CHAPTER XXXIV.

MISCELLANEOUS THEOREMS AND EXAMPLES.

506. We shall begin this chapter with some remarks on the permanence of algebraical form, briefly reviewing the fundamental laws which have been established in the course of the work.

507. In the exposition of algebraical principles we proceed analytically: at the outset we do not lay down new names and new ideas, but we begin from our knowledge of abstract Arithmetic; we prove certain laws of operation which are capable of verification in every particular case, and the *general* theory of these operations constitutes the science of Algebra.

Hence it is usual to speak of *Arithmetical Algebra* and *Symbolical Algebra*, and to make a distinction between them. In the former we define our symbols in a sense arithmetically intelligible, and thence deduce fundamental laws of operation; in the latter we assume the laws of Arithmetical Algebra to be true in all cases, whatever the nature of the symbols may be, and so find out what meaning must be attached to the symbols in order that they may obey these laws. Thus gradually, as we transcend the limits of ordinary Arithmetic, new results spring up, new language has to be employed, and interpretations given to symbols which were not contemplated in the original definitions. At the same time, from the way in which the general laws of Algebra are established, we are assured of their permanence and universality, even when they are applied to quantities not arithmetically intelligible.

508. Confining our attention to positive integral values of the symbols, the following laws are easily established from *a priori* arithmetical definitions.

I. **The Law of Commutation**, which we enunciate as follows:

(i) *Additions and subtractions may be made in any order.*

Thus $$a + b - c = a - c + b = b - c + a.$$

(ii) *Multiplications and divisions may be made in any order.*

Thus $$a \times b = b \times a;$$

$$a \times b \times c = b \times c \times a = a \times c \times b; \text{ and so on.}$$

$$ab \div c = a \times b \div c = (a \div c) \times b = (b \div c) \times a.$$

II. **The Law of Distribution**, which we enunciate as follows:

Multiplications and divisions may be distributed over additions and subtractions.

Thus $$(a - b + c)\, m = am - bm + cm,$$

$$(a - b)(c - d) = ac - ad - bc + bd.$$

[See *Elementary Algebra*, Arts. 33, 35.]

And since division is the reverse of multiplication, the distributive law for division requires no separate discussion.

III. **The Laws of Indices.**

(i) $$a^m \times a^n = a^{m+n},$$

$$a^m \div a^n = a^{m-n}.$$

(ii) $$(a^m)^n = a^{mn}.$$

[See *Elementary Algebra*, Art. 233 to 235.]

These laws are *laid down* as fundamental to our subject, having been proved on the supposition that the symbols employed are positive and integral, and that they are restricted in such a way that the operations above indicated are arithmetically intelligible. If these conditions do not hold, by the principles of Symbolical Algebra we assume the laws of Arithmetical Algebra to be true in every case and accept the interpretation to which this assumption leads us. By this course we are assured that the laws of Algebraical operation are self-consistent, and that they include in their generality the particular cases of ordinary Arithmetic.

509. From the law of commutation we deduce the rules for the removal and insertion of brackets [*Elementary Algebra*, Arts. 21, 22]; and by the aid of these rules we establish the law

of distribution as in Art. 35. For example, it is proved that

$$(a-b)(c-d)=ac-ad-bc+bd,$$

with the restriction that a, b, c, d are positive integers, and a greater than b, and c greater than d. Now it is the province of Symbolical Algebra to interpret results like this when all restrictions are removed. Hence by putting $a=0$ and $c=0$, we obtain $(-b)\times(-d)=bd$, or *the product of two negative quantities is positive.* Again by putting $b=0$ and $c=0$, we obtain $a\times(-d)=-ad$, or *the product of two quantities of opposite signs is negative.*

We are thus led to the *Rule of Signs* as a direct consequence of the law of distribution, and henceforth the rule of signs is included in our fundamental laws of operation.

510. For the way in which the fundamental laws are applied to establish the properties of algebraical fractions, the reader is referred to Chapters XIX., XXI., and XXII. of the *Elementary Algebra;* it will there be seen that symbols and operations to which we cannot give any *a priori* definition are always interpreted so as to make them conform to the laws of Arithmetical Algebra.

511. The laws of indices are fully discussed in Chapter XXX. of the *Elementary Algebra.* When m and n are positive integers and $m>n$, we prove directly from the definition of an index that

$$a^m\times a^n=a^{m+n}; \qquad a^m\div a^n=a^{m-n}; \qquad (a^m)^n=a^{mn}.$$

We then assume the first of these to be true when the indices are free from all restriction, and in this way we determine meanings for symbols to which our original definition does not apply. The interpretations for $a^{\frac{p}{q}}$, a^0, a^{-n} thus derived from the first law are found to be in strict conformity with the other two laws; and henceforth the laws of indices can be applied consistently and with perfect generality.

512. In Chapter VIII. we defined the symbol i or $\sqrt{-1}$ as obeying the relation $i^2=-1$. From this definition, and by making i subject to the general laws of Algebra we are enabled to discuss the properties of expressions of the form $a+ib$, in which real and imaginary quantities are combined. Such forms are sometimes called *complex numbers*, and it will be seen by reference to Articles 92 to 105 that if we perform on a *complex number* the operations of addition, subtraction, multiplication, and division, the result is in general itself a complex number,

Also since every rational function involves no operations but those above mentioned, it follows that a rational function of a complex number is in general a complex number.

Expressions of the form a^{x+iy}, $\log(x+iy)$ cannot be fully treated without Trigonometry; but by the aid of De Moivre's theorem, it is easy to shew that such functions can be reduced to complex numbers of the form $A+iB$.

The expression e^{x+iy} is of course included in the more general form a^{x+iy}, but another mode of treating it is worthy of attention.

We have seen in Art. 220 that

$$e^x = Lim \left(1 + \frac{x}{n}\right)^n, \text{ when } n \text{ is infinite,}$$

x being any real quantity; the quantity e^{x+iy} may be similarly defined by means of the equation

$$e^{x+iy} = Lim \left(1 + \frac{x+iy}{n}\right)^n, \text{ when } n \text{ is infinite,}$$

x and y being any real quantities.

