

19. Indefinite Integrals

Exercise 19.2

1. Question

Evaluate the following integrals:

$$\int (3x\sqrt{x} + 4\sqrt{x} + 5) dx$$

Answer

Given:

$$\int (3x\sqrt{x} + 4\sqrt{x} + 5) dx$$

By Splitting, we get,

$$\Rightarrow \int ((3x\sqrt{x})dx + (4\sqrt{x})dx + 5dx)$$

$$\Rightarrow \int 3x\sqrt{x}dx + \int 4\sqrt{x}dx + \int 5dx$$

$$\Rightarrow \int 3x^{\frac{3}{2}} dx + \int 4x^{\frac{1}{2}} dx + \int 5dx$$

By using the formula, $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$\Rightarrow \frac{3x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{4x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \int 5dx$$

$$\int kdx = kx + c$$

$$\Rightarrow \frac{3x^{\frac{5}{2}}}{5/2} + \frac{4x^{\frac{3}{2}}}{5/2} + 5x + c$$

$$\Rightarrow \frac{6}{5}x^{\frac{5}{2}} + \frac{4}{5}x^{3/2} + 5x + c$$

2. Question

Evaluate the following integrals:

$$\int \left(2^x + \frac{5}{x} - \frac{1}{x^{1/3}} \right) dx$$

Answer

Given:

$$\int \left(2^x + \frac{5}{x} - \frac{1}{x^{1/3}} \right) dx$$

By Splitting them, we get,

$$\Rightarrow \int 2^x dx + \int \left(\frac{5}{x} \right) dx - \int \frac{1}{x^{1/3}} dx$$

By using the formula,

$$\int a^x dx = \frac{a^x}{\log a}$$

$$\Rightarrow \frac{2^x}{\log 2} + 5 \int \left(\frac{1}{x}\right) dx - \int x^{-1/3} dx$$

By using the formula,

$$\int \left(\frac{1}{x}\right) dx = \log x$$

$$\Rightarrow \frac{2^x}{\log 2} + 5 \log x - \int x^{-1/3} dx$$

By using the formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{2^x}{\log 2} + 5 \log x - \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}$$

$$\Rightarrow \frac{2^x}{\log 2} + 5 \log x - \frac{x^{\frac{2}{3}}}{2/3}$$

$$\Rightarrow \frac{2^x}{\log 2} + 5 \log x - \frac{3}{2} x^{2/3} + c$$

3. Question

Evaluate the following integrals:

$$\int \left\{ \sqrt{x} (ax^2 + bx + c) \right\} dx$$

Answer

Given:

$$\int \{ \sqrt{x} (ax^2 + bx + c) \} dx$$

$$\Rightarrow \int (\sqrt{x} ax^2 + \sqrt{x} bx + \sqrt{x} c) dx$$

By Splitting, we get,

$$\Rightarrow a \int x^2 \times x^{\frac{1}{2}} dx + b \int x^1 \times x^{\frac{1}{2}} dx + c \int x^{1/2} dx$$

$$\Rightarrow a \int x^{\frac{5}{2}} dx + b \int x^{\frac{3}{2}} dx + c \int x^{\frac{1}{2}} dx$$

By using the formula

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{ax^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{bx^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{cx^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$\Rightarrow \frac{ax^{\frac{7}{2}}}{7/2} + \frac{bx^{\frac{5}{2}}}{5/2} + \frac{cx^{\frac{3}{2}}}{3/2} + c$$

4. Question

Evaluate the following integrals:

$$\int (2 - 3x)(3 + 2x)(1 - 2x)dx$$

Answer

Given:

$$\Rightarrow \int (2 - 3x)(3 + 2x)(1 - 2x)dx$$

By multiplying,

$$\Rightarrow \int (6 - 4x - 9x - 6x^2) dx$$

$$\Rightarrow \int (6 - 13x - 6x^2) dx$$

By Splitting, we get,

$$\Rightarrow \int 6dx - \int 13x dx - \int 6x^2 dx$$

By using the formulas,

$$\int x^n dx = \frac{x^{n+1}}{n+1} \text{ and}$$

$$\int kdx = kx + c$$

We get,

$$\Rightarrow 6x - \frac{13x^{1+1}}{1+1} - \frac{6x^{2+1}}{2+1} + c$$

$$\Rightarrow 6x - \frac{13x^2}{2} - \frac{6x^3}{3} + c$$

5. Question

Evaluate the following integrals:

$$\int \left(\frac{m}{x} + \frac{x}{m} + m^x + x^m + mx \right) dx$$

Answer

Given:

$$\int \left(\frac{m}{x} + \frac{x}{m} + m^x + x^m + mx \right) dx$$

By Splitting, we get,

$$\Rightarrow \int \frac{m}{x} dx + \int \frac{x}{m} dx + \int x^m dx + \int m^x dx + \int mx dx$$

By using formula,

$$\int \frac{1}{x} dx = \log x + c$$

$$\Rightarrow m \log x + \frac{1}{m} \int x dx + \int x^m dx + \int m^x dx + \int mx dx$$

By using the formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow m \log x + \frac{\frac{1}{m} x^{1+1}}{1+1} + \frac{x^{m+1}}{m+1} + \int m^x dx + \frac{mx^{1+1}}{1+1}$$

By using the formula,

$$\int a^x dx = \frac{a^x}{\log a}$$

$$\Rightarrow m \log x + \frac{\frac{1}{m} x^2}{2} + \frac{x^{m+1}}{m+1} + \frac{m^x}{\log m} + \frac{mx^2}{2} + c$$

6. Question

Evaluate the following integrals:

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx$$

Answer

Given:

$$\left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 dx$$

By applying $(a - b)^2 = a^2 - 2ab + b^2$

$$\Rightarrow \int \left((\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}} \right)^2 - 2(\sqrt{x}) \left(\frac{1}{\sqrt{x}} \right) \right) dx$$

$$\Rightarrow \int \left((\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}} \right)^2 - 2(\sqrt{x}) \left(\frac{1}{\sqrt{x}} \right) \right) dx$$

After computing,

$$\Rightarrow \int \left(x + \frac{1}{x} - 2 \right) dx$$

By Splitting, we get,

$$\Rightarrow \int x dx + \int \frac{1}{x} dx - 2 \int dx$$

By applying the formulas:

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int \left(\frac{1}{x} \right) dx = \log x$$

$$\int k dx = kx + c$$

We get,

$$\Rightarrow \frac{x^{1+1}}{1+1} + \log x - 2x + c = \frac{1}{2} x^2 + \log x - 2x + c$$

7. Question

Evaluate the following integrals:

$$\int \frac{(1+x)^3}{\sqrt{x}} dx$$

Answer

Given:

$$\int \frac{(1+x)^3}{\sqrt{x}} dx$$

Applying: $(a+b)^3 = a^3 + b^3 + 3ab^2 + 3a^2b$

$$\Rightarrow \int \frac{1+x^3+3x^2 \times 1+3 \times 1^2 \times x}{\sqrt{x}} dx$$

$$\Rightarrow \int \frac{1+x^3+3x^2+3x}{\sqrt{x}} dx$$

By Splitting, we get,

$$\Rightarrow \int \frac{1}{\sqrt{x}} dx + \int \frac{x^3}{\sqrt{x}} dx + \int \frac{3x^2}{\sqrt{x}} dx + \int \frac{3x}{\sqrt{x}} dx$$

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x^3 \times x^{-\frac{1}{2}} dx + \int 3x^2 \times x^{-\frac{1}{2}} dx + \int 3x \times x^{-\frac{1}{2}} dx$$

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x^{\frac{5}{2}} dx + 3 \int x^{\frac{3}{2}} dx + 3 \int x^{\frac{1}{2}} dx$$

By applying formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + 3 \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{3x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

8. Question

Evaluate the following integrals:

$$\int \left\{ x^2 + e^{\log x} + \left(\frac{e}{2} \right)^x \right\} dx$$

Answer

Given:

$$\int \left\{ x^2 + e^{\log x} + \left(\frac{e}{2} \right)^x \right\} dx$$

By Splitting, we get,

$$\Rightarrow \int x^2 dx + \int e^{\log x} dx + \int \left(\frac{e}{2} \right)^x dx$$

By applying formula,

$$\begin{aligned}\int x^n dx &= \frac{x^{n+1}}{n+1} \\ \Rightarrow \frac{x^{2+1}}{2+1} + \int e^{\log_e x} dx + \int \left(\frac{e}{2}\right)^x dx \\ \Rightarrow \frac{x^3}{3} + \int x dx + \frac{1}{\log\left(\frac{e}{2}\right)} \log\left(\frac{e}{2}\right)^x \\ \Rightarrow \frac{x^3}{3} + \int x dx + \frac{1}{\log\left(\frac{e}{2}\right)} \log\left(\frac{e}{2}\right)^x \\ \Rightarrow \frac{x^3}{3} + \frac{x^2}{2} + \frac{1}{\log\left(\frac{e}{2}\right)} \log\left(\frac{e}{2}\right)^x + c\end{aligned}$$

9. Question

Evaluate the following integrals:

$$\int (x^e + e^x + e^e) dx$$

Answer

Given:

$$\int (x^e + e^x + e^e) dx$$

By Splitting, we get,

$$\Rightarrow \int x^e dx + \int e^x dx + \int e^e dx$$

By using the formula,

$$\begin{aligned}\int x^n dx &= \frac{x^{n+1}}{n+1} \\ \Rightarrow \frac{x^{e+1}}{e+1} + \int e^x dx + \int e^e dx\end{aligned}$$

By applying the formula,

$$\begin{aligned}\int a^x dx &= \frac{a^x}{\log a} \\ \Rightarrow \frac{x^{e+1}}{e+1} + \frac{e^x}{\log_e e} + \int e^e dx\end{aligned}$$

We know that,

$$\begin{aligned}\int k dx &= kx + c \\ \Rightarrow \frac{x^{e+1}}{e+1} + \frac{e^x}{\log_e e} + e^e x + c \\ \Rightarrow \frac{x^{e+1}}{e+1} + \frac{e^x}{\log_e e} + e^e x + c\end{aligned}$$

10. Question

Evaluate the following integrals:

$$\int \sqrt{x} \left(x^3 - \frac{2}{x} \right) dx$$

Answer

Given:

$$\int \sqrt{x} \left(x^3 - \frac{2}{x} \right) dx$$

Opening the bracket, we get,

$$\Rightarrow \int \left(x^{\frac{1}{2}} \times x^3 - x^{\frac{1}{2}} \times \frac{2}{x} \right) dx$$

$$\Rightarrow \int \left(x^{\frac{1}{2}+3} - x^{\frac{1}{2}-1} \times 2 \right) dx$$

$$\Rightarrow \int \left(x^{\frac{7}{2}} - 2x^{-\frac{1}{2}} \right) dx$$

By multiplying,

$$\Rightarrow \int x^{\frac{7}{2}} dx - 2 \int x^{-\frac{1}{2}} dx$$

By applying the formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{x^{\frac{7}{2}+1}}{\frac{7}{2}+1} - 2 \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{9}{2}}}{\frac{9}{2}} - 2 \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$\Rightarrow \frac{2x^{\frac{9}{2}}}{9} - 4x^{\frac{1}{2}} + c$$

11. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{x}} \left(1 + \frac{1}{x} \right) dx$$

Answer

Given:

$$\int \frac{1}{\sqrt{x}} \left\{ 1 + \frac{1}{x} \right\} dx$$

By multiplying $\frac{1}{\sqrt{x}}$ with inside brackets,

$$\Rightarrow \int \left\{ \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} \times \frac{1}{x} \right\} dx$$

$$\Rightarrow \int \left\{ \frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}}} \times \frac{1}{x} \right\} dx$$

$$\Rightarrow \int \left\{ \frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{1}{2}+1}} \right\} dx$$

$$\Rightarrow \int \left\{ \frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{3}{2}}} \right\} dx$$

By Splitting them, we get,

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x^{-\frac{3}{2}} dx$$

By applying the formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + c$$

$$\Rightarrow 2x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + c$$

12. Question

Evaluate the following integrals:

$$\int \frac{x^6 + 1}{x^2 + 1} dx$$

Answer

Given:

$$\int \frac{x^6 + 1}{x^2 + 1} dx$$

By applying: $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

$$\Rightarrow \int \frac{(x^2)^3 + (1)^3}{x^2 + 1} dx$$

$$\Rightarrow \int \frac{(x^2 + 1)((x^2)^2 + (1)^2 - x^2 \times 1)}{(x^2 + 1)} dx$$

$$\Rightarrow \int \frac{(x^2 + 1)(x^4 + 1 - x^2)}{x^2 + 1} dx$$

$$\Rightarrow \int (x^4 + 1 - x^2) dx$$

By Splitting

$$\Rightarrow \int x^4 dx + 1 \int dx - \int x^2 dx$$

By using the formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int k dx = kx + c$$

$$\Rightarrow \frac{x^{5+1}}{5+1} + x - \frac{x^{3+1}}{3+1} + c$$

$$\Rightarrow \frac{x^6}{6} + x - \frac{x^4}{4} + c$$

13. Question

Evaluate the following integrals:

$$\int \frac{x^{-1/3} + \sqrt{x} + 2}{\sqrt[3]{x}} dx$$

Answer

Given:

$$\int \frac{x^{-\frac{1}{3}} + \sqrt{x} + 2}{\sqrt[3]{x}} dx$$

By Splitting them,

$$\Rightarrow \int \frac{x^{-\frac{1}{3}}}{\sqrt[3]{x}} dx + \int \frac{\sqrt{x}}{\sqrt[3]{x}} dx + \int \frac{2}{\sqrt[3]{x}} dx$$

$$\Rightarrow \int x^{-\frac{1}{3}} \times x^{-\frac{1}{3}} dx + \int x^{\frac{1}{2}} \times x^{-\frac{1}{3}} dx + 2 \int x^{-\frac{1}{3}} dx$$

$$\Rightarrow \int x^{-\frac{1}{3}-\frac{1}{3}} dx + \int x^{\frac{1}{2}-\frac{1}{3}} dx + 2 \int x^{-\frac{1}{3}} dx$$

$$\Rightarrow \int x^{-\frac{2}{3}} dx + \int x^{\frac{5}{6}} dx + 2 \int x^{-\frac{1}{3}} dx$$

By applying the formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

We get,

$$\Rightarrow \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + \frac{x^{\frac{5}{6}+1}}{\frac{5}{6}+1} + \frac{2x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c$$

$$\Rightarrow \frac{x^{\frac{1}{3}}}{\frac{1}{3}} + \frac{x^{\frac{11}{6}}}{\frac{11}{6}} + \frac{2x^{\frac{2}{3}}}{\frac{2}{3}} + c$$

$$\Rightarrow 3x^{\frac{1}{3}} + \frac{6x^{\frac{11}{6}}}{11} + 3x^{\frac{2}{3}} + c$$

14. Question

Evaluate the following integrals:

$$\int \frac{(1+\sqrt{x})^2}{\sqrt{x}} dx$$

Answer

Given:

$$\int \frac{(1 + \sqrt{x})^2}{\sqrt{x}} dx$$

By applying $(a + b)^2 = a^2 + b^2 + 2ab$

$$\Rightarrow \int \frac{(1)^2 + (\sqrt{x})^2 + 2 \times 1 \times \sqrt{x}}{\sqrt{x}} dx$$

$$\Rightarrow \int \frac{1 + x + 2\sqrt{x}}{\sqrt{x}} dx$$

By Splitting, we get,

$$\Rightarrow \int \left(\frac{1}{\sqrt{x}} + \frac{x}{\sqrt{x}} + \frac{2\sqrt{x}}{\sqrt{x}} \right) dx$$

$$\Rightarrow \int x^{-\frac{1}{2}} dx + \int x \times x^{-\frac{1}{2}} dx + 2 \int dx$$

$$\Rightarrow \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \int x^{1-\frac{1}{2}} dx + 2x + c$$

$$\Rightarrow \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \int x^{\frac{1}{2}} dx + 2x + c$$

$$\Rightarrow 2x^{\frac{1}{2}} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2x + c$$

$$\Rightarrow 2x^{\frac{1}{2}} + \frac{2x^{\frac{3}{2}}}{3} + 2x + c$$

15. Question

Evaluate the following integrals:

$$\int \sqrt{x}(3 - 5x) dx$$

Answer

Given:

$$\int \sqrt{x}(3 - 5x) dx$$

By multiplying $\sqrt{x}$ inside the bracket we get,

$$\Rightarrow \int (3\sqrt{x} - 5x\sqrt{x}) dx$$

$$\Rightarrow \int \left(3x^{\frac{1}{2}} - 5x^1 \times x^{\frac{1}{2}} \right) dx$$

$$\Rightarrow \int (3x^{\frac{1}{2}} - 5x^{1+\frac{1}{2}}) dx$$

$$\Rightarrow \int (3x^{\frac{1}{2}} - 5x^{\frac{3}{2}}) dx$$

By Splitting, we get,

$$\Rightarrow 3 \int x^{\frac{1}{2}} dx - 5 \int x^{\frac{3}{2}} dx$$

By using the formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{5x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$$

$$\Rightarrow \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{5x^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$\Rightarrow 2x^{\frac{3}{2}} - 2x^{\frac{5}{2}} + c$$

16. Question

Evaluate the following integrals:

$$\int \frac{(x+1)(x-2)}{\sqrt{x}} dx$$

Answer

Given:

$$\int \frac{(x+1)(x-2)}{\sqrt{x}} dx$$

$$\Rightarrow \int \frac{x^2 - 2x + x - 2}{\sqrt{x}} dx$$

$$\Rightarrow \int \frac{x^2 - x - 2}{\sqrt{x}} dx$$

By Splitting,

$$\Rightarrow \int \frac{x^2}{\sqrt{x}} dx - \int \frac{x}{\sqrt{x}} dx - \int \frac{2}{\sqrt{x}} dx$$

$$\Rightarrow \int x^2 \times x^{-\frac{1}{2}} dx - \int x \times x^{-\frac{1}{2}} dx - 2 \int x^{-\frac{1}{2}} dx$$

$$\Rightarrow \int x^{2-\frac{1}{2}} dx - \int x^{1-\frac{1}{2}} dx - 2 \int x^{-\frac{1}{2}} dx$$

$$\Rightarrow \int x^{\frac{3}{2}} dx - \int x^{\frac{1}{2}} dx - 2 \int x^{-\frac{1}{2}} dx$$

By applying the formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{2x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$\Rightarrow \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$\Rightarrow \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + c$$

17. Question

Evaluate the following integrals:

$$\int \frac{x^5 + x^{-2} + 2}{x^2} dx$$

Answer

Given:

$$\int \frac{x^5 + x^{-2} + 2}{x^2} dx$$

By Splitting, we get,

$$\Rightarrow \int \left(\frac{x^5}{x^2} + \frac{x^{-2}}{x^2} + \frac{2}{x^2} \right) dx$$

$$\Rightarrow \int (x^5 \times x^{-2} + x^{-2} \times x^{-2} + 2 \times x^{-2}) dx$$

By applying,

$$\Rightarrow \int (x^{5-2} + x^{-2-2} + 2x^{-2}) dx$$

$$\Rightarrow \int (x^3 + x^{-4} + 2x^{-2}) dx$$

By Splitting, we get,

$$\Rightarrow \int x^3 dx + \int x^{-4} dx + 2 \int x^{-2} dx$$

By applying the formula,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{x^{3+1}}{3+1} + \frac{x^{-4+1}}{-4+1} + \frac{2x^{-2+1}}{-2+1} + c$$

$$\Rightarrow \frac{x^4}{4} + \frac{x^{-3}}{-3} + \frac{2x^{-1}}{-1} + c$$

18. Question

Evaluate the following integrals:

$$\int (3x + 4)^2 dx$$

Answer

Given:

$$\int (3x + 4)^2 dx$$

By applying,

$$(a + b)^2 = a^2 + b^2 + 2ab$$

$$\Rightarrow \int ((3x)^2 + 4^2 + 2 \times 3x \times 4) dx$$

$$\Rightarrow \int (9x^2 + 16 + 24x) dx$$

By Splitting, we get,

$$\Rightarrow \int 9x^2 dx + \int 16 dx + \int 24x dx$$

$$\Rightarrow 9 \int x^2 + 16 \int dx + 24 \int x dx$$

By applying,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int k dx = kx + c$$

$$\Rightarrow \frac{9x^{2+1}}{2+1} + 16x + \frac{24x^{1+1}}{1+1} + c$$

$$\Rightarrow \frac{9}{3}x^3 + 16x + \frac{24}{2}x^2 + c$$

$$\Rightarrow 3x^3 + 16x + 12x^2 + c$$

19. Question

Evaluate the following integrals:

$$\int \frac{2x^4 + 7x^3 + 6x^2}{x^2 + 2x} dx$$

Answer

Given:

$$\int \frac{2x^4 + 7x^3 + 6x^2}{x^2 + 2x} dx$$

Take x is common on both numerator and denominator,

$$\Rightarrow \int \frac{x(2x^3 + 7x^2 + 6x)}{x(x+2)} dx$$

$$\Rightarrow \int \frac{2x^3 + 7x^2 + 6x}{x+2} dx$$

Splitting $7x^2$ into $4x^2$ and $3x^2$

$$\Rightarrow \int \frac{2x^3 + 4x^2 + 3x^2 + 6x}{x+2} dx$$

Common the $2x^2$ from first two elements and $3x$ from next elements,

$$\Rightarrow \int \frac{2x^2(x+2) + 3x(x+2)}{x+2} dx$$

Now common the $x+2$ from the elements

$$\Rightarrow \int \frac{(x+2)(2x^2+3x)}{x+2} dx$$

$$\Rightarrow \int (2x^2+3x) dx$$

Now Splitting, we get,

$$\Rightarrow \int 2x^2 dx + \int 3x dx$$

Now applying the formula,

$$\Rightarrow \frac{2x^{2+1}}{2+1} + \frac{3x^{1+1}}{1+1} + c$$

$$\Rightarrow \frac{2x^3}{3} + 3x + c$$

20. Question

Evaluate the following integrals:

$$\int \frac{5x^4 + 12x^3 + 7x^2}{x^2 + x} dx$$

Answer

Given:

$$\int \frac{5x^4 + 12x^3 + 7x^2}{x^2 + x} dx$$

Now split $12x^3$ into $7x^3$ and $5x^3$

$$\Rightarrow \int \frac{5x^4 + 7x^3 + 5x^3 + 7x^2}{x^2 + x} dx$$

Now common $5x^3$ from two elements $7x$ from other two elements,

$$\Rightarrow \int \frac{5x^2(x+1) + 7x(x+1)}{x^2 + x} dx$$

$$\Rightarrow \frac{\int (5x^2 + 7x)(x+1)}{x(x+1)} dx$$

$$\Rightarrow \int (5x^2 + 7x) dx$$

Now Splitting, we get,

$$\Rightarrow \int 5x^2 dx + \int 7x dx$$

$$\Rightarrow \frac{5x^{2+1}}{2+1} + \frac{7x^{1+1}}{1+1} + c$$

$$\Rightarrow \frac{5x^3}{3} + \frac{7x^2}{2} + c$$

21. Question

Evaluate the following integrals:

$$\int \frac{\sin^2 x}{1 + \cos x} dx$$

Answer

Given:

$$\int \frac{\sin^2 x}{1 + \cos x} dx$$

We know that,

$$\sin^2 x = 1 - \cos^2 x$$

$$\Rightarrow \int \frac{1 - \cos^2 x}{1 + \cos x} dx$$

We treat $1 - \cos^2 x$ as $a^2 - b^2 = (a + b)(a - b)$

$$\Rightarrow \int \frac{(1)^2 - (\cos x)^2}{1 + \cos x} dx$$

$$\Rightarrow \int \frac{(1 + \cos x)(1 - \cos x)}{1 + \cos x} dx$$

$$\Rightarrow \int (1 - \cos x) dx$$

By Splitting, we get,

$$\Rightarrow \int dx - \int \cos x dx$$

We know that,

$$\int k dx = kx + c$$

$$\int \cos x dx = \sin x$$

$$\Rightarrow x - \sin x + c$$

22. Question

Evaluate the following integrals:

$$\int (\sec^2 x + \operatorname{cosec}^2 x) dx$$

Answer

Given:

$$\int (\sec^2 x + \operatorname{cosec}^2 x) dx$$

By Splitting, we get,

$$\Rightarrow \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx$$

By applying the formula,

$$\int \sec^2 x dx = \tan x$$

$$\int \sec^2 x dx = \tan x + c$$

$$\Rightarrow \tan x - \cot x + c$$

23. Question

Evaluate the following integrals:

$$\int \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x} dx$$

Answer

Given:

$$\int \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x} dx$$

By Splitting, we get,

$$\Rightarrow \int \left(\frac{\sin^3 x}{\sin^2 x \cos^2 x} - \frac{\cos^3 x}{\sin^2 x \cos^2 x} \right) dx$$

