Chapter 4

INDEFINITE INTEGRALS. BASIC METHODS OF INTEGRATION

§ 4.1. Direct Integration and the Method of Expansion

Direct integration consists in using the following table of integrals:

(1) $\int u^n\,du=\frac{u^{n+1}}{n+1}+C \quad (n\neq -1)$;

(2) $\int \frac{du}{u}=\ln|u|+C$;

(3) $\int a^u\,du=\frac{1}{\ln a}a^u+C$; $\qquad \int e^u\,du=e^u+C$;

(4) $\int \cos u\,du=\sin u+C$; $\qquad \int \sin u\,du=-\cos u+C$;

(5) $\int \cosh u\,du=\sinh u+C$; $\qquad \int \sinh u\,du=\cosh u+C$;

(6) $\int \frac{du}{\cos^2 u}=\tan u+C$; $\qquad \int \frac{du}{\sin^2 u}=-\cot u+C$;

(7) $\int \frac{du}{u^2+a^2}=\frac{1}{a}\arctan\frac{u}{a}+C=-\frac{1}{a}\operatorname{arc\,cot}\frac{u}{a}+C_1 \quad (a>0)$;

(8) $\int \frac{du}{\sqrt{a^2-u^2}}=\arcsin\frac{u}{a}+C=-\arccos\frac{u}{a}+C_1 \quad (a>0)$;

(9) $\int \frac{du}{\sqrt{u^2\pm a^2}}=\ln\left(u+\sqrt{u^2\pm a^2}\right)+C$;

(10) $\int \frac{du}{u^2-a^2}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C$.

In all these formulas the variable u is either an independent variable or a differentiable function of some variable. If

$$\int f(u)\,du=F(u)+C,$$

then

$$\int f(ax+b)\,dx=\frac{1}{a}F(ax+b)+C.$$

The method of expansion consists in expanding the integrand into a linear combination of simpler functions and using the linearity property of the integral:

$$\int \sum_{i=1}^{n} a_i f_i(x)\,dx = \sum_{i=1}^{n} a_i \int f_i(x)\,dx \quad \left(\sum_{i=1}^{n} |a_i| > 0\right).$$

4.1.1. Find the integral $I = \int \frac{x^2+5x-1}{\sqrt{x}}\,dx$.

Solution.

$$I = \int \frac{x^2+5x-1}{\sqrt{x}}\,dx = \int (x^{3/2} + 5x^{1/2} - x^{-1/2})\,dx =$$
$$= \int x^{3/2}\,dx + 5\int x^{1/2}\,dx - \int x^{-1/2}\,dx =$$
$$= \frac{2x^{5/2}}{5} + C_1 + \frac{5\cdot 2}{3} x^{3/2} + C_2 - 2x^{1/2} + C_3 =$$
$$= 2\sqrt{x}\left(\frac{x^2}{5} + \frac{5x}{3} - 1\right) + C.$$

Note. There is no need to introduce an arbitrary constant after calculating each integral (as is done in the above example). By combining all arbitrary constants we get a single arbitrary constant, denoted by letter C, which is added to the final answer.

4.1.2. $I = \int \frac{6x^3+x^2-2x+1}{2x-1}\,dx.$

4.1.3. $I = \int \frac{dx}{\sin^2 x \cos^2 x}$.

Solution. Transform the integrand in the following way:

$$\frac{1}{\sin^2 x \cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}.$$

Hence,

$$I = \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x} = \tan x - \cot x + C.$$

4.1.4. $I = \int \tan^2 x\,dx.$

Solution. Since $\tan^2 x = \sec^2 x - 1$, then

$$I = \int \tan^2 x\,dx = \int \frac{dx}{\cos^2 x} - \int dx = \tan x - x + C.$$

4.1.5. $I = \int (x^2+5)^3\,dx.$

Solution. Expanding the integrand by the binomial formula, we find

$$I = \int (x^6 + 15x^4 + 75x^2 + 125)\,dx = \frac{x^7}{7} + \frac{15x^5}{5} + \frac{75x^3}{3} + 125x + C.$$

4.1.6. $I = \int (3x+5)^{17}\,dx.$

Solution. Here it is not expedient to raise the binomial to the 17th power, since $u=3x+5$ is a linear function.

Proceeding from the tabular integral

$$\int u^{17}\,du=\frac{u^{18}}{18}+C,$$

we get

$$I=\frac{1}{3}\cdot\frac{(3x+5)^{18}}{18}+C.$$

4.1.7. $I=\int\frac{dx}{\sqrt{x+1}-\sqrt{x}}$.

4.1.8. $I=\int\cos(\pi x+1)\,dx$.

Solution. Proceeding from the tabular integral (4)

$$\int\cos u\,du=\sin u+C,$$

we obtain

$$I=\frac{1}{\pi}\sin(\pi x+1)+C.$$

4.1.9. $I=\int\cos 4x\cos 7x\,dx$.

Solution. When calculating such integrals it is advisable to use the trigonometric product formulas. Here

$$\cos 4x\cos 7x=\frac{1}{2}(\cos 3x+\cos 11x)$$

and therefore

$$I=\frac{1}{2}\int\cos 3x\,dx+\frac{1}{2}\int\cos 11x\,dx=\frac{1}{6}\sin 3x+\frac{1}{22}\sin 11x+C.$$

Note. When solving such problems it is expedient to use the following trigonometric identities:

$$\sin mx\cos nx=\frac{1}{2}[\sin(m-n)x+\sin(m+n)x];$$

$$\sin mx\sin nx=\frac{1}{2}[\cos(m-n)x-\cos(m+n)x];$$

$$\cos mx\cos nx=\frac{1}{2}[\cos(m-n)x+\cos(m+n)x].$$

4.1.10. $I=\int\cos x\cos 2x\cos 5x\,dx$.

