CHAPTER XXVI.

INDETERMINATE EQUATIONS OF THE FIRST DEGREE.

345. In Chap. X. we have shewn how to obtain the positive integral solutions of indeterminate equations with numerical coefficients; we shall now apply the properties of continued fractions to obtain the general solution of any indeterminate equation of the first degree.

346. Any equation of the first degree involving two unknowns x and y can be reduced to the form $ax \pm by = \pm c$, where a, b, c are positive integers. This equation admits of an unlimited number of solutions; but if the conditions of the problem require x and y to be positive integers, the number of solutions may be limited.

It is clear that the equation $ax + by = -c$ has no positive integral solution; and that the equation $ax - by = -c$ is equivalent to $by - ax = c$; hence it will be sufficient to consider the equations $ax \pm by = c$.

If a and b have a factor m which does not divide c, neither of the equations $ax \pm by = c$ can be satisfied by integral values of x and y; for $ax \pm by$ is divisible by m, whereas c is not.

If a, b, c have a common factor it can be removed by division; so that we shall suppose a, b, c to have no common factor, and that a and b are prime to each other.

347. *To find the general solution in positive integers of the equation* $ax - by = c$.

Let $\frac{a}{b}$ be converted into a continued fraction, and let $\frac{p}{q}$ denote the convergent just preceding $\frac{a}{b}$; then $aq - bp = \pm 1$. [Art. 338.]

I. If $aq - bp = 1$, the given equation may be written

$$ax - by = c(aq - bp);$$
$$\therefore \quad a(x - cq) = b(y - cp).$$

Now since a and b have no common factor, $x - cq$ must be divisible by b; hence $x - cq = bt$, where t is an integer,

$$\therefore \quad \frac{x - cq}{b} = t = \frac{y - cp}{a};$$

that is, $$x = bt + cq, \quad y = at + cp;$$

from which positive integral solutions may be obtained by giving to t any positive integral value, or any negative integral value numerically smaller than the less of the two quantities $\frac{cq}{b}$, $\frac{cp}{a}$; also t may be zero; thus the number of solutions is unlimited.

II. If $aq - bp = -1$, we have

$$ax - by = -c(aq - bp);$$
$$\therefore \quad a(x + cq) = b(y + cp);$$
$$\therefore \quad \frac{x + cq}{b} = \frac{y + cp}{a} = t, \text{ an integer};$$

hence $$x = bt - cq, \quad y = at - cp;$$

from which positive integral solutions may be obtained by giving to t any positive integral value which exceeds the greater of the two quantities $\frac{cq}{b}$, $\frac{cp}{a}$; thus the number of solutions is unlimited.

III. If either a or b is unity, the fraction $\frac{a}{b}$ cannot be converted into a continued fraction with unit numerators, and the investigation fails. In these cases, however, the solutions may be written down by inspection; thus if $b = 1$, the equation becomes $ax - y = c$; whence $y = ax - c$, and the solutions may be found by ascribing to x any positive integral value greater than $\frac{c}{a}$.

NOTE. It should be observed that the series of values for x and y form two arithmetical progressions in which the common differences are b and a respectively.

Example. Find the general solution in positive integers of $29x - 42y = 5$.

In converting $\frac{42}{29}$ into a continued fraction the convergent just before $\frac{42}{29}$ is $\frac{13}{9}$; we have therefore

$$29 \times 13 - 42 \times 9 = -1;$$

$$\therefore\ 29 \times 65 - 42 \times 45 = -5;$$

combining this with the given equation, we obtain

$$29(x + 65) = 42(y + 45);$$

$$\therefore\ \frac{x+65}{42} = \frac{y+45}{29} = t, \text{ an integer};$$

hence the general solution is

$$x = 42t - 65,\quad y = 29t - 45.$$

348. *Given one solution in positive integers of the equation* ax − by = c, *to find the general solution.*

Let h, k be a solution of $ax - by = c$; then $ah - bk = c$.

$$\therefore\quad ax - by = ah - bk;$$

$$\therefore\quad a(x - h) = b(y - k);$$

$$\therefore\quad \frac{x-h}{b} = \frac{y-k}{a} = t, \text{ an integer};$$

$$\therefore\quad x = h + bt,\quad y = k + at;$$

which is the general solution.

349. *To find the general solution in positive integers of the equation* ax + by = c.

Let $\frac{a}{b}$ be converted into a continued fraction, and let $\frac{p}{q}$ be the convergent just preceding $\frac{a}{b}$; then $aq - bp = \pm 1$.

