CHAPTER XXV.

CONTINUED FRACTIONS.

331. An expression of the form $a + \cfrac{b}{c + \cfrac{d}{e + \dots}}$ is called a **continued fraction**; here the letters $a, b, c, \dots\dots$ may denote any quantities whatever, but for the present we shall only consider the simpler form $a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}$, where $a_1, a_2, a_3, \dots$ are positive integers. This will be usually written in the more compact form

$$a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\;\dots\dots$$

332. When the number of *quotients* $a_1, a_2, a_3, \dots$ is finite the continued fraction is said to be *terminating*; if the number of quotients is unlimited the fraction is called an *infinite continued fraction*.

It is possible to reduce every terminating continued fraction to an ordinary fraction by simplifying the fractions in succession beginning from the lowest.

333. *To convert a given fraction into a continued fraction.*

Let $\frac{m}{n}$ be the given fraction; divide m by n, let a_1 be the quotient and p the remainder; thus

$$\frac{m}{n} = a_1 + \frac{p}{n} = a_1 + \cfrac{1}{\cfrac{n}{p}};$$

divide n by p, let a_2 be the quotient and q the remainder; thus

$$\frac{n}{p} = a_2 + \frac{q}{p} = a_2 + \frac{1}{\frac{p}{q}};$$

divide p by q, let a_3 be the quotient and r the remainder; and so on. Thus

$$\frac{m}{n} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}} = a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\,\dots\dots$$

If m is less than n, the first quotient is zero, and we put

$$\frac{m}{n} = \frac{1}{\frac{n}{m}}$$

and proceed as before.

It will be observed that the above process is the same as that of finding the greatest common measure of m and n; hence if m and n are *commensurable* we shall at length arrive at a stage where the division is exact and the process terminates. Thus every fraction whose numerator and denominator are positive integers can be converted into a terminating continued fraction.

Example. Reduce $\frac{251}{802}$ to a continued fraction.

Finding the greatest common measure of 251 and 802 by the usual process, we have

$$\begin{array}{r|r|r|l} 5 & 251 & 802 & 3 \\ 6 & 6 & 49 & 8 \\ & & 1 & \end{array}$$

and the successive quotients are 3, 5, 8, 6; hence

$$\frac{251}{802} = \frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{8+}\,\frac{1}{6}.$$

334. The fractions obtained by stopping at the first, second, third,...... quotients of a continued fraction are called the first, second, third,...... **convergents**, because, as will be shewn in Art. 339, each successive convergent is a nearer approximation to the true value of the continued fraction than any of the preceding convergents.

335. *To shew that the convergents are alternately less and greater than the continued fraction.*

Let the continued fraction be $a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\,\ldots\ldots$

The first convergent is a_1, and is too small because the part $\frac{1}{a_2 +}\,\frac{1}{a_3 +}\ldots$ is omitted. The second convergent is $a_1 + \frac{1}{a_2}$, and is too great, because the denominator a_2 is too small. The third convergent is $a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3}$, and is too small because $a_2 + \frac{1}{a_3}$ is too great; and so on.

When the given fraction is a proper fraction $a_1 = 0$; if in this case we agree to consider zero as the first convergent, we may enunciate the above results as follows:

The convergents of an odd order are all less, and the convergents of an even order are all greater, than the continued fraction.

336. *To establish the law of formation of the successive convergents.*

Let the continued fraction be denoted by

$$a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\,\frac{1}{a_4 +}\,\ldots\ldots;$$

then the first three convergents are

$$\frac{a_1}{1},\quad \frac{a_1 a_2 + 1}{a_2},\quad \frac{a_3(a_1 a_2 + 1) + a_1}{a_3 \cdot a_2 + 1};$$

and we see that the numerator of the third convergent may be formed by multiplying the numerator of the second convergent by the third quotient, and adding the numerator of the first convergent; also that the denominator may be formed in a similar manner.