The development of the theory of complex numbers will be found fully discussed in Chapters x. and xi. of Schlömilch's *Handbuch der algebraischen Analysis.*

513. We shall now give some theorems and examples illustrating methods which will often be found useful in proving identities, and in the Theory of Equations.

514. *To find the remainder when any rational integral function of* x *is divided by* x − a.

Let $f(x)$ denote any rational integral function of x; divide $f(x)$ by $x-a$ until a remainder is obtained which does not involve x; let Q be the quotient, and R the remainder; then

$$f(x) = Q(x-a) + R.$$

Since R does not involve x it will remain unaltered whatever value we give to x; put $x=a$, then

$$f(a) = Q \times 0 + R;$$

now Q is finite for finite values of x, hence

$$R = f(a).$$

COR. If $f(x)$ is exactly divisible by $x-a$, then $R=0$, that is $f(a)=0$; hence *if a rational integral function of* x *vanishes when* x = a, *it is divisible by* x − a.

515. The proposition contained in the preceding article is so useful that we give another proof of it which has the advantage of exhibiting the form of the quotient.

Suppose that the function is of n dimensions, and let it be denoted by

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \dots + p_n,$$

then the quotient will be of $n-1$ dimensions; denote it by

$$q_0x^{n-1} + q_1x^{n-2} + q_2x^{n-3} + \dots + q_{n-1};$$

let R be the remainder not containing x; then

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \dots + p_n$$
$$= (x-a)(q_0x^{n-1} + q_1x^{n-2} + q_2x^{n-3} + \dots + q_{n-1}) + R.$$

Multiplying out and equating the coefficients of like powers of x, we have

$$q_0 = p_0;$$
$$q_1 - aq_0 = p_1, \text{ or } q_1 = aq_0 + p_1;$$
$$q_2 - aq_1 = p_2, \text{ or } q_2 = aq_1 + p_2;$$
$$q_3 - aq_2 = p_3, \text{ or } q_3 = aq_2 + p_3;$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$R - aq_{n-1} = p_n, \text{ or } R = aq_{n-1} + p_n;$$

thus each successive coefficient in the quotient is formed by multiplying by a the coefficient last formed, and adding the next coefficient in the dividend. The process of finding the successive terms of the quotient and the remainder may be arranged thus:

$$\begin{array}{cccccc} p_0 & p_1 & p_2 & p_3 & p_{n-1} & p_n \\ & aq_0 & aq_1 & aq_2 & aq_{n-2} & aq_{n-1} \\ \hline q_0 & q_1 & q_2 & q_3 & q_{n-1} & R \end{array}$$

Thus $R = aq_{n-1} + p_n = a(aq_{n-2} + p_{n-1}) + p_n = \dots\dots$
$= p_0a^n + p_1a^{n-1} + p_2a^{n-2} + \dots + p_n.$

If the divisor is $x + a$ the same method can be used, only in this case the multiplier is $-a$.

Example. Find the quotient and remainder when $3x^7 - x^6 + 31x^4 + 21x + 5$ is divided by $x + 2$.

Here the multiplier is -2, and we have

$$\begin{array}{rrrrrrrr} 3 & -1 & 0 & 31 & 0 & 0 & 21 & 5 \\ & -6 & 14 & -28 & -6 & 12 & -24 & 6 \\ \hline 3 & -7 & 14 & 3 & -6 & 12 & -3 & 11 \end{array}$$

Thus the quotient is $3x^6 - 7x^5 + 14x^4 + 3x^3 - 6x^2 + 12x - 3$, and the remainder is 11.

516. In the preceding example the work has been abridged by writing down only the coefficients of the several terms, zero coefficients being used to represent terms corresponding to powers of x which are absent. This method of *Detached Coefficients* may frequently be used to save labour in elementary algebraical processes, particularly when the functions we are dealing with are rational and integral. The following is another illustration.

Example. Divide $3x^5 - 8x^4 - 5x^3 + 26x^2 - 33x + 26$ by $x^3 - 2x^2 - 4x + 8$.

$$\begin{array}{r} 1+2+4-8)\,3-8-\ \ 5+26-33+26\,(3-2+3 \\ 3+6+12-24\qquad\qquad\qquad\qquad \\ \hline -2+\ 7+\ 2-33\qquad\qquad\qquad \\ -2-\ 4-\ 8+16\qquad\qquad\qquad \\ \hline 3-\ 6-17+26\qquad\qquad \\ 3+\ 6+12-24\qquad\qquad \\ \hline -\ 5+\ 2\qquad\qquad \end{array}$$

Thus the quotient is $3x^2 - 2x + 3$ and the remainder is $-5x + 2$.

It should be noticed that in writing down the divisor, the sign of every term *except the first* has been changed; this enables us *to replace the process of subtraction by that of addition* at each successive stage of the work.

517. The work may be still further abridged by the following arrangement, which is known as Horner's *Method of Synthetic Division.*

$$\begin{array}{r|rrrrrr} 1 & 3 & -8 & -5 & +26 & -33 & +26 \\ 2 & & 6 & +12 & -24 & & \\ 4 & & & -4 & -8 & +16 & \\ -8 & & & & 6 & +12 & -24 \\ \hline & 3 & -2 & +3 & +0 & -5 & +2 \end{array}$$

[*Explanation.* The column of figures to the left of the vertical line consists of the coefficients of the divisor, the sign of each after the first being changed; the second horizontal line is obtained by multiplying 2, 4, -8 by 3, the first term of the quotient. We then add the terms in the second column to the right of the vertical line; this gives -2, which is the coefficient of the second term of the quotient. With the coefficient thus obtained we form the next horizontal line, and add the terms in the third column; this gives 3, which is the coefficient of the third term of the quotient.

By adding up the other columns we get the coefficients of the terms in the remainder.]

Example. Divide $6a^5+5a^4b-8a^3b^2-6a^2b^3-6ab^4$ by $2a^3+3a^2b-b^3$ to four terms in the quotient.

$$
\begin{array}{r|lllllll|lll}
2 & 6+ & 5- & 8- & 6- & 6 & & & & & \\
-3 & & -9+ & 0+ & 3 & & & & & & \\
0 & & & 6+ & 0- & 2 & & & & & \\
1 & & & & 3+ & 0 & & & -1 & & \\
 & & & & & & & & 12+ & 0- & 4 \\
\hline
 & 3- & 2- & 1+ & 0- & 4 & & & +11+ & 0- & 4
\end{array}
$$

Thus the quotient is $3a^2-2ab-b^2-4a^{-2}b^4$, and $11b^5-4a^{-2}b^7$ is the remainder.