By cancelling the $\sin^2 x$ on first and $\cos^2 x$ on second,

$$\Rightarrow \int \left(\frac{\sin x}{\cos^2 x} - \frac{\cos x}{\sin^2 x} \right) dx$$

We know that,

$$\frac{\sin x}{\cos x} = \tan x$$

$$\frac{\cos x}{\sin x} = \cot x$$

$$\frac{1}{\cos x} = \sec x$$

$$\frac{1}{\sin x} = \csc x$$

$$\Rightarrow \int (\tan x \sec x - \cot x \csc x) dx$$

We know that,

$$\int \tan x \sec x dx = \sec x$$

$$\int \cot x \csc x dx = -\cot x$$

$$\Rightarrow \sec x - (-\cot x) + c$$

$$\Rightarrow \sec x + \cot x + c$$

24. Question

Evaluate the following integrals:

$$\int \frac{5 \cos^3 x + 6 \sin^3 x}{2 \sin^2 x \cos^2 x} dx$$

Answer

Given:

$$\int \frac{5\cos^3 x + 6\sin^3 x}{2\sin^2 x \cos^2 x} dx$$

By Splitting we get,

$$\begin{aligned} &\Rightarrow \int \frac{5\cos^3 x}{2\sin^2 x \cos^2 x} dx + \int \frac{6\sin^3 x}{2\sin^2 x \cos^2 x} dx \\ &\Rightarrow \frac{5}{2} \int \frac{\cos x \cos^2 x}{\sin^2 x \cos^2 x} dx + 3 \int \frac{\sin^2 x \sin^1 x}{\sin^2 x \cos^2 x} dx \\ &\Rightarrow \frac{5}{2} \int \frac{\cos x}{\sin^2 x} dx + 3 \int \frac{\sin^1 x}{1 \cos^2 x} dx \end{aligned}$$

We know that,

$$\int 1 \frac{\cos x}{\sin x} dx = \cot x$$

$$\int \frac{\sin x}{\cos x} dx = \tan x$$

$$\int 1 \frac{1}{\sin x} dx = \sec x$$

$$\int 1 \frac{1}{\sin x} dx = \operatorname{cosec} x$$

$$\Rightarrow \frac{5}{2} \int \cot x \operatorname{cosec} x dx + 3 \int \sec x \tan x dx$$

We know that,

$$\int \cot x \operatorname{cosec} x dx = -\operatorname{cosec} x$$

$$\int \sec x \tan x dx = \sec x$$

$$\Rightarrow \frac{5}{2} (-\operatorname{cosec} x) + 3 \sec x + c$$

$$I = -\frac{5}{2} \operatorname{cosec} x + 3 \sec x + c$$

25. Question

Evaluate the following integrals:

$$\int (\tan x + \cot x)^2 dx$$

Answer

Given:

$$I = \int (\tan x + \cot x)^2 dx$$

$$\Rightarrow \int (\tan^2 x + \cot^2 x + 2 \tan x \cot x) dx$$

We know that,

$$\tan^2 x = \sec^2 x - 1$$

$$\cot^2 x = \operatorname{cosec}^2 x - 1$$

$$\tan x = \frac{1}{\cot x}$$

$$\Rightarrow \int \left(\sec^2 x - 1 + \operatorname{cosec}^2 x - 1 + \frac{2}{\cot x} \cot x \right) dx$$

$$\Rightarrow \int (\sec^2 x + \operatorname{cosec}^2 x - 2 + 2) dx$$

$$\Rightarrow \int (\sec^2 x + \operatorname{cosec}^2 x) dx$$

$$\Rightarrow \int \sec^2 x + \int \operatorname{cosec}^2 x dx$$

We know that,

$$\int \sec^2 x dx = \tan x$$

$$\int \operatorname{cosec}^2 x dx = -\cot x$$

$$I = \tan x - \cot x - c$$

26. Question

Evaluate the following integrals:

$$\int \frac{1 - \cos 2x}{1 + \cos 2x} dx$$

Answer

$$\text{Let } I = \int \frac{1 - \cos 2x}{1 + \cos 2x} dx$$

$$\text{We know } \cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\text{Hence, in the numerator, we can write } 1 - \cos 2x = 2\sin^2 x$$

$$\text{In the denominator, we can write } 1 + \cos 2x = 2\cos^2 x$$

Therefore, we can write the integral as

$$I = \int \frac{2\sin^2 x}{2\cos^2 x} dx$$

$$\Rightarrow I = \int \frac{\sin^2 x}{\cos^2 x} dx$$

$$\Rightarrow I = \int \tan^2 x dx$$

$$\Rightarrow I = \int (\sec^2 x - 1) dx \quad [\because \sec^2 \theta - \tan^2 \theta = 1]$$

$$\Rightarrow I = \int \sec^2 x dx - \int dx$$

$$\text{Recall } \int \sec^2 x dx = \tan x + c \text{ and } \int dx = x + c$$

$$\therefore I = \tan x - x + c$$

$$\text{Thus, } \int \frac{1 - \cos 2x}{1 + \cos 2x} dx = \tan x - x + c$$

27. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{1 - \cos x} dx$$

Answer

$$\text{Let } I = \int \frac{\cos x}{1 - \cos x} dx$$

On multiplying and dividing $(1 + \cos x)$, we can write the integral as

$$I = \int \frac{\cos x}{1 - \cos x} \left(\frac{1 + \cos x}{1 + \cos x} \right) dx$$

$$\Rightarrow I = \int \frac{\cos x (1 + \cos x)}{(1 - \cos x)(1 + \cos x)} dx$$

$$\Rightarrow I = \int \frac{\cos x + \cos^2 x}{1 - \cos^2 x} dx$$

$$\Rightarrow I = \int \frac{\cos x + \cos^2 x}{\sin^2 x} dx \quad [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$\Rightarrow I = \int \left(\frac{\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} \right) dx$$

$$\Rightarrow I = \int \left(\frac{1}{\sin x} \times \frac{\cos x}{\sin x} + \frac{\cos^2 x}{\sin^2 x} \right) dx$$

$$\Rightarrow I = \int (\operatorname{cosec} x \cot x + \cot^2 x) dx$$

$$\Rightarrow I = \int (\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x - 1) dx \quad [\because \operatorname{cosec}^2 \theta - \cot^2 \theta = 1]$$

$$\Rightarrow I = \int \operatorname{cosec} x \cot x dx + \int \operatorname{cosec}^2 x dx - \int dx$$

Recall $\int \operatorname{cosec}^2 x dx = -\cot x + c$ and $\int dx = x + c$

We also have $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$

$$\therefore I = -\operatorname{cosec} x - \cot x - x + c$$

$$\text{Thus, } \int \frac{\cos x}{1 - \cos x} dx = -\operatorname{cosec} x - \cot x - x + c$$

28. Question

Evaluate the following integrals:

$$\int \frac{\cos^2 x - \sin^2 x}{\sqrt{1 + \cos 4x}} dx$$

Answer

$$\text{Let } I = \int \frac{\cos^2 x - \sin^2 x}{\sqrt{1 + \cos 4x}} dx$$

We know $\cos 2\theta = 2\cos^2 \theta - 1 = \cos^2 \theta - \sin^2 \theta$

Hence, in the numerator, we can write $\cos^2 x - \sin^2 x = \cos 2x$

In the denominator, we can write $4x = 2 \times 2x$

$$\Rightarrow 1 + \cos 4x = 1 + \cos(2 \times 2x)$$

$$\Rightarrow 1 + \cos 4x = 2\cos^2 2x$$

Therefore, we can write the integral as

$$I = \int \frac{\cos 2x}{\sqrt{2} \cos^2 2x} dx$$

$$\Rightarrow I = \int \frac{\cos 2x}{\sqrt{2} \cos 2x} dx$$

$$\Rightarrow I = \int \frac{1}{\sqrt{2}} dx$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \int dx$$

Recall $\int dx = x + c$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \times x + c$$

$$\therefore I = \frac{x}{\sqrt{2}} + c$$

Thus, $\int \frac{\cos^2 x - \sin^2 x}{\sqrt{1 + \cos 4x}} dx = \frac{x}{\sqrt{2}} + c$

29. Question

Evaluate the following integrals:

$$\int \frac{1}{1 - \cos x} dx$$

Answer

$$\text{Let } I = \int \frac{1}{1 - \cos x} dx$$

On multiplying and dividing $(1 + \cos x)$, we can write the integral as

$$I = \int \frac{1}{1 - \cos x} \left(\frac{1 + \cos x}{1 + \cos x} \right) dx$$

$$\Rightarrow I = \int \frac{1 + \cos x}{(1 - \cos x)(1 + \cos x)} dx$$

$$\Rightarrow I = \int \frac{1 + \cos x}{1 - \cos^2 x} dx$$

$$\Rightarrow I = \int \frac{1 + \cos x}{\sin^2 x} dx \quad [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$\Rightarrow I = \int \left(\frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x} \right) dx$$

$$\Rightarrow I = \int \left(\frac{1}{\sin^2 x} + \frac{1}{\sin x} \times \frac{\cos x}{\sin x} \right) dx$$

$$\Rightarrow I = \int (\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x) dx$$

$$\Rightarrow I = \int \operatorname{cosec}^2 x dx + \int \operatorname{cosec} x \cot x dx$$

Recall $\int \operatorname{cosec}^2 x dx = -\cot x + c$

We also have $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$

$$\therefore I = -\cot x - \operatorname{cosec} x + c$$

Thus, $\int \frac{1}{1-\cos x} dx = -\cot x - \operatorname{cosec} x + c$

30. Question

Evaluate the following integrals:

$$\int \frac{1}{1-\sin x} dx$$

Answer

$$\text{Let } I = \int \frac{1}{1-\sin x} dx$$

On multiplying and dividing $(1 + \sin x)$, we can write the integral as

$$I = \int \frac{1}{1-\sin x} \left(\frac{1+\sin x}{1+\sin x} \right) dx$$

$$\Rightarrow I = \int \frac{1+\sin x}{(1-\sin x)(1+\sin x)} dx$$

$$\Rightarrow I = \int \frac{1+\sin x}{1-\sin^2 x} dx$$

$$\Rightarrow I = \int \frac{1+\sin x}{\cos^2 x} dx \quad [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$\Rightarrow I = \int \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \right) dx$$

$$\Rightarrow I = \int \left(\frac{1}{\cos^2 x} + \frac{1}{\cos x} \times \frac{\sin x}{\cos x} \right) dx$$

$$\Rightarrow I = \int (\sec^2 x + \sec x \tan x) dx$$

$$\Rightarrow I = \int \sec^2 x dx + \int \sec x \tan x dx$$

Recall $\int \sec^2 x dx = \tan x + c$

We also have $\int \sec x \tan x dx = \sec x + c$

$$\therefore I = \tan x + \sec x + c$$

$$\text{Thus, } \int \frac{1}{1-\sin x} dx = \tan x + \sec x + c$$

31. Question

Evaluate the following integrals:

$$\int \frac{\tan x}{\sec x + \tan x} dx$$

Answer

$$\text{Let } I = \int \frac{\tan x}{\sec x + \tan x} dx$$

On multiplying and dividing $(\sec x - \tan x)$, we can write the integral as

$$I = \int \frac{\tan x}{\sec x + \tan x} \left(\frac{\sec x - \tan x}{\sec x - \tan x} \right) dx$$

$$\Rightarrow I = \int \frac{\tan x (\sec x - \tan x)}{(\sec x + \tan x)(\sec x - \tan x)} dx$$

$$\Rightarrow I = \int \frac{\sec x \tan x - \tan^2 x}{\sec^2 x - \tan^2 x} dx$$

$$\Rightarrow I = \int (\sec x \tan x - \tan^2 x) dx \quad [\because \sec^2 \theta - \tan^2 \theta = 1]$$

$$\Rightarrow I = \int (\sec x \tan x - (\sec^2 x - 1)) dx$$

$$\Rightarrow I = \int (\sec x \tan x - \sec^2 x + 1) dx$$

$$\Rightarrow I = \int \sec x \tan x dx - \int \sec^2 x dx + \int dx$$

$$\text{Recall } \int \sec^2 x dx = \tan x + c \text{ and } \int dx = x + c$$

$$\text{We also have } \int \sec x \tan x dx = \sec x + c$$

$$\therefore I = \sec x - \tan x + x + c$$

$$\text{Thus, } \int \frac{\tan x}{\sec x + \tan x} dx = \sec x - \tan x + x + c$$

32. Question

Evaluate the following integrals:

$$\int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x} dx$$

Answer

$$\text{Let } I = \int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x} dx$$

On multiplying and dividing $(\operatorname{cosec} x + \cot x)$, we can write the integral as

$$I = \int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x} \left(\frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x + \cot x} \right) dx$$

$$\Rightarrow I = \int \frac{\operatorname{cosec} x (\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x - \cot x)(\operatorname{cosec} x + \cot x)} dx$$

$$\Rightarrow I = \int \frac{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x}{\operatorname{cosec}^2 x - \cot^2 x} dx$$

$$\Rightarrow I = \int (\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x) dx \quad [\because \operatorname{cosec}^2 \theta - \cot^2 \theta = 1]$$

$$\Rightarrow I = \int \operatorname{cosec}^2 x dx + \int \operatorname{cosec} x \cot x dx$$

$$\text{Recall } \int \operatorname{cosec}^2 x dx = -\cot x + c$$

$$\text{We also have } \int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$$

$$\therefore I = -\cot x - \operatorname{cosec} x + c$$

$$\text{Thus, } \int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x} dx = -\cot x - \operatorname{cosec} x + c$$

33. Question

Evaluate the following integrals:

$$\int \frac{1}{1 + \cos 2x} dx$$

Answer

$$\text{Let } I = \int \frac{1}{1+\cos 2x} dx$$

$$\text{We know } \cos 2\theta = 2\cos^2\theta - 1$$

$$\text{Hence, in the denominator, we can write } 1 + \cos 2x = 2\cos^2 x$$

Therefore, we can write the integral as

$$I = \int \frac{1}{2\cos^2 x} dx$$

$$\Rightarrow I = \frac{1}{2} \int \frac{1}{\cos^2 x} dx$$

$$\Rightarrow I = \frac{1}{2} \int \sec^2 x dx$$

$$\text{Recall } \int \sec^2 x dx = \tan x + c$$

$$\therefore I = \frac{1}{2} \tan x + c$$

$$\text{Thus, } \int \frac{1}{1+\cos 2x} dx = \frac{1}{2} \tan x + c$$

34. Question

Evaluate the following integrals:

$$\int \frac{1}{1-\cos 2x} dx$$

Answer

$$\text{Let } I = \int \frac{1}{1-\cos 2x} dx$$

$$\text{We know } \cos 2\theta = 1 - 2\sin^2\theta$$

$$\text{Hence, in the denominator, we can write } 1 - \cos 2x = 2\sin^2 x$$

Therefore, we can write the integral as

$$I = \int \frac{1}{2\sin^2 x} dx$$

$$\Rightarrow I = \frac{1}{2} \int \frac{1}{\sin^2 x} dx$$

$$\Rightarrow I = \frac{1}{2} \int \operatorname{cosec}^2 x dx$$

$$\text{Recall } \int \operatorname{cosec}^2 x dx = -\cot x + c$$

$$\Rightarrow I = \frac{1}{2} (-\cot x) + c$$

$$\therefore I = -\frac{1}{2} \cot x + c$$

$$\text{Thus, } \int \frac{1}{1-\cos 2x} dx = -\frac{1}{2} \cot x + c$$

35. Question

Evaluate the following integrals:

$$\int \tan^{-1} \left(\frac{\sin 2x}{1 + \cos 2x} \right) dx$$

Answer

$$\text{Let } I = \int \tan^{-1} \left(\frac{\sin 2x}{1 + \cos 2x} \right) dx$$

$$\text{We know } \cos 2\theta = 2\cos^2\theta - 1$$

$$\text{Hence, in the denominator, we can write } 1 + \cos 2x = 2\cos^2 x$$

$$\text{In the numerator, we have } \sin 2x = 2\sin x \cos x$$

Therefore, we can write the integral as

$$I = \int \tan^{-1} \left(\frac{2 \sin x \cos x}{2 \cos^2 x} \right) dx$$

$$\Rightarrow I = \int \tan^{-1} \left(\frac{\sin x}{\cos x} \right) dx$$

$$\Rightarrow I = \int \tan^{-1}(\tan x) dx$$

$$\Rightarrow I = \int x dx$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I = \frac{x^{1+1}}{1+1} + c$$

$$\therefore I = \frac{x^2}{2} + c$$

$$\text{Thus, } \int \tan^{-1} \left(\frac{\sin 2x}{1 + \cos 2x} \right) dx = \frac{x^2}{2} + c$$

36. Question

Evaluate the following integrals:

$$\int \cos^{-1}(\sin x) dx$$

Answer

$$\text{Let } I = \int \cos^{-1}(\sin x) dx$$

$$\text{We know } \sin \theta = \cos(90^\circ - \theta)$$

Therefore, we can write the integral as

$$I = \int \cos^{-1} \left[\cos \left(\frac{\pi}{2} - x \right) \right] dx$$

$$\Rightarrow I = \int \left(\frac{\pi}{2} - x \right) dx$$

$$\Rightarrow I = \int \frac{\pi}{2} dx - \int x dx$$

$$\Rightarrow I = \frac{\pi}{2} \int dx - \int x dx$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ and } \int dx = x + c$$

$$\Rightarrow I = \frac{\pi}{2} \times x - \frac{x^{1+1}}{1+1} + c$$

$$\therefore I = \frac{\pi x}{2} - \frac{x^2}{2} + c$$

$$\text{Thus, } \int \cos^{-1}(\sin x) dx = \frac{\pi x}{2} - \frac{x^2}{2} + c$$

37. Question

Evaluate the following integrals:

$$\int \cot^{-1}\left(\frac{\sin 2x}{1 - \cos 2x}\right) dx$$

Answer

$$\text{Let } I = \int \cot^{-1}\left(\frac{\sin 2x}{1 - \cos 2x}\right) dx$$

$$\text{We know } \cos 2\theta = 1 - 2\sin^2\theta$$

$$\text{Hence, in the denominator, we can write } 1 - \cos 2x = 2\sin^2 x$$

$$\text{In the numerator, we have } \sin 2x = 2\sin x \cos x$$

Therefore, we can write the integral as

$$I = \int \cot^{-1}\left(\frac{2 \sin x \cos x}{2 \sin^2 x}\right) dx$$

$$\Rightarrow I = \int \cot^{-1}\left(\frac{\cos x}{\sin x}\right) dx$$

$$\Rightarrow I = \int \cot^{-1}(\cot x) dx$$

$$\Rightarrow I = \int x dx$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I = \frac{x^{1+1}}{1+1} + c$$

$$\therefore I = \frac{x^2}{2} + c$$

$$\text{Thus, } \int \cot^{-1}\left(\frac{\sin 2x}{1 - \cos 2x}\right) dx = \frac{x^2}{2} + c$$

38. Question

Evaluate the following integrals:

$$\int \sin^{-1}\left(\frac{2 \tan x}{1 + \tan^2 x}\right) dx$$

Answer

$$\text{Let } I = \int \sin^{-1}\left(\frac{2 \tan x}{1 + \tan^2 x}\right) dx$$

$$\text{We know } \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

Therefore, we can write the integral as

$$I = \int \sin^{-1}(\sin 2x) dx$$

$$\Rightarrow I = \int 2x dx$$

$$\Rightarrow I = 2 \int x dx$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I = 2 \times \frac{x^{1+1}}{1+1} + c$$

$$\Rightarrow I = 2 \times \frac{x^2}{2} + c$$

$$\therefore I = x^2 + c$$

$$\text{Thus, } \int \sin^{-1}\left(\frac{2\tan x}{1+\tan^2 x}\right) dx = x^2 + c$$

39. Question

Evaluate the following integrals:

$$\int \frac{(x^3 + 8)(x - 1)}{x^2 - 2x + 4} dx$$

Answer

$$\text{Let } I = \int \frac{(x^3 + 8)(x - 1)}{x^2 - 2x + 4} dx$$

$$\text{We know } a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Hence, in the numerator, we can write

$$x^3 + 8 = x^3 + 2^3$$

$$\Rightarrow x^3 + 8 = (x + 2)(x^2 - x \times 2 + 2^2)$$

$$\Rightarrow x^3 + 8 = (x + 2)(x^2 - 2x + 4)$$

Therefore, we can write the integral as

$$I = \int \frac{(x + 2)(x^2 - 2x + 4)(x - 1)}{x^2 - 2x + 4} dx$$

$$\Rightarrow I = \int (x + 2)(x - 1) dx$$

$$\Rightarrow I = \int (x^2 + x - 2) dx$$

$$\Rightarrow I = \int x^2 dx + \int x dx - \int 2 dx$$

$$\Rightarrow I = \int x^2 dx + \int x dx - 2 \int dx$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ and } \int dx = x + c$$

$$\Rightarrow I = \frac{x^{2+1}}{2+1} + \frac{x^{1+1}}{1+1} - 2 \times x + c$$

$$\therefore I = \frac{x^3}{3} + \frac{x^2}{2} - 2x + c$$

Thus, $\int \frac{(x^2+8)(x-1)}{x^2-2x+4} dx = \frac{x^3}{3} + \frac{x^2}{2} - 2x + c$