Solution. We have

$$(\cos x\cos 2x)\cos 5x=\frac{1}{2}(\cos x+\cos 3x)\cos 5x=$$

$$=\frac{1}{4}[\cos 4x+\cos 6x]+\frac{1}{4}(\cos 2x+\cos 8x).$$

Thus,

$$I=\frac{1}{4}\left[\int \cos 2x\,dx+\int \cos 4x\,dx+\int \cos 6x\,dx+\int \cos 8x\,dx\right]=$$
$$=\frac{1}{8}\sin 2x+\frac{1}{16}\sin 4x+\frac{1}{24}\sin 6x+\frac{1}{32}\sin 8x+C.$$

4.1.11. $I=\int \sin^2 3x\,dx.$

Solution. Since $\sin^2 3x=\frac{1-\cos 6x}{2}$, then

$$I=\frac{1}{2}\int (1-\cos 6x)\,dx=\frac{1}{2}x-\frac{1}{12}\sin 6x+C.$$

4.1.12. $I=\int \cosh^2 (8x+5)\,dx.$

Solution. Since $\cosh^2 u=\frac{\cosh 2u+1}{2}$, then

$$I=\frac{1}{2}\int [1+\cosh (16x+10)]\,dx=\frac{1}{2}x+\frac{1}{32}\sinh (16x+10)+C.$$

4.1.13. $I=\int \frac{dx}{x^2+4x+5}.$

Solution. $I=\int \frac{dx}{x^2+4x+5}=\int \frac{dx}{(x+2)^2+1}=\arctan (x+2)+C.$

4.1.14. $I=\int \frac{dx}{4x^2+25}.$

4.1.15. $I=\int \frac{dx}{x^2+x+1}.$

4.1.16. $I=\int \frac{dx}{\sqrt{4-9x^2}}.$

Solution. $I=\int \frac{dx}{\sqrt{4-9x^2}}=\frac{1}{3}\int \frac{dx}{\sqrt{4/9-x^2}}=\frac{1}{3}\arcsin \frac{3x}{2}+C.$

4.1.17. $I=\int \frac{dx}{\sqrt{5-x^2-4x}}.$

Solution. $I=\int \frac{dx}{\sqrt{5-x^2-4x}}=\int \frac{dx}{\sqrt{9-(x+2)^2}}=\arcsin \frac{x+2}{3}+C.$

4.1.18. $I=\int \frac{dx}{\sqrt{x^2+6x+1}}.$

4.1.19. $I=\int \frac{dx}{4-x^2-4x}.$

Solution.

$$I=\int \frac{dx}{4-x^2-4x}=\int \frac{dx}{8-(x+2)^2}=\frac{1}{4\sqrt{2}}\ln\left|\frac{2\sqrt{2}+x+2}{2\sqrt{2}-(x+2)}\right|+C.$$

4.1.20. $I=\int \frac{dx}{10x^2-7}.$

4.1.21. Evaluate the following integrals:

(a) $\int \frac{dx}{x^2-6x+13}$; (b) $\int \frac{x-1}{\sqrt[3]{x^2}} dx$;

(c) $\int \frac{3-2\cot^2 x}{\cos^2 x} dx$; (d) $\int \frac{2+3x^2}{x^2(1+x^2)} dx$.

4.1.22. Integrate:

(a) $\int \frac{\sqrt{1-x^2}+\sqrt{1+x^2}}{\sqrt{1-x^4}} dx$; (b) $\int \frac{\cos 2x}{\cos x - \sin x} dx$;

(c) $\int \frac{2^{x+1}-5^{x-1}}{10^x} dx$; (d) $\int (\sin 5x - \sin 5\alpha)\, dx$.

§ 4.2. Integration by Substitution

The *method of substitution* (or *change of variable*) consists in substituting $\varphi(t)$ for x where $\varphi(t)$ is a continuously differentiable function. On substituting we have:

$$\int f(x)\,dx = \int f[\varphi(t)]\,\varphi'(t)\,dt,$$

and after integration we return to the old variable by inverse substitution $t=\varphi^{-1}(x)$.

The indicated formula is also used in the reverse direction:

$$\int f[\varphi(t)]\,\varphi'(t)\,dt = \int f(x)\,dx, \text{ where } x=\varphi(t).$$

4.2.1. $I=\int x\sqrt{x-5}\,dx$.

Solution. Make the substitution

$$\sqrt{x-5}=t.$$

Whence

$$x-5=t^2, \quad x=t^2+5, \quad dx=2t\,dt.$$

Substituting into the integral we get

$$I=\int (t^2+5)\,t\cdot 2t\,dt = 2\int (t^4+5t^2)\,dt = 2\,\frac{t^5}{5}+\frac{10t^3}{3}+C.$$

Now return to the initial variable x:

$$I=\frac{2(x-5)^{5/2}}{5}+\frac{10(x-5)^{3/2}}{3}+C.$$

4.2.2. $I=\int \frac{dx}{1+e^x}$.