Here we add the terms in the several columns as before, but each sum has to be divided by 2, the first coefficient in the divisor. When the requisite number of terms in the *quotient* has been so obtained, the remainder is found by merely adding up the rest of the columns, and setting down the results without division.

The student may easily verify this rule by working the division by detached coefficients.

518. The principle of Art. 514 is often useful in proving algebraical identities; but before giving any illustrations of it we shall make some remarks upon *Symmetrical and Alternating Functions.*

A function is said to be *symmetrical* with respect to its variables when its value is unaltered by the interchange of any pair of them; thus $x+y+z$, $bc+ca+ab$, $x^3+y^3+z^3-xyz$ are symmetrical functions of the first, second, and third degrees respectively.

It is worthy of notice that the only symmetrical function of the first degree in x, y, z is of the form $M(x+y+z)$, where M is independent of x, y, z.

519. It easily follows from the definition that the sum, difference, product, and quotient of any two symmetrical expressions must also be symmetrical expressions. The recognition of this principle is of great use in checking the accuracy of algebraical work, and in some cases enables us to dispense with much of the labour of calculation.

For example, we know that the expansion of $(x+y+z)^3$ must be a homogeneous function of three dimensions, and therefore of the form $x^3+y^3+z^3+A(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)+Bxyz$, where A and B are quantities independent of x, y, z.

Put $z=0$, then $A=3$, being the coefficient of x^2y in the expansion of $(x+y)^3$.

Put $x=y=z=1$, and we get $27=3+(3\times 6)+B$; whence $B=6$.

Thus $(x+y+z)^3$

$$=x^3+y^3+z^3+3x^2y+3xy^2+3y^2z+3yz^2+3z^2x+3zx^2+6xyz.$$

520. A function is said to be *alternating* with respect to its variables, when its sign but not its value is altered by the interchange of any pair of them. Thus $x-y$ and

$$a^2(b-c)+b^2(c-a)+c^2(a-b)$$

are alternating functions.

It is evident that there can be no linear alternating function involving more than two variables, and also that the product of a symmetrical function and an alternating function must be an alternating function.

521. Symmetrical and alternating functions may be concisely denoted by writing down one of the terms and prefixing the symbol Σ; thus Σa stands for the sum of all the terms of which a is the type, Σab stands for the sum of all the terms of which ab is the type; and so on. For instance, if the function involves four letters a, b, c, d,

$$\Sigma a=a+b+c+d;$$

$$\Sigma ab=ab+ac+ad+bc+bd+cd;$$

and so on.

Similarly if the function involves three letters a, b, c,

$$\Sigma a^2(b-c)=a^2(b-c)+b^2(c-a)+c^2(a-b);$$

$$\Sigma a^2bc=a^2bc+b^2ca+c^2ab;$$

and so on.

It should be noticed that when there are three letters involved Σa^2b does not consist of three terms, but of six: thus

$$\Sigma a^2b=a^2b+a^2c+b^2c+b^2a+c^2a+c^2b.$$

The symbol Σ may also be used to imply summation with regard to two or more sets of letters; thus

$$\Sigma yz(b-c)=yz(b-c)+zx(c-a)+xy(a-b).$$

522. The above notation enables us to express in an abridged form the products and powers of symmetrical expressions: thus

$$(a+b+c)^3 = \Sigma a^3 + 3\Sigma a^2 b + 6abc;$$

$$(a+b+c+d)^3 = \Sigma a^3 + 3\Sigma a^2 b + 6\Sigma abc;$$

$$(a+b+c)^4 = \Sigma a^4 + 4\Sigma a^3 b + 6\Sigma a^2 b^2 + 12\Sigma a^2 bc;$$

$$\Sigma a \times \Sigma a^2 = \Sigma a^3 + \Sigma a^2 b.$$

Example 1. Prove that

$$(a+b)^5 - a^5 - b^5 = 5ab(a+b)(a^2+ab+b^2).$$

Denote the expression on the left by E; then E is a function of a which vanishes when $a=0$; hence a is a factor of E; similarly b is a factor of E. Again E vanishes when $a=-b$, that is $a+b$ is a factor of E; and therefore E contains $ab(a+b)$ as a factor. The remaining factor must be of two dimensions, and, since it is symmetrical with respect to a and b, it must be of the form $Aa^2+Bab+Ab^2$; thus

$$(a+b)^5 - a^5 - b^5 = ab(a+b)(Aa^2+Bab+Ab^2),$$

where A and B are independent of a and b.

Putting $a=1$, $b=1$, we have $15=2A+B$;

putting $a=2$, $b=-1$, we have $15=5A-2B$;

whence $A=5$, $B=5$; and thus the required result at once follows.

Example 2. Find the factors of

$$(b^3+c^3)(b-c)+(c^3+a^3)(c-a)+(a^3+b^3)(a-b).$$

Denote the expression by E; then E is a function of a which vanishes when $a=b$, and therefore contains $a-b$ as a factor [Art. 514]. Similarly it contains the factors $b-c$ and $c-a$; thus E contains $(b-c)(c-a)(a-b)$ as a factor.

Also since E is of the fourth degree the remaining factor must be of the first degree; and since it is a symmetrical function of a, b, c, it must be of the form $M(a+b+c)$. [Art. 518];

$$\therefore E = M(b-c)(c-a)(a-b)(a+b+c).$$

To obtain M we may give to a, b, c any values that we find most convenient; thus by putting $a=0$, $b=1$, $c=2$, we find $M=1$, and we have the required result.