40. Question

Evaluate the following integrals:

$$\int (a \tan x + b \cot x)^2 dx$$

Answer

Let $I = \int (a \tan x + b \cot x)^2 dx$

We know $(a + b)^2 = a^2 + 2ab + b^2$

Therefore, we can write the integral as

$$I = \int [(a \tan x)^2 + 2(a \tan x)(b \cot x) + (b \cot x)^2] dx$$

$$\Rightarrow I = \int (a^2 \tan^2 x + 2ab \tan x \cot x + b^2 \cot^2 x) dx$$

$$\Rightarrow I = \int (a^2 \tan^2 x + 2ab + b^2 \cot^2 x) dx \left[\because \cot \theta = \frac{1}{\tan \theta} \right]$$

We have $\sec^2 \theta - \tan^2 \theta = \operatorname{cosec}^2 \theta - \cot^2 \theta = 1$

$$\Rightarrow I = \int [a^2 (\sec^2 x - 1) + 2ab + b^2 (\operatorname{cosec}^2 x - 1)] dx$$

$$\Rightarrow I = \int (a^2 \sec^2 x - a^2 + 2ab + b^2 \operatorname{cosec}^2 x - b^2) dx$$

$$\Rightarrow I = \int (a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x - a^2 + 2ab - b^2) dx$$

$$\Rightarrow I = \int (a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x - (a^2 - 2ab + b^2)) dx$$

$$\Rightarrow I = \int (a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x - (a - b)^2) dx$$

$$\Rightarrow I = \int a^2 \sec^2 x dx + \int b^2 \operatorname{cosec}^2 x dx - \int (a - b)^2 dx$$

$$\Rightarrow I = a^2 \int \sec^2 x dx + b^2 \int \operatorname{cosec}^2 x dx - (a - b)^2 \int dx$$

Recall $\int \sec^2 x dx = \tan x + c$ and $\int dx = x + c$

We also have $\int \operatorname{cosec}^2 x dx = -\cot x + c$

$$\Rightarrow I = a^2 \tan x + b^2 (-\cot x) - (a - b)^2 \times x + c$$

$$\therefore I = a^2 \tan x - b^2 \cot x - (a - b)^2 x + c$$

Thus, $\int (a \tan x + b \cot x)^2 dx = a^2 \tan x - b^2 \cot x - (a - b)^2 x + c$

41. Question

Evaluate the following integrals:

$$\int \frac{x^3 - 3x^2 + 5x - 7 + x^2 a^x}{2x^2} dx$$

Answer

$$\text{Let } I = \int \frac{x^3 - 3x^2 + 5x - 7 + x^2 a^x}{2x^2} dx$$

$$\Rightarrow I = \frac{1}{2} \int \frac{x^3 - 3x^2 + 5x - 7 + x^2 a^x}{x^2} dx$$

$$\Rightarrow I = \frac{1}{2} \int \left(\frac{x^3}{x^2} - \frac{3x^2}{x^2} + \frac{5x}{x^2} - \frac{7}{x^2} + \frac{x^2 a^x}{x^2} \right) dx$$

$$\Rightarrow I = \frac{1}{2} \int \left(x - 3 + \frac{5}{x} - \frac{7}{x^2} + a^x \right) dx$$

$$\Rightarrow I = \frac{1}{2} \int \left(x - 3 + \frac{5}{x} - 7x^{-2} + a^x \right) dx$$

$$\Rightarrow I = \frac{1}{2} \left[\int x dx - \int 3 dx + \int \frac{5}{x} dx - \int 7x^{-2} dx + \int a^x dx \right]$$

$$\Rightarrow I = \frac{1}{2} \left[\int x dx - 3 \int dx + 5 \int \frac{1}{x} dx - 7 \int x^{-2} dx + \int a^x dx \right]$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ and } \int dx = x + c$$

$$\text{We also have } \int a^x dx = \frac{a^x}{\log a} + c \text{ and } \int \frac{1}{x} dx = \log x + c$$

$$\Rightarrow I = \frac{1}{2} \left[\frac{x^{1+1}}{1+1} - 3 \times x + 5 \times \log x - 7 \left(\frac{x^{-2+1}}{-2+1} \right) + \frac{a^x}{\log a} \right] + c$$

$$\Rightarrow I = \frac{1}{2} \left[\frac{x^2}{2} - 3x + 5 \log x + 7x^{-1} + \frac{a^x}{\log a} \right] + c$$

$$\therefore I = \frac{1}{2} \left[\frac{x^2}{2} - 3x + 5 \log x + \frac{7}{x} + \frac{a^x}{\log a} \right] + c$$

$$\text{Thus, } \int \frac{x^3 - 3x^2 + 5x - 7 + x^2 a^x}{2x^2} dx = \frac{1}{2} \left[\frac{x^2}{2} - 3x + 5 \log x + \frac{7}{x} + \frac{a^x}{\log a} \right] + c$$

42. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{1 + \cos x} dx$$

Answer

$$\text{Let } I = \int \frac{\cos x}{1 + \cos x} dx$$

On multiplying and dividing $(1 - \cos x)$, we can write the integral as

$$I = \int \frac{\cos x}{1 + \cos x} \left(\frac{1 - \cos x}{1 - \cos x} \right) dx$$

$$\Rightarrow I = \int \frac{\cos x (1 - \cos x)}{(1 + \cos x)(1 - \cos x)} dx$$

$$\Rightarrow I = \int \frac{\cos x - \cos^2 x}{1 - \cos^2 x} dx$$

$$\Rightarrow I = \int \frac{\cos x - \cos^2 x}{\sin^2 x} dx \quad [\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$\Rightarrow I = \int \left(\frac{\cos x}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x} \right) dx$$

$$\Rightarrow I = \int \left(\frac{1}{\sin x} \times \frac{\cos x}{\sin x} - \frac{\cos^2 x}{\sin^2 x} \right) dx$$

$$\Rightarrow I = \int (\operatorname{cosec} x \cot x - \cot^2 x) dx$$

$$\Rightarrow I = \int (\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x + 1) dx \quad [\because \operatorname{cosec}^2 \theta - \cot^2 \theta = 1]$$

$$\Rightarrow I = \int \operatorname{cosec} x \cot x dx - \int \operatorname{cosec}^2 x dx + \int dx$$

Recall $\int \operatorname{cosec}^2 x dx = -\cot x + c$ and $\int dx = x + c$

We also have $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$

$$\Rightarrow I = -\operatorname{cosec} x - (-\cot x) + x + c$$

$$\Rightarrow I = -\operatorname{cosec} x + \cot x + x + c$$

Thus, $\int \frac{\cos x}{1+\cos x} dx = -\operatorname{cosec} x + \cot x + x + c$

43. Question

Evaluate the following integrals:

$$\int \frac{1 - \cos x}{1 + \cos x} dx$$

Answer

Let $I = \int \frac{1 - \cos x}{1 + \cos x} dx$

We have $\cos x = \cos \left(2 \times \frac{x}{2} \right)$

We know $\cos 2\theta = 1 - 2\sin^2 \theta = 2\cos^2 \theta - 1$

Hence, in the numerator, we can write $1 - \cos x = 2 \sin^2 \frac{x}{2}$

In the denominator, we can write $1 + \cos x = 2 \cos^2 \frac{x}{2}$

Therefore, we can write the integral as

$$I = \int \frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} dx$$

$$\Rightarrow I = \int \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} dx$$

$$\Rightarrow I = \int \tan^2 \frac{x}{2} dx$$

$$\Rightarrow I = \int \left(\sec^2 \frac{x}{2} - 1 \right) dx \quad [\because \sec^2 \theta - \tan^2 \theta = 1]$$

$$\Rightarrow I = \int \sec^2 \frac{x}{2} dx - \int dx$$

Recall $\int \sec^2 x dx = \tan x + c$ and $\int dx = x + c$

$$\Rightarrow I = \frac{\tan \frac{x}{2}}{\frac{1}{2}} - x + c$$

$$\therefore I = 2 \tan \frac{x}{2} - x + c$$

$$\text{Thus, } \int \frac{1 - \cos x}{1 + \cos x} dx = 2 \tan \frac{x}{2} - x + c$$

44. Question

Evaluate the following integrals:

$$\int \left\{ 3 \sin x - 4 \cos x + \frac{5}{\cos^2 x} - \frac{6}{\sin^2 x} + \tan^2 x - \cot^2 x \right\} dx$$

Answer

$$\text{Let } I = \int \left\{ 3 \sin x - 4 \cos x + \frac{5}{\cos^2 x} - \frac{6}{\sin^2 x} + \tan^2 x - \cot^2 x \right\} dx$$

$$\Rightarrow I = \int \{ 3 \sin x - 4 \cos x + 5 \sec^2 x - 6 \operatorname{cosec}^2 x + \tan^2 x - \cot^2 x \} dx$$

$$\text{We have } \sec^2 \theta - \tan^2 \theta = \operatorname{cosec}^2 \theta - \cot^2 \theta = 1$$

$$\Rightarrow I = \int \{ 3 \sin x - 4 \cos x + 5 \sec^2 x - 6 \operatorname{cosec}^2 x + (\sec^2 x - 1) - (\operatorname{cosec}^2 x - 1) \} dx$$

$$\Rightarrow I = \int \{ 3 \sin x - 4 \cos x + 5 \sec^2 x - 6 \operatorname{cosec}^2 x + \sec^2 x - 1 - \operatorname{cosec}^2 x + 1 \} dx$$

$$\Rightarrow I = \int \{ 3 \sin x - 4 \cos x + 6 \sec^2 x - 7 \operatorname{cosec}^2 x \} dx$$

$$\Rightarrow I = \int 3 \sin x dx - \int 4 \cos x dx + \int 6 \sec^2 x dx - \int 7 \operatorname{cosec}^2 x dx$$

$$\Rightarrow I = 3 \int \sin x dx - 4 \int \cos x dx + 6 \int \sec^2 x dx - 7 \int \operatorname{cosec}^2 x dx$$

$$\text{Recall } \int \sec^2 x dx = \tan x + c \text{ and } \int \sin x dx = -\cos x + c$$

$$\text{We also have } \int \operatorname{cosec}^2 x dx = -\cot x + c \text{ and } \int \cos x dx = \sin x + c$$

$$\Rightarrow I = 3(-\cos x) - 4(\sin x) + 6(\tan x) - 7(-\cot x) + c$$

$$\therefore I = -3 \cos x - 4 \sin x + 6 \tan x + 7 \cot x + c$$

$$\text{Thus, } \int \left\{ 3 \sin x - 4 \cos x + \frac{5}{\cos^2 x} - \frac{6}{\sin^2 x} + \tan^2 x - \cot^2 x \right\} dx = -3 \cos x - 4 \sin x + 6 \tan x + 7 \cot x + c$$

45. Question

$$\text{If } f'(x) = x - \frac{1}{x^2} \text{ and } f(1) = \frac{1}{2}, \text{ find } f(x).$$

Answer

$$\text{Given } f'(x) = x - \frac{1}{x^2} \text{ and } f(1) = \frac{1}{2}$$

On integrating the given equation, we have

$$\int f'(x) dx = \int \left(x - \frac{1}{x^2} \right) dx$$

$$\text{We know } \int f'(x) dx = f(x)$$

$$\Rightarrow f(x) = \int \left(x - \frac{1}{x^2} \right) dx$$

$$\Rightarrow f(x) = \int (x - x^{-2}) dx$$

$$\Rightarrow f(x) = \int x dx - \int x^{-2} dx$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow f(x) = \frac{x^{1+1}}{1+1} - \frac{x^{-2+1}}{-2+1} + c$$

$$\Rightarrow f(x) = \frac{x^2}{2} - \frac{x^{-1}}{-1} + c$$

$$\Rightarrow f(x) = \frac{x^2}{2} + \frac{1}{x} + c$$

On substituting $x = 1$ in $f(x)$, we get

$$f(1) = \frac{1^2}{2} + \frac{1}{1} + c$$

$$\Rightarrow \frac{1}{2} = \frac{1}{2} + 1 + c$$

$$\Rightarrow 0 = 1 + c$$

$$\Rightarrow 1 + c = 0$$

$$\therefore c = -1$$

On substituting the value of c in $f(x)$, we get

$$f(x) = \frac{x^2}{2} + \frac{1}{x} + (-1)$$

$$\therefore f(x) = \frac{x^2}{2} + \frac{1}{x} - 1$$

$$\text{Thus, } f(x) = \frac{x^2}{2} + \frac{1}{x} - 1$$

46. Question

If $f'(x) = x + b$, $f(1) = 5$, $f(2) = 13$, find $f(x)$.

Answer

Given $f'(x) = x + b$, $f(1) = 5$ and $f(2) = 13$

On integrating the given equation, we have

$$\int f'(x) dx = \int (x + b) dx$$

$$\text{We know } \int f'(x) dx = f(x)$$

$$\Rightarrow f(x) = \int (x + b) dx$$

$$\Rightarrow f(x) = \int x dx + \int b dx$$

$$\Rightarrow f(x) = \int x dx + b \int dx$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ and $\int dx = x + c$

$$\Rightarrow f(x) = \frac{x^{1+1}}{1+1} + b(x) + c$$

$$\Rightarrow f(x) = \frac{x^2}{2} + bx + c$$

On substituting $x = 1$ in $f(x)$, we get

$$f(1) = \frac{1^2}{2} + b(1) + c$$

$$\Rightarrow 5 = \frac{1}{2} + b + c$$

$$\Rightarrow 5 - \frac{1}{2} = b + c$$

$$\Rightarrow b + c = \frac{9}{2} \dots\dots (1)$$

On substituting $x = 2$ in $f(x)$, we get

$$f(2) = \frac{2^2}{2} + b(2) + c$$

$$\Rightarrow 13 = 2 + 2b + c$$

$$\Rightarrow 13 - 2 = 2b + c$$

$$\Rightarrow 2b + c = 11 \dots\dots (2)$$

By subtracting equation (1) from equation (2), we have

$$(2b + c) - (b + c) = 11 - \frac{9}{2}$$

$$\Rightarrow 2b + c - b - c = \frac{13}{2}$$

$$\therefore b = \frac{13}{2}$$

On substituting the value of b in equation (1), we get

$$\frac{13}{2} + c = \frac{9}{2}$$

$$\Rightarrow c = \frac{9}{2} - \frac{13}{2}$$

$$\therefore c = -2$$

On substituting the values of b and c in $f(x)$, we get

$$f(x) = \frac{x^2}{2} + \frac{13}{2}x + (-2)$$

$$\therefore f(x) = \frac{x^2}{2} + \frac{13}{2}x - 2$$

$$\text{Thus, } f(x) = \frac{x^2}{2} + \frac{13}{2}x - 2$$

47. Question

If $f'(x) = 8x^3 - 2x$, $f(2) = 8$, find $f(x)$.

Answer

Given $f'(x) = 8x^3 - 2x$ and $f(2) = 8$

On integrating the given equation, we have

$$\int f'(x)dx = \int (8x^3 - 2x)dx$$

We know $\int f'(x)dx = f(x)$

$$\Rightarrow f(x) = \int (8x^3 - 2x)dx$$

$$\Rightarrow f(x) = \int 8x^3 dx - \int 2x dx$$

$$\Rightarrow f(x) = 8 \int x^3 dx - 2 \int x dx$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow f(x) = 8 \left(\frac{x^{3+1}}{3+1} \right) - 2 \left(\frac{x^{1+1}}{1+1} \right) + c$$

$$\Rightarrow f(x) = 8 \left(\frac{x^4}{4} \right) - 2 \left(\frac{x^2}{2} \right) + c$$

$$\Rightarrow f(x) = 2x^4 - x^2 + c$$

On substituting $x = 2$ in $f(x)$, we get

$$f(2) = 2(2^4) - 2^2 + c$$

$$\Rightarrow 8 = 32 - 4 + c$$

$$\Rightarrow 8 = 28 + c$$

$$\therefore c = -20$$

On substituting the value of c in $f(x)$, we get

$$f(x) = 2x^4 - x^2 + (-20)$$

$$\therefore f(x) = 2x^4 - x^2 - 20$$

$$\text{Thus, } f(x) = 2x^4 - x^2 - 20$$

48. Question

If $f'(x) = a \sin x + b \cos x$ and $f'(0) = 4$, $f(0) = 3$, $f\left(\frac{\pi}{2}\right) = 5$, find $f(x)$.

Answer

Given $f'(x) = a \sin x + b \cos x$ and $f'(0) = 4$

On substituting $x = 0$ in $f'(x)$, we get

$$f'(0) = a \sin 0 + b \cos 0$$

$$\Rightarrow 4 = a \times 0 + b \times 1$$

$$\Rightarrow 4 = 0 + b$$

$$\therefore b = 4$$

Hence, $f'(x) = a \sin x + 4 \cos x$

On integrating this equation, we have

$$\int f'(x)dx = \int (a \sin x + 4 \cos x)dx$$

We know $\int f'(x)dx = f(x)$

$$\Rightarrow f(x) = \int (a \sin x + 4 \cos x)dx$$

$$\Rightarrow f(x) = \int a \sin x dx + \int 4 \cos x dx$$

$$\Rightarrow f(x) = a \int \sin x dx + 4 \int \cos x dx$$

Recall $\int \sin x dx = -\cos x + c$ and $\int \cos x dx = \sin x + c$

$$\Rightarrow f(x) = a(-\cos x) + 4(\sin x) + c$$

$$\Rightarrow f(x) = -a \cos x + 4 \sin x + c$$

On substituting $x = 0$ in $f(x)$, we get

$$f(0) = -a \cos 0 + 4 \sin 0 + c$$

$$\Rightarrow 3 = -a \times 1 + 4 \times 0 + c$$

$$\Rightarrow 3 = -a + c$$

$$\Rightarrow c - a = 3 \text{ ----- (1)}$$

On substituting $x = \frac{\pi}{2}$ in $f(x)$, we get

$$f\left(\frac{\pi}{2}\right) = -a \cos \frac{\pi}{2} + 4 \sin \frac{\pi}{2} + c$$

$$\Rightarrow 5 = -a \times 0 + 4 \times 1 + c$$

$$\Rightarrow 5 = 0 + 4 + c$$

$$\Rightarrow 5 = 4 + c$$

$$\therefore c = 1$$

On substituting $c = 1$ in equation (1), we get

$$1 - a = 3$$

$$\Rightarrow a = 1 - 3$$

$$\therefore a = -2$$

On substituting the values of c and a in $f(x)$, we get

$$f(x) = -(-2)\cos x + 4\sin x + 1$$

$$\therefore f(x) = 2\cos x + 4\sin x + 1$$

$$\text{Thus, } f(x) = 2\cos x + 4\sin x + 1$$

49. Question

Write the primitive or anti-derivative of $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$.

Answer

$$\text{Given } f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$$

$$\text{Let } I = \int f(x)dx$$

$$\Rightarrow I = \int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx$$

$$\Rightarrow I = \int \left(x^{\frac{1}{2}} + \frac{1}{x^{\frac{1}{2}}} \right) dx$$

$$\Rightarrow I = \int \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}} \right) dx$$

$$\Rightarrow I = \int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{2}} dx$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$\Rightarrow I = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$\Rightarrow I = \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$$

$$\therefore I = \frac{2}{3} x\sqrt{x} + 2\sqrt{x} + c$$

Thus, the primitive of $f(x)$ is $\frac{2}{3}x\sqrt{x} + 2\sqrt{x} + c$

Exercise 19.3

1. Question

Evaluate: $\int (2x-3)^5 + \sqrt{3x+2} \, dx$

Answer

Let $I = \int (2x-3)^5 + \sqrt{3x+2}$ then,

$$I = \int (2x-3)^5 + (3x+2)^{\frac{1}{2}}$$

$$= \frac{(2x-3)^{5+1}}{2(5+1)} + \frac{(3x+2)^{\frac{1}{2}+1}}{3(\frac{1}{2}+1)}$$

$$= \frac{(2x-3)^6}{2(6)} + \frac{(3x+2)^{\frac{3}{2}}}{3(\frac{3}{2})}$$

$$= \frac{(2x-3)^6}{12} + \frac{2(3x+2)^{\frac{3}{2}}}{9}$$

$$\text{Hence, } I = \frac{(2x-3)^6}{12} + \frac{2(3x+2)^{\frac{3}{2}}}{9} + C$$

2. Question

Evaluate: $\int \frac{1}{(7x-5)^3} + \frac{1}{\sqrt{5x-4}} \, dx$

Answer

Let $I = \int \frac{1}{(7x-5)^3} + \frac{1}{\sqrt{5x-4}} dx$ then,

$$I = \int (7x-5)^{-3} + (5x-4)^{-\frac{1}{2}}$$

$$= \frac{(7x-5)^{-3+1}}{7(-3+1)} + \frac{(5x-4)^{-\frac{1}{2}+1}}{5(-\frac{1}{2}+1)}$$

$$= \frac{(7x-5)^{-2}}{-14} + \frac{(5x-4)^{\frac{1}{2}}}{5(\frac{1}{2})}$$

$$\text{Hence, } I = -\frac{1}{14}(7x-5)^{-2} + \frac{2}{5}\sqrt{5x-4} + C$$

3. Question

$$\text{Evaluate: } \int \frac{1}{2-3x} + \frac{1}{\sqrt{3x-2}} dx$$

Answer

$$\text{Let } I = \int \frac{1}{2-3x} + \frac{1}{\sqrt{3x-2}} dx$$

$$I = \int \frac{1}{2-3x} + \frac{1}{\sqrt{3x-2}} dx$$

$$\text{We know } \int \frac{1}{x} dx = \log|x| + C$$

$$= \frac{\log|2-3x|}{-3} + \frac{2}{3}(3x-2)^{\frac{1}{2}}$$

$$= -\frac{1}{3}x \cdot \log|2x-3| + \frac{2}{3}\sqrt{3x-3} + C$$

4. Question

$$\text{Evaluate: } \int \frac{x+3}{(x+1)^4} dx$$

Answer

$$\text{Let } I = \int \frac{x+3}{(x+1)^4} dx$$

$$I = \int \frac{x+3}{(x+1)^4} dx$$

$$= \int \frac{x+1}{x+1^4} dx + \int \frac{2}{(x+1)^4} dx$$

$$= \int \frac{1}{(x+1)^3} dx + \int \frac{2}{(x+1)^4} dx$$

$$= \int (x+1)^{-3} dx + \int 2(x+1)^{-4} dx$$

$$= \frac{[x+1]^{-3+1}}{-3+1} + \frac{2[x+1]^{-4+1}}{-4+1}$$

$$= \frac{[x+1]^{-2}}{-2} + \frac{2[x+1]^{-3}}{-3}$$

$$\text{Hence, } I = -\frac{1}{2(x+1)^2} - \frac{2}{3(x+1)^3} + C$$

5. Question

Evaluate: $\int \frac{1}{\sqrt{x+1} + \sqrt{x}} dx$

Answer

$$\text{Let } I = \int \frac{1}{\sqrt{x+1} + \sqrt{x}} dx$$

$$= \int \frac{1}{\sqrt{x+1} + \sqrt{x}} dx$$

Now Multiply with the conjugate, we get

$$= \int \frac{1}{\sqrt{x+1} + \sqrt{x}} \cdot \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x+1} - \sqrt{x}} dx$$

$$= \int \frac{\sqrt{x+1} - \sqrt{x}}{x+1-x} dx$$

$$= \int \sqrt{x+1} - \sqrt{x} dx$$

$$= \int (x+1)^{\frac{1}{2}} - x^{\frac{1}{2}}$$

$$= \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}}$$

$$\text{Hence } I = \frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3}(x)^{\frac{3}{2}} + C$$

