CHAPTER III.

VARIATION.

29. Definition. One quantity A is said to **vary directly** as another B, when the two quantities depend upon each other in such a manner that if B is changed, A is changed *in the same ratio.*

Note. The word *directly* is often omitted, and A is said to vary as B.

For instance: if a train moving at a uniform rate travels 40 miles in 60 minutes, it will travel 20 miles in 30 minutes, 80 miles in 120 minutes, and so on; the distance in each case being increased or diminished in the same ratio as the time. This is expressed by saying that when the velocity is uniform *the distance is proportional to the time*, or *the distance varies as the time.*

30. The symbol $\propto$ is used to denote variation; so that $A \propto B$ is read "A varies as B."

31. *If* A *varies as* B, *then* A *is equal to* B *multiplied by some constant quantity.*

For suppose that $a, a_1, a_2, a_3 \ldots, b, b_1, b_2, b_3 \ldots$ are corresponding values of A and B.

Then, by definition, $\dfrac{a}{a_1} = \dfrac{b}{b_1}$; $\dfrac{a}{a_2} = \dfrac{b}{b_2}$; $\dfrac{a}{a_3} = \dfrac{b}{b_3}$; and so on,

$$\therefore \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \ldots, \text{ each being equal to } \frac{a}{b}.$$

Hence $\dfrac{\text{any value of } A}{\text{the corresponding value of } B}$ is always the same;

that is, $\dfrac{A}{B} = m$, where m is constant.

$$\therefore A = mB.$$

If any pair of corresponding values of A and B are known, the constant m can be determined. For instance, if $A=3$ when $B=12$,

we have $$3 = m \times 12;$$

$$\therefore\ m = \tfrac{1}{4},$$

and $$A = \tfrac{1}{4} B.$$

32. Definition. One quantity A is said to **vary inversely** as another B, when A varies *directly* as the reciprocal of B.

Thus if A varies inversely as B, $A = \dfrac{m}{B}$, where m is constant.

The following is an illustration of inverse variation: If 6 men do a certain work in 8 hours, 12 men would do the same work in 4 hours, 2 men in 24 hours; and so on. Thus it appears that when the number of men is increased, the time is proportionately decreased; and vice-versâ.

Example 1. The cube root of x varies inversely as the square of y; if $x=8$ when $y=3$, find x when $y=1\frac{1}{2}$.

By supposition $\sqrt[3]{x}=\dfrac{m}{y^2}$, where m is constant.

Putting $x=8$, $y=3$, we have $$2=\frac{m}{9},$$

$$\therefore\ m=18,$$

and $$\sqrt[3]{x}=\frac{18}{y^2};$$

hence, by putting $y=\dfrac{3}{2}$, we obtain $x=512$.

Example 2. The square of the time of a planet's revolution varies as the cube of its distance from the Sun; find the time of Venus' revolution, assuming the distances of the Earth and Venus from the Sun to be $91\frac{1}{4}$ and 66 millions of miles respectively.

Let P be the periodic time measured in days, D the distance in millions of miles; we have $$P^2 \propto D^3,$$

or $$P^2 = kD^3,$$

where k is some constant.

For the Earth, $$365 \times 365 = k \times 91\tfrac{1}{4} \times 91\tfrac{1}{4} \times 91\tfrac{1}{4},$$

whence $$k = \frac{4 \times 4 \times 4}{365};$$

$$\therefore\ P^2 = \frac{4 \times 4 \times 4}{365} D^3.$$

For Venus, $$P^2 = \frac{4 \times 4 \times 4}{365} \times 66 \times 66 \times 66;$$

whence $$P = 4 \times 66 \times \sqrt{\frac{264}{365}}$$
$$= 264 \times \sqrt{\cdot 7233}, \text{ approximately},$$
$$= 264 \times \cdot 85$$
$$= 224 \cdot 4.$$

Hence the time of revolution is nearly $224\frac{1}{2}$ days.

33. Definition. One quantity is said to **vary jointly** as a number of others, when it varies directly as their product.

Thus A varies jointly as B and C, when $A = mBC$. For instance, the interest on a sum of money varies jointly as the principal, the time, and the rate per cent.

34. Definition. A is said to vary directly as B and inversely as C, when A varies as $\frac{B}{C}$.

35. *If* A *varies as* B *when* C *is constant, and* A *varies as* C *when* B *is constant, then will* A *vary as* BC *when both* B *and* C *vary.*

The variation of A depends partly on that of B and partly on that of C. Suppose these latter variations to take place separately, each in its turn producing its own effect on A; also let a, b, c be certain simultaneous values of A, B, C.