Example 3. Shew that

$$(x+y+z)^5 - x^5 - y^5 - z^5 = 5(y+z)(z+x)(x+y)(x^2+y^2+z^2+yz+zx+xy).$$

Denote the expression on the left by E; then E vanishes when $y=-z$, and therefore $y+z$ is a factor of E; similarly $z+x$ and $x+y$ are factors; therefore E contains $(y+z)(z+x)(x+y)$ as a factor. Also since E is of the fifth degree the remaining factor is of the second degree, and, since it is symmetrical in x, y, z, it must be of the form

$$A(x^2+y^2+z^2)+B(yz+zx+xy).$$

Put $x=y=z=1$; thus $10=A+B$;

put $x=2$, $y=1$, $z=0$; thus $35=5A+2B$;

whence $A=B=5$,

and we have the required result.

523. We collect here for reference a list of identities which are useful in the transformation of algebraical expressions; many of these have occurred in Chap. XXIX. of the *Elementary Algebra*.

$$\Sigma bc(b-c)=-(b-c)(c-a)(a-b).$$

$$\Sigma a^2(b-c)=-(b-c)(c-a)(a-b).$$

$$\Sigma a(b^2-c^2)=(b-c)(c-a)(a-b).$$

$$\Sigma a^3(b-c)=-(b-c)(c-a)(a-b)(a+b+c).$$

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab).$$

This identity may be given in another form,

$$a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}.$$

$$(b-c)^3+(c-a)^3+(a-b)^3=3(b-c)(c-a)(a-b).$$

$$(a+b+c)^3-a^3-b^3-c^3=3(b+c)(c+a)(a+b).$$

$$\Sigma bc(b+c)+2abc=(b+c)(c+a)(a+b).$$

$$\Sigma a^2(b+c)+2abc=(b+c)(c+a)(a+b).$$

$$(a+b+c)(bc+ca+ab)-abc=(b+c)(c+a)(a+b).$$

$$2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$$
$$=(a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

EXAMPLES. XXXIV. a.

1. Find the remainder when $3x^5+11x^4+90x^2-19x+53$ is divided by $x+5$.

2. Find the equation connecting a and b in order that

$$2x^4-7x^3+ax+b$$

may be divisible by $x-3$.

3. Find the quotient and remainder when $x^5-5x^4+9x^3-6x^2-16x+13$ is divided by x^2-3x+2.

4. Find a in order that x^3-7x+5 may be a factor of

$$x^5-2x^4-4x^3+19x^2-31x+12+a.$$

5. Expand $\dfrac{1}{x^4-5x^3+7x^2+x-8}$ in descending powers of x to four terms, and find the remainder.

Find the factors of

6. $a(b-c)^3+b(c-a)^3+c(a-b)^3$.

7. $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$.

8. $(a+b+c)^3-(b+c-a)^3-(c+a-b)^3-(a+b-c)^3$.

9. $a(b-c)^2+b(c-a)^2+c(a-b)^2+8abc$.

10. $a(b^4-c^4)+b(c^4-a^4)+c(a^4-b^4)$.

11. $(bc+ca+ab)^3-b^3c^3-c^3a^3-a^3b^3$.

12. $(a+b+c)^4-(b+c)^4-(c+a)^4-(a+b)^4+a^4+b^4+c^4$.

13. $(a+b+c)^5-(b+c-a)^5-(c+a-b)^5-(a+b-c)^5$.

14. $(x-a)^3(b-c)^3+(x-b)^3(c-a)^3+(x-c)^3(a-b)^3$.

Prove the following identities:

15. $\Sigma(b+c-2a)^3=3(b+c-2a)(c+a-2b)(a+b-2c)$.

16. $\dfrac{a(b-c)^2}{(c-a)(a-b)}+\dfrac{b(c-a)^2}{(a-b)(b-c)}+\dfrac{c(a-b)^2}{(b-c)(c-a)}=a+b+c$.

17. $\dfrac{2a}{a+b}+\dfrac{2b}{b+c}+\dfrac{2c}{c+a}+\dfrac{(b-c)(c-a)(a-b)}{(b+c)(c+a)(a+b)}=3$.

18. $\Sigma a^2(b+c)-\Sigma a^3-2abc=(b+c-a)(c+a-b)(a+b-c)$.

19. $\dfrac{a^3(b+c)}{(a-b)(a-c)}+\dfrac{b^3(c+a)}{(b-c)(b-a)}+\dfrac{c^3(a+b)}{(c-a)(c-b)}=bc+ca+ab$.

20. $4\Sigma(b-c)(b+c-2a)^2=9\Sigma(b-c)(b+c-a)^2$.

21. $(y+z)^2(z+x)^2(x+y)^2=\Sigma x^4(y+z)^2+2(\Sigma yz)^3-2x^2y^2z^2$.

22. $\Sigma(ab-c^2)(ac-b^2)=(\Sigma bc)(\Sigma bc-\Sigma a^2)$.

23. $abc(\Sigma a)^3-(\Sigma bc)^3=abc\Sigma a^3-\Sigma b^3c^3=(a^2-bc)(b^2-ca)(c^2-ab)$.

24. $\Sigma(b-c)^3(b+c-2a)=0$; hence deduce $\Sigma(\beta-\gamma)(\beta+\gamma-2\alpha)^3=0$.

25. $(b+c)^3+(c+a)^3+(a+b)^3-3(b+c)(c+a)(a+b)$
$$=2(a^3+b^3+c^3-3abc).$$

26. If $x=b+c-a$, $y=c+a-b$, $z=a+b-c$, shew that
$$x^3+y^3+z^3-3xyz=4(a^3+b^3+c^3-3abc).$$

27. Prove that the value of $a^3+b^3+c^3-3abc$ is unaltered if we substitute $s-a$, $s-b$, $s-c$ for a, b, c respectively, where
$$3s=2(a+b+c).$$

Find the value of

28. $\dfrac{a}{(a-b)(a-c)(x-a)}+\dfrac{b}{(b-c)(b-a)(x-b)}+\dfrac{c}{(c-a)(c-b)(x-c)}.$

29. $\dfrac{a^2-b^2-c^2}{(a-b)(a-c)}+\dfrac{b^2-c^2-a^2}{(b-c)(b-a)}+\dfrac{c^2-a^2-b^2}{(c-a)(c-b)}.$

30. $\dfrac{(a+p)(a+q)}{(a-b)(a-c)(a+x)}+\dfrac{(b+p)(b+q)}{(b-c)(b-a)(b+x)}+\dfrac{(c+p)(c+q)}{(c-a)(c-b)(c+x)}.$

31. $\Sigma\dfrac{bcd}{(a-b)(a-c)(a-d)}.$ 32. $\Sigma\dfrac{a^4}{(a-b)(a-c)(a-d)}.$

33. If $x+y+z=s$, and $xyz=p^2$, shew that
$$\left(\frac{p}{ys}-\frac{y}{p}\right)\left(\frac{p}{zs}-\frac{z}{p}\right)+\left(\frac{p}{zs}-\frac{z}{p}\right)\left(\frac{p}{xs}-\frac{x}{p}\right)+\left(\frac{p}{xs}-\frac{x}{p}\right)\left(\frac{p}{ys}-\frac{y}{p}\right)=\frac{4}{s}.$$

Miscellaneous Identities.

