CHAPTER IX.

THE THEORY OF QUADRATIC EQUATIONS.

111. AFTER suitable reduction every quadratic equation may be written in the form

$$ax^2 + bx + c = 0 \qquad \text{.....................} \qquad (1),$$

and the solution of the equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{.......................} \qquad (2).$$

We shall now prove some important propositions connected with the roots and coefficients of all equations of which (1) is the type.

112. *A quadratic equation cannot have more than two roots.*

For, if possible, let the equation $ax^2 + bx + c = 0$ have three *different* roots α, β, γ. Then since each of these values must satisfy the equation, we have

$$a\alpha^2 + b\alpha + c = 0 \qquad \text{.......................} \qquad (1),$$

$$a\beta^2 + b\beta + c = 0 \qquad \text{.......................} \qquad (2),$$

$$a\gamma^2 + b\gamma + c = 0 \qquad \text{.......................} \qquad (3).$$

From (1) and (2), by subtraction,

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0;$$

divide out by $\alpha - \beta$ which, by hypothesis, is not zero; then

$$a(\alpha + \beta) + b = 0.$$

Similarly from (2) and (3)

$$a(\beta + \gamma) + b = 0;$$

$\therefore$ by subtraction $\qquad a(\alpha - \gamma) = 0;$

which is impossible, since, by hypothesis, a is not zero, and α is not equal to γ. Hence there cannot be three different roots.

113. In Art. 111 let the two roots in (2) be denoted by α and β, so that

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad \beta = \frac{-b-\sqrt{b^2-4ac}}{2a};$$

then we have the following results:

(1) If b^2-4ac (the quantity under the radical) is positive, α and β are real and unequal.

(2) If b^2-4ac is zero, α and β are real and equal, each reducing in this case to $-\dfrac{b}{2a}$.

(3) If b^2-4ac is negative, α and β are imaginary and unequal.

(4) If b^2-4ac is a perfect square, α and β are rational and unequal.

By applying these tests the nature of the roots of any quadratic may be determined without solving the equation.

Example 1. Shew that the equation $2x^2-6x+7=0$ cannot be satisfied by any real values of x.

Here $\quad a=2,\ b=-6,\ c=7$; so that

$$b^2-4ac=(-6)^2-4\,.\,2\,.\,7=-20.$$

Therefore the roots are imaginary.

Example 2. If the equation $x^2+2(k+2)x+9k=0$ has equal roots, find k.

The condition for equal roots gives

$$(k+2)^2=9k,$$
$$k^2-5k+4=0,$$
$$(k-4)(k-1)=0;$$
$$\therefore\ k=4, \text{ or } 1.$$

Example 3. Shew that the roots of the equation

$$x^2-2px+p^2-q^2+2qr-r^2=0$$

are rational.

The roots will be rational provided $(-2p)^2-4(p^2-q^2+2qr-r^2)$ is a perfect square. But this expression reduces to $4(q^2-2qr+r^2)$, or $4(q-r)^2$. Hence the roots are rational.

114. Since $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}, \quad \beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a},$

we have by addition

$$\alpha+\beta = \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$= -\frac{2b}{2a} = -\frac{b}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1);$$

and by multiplication we have

$$\alpha\beta = \frac{(-b+\sqrt{b^2-4ac})(-b-\sqrt{b^2-4ac})}{4a^2}$$

$$= \frac{(-b)^2-(b^2-4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a} \qquad\ldots\ldots\ldots\ldots\ldots(2).$$

By writing the equation in the form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

these results may also be expressed as follows.

In a quadratic equation *where the coefficient of the first term is unity,*

(i) the sum of the roots is equal to the coefficient of x with its sign changed;

(ii) the product of the roots is equal to the third term.

NOTE. In any equation the term which does not contain the unknown quantity is frequently called *the absolute term.*

115. Since $\qquad -\dfrac{b}{a} = \alpha + \beta$, and $\dfrac{c}{a} = \alpha\beta$,

the equation $x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$ may be written

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0 \qquad\ldots\ldots\ldots\ldots(1).$$

Hence any quadratic may also be expressed in the form

$$x^2 - (\text{sum of roots})\,x + \text{product of roots} = 0 \;\ldots\ldots(2).$$

Again, from (1) we have

$$(x-\alpha)(x-\beta) = 0 \qquad\ldots\ldots\ldots\ldots(3).$$

We may now easily form an equation with given roots.

Example 1. Form the equation whose roots are 3 and -2.

