,\ a = \sqrt{5}$$

$$= \int \frac{du}{u\sqrt{u^2 - a^2}} \qquad \text{The 2's cancel.}$$

$$= \frac{1}{a} \sec^{-1}\left|\frac{u}{a}\right| + C \qquad \text{Eq. (4)}$$

$$= \frac{1}{\sqrt{5}} \sec^{-1}\left(\frac{2|x|}{\sqrt{5}}\right) + C \qquad a = \sqrt{5},\quad u = 2x$$

❑

EXAMPLE 9 Evaluate $\displaystyle\int \frac{dx}{\sqrt{e^{2x} - 6}}$.

Solution

$$\int \frac{dx}{\sqrt{e^{2x} - 6}} = \int \frac{du/u}{\sqrt{u^2 - a^2}} \qquad u = e^x,\ du = e^x dx,\ dx = du/e^x = du/u,\ a = \sqrt{6}$$

$$= \int \frac{du}{u\sqrt{u^2 - a^2}}$$

$$= \frac{1}{a} \sec^{-1}\left|\frac{u}{a}\right| + C \qquad \text{Eq. (4)}$$

$$= \frac{1}{\sqrt{6}} \sec^{-1}\left(\frac{e^x}{\sqrt{6}}\right) + C$$

❑

Exercises 6.9

Finding Derivatives

In Exercises 1–22, find the derivative of y with respect to the appropriate variable.

1. $y = \cos^{-1}(x^2)$

2. $y = \cos^{-1}(1/x)$

3. $y = \sin^{-1}\sqrt{2}\,t$

4. $y = \sin^{-1}(1 - t)$

5. $y = \sec^{-1}(2s + 1)$

6. $y = \sec^{-1} 5s$

7. $y = \csc^{-1}(x^2 + 1), \quad x > 0$

8. $y = \csc^{-1}\dfrac{x}{2}$

9. $y = \sec^{-1}\dfrac{1}{t}, \quad 0 < t < 1$

10. $y = \sin^{-1}\dfrac{3}{t^2}$

11. $y = \cot^{-1}\sqrt{t}$

12. $y = \cot^{-1}\sqrt{t - 1}$

13. $y = \ln(\tan^{-1} x)$

14. $y = \tan^{-1}(\ln x)$

15. $y = \csc^{-1}(e^t)$

16. $y = \cos^{-1}(e^{-t})$

17. $y = s\sqrt{1 - s^2} + \cos^{-1} s$

18. $y = \sqrt{s^2 - 1} - \sec^{-1} s$

19. $y = \tan^{-1}\sqrt{x^2 - 1} + \csc^{-1} x, \quad x > 1$

20. $y = \cot^{-1}\dfrac{1}{x} - \tan^{-1} x$

21. $y = x\sin^{-1} x + \sqrt{1 - x^2}$

22. $y = \ln(x^2 + 4) - x\tan^{-1}\left(\dfrac{x}{2}\right)$

Evaluating Integrals

Evaluate the integrals in Exercises 23–46.

23. $\displaystyle\int \frac{dx}{\sqrt{9 - x^2}}$

24. $\displaystyle\int \frac{dx}{\sqrt{1 - 4x^2}}$

25. $\displaystyle\int \frac{dx}{17 + x^2}$

26. $\displaystyle\int \frac{dx}{9 + 3x^2}$

27. $\displaystyle\int \frac{dx}{x\sqrt{25x^2 - 2}}$

28. $\displaystyle\int \frac{dx}{x\sqrt{5x^2 - 4}}$

29. $\displaystyle\int_0^1 \frac{4\,ds}{\sqrt{4 - s^2}}$

30. $\displaystyle\int_0^{3\sqrt{2}/4} \frac{ds}{\sqrt{9 - 4s^2}}$

31. $\displaystyle\int_0^2 \frac{dt}{8 + 2t^2}$

32. $\displaystyle\int_{-2}^2 \frac{dt}{4 + 3t^2}$

33. $\displaystyle\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2 - 1}}$

34. $\displaystyle\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2 - 1}}$

35. $\displaystyle\int \frac{3\,dr}{\sqrt{1 - 4(r - 1)^2}}$

36. $\displaystyle\int \frac{6\,dr}{\sqrt{4 - (r + 1)^2}}$

37. $\displaystyle\int \frac{dx}{2 + (x - 1)^2}$

38. $\displaystyle\int \frac{dx}{1 + (3x + 1)^2}$

39. $\displaystyle\int \frac{dx}{(2x - 1)\sqrt{(2x - 1)^2 - 4}}$

40. $\displaystyle\int \frac{dx}{(x + 3)\sqrt{(x + 3)^2 - 25}}$

41. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\,d\theta}{1 + (\sin\theta)^2}$

42. $\displaystyle\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\,dx}{1 + (\cot x)^2}$

43. $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x\,dx}{1 + e^{2x}}$

44. $\displaystyle\int_1^{e^{\pi/4}} \frac{4\,dt}{t(1 + \ln^2 t)}$

45. $\displaystyle\int \frac{y\,dy}{\sqrt{1 - y^4}}$

46. $\displaystyle\int \frac{\sec^2 y\,dy}{\sqrt{1 - \tan^2 y}}$

Evaluate the integrals in Exercises 47–56.

47. $\displaystyle\int \frac{dx}{\sqrt{-x^2 + 4x - 3}}$

48. $\displaystyle\int \frac{dx}{\sqrt{2x - x^2}}$

49. $\displaystyle\int_{-1}^0 \frac{6\,dt}{\sqrt{3 - 2t - t^2}}$

50. $\displaystyle\int_{1/2}^1 \frac{6\,dt}{\sqrt{3 + 4t - 4t^2}}$

51. $\displaystyle\int \frac{dy}{y^2 - 2y + 5}$

52. $\displaystyle\int \frac{dy}{y^2 + 6y + 10}$

53. $\displaystyle\int_1^2 \frac{8\,dx}{x^2 - 2x + 2}$

54. $\displaystyle\int_2^4 \frac{2\,dx}{x^2 - 6x + 10}$

55. $\displaystyle\int \frac{dx}{(x + 1)\sqrt{x^2 + 2x}}$

56. $\displaystyle\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 3}}$

Evaluate the integrals in Exercises 57–64.

57. $\displaystyle\int \frac{e^{\sin^{-1} x}dx}{\sqrt{1 - x^2}}$

58. $\displaystyle\int \frac{e^{\cos^{-1} x}dx}{\sqrt{1 - x^2}}$

59. $\displaystyle\int \frac{(\sin^{-1} x)^2 dx}{\sqrt{1 - x^2}}$

60. $\displaystyle\int \frac{\sqrt{\tan^{-1} x}\,dx}{1 + x^2}$

61. $\displaystyle\int \frac{dy}{(\tan^{-1} y)(1 + y^2)}$

62. $\displaystyle\int \frac{dy}{(\sin^{-1} y)\sqrt{1 - y^2}}$

63. $\displaystyle\int_{\sqrt{2}}^2 \frac{\sec^2(\sec^{-1} x)\,dx}{x\sqrt{x^2 - 1}}$

64. $\displaystyle\int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1} x)\,dx}{x\sqrt{x^2 - 1}}$

Limits

Find the limits in Exercises 65–68.

65. $\displaystyle\lim_{x\to 0} \frac{\sin^{-1} 5x}{x}$

66. $\displaystyle\lim_{x\to 1^+} \frac{\sqrt{x^2 - 1}}{\sec^{-1} x}$

67. $\displaystyle\lim_{x\to\infty} x\tan^{-1}\frac{2}{x}$

68. $\displaystyle\lim_{x\to 0} \frac{2\tan^{-1} 3x^2}{7x^2}$

Integration Formulas

Verify the integration formulas in Exercises 69–72.

69. $\displaystyle\int \frac{\tan^{-1} x}{x^2}\,dx = \ln x - \frac{1}{2}\ln(1 + x^2) - \frac{\tan^{-1} x}{x} + C$

70. $\displaystyle\int x^3 \cos^{-1} 5x\, dx = \frac{x^4}{4}\cos^{-1} 5x + \frac{5}{4}\int \frac{x^4\, dx}{\sqrt{1-25x^2}}$

71. $\displaystyle\int (\sin^{-1} x)^2\, dx = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1-x^2}\sin^{-1} x + C$

72. $\displaystyle\int \ln\,(a^2+x^2)\, dx = x\ln\,(a^2+x^2) - 2x + 2a\tan^{-1}\frac{x}{a} + C$

Initial Value Problems

Solve the initial value problems in Exercises 73–76.

73. $\displaystyle\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad y(0) = 0$

74. $\displaystyle\frac{dy}{dx} = \frac{1}{x^2+1} - 1, \quad y(0) = 1$

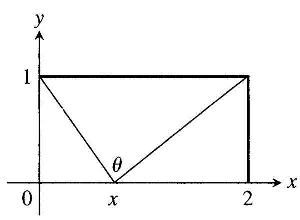
75. $\displaystyle\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}, \quad x > 1; \quad y(2) = \pi$

76. $\displaystyle\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}}, \quad y(0) = 2$

Theory and Examples

77. (*Continuation of Exercise 49, Section 6.8.*) You want to position your chair along the wall to maximize your viewing angle α. How far from the front of the room should you sit?

78. What value of x maximizes the angle θ shown here? How large is θ at that point? Begin by showing that $\theta = \pi - \cot^{-1} x - \cot^{-1}(2-x)$.



79. Can the integrations in (a) and (b) both be correct? Explain.

a) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

b) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = -\int -\frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$

80. Can the integrations in (a) and (b) both be correct? Explain.

a) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = -\int -\frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$

b) $$\begin{aligned}\int \frac{dx}{\sqrt{1-x^2}} &= \int \frac{-du}{\sqrt{1-(-u)^2}} && \begin{array}{l} x = -u, \\ dx = -du \end{array} \\ &= \int \frac{-du}{\sqrt{1-u^2}} \\ &= \cos^{-1} u + C \\ &= \cos^{-1}(-x) + C && u = -x \end{aligned}$$

81. Use the identity
$$\cos^{-1} u = \frac{\pi}{2} - \sin^{-1} u$$
to derive the formula for the derivative of $\cos^{-1} u$ in Table 6.5 from the formula for the derivative of $\sin^{-1} u$.

82. Use the identity
$$\cot^{-1} u = \frac{\pi}{2} - \tan^{-1} u$$
to derive the formula for the derivative of $\cot^{-1} u$ in Table 6.5 from the formula for the derivative of $\tan^{-1} u$.

83. Use the identity
$$\csc^{-1} u = \frac{\pi}{2} - \sec^{-1} u$$
to derive the formula for the derivative of $\csc^{-1} u$ in Table 6.5 from the formula for the derivative of $\sec^{-1} u$.

84. Derive the formula
$$\frac{dy}{dx} = \frac{1}{1+x^2}$$
for the derivative of $y = \tan^{-1} x$ by differentiating both sides of the equivalent equation $\tan y = x$.

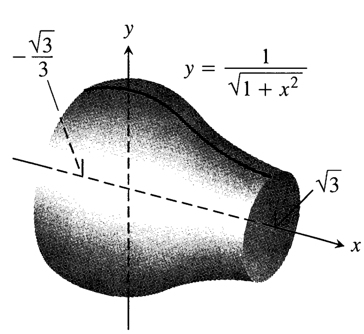
85. Use the Derivative Rule in Section 6.1, Theorem 1, to derive
$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1.$$

86. Use the Derivative Rule in Section 6.1, Theorem 1, to derive
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}.$$

87. What is special about the functions
$$f(x) = \sin^{-1}\frac{x-1}{x+1}, \quad x \ge 0, \quad \text{and} \quad g(x) = 2\tan^{-1}\sqrt{x}?$$
Explain.

88. What is special about the functions
$$f(x) = \sin^{-1}\frac{1}{\sqrt{x^2+1}} \quad \text{and} \quad g(x) = \tan^{-1}\frac{1}{x}?$$
Explain.

89. Find the volume of the solid of revolution shown here.



90. Find the length of the curve $y = \sqrt{1-x^2}$, $-1/2 \le x \le 1/2$.

Volumes by Slicing

Find the volumes of the solids in Exercises 91 and 92.

91. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross sections perpendicular to the x-axis are

a) circles whose diameters stretch from the curve $y = -1/\sqrt{1+x^2}$ to the curve $y = 1/\sqrt{1+x^2}$;

b) vertical squares whose base edges run from the curve $y = -1/\sqrt{1+x^2}$ to the curve $y = 1/\sqrt{1+x^2}$.

92. The solid lies between planes perpendicular to the x-axis at $x = -\sqrt{2}/2$ and $x = \sqrt{2}/2$. The cross sections are

a) circles whose diameters stretch from the x-axis to the curve $y = 2/\sqrt[4]{1-x^2}$.

b) squares whose diagonals stretch from the x-axis to the curve $y = 2/\sqrt[4]{1-x^2}$.

Calculator and Grapher Explorations

93. CALCULATOR Use numerical integration to estimate the value of

$$\sin^{-1} 0.6 = \int_0^{0.6} \frac{dx}{\sqrt{1-x^2}}.$$

For reference, $\sin^{-1} 0.6 = 0.64350$ to 5 places.

94. CALCULATOR Use numerical integration to estimate the value of

$$\pi = 4\int_0^1 \frac{1}{1+x^2}\,dx.$$

95. GRAPHER Graph $f(x) = \sin^{-1} x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f''.

96. GRAPHER Graph $f(x) = \tan^{-1} x$ together with its first two derivatives. Comment on the behavior of f and the shape of its graph in relation to the signs and values of f' and f''.

6.10 Hyperbolic Functions

Every function f that is defined on an interval centered at the origin can be written in a unique way as the sum of one even function and one odd function. The decomposition is

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd part}}.$$

If we write e^x this way, we get

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{even part}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{odd part}}.$$

The notation $\cosh x$ is often read "kosh x," rhyming with either "gosh x" or "gauche x," and $\sinh x$ is pronounced as if spelled "cinch x" or "shine x."

The even and odd parts of e^x, called the hyperbolic cosine and hyperbolic sine of x, respectively, are useful in their own right. They describe the motions of waves in elastic solids, the shapes of hanging electric power lines, and the temperature distributions in metal cooling fins. The center line of the Gateway Arch to the West in St. Louis is a weighted hyperbolic cosine curve.

Definitions and Identities

The hyperbolic cosine and hyperbolic sine functions are defined by the first two equations in Table 6.7. The table also lists the definitions of the hyperbolic tangent, cotangent, secant, and cosecant. As we will see, the hyperbolic functions bear a number of similarities to the trigonometric functions after which they are named. (See Exercise 86 as well.)

Table 6.7 The six basic hyperbolic functions (See Fig. 6.31 for graphs.)

Hyperbolic cosine of x:	$\cosh x = \dfrac{e^x + e^{-x}}{2}$
Hyperbolic sine of x:	$\sinh x = \dfrac{e^x - e^{-x}}{2}$
Hyperbolic tangent:	$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
Hyperbolic cotangent:	$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
Hyperbolic secant:	$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$
Hyperbolic cosecant:	$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

Table 6.8 Identities for hyperbolic functions

$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
$$\cosh^2 x - \sinh^2 x = 1$$
$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Table 6.9 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$
$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$
$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$
$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$
$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$
$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

Table 6.10 Integral formulas for hyperbolic functions

$$\int \sinh u \, du = \cosh u + C$$
$$\int \cosh u \, du = \sinh u + C$$
$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$
$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$
$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$
$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

Identities

Hyperbolic functions satisfy the identities in Table 6.8. Except for differences in sign, these are identities we already know for trigonometric functions.

Derivatives and Integrals

The six hyperbolic functions, being rational combinations of the differentiable functions e^x and e^{-x}, have derivatives at every point at which they are defined (Table 6.9). Again, there are similarities with trigonometric functions. The derivative formulas in Table 6.9 lead to the integral formulas in Table 6.10.

EXAMPLE 1

$$\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \operatorname{sech}^2\sqrt{1+t^2} \cdot \frac{d}{dt}\left(\sqrt{1+t^2}\right) = \frac{t}{\sqrt{1+t^2}} \operatorname{sech}^2\sqrt{1+t^2}$$

❑

EXAMPLE 2

$$\int \coth 5x \, dx = \int \frac{\cosh 5x}{\sinh 5x} dx = \frac{1}{5}\int \frac{du}{u} \qquad \begin{aligned} u &= \sinh 5x,\\ du &= 5\cosh 5x\, dx \end{aligned}$$
$$= \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\sinh 5x| + C$$

❑

EXAMPLE 3

$$\int_0^1 \sinh^2 x \, dx = \int_0^1 \frac{\cosh 2x - 1}{2} dx \qquad \text{Table 6.8}$$
$$= \frac{1}{2}\int_0^1 (\cosh 2x - 1)dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$$
$$= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672$$

❑

Evaluating hyperbolic functions

Like many standard functions, hyperbolic functions and their inverses are easily evaluated with calculators, which have special keys or keystroke sequences for that purpose.