524. Many identities can be readily established by making use of the properties of the cube roots of unity; as usual these will be denoted by 1, ω, ω^2.

Example. Shew that
$$(x+y)^7-x^7-y^7=7xy(x+y)(x^2+xy+y^2)^2.$$

The expression, E, on the left vanishes when $x=0$, $y=0$, $x+y=0$; hence it must contain $xy(x+y)$ as a factor.

Putting $x=\omega y$, we have
$$E=\{(1+\omega)^7-\omega^7-1\}y^7=\{(-\omega^2)^7-\omega^7-1\}y^7$$
$$=(-\omega^2-\omega-1)y^7=0;$$

hence E contains $x-\omega y$ as a factor; and similarly we may shew that it contains $x-\omega^2 y$ as a factor; that is, E is divisible by
$$(x-\omega y)(x-\omega^2 y),\text{ or } x^2+xy+y^2.$$

Further, E being of seven, and $xy(x+y)(x^2+xy+y^2)$ of five dimensions, the remaining factor must be of the form $A(x^2+y^2)+Bxy$; thus

$$(x+y)^7-x^7-y^7=xy(x+y)(x^2+xy+y^2)(Ax^2+Bxy+Ay^2).$$

Putting $x=1$, $y=1$, we have $21=2A+B$;

putting $x=2$, $y=-1$, we have $21=5A-2B$;

whence $A=7$, $B=7$;

$$\therefore\ (x+y)^7-x^7-y^7=7xy(x+y)(x^2+xy+y^2)^2.$$

525. We know from elementary Algebra that

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab);$$

also we have seen in Ex. 3, Art. 110, that

$$a^2+b^2+c^2-bc-ca-ab=(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c);$$

hence $a^3+b^3+c^3-3abc$ can be resolved into three linear factors; thus

$$a^3+b^3+c^3-3abc=(a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c).$$

Example. Shew that the product of

$$a^3+b^3+c^3-3abc \text{ and } x^3+y^3+z^3-3xyz$$

can be put into the form $A^3+B^3+C^3-3ABC$.

The product $=(a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)$
$\times(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$.

By taking these six factors in the pairs $(a+b+c)(x+y+z)$;

$(a+\omega b+\omega^2 c)(x+\omega^2 y+\omega z)$; and $(a+\omega^2 b+\omega c)(x+\omega y+\omega^2 z)$,

we obtain the three partial products

$$A+B+C,\ A+\omega B+\omega^2 C,\ A+\omega^2 B+\omega C,$$

where $A=ax+by+cz$, $B=bx+cy+az$, $C=cx+ay+bz$.

Thus the product $=(A+B+C)(A+\omega B+\omega^2 C)(A+\omega^2 B+\omega C)$
$=A^3+B^3+C^3-3ABC$.

526. In order to find the values of expressions involving a, b, c when these quantities are connected by the equation $a+b+c=0$, we might employ the substitution

$$a=h+k,\quad b=\omega h+\omega^2 k,\quad c=\omega^2 h+\omega k.$$

If however the expressions involve a, b, c symmetrically the method exhibited in the following example is preferable.

Example. If $a+b+c=0$, shew that

$$6(a^5+b^5+c^5)=5(a^3+b^3+c^3)(a^2+b^2+c^2).$$

We have identically

$$(1+ax)(1+bx)(1+cx)=1+px+qx^2+rx^3,$$

where $$p=a+b+c, \quad q=bc+ca+ab, \quad r=abc.$$

Hence, using the condition given,

$$(1+ax)(1+bx)(1+cx)=1+qx^2+rx^3.$$

Taking logarithms and equating the coefficients of x^n, we have
$\frac{(-1)^{n-1}}{n}(a^n+b^n+c^n)=$ coefficient of x^n in the expansion of $\log(1+qx^2+rx^3)$

$=$ coefficient of x^n in $\quad (qx^2+rx^3)-\frac{1}{2}(qx^2+rx^3)^2+\frac{1}{3}(qx^2+rx^3)^3-\dots$

By putting $n=2, 3, 5$ we obtain

$$-\frac{a^2+b^2+c^2}{2}=q, \quad \frac{a^3+b^3+c^3}{3}=r, \quad \frac{a^5+b^5+c^5}{5}=-qr;$$

whence $$\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\cdot\frac{a^2+b^2+c^2}{2},$$

and the required result at once follows.

If $a=\beta-\gamma$, $b=\gamma-\alpha$, $c=\alpha-\beta$, the given condition is satisfied; hence we have identically for all values of α, β, γ

$$6\{(\beta-\gamma)^5+(\gamma-\alpha)^5+(\alpha-\beta)^5\}$$
$$=5\{(\beta-\gamma)^3+(\gamma-\alpha)^3+(\alpha-\beta)^3\}\{(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2\}$$

that is,

$$(\beta-\gamma)^5+(\gamma-\alpha)^5+(\alpha-\beta)^5=5(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta);$$

compare Ex. 3, Art. 522.

EXAMPLES. XXXIV. b.