The equation is $\qquad (x-3)(x+2)=0,$

or $\qquad x^2 - x - 6 = 0.$

When the roots are irrational it is easier to use the following method.

Example 2. Form the equation whose roots are $2+\sqrt{3}$ and $2-\sqrt{3}$.

We have
$$\text{sum of roots}=4,$$
$$\text{product of roots}=1;$$
$\therefore$ the equation is
$$x^2-4x+1=0,$$
by using formula (2) of the present article.

116. By a method analogous to that used in Example 1 of the last article we can form an equation with three or more given roots.

Example 1. Form the equation whose roots are 2, -3, and $\frac{7}{5}$.

The required equation must be satisfied by each of the following suppositions:
$$x-2=0,\ \ x+3=0,\ \ x-\frac{7}{5}=0;$$
therefore the equation must be
$$(x-2)(x+3)\left(x-\frac{7}{5}\right)=0;$$
that is,
$$(x-2)(x+3)(5x-7)=0,$$
or
$$5x^3-2x^2-37x+42=0.$$

Example 2. Form the equation whose roots are 0, $\pm a$, $\frac{c}{b}$.

The equation has to be satisfied by
$$x=0,\ \ x=a,\ \ x=-a,\ \ x=\frac{c}{b};$$
therefore it is
$$x(x+a)(x-a)\left(x-\frac{c}{b}\right)=0;$$
that is,
$$x(x^2-a^2)(bx-c)=0,$$
or
$$bx^4-cx^3-a^2bx^2+a^2cx=0.$$

117. The results of Art. 114 are most important, and they are generally sufficient to solve problems connected with the roots of quadratics. In such questions *the roots should never be considered singly*, but use should be made of the relations obtained by writing down the sum of the roots, and their product, in terms of the coefficients of the equation.

Example 1. If α and β are the roots of $x^2-px+q=0$, find the value of (1) $\alpha^2+\beta^2$, (2) $\alpha^3+\beta^3$.

We have
$$\alpha+\beta=p,$$
$$\alpha\beta=q.$$
$$\therefore\ \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$
$$=p^2-2q.$$

Again,
$$\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)$$
$$=p\{(\alpha+\beta)^2-3\alpha\beta\}$$
$$=p(p^2-3q).$$

Example 2. If α, β are the roots of the equation $lx^2+mx+n=0$, find the equation whose roots are $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$.

We have
$$\text{sum of roots}=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta},$$
$$\text{product of roots}=\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}=1;$$

$\therefore$ by Art. 115 the required equation is
$$x^2-\left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)x+1=0,$$
or
$$\alpha\beta x^2-(\alpha^2+\beta^2)x+\alpha\beta=0.$$

As in the last example $\alpha^2+\beta^2=\dfrac{m^2-2nl}{l^2}$, and $\alpha\beta=\dfrac{n}{l}$.

$\therefore$ the equation is
$$\frac{n}{l}x^2-\frac{m^2-2nl}{l^2}x+\frac{n}{l}=0,$$
or
$$nlx^2-(m^2-2nl)x+nl=0.$$

Example 3. When $x=\dfrac{3+5\sqrt{-1}}{2}$, find the value of $2x^3+2x^2-7x+72$; and shew that it will be unaltered if $\dfrac{3-5\sqrt{-1}}{2}$ be substituted for x.

Form the quadratic equation whose roots are $\dfrac{3\pm5\sqrt{-1}}{2}$;

the sum of the roots $=3$;

the product of the roots $=\dfrac{17}{2}$;

hence the equation is $2x^2-6x+17=0$;

$\therefore$ $2x^2-6x+17$ is a quadratic *expression* which vanishes for *either* of the values $\dfrac{3\pm5\sqrt{-1}}{2}$.

Now
$$2x^3+2x^2-7x+72=x(2x^2-6x+17)+4(2x^2-6x+17)+4$$
$$=x\times0+4\times0+4$$
$$=4;$$

which is the numerical value of the expression in each of the supposed cases.

118. *To find the condition that the roots of the equation* $ax^2+bx+c=0$ *should be* (1) *equal in magnitude and opposite in sign,* (2) *reciprocals.*

The roots will be equal in magnitude and opposite in sign if their sum is zero; hence the required condition is

$$-\frac{b}{a}=0, \text{ or } b=0.$$

Again, the roots will be reciprocals when their product is unity; hence we must have

$$\frac{c}{a}=1, \text{ or } c=a.$$

The first of these results is of frequent occurrence in Analytical Geometry, and the second is a particular case of a more general condition applicable to equations of any degree.