EXAMPLE 4

$$\int_0^{\ln 2} 4e^x \sinh x\,dx = \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2}\,dx = \int_0^{\ln 2} (2e^{2x} - 2)\,dx$$

$$= \left[e^{2x} - 2x\right]_0^{\ln 2} = (e^{2\ln 2} - 2\ln 2) - (1 - 0)$$

$$= 4 - 2\ln 2 - 1$$

$$\approx 1.6137$$

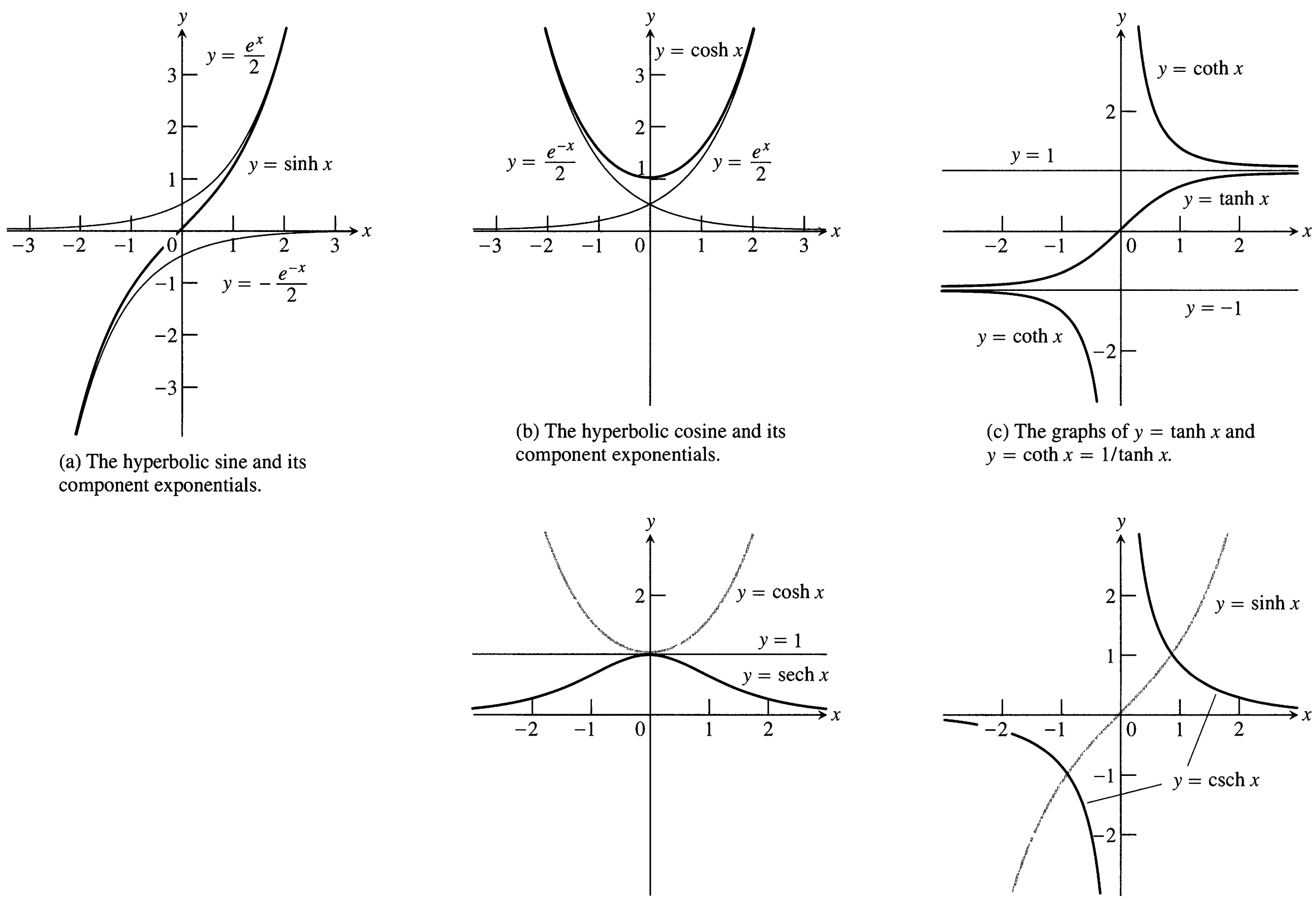
❑

The Inverse Hyperbolic Functions

We use the inverses of the six basic hyperbolic functions in integration. Since $d(\sinh x)/dx = \cosh x > 0$, the hyperbolic sine is an increasing function of x. We denote its inverse by

$$y = \sinh^{-1} x.$$

For every value of x in the interval $-\infty < x < \infty$, the value of $y = \sinh^{-1} x$ is the number whose hyperbolic sine is x. The graphs of $y = \sinh x$ and $y = \sinh^{-1} x$ are shown in Fig. 6.32(a).



(a) The hyperbolic sine and its component exponentials.

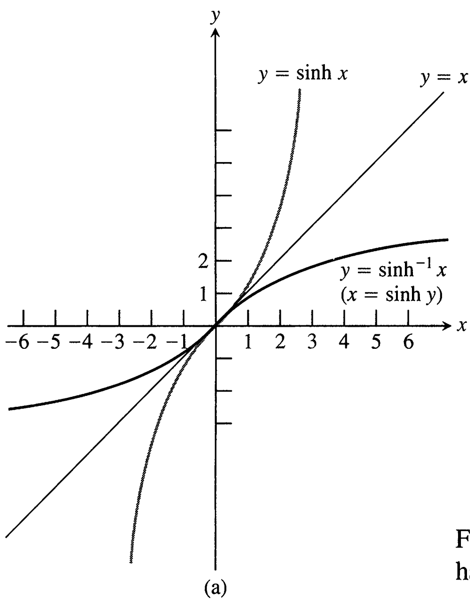
(b) The hyperbolic cosine and its component exponentials.

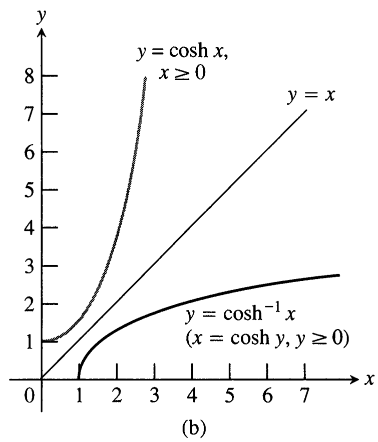
(c) The graphs of $y = \tanh x$ and $y = \coth x = 1/\tanh x$.

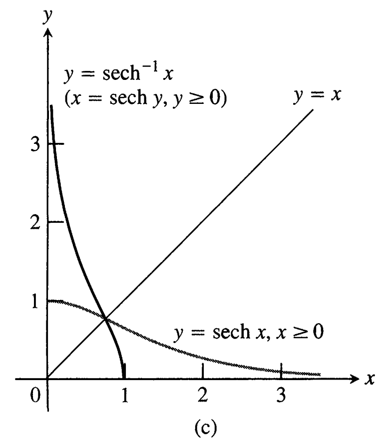
(d) The graphs of $y = \cosh x$ and $y = \operatorname{sech} x = 1/\cosh x$.

(e) The graphs of $y = \sinh x$ and $y = \operatorname{csch} x = 1/\sinh x$.

6.31 The graphs of the six hyperbolic functions.







6.32 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

The function $y = \cosh x$ is not one-to-one, as we can see from the graph in Fig. 6.31. But the restricted function $y = \cosh x$, $x \geq 0$, is one-to-one and therefore has an inverse, denoted by

$$y = \cosh^{-1} x.$$

For every value of $x \geq 1$, $y = \cosh^{-1} x$ is the number in the interval $0 \leq y < \infty$ whose hyperbolic cosine is x. The graphs of $y = \cosh x$, $x \geq 0$, and $y = \cosh^{-1} x$ are shown in Fig. 6.32(b).

Like $y = \cosh x$, the function $y = \operatorname{sech} x = 1/\cosh x$ fails to be one-to-one, but its restriction to nonnegative values of x does have an inverse, denoted by

$$y = \operatorname{sech}^{-1} x.$$

For every value of x in the interval $(0, 1]$, $y = \operatorname{sech}^{-1} x$ is the nonnegative number whose hyperbolic secant is x. The graphs of $y = \operatorname{sech} x$, $x \geq 0$, and $y = \operatorname{sech}^{-1} x$ are shown in Fig. 6.32(c).

The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

$$y = \tanh^{-1} x, \qquad y = \coth^{-1} x, \qquad y = \operatorname{csch}^{-1} x.$$

These functions are graphed in Fig. 6.33.

6.33 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

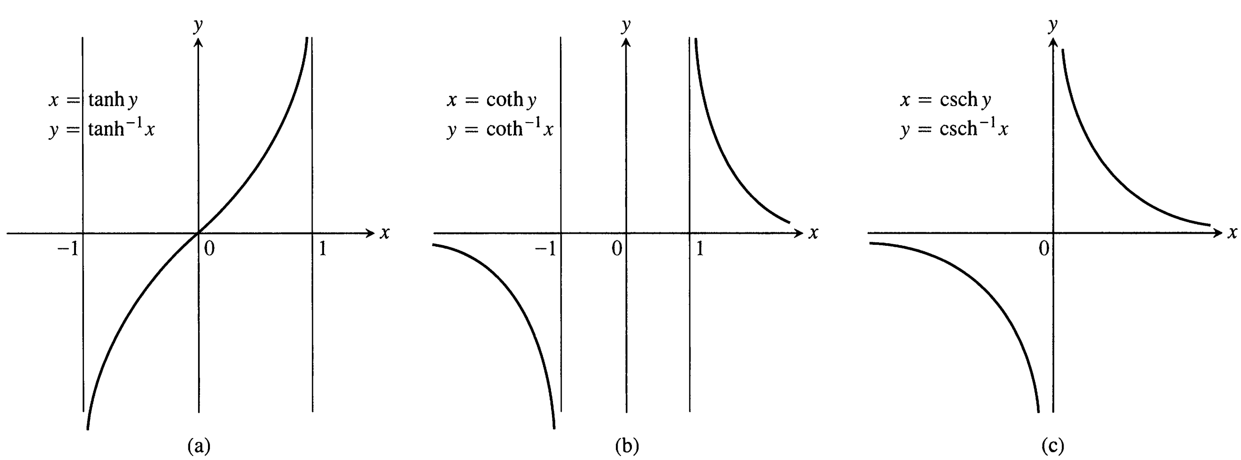


Table 6.11 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

Table 6.12 Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}, \quad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx}, \quad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx}, \quad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = \frac{-du/dx}{u\sqrt{1-u^2}}, \quad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{1+u^2}}, \quad u \neq 0$$

Useful Identities

We use the identities in Table 6.11 to calculate the values of $\operatorname{sech}^{-1} x$, $\operatorname{csch}^{-1} x$, and $\coth^{-1} x$ on calculators that give only $\cosh^{-1} x$, $\sinh^{-1} x$, and $\tanh^{-1} x$.

Derivatives and Integrals

The chief use of inverse hyperbolic functions lies in integrations that reverse the derivative formulas in Table 6.12.

The restrictions $|u| < 1$ and $|u| > 1$ on the derivative formulas for $\tanh^{-1} u$ and $\coth^{-1} u$ come from the natural restrictions on the values of these functions. (See Figs. 6.33a and b.) The distinction between $|u| < 1$ and $|u| > 1$ becomes important when we convert the derivative formulas into integral formulas. If $|u| < 1$, the integral of $1/(1-u^2)$ is $\tanh^{-1} u + C$. If $|u| > 1$, the integral is $\coth^{-1} u + C$.

EXAMPLE 5 Show that if u is a differentiable function of x whose values are greater than 1, then

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}.$$

Solution First we find the derivative of $y = \cosh^{-1} x$ for $x > 1$:

$$
\begin{aligned}
y &= \cosh^{-1} x \\
x &= \cosh y && \text{Equivalent equation} \\
1 &= \sinh y \frac{dy}{dx} && \text{Differentiation with respect to } x \\
\frac{dy}{dx} &= \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} && \text{Since } x > 1, y > 0 \text{ and } \sinh y > 0 \\
&= \frac{1}{\sqrt{x^2-1}} && \cosh y = x
\end{aligned}
$$

In short, $\dfrac{d}{dx}(\cosh^{-1} x) = \dfrac{1}{\sqrt{x^2-1}}$. The Chain Rule gives the final result:

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}.$$

❑

With appropriate substitutions, the derivative formulas in Table 6.12 lead to the integration formulas in Table 6.13.

EXAMPLE 6 Evaluate $\displaystyle\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}$.

Solution The indefinite integral is

$$
\begin{aligned}
\int \frac{2\,dx}{\sqrt{3+4x^2}} &= \int \frac{du}{\sqrt{a^2+u^2}} && u = 2x, \quad du = 2\,dx, \quad a = \sqrt{3} \\
&= \sinh^{-1}\left(\frac{u}{a}\right) + C && \text{Formula from Table 6.13} \\
&= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.
\end{aligned}
$$

Table 6.13 Integrals leading to inverse hyperbolic functions

1. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \quad a > 0$

2. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \quad u > a > 0$

3. $\displaystyle\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if} \quad u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C & \text{if} \quad u^2 > a^2 \end{cases}$

4. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \quad 0 < u < a$

5. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \quad u \neq 0$

Therefore,

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Bigg]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$
$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.$$

❑

Exercises 6.10

Hyperbolic Function Values and Identities

Each of Exercises 1–4 gives a value of $\sinh x$ or $\cosh x$. Use the definitions and the identity $\cosh^2 x - \sinh^2 x = 1$ to find the values of the remaining five hyperbolic functions.

1. $\sinh x = -\dfrac{3}{4}$

2. $\sinh x = \dfrac{4}{3}$

3. $\cosh x = \dfrac{17}{15}, \quad x > 0$

4. $\cosh x = \dfrac{13}{5}, \quad x > 0$

Rewrite the expressions in Exercises 5–10 in terms of exponentials and simplify the results as much as you can.

5. $2\cosh(\ln x)$

6. $\sinh(2\ln x)$

7. $\cosh 5x + \sinh 5x$

8. $\cosh 3x - \sinh 3x$

9. $(\sinh x + \cosh x)^4$

10. $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$

11. Use the identities

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

to show that

a) $\sinh 2x = 2\sinh x \cosh x$;
b) $\cosh 2x = \cosh^2 x + \sinh^2 x$.

12. Use the definitions of $\cosh x$ and $\sinh x$ to show that

$$\cosh^2 x - \sinh^2 x = 1.$$

Derivatives

In Exercises 13–24, find the derivative of y with respect to the appropriate variable.

13. $y = 6\sinh\dfrac{x}{3}$

14. $y = \dfrac{1}{2}\sinh(2x+1)$

15. $y = 2\sqrt{t}\tanh\sqrt{t}$

16. $y = t^2\tanh\dfrac{1}{t}$

17. $y = \ln(\sinh z)$

18. $y = \ln(\cosh z)$

19. $y = \operatorname{sech}\theta(1 - \ln\operatorname{sech}\theta)$

20. $y = \operatorname{csch}\theta(1 - \ln\operatorname{csch}\theta)$

21. $y = \ln\cosh v - \dfrac{1}{2}\tanh^2 v$

22. $y = \ln\sinh v - \dfrac{1}{2}\coth^2 v$

23. $y = (x^2 + 1)\ \text{sech}\ (\ln x)$
(*Hint:* Before differentiating, express in terms of exponentials and simplify.)

24. $y = (4x^2 - 1)\ \text{csch}\ (\ln 2x)$

In Exercises 25–36, find the derivative of y with respect to the appropriate variable.

25. $y = \sinh^{-1}\sqrt{x}$

26. $y = \cosh^{-1} 2\sqrt{x+1}$

27. $y = (1-\theta)\tanh^{-1}\theta$

28. $y = (\theta^2 + 2\theta)\tanh^{-1}(\theta + 1)$

29. $y = (1-t)\coth^{-1}\sqrt{t}$

30. $y = (1-t^2)\coth^{-1} t$

31. $y = \cos^{-1} x - x\ \text{sech}^{-1} x$

32. $y = \ln x + \sqrt{1-x^2}\ \text{sech}^{-1} x$

33. $y = \text{csch}^{-1}\left(\frac{1}{2}\right)^{\theta}$

34. $y = \text{csch}^{-1} 2^{\theta}$

35. $y = \sinh^{-1}(\tan x)$

36. $y = \cosh^{-1}(\sec x),\quad 0 < x < \pi/2$

Integration Formulas

Verify the integration formulas in Exercises 37–40.

37. a) $\displaystyle\int \text{sech}\ x\,dx = \tan^{-1}(\sinh x) + C$

b) $\displaystyle\int \text{sech}\ x\,dx = \sin^{-1}(\tanh x) + C$

38. $\displaystyle\int x\ \text{sech}^{-1} x\,dx = \frac{x^2}{2}\ \text{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C$

39. $\displaystyle\int x\coth^{-1} x\,dx = \frac{x^2-1}{2}\coth^{-1} x + \frac{x}{2} + C$

40. $\displaystyle\int \tanh^{-1} x\,dx = x\tanh^{-1} x + \frac{1}{2}\ln(1-x^2) + C$

Indefinite Integrals

Evaluate the integrals in Exercises 41–50.