1. If $(a+b+c)^3=a^3+b^3+c^3$, shew that when n is a positive integer $(a+b+c)^{2n+1}=a^{2n+1}+b^{2n+1}+c^{2n+1}$.

2. Shew that

$$(a+\omega b+\omega^2 c)^3+(a+\omega^2 b+\omega c)^3=(2a-b-c)(2b-c-a)(2c-a-b).$$

3. Shew that $(x+y)^n-x^n-y^n$ is divisible by $xy(x^2+xy+y^2)$, if n is an odd positive integer not a multiple of 3.

4. Shew that

$$a^3(bz-cy)^3+b^3(cx-az)^3+c^3(ay-bx)^3=3abc(bz-cy)(cx-az)(ay-bx).$$

5. Find the value of

$$(b-c)(c-a)(a-b)+(b-\omega c)(c-\omega a)(a-\omega b)+(b-\omega^2 c)(c-\omega^2 a)(a-\omega^2 b).$$

6. Shew that $(a^2+b^2+c^2-bc-ca-ab)(x^2+y^2+z^2-yz-zx-xy)$ may be put into the form $A^2+B^2+C^2-BC-CA-AB$.

7. Shew that $(a^2+ab+b^2)(x^2+xy+y^2)$ can be put into the form A^2+AB+B^2, and find the values of A and B.

Shew that

8. $\Sigma(a^2+2bc)^3-3(a^2+2bc)(b^2+2ca)(c^2+2ab)=(a^3+b^3+c^3-3abc)^2$.

9. $\Sigma(a^2-bc)^3-3(a^2-bc)(b^2-ca)(c^2-ab)=(a^3+b^3+c^3-3abc)^2$.

10. $(a^2+b^2+c^2)^3+2(bc+ca+ab)^3-3(a^2+b^2+c^2)(bc+ca+ab)^2$
$$=(a^3+b^3+c^3-3abc)^2.$$

If $a+b+c=0$, prove the identities in questions 11—17.

11. $2(a^4+b^4+c^4)=(a^2+b^2+c^2)^2$.

12. $a^5+b^5+c^5=-5abc(bc+ca+ab)$.

13. $a^6+b^6+c^6=3a^2b^2c^2-2(bc+ca+ab)^3$.

14. $3(a^2+b^2+c^2)(a^5+b^5+c^5)=5(a^3+b^3+c^3)(a^4+b^4+c^4)$.

15. $\dfrac{a^7+b^7+c^7}{7}=\dfrac{a^5+b^5+c^5}{5}\cdot\dfrac{a^2+b^2+c^2}{2}$.

16. $\left(\dfrac{b-c}{a}+\dfrac{c-a}{b}+\dfrac{a-b}{c}\right)\left(\dfrac{a}{b-c}+\dfrac{b}{c-a}+\dfrac{c}{a-b}\right)=9$.

17. $(b^2c+c^2a+a^2b-3abc)(bc^2+ca^2+ab^2-3abc)$
$$=(bc+ca+ab)^3+27a^2b^2c^2.$$

18. $25\{(y-z)^7+(z-x)^7+(x-y)^7\}\{(y-z)^3+(z-x)^3+(x-y)^3\}$
$$=21\{(y-z)^5+(z-x)^5+(x-y)^5\}^2.$$

19. $\{(y-z)^2+(z-x)^2+(x-y)^2\}^3-54(y-z)^2(z-x)^2(x-y)^2$
$$=2(y+z-2x)^2(z+x-2y)^2(x+y-2z)^2.$$

20. $(b-c)^6+(c-a)^6+(a-b)^6-3(b-c)^2(c-a)^2(a-b)^2$
$$=2(a^2+b^2+c^2-bc-ca-ab)^3.$$

21. $(b-c)^7+(c-a)^7+(a-b)^7$
$$=7(b-c)(c-a)(a-b)(a^2+b^2+c^2-bc-ca-ab)^2.$$

22. If $a+b+c=0$, and $x+y+z=0$, shew that

$$4(ax+by+cz)^3-3(ax+by+cz)(a^2+b^2+c^2)(x^2+y^2+z^2)$$
$$-2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y)=54abcxyz.$$

If $a+b+c+d=0$, shew that

23. $$\frac{a^5+b^5+c^5+d^5}{5}=\frac{a^3+b^3+c^3+d^3}{3}\cdot\frac{a^2+b^2+c^2+d^2}{2}.$$

24. $$(a^3+b^3+c^3+d^3)^2=9(bcd+cda+dab+abc)^2$$
$$=9(bc-ad)(ca-bd)(ab-cd).$$

25. If $2s=a+b+c$ and $2\sigma^2=a^2+b^2+c^2$, prove that
$$\Sigma(s-b)(s-c)(\sigma^2-a^2)+5abcs=(s^2-\sigma^2)(4s^2+\sigma^2).$$

26. Shew that $(x^3+6x^2y+3xy^2-y^3)^3+(y^3+6xy^2+3x^2y-x^3)^3$
$$=27xy(x+y)(x^2+xy+y^2)^3.$$

27. Shew that $\Sigma\dfrac{a^5}{(a-b)(a-c)(a-d)}$
$$=a^2+b^2+c^2+d^2+ab+ac+ad+bc+bd+cd.$$

28. Resolve into factors
$$2a^2b^2c^2+(a^3+b^3+c^3)abc+b^3c^3+c^3a^3+a^3b^3.$$

Elimination.

527. In Chapter XXXIII. we have seen that the eliminant of a system of linear equations may at once be written down in the form of a determinant. General methods of elimination applicable to equations of any degree will be found discussed in treatises on the Theory of Equations; in particular we may refer the student to Chapters IV. and VI. of Dr Salmon's *Lessons Introductory to the Modern Higher Algebra*, and to Chap. XIII. of Burnside and Panton's *Theory of Equations*.

These methods, though theoretically complete, are not always the most convenient in practice. We shall therefore only give a brief explanation of the general theory, and shall then illustrate by examples some methods of elimination that are more practically useful.

528. Let us first consider the elimination of one unknown quantity between two equations.

Denote the equations by $f(x)=0$ and $\phi(x)=0$, and suppose that, if necessary, the equations have been reduced to a form in which $f(x)$ and $\phi(x)$ represent rational integral functions of x. Since these two functions vanish simultaneously there must be some value of x which satisfies both the given equations; hence the eliminant expresses the condition that must hold between the coefficients in order that the equations may have a common root.

