On Conjugates :- $\{(a+ib)(a-ib)=a^2+b^2\}$

Q.11) If $x+iy=\sqrt{\frac{a+ib}{c+id}}$ show that $(x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$.

Sol.11) We have $x+iy=\sqrt{\frac{a+ib}{c+id}}$ ………… (i)

Taking conjugate on both sides

$\Rightarrow \overline{x+iy}=\sqrt{\overline{\frac{a+ib}{c+id}}}$

$\Rightarrow x-iy=\sqrt{\frac{a-ib}{c-id}}$ ………… (ii)

Equation (i) x (ii)

$\Rightarrow (x+iy)(x-iy)=\sqrt{\frac{a+ib}{c+id}}\times\sqrt{\frac{a-ib}{c-id}}$

$\Rightarrow x^2+y^2=\sqrt{\frac{(a+ib)(a-ib)}{(c+id)(c-id)}}$

$\Rightarrow x^2+y^2=\sqrt{\frac{a^2+b^2}{c^2+d^2}}$ ……………… $\{(a+ib)(a-ib)=a^2+b^2\}$

Squaring

$(x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$ (proved)

Q.12) If $\frac{(a+i)^2}{2a-i}=p+iq$ show that $p^2+q^2=\frac{(a^2+1)^2}{4a^2+1}$

Sol.12) We have $\frac{(a+i)^2}{2a-i}=p+iq$ …………… (i)

Taking conjugate on both sides

$\Rightarrow \overline{\frac{(a+i)^2}{2a-i}}=\overline{p+iq}$

$\Rightarrow \frac{(a-i)^2}{2a+i}=p-iq$ ………… (ii)

Equation (i) x (ii)

$\Rightarrow \frac{(a+i)^2}{2a-i}\times\frac{(a-i)^2}{2a+i}=(p-iq)\,(p-iq)$

$\Rightarrow \frac{(a+i)(a+i)^2}{4a^2+1}=p^2+q^2$……………… $\{(a+ib)(a-ib)=a^2+b^2\}$

Squaring

$\frac{(a^2+1)^2}{4a^2+1}=p^2+q^2$ (proved)

Q.13) If $(1+i)(2+2i)(1+3i)\ldots(1+ni)=x+iy$ show that $2.5.10\ldots.(1+n^2)=x^2+y^2$.

Sol.13) We have $(1+i)(2+2i)(1+3i)\ldots\ldots\ldots(1+ni)=x+iy$………… (i)

Taking conjugate on both sides

$\Rightarrow (\overline{1+i})(\overline{2+2i})(\overline{1+3i})\ldots\ldots\ldots(\overline{1+ni})=\overline{x+iy}$

$\Rightarrow (1-i)(2-2i)(1-3i)\ldots\ldots\ldots(1-ni)=x-iy$ ………… (ii)

Equation (i) x (ii)

$\Rightarrow (1+i)(1-i)(2+2i)(2-2i)(1+3i)(1-3i)\ldots\ldots\ldots..(1+ni)(1-ni)$

$=(x+iy)(x-iy)$

$\Rightarrow (1+1)(1-4)(1+9)\ldots\ldots(1+n^2)=x^2+y^2\ldots\ldots\ldots$ $\{(a+ib)(a-ib)=a^2+b^2\}$

Squaring

$2.5.10\ldots.(1+n^2)=x^2+y^2$ (proved)