6. Question

Evaluate: $\int \frac{1}{\sqrt{2x+3} + \sqrt{2x-3}} dx$

Answer

$$\text{Let } I = \int \frac{1}{\sqrt{2x+3} + \sqrt{2x-3}} dx$$

$$I = \int \frac{1}{\sqrt{2x+3} + \sqrt{2x-3}} dx$$

Now, Multiply with the conjugate, we get

$$= \int \frac{1}{\sqrt{2x+3} + \sqrt{2x-3}} \times \frac{(\sqrt{2x+3} - \sqrt{2x-3})}{(\sqrt{2x+3} - \sqrt{2x-3})} dx$$

$$= \int \frac{(\sqrt{2x+3} - \sqrt{2x-3})}{(\sqrt{2x+3})^2 - (\sqrt{2x-3})^2} dx$$

$$= \int \frac{(\sqrt{2x+3} - \sqrt{2x-3})}{2x+3-2x-3} dx$$

$$= \int \frac{\sqrt{2x+3}}{6} dx - \int \frac{\sqrt{2x-3}}{6} dx$$

$$= \frac{1}{6} \int (2x+3)^{\frac{1}{2}} dx - \frac{1}{6} \int (2x-3)^{\frac{1}{2}} dx$$

$$= \frac{1}{6} \left(\frac{2x+3}{2} \right)^{\frac{1}{2}+1} - \frac{1}{6} \left[\frac{2x-3}{2} \right]^{\frac{1}{2}+1}$$

$$= \frac{1}{6} \left(\frac{2x+3}{2 \times \frac{3}{2}} \right)^{\frac{3}{2}} - \frac{1}{6} \left(\frac{2x-3}{2 \times \frac{3}{2}} \right)^{\frac{3}{2}}$$

$$\text{Hence, } I = \frac{1}{18} (2x+3)^{\frac{3}{2}} - \frac{1}{18} (2x-3)^{\frac{3}{2}} + C$$

7. Question

Evaluate: $\int \frac{2x}{(2x+1)^2} dx$

Answer

$$\begin{aligned}\text{Let } I &= \int \frac{2x}{(2x+1)^2} dx \\&= \int \frac{2x+1-1}{(2x+1)^2} dx \\&= \int \frac{2x+1}{(2x+1)^2} - \frac{1}{(2x+1)^2} dx \\&= \int \frac{1}{(2x+1)} - (2x+1)^{-2} dx \\&= \frac{1}{2} \log|2x+1| - \frac{(2x+1)^{-2+1}}{-2+1(2)} \\&= \frac{1}{2} \log|2x+1| - \frac{(2x+1)^{-1}}{-2}\end{aligned}$$

$$\text{Hence, } I = \frac{1}{2} \log|2x+1| + \frac{1}{2(2x+1)} + C$$

8. Question

Evaluate: $\int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} dx$

Answer

$$\begin{aligned}\text{Let } I &= \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} dx \\&= \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} dx\end{aligned}$$

Now, Multiply with conjugate, we get

$$\begin{aligned}&= \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} \times \frac{(\sqrt{x+a} - \sqrt{x+b})}{(\sqrt{x+a} - \sqrt{x+b})} dx \\&= \int \frac{(\sqrt{x+a} - \sqrt{x+b})}{(\sqrt{x+a})^2 - (\sqrt{x+b})^2} dx \\&= \int \frac{(\sqrt{x+a} - \sqrt{x+b})}{a-b} dx \\&= \frac{1}{a-b} \left[\frac{2}{3} (x+a)^{\frac{3}{2}} - \frac{2}{3} (x+b)^{\frac{3}{2}} \right]\end{aligned}$$

$$\text{Hence, } I = \frac{2}{3(a-b)} \left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}} \right] + C$$

9. Question

Evaluate: $\int \sin x \sqrt{1 + \cos 2x} dx$

Answer

$$\begin{aligned}\text{Let } I &= \int \sin x \sqrt{1 + \cos 2x} dx \\&= \int \sin x \sqrt{1 + \cos 2x} dx\end{aligned}$$

$$= \int \sin x \sqrt{2 \cos^2 x} dx$$

$$= \int \sin x \sqrt{2} \cos x dx$$

$$= \sqrt{2} \int \sin x \cos x dx$$

Now, Multiply and Divide by 2 we get,

$$= \frac{\sqrt{2}}{2} \int 2 \sin x \cos x dx$$

$$= \frac{\sqrt{2}}{2} \int \sin 2x dx$$

$$= \frac{\sqrt{2}}{2} \frac{-\cos 2x}{2}$$

$$\text{Hence, } I = -\frac{1}{2\sqrt{2}} \cos 2x + C$$

10. Question

$$\text{Evaluate: } \int \frac{1 + \cos x}{1 - \cos x} dx$$

Answer

$$\text{Let } I = \int \frac{1 + \cos x}{1 - \cos x} dx$$

$$\Rightarrow \int \frac{1 + \cos x}{1 - \cos x} dx$$

$$\Rightarrow \int \frac{2 \cos^2 \frac{x}{2}}{2 \sin^2 \frac{x}{2}} dx$$

$$\Rightarrow \int \cot^2 \frac{x}{2} dx$$

$$\Rightarrow \int \left(\operatorname{cosec}^2 \frac{x}{2} - 1 \right) dx$$

$$\Rightarrow \frac{\left(-\cot \frac{x}{2} \right)}{\frac{1}{2}} - x$$

$$\text{Hence, } I = -2 \cot \frac{x}{2} - x + C$$

11. Question

$$\text{Evaluate: } \int \frac{1 - \cos x}{1 + \cos x} dx$$

Answer

$$\text{Let } I = \int \frac{(1 - \cos x)}{(1 + \cos x)} dx$$

$$= \int \frac{(1 - \cos x)}{(1 + \cos x)} dx$$

$$= \int \frac{\left(2 \sin^2 \frac{x}{2} \right)}{2 \cos^2 \frac{x}{2}} dx$$

$$= \int \tan^2 \frac{x}{2} dx$$

$$= \int \left(\sec^2 \frac{x}{2} - 1 \right) dx$$

$$= \frac{\left(\tan \frac{x}{2}\right)}{\frac{1}{2}} - x$$

$$\text{Hence, } I = 2 \tan \frac{x}{2} - x + C$$

12. Question

$$\text{Evaluate: } \int \frac{1}{1 - \sin \frac{x}{2}} dx$$

Answer

$$\text{Let } I = \int \frac{1}{1 - \sin \frac{x}{2}} dx$$

$$= \int \frac{1}{1 - \sin \frac{x}{2}} dx$$

Now, Multiply with the conjugate we get,

$$= \int \frac{1}{1 - \sin \frac{x}{2}} \times \frac{1 + \sin \frac{x}{2}}{1 + \sin \frac{x}{2}} dx$$

$$= \int \frac{1 + \sin \frac{x}{2}}{1 - \sin^2 \frac{x}{2}} dx$$

$$= \int \frac{1 + \sin \frac{x}{2}}{\cos^2 \frac{x}{2}} dx$$

$$= \int \frac{1}{\cos^2 \frac{x}{2}} dx + \int \frac{\sin \frac{x}{2}}{\cos^2 \frac{x}{2}} dx$$

$$= \int \sec^2 \frac{x}{2} dx + \int \tan \frac{x}{2} \cdot \sec \frac{x}{2} dx$$

$$= \frac{\left(\tan \frac{x}{2}\right)}{\frac{1}{2}} + \frac{\left(\sec \frac{x}{2}\right)}{\frac{1}{2}}$$

$$\text{Hence, } I = 2 \tan \frac{x}{2} + 2 \sec \frac{x}{2} + C$$

13. Question

$$\text{Evaluate: } \int \frac{1}{1 + \cos 3x} dx$$

Answer

$$\text{Let } I = \int \frac{1}{1 + \cos 3x} dx$$

$$= \int \frac{1}{1 + \cos 3x} dx$$

Now Multiply with Conjugate,

$$= \int \frac{1}{1 + \cos 3x} \times \frac{1 - \cos 3x}{1 - \cos 3x} dx$$

$$= \int \frac{1 - \cos 3x}{1 - \cos^2 3x} dx$$

$$= \int \frac{1 - \cos 3x}{\sin^2 3x} dx$$

$$= \int \frac{1}{\sin^2 3x} dx - \int \frac{\cos 3x}{\sin^2 3x} dx$$

$$= \int (\operatorname{cosec}^2 3x - \operatorname{cosec} 3x \cot 3x) dx$$

$$= -\frac{\cot 3x}{3} + \frac{\operatorname{cosec} 3x}{3}$$

$$= -\frac{1}{3} \cdot \frac{\cos 3x}{\sin 3x} + \frac{1}{3} \cdot \frac{1}{\sin 3x}$$

$$\text{Hence, } I = \frac{1 - \cos 3x}{3 \sin 3x} + C$$

14. Question

Evaluate: $\int (e^x + 1)^2 e^x dx$

Answer

$$\text{Let } I = \int (e^x + 1)^2 e^x dx$$

$$\text{Let } e^x + 1 = t \Rightarrow e^x dx = dt$$

$$I = \int (e^x + 1)^2 e^x dx$$

$$= \int t^2 dt$$

$$= \frac{t^3}{3}$$

Now, substitute the value of t

$$\text{Hence, } I = \frac{(e^x + 1)^3}{3} + C$$

15. Question

$$\text{Evaluate: } \int \left(e^x + \frac{1}{e^x} \right)^2 dx$$

Answer

$$\text{Let } I = \int \left(e^x + \frac{1}{e^x} \right)^2 dx$$

$$= \int \left(e^{2x} + \frac{1}{e^{2x}} + 2 \right) dx$$

$$= \frac{e^{2x}}{2} - \frac{1}{2} e^{-2x} + 2x$$

$$\text{Hence, } I = \frac{1}{2} e^x + 2x - \frac{1}{2} e^{-2x} + C$$

16. Question

$$\text{Evaluate: } \int \frac{1 + \cos 4x}{\cot x - \tan x} dx$$

Answer

$$\text{Let } I = \int \frac{1 + \cos 4x}{\cot x - \tan x} dx$$

$$= \int \frac{1 + \cos 4x}{\cot x - \tan x} dx$$

$$= \int \frac{1 + \cos^2 2x}{\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}} dx$$

$$= \int \frac{2 \cos^2 2x}{\frac{\cos^2 x - \sin^2 x}{\sin x \cos x}} dx$$

$$= \int \frac{2\cos^2 2x \sin x \cos x}{\cos^2 x - \sin^2 x} dx$$

$$= \int \frac{\cos^2 2x \sin 2x}{\cos^2 2x} dx$$

$$= \int \cos 2x \sin 2x dx$$

$$= \frac{1}{2} \int [2 \sin 2x \cos 2x] dx$$

$$= \frac{1}{2} \int \sin(2x + 2x) + \sin(2x - 2x) dx$$

$$= \frac{1}{2} \int \sin 4x + 0 dx$$

$$= \frac{1}{2} - \frac{\cos 4x}{4}$$

$$\text{Hence, } I = -\frac{1}{8} \cos 4x + C$$

17. Question

$$\text{Evaluate: } \int \frac{1}{\sqrt{x+3} - \sqrt{x+2}} dx$$

Answer

$$\text{Let } I = \int \frac{1}{\sqrt{x+3} - \sqrt{x+2}} dx$$

$$= \int \frac{1}{\sqrt{x+3} - \sqrt{x+2}} dx$$

Now, Multiply with the conjugate

$$= \int \frac{1}{\sqrt{x+3} - \sqrt{x+2}} \times \frac{\sqrt{x+3} + \sqrt{x+2}}{\sqrt{x+3} + \sqrt{x+2}} dx$$

$$= \int \frac{\sqrt{x+3} + \sqrt{x+2}}{(\sqrt{x+3})^2 - (\sqrt{x+2})^2} dx$$

$$= \int \frac{\sqrt{x+3} + \sqrt{x+2}}{x+3-x-2} dx$$

$$= \int (x+3)^{\frac{1}{2}} + (x+2)^{\frac{1}{2}} dx$$

$$= \frac{(x+3)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}}$$

$$\text{Hence, } I = \frac{2}{3}(x+3)^{\frac{3}{2}} + \frac{2}{3}(x+2)^{\frac{3}{2}} + C$$

18. Question

$$\int \tan^2(2x - 3) dx$$

Answer

$$\text{Let } I = \int \tan^2(2x - 3) dx$$

$$= \int \tan^2(2x - 3) dx$$

$$= \int \sec^2(2x - 3) - 1 dx$$

$$\text{Let } 2x - 3 = t \quad dx = dt/2$$

$$= \frac{1}{2} \int \sec^2 t - 1 \, dt$$

$$= \frac{1}{2} \tan t - x$$

Substitute the value of t

$$\text{Hence, } I = \frac{1}{2} \tan(2x - 3) - x + C$$

19. Question

$$\text{Evaluate: } \int \frac{1}{\cos^2 x (1 - \tan x)^2} dx$$

Answer

$$\text{Let } I = \int \frac{1}{\cos^2 x (1 - \tan x)^2} dx$$

$$= \int \frac{1}{\cos^2 x (1 - \tan x)^2} dx$$

$$= \int \frac{1}{\cos^2 x \left(1 - \frac{\sin x}{\cos x}\right)^2} dx$$

$$= \int \frac{1}{(\cos x - \sin x)^2} dx$$

$$= \int \frac{1}{1 - \sin 2x} dx$$

$$= \int \frac{1}{1 + \cos\left(\frac{\pi}{2} + 2x\right)} dx$$

$$= \int \frac{1}{2 \cos^2\left(\frac{\pi}{4} + x\right)} dx$$

$$= \frac{1}{2} \int \sec^2\left(\frac{\pi}{4} + x\right) dx$$

$$\text{Hence, } I = \frac{1}{8} \left[\tan\left(\frac{\pi}{4} + x\right) \right] + 1 + C$$

Exercise 19.4

1. Question

$$\text{Evaluate: } \int \frac{x^2 + 5x + 2}{x + 2} dx$$

Answer

By doing long division of the given equation we get

$$\text{Quotient} = x + 3$$

$$\text{Remainder} = -4$$

∴ We can write the above equation as

$$\Rightarrow x + 3 - \frac{4}{x+2}$$

∴ The above equation becomes

$$\Rightarrow \int x + 3 - \frac{4}{x+2} dx$$

$$\Rightarrow \int x dx + 3 \int dx - 4 \int \frac{1}{x+2} dx$$

We know $\int x^n dx = \frac{x^{n+1}}{n+1}$; $\int \frac{1}{x} dx = \ln x$

$$\Rightarrow \frac{x^3}{3} + 3x - 4\ln(x+2) + c. \text{ (Where } c \text{ is some arbitrary constant)}$$

2. Question

Evaluate: $\int \frac{x^3}{x-2} dx$

Answer

By doing long division of the given equation we get

$$\text{Quotient} = x^2 + 2x + 4$$

$$\text{Remainder} = 8$$

$\therefore$ We can write the above equation as

$$\Rightarrow x^2 + 2x + 4 + \frac{8}{x-2}$$

$\therefore$ The above equation becomes

$$\Rightarrow \int x^2 + 2x + 4 + \frac{8}{x-2} dx$$

$$\Rightarrow \int x^2 dx + 2 \int x dx + 4 \int dx + 8 \int \frac{1}{x-2} dx$$

We know $\int x^n dx = \frac{x^{n+1}}{n+1}$; $\int \frac{1}{x} dx = \ln x$

$$\Rightarrow \frac{x^3}{3} + 2 \frac{x^2}{2} + 4x + 8 \ln(x-2) + c$$

$$\Rightarrow \frac{x^3}{3} + x^2 + 4x + 8 \ln(x-2) + c. \text{ (Where } c \text{ is some arbitrary constant)}$$

3. Question

Evaluate: $\int \frac{x^2 + x + 5}{3x + 2} dx$

Answer

By doing long division of the given equation we get

$$\text{Quotient} = \frac{x}{3} + \frac{1}{9}$$

$$\text{Remainder} = \frac{43}{9}$$

$\therefore$ We can write the above equation as

$$\Rightarrow \frac{x}{3} + \frac{1}{9} + \frac{43}{9} \left(\frac{1}{3x+2} \right)$$

$\therefore$ The above equation becomes

$$\Rightarrow \int \frac{x}{3} + \frac{1}{9} + \frac{43}{9} \left(\frac{1}{3x+2} \right) dx$$

$$\Rightarrow \frac{1}{3} \int x dx + \frac{1}{9} \int dx + \frac{43}{9} \int \frac{1}{3x+2} dx$$

We know $\int x^n dx = \frac{x^{n+1}}{n+1}$; $\int \frac{1}{x} dx = \ln x$

$$\Rightarrow \frac{1}{3} \times \frac{x^2}{2} + \frac{1}{9} \times x + \frac{43}{9} \ln(3x+2) + c$$

$$\Rightarrow \frac{x^3}{6} + \frac{x^2}{18} + \frac{43}{9} \ln(3x+2) + c. \text{ (Where } c \text{ is some arbitrary constant)}$$

4. Question

Evaluate: $\int \frac{2x+3}{(x-1)^2} dx$

Answer

The above equation can be written as

$$\Rightarrow \int \frac{2x-2+2+3}{(x-1)^2}$$

$$\Rightarrow \int \frac{2(x-1)+5}{(x-1)^2}$$

$$\Rightarrow 2 \int \frac{1 \cdot dx}{(x-1)} + 5 \int \frac{1 \cdot dx}{(x-1)^2}$$

We know $\int x dx = \frac{x^n}{n+1}; \int \frac{1}{x} dx = \ln x$

$$\Rightarrow 2 \ln(x-1) + 5 \int (x-1)^{-2} dx$$

$$\Rightarrow 2 \ln(x-1) + 5 \int \frac{(x-1)^{-1}}{-1} dx$$

$$\Rightarrow 2 \ln(x-1) - \frac{5}{(x-1)} + c. \text{ (Where } c \text{ is an arbitrary constant)}$$

5. Question

Evaluate: $\int \frac{x^2+3x-1}{(x+1)^2} dx$

Answer

$$\Rightarrow \int \frac{x^2+x+2x-1}{(x+1)^2} dx$$

$$\Rightarrow \int \frac{x(x+1)+2x-1}{(x+1)^2} dx$$

$$\Rightarrow \int \frac{x(x+1)}{(x+1)^2} dx + \int \frac{2x-1}{(x+1)^2} dx$$

$$\Rightarrow \int \frac{x}{x+1} dx + \int \frac{2x+2-2-1}{(x+1)^2} dx$$

$$\Rightarrow \int \frac{x+1-1}{x+1} dx + \int \frac{2(x+1)-3}{(x+1)^2} dx$$

$$\Rightarrow \int dx - \int \frac{1}{x+1} dx + \int \frac{2}{x+1} dx - \int \frac{3}{(x+1)^2} dx$$

We know $\int x dx = \frac{x^n}{n+1}; \int \frac{1}{x} dx = \ln x$

$$\Rightarrow x - \ln(x+1) + 2 \ln(x+1) - \int 3(x+1)^{-2} dx$$

$$\Rightarrow x - \ln(x+1) + 2 \ln(x+1) + \frac{3}{x+1} + c$$

$$\Rightarrow x + \ln(x+1) + \frac{3}{x+1} + c. \text{ (Where } c \text{ is some arbitrary constant)}$$

6. Question

Evaluate: $\int \frac{2x-1}{(x-1)^2} dx$

Answer

In this question degree of denominator is larger than that of numerator so we need to manipulate numerator.

$$\Rightarrow \int \frac{2x+2-2-1}{(x-1)^2}$$

$$\Rightarrow \int \frac{2(x-1)-1}{(x-1)^2}$$

$$\Rightarrow \int \frac{2}{x-1} dx - \frac{1}{(x-1)^2} dx$$

We know $\int x dx = \frac{x^n}{n+1}$; $\int \frac{1}{x} dx = \ln x$

$$\Rightarrow 2 \ln(x-1) - \int (x-1)^{-2} dx$$

$$\Rightarrow 2 \ln(x-1) - \frac{1}{x-1} + c. \text{ (where } c \text{ is some arbitrary constant)}$$

Exercise 19.5

1. Question

Evaluate: $\int \frac{x+1}{\sqrt{2x+3}} dx$

Answer

In these questions, little manipulation makes the questions easier to solve

Here multiply and divide by 2 we get

$$\Rightarrow \frac{1}{2} \int \frac{2x+2}{\sqrt{2x+3}} dx$$

Add and subtract 1 from the numerator

$$\Rightarrow \frac{1}{2} \int \frac{2x+2+1-1}{\sqrt{2x+3}} dx$$

$$\Rightarrow \frac{1}{2} \int \frac{2x+3-1}{\sqrt{2x+3}} dx$$

$$\Rightarrow \frac{1}{2} \int \frac{2x+3}{\sqrt{2x+3}} dx - \frac{1}{2} \int \frac{1}{\sqrt{2x+3}} dx$$

$$\Rightarrow \frac{1}{2} \left(\int \sqrt{2x+3} dx - \int (2x+3)^{-\frac{1}{2}} dx \right)$$

$$\Rightarrow \frac{1}{2} \times \frac{(2x+3)^{\frac{3}{2}}}{2 \times \frac{3}{2}} - \frac{1}{2} \times \frac{(2x+3)^{\frac{1}{2}}}{2 \times \frac{1}{2}} + c$$

$$\Rightarrow \frac{(2x+3)^{\frac{3}{2}}}{6} - \frac{(2x+3)^{\frac{1}{2}}}{2} + c$$

2. Question

Evaluate: $\int x\sqrt{x+2} dx$

Answer

Here Add and subtract 2 from x

We get

$$\Rightarrow \int (x + 2 - 2)\sqrt{x + 2} dx$$

$$\Rightarrow \int (x + 2)^{\frac{3}{2}} dx - \int 2\sqrt{x + 2} dx$$

$$\Rightarrow \frac{2(x+2)^{\frac{5}{2}}}{\frac{5}{2}} - \frac{4(x+2)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

3. Question

Evaluate: $\int \frac{x-1}{\sqrt{x+4}} dx$

Answer

In these questions, little manipulation makes the questions easier to solve

Add and subtract 5 from the numerator

$$\Rightarrow \int \frac{x+5-5-1}{\sqrt{x+4}} dx$$

$$\Rightarrow \int \frac{x+4-5}{\sqrt{x+4}} dx$$

$$\Rightarrow \int \frac{x+4}{\sqrt{x+4}} dx - \int \frac{5}{\sqrt{x+4}} dx$$

$$\Rightarrow \left(\int \sqrt{x+4} dx - 5 \int (x+4)^{-\frac{1}{2}} dx \right)$$

$$\Rightarrow \frac{(x+4)^{\frac{3}{2}}}{\frac{3}{2}} - 5 \times \frac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$\Rightarrow \frac{2(x+4)^{\frac{3}{2}}}{3} - 10(x+4)^{\frac{1}{2}} + c$$