Suppose that the successive convergents are formed, in a similar way; let the numerators be denoted by $p_1, p_2, p_3, \ldots$, and the denominators by $q_1, q_2, q_3, \ldots$

Assume that the law of formation holds for the n^{th} convergent; that is, suppose

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

The $(n+1)^{\text{th}}$ convergent differs from the n^{th} only in having the quotient $a_n + \frac{1}{a_{n+1}}$ in the place of a_n; hence the $(n+1)^{\text{th}}$ convergent

$$= \frac{\left(a_n + \frac{1}{a_{n+1}}\right) p_{n-1} + p_{n-2}}{\left(a_n + \frac{1}{a_{n+1}}\right) q_{n-1} + q_{n-2}} = \frac{a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}}$$

$$= \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}}, \text{ by supposition.}$$

If therefore we put

$$p_{n+1} = a_{n+1} p_n + p_{n-1}, \quad q_{n+1} = a_{n+1} q_n + q_{n-1},$$

we see that the numerator and denominator of the $(n+1)^{\text{th}}$ convergent follow the law which was supposed to hold in the case of the n^{th}. But the law does hold in the case of the third convergent, hence it holds for the fourth, and so on; therefore it holds universally.

337. It will be convenient to call a_n the n^{th} *partial* quotient; the *complete* quotient at this stage being $a_n + \frac{1}{a_{n+1}+}\,\frac{1}{a_{n+2}+}\cdots$. We shall usually denote the complete quotient at any stage by k.

We have seen that

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}};$$

let the continued fraction be denoted by x; then x differs from $\frac{p_n}{q_n}$ only in taking the complete quotient k instead of the partial quotient a_n; thus

$$x = \frac{k p_{n-1} + p_{n-2}}{k q_{n-1} + q_{n-2}}.$$

338. *If* $\frac{p_n}{q_n}$ *be the* n^{th} *convergent to a continued fraction, then*

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^n.$$

Let the continued fraction be denoted by

$$a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\cdots\cdots;$$

then

$$\begin{aligned} p_n q_{n-1} - p_{n-1} q_n &= (a_n p_{n-1} + p_{n-2}) q_{n-1} - p_{n-1} (a_n q_{n-1} + q_{n-2}) \\ &= (-1)(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) \\ &= (-1)^2 (p_{n-2} q_{n-3} - p_{n-3} q_{n-2}), \text{ similarly,} \\ &= \dots\dots\dots\dots\dots\dots\dots\dots \\ &= (-1)^{n-2}(p_2 q_1 - p_1 q_2). \end{aligned}$$

But $\quad p_2 q_1 - p_1 q_2 = (a_1 a_2 + 1) - a_1 . a_2 = 1 = (-1)^2;$

hence $\quad p_n q_{n-1} - p_{n-1} q_n = (-1)^n.$

When the continued fraction is *less* than unity, this result will still hold if we suppose that $a_1 = 0$, and that the first convergent is zero.

Note. When we are calculating the numerical value of the successive convergents, the above theorem furnishes an easy test of the accuracy of the work.

Cor. 1. Each convergent is in its lowest terms; for if p_n and q_n had a common divisor it would divide $p_n q_{n-1} - p_{n-1} q_n$, or unity; which is impossible.

Cor. 2. The difference between two successive convergents is a fraction whose numerator is unity; for

$$\frac{p_n}{q_n} \sim \frac{p_{n-1}}{q_{n-1}} = \frac{p_n q_{n-1} \sim p_{n-1} q_n}{q_n q_{n-1}} = \frac{1}{q_n q_{n-1}}.$$

EXAMPLES. XXV. a.

Calculate the successive convergents to

1. $2 + \frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{11+}\,\frac{1}{2}.$

2. $\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{6}.$

3. $3 + \frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{9}.$

Express the following quantities as continued fractions and find the fourth convergent to each.

4. $\frac{253}{179}$.   5. $\frac{832}{159}$.   6. $\frac{1189}{3927}$.   7. $\frac{729}{2318}$.

8. $\cdot 37$.   9. $1\cdot 139$.   10. $\cdot 3029$.   11. $4\cdot 31\dot{6}$.

12. A metre is 39·37079 inches, shew by the theory of continued fractions that 32 metres is nearly equal to 35 yards.

13. Find a series of fractions converging to ·24226, the excess in days of the true tropical year over 365 days.

14. A kilometre is very nearly equal to ·62138 miles; shew that the fractions $\frac{5}{8}, \frac{18}{29}, \frac{23}{37}, \frac{64}{103}$ are successive approximations to the ratio of a kilometre to a mile.