Suppose that $x=\alpha$, $x=\beta$, $x=\gamma$,... are the roots of $f(x)=0$, then *one* at least of the quantities $\phi(\alpha)$, $\phi(\beta)$, $\phi(\gamma)$, ... must be equal to zero; hence the eliminant is

$$\phi(\alpha)\,\phi(\beta)\,\phi(\gamma)\ldots\ldots=0.$$

The expression on the left is a symmetrical function of the roots of the equation $f(x)=0$, and its value can be found by the methods explained in treatises on the *Theory of Equations.*

529. We shall now explain three general methods of elimination: it will be sufficient for our purpose to take a simple example, but it will be seen that in each case the process is applicable to equations of any degree.

The principle illustrated in the following example is due to Euler.

Example. Eliminate x between the equations

$$ax^3+bx^2+cx+d=0,\ fx^2+gx+h=0.$$

Let $x+k$ be the factor corresponding to the root common to both equations, and suppose that

$$ax^3+bx^2+cx+d=(x+k)\,(ax^2+lx+m),$$

and
$$fx^2+gx+h=(x+k)\,(fx+n),$$

k, l, m, n being unknown quantities.

From these equations, we have identically

$$(ax^3+bx^2+cx+d)\,(fx+n)=(ax^2+lx+m)\,(fx^2+gx+h).$$

Equating coefficients of like powers of x, we obtain

$$\begin{aligned} fl \qquad\quad -an+ag-bf&=0,\\ gl+fm-bn+ah-cf&=0,\\ hl+gm-cn \qquad -df&=0,\\ hm-dn \qquad\qquad &=0. \end{aligned}$$

From these linear equations by eliminating the unknown quantities l, m, n, we obtain the determinant

$$\begin{vmatrix} f & 0 & a & ag-bf \\ g & f & b & ah-cf \\ h & g & c & -df \\ 0 & h & d & 0 \end{vmatrix}=0.$$

530. The eliminant of the equations $f(x)=0$, $\phi(x)=0$ can be very easily expressed as a determinant by Sylvester's *Dialytic Method of Elimination.* We shall take the same example as before.

Example. Eliminate x between the equations

$$ax^3+bx^2+cx+d=0, \quad fx^2+gx+h=0.$$

Multiply the first equation by x, and the second equation by x and x^2 in succession; we thus have 5 equations between which we can eliminate the 4 quantities x^4, x^3, x^2, x regarded as distinct variables. The equations are

$$\begin{aligned} ax^3+bx^2+cx+d&=0,\\ ax^4+bx^3+cx^2+dx\quad&=0,\\ fx^2+gx+h&=0,\\ fx^3+gx^2+hx\quad&=0,\\ fx^4+gx^3+hx^2\quad&=0.\end{aligned}$$

Hence the eliminant is

$$\begin{vmatrix} 0 & a & b & c & d \\ a & b & c & d & 0 \\ 0 & 0 & f & g & h \\ 0 & f & g & h & 0 \\ f & g & h & 0 & 0 \end{vmatrix}=0.$$

531. The principle of the following method is due to Bezout; it has the advantage of expressing the result as a determinant of lower order than either of the determinants obtained by the preceding methods. We shall choose the same example as before, and give Cauchy's mode of conducting the elimination.

Example. Eliminate x between the equations

$$ax^3+bx^2+cx+d=0, \quad fx^2+gx+h=0.$$

From these equations, we have

$$\frac{a}{f}=\frac{bx^2+cx+d}{gx^2+hx},$$

$$\frac{ax+b}{fx+g}=\frac{cx+d}{hx};$$

whence $$(ag-bf)x^2+(ah-cf)x-df=0,$$

and $$(ah-cf)x^2+(bh-cg-df)x-dg=0.$$

Combining these two equations with

$$fx^2+gx+h=0,$$

and regarding x^2 and x as distinct variables, we obtain for the eliminant

$$\begin{vmatrix} f & g & h \\ ag-bf & ah-cf & -df \\ ah-cf & bh-cg-df & -dg \end{vmatrix} = 0.$$

532. If we have two equations of the form $\phi_1(x, y) = 0$, $\phi_2(x, y) = 0$, then y may be eliminated by any of the methods already explained; in this case the eliminant will be a function of x.

If we have three equations of the form

$$\phi_1(x, y, z) = 0, \qquad \phi_2(x, y, z) = 0, \qquad \phi_3(x, y, z) = 0,$$

by eliminating z between the first and second equations, and then between the first and third, we obtain two equations of the form

$$\psi_1(x, y) = 0, \qquad \psi_2(x, y) = 0.$$

If we eliminate y from these equations we have a result of the form $f(x) = 0$.

By reasoning in this manner it follows that we can eliminate n variables between $n+1$ equations.

533. The general methods of elimination already explained may occasionally be employed with advantage, but the eliminants so obtained are rarely in a simple form, and it will often happen that the equations themselves suggest some special mode of elimination. This will be illustrated in the following examples.

Example 1. Eliminate l, m between the equations

$$lx + my = a, \; mx - ly = b, \; l^2 + m^2 = 1.$$

By squaring the first two equations and adding,

$$l^2x^2 + m^2x^2 + m^2y^2 + l^2y^2 = a^2 + b^2,$$

that is, $$(l^2 + m^2)(x^2 + y^2) = a^2 + b^2;$$

hence the eliminant is $$x^2 + y^2 = a^2 + b^2.$$

If $l = \cos\theta$, $m = \sin\theta$, the third equation is satisfied identically; that is, the eliminant of

$$x\cos\theta + y\sin\theta = a, \; x\sin\theta - y\cos\theta = b$$

is $$x^2 + y^2 = a^2 + b^2.$$

Example 2. Eliminate x, y, z between the equations

$$y^2+z^2=ayz,\ z^2+x^2=bzx,\ x^2+y^2=cxy.$$

We have $$\frac{y}{z}+\frac{z}{y}=a,\ \frac{z}{x}+\frac{x}{z}=b,\ \frac{x}{y}+\frac{y}{x}=c;$$

by multiplying together these three equations we obtain,

$$2+\frac{y^2}{z^2}+\frac{z^2}{y^2}+\frac{z^2}{x^2}+\frac{x^2}{z^2}+\frac{x^2}{y^2}+\frac{y^2}{x^2}=abc;$$