4. Question

Evaluate: $\int (x+2)\sqrt{3x+5} dx$

Answer

Here multiply and divide the question by 3

We get

$$\Rightarrow \frac{1}{3} \int 3(x+2)\sqrt{3x+5} dx$$

$$\Rightarrow \frac{1}{3} \int (3x+6)\sqrt{3x+5} dx$$

Add and subtract 1 from above equation

$$\Rightarrow \frac{1}{3} \int (3x+6+1-1)\sqrt{3x+5} dx$$

$$\Rightarrow \frac{1}{3} \int (3x+5-1)\sqrt{3x+5} dx$$

$$\Rightarrow \frac{1}{3} \int (3x+5)^{\frac{3}{2}} dx - \int \frac{1}{3} \sqrt{3x+5} dx$$

$$\Rightarrow \frac{1}{3} \times \frac{2(3x+5)^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2(3x+5)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\Rightarrow \frac{2(3x+5)^{\frac{5}{2}}}{45} - \frac{2(3x+5)^{\frac{3}{2}}}{9} + c$$

5. Question

Evaluate: $\int \frac{2x+1}{\sqrt{3x+2}} dx$

Answer

Let $2x + 1 = \lambda(3x + 2) + \mu$

$2x + 1 = 3x\lambda + 2\lambda + \mu$

comparing coefficients we get

$3\lambda = 2 ; 2\lambda + \mu = 1$

$\Rightarrow \lambda = \frac{2}{3}; \mu = \frac{-1}{3}$

Replacing $2x + 1$ by $\lambda(3x + 2) + \mu$ in the given equation we get

$\Rightarrow \int \frac{\lambda(3x+2)+\mu}{\sqrt{3x+2}} dx$

$\Rightarrow \lambda \int \frac{3x+2}{\sqrt{3x+2}} dx + \mu \int \frac{1}{\sqrt{3x+2}} dx$

$\Rightarrow \left(\lambda \int \sqrt{3x+2} dx - \mu \int (3x+2)^{-\frac{1}{2}} dx \right)$

$\Rightarrow \frac{2}{3} \times \frac{(3x+2)^{\frac{3}{2}}}{3 \times \frac{3}{2}} - \frac{1}{3} \times \frac{(3x+2)^{\frac{1}{2}}}{3 \times \frac{1}{2}} + c$

$\Rightarrow \frac{4(3x+2)^{\frac{3}{2}}}{27} - \frac{2(3x+2)^{\frac{1}{2}}}{9} + c$

6. Question

Evaluate: $\int \frac{3x+5}{\sqrt{7x+9}} dx$

Answer

Let $3x + 5 = \lambda(7x + 9) + \mu$

$3x + 5 = 7x\lambda + 9\lambda + \mu$

comparing coefficients, we get

$7\lambda = 3 ; 9\lambda + \mu = 1$

$\Rightarrow \lambda = \frac{3}{7}; \mu = \frac{8}{7}$

Replacing $3x + 5$ by $\lambda(7x + 9) + \mu$ in the given equation we get

$\Rightarrow \int \frac{\lambda(7x+9)+\mu}{\sqrt{7x+9}} dx$

$\Rightarrow \lambda \int \frac{7x+9}{\sqrt{7x+9}} dx + \mu \int \frac{1}{\sqrt{7x+9}} dx$

$\Rightarrow \left(\lambda \int \sqrt{7x+9} dx + \mu \int (7x+9)^{-\frac{1}{2}} dx \right)$

$\Rightarrow \frac{3}{7} \times \frac{(7x+9)^{\frac{3}{2}}}{7 \times \frac{3}{2}} + \frac{8}{7} \times \frac{(7x+9)^{\frac{1}{2}}}{7 \times \frac{1}{2}} + c$

$\Rightarrow \frac{6(7x+9)^{\frac{3}{2}}}{147} - \frac{16(7x+9)^{\frac{1}{2}}}{49} + c$

7. Question

Evaluate: $\int \frac{x}{\sqrt{x+4}} dx$

Answer

In these questions, little manipulation makes the questions easier to solve

Add and subtract 4 from the numerator

$$\Rightarrow \int \frac{x+4-4}{\sqrt{x+4}} dx$$

$$\Rightarrow \int \frac{x+4-4}{\sqrt{x+4}} dx$$

$$\Rightarrow \int \frac{x+4}{\sqrt{x+4}} dx - \int \frac{4}{\sqrt{x+4}} dx$$

$$\Rightarrow \left(\int \sqrt{x+4} dx - 4 \int (x+4)^{-\frac{1}{2}} dx \right)$$

$$\Rightarrow \frac{(x+4)^{\frac{3}{2}}}{\frac{3}{2}} - 4 \times \frac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$\Rightarrow \frac{2(x+4)^{\frac{3}{2}}}{3} - 8(x+4)^{\frac{1}{2}} + c$$

8. Question

Evaluate: $\int \frac{2-3x}{\sqrt{1+3x}} dx$

Answer

Let $2-3x = \lambda(3x+1) + \mu$

$$2-3x = 3x\lambda + \lambda + \mu$$

comparing coefficients we get

$$3\lambda = -3; \lambda + \mu = 2$$

$$\Rightarrow \lambda = -1; \mu = 3$$

Replacing $2-3x$ by $\lambda(3x+1) + \mu$ in given equation we get

$$\Rightarrow \int \frac{\lambda(3x+1) + \mu}{\sqrt{3x+1}} dx$$

$$\Rightarrow \lambda \int \frac{3x+1}{\sqrt{3x+1}} dx + \mu \int \frac{1}{\sqrt{3x+1}} dx$$

$$\Rightarrow \left(\lambda \int \sqrt{3x+1} dx + \mu \int (3x+1)^{-\frac{1}{2}} dx \right)$$

$$\Rightarrow -1 \times \frac{(3x+1)^{\frac{3}{2}}}{3 \times \frac{3}{2}} + 3 \times \frac{(3x+1)^{\frac{1}{2}}}{3 \times \frac{1}{2}} + c$$

$$\Rightarrow \frac{-2(3x+1)^{\frac{3}{2}}}{9} - 2(3x+1)^{\frac{1}{2}} + c$$

9. Question

Evaluate: $\int (5x+3)\sqrt{2x-1} dx$

Answer

Let $5x+3 = \lambda(2x-1) + \mu$

$$5x + 3 = 2x\lambda - \lambda + \mu$$

comparing coefficients we get

$$2\lambda = 5 ; -\lambda + \mu = 3$$

$$\Rightarrow \lambda = \frac{5}{2}; \mu = \frac{11}{2}$$

Replacing $5x + 3$ by $\lambda(2x - 1) + \mu$ in the given equation we get

$$\Rightarrow \int \sqrt{2x-1} \lambda(2x-1) + \mu dx$$

$$\Rightarrow \lambda \int (2x-1) \sqrt{2x-1} dx + \int \sqrt{2x-1} \mu dx$$

$$\Rightarrow \left(\lambda \int (2x-1)^{\frac{3}{2}} dx - \mu \int (2x-1)^{\frac{1}{2}} dx \right)$$

$$\Rightarrow \frac{5}{2} \times \frac{(2x-1)^{\frac{5}{2}}}{2 \times \frac{5}{2}} - \frac{11}{2} \times \frac{(2x-1)^{\frac{3}{2}}}{2 \times \frac{3}{2}} + c$$

$$\Rightarrow \frac{(2x-1)^{\frac{5}{2}}}{2} - \frac{11(2x-1)^{\frac{3}{2}}}{6} + c$$

10. Question

Evaluate: $\int \frac{x}{\sqrt{x+a} - \sqrt{x+b}} dx$

Answer

Rationalise the given equation we get

$$\Rightarrow \int \frac{x}{\sqrt{x+a} - \sqrt{x+b}} \times \frac{\sqrt{x+a} + \sqrt{x+b}}{\sqrt{x+a} + \sqrt{x+b}} dx$$

$$\Rightarrow \int \frac{x(\sqrt{x+a} - \sqrt{x+b})}{x+a-x-b} dx$$

$$\Rightarrow \int \frac{x(\sqrt{x+a} - \sqrt{x+b})}{a-b} dx$$

$$\Rightarrow \frac{1}{a-b} \int x(\sqrt{x+a} - \sqrt{x+b}) dx$$

Assume $x = \sqrt{t}$

$$\Rightarrow dx = \frac{dt}{2\sqrt{t}}$$

Substituting t and dt

$$\Rightarrow \int \sqrt{t} \frac{(\sqrt{\sqrt{t}+a} - \sqrt{\sqrt{t}-b})}{2\sqrt{t}(a-b)} dt$$

$$\Rightarrow \frac{1}{2(a-b)} \int (\sqrt{\sqrt{t}+a} - \sqrt{\sqrt{t}-b}) dt$$

$$\Rightarrow \frac{1}{2(a-b)} \int (\sqrt{t} + a)^{1/2} dt - \int (\sqrt{t} - b)^{1/2} dt$$

$$\Rightarrow \frac{1}{2(a-b)} \left(\frac{4}{3} (\sqrt{t} + a)^{\frac{3}{2}} - \frac{4}{3} (t - a^2)^{\frac{3}{2}} \right)$$

But $x = \sqrt{t}$

$$\Rightarrow \frac{1}{2(a-b)} \left(\frac{2}{3} (x + a)^{\frac{3}{2}} - \frac{2}{3} (x - b)^{\frac{3}{2}} \right)$$

Exercise 19.6

1. Question

Evaluate: $\int \sin^2(2x + 5) dx$

Answer

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$\therefore$ The given equation becomes,

$$\Rightarrow \int \frac{1 - \cos 2(2x+5)}{2} dx$$

$$\text{We know } \int \cos ax dx = \frac{1}{a} \sin ax + c$$

$$\Rightarrow \frac{1}{2} \int dx - \frac{1}{2} \int \cos(4x + 10) dx$$

$$\Rightarrow \frac{x}{2} - \frac{1}{8} \sin(4x + 10) + c$$

2. Question

Evaluate: $\int \sin^3(2x + 1) dx$

Answer

$$\text{We know } \sin 3x = -4\sin^3 x + 3\sin x$$

$$\Rightarrow 4\sin^3 x = 3\sin x - \sin 3x$$

$$\Rightarrow \sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

$$\Rightarrow \int \sin^3(2x + 1) dx = \int \frac{3\sin(2x+1) - \sin 3(2x+1)}{4} dx$$

$$\Rightarrow \text{We know } \int \sin ax dx = \frac{-1}{a} \cos ax + c$$

$$\Rightarrow \frac{3}{8} \int \sin(2x + 1) dx - \frac{1}{4} \int \sin(6x + 3) dx$$

$$\Rightarrow \frac{-3}{8} \cos(2x + 1) + \frac{1}{24} \cos(6x + 3) + c.$$

3. Question

Evaluate: $\int \cos^4 2x dx$

Answer

$$\cos^4 2x = (\cos^2 2x)^2$$

$$\Rightarrow \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\Rightarrow (\cos^2 2x)^2 = \left(\frac{1 + \cos 4x}{2} \right)^2$$

$$\Rightarrow \left(\frac{1 + \cos 4x}{2} \right)^2 = \left(\frac{1 + 2\cos 4x + \cos^2 4x}{4} \right)$$

$$\Rightarrow \cos^2 4x = \frac{1 + \cos 8x}{2}$$

$$\Rightarrow \left(\frac{1 + 2\cos 4x + \cos^2 4x}{4} \right) = \frac{1}{4} + \frac{\cos 4x}{2} + \frac{1 + \cos 8x}{8}$$

Now the question becomes

$$\Rightarrow \frac{1}{4} \int dx + \frac{1}{2} \int \cos 4x dx + \frac{1}{8} \int dx + \frac{1}{8} \int \cos 8x dx$$

We know $\int \cos ax \, dx = \frac{1}{a} \sin ax + c$

$$\Rightarrow \frac{x}{4} + \frac{1}{8} \sin 4x + \frac{x}{8} + \frac{\sin 8x}{64} + c$$

$$\Rightarrow \frac{24x + 8 \sin 4x + \sin 8x}{64} + c$$

4. Question

Evaluate: $\int \sin^2 b x \, dx$

Answer

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$\therefore$ The given equation becomes,

$$\Rightarrow \int \frac{1 - \cos 2b}{2} \, dx$$

We know $\int \cos ax \, dx = \frac{1}{a} \sin ax + c$

$$\Rightarrow \frac{1}{2} \int dx - \frac{1}{2} \int \cos(2b) \, dx$$

$$\Rightarrow \frac{x}{2} - \frac{1}{4b} \sin(2bx) + c$$

5. Question

Evaluate: $\int \sin^2 \frac{x}{2} \, dx$

Answer

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$\therefore$ The given equation becomes,

$$\Rightarrow \int \frac{1 - \cos 2 \cdot \frac{x}{2}}{2} \, dx = \int \frac{1 - \cos x}{2} \, dx$$

We know $\int \cos ax \, dx = \frac{1}{a} \sin ax + c$

$$\Rightarrow \frac{1}{2} \int dx - \frac{1}{2} \int \cos(x) \, dx$$

$$\Rightarrow \frac{x}{2} - \frac{1}{2} \sin(x) + c$$

6. Question

Evaluate: $\int \cos^2 \frac{x}{2} \, dx$

Answer

$$\text{We know, } \cos^2 x = \frac{1 + \cos 2x}{2}$$

$\therefore$ The given equation becomes,

$$\Rightarrow \int \frac{1 + \cos 2 \cdot \frac{x}{2}}{2} \, dx = \int \frac{1 + \cos x}{2} \, dx$$

We know $\int \cos ax \, dx = \frac{1}{a} \sin ax + c$

$$\Rightarrow \frac{1}{2} \int dx + \frac{1}{2} \int \cos(x) \, dx$$

$$\Rightarrow \frac{x}{2} + \frac{1}{2} \sin(x) + c$$

7. Question

Evaluate: $\int \cos^2 nx \, dx$

Answer

$$\text{We know, } \cos^2 x = \frac{1 + \cos 2x}{2}$$

$\therefore$ The given equation becomes,

$$\Rightarrow \int \frac{1 + \cos nx}{2} dx = \int \frac{1 + \cos 2nx}{2} dx$$

$$\text{We know } \int \cos ax \, dx = \frac{1}{a} \sin ax + c$$

$$\Rightarrow \frac{1}{2} \int dx + \frac{1}{2} \int \cos(2nx) \, dx$$

$$\Rightarrow \frac{x}{2} + \frac{1}{4n} \sin(2nx) + c$$

8. Question

Evaluate: $\int \sin x \sqrt{1 - \cos 2x} \, dx$

Answer

$$\Rightarrow 2\sin^2 x = 1 - \cos 2x$$

We can substitute the above result in the given equation

$\therefore$ The given equation becomes

$$\Rightarrow \int \sin x \sqrt{2 \sin^2 x} \, dx$$

$$\Rightarrow \int \sqrt{2} \sin^2 x \, dx$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\Rightarrow \frac{\sqrt{2}}{2} \int 1 - \cos 2x \, dx$$

$$\Rightarrow \frac{1}{\sqrt{2}} \int dx - \frac{1}{\sqrt{2}} \int \cos 2x \, dx$$

$$\Rightarrow \frac{x}{\sqrt{2}} - \frac{1}{2\sqrt{2}} \sin(2x) + c$$

Exercise 19.7

1. Question

$\int \sin 4x \cos 7x \, dx$

Answer

We know $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

$$\therefore \sin 4x \cos 7x = \frac{\sin 11x + \sin(-3x)}{2}$$

We know $\sin(-\theta) = -\sin\theta$

$$\therefore \sin(-3x) = -\sin 3x$$

$\therefore$ The above equation becomes

$$\Rightarrow \int \frac{1}{2}(\sin 11x - \sin 3x) dx$$

$$\Rightarrow \frac{1}{2}(\int \sin 11x dx - \int \sin 3x dx)$$

$$\text{We know } \int \sin ax dx = \frac{-1}{a} \cos ax + c$$

$$\Rightarrow \frac{1}{2} \left(\frac{-1}{11} \cos 11x + \frac{1}{3} \cos 3x \right)$$

$$\Rightarrow \frac{11 \cos 3x - 3 \cos 11x}{66} + c$$

2. Question

$$\int \cos 3x \cos 4x dx$$

Answer

$$\text{We know } 2\cos A \cos B = \cos(A - B) + \cos(A + B)$$

$$\therefore \cos 4x \cos 3x = \frac{\cos x + \cos 7x}{2}$$

$\therefore$ The above equation becomes

$$\Rightarrow \int \frac{1}{2}(\cos x - \cos 7x) dx$$

$$\Rightarrow \frac{1}{2}(\int \cos x dx - \int \cos 7x dx)$$

$$\text{We know } \int \cos ax dx = \frac{1}{a} \sin ax + c$$

$$\Rightarrow \frac{1}{2} \left(\sin x - \frac{1}{7} \sin 7x \right)$$

$$\Rightarrow \frac{7 \sin x - \sin 7x}{14} + c$$

3. Question

$$\int \cos mx \cos nx dx, m \neq n$$

Answer

$$\text{We know } 2\cos A \cos B = \cos(A - B) + \cos(A + B)$$

$$\therefore \cos mx \cos nx = \frac{\cos(m-n)x + \cos(m+n)x}{2}$$

$\therefore$ The above equation becomes

$$\Rightarrow \int \frac{1}{2}(\cos(m-n)x + \cos(m+n)x) dx$$

$$\text{We know } \int \cos ax dx = \frac{1}{a} \sin ax + c$$

$$\Rightarrow \frac{1}{2} \left(\frac{1}{m-n} \sin(m-n)x + \frac{1}{m+n} \sin(m+n)x \right)$$

$$\Rightarrow \frac{1}{2} \left(\frac{(m+n) \sin(m-n)x + (m-n) \sin(m+n)x}{m^2 - n^2} \right) + c$$

4. Question

$$\int \sin mx \cos nx dx, m \neq n$$

Answer

$$\text{We know } 2\sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$\therefore \sin mx \cos nx = \frac{\sin(m+n)x + \sin(m-n)x}{2}$$

∴ The above equation becomes

$$\Rightarrow \int \frac{1}{2} (\sin(m+n)x + \sin(m-n)x) dx$$

$$\text{We know } \int \sin ax \, dx = \frac{-1}{a} \cos ax + c$$

$$\Rightarrow \frac{1}{2} \left(\frac{-1}{m+n} \cos(m+n)x - \frac{1}{(m-n)} \cos(m-n)x \right)$$

$$\Rightarrow \frac{1}{2} \left(\frac{-(m-n) \cos(m+n)x - (m+n) \cos(m-n)x}{m^2 - n^2} \right)$$

5. Question

$$\int \sin 2x \sin 4x \sin 6x \, dx$$

Answer

We need to simplify the given equation to make it easier to solve

$$\text{We know } 2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$\therefore \sin 4x \sin 2x = \frac{\cos 2x - \cos 6x}{2}$$

∴ The above equation becomes

$$\Rightarrow \int \frac{1}{2} (\cos 2x - \cos 6x) \sin 6x \, dx$$

$$\Rightarrow \frac{1}{2} \int ((\cos 2x \sin 6x) - (\cos 6x \sin 6x)) \, dx$$

$$\text{We know } 2\sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$\therefore \sin 6x \cos 2x = \frac{\sin 8x + \sin 4x}{2}$$

$$\text{Also } 2\sin x \cos x = \sin 2x$$

$$\therefore \sin 6x \cos 6x = \frac{\sin 12x}{2}$$

∴ The above equation simplifies to

$$\Rightarrow \frac{1}{2} \int \frac{1}{2} (\sin 8x + \sin 4x) \, dx - \int \frac{1}{2} \sin 12x \, dx$$

$$\Rightarrow \frac{1}{4} (\int \sin 8x \, dx + \int \sin 4x \, dx - \int \sin 12x \, dx)$$

$$\text{We know } \int \sin ax \, dx = \frac{-1}{a} \cos ax + c$$

$$\Rightarrow \frac{1}{4} \left(\frac{-1}{8} \cos 8x + \frac{(-1)}{4} \cos 4x + \frac{1}{12} \cos 12x + c \right)$$

$$\Rightarrow \frac{1}{4} \left(\frac{2\cos 12x - 3\cos 8x - 6\cos 4x}{24} + c \right)$$

$$\Rightarrow \frac{2\cos 12x - 3\cos 8x - 6\cos 4x}{96} + c \text{ (where } c \text{ is some arbitrary constant)}$$

6. Question

$$\int \sin x \cos 2x \sin 3x \, dx$$

Answer

$$\text{We know } 2\sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$\therefore \sin 3x \cos 2x = \frac{\sin 5x + \sin x}{2}$$

∴ The given equation becomes

$$\Rightarrow \int \frac{1}{2} (\sin 5x - \sin x) \sin x \, dx$$

$$\Rightarrow \int \frac{1}{2} (\sin 5x \sin x \, dx - \sin^2 x \, dx)$$

We know $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

$$\therefore \sin 5x \sin x = \frac{\cos 4x - \cos 6x}{2}$$

$$\text{Also } \sin^2 x = \frac{1 - \cos 2x}{2}$$

$\therefore$ Above equation can be written as

$$\Rightarrow \int \frac{1}{2} \left(\frac{1}{2} (\cos 4x - \cos 6x) \, dx - \frac{1}{2} (1 - \cos 2x) \, dx \right)$$

$$\Rightarrow \frac{1}{4} \int \cos 4x \, dx - \int \cos 6x \, dx - \int dx + \int \cos 2x \, dx$$

We know $\int \cos ax \, dx = \frac{1}{a} \sin ax + c$

$$\Rightarrow \frac{1}{4} \left(\frac{1}{4} \sin 4x - \frac{1}{6} \sin 6x - x + \frac{1}{2} \sin 2x + c \right)$$

$$\Rightarrow \frac{1}{4} \left(\frac{3 \sin 4x - 2 \sin 6x - 12 + 6 \sin 2x}{12} + c \right)$$

$$\Rightarrow \frac{3 \sin 4x - 2 \sin 6x - 12 + 6 \sin 2x}{48} + c$$

NOTE: - Whenever you are solving integral questions having trigonometric functions in the product then the first thing that should be done is convert them in the form of addition or subtraction.

Exercise 19.8

1. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{1 - \cos 2x}} \, dx$$

Answer

In the given equation $\cos 2x = \cos^2 x - \sin^2 x$

Also we know $\cos^2 x + \sin^2 x = 1$.