41. $\displaystyle\int \sinh 2x\,dx$

42. $\displaystyle\int \sinh\frac{x}{5}\,dx$

43. $\displaystyle\int 6\cosh\left(\frac{x}{2} - \ln 3\right)dx$

44. $\displaystyle\int 4\cosh(3x - \ln 2)\,dx$

45. $\displaystyle\int \tanh\frac{x}{7}\,dx$

46. $\displaystyle\int \coth\frac{\theta}{\sqrt{3}}\,d\theta$

47. $\displaystyle\int \text{sech}^2\left(x - \frac{1}{2}\right)dx$

48. $\displaystyle\int \text{csch}^2(5-x)\,dx$

49. $\displaystyle\int \frac{\text{sech}\sqrt{t}\tanh\sqrt{t}\,dt}{\sqrt{t}}$

50. $\displaystyle\int \frac{\text{csch}(\ln t)\coth(\ln t)\,dt}{t}$

Definite Integrals

Evaluate the integrals in Exercises 51–60.

51. $\displaystyle\int_{\ln 2}^{\ln 4} \coth x\,dx$

52. $\displaystyle\int_0^{\ln 2} \tanh 2x\,dx$

53. $\displaystyle\int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta\,d\theta$

54. $\displaystyle\int_0^{\ln 2} 4e^{-\theta}\sinh\theta\,d\theta$

55. $\displaystyle\int_{-\pi/4}^{\pi/4} \cosh(\tan\theta)\sec^2\theta\,d\theta$

56. $\displaystyle\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta\,d\theta$

57. $\displaystyle\int_1^2 \frac{\cosh(\ln t)}{t}dt$

58. $\displaystyle\int_1^4 \frac{8\cosh\sqrt{x}}{\sqrt{x}}dx$

59. $\displaystyle\int_{-\ln 2}^0 \cosh^2\left(\frac{x}{2}\right)dx$

60. $\displaystyle\int_0^{\ln 10} 4\sinh^2\left(\frac{x}{2}\right)dx$

Evaluating Inverse Hyperbolic Functions and Related Integrals

When hyperbolic function keys are not available on a calculator, it is still possible to evaluate the inverse hyperbolic functions by expressing them as logarithms as shown in the table below.

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right),\quad -\infty < x < \infty$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right),\quad x \ge 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x},\quad |x| < 1$$

$$\text{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right),\quad 0 < x \le 1$$

$$\text{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right),\quad x \ne 0$$

$$\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1},\quad |x| > 1$$

Use the formulas in the table here to express the numbers in Exercises 61–66 in terms of natural logarithms.

61. $\sinh^{-1}(-5/12)$

62. $\cosh^{-1}(5/3)$

63. $\tanh^{-1}(-1/2)$

64. $\coth^{-1}(5/4)$

65. $\text{sech}^{-1}(3/5)$

66. $\text{csch}^{-1}\left(-1/\sqrt{3}\right)$

Evaluate the integrals in Exercises 67–74 in terms of (a) inverse hyperbolic functions, (b) natural logarithms.

67. $\displaystyle\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}}$

68. $\displaystyle\int_0^{1/3} \frac{6\,dx}{\sqrt{1+9x^2}}$

69. $\displaystyle\int_{5/4}^2 \frac{dx}{1-x^2}$

70. $\displaystyle\int_0^{1/2} \frac{dx}{1-x^2}$

71. $\displaystyle\int_{1/5}^{3/13} \frac{dx}{\sqrt{1-16x^2}}$

72. $\displaystyle\int_1^2 \frac{dx}{x\sqrt{4+x^2}}$

73. $\int_0^{\pi} \frac{\cos x\,dx}{\sqrt{1+\sin^2 x}}$

74. $\int_1^{e} \frac{dx}{x\sqrt{1+(\ln x)^2}}$

Applications and Theory

75. a) Show that if a function f is defined on an interval symmetric about the origin (so that f is defined at $-x$ whenever it is defined at x), then

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}. \qquad (1)$$

Then show that $(f(x)+f(-x))/2$ is even and that $(f(x)-f(-x))/2$ is odd.

b) Equation (1) simplifies considerably if f itself is (i) even or (ii) odd. What are the new equations? Give reasons for your answers.

76. Derive the formula $\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$, $-\infty < x < \infty$. Explain in your derivation why the plus sign is used with the square root instead of the minus sign.

77. *Skydiving.* If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of the velocity, then the body's velocity t seconds into the fall satisfies the differential equation

$$m\frac{dv}{dt} = mg - kv^2,$$

where k is a constant that depends on the body's aerodynamic properties and the density of the air. (We assume that the fall is short enough so that the variation in the air's density will not affect the outcome.)

a) Show that

$$v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}\,t\right)$$

satisfies the differential equation and the initial condition that $v = 0$ when $t = 0$.

b) Find the body's *limiting velocity*, $\lim_{t\to\infty} v$.

c) CALCULATOR For a 160-lb skydiver ($mg = 160$), with time in seconds and distance in feet, a typical value for k is 0.005. What is the diver's limiting velocity?

78. *Accelerations whose magnitudes are proportional to displacement.* Suppose that the position of a body moving along a coordinate line at time t is

a) $s = a\cos kt + b\sin kt$,

b) $s = a\cosh kt + b\sinh kt$.

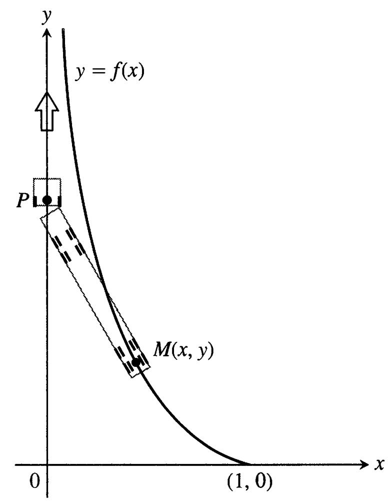
Show in both cases that the acceleration d^2s/dt^2 is proportional to s but that in the first case it is directed toward the origin while in the second case it is directed away from the origin.

79. *Tractor trailers and the tractrix.* When a tractor trailer turns into a cross street or driveway, its rear wheels follow a curve like the one shown here. (This is why the rear wheels sometimes ride up over the curb.) We can find an equation for the curve if we picture the rear wheels as a mass M at the point (1, 0) on the x-axis attached by a rod of unit length to a point P representing the cab at the origin. As the point P moves up the y-axis, it drags M along behind it. The curve traced by M, called a *tractrix* from the Latin word *tractum* for "drag," can be shown to be the graph of the function $y = f(x)$ that solves the initial value problem

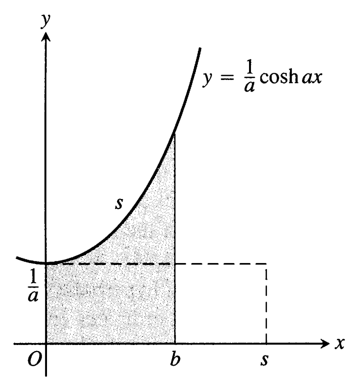
Differential equation: $\frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}$,

Initial condition: $y = 0$ when $x = 1$.

Solve the initial value problem to find an equation for the curve. (You need an inverse hyperbolic function.)



80. Show that the area of the region in the first quadrant enclosed by the curve $y = (1/a)\cosh ax$, the coordinate axes, and the line $x = b$ is the same as the area of a rectangle of height $1/a$ and length s, where s is the length of the curve from $x = 0$ to $x = b$.



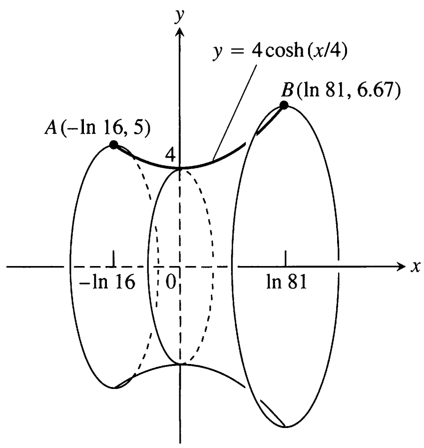
81. A region in the first quadrant is bounded above by the curve $y = \cosh x$, below by the curve $y = \sinh x$, and on the left and right by the y-axis and the line $x = 2$, respectively. Find the volume of the solid generated by revolving the region about the x-axis.

82. The region enclosed by the curve $y = \operatorname{sech} x$, the x-axis, and the lines $x = \pm \ln \sqrt{3}$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

83. a) Find the length of the segment of the curve $y = (1/2) \cosh 2x$ from $x = 0$ to $x = \ln \sqrt{5}$.

b) Find the length of the segment of the curve $y = (1/a) \cosh ax$ from $x = 0$ to $x = b > 0$.

84. *A minimal surface.* Find the area of the surface swept out by revolving about the x-axis the curve $y = 4 \cosh (x/4)$, $-\ln 16 \le x \le \ln 81$.



It can be shown that, of all continuously differentiable curves joining points A and B in the figure, the curve $y = 4 \cosh (x/4)$ generates the surface of least area. If you made a rigid wire frame of the end-circles through A and B and dipped them in a soap-film solution, the surface spanning the circles would be the one generated by the curve.

85. a) Find the centroid of the curve $y = \cosh x$, $-\ln 2 \le x \le \ln 2$.

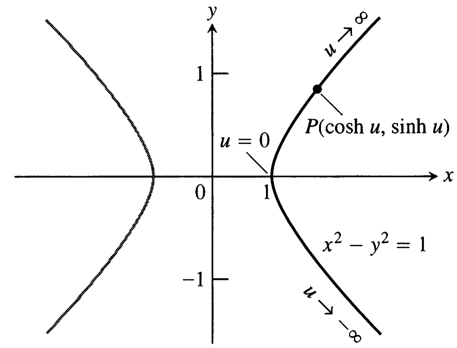
b) CALCULATOR Evaluate the coordinates to 2 decimal places. Then sketch the curve and plot the centroid to show its relation to the curve.

86. *The hyperbolic in hyperbolic functions.* In case you are wondering where the name *hyperbolic* comes from, here is the answer: Just as $x = \cos u$ and $y = \sin u$ are identified with points (x, y) on the unit circle, the functions $x = \cosh u$ and $y = \sinh u$ are identified with points (x, y) on the right-hand branch of the unit hyperbola, $x^2 - y^2 = 1$ (Fig. 6.34).

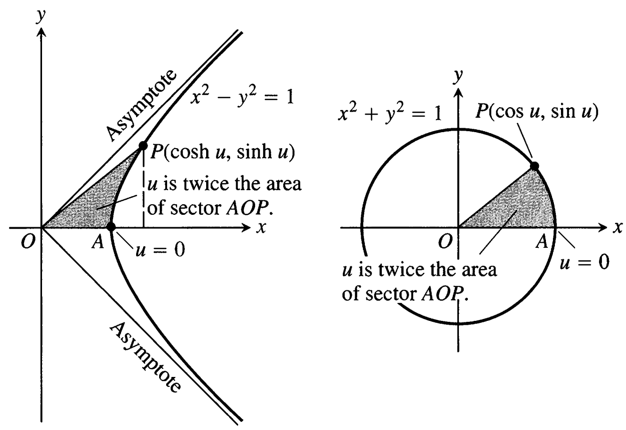
Another analogy between hyperbolic and circular functions is that the variable u in the coordinates $(\cosh u, \sinh u)$ for the points of the right-hand branch of the hyperbola $x^2 - y^2 = 1$ is twice the area of the sector AOP pictured in Fig. 6.35. To see why this is so, carry out the following steps.

a) Show that the area $A(u)$ of sector AOP is given by the formula

$$A(u) = \frac{1}{2} \cosh u \sinh u - \int_1^{\cosh u} \sqrt{x^2 - 1}\, dx.$$



6.34 Since $\cosh^2 u - \sinh^2 u = 1$, the point $(\cosh u, \sinh u)$ lies on the right-hand branch of the hyperbola $x^2 - y^2 = 1$ for every value of u (Exercise 86).



6.35 One of the analogies between hyperbolic and circular functions is revealed by these two diagrams (Exercise 86).

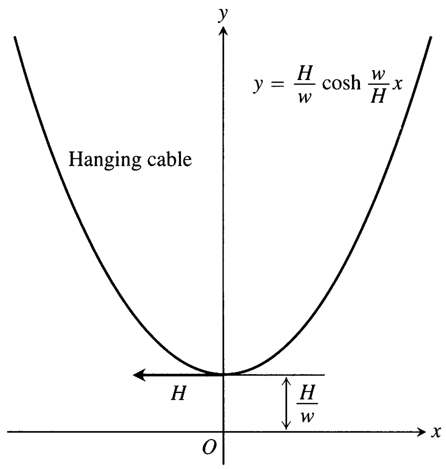
a) Differentiate both sides of the equation in (a) with respect to u to show that

$$A'(u) = \frac{1}{2}.$$

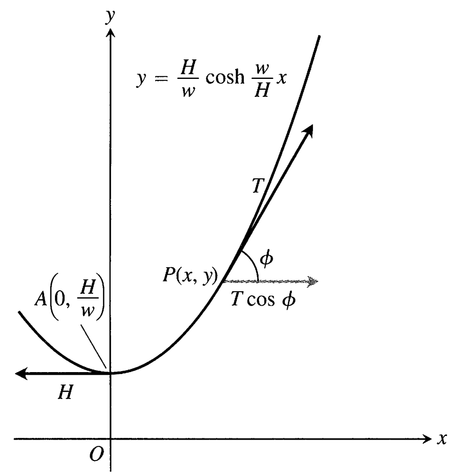
b) Solve this last equation for $A(u)$. What is the value of $A(0)$? What is the value of the constant of integration C in your solution? With C determined, what does your solution say about the relationship of u to $A(u)$?

Hanging Cables

87. Imagine a cable, like a telephone line or TV cable, strung from one support to another and hanging freely. The cable's weight per unit length is w and the horizontal tension at its lowest point is a vector of length H. If we choose a coordinate system for the plane of the cable in which the x-axis is horizontal, the force of gravity is straight down, the positive y-axis points straight up, and the lowest point of the cable lies at the point $y = H/w$ on the y-axis (Fig. 6.36), then it can be shown that the cable lies



6.36 In a coordinate system chosen to match H and w in the manner shown, a hanging cable lies along the hyperbolic cosine $y = (H/w) \cosh (wx/H)$.



6.37 As discussed in Exercise 87, $T = wy$ in this coordinate system.

along the graph of the hyperbolic cosine

$$y = \frac{H}{w} \cosh \frac{w}{H} x.$$

Such a curve is sometimes called a **chain curve** or a **catenary**, the latter deriving from the Latin *catena,* meaning "chain."

a) Let $P(x, y)$ denote an arbitrary point on the cable. Figure 6.37 displays the tension at P as a vector of length (magnitude) T, as well as the tension H at the lowest point A. Show that the cable's slope at P is

$$\tan \phi = \frac{dy}{dx} = \sinh \frac{w}{H} x.$$

b) Using the result from part (a) and the fact that the tension at P must equal H (the cable is not moving), show that $T = wy$. This means that the magnitude of the tension at $P(x, y)$ is exactly equal to the weight of y units of cable.

88. (*Continuation of Exercise 87.*) The length of arc AP in Fig. 6.37 is $s = (1/a) \sinh ax$, where $a = w/H$. Show that the coordinates of P may be expressed in terms of s as

$$x = \frac{1}{a} \sinh^{-1} as, \quad y = \sqrt{s^2 + \frac{1}{a^2}}.$$

89. *The sag and horizontal tension in a cable.* The ends of a cable 32 ft long and weighing 2 lb/ft are fastened at the same level to posts 30 ft apart.

a) Model the cable with the equation

$$y = \frac{1}{a} \cosh ax, \quad -15 \le x \le 15.$$

Use information from Exercise 88 to show that a satisfies the equation

$$16a = \sinh 15a. \tag{2}$$

b) GRAPHER Solve Eq. (2) graphically by estimating the coordinates of the points where the graphs of the equations $y = 16a$ and $y = \sinh 15a$ intersect in the ay-plane.

c) EQUATION SOLVER or ROOT FINDER Solve Eq. (2) for a numerically. Compare your solution with the value you found in (b).

d) Estimate the horizontal tension in the cable at the cable's lowest point.

e) GRAPHER Graph the catenary

$$y = \frac{1}{a} \cosh ax$$

over the interval $-15 \le x \le 15$. Estimate the sag in the cable at its center.

6.11 First Order Differential Equations

In Section 6.5 we derived the law of exponential change, $y = y_0 e^{kt}$, as the solution of the initial value problem $dy/dt = ky$, $y(0) = y_0$. As we saw, this problem models population growth, radioactive decay, heat transfer, and a great many other phenomena. In the present section, we study initial value problems based on the equation

$dy/dx = f(x, y)$, in which f is a function of both the independent and dependent variables. The applications of this equation, a generalization of $dy/dt = ky$ (think of t as x), are broader still.

First Order Differential Equations

A **first order** differential equation is a relation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane. A **solution** of Eq. (1) is a differentiable function $y = y(x)$ defined on an interval of x-values (perhaps infinite) such that

$$\frac{d}{dx}y(x) = f(x, y(x))$$

on that interval. The initial condition that $y(x_0) = y_0$ amounts to requiring the solution curve $y = y(x)$ to pass through the point (x_0, y_0).