Solution. Let us make the substitution $1+e^x=t$. Whence

$$e^x=t-1, \quad x=\ln(t-1), \quad dx=dt/(t-1).$$

Substituting into the integral, we get

$$I=\int \frac{dx}{1+e^x}=\int \frac{dt}{t(t-1)}.$$

But

$$\frac{1}{t(t-1)}=\frac{1}{t-1}-\frac{1}{t},$$

therefore

$$I=\int \frac{dt}{t-1}-\int \frac{dt}{t}=\ln|t-1|-\ln|t|+C.$$

Coming back to the variable x, we obtain

$$I=\ln \frac{e^x}{1+e^x}+C=x-\ln(1+e^x)+C.$$

Note. This integral can be calculated in a simpler way by multiplying both the numerator and denominator by e^{-x}:

$$\int \frac{e^{-x}}{e^{-x}+1}\,dx=-\int \frac{-e^{-x}}{e^{-x}+1}\,dx=-\ln(e^{-x}+1)+C=$$
$$=-\ln \frac{e^x+1}{e^x}=x-\ln(e^x+1)+C.$$

4.2.3. $I=\int \frac{x^2+3}{\sqrt{(2x-5)^3}}\,dx.$

4.2.4. $I=\int \frac{(x^2-1)\,dx}{(x^4+3x^2+1)\arctan \frac{x^2+1}{x}}.$

Solution. Transform the integrand

$$I=\int \frac{(1-1/x^2)\,dx}{[(x+1/x)^2+1]\arctan(x+1/x)}.$$

Make the substitution $x+\frac{1}{x}=t$; differentiating, we get

$$\left(1-\frac{1}{x^2}\right)dx=dt.$$

Whence

$$I=\int \frac{dt}{(t^2+1)\arctan t}.$$

Make one more substitution: $\arctan t=u$. Then

$$\frac{dt}{t^2+1}=du$$

and

$$I=\int \frac{du}{u}=\ln|u|+C.$$

Returning first to t, and then to x, we have

$$I = \ln|\arc\tan t| + C = \ln\left|\arc\tan\left(x + \frac{1}{x}\right)\right| + C.$$

4.2.5. $I = \int \frac{\sqrt{a^2 - x^2}}{x^4}\, dx.$

Solution. Make the substitution:

$$x = \frac{1}{t}; \quad dx = -\frac{dt}{t^2}.$$

Hence,

$$I = -\int \frac{\sqrt{a^2 - 1/t^2}}{(1/t^4)\, t^2}\, dt = -\int t\sqrt{a^2t^2 - 1}\, dt.$$

Now make one more substitution: $\sqrt{a^2t^2 - 1} = z$. Then $2a^2t\, dt = 2z\, dz$ and

$$I = -\frac{1}{a^2}\int z^2\, dz = -\frac{1}{3a^2} z^3 + C.$$

Returning to t and then to x, we obtain

$$I = -\frac{(a^2 - x^2)^{3/2}}{3a^2x^3} + C.$$

4.2.6. $I = \int \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}.$

Solution.

$$I = \int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{1}{b^2}\int \frac{1}{\frac{a^2}{b^2}\tan^2 x + 1} \cdot \frac{dx}{\cos^2 x}.$$

Make the substitution $\frac{a}{b}\tan x = t$; $dt = \frac{a}{b}\frac{dx}{\cos^2 x}$. Then

$$I = \frac{1}{ab}\int \frac{dt}{1 + t^2} = \frac{1}{ab}\arc\tan t + C.$$

Returning to x, we obtain

$$I = \frac{1}{ab}\arc\tan\left(\frac{a}{b}\tan x\right) + C.$$

4.2.7. $I = \int \sqrt[3]{1 + 3\sin x}\cos x\, dx.$

Solution. Make the substitution $1 + 3\sin x = t$, $3\cos x\, dx = dt$. Then

$$I = \frac{1}{3}\int \sqrt[3]{t}\, dt = \frac{1}{3}\int t^{1/3}\, dt = \frac{1}{3}\cdot\frac{3}{4} t^{4/3} + C = \frac{(1 + 3\sin x)^{4/3}}{4} + C.$$

4.2.8. $I = \int \frac{\sin x\, dx}{\sqrt{\cos x}}.$

4.2.9. $I=\int \frac{dx}{(\arccos x)^5 \sqrt{1-x^2}}$.

Solution. Make the substitution: $\arccos x=t$; $-\frac{dx}{\sqrt{1-x^2}}=dt$. Then

$$I=-\int \frac{dt}{t^5}=-\int t^{-5}\,dt=\frac{1}{4}t^{-4}+C=\frac{1}{4\arccos^4 x}+C.$$

4.2.10. $I=\int \frac{x^2+1}{\sqrt[3]{x^3+3x+1}}\,dx$.

4.2.11. $I=\int \frac{\sin 2x}{1+\sin^2 x}\,dx$.

Solution. Make the substitution:

$$1+\sin^2 x=t;\ 2\sin x\cos x\,dx=\sin 2x\,dx=dt.$$

Then

$$I=\int \frac{dt}{t}=\ln t+C=\ln(1+\sin^2 x)+C.$$

4.2.12. $I=\int \frac{1+\ln x}{3+x\ln x}\,dx$.

