*CHAPTER XXVIII.

INDETERMINATE EQUATIONS OF THE SECOND DEGREE.

*366. The solution in positive integers of indeterminate equations of a degree higher than the first, though not of much practical importance, is interesting because of its connection with the *Theory of Numbers*. In the present chapter we shall confine our attention to equations of the second degree involving two variables.

*367. *To shew how to obtain the positive integral values of* x *and* y *which satisfy the equation*

$$\mathrm{ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0},$$

a, b, c, f, g, h *being integers.*

Solving this equation as a quadratic in x, as in Art. 127, we have

$$ax + hy + g = \pm\sqrt{(h^2 - ab)\, y^2 + 2\,(hg - af)\, y + (g^2 - ac)} \ldots (1).$$

Now in order that the values of x and y may be positive integers, the expression under the radical, which we may denote by $py^2 + 2qy + r$, must be a perfect square; that is

$$py^2 + 2qy + r = z^2, \text{ suppose.}$$

Solving this equation as a quadratic in y, we have

$$py + q = \pm\sqrt{q^2 - pr + pz^2};$$

and, as before, the expression under the radical must be a perfect square; suppose that it is equal to t^2; then

$$t^2 - pz^2 = q^2 - pr,$$

where t and z are variables, and p, q, r are constants.

Unless this equation can be solved in positive integers, the original equation does not admit of a positive integral solution. We shall return to this point in Art. 374.

If a, b, h are all positive, it is clear that the number of solutions is limited, because for large values of x and y the sign of the expression on the left depends upon that of $ax^2+2hxy+by^2$ [Art. 269], and thus cannot be zero for large positive integral values of x and y.

Again, if h^2-ab is negative, the coefficient of y^2 in (1) is negative, and by similar reasoning we see that the number of solutions is limited.

Example. Solve in positive integers the equation

$$x^2-4xy+6y^2-2x-20y=29.$$

Solving as a quadratic in x, we have

$$x=2y+1\pm\sqrt{30+24y-2y^2}.$$

But $30+24y-2y^2=102-2(y-6)^2$; hence $(y-6)^2$ cannot be greater than 51. By trial we find that the expression under the radical becomes a perfect square when $(y-6)^2=1$ or 49; thus the positive integral values of y are 5, 7, 13.

When $y=5$, $x=21$ or 1; when $y=7$, $x=25$ or 5; when $y=13$, $x=29$ or 25.

*368. We have seen that the solution in positive integers of the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

can be made to depend upon the solution of an equation of the form

$$x^2\pm Ny^2=\pm a,$$

where N and a are positive integers.

The equation $x^2+Ny^2=-a$ has no real roots, whilst the equation $x^2+Ny^2=a$ has a limited number of solutions, which may be found by trial; we shall therefore confine our attention to equations of the form $x^2-Ny^2=\pm a$.

*369. *To shew that the equation* $x^2-Ny^2=1$ *can always be solved in positive integers.*

Let $\sqrt{N}$ be converted into a continued fraction, and let $\frac{p}{q}$, $\frac{p'}{q'}$, $\frac{p''}{q''}$ be any three consecutive convergents; suppose that

$\dfrac{\sqrt{N}+a_n}{r_n}$ is the complete quotient corresponding to $\dfrac{p''}{q''}$; then

$$r_n(pq'-p'q) = Nq'^2 - p'^2 \qquad \text{[Art. 358].}$$

But $r_n = 1$ at the end of any period [Art. 361];

$$\therefore p'^2 - Nq'^2 = p'q - pq';$$

$\dfrac{p'}{q'}$ being the penultimate convergent of any recurring period.

If the number of quotients in the period is even, $\dfrac{p'}{q'}$ is an even convergent, and is therefore greater than $\sqrt{N}$, and therefore greater than $\dfrac{p}{q}$; thus $p'q - pq' = 1$. In this case $p'^2 - Nq'^2 = 1$, and therefore $x = p'$, $y = q'$ is a solution of the equation $x^2 - Ny^2 = 1$.

Since $\dfrac{p'}{q'}$ is the penultimate convergent of any recurring period, the number of solutions is unlimited.

If the number of quotients in the period is odd, the penultimate convergent in the first period is an *odd* convergent, but the penultimate convergent in the second period is an *even* convergent. Thus integral solutions will be obtained by putting $x = p'$, $y = q'$, where $\dfrac{p'}{q'}$ is the penultimate convergent in the second, fourth, sixth,......recurring periods. Hence also in this case the number of solutions is unlimited.