15. Two scales of equal length are divided into 162 and 209 equal parts respectively; if their zero points be coincident shew that the 31^{st} division of one nearly coincides with the 40^{th} division of the other.

16. If $\dfrac{n^4+n^2-1}{n^3+n^2+n+1}$ is converted into a continued fraction, shew that the quotients are $n-1$ and $n+1$ alternately, and find the successive convergents.

17. Shew that

(1) $\dfrac{p_{n+1}-p_{n-1}}{q_{n+1}-q_{n-1}}=\dfrac{p_n}{q_n}$,

(2) $\left(\dfrac{p_{n+2}}{p_n}-1\right)\left(1-\dfrac{p_{n-1}}{p_{n+1}}\right)=\left(\dfrac{q_{n+2}}{q_n}-1\right)\left(1-\dfrac{q_{n-1}}{q_{n+1}}\right)$.

18. If $\dfrac{p_n}{q_n}$ is the n^{th} convergent to a continued fraction, and a_n the corresponding quotient, shew that

$$p_{n+2}q_{n-2} \sim p_{n-2}q_{n+2} = a_{n+2} \cdot a_{n+1} \cdot a_n + a_{n+2} + a_n.$$

339. *Each convergent is nearer to the continued fraction than any of the preceding convergents.*

Let x denote the continued fraction, and $\dfrac{p_n}{q_n}, \dfrac{p_{n+1}}{q_{n+1}}, \dfrac{p_{n+2}}{q_{n+2}}$ three consecutive convergents; then x differs from $\dfrac{p_{n+2}}{q_{n+2}}$ only in taking the *complete* $(n+2)^{\text{th}}$ quotient in the place of a_{n+2}; denote this by k; thus
$$x=\frac{kp_{n+1}+p_n}{kq_{n+1}+q_n};$$

$$\therefore\ x \sim \frac{p_n}{q_n}=\frac{k\,(p_{n+1}q_n \sim p_nq_{n+1})}{q_n\,(kq_{n+1}+q_n)}=\frac{k}{q_n\,(kq_{n+1}+q_n)},$$

and
$$\frac{p_{n+1}}{q_{n+1}} \sim x=\frac{p_{n+1}\,q_n \sim p_n\,q_{n+1}}{q_{n+1}\,(kq_{n+1}+q_n)}=\frac{1}{q_{n+1}\,(kq_{n+1}+q_n)}.$$

Now k is greater than unity, and q_n is less than q_{n+1}; hence on both accounts the difference between $\frac{p_{n+1}}{q_{n+1}}$ and x is less than the difference between $\frac{p_n}{q_n}$ and x; that is, every convergent is nearer to the continued fraction than the next preceding convergent, and therefore *a fortiori* than any preceding convergent.

Combining the result of this article with that of Art. 335, it follows that

the convergents of an odd order continually increase, but are always less than the continued fraction;

the convergents of an even order continually decrease, but are always greater than the continued fraction.

340. *To find limits to the error made in taking any convergent for the continued fraction.*

Let $\frac{p_n}{q_n}$, $\frac{p_{n+1}}{q_{n+1}}$, $\frac{p_{n+2}}{q_{n+2}}$ be three consecutive convergents, and let k denote the complete $(n+2)^{\text{th}}$ quotient;

then
$$x = \frac{kp_{n+1} + p_n}{kq_{n+1} + q_n},$$

$$\therefore \; x \sim \frac{p_n}{q_n} = \frac{k}{q_n(kq_{n+1} + q_n)} = \frac{1}{q_n\left(q_{n+1} + \dfrac{q_n}{k}\right)}.$$

Now k is greater than 1, therefore the difference between x and $\frac{p_n}{q_n}$ is less than $\frac{1}{q_n q_{n+1}}$, and greater than $\frac{1}{q_n(q_{n+1} + q_n)}$.

Again, since $q_{n+1} > q_n$, the error in taking $\frac{p_n}{q_n}$ instead of x is less than $\frac{1}{q_n^2}$ and greater than $\frac{1}{2q^2_{n+1}}$.