$\therefore$ Substituting the values in the above equation we get

$$\Rightarrow \int \frac{1}{\sqrt{\sin^2 x + \cos^2 x - (-\sin^2 x + \cos^2 x)}} \, dx$$

$$\Rightarrow \int \frac{1}{\sqrt{\sin^2 x + \cos^2 x + \sin^2 x - \cos^2 x}} \, dx$$

$$\Rightarrow \int \frac{1}{\sqrt{2 \sin^2 x}} \, dx$$

$$\Rightarrow \int \frac{1}{\sqrt{2} \sin x} \, dx$$

$$\Rightarrow \int \frac{\csc x}{\sqrt{2}} \, dx$$

$$\Rightarrow \frac{1}{\sqrt{2}} \int \csc x \, dx$$

$$\Rightarrow \frac{1}{\sqrt{2}} \log \left| \frac{\tan x}{2} \right| + c$$

2. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{1 + \cos x}} dx$$

Answer

In the given equation

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$$

$$\text{Also, } \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} = 1$$

Substituting in the above equation we get,

$$\Rightarrow \int \frac{1}{\sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + (\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2})}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{2 \cos^2 \frac{x}{2}}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{2} \cos \frac{x}{2}} dx$$

$$\Rightarrow \frac{1}{\sqrt{2}} \int \sec \frac{x}{2} dx$$

$$\Rightarrow \frac{1}{\sqrt{2}} \ln \left| \sec \frac{x}{2} + \tan \frac{x}{2} \right| + c$$

3. Question

Evaluate the following integrals:

$$\int \sqrt{\frac{1 + \cos 2x}{1 - \cos 2x}} dx$$

Answer

$$1 + \cos 2x = 2 \cos^2 x$$

$$1 - \cos 2x = 2 \sin^2 x$$

(both of them are trigonometric formulae)

$$\Rightarrow \int \sqrt{\frac{2 \cos^2 x}{2 \sin^2 x}} dx$$

$$\Rightarrow \int \sqrt{\cot^2 x} dx$$

$$\Rightarrow \int \cot x dx$$

$$\Rightarrow \ln |\sin x| + c$$

4. Question

Evaluate the following integrals:

$$\int \sqrt{\frac{1 - \cos x}{1 + \cos x}} dx$$

Answer

$$1 - \cos x = 2 \sin^2 \frac{x}{2}$$

$$1 + \cos x = 2 \cos^2 \frac{x}{2}$$

$$\Rightarrow \int \sqrt{\frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}}} dx$$

$$\Rightarrow \int \sqrt{\tan^2 \frac{x}{2}} dx$$

$$\Rightarrow \int \tan \frac{x}{2} dx$$

$$\Rightarrow -2 \ln \left| \cos \frac{x}{2} \right| + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{\sec x}{\sec 2x} dx$$

Answer

Here first of all convert secx in terms of cosx

∴ We get

$$\Rightarrow \sec x = \frac{1}{\cos x}, \sec 2x = \frac{1}{\cos 2x}$$

∴ We get

$$\Rightarrow \frac{\frac{1}{\cos x}}{\frac{1}{\cos 2x}} = \frac{\cos 2x}{\cos x}$$

∴ The equation now becomes

$$\Rightarrow \int \frac{\cos 2x}{\cos x} dx$$

We know

$$\cos 2x = 2 \cos^2 x - 1$$

∴ We can write the above equation as

$$\Rightarrow \int \frac{2 \cos^2 x - 1}{\cos x} dx$$

$$\Rightarrow \int 2 \cos x dx - \int \frac{1}{\cos x} dx$$

$$\Rightarrow 2 \sin x - \int \sec x dx$$

$$(\int \sec x dx = \ln |\sec x + \tan x| + c)$$

$$\Rightarrow 2 \sin x - \ln |\sec x + \tan x| + c$$

6. Question

Evaluate the following integrals:

$$\int \frac{\cos 2x}{(\cos x + \sin x)^2} dx$$

Answer

Expanding $(\cos x + \sin x)^2 = \cos^2 x + \sin^2 x + 2 \sin x \cos x$

We know $\cos^2 x + \sin^2 x = 1$, $2 \sin x \cos x = \sin 2x$

$$\therefore (\cos x + \sin x)^2 = 1 + \sin 2x$$

$\therefore$ we can write the given equation as

$$\Rightarrow \int \frac{\cos 2x}{1 + \sin 2x} dx$$

Assume $1 + \sin 2x = t$

$$\Rightarrow \frac{d(1 + \sin 2x)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow 2 \cos 2x dx = dt$$

$$\therefore \cos 2x dx = \frac{dt}{2}$$

Substituting these values in the above equation we get

$$\Rightarrow \int \frac{1}{2t} dt$$

$$\Rightarrow \frac{1}{2} \ln t + c$$

substituting $t = 1 + 2 \sin x$ in above equation

$$\Rightarrow \frac{1}{2} \ln(1 + 2 \sin x) + c$$

7. Question

Evaluate the following integrals:

$$\int \frac{\sin(x-a)}{\sin(x-b)} dx$$

Answer

While solving these types of questions, it is better to eliminate the denominator.

$$\Rightarrow \int \frac{\sin(x-a)}{\sin(x-b)} dx$$

Add and subtract b in $(x - a)$

$$\Rightarrow \int \frac{\sin(x-a+b-b)}{\sin(x-b)} dx$$

$$\Rightarrow \int \frac{\sin(x-b+b-a)}{\sin(x-b)} dx$$

Numerator is of the form $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Where $A = x - b$; $B = b - a$

$$\Rightarrow \int \frac{\sin(x-b) \cos(b-a) + \cos(x-b) \sin(b-a)}{\sin(x-b)} dx$$

$$\Rightarrow \int \frac{\sin(x-b) \cos(b-a)}{\sin(x-b)} dx + \int \frac{\cos(x-b) \sin(b-a)}{\sin(x-b)} dx$$

$$\Rightarrow \int \cos(b-a) dx + \int \cot(x-b) \sin(b-a) dx$$

$$\Rightarrow \cos(b-a) \int dx + \sin(b-a) \int \cot(x-b) dx$$

$$\text{As } \int \cot(x) dx = \ln |\sin x|$$

$$\Rightarrow \cos(b-a)x + \sin(b-a) \ln |\sin(x-b)|$$

8. Question

Evaluate the following integrals:

$$\int \frac{\sin(x-\alpha)}{\sin(x+\alpha)} dx$$

Answer

Add and subtract α in the numerator

$$\Rightarrow \int \frac{\sin(x-\alpha+\alpha-\alpha)}{\sin(x+\alpha)} dx$$

$$\Rightarrow \int \frac{\sin(x+\alpha-2\alpha)}{\sin(x+\alpha)} dx$$

Numerator is of the form $\sin(A-B) = \sin A \cos B - \cos A \sin B$

Where $A = x + \alpha$; $B = 2\alpha$

$$\Rightarrow \int \frac{\sin(x+\alpha)\cos(2\alpha) - \cos(x+\alpha)\sin(2\alpha)}{\sin(x+\alpha)} dx$$

$$\Rightarrow \int \frac{\sin(x+\alpha)\cos(2\alpha)}{\sin(x+\alpha)} dx + \int \frac{\cos(x+\alpha)\sin(2\alpha)}{\sin(x+\alpha)} dx$$

$$\Rightarrow \int \cos(2\alpha) dx + \int \cot(x+\alpha) \sin(2\alpha) dx$$

$$\Rightarrow \cos(2\alpha) \int dx + \sin(2\alpha) \int \cot(x+\alpha) dx$$

$$\text{As } \int \cot(x) dx = \ln |\sin x|$$

$$\Rightarrow \cos(2\alpha)x + \sin(2\alpha) \ln |\sin(x+\alpha)|$$

9. Question

Evaluate the following integrals:

$$\int \frac{1+\tan x}{1-\tan x} dx$$

Answer

Convert $\tan x$ in form of $\sin x$ and $\cos x$.

$$\Rightarrow \tan x = \frac{\sin x}{\cos x}$$

$\therefore$ The equation now becomes

$$\Rightarrow \int \frac{1 + \frac{\sin x}{\cos x}}{1 - \frac{\sin x}{\cos x}} dx$$

$$\Rightarrow \int \frac{\frac{\cos x + \sin x}{\cos x}}{\frac{\cos x - \sin x}{\cos x}} dx$$

$$\Rightarrow \int \frac{\cos x + \sin x}{\cos x - \sin x} dx$$

Let $\cos x - \sin x = t$

$$\therefore \frac{d(\cos x - \sin x)}{dx} = \frac{dt}{dx}$$

$$\Rightarrow -(\cos x + \sin x)dx = dt$$

Substituting dt and t

We get

$$\Rightarrow \int -\frac{dt}{t}$$

$$\Rightarrow -\ln t + c$$

$$t = \cos x - \sin x$$

$$\therefore -\ln|\cos x - \sin x| + c$$

10. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{\cos(x-a)} dx$$

Answer

Add and subtract a from x in the numerator

$\therefore$ The equation becomes

$$\Rightarrow \int \frac{\cos(x-a+a)}{\cos(x-a)} dx$$

Numerator is of the form $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Where $A = x - a$; $B = a$

$$\Rightarrow \int \frac{\cos(x-a)\cos a}{\cos(x-a)} dx - \int \frac{\sin(x-a)\sin a}{\cos(x-a)} dx$$

$$\Rightarrow \cos a \int dx - \sin a \int \tan(x-a) dx$$

$$\text{As } \int \tan x = \ln|\sec x| + c$$

$$\Rightarrow x \cos a - \sin a \frac{\ln|\sec(x-a)|}{(x-a)} + c$$

11. Question

Evaluate the following integrals:

$$\int \sqrt{\frac{1-\sin 2x}{1+\sin 2x}} dx$$

Answer

We know $\cos^2 x + \sin^2 x = 1$.

Also, $2\sin x \cos x = \sin 2x$

$$1 + \sin 2x = \cos^2 x + \sin^2 x + 2\sin x \cos x = (\cos x + \sin x)^2$$

$$1 - \sin 2x = \cos^2 x + \sin^2 x - 2\sin x \cos x = (\cos x - \sin x)^2$$

$\therefore$ The equation becomes

$$\Rightarrow \int \sqrt{\frac{(\cos x - \sin x)^2}{(\cos x + \sin x)^2}} dx$$

$$\Rightarrow \int \frac{(\cos x - \sin x)}{(\cos x + \sin x)} dx$$

Assume $\cos x + \sin x = t$

$$\therefore d(\cos x + \sin x) = dt$$

$$= \cos x - \sin x$$

$$\therefore dt = \cos x - \sin x$$

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c$$

But $t = \cos x + \sin x$

$$\therefore \ln|\cos x + \sin x| + c.$$

12. Question

Evaluate the following integrals:

$$\int \frac{e^{3x}}{e^{3x} + 1} dx$$

Answer

Assume $e^{3x} + 1 = t$

$$\Rightarrow d(e^{3x} + 1) = dt$$

$$\Rightarrow 3e^{3x} = dt$$

$$\Rightarrow e^{3x} = \frac{dt}{3}$$

Substituting t and dt in the given equation we get

$$\Rightarrow \int \frac{dt}{3t}$$

$$\Rightarrow \frac{1}{3} \int \frac{dt}{t}$$

$$\Rightarrow \frac{1}{3} \ln|t| + c$$

But $t = e^{3x} + 1$

$\therefore$ The above equation becomes

$$\Rightarrow \frac{1}{3} \ln|e^{3x} + 1| + c.$$

13. Question

Evaluate the following integrals:

$$\int \frac{\sec x \tan x}{3 \sec x + 5} dx$$

Answer

Assume $3 \sec x + 5 = t$

$$d(3 \sec x + 5) = dt$$

$$3 \sec x \tan x = dt$$

$$\sec x \tan x = \frac{dt}{3}$$

Substitute t and dt

We get

$$\Rightarrow \frac{1}{3} \int \frac{dt}{t}$$

$$\Rightarrow \frac{1}{3} \ln|t| + c$$

But $t = 3\sec x + 5$

$\therefore$ the equation becomes

$$\Rightarrow \frac{1}{3} \ln|3\sec x + 5| + c.$$

14. Question

Evaluate the following integrals:

$$\int \frac{1 - \cot x}{1 + \cot x} dx$$

Answer

Convert $\cot x$ in form of $\sin x$ and $\cos x$.

$$\Rightarrow \cot x = \frac{\cos x}{\sin x}$$

$\therefore$ The equation now becomes

$$\Rightarrow \int \frac{1 - \frac{\cos x}{\sin x}}{1 + \frac{\cos x}{\sin x}} dx$$

$$\Rightarrow \int \frac{\frac{\sin x - \cos x}{\sin x}}{\frac{\sin x + \cos x}{\sin x}} dx$$

$$\Rightarrow \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$$

Assume $\cos x + \sin x = t$

$$\therefore d(\cos x + \sin x) = dt$$

$$= \cos x - \sin x$$

$$\therefore dt = \cos x - \sin x$$

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c$$

But $t = \cos x + \sin x$

$$\therefore \ln|\cos x + \sin x| + c.$$

15. Question

Evaluate the following integrals:

$$\int \frac{\sec x \operatorname{cosec} x}{\log(\tan x)} dx$$

Answer

Assume $\log(\tan x) = t$

$$d(\log(\tan x)) = dt$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} dx = dt$$

$$\Rightarrow \sec x \cdot \operatorname{cosec} x \cdot dx = dt$$

Put t and dt in given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c.$$

$$\text{But } t = \log(\tan x)$$

$$= \ln|\log(\tan x)| + c.$$

16. Question

Evaluate the following integrals:

$$\int \frac{1}{x(3 + \log x)} dx$$

Answer

$$\text{Assume } 3 + \log x = t$$

$$d(3 + \log x) = dt$$

$$\Rightarrow \frac{1}{x} dx = dt$$

Put t and dt in given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c.$$

$$\text{But } t = 3 + \log x$$

$$= \ln|3 + \log x| + c$$

17. Question

Evaluate the following integrals:

$$\int \frac{e^x + 1}{e^x + x} dx$$

Answer

$$\text{Assume } e^x + x = t$$

$$d(e^x + x) = dt$$

$$e^x + 1 = dt$$

Put t and dt in given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c.$$

$$\text{But } t = e^x + x$$

$$= \ln|e^x + 1| + c$$

18. Question

Evaluate the following integrals:

$$\int \frac{1}{x \log x} dx$$

Answer

Assume $\log x = t$

$$d(\log x) = dt$$

$$\frac{1}{x} dx = dt$$

Put t and dt in given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c.$$

But $t = \log x$

$$= \ln|\log x| + c$$

19. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{a \cos^2 x + b \sin^2 x} dx$$

Answer

Assume $a \cos^2 x + b \sin^2 x = t$

$$d(a \cos^2 x + b \sin^2 x) = dt$$

$$(-2a \cos x \sin x + 2b \sin x \cos x) dx = dt$$

$$(b \sin 2x - a \sin 2x) dx = dt$$

$$(b - a) \sin 2x dx = dt$$

$$\sin 2x dx = \frac{dt}{(b-a)}$$

Put t and dt in given equation we get

$$\Rightarrow \frac{1}{(b-a)} \int \frac{dt}{t}$$

$$= \frac{1}{b-a} \ln|t| + c.$$

But $t = a \cos^2 x + b \sin^2 x$

$$= \frac{1}{b-a} \ln|a \cos^2 x + b \sin^2 x| + c.$$

20. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{2 + 3 \sin x} dx$$

Answer

Assume $2 + 3 \sin x = t$

$$d(2 + 3 \sin x) = dt$$

$$3\cos x dx = dt$$

$$\cos x dx = \frac{dt}{3}$$

Put t and dt in given equation we get

$$\Rightarrow \frac{1}{3} \int \frac{dt}{t}$$

$$= \frac{1}{3} \ln|t| + c$$

But $t = 2 + 3\sin x$

$$= \frac{1}{3} \ln|2 + 3\sin x| + c.$$

21. Question

Evaluate the following integrals:

$$\int \frac{1 - \sin x}{x + \cos x} dx$$

Answer

Assume $x + \cos x = t$

$$d(x + \cos x) = dt$$

$$\Rightarrow 1 - \sin x dx = dt$$

Put t and dt in given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c.$$

But $t = x + \cos x$

$$= \ln|x + \cos x| + c$$

22. Question

Evaluate the following integrals:

$$\int \frac{a}{b + ce^x} dx$$

Answer

First of all take e^x common from denominator so we get

$$\Rightarrow \int \frac{a}{e^x \left(\frac{b}{e^x} + c \right)} \cdot dx$$

$$\Rightarrow \int \frac{a \cdot e^{-x}}{b e^{-x} + c} dx$$

Assume $b e^{-x} + c = t$

$$d(b e^{-x} + c) = dt$$

$$\Rightarrow -b e^{-x} dx = dt$$

$$\Rightarrow e^{-x} dx = \frac{-dt}{b}$$

Substituting t and dt we get

$$\Rightarrow \int \frac{-adt}{bt}$$

$$\Rightarrow \frac{-a}{b} \ln|t| + c$$

But $t = (be^{-x} + c)$

$$\Rightarrow \frac{-a}{b} \ln|be^{-x} + c| + c$$

23. Question

Evaluate the following integrals:

$$\int \frac{1}{e^x + 1} dx$$

Answer

First of all, take e^x common from the denominator, so we get

$$\Rightarrow \int \frac{1}{e^x(e^{\frac{1}{e^x}} + 1)} \cdot dx$$

$$\Rightarrow \int \frac{1 \cdot e^{-x}}{e^{-x} + 1} dx$$

Assume $e^{-x} + 1 = t$

$$d(e^{-x} + 1) = dt$$

$$\Rightarrow -e^{-x} dx = dt$$

$$\Rightarrow e^{-x} dx = -dt$$

Substituting t and dt we get

$$\Rightarrow \int \frac{-dt}{t}$$

$$\Rightarrow \ln|t| + c$$

But $t = (e^{-x} + 1)$

$$\Rightarrow \ln|e^{-x} + 1| + c.$$

24. Question

Evaluate the following integrals:

$$\int \frac{\cot x}{\log \sin x} dx$$

Answer

Assume $\log(\sin x) = t$

$$d(\log(\sin x)) = dt$$

$$\Rightarrow \frac{\cos x}{\sin x} dx = dt$$

$$\Rightarrow \cot x dx = dt$$

Put t and dt in given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c.$$

But $t = \log(\sin x)$

$$= \ln | \log(\sin x) | + c$$

25. Question

Evaluate the following integrals:

$$\int \frac{e^{2x}}{e^{2x} - 2} dx$$

Answer

Assume $e^{2x} - 2 = t$

$$d(e^{2x} - 2) = dt$$

$$\Rightarrow 2e^{2x} dx = dt$$

$$\Rightarrow e^{2x} dx = \frac{dt}{2}$$

Put t and dt in the given equation we get

$$\Rightarrow \frac{1}{2} \int \frac{dt}{t}$$

$$= \frac{1}{2} \ln |t| + c$$

But $t = e^{2x} - 2$

$$= \frac{1}{2} \ln |e^{2x} - 2| + c$$

26. Question

Evaluate the following integrals:

$$\int \frac{2 \cos x - 3 \sin x}{6 \cos x + 4 \sin x} dx$$

Answer

Taking 2 common in denominator we get

$$\Rightarrow \int \frac{2 \cos x - 3 \sin x}{2(3 \cos x + 2 \sin x)} dx$$

Now assume

$$3 \cos x + 2 \sin x = t$$

$$(-3 \sin x + 2 \cos x) dx = dt$$

Put t and dt in given equation we get

$$\Rightarrow \frac{1}{2} \int \frac{dt}{t}$$

$$= \frac{1}{2} \ln |t| + c$$

But $t = 3 \cos x + 2 \sin x$

$$= \frac{1}{2} \ln |3 \cos x + 2 \sin x| + c$$

27. Question

Evaluate the following integrals:

$$\int \frac{\cos 2x + x + 1}{x^2 + \sin 2x + 2x} dx$$

Answer

Assume $x^2 + \sin 2x + 2x = t$

$$d(x^2 + \sin 2x + 2x) = dt$$

$$(2x + 2\cos 2x + 2)dx = dt$$

$$2(x + \cos 2x + 1)dx = dt$$

$$(x + \cos 2x + 1)dx = \frac{1}{2}dt$$

Put t and dt in given equation we get

$$\Rightarrow \frac{1}{2} \int \frac{dt}{t}$$

$$= \frac{1}{2} \ln|t| + c$$

But $t = x^2 + \sin 2x + 2x$

$$= \frac{1}{2} \ln|x^2 + \sin 2x + 2x| + c$$

28. Question

Evaluate the following integrals:

$$\int \frac{1}{\cos(x+a)\cos(x+b)} dx$$

Answer

$$\text{Let } I = \int \frac{1}{\cos(x+a)\cos(x+b)} dx$$

Dividing and multiplying I by $\sin(a-b)$ we get,

$$I = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x+a)\cos(x+b)} dx$$

$$I = \frac{1}{\sin(a-b)} \int \frac{\sin\{(x+a)-(x+b)\}}{\cos(x+a)\cos(x+b)} dx$$

$$I = \frac{1}{\sin(a-b)} \int \frac{\sin(x+a)\cos(x+b) - \cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)} dx$$

$$I = \frac{1}{\sin(a-b)} \int \{\tan(x+a) - \tan(x+b)\} dx$$

We know that,

$$\int \tan x dx = |\log \sec x| + c$$

Therefore,

$$I = \frac{1}{\sin(a-b)} \left\{ \frac{\log(\sec(x+a))}{x+a} - \frac{\log(\sec(x+b))}{x+b} \right\} + c$$

29. Question

Evaluate the following integrals:

$$\int \frac{-\sin x + 2\cos x}{2\sin x + \cos x} dx$$

Answer

Assume $2\sin x + \cos x = t$

$$d(2\sin x + \cos x) = dt$$

$$(2\cos x - \sin x)dx = dt$$

Put t and dt in given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c$$

But $t = 2\sin x + \cos x$

$$= \ln|2\sin x + \cos x| + c.$$

30. Question

Evaluate the following integrals:

$$\int \frac{\cos 4x - \cos 2x}{\sin 4x - \sin 2x} dx$$

Answer

Assume $\sin 4x - \sin 2x = t$

$$d(\sin 4x - \sin 2x) = dt$$

$$(\cos 4x - \cos 2x)dx = dt$$

Put t and dt in given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c$$

But $t = \sin 4x - \sin 2x$

$$= \ln|\sin 4x - \sin 2x| + c.$$

31. Question

Evaluate the following integrals:

$$\int \frac{\sec x}{\log(\sec x + \tan x)} dx$$

Answer

Assume $\log(\sec x + \tan x) = t$

$$d(\log(\sec x + \tan x)) = dt$$

(use chain rule to differentiate first differentiate $\log(\sec x + \tan x)$ then $(\sec x + \tan x)$)

$$\Rightarrow \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx = dt$$

$$\Rightarrow \frac{\sec x (\tan x + \sec x)}{\sec x + \tan x} dx = dt$$

$$\Rightarrow \sec x dx = dt$$

Put t and dt in the given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c$$

$$\text{But } t = \log(\sec x + \tan x)$$

$$= \ln | \log(\sec x + \tan x) | + c.$$

32. Question

Evaluate the following integrals:

$$\int \frac{\operatorname{cosec} x}{\log \tan \frac{x}{2}} dx$$

Answer

$$\text{Assume } \log(\tan \frac{x}{2}) = t$$

$$d(\log(\tan \frac{x}{2})) = dt$$

(use chain rule to differentiate)

$$\Rightarrow \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2}} dx = dt$$

$$\Rightarrow \frac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2}} dx = dt$$

$$\Rightarrow \frac{1}{\sin x} dx = dt$$

$$\Rightarrow \operatorname{cosec} x \, dx = dt$$

Put t and dt in the given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c$$

$$\text{But } t = \log(\tan \frac{x}{2})$$

$$= \ln | \log(\tan \frac{x}{2}) | + c.$$

33. Question

Evaluate the following integrals:

$$\int \frac{1}{x \log x \log(\log x)} dx$$

Answer

$$\text{Assume } \log(\log x) = t$$

$$d(\log(\log x)) = dt$$

(use chain rule to differentiate first)

$$\Rightarrow \frac{1}{x \log x} dx = dt$$

Put t and dt in given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c$$

$$\begin{aligned}\text{But } t &= \log(\log(x)) \\ &= \ln|\log(\log(x))| + c.\end{aligned}$$