Example. Find the condition that the roots of $ax^2+bx+c=0$ may be (1) both positive, (2) opposite in sign, but the greater of them negative.

We have $$\alpha+\beta=-\frac{b}{a}, \quad \alpha\beta=\frac{c}{a}.$$

(1) If the roots are both positive, $\alpha\beta$ is positive, and therefore c and a have like signs.

Also, since $\alpha+\beta$ is positive, $\frac{b}{a}$ is negative; therefore b and a have unlike signs.

Hence the required condition is that the signs of a and c should be like, and opposite to the sign of b.

(2) If the roots are of opposite signs, $\alpha\beta$ is negative, and therefore c and a have unlike signs.

Also since $\alpha+\beta$ has the sign of the greater root it is negative, and therefore $\frac{b}{a}$ is positive; therefore b and a have like signs.

Hence the required condition is that the signs of a and b should be like, and opposite to the sign of c.

EXAMPLES. IX. a.

Form the equations whose roots are

1. $-\frac{4}{5}, \ \frac{3}{7}$. 2. $\frac{m}{n}, \ -\frac{n}{m}$. 3. $\frac{p-q}{p+q}, \ -\frac{p+q}{p-q}$.

4. $7\pm2\sqrt{5}$. 5. $\pm2\sqrt{3}-5$. 6. $-p\pm2\sqrt{2q}$.

7. $-3 \pm 5i$. 8. $-a \pm ib$. 9. $\pm i(a-b)$.

10. $-3, \frac{2}{3}, \frac{1}{2}$. 11. $\frac{a}{2}, 0, -\frac{2}{a}$. 12. $2 \pm \sqrt{3}, 4$.

13. Prove that the roots of the following equations are real:

(1) $x^2 - 2ax + a^2 - b^2 - c^2 = 0$,

(2) $(a-b+c)x^2 + 4(a-b)x + (a-b-c) = 0$.

14. If the equation $x^2 - 15 - m(2x-8) = 0$ has equal roots, find the values of m.

15. For what values of m will the equation

$$x^2 - 2x(1+3m) + 7(3+2m) = 0$$

have equal roots?

16. For what value of m will the equation

$$\frac{x^2 - bx}{ax - c} = \frac{m-1}{m+1}$$

have roots equal in magnitude but opposite in sign?

17. Prove that the roots of the following equations are rational:

(1) $(a+c-b)x^2 + 2cx + (b+c-a) = 0$,

(2) $abc^2x^2 + 3a^2cx + b^2cx - 6a^2 - ab + 2b^2 = 0$.

If α, β are the roots of the equation $ax^2 + bx + c = 0$, find the values of

18. $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$. 19. $\alpha^4\beta^7 + \alpha^7\beta^4$. 20. $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$.

Find the value of

21. $x^3 + x^2 - x + 22$ when $x = 1 + 2i$.

22. $x^3 - 3x^2 - 8x + 15$ when $x = 3 + i$.

23. $x^3 - ax^2 + 2a^2x + 4a^3$ when $\frac{x}{a} = 1 - \sqrt{-3}$.

24. If α and β are the roots of $x^2 + px + q = 0$, form the equation whose roots are $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$.

25. Prove that the roots of $(x-a)(x-b) = h^2$ are always real.

26. If x_1, x_2 are the roots of $ax^2 + bx + c = 0$, find the value of

(1) $(ax_1 + b)^{-2} + (ax_2 + b)^{-2}$,

(2) $(ax_1 + b)^{-3} + (ax_2 + b)^{-3}$.

27. Find the condition that one root of $ax^2+bx+c=0$ shall be n times the other.

28. If α, β are the roots of $ax^2+bx+c=0$, form the equation whose roots are $\alpha^2+\beta^2$ and $\alpha^{-2}+\beta^{-2}$.

29. Form the equation whose roots are the squares of the sum and of the difference of the roots of

$$2x^2+2(m+n)x+m^2+n^2=0.$$

30. Discuss the signs of the roots of the equation

$$px^2+qx+r=0.$$

119. The following example illustrates a useful application of the results proved in Art. 113.

Example. If x is a real quantity, prove that the expression $\dfrac{x^2+2x-11}{2(x-3)}$ can have all numerical values except such as lie between 2 and 6.