I. If $aq - bp = 1$, we have

$$ax + by = c(aq - bp);$$

$$\therefore\quad a(cq - x) = b(y + cp);$$

$$\therefore\quad \frac{cq-x}{b} = \frac{y+cp}{a} = t, \text{ an integer};$$

$$\therefore\quad x = cq - bt,\quad y = at - cp;$$

from which positive integral solutions may be obtained by giving to t positive integral values greater than $\frac{cp}{a}$ and less than $\frac{cq}{b}$. Thus the number of solutions is limited, and if there is no integer fulfilling these conditions there is no solution.

II. If $aq - bp = -1$, we have

$$ax + by = -c(aq - bp);$$

$$\therefore \quad a(x + cq) = b(cp - y);$$

$$\therefore \quad \frac{x + cq}{b} = \frac{cp - y}{a} = t, \text{ an integer};$$

$$\therefore \quad x = bt - cq, \quad y = cp - at;$$

from which positive integral solutions may be obtained by giving to t positive integral values greater than $\frac{cq}{b}$ and less than $\frac{cp}{a}$. As before, the number of solutions is limited, and there may be no solution.

III. If either a or b is equal to unity, the solution may be found by inspection as in Art. 347.

350. *Given one solution in positive integers of the equation* ax + by = c, *to find the general solution.*

Let h, k be a solution of $ax + by = c$; then $ah + bk = c$.

$$\therefore \quad ax + by = ah + bk;$$

$$\therefore \quad a(x - h) = b(k - y);$$

$$\therefore \quad \frac{x - h}{b} = \frac{k - y}{a} = t, \text{ an integer};$$

$$\therefore \quad x = h + bt, \quad y = k - at;$$

which is the general solution.

351. *To find the number of solutions in positive integers of the equation* ax + by = c.

Let $\frac{a}{b}$ be converted into a continued fraction, and let $\frac{p}{q}$ be the convergent just preceding $\frac{a}{b}$; then $aq - bp = \pm 1$.

I. Let $aq - bp = 1$; then the general solution is

$$x = cq - bt, \quad y = at - cp. \qquad \text{[Art. 349.]}$$

Positive integral solutions will be obtained by giving to t positive integral values not greater than $\frac{cq}{b}$, and not less than $\frac{cp}{a}$.

(i) Suppose that $\frac{c}{a}$ and $\frac{c}{b}$ are not integers.

Let $$\frac{cp}{a} = m + f, \quad \frac{cq}{b} = n + g,$$

where m, n are positive integers and f, g proper fractions; then the least value t can have is $m + 1$, and the greatest value is n; therefore the number of solutions is

$$n - m = \frac{cq}{b} - \frac{cp}{a} + f - g = \frac{c}{ab} + f - g.$$

Now this is an integer, and may be written $\frac{c}{ab}$ + a fraction, or $\frac{c}{ab}$ − a fraction, according as f is greater or less than g. Thus the number of solutions is the integer nearest to $\frac{c}{ab}$, greater or less according as f or g is the greater.

(ii) Suppose that $\frac{c}{b}$ is an integer.

In this case $g = 0$, and one value of x is zero. If we include this, the number of solutions is $\frac{c}{ab} + f$, which must be an integer. Hence the number of solutions is the greatest integer in $\frac{c}{ab} + 1$ or $\frac{c}{ab}$, according as we include or exclude the zero solution.

(iii) Suppose that $\frac{c}{a}$ is an integer.

In this case $f = 0$, and one value of y is zero. If we include this, the least value of t is m and the greatest is n; hence the number of solutions is $n - m + 1$, or $\frac{c}{ab} - g + 1$. Thus the number of solutions is the greatest integer in $\frac{c}{ab}+1$ or $\frac{c}{ab}$, according as we include or exclude the zero solution.

(iv) Suppose that $\frac{c}{a}$ and $\frac{c}{b}$ are both integers.

In this case $f=0$ and $g=0$, and both x and y have a zero value. If we include these, the least value t can have is m, and the greatest is n; hence the number of solutions is $n-m+1$, or $\frac{c}{ab}+1$. If we exclude the zero values the number of solutions is $\frac{c}{ab}-1$.

II. If $aq-bp=-1$, the general solution is

$$x=bt-cq, \quad y=cp-at,$$

and similar results will be obtained.

352. To find the solutions in positive integers of the equation $ax+by+cz=d$, we may proceed as follows.

By transposition $ax+by=d-cz$; from which by giving to z in succession the values 0, 1, 2, 3,...... we obtain equations of the form $ax+by=c'$, which may be solved as already explained.