34. Question

Evaluate the following integrals:

$$\int \frac{\operatorname{cosec}^2 x}{1 + \cot x} dx$$

Answer

$$\text{Assume } 1 + \cot x = t$$

$$d(1 + \cot x) = dt$$

$$\Rightarrow \operatorname{cosec}^2 x = dt$$

Put t and dt in given equation we get

$$\begin{aligned}\Rightarrow \int \frac{dt}{t} \\ = \ln|t| + c\end{aligned}$$

$$\text{But } t = 1 + \cot x$$

$$= \ln|1 + \cot x| + c.$$

35. Question

Evaluate the following integrals:

$$\int \frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}} dx$$

Answer

$$\text{Assume } 10^x + x^{10} = t$$

$$d(10^x + x^{10}) = dt$$

$$a^x = \log_e a$$

$$\Rightarrow 10x^9 + 10^x \log_e 10 = dt$$

Put t and dt in given equation we get

$$\begin{aligned}\Rightarrow \int \frac{dt}{t} \\ = \ln|t| + c\end{aligned}$$

$$\text{But } t = 10^x + x^{10}$$

$$= \ln|10^x + x^{10}| + c.$$

36. Question

Evaluate the following integrals:

$$\int \frac{1 - \sin 2x}{x + \cos^2 x} dx$$

Answer

$$\text{Assume } x + \cos^2 x = t$$

$$d(x + \cos^2 x) = dt$$

$$(1 + (-2\cos x \cdot \sin x))dx = dt$$

$$2\sin x \cdot \cos x = \sin 2x$$

$$(1 - \sin 2x)dx = dt$$

Put t and dt in given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c$$

$$\text{But } t = x + \cos^2 x$$

$$= \ln|x + \cos^2 x| + c.$$

37. Question

Evaluate the following integrals:

$$\int \frac{1 + \tan x}{x + \log x \sec x} dx$$

Answer

$$\text{Assume } x + \log x \sec x = t$$

$$d(x + \log x \sec x) = dt$$

$$1 + \frac{\sec x \tan x}{\sec x} dx = dt$$

$$(1 + \tan x)dx = dt$$

Put t and dt in given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c$$

$$\text{But } t = x + \log x \sec x$$

$$= \ln|x + \log x \sec x| + c.$$

38. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{a^2 + b^2 \sin^2 x} dx$$

Answer

$$\text{Assume } a^2 + b^2 \sin^2 x = t$$

$$d(a^2 + b^2 \sin^2 x) = dt$$

$$2b^2 \cdot \sin x \cdot \cos x \cdot dx = dt$$

$$(2\sin x \cdot \cos x = \sin 2x)$$

$$\sin 2x dx = \frac{dt}{b^2}$$

Put t and dt in the given equation we get

$$\Rightarrow \frac{1}{b^2} \int \frac{dt}{t}$$

$$= \frac{1}{b^2} \ln|t| + c$$

$$\text{But } t = a^2 + b^2 \sin^2 x$$

$$= \frac{1}{b^2} \ln|a^2 + b^2 \sin^2 x| + c.$$

39. Question

Evaluate the following integrals:

$$\int \frac{x+1}{x(x+\log x)} dx$$

Answer

$$\text{Assume } x + \log x = t$$

$$d(x + \log x) = dt$$

$$\Rightarrow \left(1 + \frac{1}{x}\right) dx = dt$$

$$\Rightarrow \left(\frac{x+1}{x}\right) dx = dt$$

Put t and dt in the given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c$$

$$\text{But } t = x + \log x$$

$$= \ln|x + \log x| + c.$$

40. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{1-x^2} (2 + 3 \sin^{-1} x)} dx$$

Answer

$$\text{Assume } 2 + 3 \sin^{-1} x = t$$

$$d(2 + 3 \sin^{-1} x) = dt$$

$$\Rightarrow \frac{3}{\sqrt{1-x^2}} dx = dt$$

$$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = \frac{dt}{3}$$

Put t and dt in the given equation we get

$$\Rightarrow \frac{1}{3} \int \frac{dt}{t}$$

$$= \frac{1}{3} \ln|t| + c$$

$$\text{But } t = 2 + 3 \sin^{-1} x$$

$$= \frac{1}{3} \ln|2 + 3 \sin^{-1} x| + c.$$

41. Question

Evaluate the following integrals:

$$\int \frac{\sec^2 x}{\tan x + 2} dx$$

Answer

Assume $\tan x + 2 = t$

$$d(\tan x + 2) = dt$$

$$(\sec^2 x dx) = dt$$

Put t and dt in given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c$$

But $t = \tan x + 2$

$$= \ln|\tan x + 2| + c.$$

42. Question

Evaluate the following integrals:

$$\int \frac{2\cos 2x + \sec^2 x}{\sin 2x + \tan x - 5} dx$$

Answer

Assume $\sin 2x + \tan x - 5 = t$

$$d(\sin 2x + \tan x - 5) = dt$$

$$(2\cos 2x + \sec^2 x)dx = dt$$

Put t and dt in given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c$$

But $t = \sin 2x + \tan x - 5$

$$= \ln|\sin 2x + \tan x - 5| + c.$$

43. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$$

Answer

$\sin 2x$ can be written as $\sin(5x - 3x)$

$\therefore$ The equation now becomes

$$\Rightarrow \int \frac{\sin(5x-3x)}{\sin 5x \sin 3x} dx$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\Rightarrow \int \frac{\sin 5x \cos 3x - \cos 5x \sin 3x}{\sin 5x \sin 3x} dx$$

$$\Rightarrow \int \frac{\sin 5x \cos 3x}{\sin 5x \sin 3x} dx - \int \frac{\cos 5x \sin 3x}{\sin 5x \sin 3x} dx$$

$$\Rightarrow \int \frac{\cos 3x}{\sin 3x} dx - \int \frac{\cos 5x}{\sin 5x} dx$$

$$\Rightarrow \int \cot 3x dx - \int \cot 5x dx$$

$$\Rightarrow \frac{1}{3} \ln |\sin 3x| - \frac{1}{5} \ln |\sin 5x| + c.$$

44. Question

Evaluate the following integrals:

$$\int \frac{1 + \cot x}{x + \log \sin x} dx$$

Answer

Assume $x + \log(\sin x) = t$

$$d(x + \log(\sin x)) = dt$$

$$1 + \frac{\cos x}{\sin x} dx = dt$$

$$(1 + \cot) dx = dt$$

Put t and dt in given equation we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c$$

But $t = x + \log(\sin x)$

$$= \ln|x + \log(\sin x)| + c.$$

45. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{x}(\sqrt{x} + 1)} dx$$

Answer

Assume $\sqrt{x} + 1 = t$

$$d(\sqrt{x} + 1) = dt$$

$$\Rightarrow \frac{1}{2\sqrt{x}} dx = dt$$

$$\Rightarrow \frac{1}{\sqrt{x}} dx = 2dt$$

Put t and dt in given equation we get

$$\Rightarrow \int 2 \frac{dt}{t}$$

$$= \ln|t| + c$$

But $t = \sqrt{x} + 1$

$$= 2 \ln|\sqrt{x} + 1| + c.$$

46. Question

Evaluate the following integrals:

$$\int \tan 2x \tan 3x \tan 5x \, dx$$

Answer

We know $\tan 5x = \tan(2x + 3x)$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\therefore \tan(2x + 3x) = \frac{\tan 2x + \tan 3x}{1 - \tan 2x \tan 3x}$$

$$\therefore \tan(5x) = \frac{\tan 2x + \tan 3x}{1 - \tan 2x \tan 3x}$$

$$\Rightarrow \tan(5x)(1 - \tan 2x \tan 3x) = \tan(2x) + \tan(3x)$$

$$\Rightarrow \tan(5x) - \tan 2x \tan 3x \tan 5x = \tan(2x) + \tan(3x)$$

$$\Rightarrow \tan(5x) - \tan(2x) - \tan(3x) = \tan 2x \tan 3x \tan 5x$$

Substituting the above result in given equation we get

$$\Rightarrow \int \tan 5x - \tan 3x - \tan 2x \, dx$$

$$\Rightarrow \int \tan 5x \, dx - \int \tan 3x \, dx - \int \tan 2x \, dx$$

$$\Rightarrow \frac{-1}{5} \ln|\cos 5x| - \frac{(-1)}{3} \ln|\cos 3x| - \frac{(-1)}{2} \ln|\cos 2x| + c.$$

$$\Rightarrow \frac{-1}{5} \ln|\cos 5x| + \frac{1}{3} \ln|\cos 3x| + \frac{1}{2} \ln|\cos 2x| + c.$$

47. Question

Evaluate the following integrals:

$$\int \{1 + \tan x \tan(x + \theta)\} \, dx$$

Answer

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\therefore \tan(x - (x + \theta)) = \frac{\tan x - \tan(x + \theta)}{1 + \tan x \tan(x + \theta)}$$

$$\therefore \tan(\theta) = \frac{\tan x - \tan(x + \theta)}{1 + \tan x \tan(x + \theta)}$$

$$\Rightarrow \tan(\theta)(1 + \tan x \tan(x + \theta)) = \tan(x) - \tan(x + \theta)$$

$$\Rightarrow (1 + \tan x \tan(x + \theta)) = \frac{1}{\tan \theta} (\tan x - \tan(x + \theta))$$

$$\Rightarrow \int \frac{1}{\tan \theta} (\tan x - \tan(x + \theta)) \, dx$$

$$\Rightarrow \frac{1}{\tan \theta} \int \tan x \, dx - \int \tan(x + \theta) \, dx$$

$$\Rightarrow \frac{1}{\tan \theta} (-\ln|\cos x| - (-\ln|\cos(x + \theta)|) + c.$$

$$\Rightarrow \frac{1}{\tan \theta} (-\ln|\cos x| + \ln|\cos(x + \theta)|) + c.$$

48. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{\sin\left(x - \frac{\pi}{6}\right)\sin\left(x + \frac{\pi}{6}\right)} dx$$

Answer

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\therefore \text{We can write } \sin\left(x - \frac{\pi}{6}\right) = \sin x \cos \frac{\pi}{6} - \cos x \sin \frac{\pi}{6}$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\therefore \text{We can write } \sin\left(x + \frac{\pi}{6}\right) = \sin x \cos \frac{\pi}{6} + \cos x \sin \frac{\pi}{6}$$

$\therefore$ The given equation becomes

$$\Rightarrow \int \frac{\sin 2x}{\left(\sin x \cos \frac{\pi}{6} - \cos x \sin \frac{\pi}{6}\right)\left(\sin x \cos \frac{\pi}{6} + \cos x \sin \frac{\pi}{6}\right)} dx$$

$$\Rightarrow \int \frac{\sin 2x}{\left(\sin x \frac{\sqrt{3}}{2} - \cos x \frac{1}{2}\right)\left(\sin x \frac{\sqrt{3}}{2} + \cos x \frac{1}{2}\right)} dx$$

Denominator is of the form $(a - b)(a + b) = a^2 - b^2$

$$\Rightarrow \int \frac{\sin 2x}{\left(\frac{3}{4} \sin^2 x - \cos^2 x \frac{1}{4}\right)} dx \dots (1)$$

$$\text{We know } \sin^2 x + \cos^2 x = 1$$

$$\therefore \sin^2 x = 1 - \cos^2 x$$

Substituting the above result in (1) we get

$$\Rightarrow \int \frac{\sin 2x}{\left(\frac{3}{4}(1 - \cos^2 x) - \cos^2 x \frac{1}{4}\right)} dx$$

$$\Rightarrow \int \frac{\sin 2x}{\left(\frac{3}{4} - \cos^2 x\right)} dx \dots (2)$$

$$\text{Let us assume } \left(\frac{3}{4} - \cos^2 x\right) = t$$

$$\Rightarrow d\left(\frac{3}{4} - \cos^2 x\right) = dt$$

$$\Rightarrow 2 \sin x \cdot \cos x \cdot dx = dt$$

$$\Rightarrow \sin 2x \cdot dx = dt$$

Substituting dt and t in (2) we get

$$\Rightarrow \int \frac{dt}{t}$$

$$= \ln|t| + c$$

$$\text{But } t = \left(\frac{3}{4} - \cos^2 x\right)$$

$$\therefore \ln\left|\left(\frac{3}{4} - \cos^2 x\right)\right| + c.$$

49. Question

Evaluate the following integrals:

$$\int \frac{e^{x-1} + x^{e-1}}{e^x + x^e} dx$$

Answer

Multiplying and dividing the numerator by e we get the given as

$$\Rightarrow \frac{1}{e} \int \frac{e^x + ex^{e-1}}{e^x + x^e} dx \dots (1)$$

Assume $e^x + x^e = t$

$$\Rightarrow d(e^x + x^e) = dt$$

$$\Rightarrow e^x + ex^{e-1} = dt$$

Substituting t and dt in equation 1 we get

$$\Rightarrow \frac{1}{e} \int \frac{dt}{t}$$

$$= \ln|t| + c$$

$$\text{But } t = e^x + x^e$$

$$\therefore \ln|e^x + x^e| + c.$$

50. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin x \cos^2 x} dx$$

Answer

We know $\sin^2 x + \cos^2 x = 1$

$$\Rightarrow \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^2 x} dx$$

$$\Rightarrow \int \frac{\sin^2 x}{\sin x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin x \cos^2 x} dx$$

$$\Rightarrow \int \frac{\sin x}{\cos^2 x} dx + \int \frac{1}{\sin x} dx$$

$$\Rightarrow \int \tan x \sec x dx + \int \csc x dx$$

$$d(\sec x) = \tan x \cdot \sec x$$

$$\therefore \int \tan x \sec x dx = \sec x + c$$

$$\therefore \int \tan x \sec x dx + \int \csc x dx$$

$$\therefore \int \csc x dx = \log \left| \tan \frac{x}{2} \right| + c$$

$$\Rightarrow \sec x + \log \left| \tan \frac{x}{2} \right| + c.$$

51. Question

Evaluate the following integrals:

$$\int \frac{1}{\cos 3x - \cos x} dx$$

Answer

The denominator is of the form $\cos C - \cos D = -2 \sin \left(\frac{C+D}{2} \right) \cdot \sin \left(\frac{C-D}{2} \right)$

$$\therefore \cos 3x - \cos x = -2 \sin \left(\frac{3+1}{2} x \right) \sin \left(\frac{3-1}{2} x \right)$$

$$\therefore \cos 3x - \cos x = -2 \sin 2x \cdot \sin x$$

$$-2 \sin 2x \cdot \sin x = -2 \cdot 2 \cdot \sin x \cdot \cos x \cdot \sin x$$

$$-2 \sin 2x \cdot \sin x = -4 \sin^2 x \cdot \cos x$$

$$\text{Also } \sin^2 x + \cos^2 x = 1$$

$$\Rightarrow \int \frac{\sin^2 x + \cos^2 x}{-4 \sin^2 x \cos x} dx$$

$$\Rightarrow \frac{-1}{4} \int \frac{\sin^2 x}{\sin^2 x \cos x} dx + \frac{-1}{4} \int \frac{\cos^2 x}{\sin^2 x \cos x} dx$$

$$\Rightarrow \frac{-1}{4} \left(\int \frac{1}{\cos x} dx + \int \frac{\cos x}{\sin^2 x} dx \right)$$

$$\Rightarrow \frac{-1}{4} \int \sec x dx + \int \csc x \cdot \cot x dx$$

$$d(\csc x) = \csc x \cdot \cot x$$

$$\therefore \int \csc x \cot x dx = \csc x + c$$

$$\therefore \int \sec x dx + \int \csc x \cdot \cot x dx$$

$$\therefore \int \sec x dx = \log |\sec x + \tan x| + c$$

$$\Rightarrow \frac{-1}{4} (\csc x + \log |\sec x + \tan x|) + c$$

Exercise 19.9

1. Question

Evaluate the following integrals:

$$\int \frac{\log x}{x} dx$$

Answer

Assume $\log x = t$

$$\Rightarrow d(\log x) = dt$$

$$\Rightarrow \frac{1}{x} dx = dt$$

Substituting t and dt in above equation we get

$$\Rightarrow \int t \cdot dt$$

$$\Rightarrow \frac{t^2}{2} + c$$

But $t = \log(x)$

$$\Rightarrow \frac{\log^2 x}{2} + c.$$

2. Question

Evaluate the following integrals:

$$\int \frac{\log \left(1 + \frac{1}{x} \right)}{x(1+x)} dx$$

Answer

$$\text{Assume } \log\left(1 + \frac{1}{x}\right) = t$$

$$\Rightarrow d\left(\log\left(1 + \frac{1}{x}\right)\right) = dt$$

$$\Rightarrow \frac{1}{1 + \frac{1}{x}} \times \frac{-1}{x^2} dx = dt$$

$$\Rightarrow \frac{x}{x+1} \times \frac{-1}{x^2} dx = dt$$

$$\Rightarrow \frac{-1 \cdot dx}{x(x+1)} = dt$$

$$\Rightarrow \frac{dx}{x(x+1)} = -dt$$

$\therefore$ Substituting t and dt in the given equation we get

$$\Rightarrow \int -t \cdot dt$$

$$\Rightarrow -\int t \cdot dt$$

$$\Rightarrow \frac{-t^2}{2} + c$$

$$\text{But } \log\left(1 + \frac{1}{x}\right) = t$$

$$\Rightarrow -\frac{1}{2} \log^2\left(1 + \frac{1}{x}\right) + c$$

3. Question

Evaluate the following integrals:

$$\int \frac{(1 + \sqrt{x})^2}{\sqrt{x}} dx$$

Answer

$$\text{Assume } 1 + \sqrt{x} = t$$

$$\Rightarrow d(1 + \sqrt{x}) = dt$$

$$\Rightarrow \frac{1}{2\sqrt{x}} dx = dt$$

$$\Rightarrow \frac{1}{\sqrt{x}} dx = 2dt$$

$\therefore$ Substituting t and dt in the given equation we get

$$\Rightarrow \int 2t^2 \cdot dt$$

$$\Rightarrow 2 \int t^2 \cdot dt$$

$$\Rightarrow \frac{2t^3}{3} + c$$

$$\text{But } 1 + \sqrt{x} = t$$

$$\Rightarrow \frac{2(1 + \sqrt{x})^3}{3} + c$$

4. Question

Evaluate the following integrals:

$$\int \sqrt{1 + e^x} e^x dx$$

Answer

Assume $1 + e^x = t$

$$\Rightarrow d(1 + e^x) = dt$$

$$\Rightarrow e^x dx = dt$$

$\therefore$ Substituting t and dt in given equation we get

$$\Rightarrow \int \sqrt{t} \cdot dt$$

$$\Rightarrow \int t^{1/2} \cdot dt$$

$$\Rightarrow \frac{2t^{3/2}}{3} + c$$

But $1 + e^x = t$

$$\Rightarrow \frac{2(1 + e^x)^{3/2}}{3} + c.$$

5. Question

Evaluate the following integrals:

$$\int \sqrt[3]{\cos^2 x} \sin x \, dx$$

Answer

Assume $\cos x = t$

$$\Rightarrow d(\cos x) = dt$$

$$\Rightarrow -\sin x \, dx = dt$$

$$\Rightarrow dx = \frac{-dt}{\sin x}$$

$\therefore$ Substituting t and dt in the given equation we get

$$\Rightarrow \int \sqrt[3]{t^2} \sin x \cdot \frac{dt}{\sin x}$$

$$\Rightarrow \int t^{2/3} \cdot dt$$

$$\Rightarrow \frac{3t^{5/3}}{5} + c$$

But $\cos x = t$

$$\Rightarrow \frac{3(\cos x)^{5/3}}{5} + c.$$

6. Question

Evaluate the following integrals:

$$\int \frac{e^x}{(1 + e^x)^2} \, dx$$

Answer

Assume $1 + e^x = t$

$$\Rightarrow d(1 + e^x) = dt$$

$$\Rightarrow e^x dx = dt$$

∴ Substituting t and dt in given equation we get

$$\Rightarrow \int \frac{1}{t^2} dt$$

$$\Rightarrow \int t^{-2} . dt$$

$$\Rightarrow \frac{-1}{t} + c$$

But $1 + e^x = t$

$$\Rightarrow \frac{-1}{1 + e^x} + c.$$

7. Question

Evaluate the following integrals:

$$\int \cot^3 x \operatorname{cosec}^2 x \, dx$$

Answer

Assume $\cot x = t$

$$\Rightarrow d(\cot x) = dt$$

$$\Rightarrow -\operatorname{cosec}^2 x . dx = dt$$

$$\Rightarrow dt = \frac{-dx}{\operatorname{csc}^2 x}$$

∴ Substituting t and dt in the given equation we get

$$\Rightarrow \int t^3 \operatorname{csc}^2 x . \frac{-dx}{\operatorname{csc}^2 x}$$

$$\Rightarrow \int -t^3 . dx$$

$$\Rightarrow -\int t^3 . dt$$

$$\Rightarrow \frac{-t^4}{4} + c$$

But $t = \cot x$

$$\Rightarrow \frac{-\cot^4 x}{4} + c.$$

8. Question

Evaluate the following integrals:

$$\int \frac{\left\{ e^{\sin^{-1} x} \right\}^2}{\sqrt{1-x^2}} dx$$

Answer

Assume $\sin^{-1} x = t$

$$\Rightarrow d(\sin^{-1} x) = dt$$

$$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$$

∴ Substituting t and dt in the given equation we get

$$\Rightarrow \int e^{t^2} dt$$

$$\Rightarrow \int e^{2t} . dt$$

$$\Rightarrow \frac{e^{2t}}{2} + c$$

$$\text{But } t = \sin^{-1}x$$

$$\Rightarrow \frac{e^{2(\sin^{-1}x)}}{2} + c$$

9. Question

Evaluate the following integrals:

$$\int \frac{1 + \sin x}{\sqrt{x - \cos x}} dx$$

Answer

$$\text{Assume } x - \cos x = t$$

$$\Rightarrow d(x - \cos x) = dt$$

$$\Rightarrow (1 + \sin x) dx = dt$$

$\therefore$ Substituting t and dt in given equation we get

$$\Rightarrow \int \frac{1}{\sqrt{t}} dt$$

$$\Rightarrow \int t^{-1/2} . dt$$

$$\Rightarrow 2t^{1/2} + c$$

$$\text{But } t = x - \cos x.$$

$$\Rightarrow 2(x - \cos x)^{1/2} + c.$$

10. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{1-x^2} (\sin^{-1} x)^2} dx$$