Let the given expression be represented by y, so that

$$\frac{x^2+2x-11}{2(x-3)}=y;$$

then multiplying up and transposing, we have

$$x^2+2x(1-y)+6y-11=0.$$

This is a quadratic equation, and in order that x may have real values $4(1-y)^2-4(6y-11)$ must be positive; or dividing by 4 and simplifying, $y^2-8y+12$ must be positive; that is, $(y-6)(y-2)$ must be positive. Hence the factors of this product must be both positive, or both negative. In the former case y is greater than 6; in the latter y is less than 2. Therefore y cannot lie between 2 and 6, but may have any other value.

In this example it will be noticed that the *quadratic expression* $y^2-8y+12$ is positive so long as y does not lie between the roots of the corresponding *quadratic equation* $y^2-8y+12=0$.

This is a particular case of the general proposition investigated in the next article.

120. *For all real values of* x *the expression* ax^2+bx+c *has the same sign as* a, *except when the roots of the equation* $ax^2+bx+c=0$ *are real and unequal, and* x *has a value lying between them.*

Case I. Suppose that the roots of the equation

$$ax^2+bx+c=0$$

are real; denote them by α and β, and let α be the greater.

Then $\quad ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$

$$= a\{x^2 - (\alpha + \beta)x + \alpha\beta\}$$

$$= a(x - \alpha)(x - \beta).$$

Now if x is greater than α, the factors $x - \alpha$, $x - \beta$ are both positive; and if x is less than β, the factors $x - \alpha$, $x - \beta$ are both negative; therefore in each case the expression $(x - \alpha)(x - \beta)$ is positive, and $ax^2 + bx + c$ has the same sign as a. But if x has a value lying between α and β, the expression $(x - \alpha)(x - \beta)$ is negative, and the sign of $ax^2 + bx + c$ is opposite to that of a.

Case II. If α and β are equal, then

$$ax^2 + bx + c = a(x - \alpha)^2,$$

and $(x - \alpha)^2$ is positive for all real values of x; hence $ax^2 + bx + c$ has the same sign as a.

Case III. Suppose that the equation $ax^2 + bx + c = 0$ has imaginary roots; then

$$ax^2 + bx + c = a\left\{x^2 + \frac{b}{a}x + \frac{c}{a}\right\}$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}.$$

But $b^2 - 4ac$ is negative since the roots are imaginary; hence $\frac{4ac - b^2}{4a^2}$ is positive, and the expression

$$\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$$

is positive for all real values of x; therefore $ax^2 + bx + c$ has the same sign as a. This establishes the proposition.

121. From the preceding article it follows that the expression $ax^2 + bx + c$ will always have the same sign whatever real value x may have, provided that $b^2 - 4ac$ is negative or zero; and if this condition is satisfied the expression is positive or negative according as a is positive or negative.

Conversely, in order that the expression $ax^2 + bx + c$ may be always positive, $b^2 - 4ac$ must be negative or zero, and a must be positive; and in order that $ax^2 + bx + c$ may be always negative $b^2 - 4ac$ must be negative or zero, and a must be negative.

Example. Find the limits between which a must lie in order that

$$\frac{ax^2-7x+5}{5x^2-7x+a}$$

may be capable of all values, x being any real quantity.

Put $$\frac{ax^2-7x+5}{5x^2-7x+a}=y;$$

then $$(a-5y)x^2-7x(1-y)+(5-ay)=0.$$

In order that the values of x found from this quadratic may be real, the expression

$$49(1-y)^2-4(a-5y)(5-ay) \text{ must be positive,}$$

that is, $(49-20a)y^2+2(2a^2+1)y+(49-20a)$ must be positive;

hence $(2a^2+1)^2-(49-20a)^2$ must be negative or zero, and $49-20a$ must be positive.

Now $(2a^2+1)^2-(49-20a)^2$ is negative or zero, according as

$$2(a^2-10a+25)\times 2(a^2+10a-24) \text{ is negative or zero;}$$

that is, according as $4(a-5)^2(a+12)(a-2)$ is negative or zero.

This expression is negative as long as a lies between 2 and -12, and for such values $49-20a$ is positive; the expression is zero when $a=5$, -12, or 2, but $49-20a$ is negative when $a=5$. Hence the limiting values are 2 and -12, and a may have any intermediate value.

EXAMPLES. IX. b.

1. Determine the limits between which n must lie in order that the equation

$$2ax(ax+nc)+(n^2-2)c^2=0$$

may have real roots.