*370. *To obtain a solution in positive integers of the equation* $x^2 - Ny^2 = -1$.

As in the preceding article, we have

$$p'^2 - Nq'^2 = p'q - pq'.$$

If the number of quotients in the period is *odd*, and if $\dfrac{p'}{q'}$ is an *odd* penultimate convergent in any recurring period, $\dfrac{p'}{q'} < \dfrac{p}{q}$, and therefore $p'q - pq' = -1$.

In this case $p'^2 - Nq'^2 = -1$, and integral solutions of the equation $x^2 - Ny^2 = -1$ will be obtained by putting $x = p'$, $y = q'$, where $\dfrac{p'}{q'}$ is the penultimate convergent in the first, third, fifth... recurring periods.

Example. Solve in positive integers $x^2-13y^2=\pm 1$.

We can shew that

$$\sqrt{13}=3+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{6+}\dots\dots$$

Here the number of quotients in the period is odd; the penultimate convergent in the first period is $\frac{18}{5}$; hence $x=18$, $y=5$ is a solution of

$$x^2-13y^2=-1.$$

By Art. 364, the penultimate convergent in the second recurring period is

$$\frac{1}{2}\left(\frac{18}{5}+\frac{5}{18}\times 13\right), \text{ that is, } \frac{649}{180};$$

hence $x=649$, $y=180$ is a solution of $x^2-13y^2=1$.

By forming the successive penultimate convergents of the recurring periods we can obtain any number of solutions of the equations

$$x^2-13y^2=-1, \text{ and } x^2-13y^2=+1.$$

*371. When one solution in positive integers of $x^2-Ny^2=1$ has been found, we may obtain as many as we please by th following method.

Suppose that $x=h$, $y=k$ is a solution, h and k being positive integers; then $(h^2-Nk^2)^n=1$, where n is any positive integer. Thus

$$x^2-Ny^2=(h^2-Nk^2)^n.$$

$$\therefore\ (x+y\sqrt{N})(x-y\sqrt{N})=(h+k\sqrt{N})^n(h-k\sqrt{N})^n.$$

Put $x+y\sqrt{N}=(h+k\sqrt{N})^n$, $x-y\sqrt{N}=(h-k\sqrt{N})^n$;

$$\therefore\ 2x=(h+k\sqrt{N})^n+(h-k\sqrt{N})^n;$$

$$2y\sqrt{N}=(h+k\sqrt{N})^n-(h-k\sqrt{N})^n.$$

The values of x and y so found are positive integers, and by ascribing to n the values 1, 2, 3,..., as many solutions as we please can be obtained.

Similarly if $x=h$, $y=k$ is a solution of the equation $x^2-Ny^2=-1$, and if n is any *odd* positive integer,

$$x^2-Ny^2=(h^2-Nk^2)^n.$$

Thus the values of x and y are the same as already found, but n is restricted to the values 1, 3, 5,.......

*372. By putting $x=ax'$, $y=ay'$ the equations $x^2-Ny^2=\pm a^2$ become $x'^2-Ny'^2=\pm 1$, which we have already shewn how to solve.

*373. We have seen in Art. 369 that

$$p'^2 - Nq'^2 = -r_n(pq' - p'q) = \pm r_n.$$

Hence if a is a denominator of any complete quotient which occurs in converting $\sqrt{N}$ into a continued fraction, and if $\frac{p'}{q'}$ is the convergent obtained by stopping short of this complete quotient, *one* of the equations $x^2 - Ny^2 = \pm a$ is satisfied by the values $x = p'$, $y = q'$.

Again, the odd convergents are all less than $\sqrt{N}$, and the even convergents are all greater than $\sqrt{N}$; hence if $\frac{p'}{q'}$ is an even convergent, $x = p'$, $y = q'$ is a solution of $x^2 - Ny^2 = a$; and if $\frac{p'}{q'}$ is an odd convergent, $x = p'$, $y = q'$ is a solution of $x^2 - Ny^2 = -a$.

*374. The method explained in the preceding article enables us to find a solution of *one* of the equations $x^2 - Ny^2 = \pm a$ only when a is one of the denominators which occurs in the process of converting $\sqrt{N}$ into a continued fraction. For example, if we convert $\sqrt{7}$ into a continued fraction, we shall find that

$$\sqrt{7} = 2 + \frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{4+}\cdots\cdots$$

and that the denominators of the complete quotients are 3, 2, 3, 1.