	On Standard Form:-	
Q.14)	Find θ such that $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is purely real.	
Sol.14)	Let $z=\frac{3+2i\sin\theta}{1-2i\sin\theta}$ Rationalize $\Rightarrow z=\frac{3+2i\sin\theta}{1-2i\sin\theta}\times\frac{1+2i\sin\theta}{1+2i\sin\theta}$ $\Rightarrow z=\frac{3+6i\sin\theta+2i\sin\theta+4i^2\sin^2\theta}{1-4i^2\sin^2\theta}$ $\Rightarrow z=\frac{3+8i\sin\theta-4\sin^2\theta}{1+4\sin^2\theta}$ $\Rightarrow z=\frac{(3-4\sin^2\theta)}{1+4\sin^2\theta}+\frac{8i\sin\theta}{1+4\sin^2\theta}$ It is given that z is purely real $\therefore Im(z)=0$ $\Rightarrow \frac{8\sin\theta}{1+4\sin^2\theta}=0$ $\Rightarrow 8\sin\theta=0$ $\Rightarrow \sin\theta=0$ $\Rightarrow \theta=n\pi\ ;\ n\in z$ ans.	
Q.15)	If z is complex number such that $\lvert z\rvert=1$ prove that $\left(\frac{z-1}{z+1}\right)$ is purely imaginary. What will be your conclusion of $z=1$.	
Sol.15)	Let $zx+iy$ $\lvert z\rvert=1$ (given) $\Rightarrow \sqrt{x^2+y^2}=1$ $\Rightarrow x^2+y^2=1$ .... (i) Now, $\frac{z-1}{z+1}=\frac{x+iy-1}{x+iy+1}=\frac{(x-1)+iy}{(x+1)+iy}$ $\Rightarrow \frac{z-1}{z+1}=\frac{[(x-1)+iy][(x+1)-iy]}{[(x+1)+iy][(x+1)-iy]}$................(rationalize) $=\frac{(x^2-1)-iy(x-1)+iy(x+1)-i^2y^2}{(x+1)^2-i^2y^2}$ $=\frac{(x^2-1)+i(-yx+y+yx+y)+y^2}{(x+1)^2+y^2}$ $=\frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$ $=\frac{0+2iy}{(x+1)^2+y^2}$ .... ($x^2+y^2=1$ from eq. (i)) $\Rightarrow \frac{z-1}{z+1}=\frac{2iy}{(x+1)^2+y^2}$ which is purely imaginary Now when $z=1$ Then $\frac{z-1}{z+1}=\frac{1-1}{1+1}=\frac{0}{2}=0$ which is purely real ans.	
	Natural Number:-	
Q.16)	Find the least +ve integral value of m, if $\left(\frac{1+i}{1-i}\right)^m=1$	
Sol.16)	We have $\left(\frac{1+i}{1-i}\right)^m=1$ $\Rightarrow \left(\frac{1+i}{1-i}\times\frac{1+i}{1+i}\right)^m=1$ $\Rightarrow \left(\frac{1+i^2+2i}{1-i^2}\right)^m=1$ $\Rightarrow \left(\frac{1-1+2i}{1+1}\right)^m=1$	

	$\Rightarrow \left(\frac{2i}{2}\right)^m = 1$ $\Rightarrow (i)^m = 1$ $\therefore$ the least +ve integral value of $m = 4$ ans. ……. (Since $i^4 = 1$)	
Q.17)	Find the smallest +ve integer value of n for which $\frac{(1+i)^n}{(1-i)^{n-2}}$ is a real number.	
Sol.17)	We have $\frac{(1+i)^n}{(1-i)^{n-2}}$ $= \frac{(1+i)^n}{(1+i)^n(1-i)^{-2}}$ $= \left(\frac{1+i}{1-i}\right)^n (1-i)^2$ $= \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^n (1-i)^2$ $= \left(\frac{1+i^2+2i}{1-i^2}\right)^n (1-i^2-2i)$ $= \left(\frac{1-1+2i}{1+1}\right)^n (1-1-2i)$ $= (i)^n(-2i)$ $= -2(i)^{n+1}$ For real number : n should be equal to 1 Since $-2(i)^{1+1} = -2(i)^2 = -2(-1) = 2$ which is a real number $\therefore n = 1$ ans.	
Q.18)	Find the number of non-zero integral solutions if $\lvert 1-i\rvert^x = 2^x$	
Sol.18)	We have $\lvert 1-i\rvert^x = 2^x$ $\Rightarrow \left(\sqrt{1+1}\right)^x = 2^x$………… $\{\lvert z\rvert = \sqrt{a^2+b^2}\}$ $\Rightarrow \left(\sqrt{2}\right)^x = 2^x$ $\Rightarrow (2)^{\frac{x}{2}} = 2^x$ $\Rightarrow \frac{x}{2} = x$ $\Rightarrow x = 2x$ $\Rightarrow 2x - x = 0$ $\Rightarrow x = 0$ Hence, there is no non-zero integral solution	
Q.19)	If α & β are different complex numbers with $\lvert\beta\rvert = 1$. Find $\left\lvert\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right\rvert$	
Sol.19)	We have $\left\lvert\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right\rvert^2 = \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\frac{\overline{\beta-\alpha}}{1-\bar{\alpha}\beta}\right)$ ………….. $\lvert z\rvert^2 = z\bar{z}$ $= \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\frac{\bar{\beta}-\bar{\alpha}}{1-\alpha\bar{\beta}}\right)$ ………. $\{\overline{(\bar{z})} = z\}$ $= \frac{\beta\bar{\beta}-\beta\bar{\alpha}-\alpha\bar{\beta}+\alpha\bar{\alpha}}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+\alpha\bar{\alpha}\beta\bar{\beta}}$ $= \frac{\lvert\beta\rvert^2-\beta\bar{\alpha}-\alpha\bar{\beta}+\lvert\alpha\rvert^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+\lvert\alpha\rvert^2\lvert\beta\rvert^2}$ ………….. $\lvert z\rvert^2 = z\bar{z}$ $= \frac{1-\beta\bar{\alpha}-\alpha\bar{\beta}+\lvert\alpha\rvert^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+\lvert\alpha\rvert^2}$ …………… $given\ \lvert\beta\rvert = 1$ $\Rightarrow \left\lvert\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right\rvert^2 = 1$ $\Rightarrow \left\lvert\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right\rvert = 1$ ans.	
Q.20)	If $z_1$and z_2 are any two complex numbers then show that	