1. *Let C be constant* while B changes to b; then A must undergo a partial change and will assume some intermediate value a', where

$$\frac{A}{a'} = \frac{B}{b} \quad \text{.............................. (1).}$$

2. *Let B be constant*, that is, let it retain its value b, while C changes to c; then A must complete its change and pass from its intermediate value a' to its final value a, where

$$\frac{a'}{a} = \frac{C}{c} \quad \text{.................................. (2).}$$

From (1) and (2) $$\frac{A}{a'} \times \frac{a'}{a} = \frac{B}{b} \times \frac{C}{c};$$

that is, $$A = \frac{a}{bc} \cdot BC,$$

or $$A \text{ varies as } BC.$$

36. The following are illustrations of the theorem proved in the last article.

The amount of work done by *a given number of men* varies directly as the number of days they work, and the amount of work done *in a given time* varies directly as the number of men; therefore when the number of days and the number of men are both variable, the amount of work will vary as the product of the number of men and the number of days.

Again, in Geometry the area of a triangle varies directly as its base when the height is constant, and directly as the height when the base is constant; and when both the height and base are variable, the area varies as the product of the numbers representing the height and the base.

Example. The volume of a right circular cone varies as the square of the radius of the base when the height is constant, and as the height when the base is constant. If the radius of the base is 7 feet and the height 15 feet, the volume is 770 cubic feet; find the height of a cone whose volume is 132 cubic feet and which stands on a base whose radius is 3 feet.

Let h and r denote respectively the height and radius of the base measured in feet; also let V be the volume in cubic feet.

Then $V=mr^2h$, where m is constant.

By supposition, $$770=m\times 7^2\times 15;$$

whence $$m=\frac{22}{21};$$

$$\therefore\ V=\frac{22}{21}r^2h.$$

$\therefore$ by substituting $V=132$, $r=3$, we get

$$132=\frac{22}{21}\times 9\times h;$$

whence $$h=14;$$

and therefore the height is 14 feet.

37. The proposition of Art. 35 can easily be extended to the case in which the variation of A depends upon that of more than two variables. Further, the variations may be either direct or inverse. The principle is interesting because of its frequent occurrence in Physical Science. For example, in the theory of gases it is found by experiment that the pressure (p) of a gas varies as the "absolute temperature" (t) when its volume (v) is constant, and that the pressure varies inversely as the volume when the temperature is constant; that is

$$p\propto t, \text{ when } v \text{ is constant};$$

and $$p \propto \frac{1}{v}, \text{ when } t \text{ is constant.}$$

From these results we should expect that, when both t and v are variable, we should have the formula

$$p \propto \frac{t}{v}, \text{ or } pv = kt, \text{ where } k \text{ is constant;}$$

and by actual experiment this is found to be the case.

Example. The duration of a railway journey varies directly as the distance and inversely as the velocity; the velocity varies directly as the square root of the quantity of coal used per mile, and inversely as the number of carriages in the train. In a journey of 25 miles in half an hour with 18 carriages 10 cwt. of coal is required; how much coal will be consumed in a journey of 21 miles in 28 minutes with 16 carriages?

Let t be the time expressed in hours,
d the distance in miles,
v the velocity in miles per hour,
q the quantity of coal in cwt.,
c the number of carriages.

We have $$t \propto \frac{d}{v},$$

and $$v \propto \frac{\sqrt{q}}{c},$$

whence $$t \propto \frac{cd}{\sqrt{q}},$$

or $$t = \frac{kcd}{\sqrt{q}}, \text{ where } k \text{ is constant.}$$

Substituting the values given, we have

$$\frac{1}{2} = \frac{k \times 18 \times 25}{\sqrt{10}};$$

that is, $$k = \frac{\sqrt{10}}{25 \times 36}.$$

Hence $$t = \frac{\sqrt{10} \,.\, cd}{25 \times 36 \sqrt{q}}.$$

Substituting now the values of t, c, d given in the second part of the question, we have

$$\frac{28}{60} = \frac{\sqrt{10} \times 16 \times 21}{25 \times 36 \sqrt{q}};$$

that is, $$\sqrt{q} = \frac{\sqrt{10} \times 16 \times 21}{15 \times 28} = \frac{4}{5}\sqrt{10};$$

whence $$q = \frac{32}{5} = 6\tfrac{2}{5}.$$

Hence the quantity of coal is $6\frac{2}{5}$ cwt.

EXAMPLES. III.