Solution. Substitute

$$3+x\ln x=t, \qquad (1+\ln x)\,dx=dt$$

and get

$$I=\int \frac{dt}{t}=\ln|t|+C=\ln|3+x\ln x|+C.$$

4.2.13. Evaluate the following integrals:

(a) $\int \frac{\sqrt[3]{1+\ln x}}{x}\,dx$; (b) $\int \frac{dx}{x\ln x}$;

(c) $\int \frac{x\,dx}{\sqrt{3-x^4}}$; (d) $\int \frac{x^{n-1}}{x^{2n}+a^2}\,dx$;

(e) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$; (f) $\int \left(\ln x+\frac{1}{\ln x}\right)\frac{dx}{x}$.

4.2.14. Find the following integrals:

(a) $\int x^2\sqrt[3]{1-x}\,dx$; (b) $\int \frac{\ln x\,dx}{x\sqrt{1+\ln x}}$;

(c) $\int \cos^5 x\sqrt{\sin x}\,dx$; (d) $\int \frac{x^5}{\sqrt{1-x^2}}\,dx$.

§ 4.3. *Integration by Parts*

The formula

$$\int u\,dv=uv-\int v\,du$$

is known as the formula for *integration by parts*, where u and v are differentiable functions of x.

To use this formula the integrand should be reduced to the product of two factors: one function and the differential of another function. If the integrand is the product of a logarithmic or an inverse trigonometric function and a polynomial, then u is usually taken to be either the logarithmic or the inverse trigonometric function. But if the integrand is the product of a trigonometric or an exponential function and an algebraic one, then u usually denotes the algebraic function.

4.3.1. $I=\int \arctan x\,dx.$

Solution. Let us put here

$$u=\arctan x, \qquad dv=dx;$$

whence

$$du=\frac{dx}{1+x^2}; \qquad v=x;$$

$$I=\int \arctan x\,dx=x\arctan x-\int\frac{x\,dx}{1+x^2}=x\arctan x-\frac{1}{2}\ln(1+x^2)+C.$$

4.3.2. $I=\int \arcsin x\,dx.$

4.3.3. $I=\int x\cos x\,dx.$

Solution. Let us put

$$u=x; \qquad dv=\cos x\,dx,$$

whence

$$du=dx; \qquad v=\sin x,$$

$$I=\int x\cos x\,dx=x\sin x-\int \sin x\,dx=x\sin x+\cos x+C.$$

We will show now what would result from an unsuitable choice of the multipliers u and dv.

In the integral $\int x\cos x\,dx$ let us put

$$u=\cos x; \qquad dv=x\,dx,$$

whence

$$du=-\sin x\,dx; \qquad v=\frac{1}{2}x^2.$$

In this case

$$I=\frac{1}{2}x^2\cos x+\frac{1}{2}\int x^2\sin x\,dx.$$

As is obvious, the integral has become more complicated.

4.3.4. $I=\int x^3\ln x\,dx.$

Solution. Let us put

$$u=\ln x; \quad dv=x^3\,dx,$$

whence

$$du=\frac{dx}{x}; \quad v=\frac{1}{4}x^4,$$

$$I=\frac{1}{4}x^4\ln x-\frac{1}{4}\int x^4\frac{dx}{x}=\frac{1}{4}x^4\ln x-\frac{1}{4}\int x^3\,dx=\frac{1}{4}x^4\ln x-\frac{1}{16}x^4+C.$$

4.3.5. $I=\int (x^2-2x+5)\,e^{-x}\,dx.$

Solution. Let us put

$$u=x^2-2x+5; \quad dv=e^{-x}\,dx,$$

whence

$$du=(2x-2)\,dx; \quad v=-e^{-x};$$

$$I=\int (x^2-2x+5)\,e^{-x}\,dx=-e^{-x}(x^2-2x+5)+2\int (x-1)e^{-x}\,dx.$$

We again integrate the last integral by parts. Put

$$x-1=u; \qquad dv=e^{-x}\,dx,$$

whence

$$du=dx; \ v=-e^{-x}.$$

$$I_1=2\int (x-1)\,e^{-x}\,dx=-2e^{-x}(x-1)+2\int e^{-x}\,dx=-2xe^{-x}+C.$$

Finally we get

$$I=-e^{-x}(x^2-2x+5)-2xe^{-x}+C=-e^{-x}(x^2+5)+C.$$

Note. As a result of calculation of integrals of the form $\int P(x)\,e^{ax}\,dx$ we obtain a function of the form $Q(x)\,e^{ax}$, where $Q(x)$ is a polynomial of the same degree as the polynomial $P(x)$.

This circumstance allows us to calculate the integrals of the indicated type using the method of indefinite coefficients, the essence of which is explained by the following example.