EXAMPLE 1 The equation

$$\frac{dy}{dx} = 1 - \frac{y}{x}$$

is a first order differential equation in which $f(x, y) = 1 - (y/x)$. ❑

EXAMPLE 2 Show that the function

$$y = \frac{1}{x} + \frac{x}{2}$$

is a solution of the initial value problem

$$\frac{dy}{dx} = 1 - \frac{y}{x}, \qquad y(2) = \frac{3}{2}.$$

Solution The given function satisfies the initial condition because

$$y(2) = \left(\frac{1}{x} + \frac{x}{2}\right)_{x=2} = \frac{1}{2} + \frac{2}{2} = \frac{3}{2}.$$

To show that it satisfies the differential equation, we show that the two sides of the equation agree when we substitute $(1/x) + (x/2)$ for y.

On the left: $$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x} + \frac{x}{2}\right) = -\frac{1}{x^2} + \frac{1}{2}$$

On the right: $$1 - \frac{y}{x} = 1 - \frac{1}{x}\left(\frac{1}{x} + \frac{x}{2}\right)$$
$$= 1 - \frac{1}{x^2} - \frac{1}{2} = -\frac{1}{x^2} + \frac{1}{2}$$

The function $y = (1/x) + (x/2)$ satisfies both the differential equation and the initial condition, which is what we needed to show. ❑

We sometimes write $y' = f(x, y)$ for $dy/dx = f(x, y)$.

Separable Equations

The equation $y' = f(x, y)$ is **separable** if f can be expressed as a product of a function of x and a function of y. The differential equation then has the form

$$\frac{dy}{dx} = g(x)h(y).$$

If $h(y) \neq 0$, we can **separate the variables** by dividing both sides by h and multiplying both sides by dx, obtaining

$$\frac{1}{h(y)}\,dy = g(x)\,dx.$$

This groups the y-terms with dy on the left and the x-terms with dx on the right. We then integrate both sides, obtaining

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx.$$

The integrated equation provides the solutions we seek by expressing y either explicitly or implicitly as a function of x, up to an arbitrary constant.

EXAMPLE 3 Solve the differential equation

$$\frac{dy}{dx} = (1 + y^2)\,e^x.$$

Solution Since $1 + y^2$ is never zero, we can solve the equation by separating the variables.

$$\frac{dy}{dx} = (1 + y^2)\,e^x$$

$$dy = (1 + y^2)\,e^x\,dx \qquad \text{Treat } dy/dx \text{ as a quotient of differentials and multiply both sides by } dx.$$

$$\frac{dy}{1 + y^2} = e^x\,dx \qquad \text{Divide by } (1 + y^2).$$

$$\int \frac{dy}{1 + y^2} = \int e^x\,dx \qquad \text{Integrate both sides.}$$

$$\tan^{-1} y = e^x + C \qquad C \text{ represents the combined constants of integration.}$$

The equation $\tan^{-1} y = e^x + C$ gives y as an implicit function of x. In this case, we can solve for y as an explicit function of x by taking the tangent of both sides:

$$\tan(\tan^{-1} y) = \tan(e^x + C)$$

$$y = \tan(e^x + C).$$

❑

Linear First Order Equations

A first order differential equation that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{2}$$

where P and Q are functions of x, is a **linear** first order equation. Equation (2) is the equation's **standard form**.

EXAMPLE 4 Put the following equation in standard form

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0$$

Solution

$$x\frac{dy}{dx} = x^2 + 3y$$

$$\frac{dy}{dx} = x + \frac{3}{x}y \qquad \text{Divide by } x.$$

$$\frac{dy}{dx} - \frac{3}{x}y = x \qquad \text{Standard form with } P(x) = -3/x \text{ and } Q(x) = x$$

Notice that $P(x)$ is $-3/x$, not $+3/x$. The standard form is $y' + P(x)y = Q(x)$, so the minus sign is part of the formula for $P(x)$. ❑

EXAMPLE 5 The equation

$$\frac{dy}{dx} = ky$$

with which we modeled bacterial growth, radioactive decay, and temperature change in Section 6.5 is a linear first order equation. Its standard form is

$$\frac{dy}{dx} - ky = 0. \qquad P(x) = -k \text{ and } Q(x) = 0$$

❑

We solve the equation

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{3}$$

by multiplying both sides by a positive function $v(x)$ that transforms the left-hand side into the derivative of the product $v(x) \cdot y$. We will show how to find v in a moment, but first we want to show how, once found, it provides the solution we seek.

Here is why multiplying by v works:

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \text{Original equation is in standard form.}$$

$$v(x)\frac{dy}{dx} + P(x)v(x)y = v(x)Q(x) \qquad \text{Multiply by } v(x).$$

$$\frac{d}{dx}(v(x) \cdot y) = v(x)Q(x) \qquad v(x) \text{ is chosen to make } v\frac{dy}{dx} + Pvy = \frac{d}{dx}(v \cdot y)$$

$$v(x) \cdot y = \int v(x)Q(x)\,dx \qquad \text{Integrate with respect to } x.$$

$$y = \frac{1}{v(x)}\int v(x)Q(x)\,dx \qquad \text{Solve for } y. \tag{4}$$

We call $v(x)$ an **integrating factor** for Eq. (3) because its presence makes the equation integrable.

Equation (4) expresses the solution of Eq. (3) in terms of the functions $v(x)$ and $Q(x)$.

Why doesn't the formula for $P(x)$ appear in the solution as well? It does, but indirectly, in the construction of the positive function $v(x)$. We have

$$\frac{d}{dx}(vy) = v\frac{dy}{dx} + Pvy \qquad \text{Condition imposed on } v$$

$$v\frac{dy}{dx} + y\frac{dv}{dx} = v\frac{dy}{dx} + Pvy \qquad \text{Product Rule for derivatives}$$

$$y\frac{dv}{dx} = Pvy \qquad \text{The terms } v\frac{dy}{dx} \text{ cancel.}$$

This last equation will hold if

$$\frac{dv}{dx} = Pv$$

$$\frac{dv}{v} = P\,dx \qquad \text{Variables separated}$$

$$\int \frac{dv}{v} = \int P\,dx \qquad \text{Integrate both sides.}$$

$$\ln v = \int P\,dx \qquad \text{Since } v > 0, \text{ we do not need absolute value signs in } \ln v.$$

$$e^{\ln v} = e^{\int P\,dx} \qquad \text{Exponentiate both sides to solve for } v.$$

$$v = e^{\int P\,dx} \tag{5}$$

From this, we see that any function v that satisfies Eq. (5) will enable us to solve Eq. (3) with the formula in Eq. (4). We do not need the most general possible v, only one that will work. Therefore, it will do no harm to simplify our lives by choosing the simplest possible antiderivative of P for $\int P\,dx$.

Theorem 4

The solution of the equation

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{6}$$

is

$$y = \frac{1}{v(x)}\int v(x)\,Q(x)\,dx, \tag{7}$$

where

$$v(x) = e^{\int P(x)\,dx}. \tag{8}$$

In the formula for v, we do not need the most general antiderivative of $P(x)$. Any antiderivative will do.

EXAMPLE 6 Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \quad x > 0.$$

How to Solve a Linear First Order Equation

1. Put it in standard form.
2. Find an antiderivative of $P(x)$.
3. Find $v(x) = e^{\int P(x)\,dx}$.
4. Use Eq. (7) to find y.

Solution We solve the equation in four steps.

Step 1: *Put the equation in standard form to identify P and Q.*

$$\frac{dy}{dx} - \frac{3}{x}y = x, \qquad P(x) = -\frac{3}{x}, \qquad Q(x) = x. \qquad \text{Example 4}$$

Step 2: *Find an antiderivative of P(x) (any one will do).*

$$\int P(x)\,dx = \int -\frac{3}{x}\,dx = -3\int \frac{1}{x}\,dx = -3\ln|x| = -3\ln x \qquad (x > 0)$$

Step 3: *Find the integrating factor v(x).*

$$v(x) = e^{\int P(x)\,dx} = e^{-3\ln x} = e^{\ln x^{-3}} = \frac{1}{x^3} \qquad \text{Eq. (8)}$$

Step 4: *Find the solution.*

$$\begin{aligned} y &= \frac{1}{v(x)}\int v(x)Q(x)\,dx && \text{Eq. (7)} \\ &= \frac{1}{(1/x^3)}\int \left(\frac{1}{x^3}\right)(x)\,dx && \text{Values from steps 1–3} \\ &= x^3 \cdot \int \frac{1}{x^2}\,dx \\ &= x^3\left(-\frac{1}{x} + C\right) && \text{Don't forget the } C \ldots \\ &= -x^2 + Cx^3 && \ldots \text{it provides part of the answer.} \end{aligned}$$

The solution is $y = -x^2 + Cx^3, \quad x > 0.$ ❑

EXAMPLE 7 Solve the equation

$$xy' = x^2 + 3y, \qquad x > 0,$$

given the intial condition $y(1) = 2$.

Solution We first solve the differential equation (Example 6), obtaining

$$y = -x^2 + Cx^3, \qquad x > 0.$$

We then use the initial condition to find the right value for C:

$$\begin{aligned} y &= -x^2 + Cx^3 \\ 2 &= -(1)^2 + C(1)^3 && y = 2 \text{ when } x = 1 \\ C &= 2 + (1)^2 = 3. \end{aligned}$$

The solution of the initial value problem is the function $y = -x^2 + 3x^3$. ❑

Resistance Proportional to Velocity

In some cases it makes sense to assume that, other forces being absent, the resistance encountered by a moving object, like a car coasting to a stop, is proportional to the object's velocity. The slower the object moves, the less its forward progress is resisted by the air through which it passes. We can describe this in mathematical terms if we picture the object as a mass m moving along a coordinate line with position s and velocity v at time t. The resisting force opposing the motion is mass $\times$ acceleration $= m(dv/dt)$, and we can write

$$m\frac{dv}{dt} = -kv \qquad (k > 0) \tag{9}$$

to say that the force decreases in proportion to velocity. If we rewrite (9) as

$$\frac{dv}{dt} + \frac{k}{m}v = 0 \qquad \text{Standard form} \tag{10}$$

and let v_0 denote the object's velocity at time $t = 0$, we can apply Theorem 4 to arrive at the solution

$$v = v_0\, e^{-(k/m)t} \tag{11}$$

(Exercise 42).

What can we learn from Eq. (11)? For one thing, we can see that if m is something large, like the mass of a 20,000-ton ore boat in Lake Erie, it will take a long time for the velocity to approach zero. For another, we can integrate the equation to find s as a function of t.

Suppose a body is coasting to a stop and the only force acting on it is a resistance proportional to its speed. How far will it coast? To find out, we start with Eq. (11) and solve the initial value problem

$$\frac{ds}{dt} = v_0\, e^{-(k/m)t}, \qquad s(0) = 0.$$

Integrating with respect to t gives

$$s = -\frac{v_0 m}{k} e^{-(k/m)t} + C.$$

Substituting $s = 0$ when $t = 0$ gives

$$0 = -\frac{v_0 m}{k} + C \qquad \text{and} \qquad C = \frac{v_0 m}{k}.$$

The body's position at time t is therefore

$$s(t) = -\frac{v_0 m}{k} e^{-(k/m)t} + \frac{v_0 m}{k} = \frac{v_0 m}{k}(1 - e^{-(k/m)t}).$$

To find how far the body will coast, we find the limit of $s(t)$ as $t \to \infty$. Since $-(k/m) < 0$, we know that $e^{-(k/m)t} \to 0$ as $t \to \infty$, so that

$$\begin{aligned}\lim_{t\to\infty} s(t) &= \lim_{t\to\infty} \frac{v_0 m}{k}(1 - e^{-(k/m)t}) \\ &= \frac{v_0 m}{k}(1 - 0) = \frac{v_0 m}{k}.\end{aligned}$$

Thus,

$$\text{Distance coasted} = \frac{v_0 m}{k}. \tag{12}$$

This is an ideal figure, of course. Only in mathematics can time stretch to infinity. The number $v_0\, m/k$ is only an upper bound (albeit a useful one). It is true to life in one respect, at least—if m is large, it will take a lot of energy to stop the body. That is why ocean liners have to be docked by tugboats. Any liner of conventional design entering a slip with enough speed to steer would smash into the pier before it could stop.

Weight vs. mass

Weight is the force that results from gravity pulling on a mass. The two are related by the equation in Newton's second law,

$$\text{Weight} = \text{mass} \times \text{acceleration}.$$

To convert mass to weight, multiply by the acceleration of gravity. To convert weight to mass, divide by the acceleration of gravity. In the metric system,

$$\text{Newtons} = \text{kilograms} \times 9.8$$

and

$$\text{Newtons}/9.8 = \text{kilograms}.$$

In the English system, where weight is measured in pounds, mass is measured in **slugs.** Thus,

$$\text{Pounds} = \text{slugs} \times 32$$

and

$$\text{Pounds}/32 = \text{slugs}.$$

A skater weighing 192 lb has a mass of

$$192/32 = 6 \text{ slugs}.$$

EXAMPLE 8 For a 192-lb ice skater, the k in Eq. (11) is about 1/3 slug/sec and $m = 192/32 = 6$ slugs. How long will it take the skater to coast from 11 ft/sec (7.5 mph) to 1 ft/sec? How far will the skater coast before coming to a complete stop?

Solution We answer the first question by solving Eq. (11) for t:

$$11e^{-t/18} = 1 \qquad \text{Eq. (11) with } k = 1/3,\ m = 6,\ v_0 = 11,\ v = 1$$

$$e^{-t/18} = 1/11$$

$$-t/18 = \ln(1/11) = -\ln 11$$

$$t = 18 \ln 11 \approx 43 \text{ sec}.$$

We answer the second question with Eq. (12):

$$\text{Distance coasted} = \frac{v_0 m}{k} = \frac{11 \cdot 6}{1/3}$$

$$= 198 \text{ ft}.$$

❑

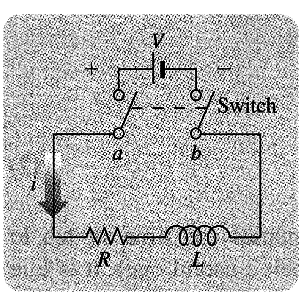
RL Circuits

The diagram in Fig. 6.38 represents an electrical circuit whose total resistance is a constant R ohms and whose self-inductance, shown as a coil, is L henries, also a constant. There is a switch whose terminals at a and b can be closed to connect a constant electrical source of V volts.

Ohm's law, $V = RI$, has to be modified for such a circuit. The modified form is

$$L\frac{di}{dt} + Ri = V, \tag{13}$$

where i is the intensity of the current in amperes and t is the time in seconds. By solving this equation, we can predict how the current will flow after the switch is closed.



6.38 The *RL* circuit in Example 9.

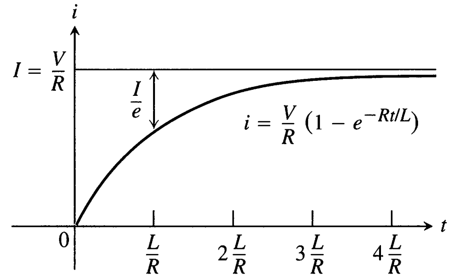
EXAMPLE 9 The switch in the RL circuit in Fig. 6.38 is closed at time $t = 0$. How will the current flow as a function of time?

Solution Equation (13) is a linear first order differential equation for i as a function of t. Its standard form is

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}, \tag{14}$$

and the corresponding solution, from Theorem 4, given that $i = 0$ when $t = 0$, is

$$i = \frac{V}{R} - \frac{V}{R}e^{-(R/L)t} \tag{15}$$



6.39 The growth of the current in the *RL* circuit in Example 9. *I* is the current's steady state value. The number $t = L/R$ is the time constant of the circuit. The current gets to within 5% of its steady state value in 3 time constants (Exercise 53).

(Exercise 54). Since R and L are positive, $-(R/L)$ is negative and $e^{-(R/L)t} \to 0$ as $t \to \infty$. Thus,

$$\lim_{t\to\infty} i = \lim_{t\to\infty}\left(\frac{V}{R} - \frac{V}{R}e^{-(R/L)t}\right) = \frac{V}{R} - \frac{V}{R}\cdot 0 = \frac{V}{R}.$$

At any given time, the current is theoretically less than V/R, but as time passes the current approaches the **steady state value** V/R. According to the equation

$$L\frac{di}{dt} + Ri = V,$$

$I = V/R$ is the current that will flow in the circuit if either $L = 0$ (no inductance) or $di/dt = 0$ (steady current, $i =$ constant) (Fig. 6.39).

Equation (15) expresses the solution of Eq. (14) as the sum of two terms: a **steady state solution** V/R and a **transient solution** $-(V/R)e^{-(R/L)t}$ that tends to zero as $t \to \infty$. ❑

Exercises 6.11

Verifying Solutions

In Exercises 1 and 2, show that each function $y = f(x)$ is a solution of the accompanying differential equation.

1. $2y' + 3y = e^{-x}$

a) $y = e^{-x}$

b) $y = e^{-x} + e^{-(3/2)x}$

c) $y = e^{-x} + Ce^{-(3/2)x}$

2. $y' = y^2$

a) $y = -\dfrac{1}{x}$ **b)** $y = -\dfrac{1}{x+3}$ **c)** $y = -\dfrac{1}{x+C}$

In Exercises 3 and 4, show that the function $y = f(x)$ is a solution of the given differential equation.