	$Re(z_1z_2) = Re(z_1)Re(z_2) - Im(z_1)Im(z_2)$.	
Sol.20)	Let $z_1 = a + ib$ and $z_2 = c + id$ $\therefore Re(z_1) = a$ and $Re(z_2) = c$ $Im(z_1) = b$ and $Im(z_2) = d$ Now, $z_1z_2 = (a + ib)(c + id)$ $= ac + iad + ibc + i^2bd$ $= ac + iad + ibc - bd$ $z_1z_2 = (ac + bd) + i(ad + bc)$ Here, $Re(z_1z_2) = ac - bd$ $= Re(z_1)Re(z_2) - Im(z_1)Im(z_2)$ RHS (proved)	
	Quadratic Equation:-	
Q.21)	Solve $2x^2 + 3ix + 2 = 0$	
Sol.21)	Here $a = 2, b = 3i\ , c = 2$ By quadratic formula, $\Rightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $\Rightarrow x = \frac{-3i\pm\sqrt{91^2-4(2)(2)}}{4}$ $\Rightarrow x = \frac{-3i\pm\sqrt{-9-16}}{4}$ $\Rightarrow x = \frac{-3i\pm\sqrt{-25}}{4}$ $\Rightarrow x = \frac{-3i\pm5i}{4}$ $\Rightarrow x = \frac{-3i\pm5i}{4}$ and $x = \frac{-3i-5i}{4}$ $\Rightarrow x = \frac{i}{2}$ and $x = -2i$ ans.	
Q.22)	Solve $2x^2 + (3 + 7i)x - (3 - 9i) = 0$.	
Sol.22)	Here $a = 2, b = -(3 + 7i)\ , c = -(3 - 9i)$ By quadratic formula, $\Rightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $\Rightarrow x = \frac{(3+7i)\pm\sqrt{(3+7i)^2-4(2)(3-9i)}}{4}$ $\Rightarrow x = \frac{(3+7i)\pm\sqrt{9+49i^2+42i+24-72i}}{4}$ $\Rightarrow x = \frac{3+7i\pm\sqrt{9-49-30i+24}}{4}$ $\Rightarrow x = \frac{3+7i\pm\sqrt{-16-30i}}{4}$ Let $a + ib = \sqrt{-16 - 30i}$ Squaring, $\Rightarrow\ a^2 + i^2b^2 + 2iab = -16 - 30i$ $\Rightarrow (a^2 - b^2) + 2iab = -16 - 30i$ $\Rightarrow a^2 - b^2 = -16$ .... (i) And $2ab = -30$ Now $(a^2 - b^2)^2 = (a^2 - b^2)^2 + (2ab)^2$ $= 256 + 900$ $= 1156$ $\Rightarrow a^2 - b^2 = 34$ .... (ii) Adding (i) & (ii) $\Rightarrow a^2 = 18$	