4.3.6. Applying the method of indefinite coefficients, evaluate

$$I=\int (3x^3-17)\,e^{2x}\,dx.$$

Solution. $\int (3x^3-17)\,e^{2x}\,dx=(Ax^3+Bx^2+Dx+E)\,e^{2x}+C.$

Differentiating the right and the left sides, we obtain

$$(3x^3-17)\,e^{2x}=2\,(Ax^3+Bx^2+Dx+E)\,e^{2x}+e^{2x}(3Ax^2+2Bx+D).$$

Cancelling e^{2x}, we have

$$3x^3-17\equiv 2Ax^3+(2B+3A)\,x^2+(2D+2B)\,x+(2E+D).$$

Equating the coefficients at the equal powers of x in the left and right sides of this identity, we get

$$3 = 2A; \qquad 0 = 2B + 3A;$$
$$0 = 2D + 2B; \qquad -17 = 2E + D.$$

Solving the system, we obtain

$$A = \frac{3}{2}; \quad B = -\frac{9}{4}; \quad D = \frac{9}{4}; \quad E = -\frac{77}{8}.$$

Hence,

$$\int (3x^3 - 17) e^{2x}\, dx = \left(\frac{3}{2}x^3 - \frac{9}{4}x^2 + \frac{9}{4}x - \frac{77}{8}\right) e^{2x} + C.$$

4.3.7. Integrate:

$$I = \int (x^3 + 1) \cos x\, dx.$$

Solution. Let us put

$$u = x^3 + 1; \quad dv = \cos x\, dx,$$

whence

$$du = 3x^2\, dx; \quad v = \sin x.$$

$$I = (x^3 + 1) \sin x - 3 \int x^2 \sin x\, dx = (x^3 + 1) \sin x - 3I_1,$$

where $I_1 = \int x^2 \sin x\, dx$.

Integrating by parts again, we get

$$I_1 = -x^2 \cos x + 2I_2,$$

where $I_2 = \int x \cos x\, dx$.

Integrating by parts again, we obtain

$$I_2 = x \sin x + \cos x + C.$$

Finally, we have:

$$I = \int (x^3 + 1) \cos x\, dx = (x^3 + 1) \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C =$$
$$= (x^3 - 6x + 1) \sin x + (3x^2 - 6) \cos x + C.$$

Note. The method of indefinite coefficients may also be applied to integrals of the form

$$\int P(x) \sin ax\, dx, \quad \int P(x) \cos ax\, dx.$$

4.3.8. $I = \int (x^2 + 3x + 5) \cos 2x\, dx.$

Solution. Let us put

$$\int (x^2 + 3x + 5) \cos 2x\, dx =$$
$$= (A_0 x^2 + A_1 x + A_2) \cos 2x + (B_0 x^2 + B_1 x + B_2) \sin 2x + C.$$

Differentiate both sides of the identity:

$$(x^2+3x+5)\cos 2x = -2(A_0x^2+A_1x+A_2)\sin 2x +$$
$$+(2A_0x+A_1)\cos 2x+2(B_0x^2+B_1x+B_2)\cos 2x+(2B_0x+B_1)\sin 2x=$$
$$=[2B_0x^2+(2B_1+2A_0)x+(A_1+2B_2)]\cos 2x+$$
$$+[-2A_0x^2+(2B_0-2A_1)x+(B_1-2A_2)]\sin 2x.$$

Equating the coefficients at equal powers of x in the multipliers $\cos 2x$ and $\sin 2x$, we get a system of equations:

$$2B_0=1; \qquad 2(B_1+A_0)=3; \qquad A_1+2B_2=5;$$
$$-2A_0=0; \qquad 2(B_0-A_1)=0; \qquad B_1-2A_2=0.$$

Solving the system, we find

$$A_0=0;\ B_0=\frac{1}{2};\ A_1=\frac{1}{2};\ B_1=\frac{3}{2};\ A_2=\frac{3}{4};\ B_2=\frac{9}{4}.$$

Thus,

$$\int (x^2+3x+5)\cos 2x\,dx=\left(\frac{x}{2}+\frac{3}{4}\right)\cos 2x+\left(\frac{1}{2}x^2+\frac{3}{2}x+\frac{9}{4}\right)\sin^2 x+C.$$

4.3.9. $I=\int (3x^2+6x+5)\arctan x\,dx.$

Solution. Let us put

$$u=\arctan x; \qquad dv=(3x^2+6x+5)\,dx,$$

whence

$$du=\frac{dx}{1+x^2}; \qquad v=x^3+3x^2+5x.$$

Hence,

$$I=(x^3+3x^2+5x)\arctan x-\int\frac{x^3+3x^2+5x}{1+x^2}\,dx.$$

Single out the integral part under the last integral by dividing the numerator by the denominator:

$$I_1=\int\frac{x^3+3x^2+5x}{1+x^2}\,dx=\int (x+3)\,dx+\int\frac{4x-3}{x^2+1}\,dx=$$
$$=\frac{x^2}{2}+3x+2\int\frac{2x\,dx}{x^2+1}-3\int\frac{dx}{x^2+1}=\frac{x^2}{2}+3x+2\ln(x^2+1)-3\arctan x+C.$$

Substituting the value of I_1, we finally get

$$I=(x^3+3x^2+5x+3)\arctan x-x^2/2-3x-2\ln(x^2+1)+C.$$

4.3.10. Find the integral

$$I=\int e^{5x}\cos 4x\,dx.$$

Solution. Let us put

$$e^{5x}=u; \qquad \cos 4x\,dx=dv,$$

whence

$$5e^{5x}\,dx = du;\quad v = \frac{1}{4}\sin 4x.$$

Hence,

$$I = \frac{1}{4}e^{5x}\sin 4x - \frac{5}{4}\int e^{5x}\sin 4x\,dx.$$

Integrating by parts again, we obtain

$$I_1 = \int e^{5x}\sin 4x\,dx = -\frac{1}{4}e^{5x}\cos 4x + \frac{5}{4}\int e^{5x}\cos 4x\,dx.$$