The successive convergents are

$$\frac{2}{1},\; \frac{3}{1},\; \frac{5}{2},\; \frac{8}{3},\; \frac{37}{14},\; \frac{45}{17},\; \frac{82}{31},\; \frac{127}{48},\cdots\cdots;$$

and if we take the cycle of equations

$$x^2 - 7y^2 = -3,\quad x^2 - 7y^2 = 2,\quad x^2 - 7y^2 = -3,\quad x^2 - 7y^2 = 1,$$

we shall find that they are satisfied by taking

for x the values 2, 3, 5, 8, 37, 45, 82, 127,......

and for y the values 1, 1, 2, 3, 14, 17, 31, 48,......

*375. It thus appears that the number of cases in which solutions in integers of the equations $x^2 - Ny^2 = \pm a$ can be obtained with certainty is very limited. In a numerical example it may, however, sometimes happen that we can discover by trial a positive integral solution of the equations $x^2 - Ny^2 = \pm a$, when a is not one of the above mentioned denominators; thus we easily find that the equation $x^2 - 7y^2 = 53$ is satisfied by $y = 2$, $x = 9$. When one solution in integers has been found, any number of solutions may be obtained as explained in the next article.

*376. Suppose that $x = f$, $y = g$ is a solution of the equation $x^2 - Ny^2 = a$; and let $x = h$, $y = k$ be *any* solution of the equation $x^2 - Ny^2 = 1$; then

$$x^2 - Ny^2 = (f^2 - Ng^2)(h^2 - Nk^2)$$

$$= (fh \pm Ngk)^2 - N(fk \pm gh)^2.$$

By putting $\quad x = fh \pm Ngk, \quad y = fk \pm gh,$

and ascribing to h, k their values found as explained in Art. 371, we may obtain any number of solutions.

*377. Hitherto it has been supposed that N is not a perfect square; if, however, N is a perfect square the equation takes the form $x^2 - n^2y^2 = a$, which may be readily solved as follows.

Suppose that $a = bc$, where b and c are two positive integers, of which b is the greater; then

$$(x + ny)(x - ny) = bc.$$

Put $x + ny = b$, $x - ny = c$; if the values of x and y found from these equations are integers we have obtained one solution of the equation; the remaining solutions may be obtained by ascribing to b and c all their possible values.

Example. Find two positive integers the difference of whose squares is equal to 60.

Let x, y be the two integers; then $x^2 - y^2 = 60$; that is, $(x+y)(x-y) = 60$.

Now 60 is the product of any of the pair of factors

1, 60; 2, 30; 3, 20; 4, 15; 5, 12; 6, 10;

and the values required are obtained from the equations

$$\begin{aligned} x+y &= 30, \\ x-y &= 2; \end{aligned} \qquad\qquad \begin{aligned} x+y &= 10, \\ x-y &= 6; \end{aligned}$$

the other equations giving fractional values of x and y.

Thus the numbers are 16, 14; or 8, 2.

COR. In like manner we may obtain the solution in positive integers of

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = k,$$

if the left-hand member can be resolved into two rational linear factors.

*378. If in the general equation a, or b, or both, are zero, instead of employing the method explained in Art. 367 it is simpler to proceed as in the following example.

Example. Solve in positive integers

$$2xy - 4x^2 + 12x - 5y = 11.$$

Expressing y in terms of x, we have

$$y = \frac{4x^2 - 12x + 11}{2x - 5} = 2x - 1 + \frac{6}{2x - 5}.$$

In order that y may be an integer $\frac{6}{2x-5}$ must be an integer; hence $2x - 5$ must be equal to ± 1, or ± 2, or ± 3, or ± 6.

The cases ± 2, ± 6 may clearly be rejected; hence the admissible values of x are obtained from $2x - 5 = \pm 1$, $2x - 5 = \pm 3$;

whence the values of x are 3, 2, 4, 1.

Taking these values in succession we obtain the solutions

$$x = 3,\ y = 11;\ x = 2,\ y = -3;\ x = 4,\ y = 9;\ x = 1,\ y = -1;$$

and therefore the admissible solutions are

$$x = 3,\ y = 11;\ x = 4,\ y = 9.$$

*379. The principles already explained enable us to discover for what values of the variables given linear or quadratic functions of x and y become perfect squares. Problems of this kind are sometimes called *Diophantine Problems* because they were first investigated by the Greek mathematician Diophantus about the middle of the fourth century.

Example 1. Find the general expressions for two positive integers which are such that if their product is taken from the sum of their squares the difference is a perfect square.