3. $y = \dfrac{1}{x}\displaystyle\int_1^x \frac{e^t}{t}\,dt, \quad x^2y' + xy = e^x$

4. $y = \dfrac{1}{\sqrt{1+x^4}}\displaystyle\int_1^x \sqrt{1+t^4}\,dt, \quad y' + \frac{2x^3}{1+x^4}y = 1$

In Exercises 5–8, show that each function is a solution of the given initial value problem.

	Differential equation	Initial condition	Solution candidate
5.	$y' + y = \dfrac{2}{1+4e^{2x}}$	$y(-\ln 2) = \dfrac{\pi}{2}$	$y = e^{-x}\tan^{-1}(2e^x)$
6.	$y' = e^{-x^2} - 2xy$	$y(2) = 0$	$y = (x-2)e^{-x^2}$
7.	$xy' + y = -\sin x$, $x > 0$	$y\left(\dfrac{\pi}{2}\right) = 0$	$y = \dfrac{\cos x}{x}$
8.	$x^2y' = xy - y^2$, $x > 1$	$y(e) = e$	$y = \dfrac{x}{\ln x}$

Separable Equations

Solve the differential equations in Exercises 9–14.

9. $\dfrac{dy}{dx} = 2(x + y^2x)$

10. $(y+1)\dfrac{dy}{dx} = y(x-1)$

11. $2\sqrt{xy}\dfrac{dy}{dx} = 1, \quad x, y > 0$

12. $\dfrac{dy}{dx} = x^2\sqrt{y}, \quad y > 0$

13. $\dfrac{dy}{dx} = e^{x-y}$

14. $\dfrac{dy}{dx} = \dfrac{2x^2+1}{xe^y}, \quad x > 0$

Linear First Order Equations

Solve the differential equations in Exercises 15–20.

15. $x\dfrac{dy}{dx} + y = e^x, \quad x > 0$

16. $e^x\dfrac{dy}{dx} + 2e^x y = 1$

17. $xy' + 3y = \dfrac{\sin x}{x^2}, \quad x > 0$

18. $y' + (\tan x)y = \cos^2 x, \quad -\pi/2 < x < \pi/2$

19. $x\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0$

20. $(1+x)y' + y = \sqrt{x}$

First Order Equations

Solve the differential equations in Exercises 21–34.

21. $2y' = e^{x/2} + y$

22. $\sqrt{x}\dfrac{dy}{dx} = e^{y+\sqrt{x}}, \quad x > 0$

23. $e^{2x}y' + 2e^{2x}y = 2x$

24. $xy' - y = 2x\ln x$

25. $\sec x\dfrac{dy}{dx} = e^{y+\sin x}$

26. $x\dfrac{dy}{dx} = \dfrac{\cos x}{x} - 2y, \quad x > 0$

27. $(t-1)^3\dfrac{ds}{dt} + 4(t-1)^2 s = t+1, \quad t > 1$

28. $(t+1)\dfrac{ds}{dt} + 2s = 3(t+1) + \dfrac{1}{(t+1)^2}, \quad t > -1$

29. $(\sec^2\sqrt{x})\dfrac{dx}{dt} = \sqrt{x}$

30. $\sin t - (x\cos^2 t)\dfrac{dx}{dt} = 0, \quad -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

31. $\sin\theta\dfrac{dr}{d\theta} + (\cos\theta)r = \tan\theta, \quad 0 < \theta < \pi/2$

32. $\tan\theta\dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \pi/2$

33. $\cosh x\dfrac{dy}{dx} + (\sinh x)y = e^{-x}$

34. $\sinh x\dfrac{dy}{dx} + 3(\cosh x)y = \cosh x\sinh x$

Solving Initial Value Problems

Solve the initial value problems in Exercises 35–40.

Differential equation	Initial condition
35. $\dfrac{dy}{dt} + 2y = 3$	$y(0) = 1$
36. $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0$	$y(2) = 1$
37. $\theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0$	$y(\pi/2) = 1$
38. $\theta\dfrac{dy}{d\theta} - 2y = \theta^3\sec\theta\tan\theta, \quad \theta > 0$	$y(\pi/3) = 2$
39. $(x+1)\dfrac{dy}{dx} - 2(x^2+x)y = \dfrac{e^{x^2}}{x+1}, \quad x > -1$	$y(0) = 5$
40. $\dfrac{dy}{dx} + xy = x$	$y(0) = -6$

41. What do you get when you use Theorem 4 to solve the following initial value problem for y as a function of t?

$$\frac{dy}{dt} = ky \quad (k \text{ constant}), \quad y(0) = y_0$$

42. Use Theorem 4 to solve the following initial value problem for v as a function of t.

$$\frac{dv}{dt} + \frac{k}{m}v = 0 \quad (k \text{ and } m \text{ positive constants}), \quad v(0) = v_0$$

Theory and Examples

43. Is either of the following equations correct? Give reasons for your answers.

a) $x\displaystyle\int \frac{1}{x}\,dx = x\ln|x| + C$

b) $x\displaystyle\int \frac{1}{x}\,dx = x\ln|x| + Cx$

44. Is either of the following equations correct? Give reasons for your answers.

a) $\dfrac{1}{\cos x}\displaystyle\int \cos x\,dx = \tan x + C$

b) $\dfrac{1}{\cos x}\displaystyle\int \cos x\,dx = \tan x + \dfrac{C}{\cos x}$

45. *Blood sugar.* If glucose is fed intravenously at a constant rate, the change in the overall concentration $c(t)$ of glucose in the blood with respect to time may be described by the differential equation

$$\frac{dc}{dt} = \frac{G}{100V} - kc.$$

In this equation, G, V, and k are positive constants, G being the rate at which glucose is admitted, in milligrams per minute, and V the volume of blood in the body, in liters (around 5 liters for an adult). The concentration $c(t)$ is measured in milligrams per centiliter. The term $-kc$ is included because the glucose is assumed to be changing continually into other molecules at a rate proportional to its concentration.

a) Solve the equation for $c(t)$, using c_0 to denote $c(0)$.
b) Find the steady state concentration, $\lim_{t\to\infty} c(t)$.

46. *Continuous compounding.* You have \$1000 with which to open an account and plan to add \$1000 per year. All funds in the account will earn 10% interest per year, compounded continuously. If the added deposits are also credited to your account continuously, the number of dollars x in your account at time t (years) will satisfy the initial value problem

$$\frac{dx}{dt} = 1000 + 0.10x, \quad x(0) = 1000.$$

a) Solve the initial value problem for x as a function of t.
b) CALCULATOR About how many years will it take for the amount in your account to reach \$100,000?

47. *How long will it take a tank to drain?* If we drain the water from a vertical cylindrical tank by opening a valve at the base of the tank, the water will flow fast when the tank is full but slow down as the tank drains. It turns out that the rate at which the water level drops is proportional to the square root of the water's depth, y. This means that

$$\frac{dy}{dt} = -k\sqrt{y}.$$

The value of k depends on the acceleration of gravity, the shape of the hole, the fluid, and the cross-section areas of the tank and drain hole.

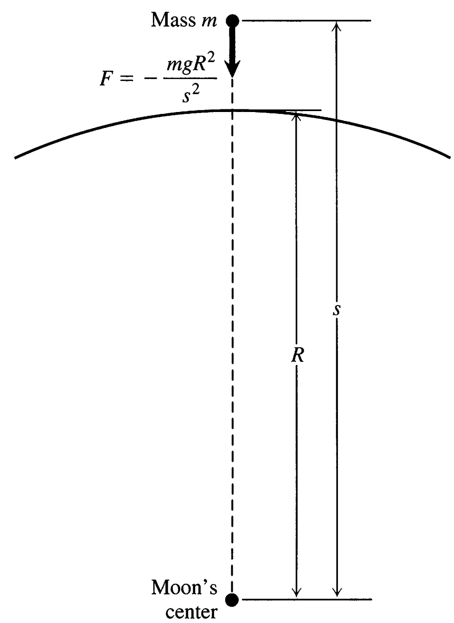
Suppose t is measured in minutes and $k = 1/10$. How long does it take the tank to drain if the water is 9 ft deep to start with?

48. *Escape velocity.* The gravitational attraction F exerted by an airless moon on a body of mass m at a distance s from the moon's center is given by the equation $F = -mg\,R^2 s^{-2}$, where g is the acceleration of gravity at the moon's surface and R is the moon's radius (Fig. 6.40). The force F is negative because it acts in the direction of decreasing s.

a) If the body is projected vertically upward from the moon's surface with an initial velocity v_0 at time $t = 0$, use Newton's second law, $F = ma$, to show that the body's velocity at position s is given by the equation

$$v^2 = \frac{2gR^2}{s} + v_0^2 - 2gR.$$

Thus, the velocity remains positive as long as $v_0 \geq \sqrt{2gR}$. The velocity $v_0 = \sqrt{2gR}$ is the moon's **escape velocity.** A body projected upward with this velocity or a greater one will escape from the moon's gravitational pull.



6.40 Diagram for Exercise 48.

b) Show that if $v_0 = \sqrt{2gR}$, then

$$s = R\left(1 + \frac{3v_0}{2R}t\right)^{2/3}.$$

RESISTANCE PROPORTIONAL TO VELOCITY

49. For a 145-lb cyclist on a 15-lb bicycle on level ground, the k in Eq. (11) is about 1/5 slug/sec and $m = 160/32 = 5$ slugs. The cyclist starts coasting at 22 ft/sec (15 mph).

a) About how far will the cyclist coast before reaching a complete stop?
b) To the nearest second, about how long will it take the cyclist's speed to drop to 1 ft/sec?

50. For a 56,000-ton Iowa class battleship, $m = 1{,}750{,}000$ slugs and the k in Eq. (11) might be 3000 slugs/sec. Suppose the battleship loses power when it is moving at a speed of 22 ft/sec (13.2 knots).

a) About how far will the ship coast before it stops?
b) About how long will it take the ship's speed to drop to 1 ft/sec?

RL CIRCUITS

51. *Current in a closed RL circuit.* How many seconds after the switch in an RL circuit is closed will it take the current i to reach half of its steady state value? Notice that the time depends on R and L and not on how much voltage is applied.

52. *Current in an open RL circuit.* If the switch is thrown open after the current in an RL circuit has built up to its steady state value, the decaying current (graphed here) obeys the equation

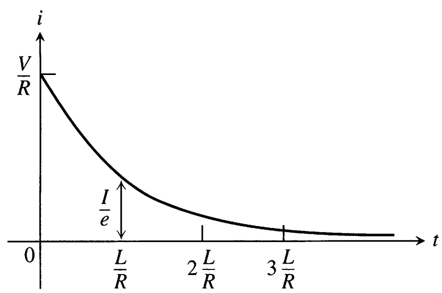
$$L\frac{di}{dt} + Ri = 0, \tag{16}$$

which is Eq. (13) with $V = 0$.

a) Solve Eq. (16) to express i as a function of t.

b) How long after the switch is thrown will it take the current to fall to half its original value?

c) What is the value of the current when $t = L/R$? (The significance of this time is explained in the next exercise.)



53. *Time constants.* Engineers call the number L/R the *time constant* of the RL circuit in Fig. 6.39. The significance of the time constant is that the current will reach 95% of its final value within 3 time constants of the time the switch is closed (Fig. 6.39). Thus, the time constant gives a built-in measure of how rapidly an individual circuit will reach equilibrium.

a) Find the value of i in Eq. (15) that corresponds to $t = 3L/R$ and show that it is about 95% of the steady state value $I = V/R$.

b) Approximately what percentage of the steady state current will be flowing in the circuit 2 time constants after the switch is closed (i.e., when $t = 2L/R$)?

54. (*Derivation of Eq. (15) in Example 9.*)

a) Use Theorem 4 to show that the solution of the equation

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$$

is

$$i = \frac{V}{R} + Ce^{-(R/L)t}.$$

b) Then use the initial condition $i(0) = 0$ to determine the value of C. This will complete the derivation of Eq. (15).

c) Show that $i = V/R$ is a solution of Eq. (14) and that $i = Ce^{-(R/L)t}$ satisfies the equation

$$\frac{di}{dt} + \frac{R}{L}i = 0.$$

MIXTURE PROBLEMS

A chemical in a liquid solution (or dispersed in a gas) runs into a container holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate. In this process it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula

$$\begin{array}{c}\text{Rate of change}\\ \text{of amount}\\ \text{in container}\end{array} = \left(\begin{array}{c}\text{rate at which}\\ \text{chemical}\\ \text{arrives}\end{array}\right) - \left(\begin{array}{c}\text{rate at which}\\ \text{chemical}\\ \text{departs.}\end{array}\right) \tag{17}$$

If $y(t)$ is the amount of chemical in the container at time t and $V(t)$ is the total volume of liquid in the container at time t, then the departure rate of the chemical at time t is

$$\begin{aligned}\text{Departure rate} &= \frac{y(t)}{V(t)} \cdot (\text{outflow rate})\\ &= \left(\begin{array}{c}\text{concentration in}\\ \text{container at time } t\end{array}\right) \cdot (\text{outflow rate}).\end{aligned} \tag{18}$$

Accordingly, Eq. (17) becomes

$$\frac{dy}{dt} = (\text{chemical's arrival rate}) - \frac{y(t)}{V(t)} \cdot (\text{outflow rate}). \tag{19}$$

If, say, y is measured in pounds, V in gallons, and t in minutes, the units in Eq. (19) are

$$\frac{\text{pounds}}{\text{min}} = \frac{\text{pounds}}{\text{min}} - \frac{\text{pounds}}{\text{gal}} \cdot \frac{\text{gal}}{\text{min}}.$$

55. A tank initially contains 100 gal of brine in which 50 lb of salt are dissolved. A brine containing 2 lb/gal of salt runs into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring and flows out of the tank at the rate of 4 gal/min.

a) At what rate (lb/min) does salt enter the tank at time t?

b) What is the volume of brine in the tank at time t?

c) At what rate (lb/min) does salt leave the tank at time t?

d) Write down and solve the initial value problem describing the mixing process.

e) Find the concentration of salt in the tank 25 min after the process starts.

56. In an oil refinery a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. Find the amount of additive in the tank 20 min after the process starts.

57. A tank contains 100 gal of fresh water. A solution containing 1 lb/gal of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at the rate of 3 gal/min. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

58. An executive conference room of a corporation contains 4500 cubic feet of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 ft^3/min. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 ft^3/min. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

6.12 Euler's Numerical Method; Slope Fields

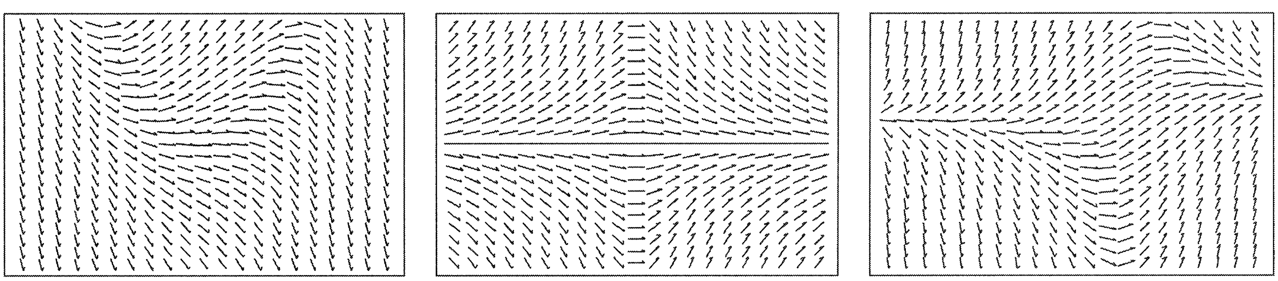
If we do not require or cannot immediately find an *exact* solution for an initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, we can probably use a computer to generate a table of approximate numerical values of y for values of x in an appropriate interval. Such a table is called a **numerical solution** of the problem and the method by which we generate the table is called a **numerical method.** Numerical methods are generally fast and accurate and are often the methods of choice when exact formulas are unnecessary, unavailable, or overly complicated. In the present section, we study one such method, called Euler's method, upon which all other numerical methods are based.

Slope Fields

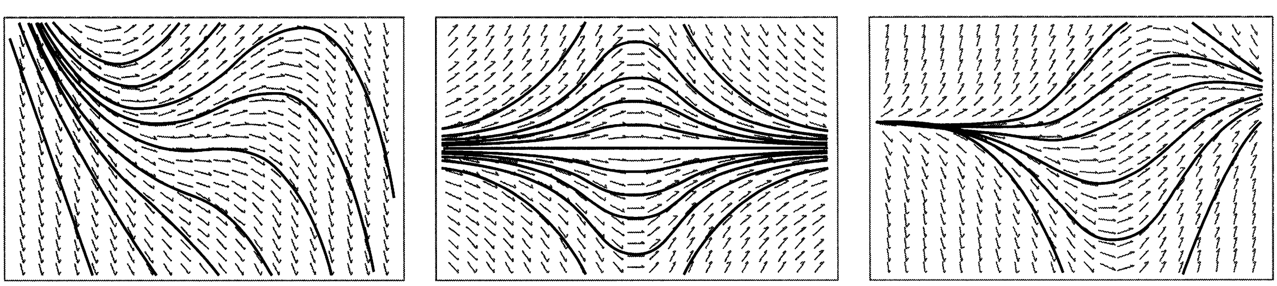
Each time we specify an initial condition $y(x_0) = y_0$ for the solution of a differential equation $y' = f(x, y)$, the solution curve is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there. We can picture these slopes graphically by drawing short line segments of slope $f(x, y)$ at selected points (x, y) in the domain of f. Each segment has the same slope as the solution curve through (x, y) and so is tangent to the curve there. We see how the curves behave by following these tangents (Fig. 6.41).