Thus,

$$I = \frac{1}{4}e^{5x}\sin 4x - \frac{5}{4}\left(-\frac{1}{4}e^{5x}\cos 4x + \frac{5}{4}\int e^{5x}\cos 4x\,dx\right),$$

i. e.

$$I = \frac{1}{4}e^{5x}\left(\sin 4x + \frac{5}{4}\cos 4x\right) - \frac{25}{16}I.$$

Whence

$$I = \frac{4}{41}e^{5x}\left(\sin 4x + \frac{5}{4}\cos 4x\right) + C.$$

4.3.11. $I = \int \cos(\ln x)\,dx.$

Solution. Let us put

$$u = \cos(\ln x);\quad dv = dx,$$

whence

$$du = -\sin(\ln x)\frac{dx}{x};\quad v = x.$$

Hence,

$$I = \int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx.$$

Integrate by parts once again

$$u = \sin(\ln x);\qquad dv = dx,$$

whence

$$du = \cos(\ln x)\frac{dx}{x};\quad v = x.$$

Hence,

$$I_1 = \int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx.$$

Thus

$$I = \int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x) - I.$$

Hence

$$I = \frac{x}{2}[\cos(\ln x) + \sin(\ln x)] + C.$$

4.3.12. $I=\int x\ln\left(1+\frac{1}{x}\right)dx.$

Solution. Let us transform the integrand

$$\ln\left(1+\frac{1}{x}\right)=\ln\frac{x+1}{x}=\ln(x+1)-\ln x.$$

Hence

$$I=\int x\ln(x+1)\,dx-\int x\ln x\,dx=I_1-I_2.$$

Let us integrate I_1 and I_2 by parts. Put

$$u=\ln(x+1);\quad dv=x\,dx,$$

whence

$$du=\frac{dx}{1+x};\quad v=\frac{1}{2}(x^2-1).$$

Hence

$$I_1=\int x\ln(x+1)\,dx=\frac{1}{2}(x^2-1)\ln(x+1)-\frac{1}{2}\int\frac{(x^2-1)\,dx}{1+x}=$$
$$=\frac{x^2-1}{2}\ln(x+1)-\frac{1}{2}\int(x-1)\,dx=\frac{x^2-1}{2}\ln(x+1)-\frac{1}{4}x^2+\frac{1}{2}x+C.$$

Analogously,

$$I_2=\int x\ln x\,dx=\frac{x^2}{2}\ln x-\frac{1}{4}x^2+C.$$

Finally we have

$$I=\int x\ln\left(1+\frac{1}{x}\right)dx=\frac{1}{2}(x^2-1)\ln(x+1)-\frac{x^2}{2}\ln x+\frac{x}{2}+C.$$

4.3.13. $I=\int\frac{\sqrt{x^2+1}\,[\ln(x^2+1)-2\ln x]}{x^4}\,dx.$

Solution. First apply the substitution

$$1+\frac{1}{x^2}=t.$$

Then

$$dt=-\frac{2dx}{x^3}\quad\text{or}\quad\frac{dx}{x^3}=-\frac{1}{2}\,dt.$$

Hence,

$$I=\int\sqrt{1+\frac{1}{x^2}}\ln\frac{x^2+1}{x^2}\cdot\frac{dx}{x^3}=-\frac{1}{2}\int\sqrt{t}\ln t\,dt.$$

The obtained integral is easily evaluated by parts. Let us put

$$u=\ln t;\quad dv=\sqrt{t}\,dt.$$

Then

$$du=\frac{dt}{t};\quad v=\frac{2}{3}t\sqrt{t}.$$

Whence

$$-\frac{1}{2}\int \sqrt{t}\ln t\,dt=-\frac{1}{2}\left[\frac{2}{3}t\sqrt{t}\ln t-\frac{2}{3}\int \sqrt{t}\,dt\right]=$$
$$=-\frac{1}{2}\left[\frac{2}{3}t\sqrt{t}\ln t-\frac{4}{9}t\sqrt{t}\right]+C.$$

Returning to x, we obtain

$$I=-\frac{1}{2}\left[\frac{2}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\ln\left(1+\frac{1}{x^2}\right)-\frac{4}{9}\left(1+\frac{1}{x^2}\right)^{3/2}\right]+C=$$
$$=\frac{(x^2+1)\sqrt{x^2+1}}{9x^3}\left[2-3\ln\left(1+\frac{1}{x^2}\right)\right]+C.$$

4.3.14. $I=\int \sin x \ln \tan x\,dx.$

4.3.15. $I=\int \ln(\sqrt{1-x}+\sqrt{1+x})\,dx.$

Solution. Let us put

$$u=\ln(\sqrt{1-x}+\sqrt{1+x});\quad dv=dx,$$

whence

$$du=\frac{1}{\sqrt{1-x}+\sqrt{1+x}}\left(-\frac{1}{2\sqrt{1-x}}+\frac{1}{2\sqrt{1+x}}\right)dx=$$
$$=\frac{1}{2}\cdot\frac{\sqrt{1-x}-\sqrt{1+x}}{\sqrt{1-x}+\sqrt{1+x}}\cdot\frac{dx}{\sqrt{1-x^2}}=\frac{1}{2}\cdot\frac{\sqrt{1-x^2}-1}{x\sqrt{1-x^2}}\,dx;$$
$$v=x.$$