2. If x be real, prove that $\dfrac{x}{x^2-5x+9}$ must lie between 1 and $-\dfrac{1}{11}$.

3. Shew that $\dfrac{x^2-x+1}{x^2+x+1}$ lies between 3 and $\dfrac{1}{3}$ for all real values of x.

4. If x be real, prove that $\dfrac{x^2+34x-71}{x^2+2x-7}$ can have no value between 5 and 9.

5. Find the equation whose roots are $\dfrac{\sqrt{a}}{\sqrt{a}\pm\sqrt{a-b}}$.

6. If α, β are roots of the equation $x^2-px+q=0$, find the value of

(1) $\alpha^2(\alpha^2\beta^{-1}-\beta)+\beta^2(\beta^2\alpha^{-1}-\alpha)$,

(2) $(\alpha-p)^{-4}+(\beta-p)^{-4}$.

7. If the roots of $lx^2+nx+n=0$ be in the ratio of $p:q$, prove that

$$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0.$$

8. If x be real, the expression $\dfrac{(x+m)^2-4mn}{2(x-n)}$ admits of all values except such as lie between $2n$ and $2m$.

9. If the roots of the equation $ax^2+2bx+c=0$ be α and β, and those of the equation $Ax^2+2Bx+C=0$ be $\alpha+\delta$ and $\beta+\delta$, prove that

$$\frac{b^2-ac}{a^2}=\frac{B^2-AC}{A^2}.$$

10. Shew that the expression $\dfrac{px^2+3x-4}{p+3x-4x^2}$ will be capable of all values when x is real, provided that p has any value between 1 and 7.

11. Find the greatest value of $\dfrac{x+2}{2x^2+3x+6}$ for real values of x.

12. Shew that if x is real, the expression

$$(x^2-bc)(2x-b-c)^{-1}$$

has no real values between b and c.

13. If the roots of $ax^2+2bx+c=0$ be possible and different, then the roots of

$$(a+c)(ax^2+2bx+c)=2(ac-b^2)(x^2+1)$$

will be impossible, and *vice versâ*.

14. Shew that the expression $\dfrac{(ax-b)(dx-c)}{(bx-a)(cx-d)}$ will be capable of all values when x is real, if a^2-b^2 and c^2-d^2 have the same sign.

*122. We shall conclude this chapter with some miscellaneous theorems and examples. It will be convenient here to introduce a phraseology and notation which the student will frequently meet with in his mathematical reading.

Definition. Any expression which involves x, and whose value is dependent on that of x, is called a **function of x.** Functions of x are usually denoted by symbols of the form $f(x)$, $F(x)$, $\phi(x)$.

Thus the equation $y=f(x)$ may be considered as equivalent to a statement that any change made in the value of x will produce a consequent change in y, and *vice versâ*. The quantities x and y are called **variables**, and are further distinguished as the **independent variable** and the **dependent variable**.

An *independent variable* is a quantity which may have any value we choose to assign to it, and the corresponding *dependent variable* has its value determined as soon as the value of the independent variable is known.

*123. An expression of the form

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n$$

where n is a positive integer, and the coefficients $p_0, p_1, p_2, \dots p_n$ do not involve x, is called a **rational and integral algebraical function of** x. In the present chapter we shall confine our attention to functions of this kind.

*124. A function is said to be **linear** when it contains no higher power of the variable than the first; thus $ax + b$ is a linear function of x. A function is said to be **quadratic** when it contains no higher power of the variable than the second; thus $ax^2 + bx + c$ is a quadratic function of x. Functions of the *third*, *fourth*,... degrees are those in which the highest power of the variable is respectively the *third*, *fourth*,.... Thus in the last article the expression is a function of x of the n^{th} degree.

*125. The symbol $f(x, y)$ is used to denote a function of two variables x and y; thus $ax + by + c$, and $ax^2 + bxy + cy^2 + dx + ey + f$ are respectively linear and quadratic functions of x, y.

The *equations* $f(x) = 0$, $f(x, y) = 0$ are said to be linear, quadratic,... according as the *functions* $f(x)$, $f(x, y)$ are linear, quadratic,....

*126. We have proved in Art. 120 that the *expression* $ax^2 + bx + c$ admits of being put in the form $a(x - \alpha)(x - \beta)$, where α and β are the roots of the *equation* $ax^2 + bx + c = 0$.

Thus a quadratic expression $ax^2 + bx + c$ is capable of being resolved into two rational factors of the first degree, whenever the equation $ax^2 + bx + c = 0$ has rational roots; that is, when $b^2 - 4ac$ is a perfect square.

*127. *To find the condition that a quadratic function of* x, y *may be resolved into two linear factors.*

Denote the function by $f(x, y)$ where

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c.$$