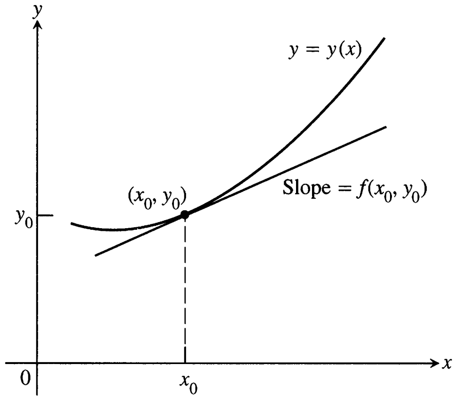
Constructing a slope field with pencil and paper can be quite tedious. All our examples were generated by a computer. Let us see how a computer might obtain one of the solution curves.

6.41 Slope fields (top row) and selected solution curves (bottom row). In computer renditions, slope segments are sometimes portrayed with vectors, as they are here. This is not to be taken as an indication that slopes have directions, however, for they do not.



(a) $y' = y - x^2$ (b) $y' = -\dfrac{2xy}{1+x^2}$ (c) $y' = (1-x)y + \dfrac{x}{2}$





6.42 The equation of the tangent line is $y = L(x) = y_0 + f(x_0, y_0)(x - x_0)$.

Using Linearizations

If we are given a differential equation $dy/dx = f(x, y)$ and an initial condition $y(x_0) = y_0$, we can approximate the solution curve $y = y(x)$ by its linearization

$$L(x) = y(x_0) + \left.\frac{dy}{dx}\right|_{x=x_0}(x - x_0)$$

or

$$L(x) = y_0 + f(x_0, y_0)(x - x_0). \tag{1}$$

The function $L(x)$ will give a good approximation to the solution $y(x)$ in a short interval about x_0 (Fig. 6.42). The basis of Euler's method is to patch together a string of linearizations to approximate the curve over a longer stretch. Here is how the method works.

We know the point (x_0, y_0) lies on the solution curve. Suppose we specify a new value for the independent variable to be $x_1 = x_0 + dx$. If the increment dx is small, then

$$y_1 = L(x_1) = y_0 + f(x_0, y_0)\,dx$$

is a good approximation to the exact solution value $y = y(x_1)$. So from the point (x_0, y_0), which lies *exactly* on the solution curve, we have obtained the point (x_1, y_1), which lies very close to the point $(x_1, y(x_1))$ on the solution curve.

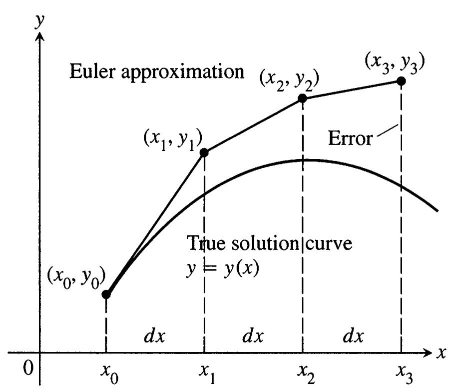
Using the point (x_1, y_1) and the slope $f(x_1, y_1)$, we take a second step. Setting $x_2 = x_1 + dx$, we calculate

$$y_2 = y_1 + f(x_1, y_1)\,dx,$$

to obtain another approximation (x_2, y_2) to values along the solution curve $y = y(x)$ (Fig. 6.43). Continuing in this fashion, we take a third step from the point (x_2, y_2) with slope $f(x_2, y_2)$ to obtain the next approximation

$$y_3 = y_2 + f(x_2, y_2)\,dx,$$

and so on.



6.43 Three steps in the Euler approximation to the solution of the initial value problem $y' = f(x, y)$, $y = y_0$ when $x = x_0$. The errors involved usually accumulate as we take more steps.

EXAMPLE 1 Find the first three approximations y_1, y_2, y_3 using the Euler approximation for the initial value problem

$$y' = 1 + y, \qquad y(0) = 1,$$

starting at $x_0 = 0$ and using $dx = 0.1$.

Solution

$$\begin{aligned}
\textit{First:}\quad y_1 &= y_0 + f(x_0, y_0)\,dx \\
&= y_0 + (1 + y_0)\,dx \\
&= 1 + (1 + 1)(0.1) = 1.2 \\
\textit{Second:}\quad y_2 &= y_1 + f(x_1, y_1)\,dx \\
&= y_1 + (1 + y_1)\,dx \\
&= 1.2 + (1 + 1.2)(0.1) = 1.42 \\
\textit{Third:}\quad y_3 &= y_2 + (1 + y_2)\,dx \\
&= 1.42 + (1 + 1.42)(0.1) = 1.662
\end{aligned}$$

❑

The Euler Method

To continue our discussion, the Euler method is a numerical process for generating a table of approximate values of the function that solves the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

If we use equally spaced values for the independent variable in the table and generate n of them, we first set

$$\begin{aligned} x_1 &= x_0 + dx, \\ x_2 &= x_1 + dx, \\ &\vdots \\ x_n &= x_{n-1} + dx. \end{aligned} \tag{2}$$

Then we calculate the solution approximations in turn:

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)\, dx, \\ y_2 &= y_1 + f(x_1, y_1)\, dx, \\ &\vdots \\ y_n &= y_{n-1} + f(x_{n-1}, y_{n-1})\, dx. \end{aligned} \tag{3}$$

The number n of steps can be as large as we like, but errors may accumulate if n is too large.

EXAMPLE 2 Investigate the accuracy of the Euler approximation method for the initial value problem

$$y' = 1 + y, \qquad y(0) = 1$$

in Example 1 over the interval $0 \le x \le 1$, starting at $x_0 = 0$ and taking $dx = 0.1$.

Solution The exact solution to the initial value problem is $y = 2e^x - 1$ (using either method discussed in Section 6.11). Table 6.14 shows the results of the Euler approximation method using Eqs. (2) and (3) and compares them to the exact results

Table 6.14 Euler solution of $y' = 1 + y$, $y(0) = 1$, increment size $dx = 0.1$

x	y (approx)	y (exact)	Error = y (exact) $-y$ (approx)
0	1	1	0
0.1	1.2	1.2103	0.0103
0.2	1.42	1.4428	0.0228
0.3	1.662	1.6997	0.0377
0.4	1.9282	1.9836	0.0554
0.5	2.2210	2.2974	0.0764
0.6	2.5431	2.6442	0.1011
0.7	2.8974	3.0275	0.1301
0.8	3.2872	3.4511	0.1639
0.9	3.7159	3.9192	0.2033
1.0	4.1875	4.4366	0.2491

rounded to 4 decimal places. By the time we reach $x = 1$ (after 10 steps), the error is about 5.6%. ❑

EXAMPLE 3 Investigate the accuracy of the Euler method for the initial value problem

$$y' = 1 + y, \qquad y(0) = 1$$

over the interval $0 \le x \le 1$, starting at $x_0 = 0$ and taking $dx = 0.05$.

Solution Table 6.15 shows the results and their comparisons with the exact solution. Notice that in doubling the number of steps from 10 to 20 we have reduced the error. This time when we reach $x = 1$ the error is only about 2.9%. ❑

It might be tempting to reduce the increment size even further to obtain greater accuracy. However, each additional calculation not only requires additional computer time but more importantly adds to the buildup of round-off errors due to the approximate representations of numbers in the calculations.

The analysis of error and the investigation of methods to reduce it when making numerical calculations is important, but appropriate for a more advanced course. There are numerical methods that are more accurate than Euler's method, as you will see when you study differential equations. In the exercises you will have the opportunity to explore the trade-offs involved in trying to reduce error by taking more but smaller increment steps.

Table 6.15 Euler solution of $y' = 1 + y$, $y(0) = 1$, increment size $dx = 0.05$

x	y (approx)	y (exact)	Error = y (exact) $-y$ (approx)
0	1	1	0
0.05	1.1	1.1025	0.0025
0.10	1.205	1.2103	0.0053
0.15	1.3153	1.3237	0.0084
0.20	1.4310	1.4428	0.0118
0.25	1.5526	1.5681	0.0155
0.30	1.6802	1.6997	0.0195
0.35	1.8142	1.8381	0.0239
0.40	1.9549	1.9836	0.0287
0.45	2.1027	2.1366	0.0339
0.50	2.2578	2.2974	0.0396
0.55	2.4207	2.4665	0.0458
0.60	2.5917	2.6442	0.0525
0.65	2.7713	2.8311	0.0598
0.70	2.9599	3.0275	0.0676
0.75	3.1579	3.2340	0.0761
0.80	3.3657	3.4511	0.0854
0.85	3.5840	3.6793	0.0953
0.90	3.8132	3.9192	0.1060
0.95	4.0539	4.1714	0.1175
1.00	4.3066	4.4366	0.1300

Exercises 6.12

Calculating Euler Approximations

In Exercises 1–6, use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round your results to 4 decimal places.

1. $y' = 1 - \dfrac{y}{x}$, $y(2) = -1$, $dx = 0.5$

2. $y' = x(1 - y)$, $y(1) = 0$, $dx = 0.2$

3. $y' = 2xy + 2y$, $y(0) = 3$, $dx = 0.2$

4. $y' = y^2(1 + 2x)$, $y(-1) = 1$, $dx = 0.5$

5. CALCULATOR $y' = 2xe^{x^2}$, $y(0) = 2$, $dx = 0.1$

6. CALCULATOR $y' = y + e^x - 2$, $y(0) = 2$, $dx = 0.5$

7. Use the Euler method with $dx = 0.2$ to estimate $y(1)$ if $y' = y$ and $y(0) = 1$. What is the exact value of $y(1)$?

8. Use the Euler method with $dx = 0.2$ to estimate $y(2)$ if $y' = y/x$ and $y(1) = 2$. What is the exact value of $y(2)$?

9. CALCULATOR Use the Euler method with $dx = 0.5$ to estimate $y(5)$ if $y' = y^2/\sqrt{x}$ and $y(1) = -1$. What is the exact value of $y(5)$?

10. CALCULATOR Use the Euler method with $dx = 1/3$ to estimate $y(2)$ if $y' = y - e^{2x}$ and $y(0) = 1$. What is the exact value of $y(2)$?

Slope Fields

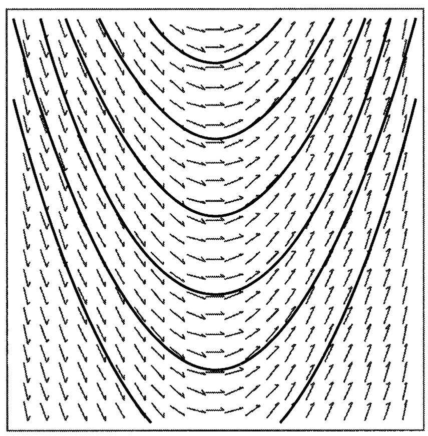
In Exercises 11–14, match the differential equations with the solution curves sketched below in the slope fields (a)–(d).

11. $y' = xy$

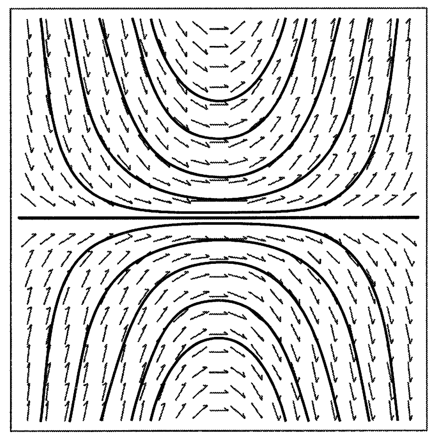
12. $y' = x + y$

13. $y' = x$

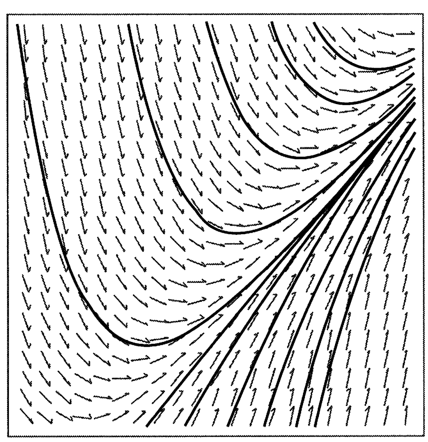
14. $y' = x - y$



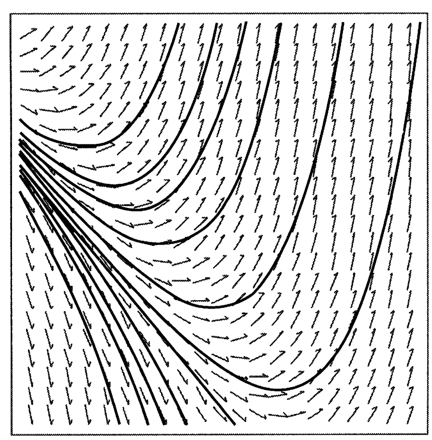
(a)



(b)



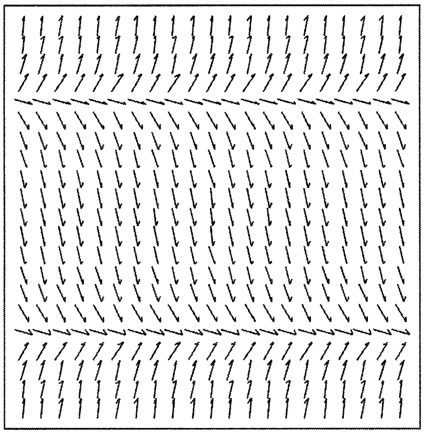
(c)



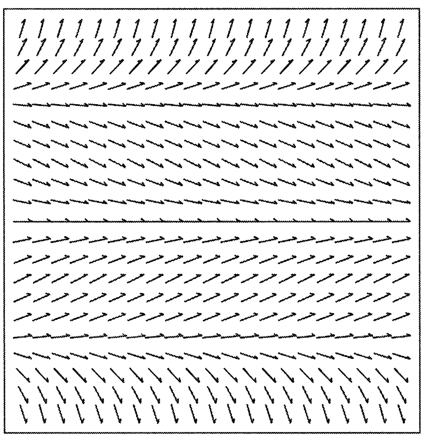
(d)

In Exercises 15 and 16, copy the slope fields and sketch in some of the solution curves.

15. $y' = (y+2)(y-2)$



16. $y' = y(y+1)(y-1)$



In Exercises 17–20, sketch part of the slope field. Using the slope field, sketch the solution curves that pass through the given points.

17. $y' = y$ with (a) $(0, 1)$, (b) $(0, 2)$, (c) $(0, -1)$

18. $y' = 2(y-4)$ with (a) $(0, 1)$, (b) $(0, 4)$, (c) $(0, 5)$

19. $y' = y(2-y)$ with (a) $(0, 1/2)$, (b) $(0, 3/2)$, (c) $(0, 2)$, (d) $(0, 3)$

20. $y' = y^2$ with (a) $(0, 1)$, (b) $(0, 2)$, (c) $(0, -1)$, (d) $(0, 0)$

CAS Explorations and Projects

Use a CAS to explore graphically each of the differential equations in Exercises 21–24. Perform the following steps to help with your explorations.

a) Plot a slope field for the differential equation in the given xy-window.

b) Find the general solution of the differential equation using your CAS DE solver.

c) Graph the solutions for the values of the arbitrary constant $C = -2, -1, 0, 1, 2$ superimposed on your slope field plot.

d) Find and graph the solution that satisfies the specified initial condition over the interval $[0, b]$.

e) Find the Euler numerical approximation to the solution of the initial value problem with 4 subintervals of the x-interval and plot the Euler approximation superimposed on the graph produced in part (d).

f) Repeat part (e) for 8, 16, and 32 subintervals. Plot these three Euler approximations superimposed on the graph from part (e).

g) Find the error y (exact) $-y$ (Euler) at the specified point $x = b$ for each of your four Euler approximations. Discuss the improvement in the percentage error.

21. $y' = x + y,\ y(0) = -7/10;\ -4 \le x \le 4,\ -4 \le y \le 4;\ b = 1$

22. $y' = -x/y,\quad y(0) = 2;\quad -3 \le x \le 3,\quad -3 \le y \le 3;\quad b = 2$

23. *A logistic equation.* $y' = y(2-y),\quad y(0) = 1/2;\quad 0 \le x \le 4,\ 0 \le y \le 3;\quad b = 3$

24. $y' = (\sin x)(\sin y),\quad y(0) = 2;\quad -6 \le x \le 6,\quad -6 \le y \le 6;\ b = 3\pi/2$

Exercises 25 and 26 have no explicit solution in terms of elementary functions. Use a CAS to explore graphically each of the differential equations, performing as many of the steps (a)–(g) above as possible.