Hence,

$$I=x\ln(\sqrt{1-x}+\sqrt{1+x})-\frac{1}{2}\int x\frac{\sqrt{1-x^2}-1}{x\sqrt{1-x^2}}\,dx=$$
$$=x\ln(\sqrt{1-x}+\sqrt{1+x})-\frac{1}{2}\int dx+\frac{1}{2}\int\frac{dx}{\sqrt{1-x^2}}=$$
$$=x\ln(\sqrt{1-x}+\sqrt{1+x})-\frac{1}{2}x+\frac{1}{2}\arcsin x+C.$$

Note. In calculating a number of integrals we had to use the method of integration by parts several times in succession. The result could be obtained more rapidly and in a more concise form by using the so-called *generalized formula for integration by parts* (or the *formula for multiple integration by parts*):

$$\int u(x)v(x)\,dx=u(x)v_1(x)-u'(x)v_2(x)+u''(x)v_3(x)-\ldots$$
$$\ldots+(-1)^{n-1}u^{(n-1)}(x)v_n(x)-(-1)^{n-1}\int u^{(n)}(x)v_n(x)\,dx,$$

where

$$v_1(x)=\int v(x)\,dx;\quad v_2(x)=\int v_1(x)\,dx;\ \ldots;\ v_n(x)=\int v_{n-1}(x)\,dx.$$

Here, of course, we assume that all derivatives and integrals appearing in this formula exist.

The use of the generalized formula for integration by parts is especially advantageous when calculating the integral $\int P_n(x)\varphi(x)\,dx$, where $P_n(x)$ is a polynomial of degree n, and the factor $\varphi(x)$ is such that it can be integrated successively $n+1$ times. For example,

$$\int P_n(x)e^{kx}\,dx = P_n(x)\frac{e^{kx}}{k} - P_n'(x)\frac{e^{kx}}{k^2} + \ldots + (-1)^n P_n^{(n)}(x)\frac{e^{kx}}{k^{n+1}} + C = e^{kx}\left[\frac{P_n(x)}{k} - \frac{1}{k^2}P_n'(x) + \ldots + \frac{(-1)^n}{k^{n+1}}P_n^{(n)}(x)\right] + C.$$

4.3.16. Applying the generalized formula for integration by parts, find the following integrals:

(a) $\int (x^3-2x^2+3x-1)\cos 2x\,dx$,

(b) $\int (2x^3+3x^2-8x+1)\sqrt{2x+6}\,dx$.

Solution.

(a) $$\int (x^3-2x^2+3x-1)\cos 2x\,dx = (x^3-2x^2+3x-1)\frac{\sin 2x}{2} - (3x^2-4x+3)\left(-\frac{\cos 2x}{4}\right) + (6x-4)\left(-\frac{\sin 2x}{8}\right) - 6\frac{\cos 2x}{16} + C = \frac{\sin 2x}{4}(2x^3-4x^2+3x) + \frac{\cos 2x}{8}(6x^2-8x+3) + C;$$

(b) $$\int (2x^3+3x^2-8x+1)\sqrt{2x+6}\,dx = (2x^3+3x^2-8x+1)\frac{(2x+6)^{3/2}}{3} - (6x^2+6x-8)\frac{(2x+6)^{5/2}}{3\cdot 5} + (12x+6)\frac{(2x+6)^{7/2}}{3\cdot 5\cdot 7} - 12\frac{(2x+6)^{9/2}}{3\cdot 5\cdot 7\cdot 9} + C = \frac{\sqrt{2x+6}}{5\cdot 7\cdot 9}(2x+6)(70x^3-45x^2-396x+897) + C.$$

Evaluate the following integrals:

4.3.17. $\int \ln(x+\sqrt{1+x^2})\,dx$.

4.3.18. $\int \sqrt[3]{x}(\ln x)^2\,dx$.

4.3.19. $\int \frac{\arcsin x\,dx}{\sqrt{1+x}}$.

4.3.20. $\int \frac{x\cos x\,dx}{\sin^3 x}$.

4.3.21. $\int 3^x\cos x\,dx$.

4.3.22. $\int (x^3-2x^2+5)\, e^{3x}\, dx$.

4.3.23. $\int (1+x^2)^2 \cos x\, dx$.

4.3.24. $\int (x^2+2x-1) \sin 3x\, dx$.

4.3.25. $\int (x^2-2x+3) \ln x\, dx$.

4.3.26. $\int x^3 \operatorname{arc\,tan} x\, dx$.

4.3.27. $\int x^2 \operatorname{arc\,cos} x\, dx$.

4.3.28. Applying the formula for multiple integration by parts, calculate the following integrals:

(a) $\int (3x^2+x-2)\sin^2(3x+1)\, dx$; (b) $\int \frac{x^2-7x+1}{\sqrt[3]{2x+1}}\, dx$.

§ 4.4. Reduction Formulas

Reduction formulas make it possible to reduce an integral depending on the index $n>0$, called the order of the integral, to an integral of the same type with a smaller index.

4.4.1. Integrating by parts, derive reduction formulas for calculating the following integrals:

(a) $I_n = \int \frac{dx}{(x^2+a^2)^n}$; (b) $I_{n,-m} = \int \frac{\sin^n x}{\cos^m x}\, dx$;

(c) $I_n = \int (a^2-x^2)^n\, dx$.