341. From the last article it appears that the error in taking $\frac{p_n}{q_n}$ instead of the continued fraction is less than $\frac{1}{q_n q_{n+1}}$, or $\frac{1}{q_n(a_{n+1}q_n + q_{n-1})}$; that is, less than $\frac{1}{a_{n+1}q_n^2}$; hence the larger a_{n+1} is, the nearer does $\frac{p_n}{q_n}$ approximate to the continued fraction;

therefore, *any convergent which immediately precedes a large quotient is a near approximation to the continued fraction.*

Again, since the error is less than $\frac{1}{q_n^2}$, it follows that in order to find a convergent which will differ from the continued fraction by less than a given quantity $\frac{1}{a}$, we have only to calculate the successive convergents up to $\frac{p_n}{q_n}$, where q_n^2 is greater than a.

342. The properties of continued fractions enable us to find two small integers whose ratio closely approximates to that of two incommensurable quantities, or to that of two quantities whose exact ratio can only be expressed by large integers.

Example. Find a series of fractions approximating to 3·14159.

In the process of finding the greatest common measure of 14159 and 100000, the successive quotients are 7, 15, 1, 25, 1, 7, 4. Thus

$$3{\cdot}14159 = 3 + \frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{25+}\,\frac{1}{1+}\,\frac{1}{7+}\,\frac{1}{4}.$$

The successive convergents are

$$\frac{3}{1},\ \frac{22}{7},\ \frac{333}{106},\ \frac{355}{113},\ \ldots\ldots;$$

this last convergent which precedes the large quotient 25 is a very near approximation, the error being less than $\frac{1}{25 \times (113)^2}$, and therefore less than $\frac{1}{25 \times (100)^2}$, or ·000004.

343. *Any convergent is nearer to the continued fraction than any other fraction whose denominator is less than that of the convergent.*

Let x be the continued fraction, $\frac{p_n}{q_n}$, $\frac{p_{n-1}}{q_{n-1}}$ two consecutive convergents, $\frac{r}{s}$ a fraction whose denominator s is less than q_n.

If possible, let $\frac{r}{s}$ be nearer to x than $\frac{p_n}{q_n}$, then $\frac{r}{s}$ must be nearer to x than $\frac{p_{n-1}}{q_{n-1}}$ [Art. 339]; and since x lies between $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$, it follows that $\frac{r}{s}$ must lie between $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$.

Hence

$$\frac{r}{s} \sim \frac{p_{n-1}}{q_{n-1}} < \frac{p_n}{q_n} \sim \frac{p_{n-1}}{q_{n-1}}, \text{ that is } < \frac{1}{q_n q_{n-1}};$$

$$\therefore rq_{n-1} \sim sp_{n-1} < \frac{s}{q_n};$$

that is, an integer less than a fraction; which is impossible. Therefore $\frac{p_n}{q_n}$ must be nearer to the continued fraction than $\frac{r}{s}$.

344. *If* $\frac{p}{q}$, $\frac{p'}{q'}$ *be two consecutive convergents to a continued fraction* x, *then* $\frac{pp'}{qq'}$ *is greater or less than* x^2, *according as* $\frac{p}{q}$ *is greater or less than* $\frac{p'}{q'}$.

Let k be the complete quotient corresponding to the convergent immediately succeeding $\frac{p'}{q'}$; then $x = \frac{kp'+p}{kq'+q}$,

$$\therefore \frac{pp'}{qq'} - x^2 = \frac{1}{qq'(kq'+q)^2}\{pp'(kq'+q)^2 - qq'(kp'+p)^2\}$$

$$= \frac{(k^2p'q' - pq)(pq' - p'q)}{qq'(kq'+q)^2}.$$

The factor $k^2p'q' - pq$ is positive, since $p' > p$, $q' > q$, and $k > 1$; hence $\frac{pp'}{qq'} >$ or $< x^2$, according as $pq' - p'q$ is positive or negative; that is, according as $\frac{p}{q} >$ or $< \frac{p'}{q'}$.

Cor. It follows from the above investigation that the expressions $pq' - p'q$, $pp' - qq'x^2$, $p^2 - q^2x^2$, $q'^2x^2 - p'^2$ have the same sign.

EXAMPLES. XXV. b.

1. Find limits to the error in taking $\frac{222}{203}$ yards as equivalent to a metre, given that a metre is equal to 1·0936 yards.

2. Find an approximation to

$$\cdot 1+\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{7+}\,\frac{1}{9+}\,\frac{1}{11+}\,\cdots\cdots$$

which differs from the true value by less than $\cdot 0001$.