353. If we have two simultaneous equations

$$ax+by+cz=d, \quad a'x+b'y+c'z=d',$$

by eliminating one of the unknowns, z say, we obtain an equation of the form $Ax+By=C$. Suppose that $x=f$, $y=g$ is a solution, then the general solution can be written

$$x=f+Bs, \quad y=g-As,$$

where s is an integer.

Substituting these values of x and y in either of the given equations, we obtain an equation of the form $Fs+Gz=H$, of which the general solution is

$$s=h+Gt, \quad z=k-Ft \text{ say.}$$

Substituting for s, we obtain

$$x=f+Bh+BGt, \quad y=g-Ah-AGt;$$

and the values of x, y, z are obtained by giving to t suitable integral values.

354. If one solution in positive integers of the equations

$$ax + by + cz = d, \quad a'x + b'y + c'z = d',$$

can be found, the general solution may be obtained as follows.

Let f, g, h be the particular solution; then

$$af + bg + ch = d, \quad a'f + b'g + c'h = d'.$$

By subtraction,

$$a(x - f) + b(y - g) + c(z - h) = 0,$$

$$a'(x - f) + b'(y - g) + c'(z - h) = 0;$$

whence

$$\frac{x - f}{bc' - b'c} = \frac{y - g}{ca' - c'a} = \frac{z - h}{ab' - a'b} = \frac{t}{k},$$

where t is an integer and k is the H.C.F. of the denominators $bc' - b'c$, $ca' - c'a$, $ab' - a'b$. Thus the general solution is

$$x = f + (bc' - b'c)\frac{t}{k}, \quad y = g + (ca' - c'a)\frac{t}{k}, \quad z = h + (ab' - a'b)\frac{t}{k}.$$

EXAMPLES. XXVI.

Find the general solution and the least positive integral solution of

1. $775x - 711y = 1$. **2.** $455x - 519y = 1$. **3.** $436x - 393y = 5$.

4. In how many ways can £1. 19*s*. 6*d*. be paid in florins and half-crowns?

5. Find the number of solutions in positive integers of

$$11x + 15y = 1031.$$

6. Find two fractions having 7 and 9 for their denominators, and such that their sum is $1\frac{10}{63}$.

7. Find two proper fractions in their lowest terms having 12 and 8 for their denominators and such that their difference is $\frac{1}{24}$.

8. A certain sum consists of x pounds y shillings, and it is half of y pounds x shillings; find the sum.

Solve in positive integers:

9. $\left.\begin{array}{r} 6x + 7y + 4z = 122 \\ 11x + 8y - 6z = 145 \end{array}\right\}.$ **10.** $\left.\begin{array}{r} 12x - 11y + 4z = 22 \\ -4x + 5y + z = 17 \end{array}\right\}.$

11. $\left.\begin{aligned} 20x-21y&=38 \\ 3y+\ 4z&=34 \end{aligned}\right\}.$ **12.** $\left.\begin{aligned} 13x+11z&=103 \\ 7z-\ 5y&=\ \ 4 \end{aligned}\right\}.$

13. $7x+4y+19z=84.$ **14.** $23x+17y+11z=130.$

15. Find the general form of all positive integers which divided by 5, 7, 8 leave remainders 3, 2, 5 respectively.

16. Find the two smallest integers which divided by 3, 7, 11 leave remainders 1, 6, 5 respectively.

17. A number of three digits in the septenary scale is represented in the nonary scale by the same three digits in reverse order; if the middle digit in each case is zero, find the value of the number in the denary scale.

18. If the integers 6, a, b are in harmonic progression, find all the possible values of a and b.

19. Two rods of equal length are divided into 250 and 243 equal parts respectively; if their ends be coincident, find the divisions which are the nearest together.

20. Three bells commenced to toll at the same time, and tolled at intervals of 23, 29, 34 seconds respectively. The second and third bells tolled 39 and 40 seconds respectively longer than the first; how many times did each bell toll if they all ceased in less than 20 minutes?

21. Find the greatest value of c in order that the equation $7x+9y=c$ may have exactly six solutions in positive integers.

22. Find the greatest value of c in order that the equation $14x+11y=c$ may have exactly five solutions in positive integers.

23. Find the limits within which c must lie in order that the equation $19x+14y=c$ may have six solutions, zero solutions being excluded.

24. Shew that the greatest value of c in order that the equation $ax+by=c$ may have exactly n solutions in positive integers is $(n+1)ab-a-b$, and that the least value of c is $(n-1)ab+a+b$, zero solutions being excluded.