Solution. (a) We integrate by parts. Let us put

$$u = \frac{1}{(x^2+a^2)^n}, \quad dv = dx,$$

whence

$$du = -\frac{2n\, x\, dx}{(x^2+a^2)^{n+1}}, \quad v = x.$$

Hence,

$$I_n = \frac{x}{(x^2+a^2)^n} + 2n \int \frac{x^2}{(x^2+a^2)^{n+1}}\, dx =$$
$$= \frac{x}{(x^2+a^2)^n} + 2n \int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^{n+1}}\, dx = \frac{x}{(x^2+a^2)^n} + 2nI_n - 2na^2 I_{n+1},$$

whence

$$I_{n+1} = \frac{1}{2na^2} \cdot \frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2n} \cdot \frac{1}{a^2} I_n.$$

The obtained formula reduces the calculation of the integral I_{n+1} to the calculation of the integral I_n and, consequently, allows us to calculate completely an integral with a natural index, since

$$I_1=\int \frac{dx}{x^2+a^2}=\frac{1}{a}\arctan\frac{x}{a}+C.$$

For instance, putting $n=1$, we obtain

$$I_2=\int \frac{dx}{(x^2+a^2)^2}=\frac{1}{2a^2}\cdot\frac{x}{x^2+a^2}+\frac{1}{2a^2}I_1=\frac{1}{2a^2}\frac{x}{x^2+a^2}+\frac{1}{2a^3}\arctan\frac{x}{a}+C;$$

putting $n=2$, we get

$$I_3=\int \frac{dx}{(x^2+a^2)^3}=\frac{1}{4a^2}\cdot\frac{x}{(x^2+a^2)^2}+\frac{3}{4a^2}I_2=$$
$$=\frac{1}{4a^2}\cdot\frac{x}{(x^2+a^2)^2}+\frac{3}{8a^4}\cdot\frac{x}{x^2+a^2}+\frac{3}{8a^5}\arctan\frac{x}{a}+C.$$

(b) Let us apply the method of integration by parts, putting

$$u=\sin^{n-1}x;\quad dv=\frac{\sin x}{\cos^m x}dx,$$

whence

$$du=(n-1)\sin^{n-2}x\cos x\,dx;\ v=\frac{1}{(m-1)\cos^{m-1}x}\quad (m\neq 1).$$

Hence,

$$I_{n,\,-m}=\frac{\sin^{n-1}x}{(m-1)\cos^{m-1}x}-\frac{n-1}{m-1}\int\frac{\sin^{n-2}x\,dx}{\cos^{m-2}x}=$$
$$=\frac{\sin^{n-1}x}{(m-1)\cos^{m-1}x}-\frac{n-1}{m-1}I_{n-2,\,2-m}\quad (m\neq 1).$$

(c) Integrate by parts, putting

$$u=(a^2-x^2)^n;\quad dv=dx,$$

whence

$$du=-2nx(a^2-x^2)^{n-1}dx;\quad v=x.$$

Hence

$$I_n=x(a^2-x^2)^n+2n\int x^2(a^2-x^2)^{n-1}dx=$$
$$=x(a^2-x^2)^n+2n\int(x^2-a^2+a^2)(a^2-x^2)^{n-1}dx=$$
$$=x(a^2-x^2)^n-2nI_n+2na^2I_{n-1}.$$

Wherefrom, reducing the similar terms, we obtain

$$(1+2n)I_n=x(a^2-x^2)^n+2na^2I_{n-1}.$$

Hence,

$$I_n=\frac{x(a^2-x^2)^n}{2n+1}+\frac{2na^2}{2n+1}I_{n-1}.$$

For instance, noting that

$$I_{-1/2}=\int \frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+C,$$

we can find successively

$$I_{1/2}=\int \sqrt{a^2-x^2}\,dx=\frac{x}{2}(a^2-x^2)^{1/2}+\frac{a^2}{2}I_{-1/2}=$$
$$=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C,$$

$$I_{3/2}=\int (a^2-x^2)^{3/2}\,dx=\frac{x}{4}(a^2-x^2)^{3/2}+\frac{3}{4}a^2I_{1/2},$$ and so on.

4.4.2. Applying integration by parts, derive the following reduction formulas:

(a) $I_n=\int (\ln x)^n\,dx=x(\ln x)^n-nI_{n-1}$;

(b) $I_n=\int x^{\alpha}(\ln x)^n\,dx=\dfrac{x^{\alpha+1}(\ln x)^n}{\alpha+1}-\dfrac{n}{\alpha+1}I_{n-1}\quad(\alpha\neq-1)$;

(c) $I_n=\int x^n e^x\,dx=x^n e^x-nI_{n-1}$;

(d) $I_n=\int e^{\alpha x}\sin^n x\,dx=$

$$=\frac{e^{\alpha x}}{\alpha^2+n^2}\sin^{n-1}x\,(\alpha\sin x-n\cos x)+\frac{n(n-1)}{\alpha^2+n^2}I_{n-2}.$$

4.4.3. Derive the reduction formula for the integration of $I_n=\int\frac{dx}{\sin^n x}$ and use it for calculating the integral $I_3=\int\frac{dx}{\sin^3 x}$.

4.4.4. Derive the reduction formulas for the following integrals:

(a) $I_n=\int \tan^n x\,dx$; (b) $I_n=\int \cot^n x\,dx$;

(c) $I_n=\int\dfrac{x^n\,dx}{\sqrt{x^2+a}}$.