3. Shew by the theory of continued fractions that $\frac{99}{70}$ differs from $1\cdot 41421$ by a quantity less than $\frac{1}{11830}$.

4. Express $\dfrac{a^3+6a^2+13a+10}{a^4+6a^3+14a^2+15a+7}$ as a continued fraction, and find the third convergent.

5. Shew that the difference between the first and n^{th} convergents is numerically equal to

$$\frac{1}{q_1q_2}-\frac{1}{q_2q_3}+\frac{1}{q_3q_4}-\ldots+\frac{(-1)^n}{q_{n-1}q_n}.$$

6. Shew that if a_n is the quotient corresponding to $\frac{p_n}{q_n}$,

(1) $$\frac{p_n}{p_{n-1}}=a_n+\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\,\frac{1}{a_{n-3}+}\,\cdots\,\frac{1}{a_3+}\,\frac{1}{a_2+}\,\frac{1}{a_1},$$

(2) $$\frac{q_n}{q_{n-1}}=a_n+\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\,\frac{1}{a_{n-3}+}\,\cdots\,\frac{1}{a_3+}\,\frac{1}{a_2}.$$

7. In the continued fraction $\frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a+}\,\cdots\cdots$, shew that

(1) $p_n{}^2+p^2_{n+1}=p_{n-1}p_{n+1}+p_np_{n+2}$,

(2) $p_n=q_{n-1}$.

8. If $\frac{p_n}{q_n}$ is the n^{th} convergent to the continued fraction

$$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\,\cdots\cdots,$$

shew that $$q_{2n}=p_{2n+1},\quad q_{2n-1}=\frac{a}{b}\,p_{2n}.$$

9. In the continued fraction

$$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\,\cdots\cdots,$$

shew that

$$p_{n+2}-(ab+2)\,p_n+p_{n-2}=0,\quad q_{n+2}-(ab+2)\,q_n+q_{n-2}=0.$$

10. Shew that

$$a\left(x_1+\frac{1}{ax_2+}\,\frac{1}{x_3+}\,\frac{1}{ax_4+}\,\ldots\ldots\text{to } 2n \text{ quotients}\right)$$

$$=ax_1+\frac{1}{x_2+}\,\frac{1}{ax_3+}\,\frac{1}{x_4+}\,\ldots\ldots\text{to } 2n \text{ quotients}.$$

11. If $\frac{M}{N}$, $\frac{P}{Q}$, $\frac{R}{S}$ are the n^{th}, $(n-1)^{\text{th}}$, $(n-2)^{\text{th}}$ convergents to the continued fractions

$$\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots, \qquad \frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\cdots, \qquad \frac{1}{a_3+}\,\frac{1}{a_4+}\,\frac{1}{a_5+}\cdots,$$

respectively, shew that

$$M=a_2P+R, \quad N=(a_1a_2+1)P+a_1R.$$

12. If $\frac{p_n}{q_n}$ is the n^{th} convergent to

$$\frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a+}\cdots,$$

shew that p_n and q_n are respectively the coefficients of x^n in the expansions of

$$\frac{x}{1-ax-x^2} \text{ and } \frac{ax+x^2}{1-ax-x^2}.$$

Hence shew that $p_n=q_{n-1}=\frac{\alpha^n-\beta^n}{\alpha-\beta}$, where α, β are the roots of the equation $t^2-at-1=0$.

13. If $\frac{p_n}{q_n}$ is the n^{th} convergent to

$$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\cdots,$$

shew that p_n and q_n are respectively the coefficients of x^n in the expansions of

$$\frac{x+bx^2-x^3}{1-(ab+2)x^2+x^4} \text{ and } \frac{ax+(ab+1)x^2-x^4}{1-(ab+2)x^2+x^4}.$$

Hence shew that

$$ap_{2n}=bq_{2n-1}=ab\,\frac{\alpha^n-\beta^n}{\alpha-\beta},$$

$$p_{2n+1}=q_{2n}=\frac{\alpha^{n+1}-\beta^{n+1}-(\alpha^n-\beta^n)}{\alpha-\beta},$$

where α, β are the values of x^2 found from the equation

$$1-(ab+2)x^2+x^4=0.$$