hence $$2+(a^2-2)+(b^2-2)+(c^2-2)=abc;$$

$$\therefore\ a^2+b^2+c^2-4=abc.$$

Example 3. Eliminate x, y between the equations

$$x^2-y^2=px-qy,\ 4xy=qx+py,\ x^2+y^2=1.$$

Multiplying the first equation by x, and the second by y, we obtain

$$x^3+3xy^2=p(x^2+y^2);$$

hence, by the third equation,

$$p=x^3+3xy^2.$$

Similarly $$q=3x^2y+y^3.$$

Thus $$p+q=(x+y)^3,\ p-q=(x-y)^3;$$

$$\therefore\ (p+q)^{\frac{2}{3}}+(p-q)^{\frac{2}{3}}=(x+y)^2+(x-y)^2$$

$$=2(x^2+y^2);$$

$$\therefore\ (p+q)^{\frac{2}{3}}+(p-q)^{\frac{2}{3}}=2.$$

Example 4. Eliminate x, y, z between the equations

$$\frac{y}{z}-\frac{z}{y}=a,\ \frac{z}{x}-\frac{x}{z}=b,\ \frac{x}{y}-\frac{y}{x}=c.$$

We have $$a+b+c=\frac{x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)}{xyz}$$

$$=\frac{(y-z)(z-x)(x-y)}{xyz}.$$

If we change the sign of x, the signs of b and c are changed, while the sign of a remains unaltered;

hence $$a-b-c=\frac{(y-z)(z+x)(x+y)}{xyz}.$$

Similarly, $$b-c-a=\frac{(y+z)(z-x)(x+y)}{xyz},$$

and $$c-a-b=\frac{(y+z)(z+x)(x-y)}{xyz}.$$

$$\therefore\ (a+b+c)(b+c-a)(c+a-b)(a+b-c)=-\frac{(y^2-z^2)^2(z^2-x^2)^2(x^2-y^2)^2}{x^4y^4z^4}$$
$$=-\left(\frac{y}{z}-\frac{z}{y}\right)^2\left(\frac{z}{x}-\frac{x}{z}\right)^2\left(\frac{x}{y}-\frac{y}{x}\right)^2$$
$$=-a^2b^2c^2.$$
$$\therefore\ 2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4+a^2b^2c^2=0.$$

EXAMPLES. XXXIV. c.

1. Eliminate m from the equations
$$m^2x-my+a=0,\quad my+x=0.$$

2. Eliminate m, n from the equations
$$m^2x-my+a=0,\quad n^2x-ny+a=0,\quad mn+1=0.$$

3. Eliminate m, n between the equations
$$mx-ny=a(m^2-n^2),\quad nx+my=2amn,\quad m^2+n^2=1.$$

4. Eliminate p, q, r from the equations
$$p+q+r=0,\quad a(qr+rp+pq)=2a-x,$$
$$apqr=y,\qquad qr=-1.$$

5. Eliminate x from the equations
$$ax^2-2a^2x+1=0,\quad a^2+x^2-3ax=0.$$

6. Eliminate m from the equations
$$y+mx=a(1+m),\quad my-x=a(1-m).$$

7. Eliminate x, y, z from the equations
$$yz=a^2,\quad zx=b^2,\quad xy=c^2,\quad x^2+y^2+z^2=d^2.$$

8. Eliminate p, q from the equations
$$x(p+q)=y,\quad p-q=k(1+pq),\quad xpq=a.$$

9. Eliminate x, y from the equations
$$x-y=a,\quad x^2-y^2=b^2,\quad x^3-y^3=c^3.$$

10. Eliminate x, y from the equations
$$x+y=a,\quad x^2+y^2=b^2,\quad x^4+y^4=c^4.$$

11. Eliminate x, y, z, u from the equations
$$x=by+cz+du,\quad y=cz+du+ax,$$
$$z=du+ax+by,\quad u=ax+by+cz.$$

12. Eliminate x, y, z from the equations
$$x+y+z=0,\quad x^2+y^2+z^2=a^2,$$
$$x^3+y^3+z^3=b^3,\quad x^5+y^5+z^5=c^5.$$

13. Eliminate x, y, z from the equations

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=a, \qquad \frac{x}{z}+\frac{y}{x}+\frac{z}{y}=b, \qquad \left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)=c.$$

14. Eliminate x, y, z from the equations

$$\frac{x^2(y+z)}{a^3}=\frac{y^2(z+x)}{b^3}=\frac{z^2(x+y)}{c^3}=\frac{xyz}{abc}=1.$$

15. Eliminate x, y from the equations

$$4(x^2+y^2)=ax+by, \quad 2(x^2-y^2)=ax-by, \quad xy=c^2.$$

16. Eliminate x, y, z from the equations

$$(y+z)^2=4a^2yz, \quad (z+x)^2=4b^2zx, \quad (x+y)^2=4c^2xy.$$

17. Eliminate x, y, z from the equations

$$(x+y-z)(x-y+z)=ayz, \quad (y+z-x)(y-z+x)=bzx,$$
$$(z+x-y)(z-x+y)=cxy.$$

18. Eliminate x, y from the equations

$$x^2y=a, \quad x(x+y)=b, \quad 2x+y=c.$$

19. Shew that $(a+b+c)^3-4(b+c)(c+a)(a+b)+5abc=0$
is the eliminant of

$$ax^2+by^2+cz^2=ax+by+cz=yz+zx+xy=0.$$

20. Eliminate x, y from the equations

$$ax^2+by^2=ax+by=\frac{xy}{x+y}=c.$$

21. Shew that $\quad b^3c^3+c^3a^3+a^3b^3=5a^2b^2c^2$
is the eliminant of

$$ax+yz=bc, \quad by+zx=ca, \quad cz+xy=ab, \quad xyz=abc.$$

22. Eliminate x, y, z from

$$x^2+y^2+z^2=x+y+z=1,$$
$$\frac{a}{x}(x-p)=\frac{b}{y}(y-q)=\frac{c}{z}(z-r).$$

23. Employ Bezout's method to eliminate x, y from

$$ax^3+bx^2y+cxy^2+dy^3=0, \quad a'x^3+b'x^2y+c'xy^2+d'y^3=0.$$