	$\Rightarrow a^2 = a$ $\Rightarrow a = \pm 3$ Put $a^2 = 9$ in eq.(ii) $\Rightarrow 9 + b^2 = 34$ $\Rightarrow b^2 = 25$ $\Rightarrow b = \pm 5$ $\therefore \sqrt{-16 - 30i} = \pm(3 - 5i)$ $\therefore$ value of x becomes $\Rightarrow x = \frac{3+7i\pm(3-5i)}{4}$ $\Rightarrow x = \frac{3+7i+(3-5i)}{4}$ and $x = \frac{3+7i-(3-5i)}{4}$ $\Rightarrow x = \frac{6+2i}{4}$ and $x = \frac{12i}{4}$ $\Rightarrow x = \frac{3}{2} + \frac{1}{2}i$ and $x = 3i$ ans.	
Q.23)	If $z = x + iy$ and $w = \frac{1-iz}{z-i}$, If $\|w\| = 1$ show that z is purely real.	
Sol.23)	We have $z = x + iy, w = \frac{1-iz}{z-i}$ and $\|w\| = 1$ $\Rightarrow \left\|\frac{1-iz}{1-iz}\right\| = 1$ $\Rightarrow \frac{\|1-iz\|}{\|z-i\|} = 1$ .... $\left\{\left\|\frac{z_1}{z_2}\right\| = \frac{\|z_1\|}{\|z_2\|}\right\}$ $\Rightarrow \|1 - iz\| = \|z - i\|$ $\Rightarrow \|1 - i(x + iy)\| = \|x + iy - i\|$ $\Rightarrow \|1 - ix - i^2y\| = \|x + i(y - i)\|$ $\Rightarrow \|(1 + y) - ix\| = \|x + i(y - i)\|$ $\Rightarrow \sqrt{(1 + y)^2 - x^2} = \sqrt{x^2 + (y - 1)^2}$ Squaring $\Rightarrow 1 + y^2 + 2y + x^2 = x^2 + y^2 + 1 - 2y$ $\Rightarrow 4y = 0$ $\Rightarrow y = 0$ Since $z = x + iy$ $\Rightarrow z = x + i(0)$ $\Rightarrow z = x$ which is purely real (proved)	
Q.24)	Convert into polar form $z = \sin\left(\frac{\pi}{5}\right) + i\left(1 - \cos\frac{\pi}{5}\right)$.	
Sol.24)	We have $z = \sin\left(\frac{\pi}{5}\right) + i\left(1 - \cos\frac{\pi}{5}\right)$ Here $a = \sin\left(\frac{\pi}{5}\right)$ and $b = 1 - \cos\frac{\pi}{5}$ $\Rightarrow r = \sqrt{a^2 + b^2} = \sqrt{\sin^2\frac{\pi}{5} + \left(1 - \cos\frac{\pi}{5}\right)^2}$ $= \sqrt{\sin^2\frac{\pi}{5} + 1 + \cos^2\frac{\pi}{5} - 2\cos\frac{\pi}{5}}$ $= \sqrt{1 + 1 - 2\cos\frac{\pi}{5}}$ $= \sqrt{2 - 2\cos\frac{\pi}{5}}$ $= \sqrt{2}\sqrt{1 - \cos\frac{\pi}{5}}$	

	$= \sqrt{2}\sqrt{2\sin^2\left(\frac{\pi}{10}\right)}$ $\Rightarrow r = 2\sin\left(\frac{\pi}{10}\right)$ Now $\tan\alpha = \left\lvert\frac{b}{a}\right\rvert = \left\lvert\frac{1-\cos\frac{\pi}{5}}{\sin\frac{\pi}{5}}\right\rvert$ $\Rightarrow \tan\alpha = \left\lvert\frac{2\sin^2\left(\frac{\pi}{10}\right)}{2\sin\left(\frac{\pi}{10}\right).\cos\frac{3}{10}}\right\rvert$ $\Rightarrow \tan\alpha = \tan\left(\frac{\pi}{10}\right)$ $\Rightarrow \alpha = \frac{\pi}{10}$ Since z is in 1st quadrant $\binom{a+ve}{b+ve}$ $\therefore \theta = \alpha$ $\Rightarrow \theta = \frac{\pi}{10}\,(argument\ is\ also\ called\ AMPLITUDE)$ Polar form is given by $\Rightarrow z = r[\cos(\theta) i \sin(\theta)]$ $\Rightarrow z = 2\sin\left(\frac{\pi}{10}\right)\left[\cos\left(\frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{10}\right)\right]$ ans.	
Q.25)	Evaluate: $(1+i)^6 + (1-i)^3$.	
Sol.25)	$\Rightarrow ((1+i)^2)^3 + (1-i)^3$ $\Rightarrow (1+i^2+2i)^3 + (1-i)^3$ $\Rightarrow (2i)^3 + 1^3 - i^3 - 3i + 3i^2$ $\Rightarrow 8i^3 + 1 + i - 3i - 3$ $\Rightarrow -8i + 1 + i - 3i - 3$ $\Rightarrow -2 - 10i$ ans.	