Denote the integers by x and y; then

$$x^2 - xy + y^2 = z^2 \text{ suppose};$$

$$\therefore\ x(x - y) = z^2 - y^2.$$

This equation is satisfied by the suppositions

$$mx = n(z + y), \quad n(x - y) = m(z - y),$$

where m and n are positive integers.

Hence $\qquad mx - ny - nz = 0, \quad nx + (m - n)y - mz = 0.$

From these equations we obtain by cross multiplication

$$\frac{x}{2mn - n^2} = \frac{y}{m^2 - n^2} = \frac{z}{m^2 - mn + n^2};$$

and since the given equation is homogeneous we may take for the general solution

$$x = 2mn - n^2, \quad y = m^2 - n^2, \quad z = m^2 - mn + n^2.$$

Here m and n are any two positive integers, m being the greater; thus if $m = 7$, $n = 4$, we have

$$x = 40, \; y = 33, \; z = 37.$$

Example 2. Find the general expression for three positive integers in arithmetic progression, and such that the sum of every two is a perfect square.

Denote the integers by $x - y$, x, $x + y$; and let

$$2x - y = p^2, \; 2x = q^2, \; 2x + y = r^2;$$

then $\qquad p^2 + r^2 = 2q^2,$

or $\qquad r^2 - q^2 = q^2 - p^2.$

This equation is satisfied by the suppositions,

$$m(r - q) = n(q - p), \; n(r + q) = m(q + p),$$

where m and n are positive integers.

From these equations we obtain by cross multiplication

$$\frac{p}{n^2 + 2mn - m^2} = \frac{q}{m^2 + n^2} = \frac{r}{m^2 + 2mn - n^2}.$$

Hence we may take for the general solution

$$p = n^2 + 2mn - m^2, \; q = m^2 + n^2, \; r = m^2 + 2mn - n^2;$$

whence $\qquad x = \frac{1}{2}(m^2 + n^2)^2, \; y = 4mn(m^2 - n^2),$

and the three integers can be found.

From the value of x it is clear that m and n are either both even or both odd; also their values must be such that x is greater than y, that is,

$$(m^2 + n^2)^2 > 8mn(m^2 - n^2),$$

or $\qquad m^3(m - 8n) + 2m^2n^2 + 8mn^3 + n^4 > 0;$

which condition is satisfied if $m > 8n$.

If $m = 9$, $n = 1$, then $x = 3362$, $y = 2880$, and the numbers are 482, 3362, 6242. The sums of these taken in pairs are 3844, 6724, 9604, which are the squares of 62, 82, 98 respectively.

*EXAMPLES. XXVIII.

Solve in positive integers:

1. $5x^2-10xy+7y^2=77$.
2. $7x^2-2xy+3y^2=27$.
3. $y^2-4xy+5x^2-10x=4$.
4. $xy-2x-y=8$.
5. $3x+3xy-4y=14$.
6. $4x^2-y^2=315$.

Find the smallest solution in positive integers of

7. $x^2-14y^2=1$.
8. $x^2-19y^2=1$.
9. $x^2=41y^2-1$.
10. $x^2-61y^2+5=0$.
11. $x^2-7y^2-9=0$.

Find the general solution in positive integers of

12. $x^2-3y^2=1$.
13. $x^2-5y^2=1$.
14. $x^2-17y^2=-1$.

Find the general values of x and y which make each of the following expressions a perfect square:

15. $x^2-3xy+3y^2$.
16. $x^2+2xy+2y^2$.
17. $5x^2+y^2$.

18. Find two positive integers such that the square of one exceeds the square of the other by 105.

19. Find a general formula for three integers which may be taken to represent the lengths of the sides of a right-angled triangle.

20. Find a general formula to express two positive integers which are such that the result obtained by adding their product to the sum of their squares is a perfect square.

21. "There came three Dutchmen of my acquaintance to see me, being lately married; they brought their wives with them. The men's names were Hendriek, Claas, and Cornelius; the women's Geertruij, Catriin, and Anna: but I forgot the name of each man's wife. They told me they had been at market to buy hogs; each person bought as many hogs as they gave shillings for one hog; Hendriek bought 23 hogs more than Catriin; and Claas bought 11 more than Geertruij; likewise, each man laid out 3 guineas more than his wife. I desire to know the name of each man's wife." (*Miscellany of Mathematical Problems*, 1743.)

22. Shew that the sum of the first n natural numbers is a perfect square, if n is equal to k^2 or k'^2-1, where k is the numerator of an odd, and k' the numerator of an even convergent to $\sqrt{2}$.