1. If x varies as y, and $x=8$ when $y=15$, find x when $y=10$.

2. If P varies inversely as Q, and $P=7$ when $Q=3$, find P when $Q=2\frac{1}{3}$.

3. If the square of x varies as the cube of y, and $x=3$ when $y=4$, find the value of y when $x=\frac{1}{\sqrt{3}}$.

4. A varies as B and C jointly; if $A=2$ when $B=\frac{3}{5}$ and $C=\frac{10}{27}$, find C when $A=54$ and $B=3$.

5. If A varies as C, and B varies as C, then $A \pm B$ and $\sqrt{AB}$ will each vary as C.

6. If A varies as BC, then B varies inversely as $\frac{C}{A}$.

7. P varies directly as Q and inversely as R; also $P=\frac{2}{3}$ when $Q=\frac{3}{7}$ and $R=\frac{9}{14}$: find Q when $P=\sqrt{48}$ and $R=\sqrt{75}$.

8. If x varies as y, prove that x^2+y^2 varies as x^2-y^2.

9. If y varies as the sum of two quantities, of which one varies directly as x and the other inversely as x; and if $y=6$ when $x=4$, and $y=3\frac{1}{3}$ when $x=3$; find the equation between x and y.

10. If y is equal to the sum of two quantities one of which varies as x directly, and the other as x^2 inversely; and if $y=19$ when $x=2$, or 3; find y in terms of x.

11. If A varies directly as the square root of B and inversely as the cube of C, and if $A=3$ when $B=256$ and $C=2$, find B when $A=24$ and $C=\frac{1}{2}$.

12. Given that $x+y$ varies as $z+\frac{1}{z}$, and that $x-y$ varies as $z-\frac{1}{z}$, find the relation between x and z, provided that $z=2$ when $x=3$ and $y=1$.

13. If A varies as B and C jointly, while B varies as D^2, and C varies inversely as A, shew that A varies as D.

14. If y varies as the sum of three quantities of which the first is constant, the second varies as x, and the third as x^2; and if $y=0$ when $x=1$, $y=1$ when $x=2$, and $y=4$ when $x=3$; find y when $x=7$.

15. When a body falls from rest its distance from the starting point varies as the square of the time it has been falling: if a body falls through $402\frac{1}{2}$ feet in 5 seconds, how far does it fall in 10 seconds? Also how far does it fall in the 10^{th} second?

16. Given that the volume of a sphere varies as the cube of its radius, and that when the radius is $3\frac{1}{2}$ feet the volume is $179\frac{2}{3}$ cubic feet, find the volume when the radius is 1 foot 9 inches.

17. The weight of a circular disc varies as the square of the radius when the thickness remains the same; it also varies as the thickness when the radius remains the same. Two discs have their thicknesses in the ratio of $9 : 8$; find the ratio of their radii if the weight of the first is twice that of the second.

18. At a certain regatta the number of races on each day varied jointly as the number of days from the beginning and end of the regatta up to and including the day in question. On three successive days there were respectively 6, 5 and 3 races. Which days were these, and how long did the regatta last?

19. The price of a diamond varies as the square of its weight. Three rings of equal weight, each composed of a diamond set in gold, have values £a, £b, £c, the diamonds in them weighing 3, 4, 5 carats respectively. Shew that the value of a diamond of one carat is

$$£\left(\frac{a+c}{2} - b\right),$$

the cost of workmanship being the same for each ring.

20. Two persons are awarded pensions in proportion to the square root of the number of years they have served. One has served 9 years longer than the other and receives a pension greater by £50. If the length of service of the first had exceeded that of the second by $4\frac{1}{4}$ years their pensions would have been in the proportion of $9 : 8$. How long had they served and what were their respective pensions?

21. The attraction of a planet on its satellites varies directly as the mass (M) of the planet, and inversely as the square of the distance (D); also the square of a satellite's time of revolution varies directly as the distance and inversely as the force of attraction. If m_1, d_1, t_1, and m_2, d_2, t_2, are simultaneous values of M, D, T respectively, prove that

$$\frac{m_1 t_1^2}{m_2 t_2^2} = \frac{d_1^3}{d_2^3}.$$

Hence find the time of revolution of that moon of Jupiter whose distance is to the distance of our Moon as $35 : 31$, having given that the mass of Jupiter is 343 times that of the Earth, and that the Moon's period is 27·32 days.

22. The consumption of coal by a locomotive varies as the square of the velocity; when the speed is 16 miles an hour the consumption of coal per hour is 2 tons: if the price of coal be 10*s.* per ton, and the other expenses of the engine be 11*s.* 3*d.* an hour, find the least cost of a journey of 100 miles.