25. $y' = \cos(2x - y),\ y(0) = 2;\ 0 \le x \le 5,\ 0 \le y \le 5;\ y(2)$

26. *A Gompertz equation.* $y' = y(1/2 - \ln y),\quad y(0) = 1/3;\ 0 \le x \le 4,\quad 0 \le y \le 3;\quad y(3)$

27. Use a CAS to find the solutions of $y' + y = f(x)$ subject to the initial condition $y(0) = 0$, if $f(x)$ is

a) $2x$

b) $\sin 2x$

c) $3e^{x/2}$

d) $2e^{-x/2}\cos 2x$.

Graph all four solutions over the interval $-2 \le x \le 6$ to compare the results.

28. **a)** Use a CAS to plot the slope field of the differential equation

$$y' = \frac{3x^2 + 4x + 2}{2(y-1)}$$

over the region $-3 \le x \le 3$ and $-3 \le y \le 3$.

b) Separate the variables and use a CAS integrator to find the general solution in implicit form.

c) Using a CAS implicit function grapher, plot solution curves for the arbitrary constant values $C = -6, -4, -2, 0, 2, 4, 6$.

d) Find and graph the solution that satisfies the initial condition $y(0) = -1$.

CHAPTER 6 QUESTIONS TO GUIDE YOUR REVIEW

1. What functions have inverses? How do you know if two functions f and g are inverses of one another? Give examples of functions that are (are not) inverses of one another.
2. How are the domains, ranges, and graphs of functions and their inverses related? Give an example.
3. How can you sometimes express the inverse of a function of x as a function of x?
4. Under what circumstances can you be sure that the inverse of a function f is differentiable? How are the derivatives of f and f^{-1} related?
5. What is the natural logarithm function? What are its domain, range, and derivative? What arithmetic properties does it have? Comment on its graph.
6. What is logarithmic differentiation? Give an example.
7. What integrals lead to logarithms? Give examples. What are the integrals of $\tan x$ and $\cot x$?
8. How is the exponential function e^x defined? What are its domain, range, and derivative? What laws of exponents does it obey? Comment on its graph.
9. How are the functions a^x and $\log_a x$ defined? Are there any restrictions on a? How is the graph of $\log_a x$ related to the graph of $\ln x$? What truth is there in the statement that there is really only one exponential function and one logarithmic function?
10. Describe some of the applications of base 10 logarithms.
11. What is the law of exponential change? How can it be derived from an initial value problem? What are some of the applications of the law?
12. How do you compare the growth rates of positive functions as $x \to \infty$?
13. What roles do the functions e^x and $\ln x$ play in growth comparisons?
14. Describe big-oh and little-oh notation. Give examples.
15. Which is more efficient—a sequential search or a binary search? Explain.
16. How are the inverse trigonometric functions defined? How can you sometimes use right triangles to find values of these functions? Give examples.
17. How can you find values of $\sec^{-1} x$, $\csc^{-1} x$, and $\cot^{-1} x$ using a calculator's keys for $\cos^{-1} x$, $\sin^{-1} x$, and $\tan^{-1} x$?
18. What are the derivatives of the inverse trigonometric functions? How do the domains of the derivatives compare with the domains of the functions?
19. What integrals lead to inverse trigonometric functions? How do substitution and completing the square broaden the application of these integrals?
20. What are the six basic hyperbolic functions? Comment on their domains, ranges, and graphs. What are some of the identities relating them?
21. What are the derivatives of the six basic hyperbolic functions? What are the corresponding integral formulas? What similarities do you see here with the six basic trigonometric functions?
22. How are the inverse hyperbolic functions defined? Comment on their domains, ranges, and graphs. How can you find values of $\operatorname{sech}^{-1}x$, $\operatorname{csch}^{-1}x$, and $\coth^{-1}x$ using a calculator's keys for $\cosh^{-1}x$, $\sinh^{-1}x$, and $\tanh^{-1}x$?
23. What integrals lead naturally to inverse hyperbolic functions?
24. What is a first order differential equation? When is a function a solution of such an equation?
25. How do you solve separable first order differential equations?
26. How do you solve linear first order differential equations?
27. What is the slope field of a differential equation $y' = f(x, y)$? What can we learn from such fields?
28. Describe Euler's method for solving the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ numerically. Give an example. Comment on the method's accuracy. Why might you want to solve an initial value problem numerically?

CHAPTER 6 PRACTICE EXERCISES

Differentiation

In Exercises 1–24, find the derivative of y with respect to the appropriate variable.

1. $y = 10e^{-x/5}$
2. $y = \sqrt{2}\,e^{\sqrt{2}x}$
3. $y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x}$
4. $y = x^2e^{-2/x}$
5. $y = \ln(\sin^2\theta)$
6. $y = \ln(\sec^2\theta)$
7. $y = \log_2(x^2/2)$
8. $y = \log_5(3x - 7)$
9. $y = 8^{-t}$
10. $y = 9^{2t}$
11. $y = 5x^{3.6}$
12. $y = \sqrt{2}\,x^{-\sqrt{2}}$
13. $y = (x + 2)^{x+2}$
14. $y = 2(\ln x)^{x/2}$
15. $y = \sin^{-1}\sqrt{1 - u^2}, \quad 0 < u < 1$
16. $y = \sin^{-1}\left(\frac{1}{\sqrt{v}}\right), \quad v > 1$
17. $y = \ln\cos^{-1}x$
18. $y = z\cos^{-1}z - \sqrt{1 - z^2}$
19. $y = t\tan^{-1}t - \frac{1}{2}\ln t$
20. $y = (1 + t^2)\cot^{-1}2t$
21. $y = z\sec^{-1}z - \sqrt{z^2 - 1}, \quad z > 1$
22. $y = 2\sqrt{x - 1}\sec^{-1}\sqrt{x}$
23. $y = \csc^{-1}(\sec\theta), \quad 0 < \theta < \pi/2$
24. $y = (1 + x^2)e^{\tan^{-1}x}$

Logarithmic Differentiation

In Exercises 25–30, use logarithmic differentiation to find the derivative of y with respect to the appropriate variable.

25. $y = \dfrac{2(x^2 + 1)}{\sqrt{\cos 2x}}$
26. $y = \sqrt[10]{\dfrac{3x + 4}{2x - 4}}$
27. $y = \left(\dfrac{(t + 1)(t - 1)}{(t - 2)(t + 3)}\right)^5, \quad t > 2$
28. $y = \dfrac{2u2^u}{\sqrt{u^2 + 1}}$
29. $y = (\sin\theta)^{\sqrt{\theta}}$
30. $y = (\ln x)^{1/(\ln x)}$

Integration

Evaluate the integrals in Exercises 31–50.

31. $\int e^x\sin(e^x)\,dx$
32. $\int e^t\cos(3e^t - 2)\,dt$
33. $\int e^x\sec^2(e^x - 7)\,dx$
34. $\int e^y\csc(e^y + 1)\cot(e^y + 1)\,dy$
35. $\int \sec^2(x)e^{\tan x}\,dx$
36. $\int \csc^2 x\,e^{\cot x}\,dx$
37. $\int_{-1}^{1}\dfrac{dx}{3x - 4}$
38. $\int_1^e \dfrac{\sqrt{\ln x}}{x}\,dx$
39. $\int_0^\pi \tan\dfrac{x}{3}\,dx$
40. $\int_{1/6}^{1/4} 2\cot\pi x\,dx$
41. $\int_0^4 \dfrac{2t}{t^2 - 25}\,dt$
42. $\int_{-\pi/2}^{\pi/6}\dfrac{\cos t}{1 - \sin t}\,dt$
43. $\int \dfrac{\tan(\ln v)}{v}\,dv$
44. $\int \dfrac{dv}{v\ln v}$
45. $\int \dfrac{(\ln x)^{-3}}{x}\,dx$
46. $\int \dfrac{\ln(x - 5)}{x - 5}\,dx$
47. $\int \dfrac{1}{r}\csc^2(1 + \ln r)\,dr$
48. $\int \dfrac{\cos(1 - \ln v)}{v}\,dv$
49. $\int x3^{x^2}\,dx$
50. $\int 2^{\tan x}\sec^2 x\,dx$

Evaluate the integrals in Exercises 51–64.

51. $\int_1^7 \dfrac{3}{x}\,dx$
52. $\int_1^{32}\dfrac{1}{5x}\,dx$
53. $\int_1^4\left(\dfrac{x}{8} + \dfrac{1}{2x}\right)dx$
54. $\int_1^8\left(\dfrac{2}{3x} - \dfrac{8}{x^2}\right)dx$
55. $\int_{-2}^{-1} e^{-(x+1)}\,dx$
56. $\int_{-\ln 2}^{0} e^{2w}\,dw$
57. $\int_0^{\ln 5} e^r(3e^r + 1)^{-3/2}\,dr$
58. $\int_0^{\ln 9} e^\theta(e^\theta - 1)^{1/2}\,d\theta$
59. $\int_1^e \dfrac{1}{x}(1 + 7\ln x)^{-1/3}\,dx$
60. $\int_e^{e^2}\dfrac{1}{x\sqrt{\ln x}}\,dx$
61. $\int_1^3 \dfrac{(\ln(v + 1))^2}{v + 1}\,dv$
62. $\int_2^4 (1 + \ln t)t\ln t\,dt$
63. $\int_1^8 \dfrac{\log_4\theta}{\theta}\,d\theta$
64. $\int_1^e \dfrac{8\ln 3\log_3\theta}{\theta}\,d\theta$

Evalute the integrals in Exercises 65–78.

65. $\int_{-3/4}^{3/4}\dfrac{6\,dx}{\sqrt{9 - 4x^2}}$
66. $\int_{-1/5}^{1/5}\dfrac{6\,dx}{\sqrt{4 - 25x^2}}$

67. $\int_{-2}^{2} \frac{3\,dt}{4+3t^2}$

68. $\int_{\sqrt{3}}^{3} \frac{dt}{3+t^2}$

69. $\int \frac{dy}{y\sqrt{4y^2-1}}$

70. $\int \frac{24\,dy}{y\sqrt{y^2-16}}$

71. $\int_{\sqrt{2}/3}^{2/3} \frac{dy}{|y|\sqrt{9y^2-1}}$

72. $\int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \frac{dy}{|y|\sqrt{5y^2-3}}$

73. $\int \frac{dx}{\sqrt{-2x-x^2}}$

74. $\int \frac{dx}{\sqrt{-x^2+4x-1}}$

75. $\int_{-2}^{-1} \frac{2\,dv}{v^2+4v+5}$

76. $\int_{-1}^{1} \frac{3\,dv}{4v^2+4v+4}$

77. $\int \frac{dt}{(t+1)\sqrt{t^2+2t-8}}$

78. $\int \frac{dt}{(3t+1)\sqrt{9t^2+6t}}$

Solving Equations with Logarithmic or Exponential Terms

In Exercises 79–84, solve for y.

79. $3^y = 2^{y+1}$

80. $4^{-y} = 3^{y+2}$

81. $9e^{2y} = x^2$

82. $3^y = 3\ln x$

83. $\ln(y-1) = x + \ln y$

84. $\ln(10\ln y) = \ln 5x$

Evaluating Limits

Find the limits in Exercises 85–96.

85. $\lim_{x\to 0} \frac{10^x - 1}{x}$

86. $\lim_{\theta\to 0} \frac{3^\theta - 1}{\theta}$

87. $\lim_{x\to 0} \frac{2^{\sin x} - 1}{e^x - 1}$

88. $\lim_{x\to 0} \frac{2^{-\sin x} - 1}{e^x - 1}$

89. $\lim_{x\to 0} \frac{5 - 5\cos x}{e^x - x - 1}$

90. $\lim_{x\to 0} \frac{4 - 4e^x}{xe^x}$

91. $\lim_{t\to 0^+} \frac{t - \ln(1+2t)}{t^2}$

92. $\lim_{x\to 4} \frac{\sin^2(\pi x)}{e^{x-4} + 3 - x}$

93. $\lim_{t\to 0^+} \left(\frac{e^t}{t} - \frac{1}{t}\right)$

94. $\lim_{y\to 0^+} e^{-1/y}\ln y$

95. $\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^x$

96. $\lim_{x\to 0^+} \left(1 + \frac{3}{x}\right)^x$

Comparing Growth Rates of Functions

97. Does f grow faster, slower, or at the same rate as g as $x \to \infty$? Give reasons for your answers.

a) $f(x) = \log_2 x, \quad g(x) = \log_3 x$
b) $f(x) = x, \quad g(x) = x + \frac{1}{x}$
c) $f(x) = x/100, \quad g(x) = xe^{-x}$
d) $f(x) = x, \quad g(x) = \tan^{-1} x$
e) $f(x) = \csc^{-1} x, \quad g(x) = 1/x$
f) $f(x) = \sinh x, \quad g(x) = e^x$

98. Does f grow faster, slower, or at the same rate as g as $x \to \infty$? Give reasons for your answers.

a) $f(x) = 3^{-x}, \quad g(x) = 2^{-x}$
b) $f(x) = \ln 2x, \quad g(x) = \ln x^2$
c) $f(x) = 10x^3 + 2x^2, \quad g(x) = e^x$
d) $f(x) = \tan^{-1}(1/x), \quad g(x) = 1/x$
e) $f(x) = \sin^{-1}(1/x), \quad g(x) = 1/x^2$
f) $f(x) = \operatorname{sech} x, \quad g(x) = e^{-x}$

99. True, or false? Give reasons for your answers.

a) $\frac{1}{x^2} + \frac{1}{x^4} = O\left(\frac{1}{x^2}\right)$
b) $\frac{1}{x^2} + \frac{1}{x^4} = O\left(\frac{1}{x^4}\right)$
c) $x = o(x + \ln x)$
d) $\ln(\ln x) = o(\ln x)$
e) $\tan^{-1} x = O(1)$
f) $\cosh x = O(e^x)$

100. True, or false? Give reasons for your answers.

a) $\frac{1}{x^4} = O\left(\frac{1}{x^2} + \frac{1}{x^4}\right)$
b) $\frac{1}{x^4} = o\left(\frac{1}{x^2} + \frac{1}{x^4}\right)$
c) $\ln x = o(x+1)$
d) $\ln 2x = O(\ln x)$
e) $\sec^{-1} x = O(1)$
f) $\sinh x = O(e^x)$

Theory and Applications

101. The function $f(x) = e^x + x$, being differentiable and one-to-one, has a differentiable inverse $f^{-1}(x)$. Find the value of df^{-1}/dx at the point $f(\ln 2)$.

102. Find the inverse of the function $f(x) = 1 + (1/x)$, $x \neq 0$. Then show that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$ and that

$$\left.\frac{df^{-1}}{dx}\right|_{f(x)} = \frac{1}{f'(x)}.$$

In Exercises 103 and 104, find the absolute maximum and minimum values of each function on the given interval.

103. $y = x\ln 2x - x, \quad \left[\frac{1}{2e}, \frac{e}{2}\right]$

104. $y = 10x(2 - \ln x), \quad (0, e^2]$

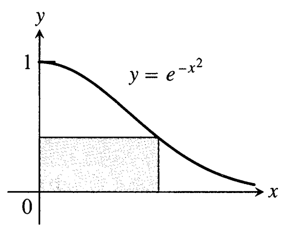
105. Find the area between the curve $y = 2(\ln x)/x$ and the x-axis from $x = 1$ to $x = e$.

106. a) Show that the area between the curve $y = 1/x$ and the x-axis from $x = 10$ to $x = 20$ is the same as the area between the curve and the x-axis from $x = 1$ to $x = 2$.
b) Show that the area between the curve $y = 1/x$ and the x-axis from ka to kb is the same as the area between the curve and the x-axis from $x = a$ to $x = b$ $(0 < a < b, k > 0)$.

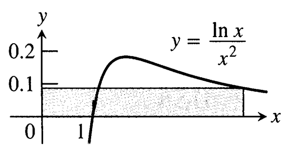
107. A particle is traveling upward and to the right along the curve $y = \ln x$. Its x-coordinate is increasing at the rate $(dx/dt) = \sqrt{x}$ m/sec. At what rate is the y-coordinate changing at the point $(e^2, 2)$?

108. A girl is sliding down a slide shaped like the curve $y = 9e^{-x/3}$. Her y-coordinate is changing at the rate $dy/dt = (-1/4)\sqrt{9-y}$ ft/sec. At approximately what rate is her x-coordinate changing when she reaches the bottom of the slide at $x = 9$ ft? (Take e^3 to be 20 and round your answer to the nearest ft/sec.)

109. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = e^{-x^2}$. What dimensions give the rectangle its largest area, and what is that area?



110. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = (\ln x)/x^2$. What dimensions give the rectangle its largest area, and what is that area?



111. The functions $f(x) = \ln 5x$ and $g(x) = \ln 3x$ differ by a constant. What constant? Give reasons for your answer.

112. **a)** If $(\ln x)/x = (\ln 2)/2$, must $x = 2$?
b) If $(\ln x)/x = -2 \ln 2$, must $x = 1/2$?

Give reasons for your answers.

113. The quotient $(\log_4 x)/(\log_2 x)$ has a constant value. What value? Give reasons for your answer.

114. *$\log_x (2)$ vs. $\log_2 (x)$.* How does $f(x) = \log_x (2)$ compare with $g(x) = \log_2(x)$? Here is one way to find out:

a) Use the equation $\log_a b = (\ln b)/(\ln a)$ to express $f(x)$ and $g(x)$ in terms of natural logarithms.

b) Graph f and g together. Comment on the behavior of f in relation to the signs and values of g.

115. GRAPHER Graph the following functions and use what you see to locate and estimate the extreme values, identify the coordinates of the inflection points, and identify the intervals on which the graphs are concave up and concave down. Then confirm your estimates by working with the functions' derivatives.

a) $y = (\ln x)/\sqrt{x}$ **b)** $y = e^{-x^2}$
c) $y = (1 + x)e^{-x}$

116. GRAPHER Graph $f(x) = x \ln x$. Does the function appear to have an absolute minimum value? Confirm your answer with calculus.

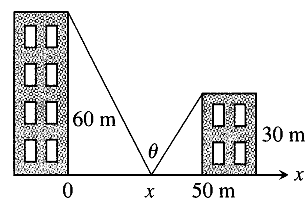
117. CALCULATOR What is the age of a sample of charcoal in which 90% of the carbon-14 originally present has decayed?

118. *Cooling a pie.* A deep-dish apple pie, whose internal temperature was 220°F when removed from the oven, was set out on a breezy 40°F porch to cool. Fifteen minutes later, the pie's internal temperature was 180°F. How long did it take the pie to cool from there to 70°F?

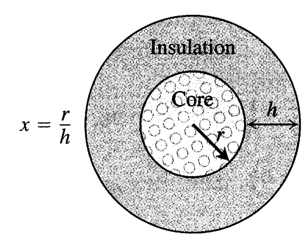
119. *Locating a solar station.* You are under contract to build a solar station at ground level on the east–west line between the two buildings shown here. How far from the taller building should you place the station to maximize the number of hours it will be in the sun on a day when the sun passes directly overhead? Begin by observing that

$$\theta = \pi - \cot^{-1} \frac{x}{60} - \cot^{-1} \frac{50 - x}{30}.$$

Then find the value of x that maximizes θ.



120. A round underwater transmission cable consists of a core of copper wires surrounded by nonconducting insulation. If x denotes the ratio of the radius of the core to the thickness of the insulation, it is known that the speed of the transmission signal is given by the equation $v = x^2 \ln (1/x)$. If the radius of the core is 1 cm, what insulation thickness h will allow the greatest transmission speed?



Initial Value Problems

Solve the initial value problems in Exercises 121–124.

	Differential equation	Initial condition
121.	$\frac{dy}{dx} = e^{-x-y-2}$	$y(0) = -2$
122.	$\frac{dy}{dx} = -\frac{y \ln y}{1 + x^2}$	$y(0) = e^2$
123.	$(x + 1)\frac{dy}{dx} + 2y = x, \quad x > -1$	$y(0) = 1$
124.	$x\frac{dy}{dx} + 2y = x^2 + 1, \quad x > 0$	$y(1) = 1$

Slope Fields and Euler's Method

In Exercises 125–128, sketch part of the equation's slope field. Then add to your sketch the solution curve that passes through the point $P(1, -1)$. Use Euler's method with $x_0 = 1$ and $dx = 0.2$ to estimate $y(2)$. Round your answers to 4 decimal places. Find the exact value of $y(2)$ for comparison.

125. $y' = x$

126. $y' = 1/x$

127. $y' = xy$

128. $y' = 1/y$

CHAPTER 6 ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Limits

Find the limits in Exercises 1–6

1. $\displaystyle\lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$

2. $\displaystyle\lim_{x\to\infty} \frac{1}{x}\int_0^x \tan^{-1} t\,dt$

3. $\displaystyle\lim_{x\to 0^+} (\cos\sqrt{x})^{1/x}$

4. $\displaystyle\lim_{x\to\infty} (x + e^x)^{2/x}$

5. $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)$

6. $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left(e^{1/n} + e^{2/n} + \cdots + e^{(n-1)/n} + e^{n/n}\right)$

7. Let $A(t)$ be the area of the region in the first quadrant enclosed by the coordinate axes, the curve $y = e^{-x}$, and the vertical line $x = t$, $t > 0$. Let $V(t)$ be the volume of the solid generated by revolving the region about the x-axis. Find the following limits.

a) $\displaystyle\lim_{t\to\infty} A(t)$ **b)** $\displaystyle\lim_{t\to\infty} V(t)/A(t)$ **c)** $\displaystyle\lim_{t\to 0^+} V(t)/A(t)$

8. *Varying a logarithm's base*

a) Find $\lim \log_a 2$ as $a \to 0^+, 1^-, 1^+$, and ∞.

b) GRAPHER Graph $y = \log_a 2$ as a function of a over the interval $0 < a \le 4$.

Determining Parameter Values

9. Find values of a and b for which

$$\lim_{x\to 0} \frac{\sin ax + bx}{x^3} = -\frac{4}{3}.$$

10. Find values of a and b for which

$$\lim_{x\to 0} \frac{a\cos x - \cos bx}{x^2} = 4.$$

Theory and Examples

11. Find the areas between the curves $y = 2(\log_2 x)/x$ and $y = 2(\log_4 x)/x$ and the x-axis from $x = 1$ to $x = e$. What is the ratio of the larger area to the smaller?

12. GRAPHER Graph $f(x) = \tan^{-1} x + \tan^{-1}(1/x)$ for $-5 \le x \le 5$. Then use calculus to explain what you see. How would you expect f to behave beyond the interval $[-5, 5]$? Give reasons for your answer.

13. For what $x > 0$ does $x^{(x^x)} = (x^x)^x$? Give reasons for your answer.

14. GRAPHER Graph $f(x) = (\sin x)^{\sin x}$ over $[0, 3\pi]$. Explain what you see.

15. Find $f'(2)$ if $f(x) = e^{g(x)}$ and $g(x) = \displaystyle\int_2^x \frac{t}{1+t^4}\,dt$.

16. **a)** Find df/dx if

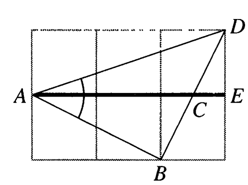
$$f(x) = \int_1^{e^x} \frac{2\ln t}{t}\,dt.$$

b) Find $f(0)$.

c) What can you conclude about the graph of f? Give reasons for your answer.

17. The figure here shows an informal proof that

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}.$$

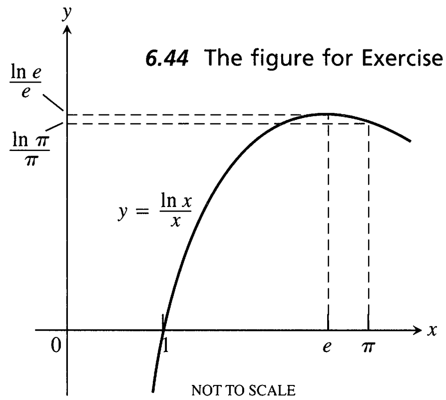


How does the argument go? (Source: "Behold! Sums of Arctan," by Edward M. Harris, *College Mathematics Journal,* Vol. 18, No. 2, March 1987, p. 141.)

18. $\pi^e < e^\pi$

a) Why does Fig. 6.44 (on the following page) "prove" that $\pi^e < e^\pi$? (Source: "Proof Without Words," by Fouad Nakhil, *Mathematics Magazine,* Vol. 60, No. 3, June 1987, p. 165.)

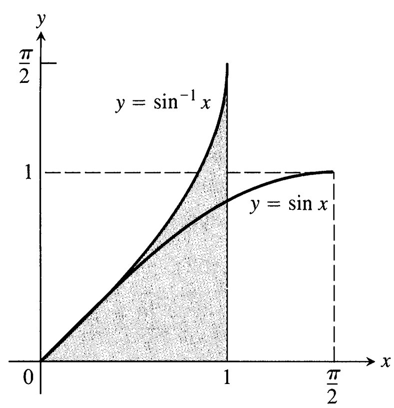
b) Figure 6.44 assumes that $f(x) = (\ln x)/x$ has an absolute maximum value at $x = e$. How do you know it does?



6.44 The figure for Exercise 18.

19. Use the accompanying figure to show that

$$\int_0^{\pi/2} \sin x \, dx = \frac{\pi}{2} - \int_0^1 \sin^{-1} x \, dx.$$

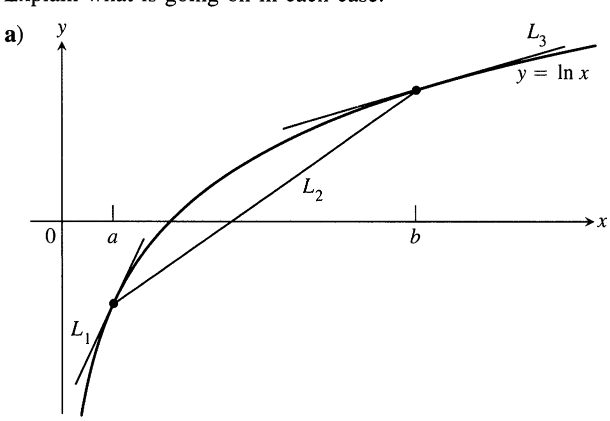


20. *Napier's inequality.* Here are two pictorial proofs that

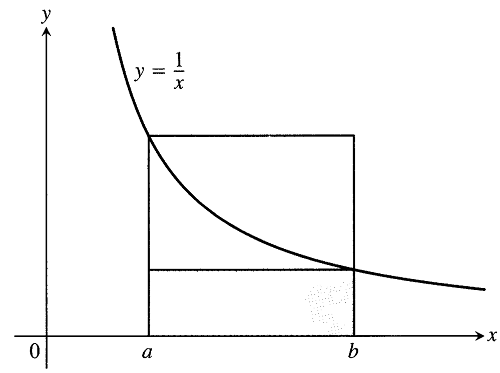
$$b > a > 0 \quad \Rightarrow \quad \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}.$$

Explain what is going on in each case.

a)



b)



(Source: Roger B. Nelson, *College Mathematics Journal,* Vol. 24, No. 2, March 1993, p. 165.)

21. *Even–odd decompositions*

a) Suppose that g is an even function of x and h is an odd function of x. Show that if $g(x) + h(x) = 0$ for all x then $g(x) = 0$ for all x and $h(x) = 0$ for all x.

b) Use the result in (a) to show that if $f(x) = f_E(x) + f_O(x)$ is the sum of an even function $f_E(x)$ and an odd function $f_O(x)$, then

$$f_E(x) = (f(x) + f(-x))/2 \quad \text{and} \quad f_O(x) = (f(x) - f(-x))/2.$$

c) What is the significance of the result in (b)?

22. Let g be a function that is differentiable throughout an open interval containing the origin. Suppose g has the following properties:

i) $g(x + y) = \dfrac{g(x) + g(y)}{1 - g(x)\,g(y)}$ for all real numbers x, y, and $x + y$ in the domain of g.

ii) $\lim_{h \to 0} g(h) = 0$

iii) $\lim_{h \to 0} \dfrac{g(h)}{h} = 1$

a) Show that $g(0) = 0$.
b) Show that $g'(x) = 1 + [g(x)]^2$.
c) Find $g(x)$ by solving the differential equation in (b).

Applications

23. Find the center of mass of a thin plate of constant density covering the region in the first and fourth quadrants enclosed by the curves $y = 1/(1 + x^2)$ and $y = -1/(1 + x^2)$ and by the lines $x = 0$ and $x = 1$.

24. The region between the curve $y = 1/(2\sqrt{x})$ and the x-axis from $x = 1/4$ to $x = 4$ is revolved about the x-axis to generate a solid.

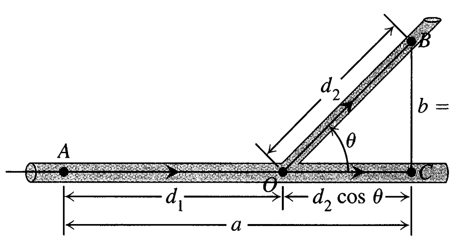
a) Find the volume of the solid.
b) Find the centroid of the region.

25. *The Rule of 70.* If you use the approximation $\ln 2 \approx 0.70$ (in place of $0.69314\ldots$), you can derive a rule of thumb that says, "To estimate how many years it will take an amount of money to double when invested at r percent compounded continuously, divide r into 70." For instance, an amount of money invested at 5% will double in about $70/5 = 14$ years. If you want it to double in 10 years instead, you have to invest it at $70/10 = 7\%$. Show how the Rule of 70 is derived. (A similar "Rule of 72" uses 72 instead of 70, because 72 has more integer factors.)

26. *Free fall in the fourteenth century.* In the middle of the fourteenth century, Albert of Saxony (1316–1390) proposed a model of free fall that assumed that the velocity of a falling body was proportional to the distance fallen. It seemed reasonable to think that a body that had fallen 20 ft might be moving twice as fast as a body that had fallen 10 ft. And besides, none of the instruments in use at the time were accurate enough to prove otherwise. Today we can see just how far off Albert of Saxony's model was by solving the initial value problem implicit in his model. Solve the problem and compare your solution graphically with the equation $s = 16t^2$. You will see that it describes a motion that starts too slowly at first and then becomes too fast too soon to be realistic.

27. *The best branching angles for blood vessels and pipes.* When a smaller pipe branches off from a larger one in a flow system, we may want it to run off at an angle that is best from some energy-saving point of view. We might require, for instance, that energy loss due to friction be minimized along the section AOB shown in Fig. 6.45. In this diagram, B is a given point to be reached by the smaller pipe, A is a point in the larger pipe upstream from B, and O is the point where the branching occurs. A law due to Poiseuille states that the loss of energy due to friction in nonturbulent flow is proportional to the length of the path and inversely proportional to the fourth power of the radius. Thus, the loss along AO is $(kd_1)/R^4$ and along OB is $(kd_2)/r^4$, where k is a constant, d_1 is the length of AO, d_2 is the length of OB, R is the radius of the larger pipe, and r is the radius of the smaller pipe. The angle θ is to be chosen to minimize the sum of these two losses:

$$L = k\,\frac{d_1}{R^4} + k\,\frac{d_2}{r^4}.$$



6.45 Diagram for Exercise 27.

In our model, we assume that $AC = a$ and $BC = b$ are fixed. Thus we have the relations

$$d_1 + d_2\cos\theta = a \qquad d_2\sin\theta = b,$$

so that

$$d_2 = b\csc\theta,$$

$$d_1 = a - d_2\cos\theta = a - b\cot\theta.$$

We can express the total loss L as a function of θ:

$$L = k\left(\frac{a - b\cot\theta}{R^4} + \frac{b\csc\theta}{r^4}\right).$$

a) Show that the critical value of θ for which $dL/d\theta$ equals zero is

$$\theta_c = \cos^{-1}\frac{r^4}{R^4}.$$

b) CALCULATOR If the ratio of the pipe radii is $r/R = 5/6$, estimate to the nearest degree the optimal branching angle given in part (a).

The mathematical analysis described here is also used to explain the angles at which arteries branch in an animal's body. (See *Introduction to Mathematics for Life Scientists,* Second Edition, by E. Batschelet [New York: Springer-Verlag, 1976].)

28. *Group blood testing.* During World War II it was necessary to administer blood tests to large numbers of recruits. There are two standard ways to administer a blood test to N people. In method 1, each person is tested separately. In method 2, the blood samples of x people are pooled and tested as one large sample. If the test is negative, this one test is enough for all x people. If the test is positive, then each of the x people is tested separately, requiring a total of $x + 1$ tests. Using the second method and some probability theory it can be shown that, on the average, the total number of tests y will be

$$y = N\left(1 - q^x + \frac{1}{x}\right).$$

With $q = 0.99$ and $N = 1000$, find the integer value of x that minimizes y. Also find the integer value of x that maximizes y. (This second result is not important to the real-life situation.) The group testing method was used in World War II with a savings of 80% over the individual testing method, but not with the given value of q.

29. *Transport through a cell membrane.* Under some conditions the result of the movement of a dissolved substance across a cell's membrane is described by the equation

$$\frac{dy}{dt} = k\,\frac{A}{V}(c - y).$$

In this equation, y is the concentration of the substance inside the cell and dy/dt is the rate at which y changes over time. The letters k, A, V, and c stand for constants, k being the *permeability coefficient* (a property of the membrane), A the surface area of the membrane, V the cell's volume, and c the concentration of the substance outside the cell. The equation says that the rate at which the concentration changes within the cell is proportional to the difference between it and the outside concentration.

a) Solve the equation for $y(t)$, using y_0 to denote $y(0)$.

b) Find the steady state concentration, $\lim_{t\to\infty} y(t)$.
(Based on *Some Mathematical Models in Biology* by R. M. Thrall, J. A. Mortimer, K. R. Rebman, R. F. Baum, Eds., Revised Edition, December 1967, PB-202 364, pp. 101–103; distributed by N.T.I.S., U.S. Department of Commerce.)