Answer

$$\text{Assume } \sin^{-1}x = t$$

$$\Rightarrow d(\sin^{-1}x) = dt$$

$$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$$

$\therefore$ Substituting t and dt in the given equation we get

$$\Rightarrow \int \frac{1}{t^2} dt$$

$$\Rightarrow \int t^{-2} . dt$$

$$\Rightarrow \frac{t^{-1}}{-1} + c$$

$$\text{But } t = \sin^{-1}x$$

$$\Rightarrow \frac{-1}{\sin^{-1}x} + c$$

11. Question

Evaluate the following integrals:

$$\int \frac{\cot x}{\sqrt{\sin x}} dx$$

Answer

We know $d(\sin x) = \cos x$, and $\cot$ can be written in terms of $\cos$ and $\sin$

$$\therefore \cot x = \frac{\cos x}{\sin x}$$

$\therefore$ The given equation can be written as

$$\Rightarrow \int \frac{\cos x}{\sin x \sqrt{\sin x}} dx$$

$$\Rightarrow \int \frac{\cos x}{\sin^{3/2} x} dx$$

Now assume $\sin x = t$

$$d(\sin x) = dt$$

$$\cos x dx = dt$$

Substitute values of t and dt in above equation

$$\Rightarrow \int \frac{dt}{t^{3/2}}$$

$$\Rightarrow \int t^{-3/2} dt$$

$$\Rightarrow -2t^{-1/2} + c$$

$$\Rightarrow -2\sin^{-1/2} x + c$$

$$\Rightarrow \frac{-2}{\sqrt{\sin x}} + c$$

12. Question

Evaluate the following integrals:

$$\int \frac{\tan x}{\sqrt{\cos x}} dx$$

Answer

We know $d(\cos x) = -\sin x$, and $\tan$ can be written in terms of $\cos$ and $\sin$

$$\therefore \tan x = \frac{\sin x}{\cos x}$$

$\therefore$ The given equation can be written as

$$\Rightarrow \int \frac{\sin x}{\cos x \sqrt{\cos x}} dx$$

$$\Rightarrow \int \frac{\sin x}{\cos^{3/2} x} dx$$

Now assume $\cos x = t$

$$d(\cos x) = -dt$$

$$\sin x dx = -dt$$

Substitute values of t and dt in above equation

$$\Rightarrow \int \frac{-dt}{t^{3/2}}$$

$$\Rightarrow -\int t^{-3/2} dt$$

$$\Rightarrow 2t^{-1/2} + c$$

$$\Rightarrow 2\cos^{-1/2} x + c$$

$$\Rightarrow \frac{2}{\sqrt{\cos x}} + c$$

13. Question

Evaluate the following integrals:

$$\int \frac{\cos^3 x}{\sqrt{\sin x}} dx$$

Answer

In this equation, we can manipulate numerator

$$\cos^3 x = \cos^2 x \cdot \cos x$$

$\therefore$ Now the equation becomes,

$$\Rightarrow \int \frac{\cos^2 x \cdot \cos x}{\sqrt{\sin x}} dx$$

$$\cos^2 x = 1 - \sin^2 x$$

$$\Rightarrow \int \frac{1 - \sin^2 x \cdot \cos x}{\sqrt{\sin x}} dx$$

Now,

Let us assume $\sin x = t$

$$d(\sin x) = dt$$

$$\cos x dx = dt$$

Substitute values of t and dt in the above equation

$$\Rightarrow \int \frac{1 - t^2}{\sqrt{t}} dt$$

$$\Rightarrow \int \frac{1}{\sqrt{t}} dt - \int \frac{t^2}{\sqrt{t}} dt$$

$$\Rightarrow \int t^{-1/2} dt - \int t^{3/2} dt$$

$$\Rightarrow 2t^{1/2} - \frac{2}{5} t^{5/2} + c$$

But $t = \sin x$

$$\Rightarrow 2 \sin x^{1/2} - \frac{2}{5} \sin x^{5/2} + c$$

14. Question

Evaluate the following integrals:

$$\int \frac{\sin^3 x}{\sqrt{\cos x}} dx$$

Answer

In this equation, we can manipulate numerator

$$\sin^3 x = \sin^2 x \cdot \sin x$$

$\therefore$ Now the equation becomes,

$$\Rightarrow \int \frac{\sin^2 x \sin x}{\sqrt{\cos x}} dx$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\Rightarrow \int \frac{1 - \cos^2 x \sin x}{\sqrt{\cos x}} dx$$

Now ,

Let us assume $\cos x = t$

$$d(\cos x) = dt$$

$$-\sin x dx = dt$$

Substitute values of t and dt in above equation

$$\Rightarrow - \int \frac{1-t^2}{\sqrt{t}} dt$$

$$\Rightarrow - \int \frac{1}{\sqrt{t}} dt - \int \frac{t^2}{\sqrt{t}} dt$$

$$\Rightarrow - \int t^{-1/2} dt + \int t^{3/2} dt$$

$$\Rightarrow -2t^{1/2} + \frac{2}{5}t^{5/2} + c$$

But $t = \cos x$

$$\Rightarrow -2 \cos x^{1/2} + \frac{2}{5} \cos x^{5/2} + c$$

15. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{\tan^{-1} x (1+x^2)}} dx$$

Answer

Assume $\tan^{-1} x = t$

$$d(\tan^{-1} x) = dt$$

$$\Rightarrow \frac{1}{1+x^2} dx = dt$$

Substituting t and dt in above equation we get

$$\Rightarrow \int \frac{1}{\sqrt{t}} dt$$

$$\Rightarrow \int t^{-1/2} dt$$

$$\Rightarrow 2t^{1/2} + c$$

But $t = \tan^{-1} x$

$$\Rightarrow 2(\tan^{-1} x)^{1/2} + c.$$

16. Question

Evaluate the following integrals:

$$\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$$

Answer

Multiply and divide by $\cos x$

$$\Rightarrow \int \frac{\sqrt{\tan x} \cdot \cos x}{\sin x \cdot \cos x \cdot \cos x} dx$$

$$\Rightarrow \int \frac{\sqrt{\tan x}}{\tan x \cdot \cos^2 x} dx$$

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan x}} dx$$

Assume $\tan x = t$

$$d(\tan x) = dt$$

$$\sec^2 x \, dx = dt$$

Substituting t and dt in above equation we get

$$\Rightarrow \int \frac{1}{\sqrt{t}} dt$$

$$\Rightarrow \int t^{-1/2} \cdot dt$$

$$\Rightarrow 2t^{1/2} + c$$

But $t = \tan x$

$$\Rightarrow 2(\tan x)^{1/2} + c.$$

17. Question

Evaluate the following integrals:

$$\int \frac{1}{x} (\log x)^2 dx$$

Answer

Assume $\log x = t$

$$d(\log(x)) = dt$$

$$\Rightarrow \frac{1}{x} dx = dt$$

$\therefore$ Substituting t and dt in given equation we get

$$\Rightarrow \int t^2 \cdot dt$$

$$\Rightarrow \int t^2 \cdot dt$$

$$\Rightarrow \frac{t^3}{3} + c$$

But $\log x = t$

$$\Rightarrow \frac{(\log(x))^3}{3} + c.$$

18. Question

Evaluate the following integrals:

$$\int \sin^5 x \cos x \, dx$$

Answer

Assume $\sin x = t$

$$d(\sin x) = dt$$

$$\cos x dx = dt$$

∴ Substituting t and dt in given equation we get

$$\Rightarrow \int t^5 dt$$

$$\Rightarrow \frac{t^6}{6} + c$$

But $t = \sin x$

$$\Rightarrow \frac{\sin^6 x}{6} + c.$$

19. Question

Evaluate the following integrals:

$$\int \tan^{3/2} x \sec^2 x dx$$

Answer

Assume $\tan x = t$

$$d(\tan x) = dt$$

$$\sec^2 x dx = dt$$

∴ Substituting t and dt in given equation we get

$$\Rightarrow \int t^{3/2} dt$$

$$\Rightarrow \frac{2t^{5/2}}{5} + c$$

But $t = \tan x$

$$\Rightarrow \frac{2\tan^{5/2} x}{5} + c$$

20. Question

Evaluate the following integrals:

$$\int \frac{x^3}{(x^2 + 1)^3} dx$$

Answer

Assume $x^2 + 1 = t$

$$\Rightarrow d(x^2 + 1) = dt$$

$$\Rightarrow 2x dx = dt$$

$$\Rightarrow x dx = \frac{dt}{2}$$

x^3 can be write as $x^2 \cdot x$

∴ Now the given equation becomes

$$\Rightarrow \int \frac{x^2 \cdot x dx}{(x^2 + 1)^3}$$

$$x^2 + 1 = t \Rightarrow x^2 = t - 1$$

$$\Rightarrow \int \frac{(t-1)dt}{2t^3}$$

$$\Rightarrow \frac{1}{2} \int \frac{t}{t^3} dt - \int \frac{1}{t^3} dt$$

$$\Rightarrow \frac{1}{2} \int t^{-2} dt - \int t^{-3} dt$$

$$\Rightarrow \frac{1}{2}(-1t^{-1} + \frac{1}{2}t^{-2}) + c$$

$$\text{But } t = (x^2 + 1)$$

$$\Rightarrow \frac{1}{2}(-1(x^2 + 1)^{-1} + \frac{1}{2}(x^2 + 1)^{-2}) + c$$

$$\Rightarrow \frac{-1}{2(x^2 + 1)} + \frac{1}{4(1 + x^2)^2} + c$$

$$\Rightarrow \frac{-4(1 + x^2)^2 + 2(1 + x^2)}{8(1 + x^2)^3} + c$$

21. Question

Evaluate the following integrals:

$$\int (4x + 2)\sqrt{x^2 + x + 1} dx$$

Answer

Here $(4x + 2)$ can be written as $2(2x + 1)$.

Now assume, $x^2 + x + 1 = t$

$$d(x^2 + x + 1) = dt$$

$$(2x + 1)dx = dt$$

$$\Rightarrow \int 2(2x + 1)\sqrt{x^2 + x + 1} dx$$

$$\Rightarrow \int 2\sqrt{t} dt$$

$$\Rightarrow \int 2t^{1/2} . dt$$

$$\Rightarrow \frac{4t^{3/2}}{3} + c$$

$$\text{But } t = x^2 + x + 1$$

$$\Rightarrow \frac{4(x^2 + x + 1)^{3/2}}{3} + c$$

22. Question

Evaluate the following integrals:

$$\int \frac{4x + 3}{\sqrt{2x^2 + 3x + 1}} dx$$

Answer

Assume, $2x^2 + 3x + 1 = t$

$$d(x^2 + x + 1) = dt$$

$$(4x + 3)dx = dt$$

Substituting t and dt in above equation we get

$$\Rightarrow \int \frac{1}{\sqrt{t}} dt$$

$$\Rightarrow \int t^{-1/2} . dt$$

$$\Rightarrow 2t^{1/2} + c$$

$$\text{But } t = 2x^2 + 3x + 1$$

$$\Rightarrow 2(2x^2 + 3x + 1)^{1/2} + c.$$

23. Question

Evaluate the following integrals:

$$\int \frac{1}{1 + \sqrt{x}} dx$$

Answer

$$x = t^2$$

$$d(x) = 2t . dt$$

$$dx = 2t . dt$$

Substituting t and dt we get

$$\Rightarrow \int \frac{2t . dt}{1 + t}$$

$$\Rightarrow 2 \int \frac{t . dt}{1 + t}$$

Add and subtract 1 from numerator

$$\Rightarrow 2 \int \frac{t + 1 - 1}{1 + t} dt$$

$$\Rightarrow 2 \left(\int \frac{t + 1}{t + 1} dt - \int \frac{1}{1 + t} dt \right)$$

$$\Rightarrow 2 \left(\int dt - \int \frac{1}{1 + t} dt \right)$$

$$\Rightarrow 2(t - \ln|1 + t|)$$

$$\text{But } t = \sqrt{x}$$

$$\Rightarrow 2(\sqrt{x} - \ln|1 + \sqrt{x}|) + c$$

24. Question

Evaluate the following integrals:

$$\int e^{\cos^2 x} \sin 2x dx$$

Answer

$$\text{Assume } \cos^2 x = t$$

$$d(\cos^2 x) = dt$$

$$- 2 \sin x \cos x dx = dt$$

$$- \sin 2x . dx = dt$$

Substituting t and dt

$$\Rightarrow \int e^t . dt$$

$$\Rightarrow e^t + c.$$

$$\text{But } t = \cos^2 x$$

$$\Rightarrow e^{\cos^2 x} + c$$

25. Question

Evaluate the following integrals:

$$\int \frac{1 + \cos x}{(x + \sin x)^3} dx$$

Answer

$$\text{Assume } x + \sin x = t$$

$$d(x + \sin x) = dt$$

$$(1 + \cos x)dx = dt$$

Substituting t and dt in given equation

$$\Rightarrow \int \frac{dt}{t^3}$$

$$\Rightarrow \int t^{-3} dt$$

$$\Rightarrow \frac{t^{-2}}{-2} + c$$

$$\Rightarrow \frac{-1}{2t^2} + c$$

$$\text{But } t = x + \sin x$$

$$\Rightarrow \frac{-1}{2(x + \sin x)^2} + c$$

26. Question

Evaluate the following integrals:

$$\int \frac{\cos x - \sin x}{1 + \sin 2x} dx$$

Answer

$$\text{We know } \cos^2 x + \sin^2 x = 1, 2\sin x \cos x = \sin 2x$$

$\therefore$ Denominator can be written as

$$\cos^2 x + \sin^2 x + 2\sin x \cos x = (\sin x + \cos x)^2$$

$\therefore$ Now the given equation becomes

$$\Rightarrow \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2} dx$$

$$\text{Assume } \cos x + \sin x = t$$

$$\therefore d(\cos x + \sin x) = dt$$

$$= \cos x - \sin x$$

$$\therefore dt = \cos x - \sin x$$

$$\Rightarrow \int \frac{dt}{t^2}$$

$$\Rightarrow \int \frac{1}{t^2} dt$$

$$\Rightarrow \int t^{-2} \cdot dt$$

$$\Rightarrow \frac{t^{-1}}{-1} + c$$

But $t = \cos x + \sin x$

$$\Rightarrow \frac{-1}{\cos x + \sin x} + c$$

27. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{(a + b \cos 2x)^2} dx$$

Answer

Assume $a + b \cos 2x = t$

$$d(a + b \cos 2x) = dt$$

$$-2b \sin 2x dx = dt$$

$$\sin 2x dx = \frac{-dt}{2b}$$

$$\Rightarrow \frac{-1}{2b} \int \frac{dt}{t^2}$$

$$\Rightarrow \frac{-1}{2b} \int \frac{1}{t^2} dt$$

$$\Rightarrow \frac{-1}{2b} \int t^{-2} \cdot dt$$

$$\Rightarrow \frac{t^{-1}}{2b} + c$$

But $t = a + b \cos 2x$

$$\Rightarrow \frac{1}{2b(a + b \cos 2x)} + c.$$

28. Question

Evaluate the following integrals:

$$\int \frac{\log x^2}{x} dx$$

Answer

Assume $\log x = t$

$$\Rightarrow d(\log x) = dt$$

$$\Rightarrow \frac{1}{x} dx = dt$$

Substituting the values of t and dt we get

$$\Rightarrow \int t^2 dt$$

$$\Rightarrow \frac{t^3}{3} + c$$

But $t = \log x$

$$\Rightarrow \frac{\log^3 x}{3} + c.$$

29. Question

Evaluate the following integrals:

$$\int \frac{\sin x}{(1 + \cos x)^2} dx$$

Answer

Assume $1 + \cos x = t$

$$\Rightarrow d(1 + \cos x) = dt$$

$$\Rightarrow -\sin x \cdot dx = dt$$

Substituting the values of t and dt we get

$$\Rightarrow -\int \frac{dt}{t^2}$$

$$\Rightarrow -\int \frac{1}{t^2} dt$$

$$\Rightarrow -\int t^{-2} \cdot dt$$

$$\Rightarrow \frac{t^{-1}}{-1} + C$$

But $t = 1 + \cos x$

$$\Rightarrow \frac{-1}{1 + \cos x} + C$$

30. Question

Evaluate the following integrals:

$$\int \cot x \log \sin x \, dx$$

Answer

Assume $\log(\sin x) = t$

$$d(\log(\sin x)) = dt$$

$$\Rightarrow \frac{\cos x}{\sin x} dx = dt$$

$$\Rightarrow \cot x \, dx = dt$$

Substituting the values of t and dt we get

$$\Rightarrow \int t \, dt$$

$$\Rightarrow \frac{t^2}{2} + C$$

But $t = \log(\sin x)$

$$\Rightarrow \frac{\log(\sin x)^2 x}{2} + C$$

31. Question

Evaluate the following integrals:

$$\int \sec x \log (\sec x + \tan x) \, dx$$

Answer

Assume $\log(\sec x + \tan x) = t$

$$d(\log(\sec x + \tan x)) = dt$$

(use chain rule to differentiate first differentiate $\log(\sec x + \tan x)$ then $(\sec x + \tan x)$)

$$\Rightarrow \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx = dt$$

$$\Rightarrow \frac{\sec x (\tan x + \sec x)}{\sec x + \tan x} dx = dt$$

$$\Rightarrow \sec x dx = dt$$

Put t and dt in given equation we get

Substituting the values of t and dt we get

$$\Rightarrow \int t dt$$

$$\Rightarrow \frac{t^2}{2} + c$$

But $t = \log(\sec x + \tan x)$

$$\Rightarrow \frac{\log^2(\sec x + \tan x)}{2} + c.$$

32. Question

Evaluate the following integrals:

$$\int \operatorname{cosec} x \log (\operatorname{cosec} x - \cot x) dx$$

Answer

Assume $\log(\operatorname{cosec} x - \cot x) = t$

$$d(\log(\operatorname{cosec} x - \cot x)) = dt$$

(use chain rule to differentiate first differentiate $\log(\sec x + \tan x)$ then $(\sec x + \tan x)$)

$$\Rightarrow \frac{-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x}{\operatorname{cosec} x - \cot x} dx = dt$$

$$\Rightarrow \frac{\operatorname{cosec} x (\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x} dx = dt$$

$$\Rightarrow \operatorname{cosec} x dx = dt$$

Put t and dt in given equation we get

Substituting the values of t and dt we get

$$\Rightarrow \int t dt$$

$$\Rightarrow \frac{t^2}{2} + c$$

But $t = \log(\operatorname{cosec} x - \cot x)$

$$\Rightarrow \frac{\log^2(\operatorname{cosec} x - \cot x)}{2} + c.$$

33. Question

Evaluate the following integrals:

$$\int x^3 \cos x^4 dx$$

Answer

Assume $x^4 = t$

$$d(x^4) = dt$$

$$4x^3 dx = dt$$

$$x^3 dx = \frac{dt}{4}$$

Substituting t and dt

$$\Rightarrow \int \frac{1}{4} \cos t \, dt$$

$$\Rightarrow \frac{1 \sin t}{4} + c$$

But $t = x^4$

$$\Rightarrow \frac{1}{4} \sin x^4 + c.$$

34. Question

Evaluate the following integrals:

$$\int x^3 \sin x^4 \, dx$$

Answer

Assume $x^4 = t$

$$d(x^4) = dt$$

$$4x^3 dx = dt$$

$$x^3 dx = \frac{dt}{4}$$

Substituting t and dt

$$\Rightarrow \int \frac{1}{4} \sin t \, dt$$

$$\Rightarrow \frac{-1 \cos t}{4} + c$$

But $t = x^4$

$$\Rightarrow \frac{-1}{4} \cos x^4 + c.$$

35. Question

Evaluate the following integrals:

$$\int \frac{x \sin^{-1} x^2}{\sqrt{1-x^4}} \, dx$$

Answer

Assume $\sin^{-1} x^2 = t$

$$\Rightarrow d(\sin^{-1} x) = dt$$

$$\Rightarrow \frac{2x dx}{\sqrt{1-x^4}} = dt$$

$$\Rightarrow \frac{x dx}{\sqrt{1-x^4}} = \frac{dt}{2}$$

$\therefore$ Substituting t and dt in given equation we get

$$\Rightarrow \int \frac{t}{2} dt$$

$$\Rightarrow \frac{1}{2} \int t \cdot dt$$

$$\Rightarrow \frac{t^2}{4} + c$$

But $t = \sin^{-1}x$

$$\Rightarrow \frac{(\sin^{-1}x)^2}{4} + c.$$

36. Question

Evaluate the following integrals:

$$\int x^3 \sin(x^4 + 1) dx$$

Answer

Assume $x^4 + 1 = t$

$$d(x^4 + 1) = dt$$

$$4x^3 dx = dt$$

$$x^3 dx = \frac{dt}{4}$$

Substituting t and dt

$$\Rightarrow \int \frac{1}{4} \sin t dt$$

$$\Rightarrow \frac{-1 \cos t}{4} + c$$

But $t = x^4 + 1$

$$\Rightarrow \frac{-1}{4} \cos(x^4 + 1) + c.$$

37. Question

Evaluate the following integrals:

$$\int \frac{(x+1)e^x}{\cos^2(xe^x)} dx$$

Answer

Assume $xe^x = t$

$$d(xe^x) = dt$$

$$(e^x + xe^x) dx = dt$$

$$e^x(1 + x) dx = dt$$

Substituting t and dt

$$\Rightarrow \int \frac{dt}{\cos^2 t}$$

$$\Rightarrow \int \sec^2 t dt$$

$$\Rightarrow \tan t + c$$

But $t = xe^x + 1$

$$\Rightarrow \tan(xe^x) + c.$$

38. Question

Evaluate the following integrals:

$$\int x^2 e^{x^3} \cos(e^{x^3}) dx$$

Answer

Assume $e^{x^3} = t$

$$\Rightarrow d(e^{x^3}) = dt$$

$$\Rightarrow 3x^2 \cdot e^{x^3} dx = dt$$

$$\Rightarrow x^2 \cdot e^{x^3} dx = \frac{dt}{3}$$

Substituting t and dt

$$\Rightarrow \int \frac{1}{3} \cos t \cdot dt$$

$$\Rightarrow \frac{1}{3} \sin t + c$$

But $t = e^{x^3}$

$$\Rightarrow \frac{1}{3} \sin e^{x^3} + c$$

39. Question

Evaluate the following integrals:

$$\int 2x \sec^3(x^2 + 3) \tan(x^2 + 3) dx$$

Answer

$\sec^3(x^2 + 3)$ can be written as $\sec^2(x^2 + 3) \cdot \sec(x^2 + 3)$

Now the question becomes

$$\Rightarrow \int 2x \cdot \sec^2(x^2 + 3) \sec(x^2 + 3) \tan(x^2 + 3) dx$$

Assume $\sec(x^2 + 3) = t$

$$d(\sec(x^2 + 3)) = dt$$

$$2x \sec(x^2 + 3) \tan(x^2 + 3) dx = dt$$

Substituting t and dt in the given equation

$$\Rightarrow \int t^2 dt$$

$$\Rightarrow \frac{t^3}{3} + c$$

$$\Rightarrow \frac{1}{3} (\sec(x^2 + 3))^3 + c.$$

40. Question

Evaluate the following integrals:

$$\int \left(\frac{x+1}{x} \right) (x + \log x)^2 dx$$

Answer

Assume $(x + \log x) = t$

$$d(x + \log x) = dt$$

$$\Rightarrow \left(1 + \frac{1}{x}\right) dx = dt$$

$$\Rightarrow \frac{x+1}{x} dx = dt$$

Substituting t and dt

$$\Rightarrow \int t^2 dt$$

$$\Rightarrow \frac{t^3}{3} + c.$$

But $t = x + \log x$

$$\Rightarrow \frac{(x + \log x)^3}{3} + c.$$

41. Question

Evaluate the following integrals:

$$\int \tan x \sec^2 x \sqrt{1 - \tan^2 x} dx$$

Answer

Assume $1 - \tan^2 x = t$

$$d(1 - \tan^2 x) = dt$$

$$2 \tan x \sec^2 x dx = dt$$

Substituting t and dt we get

$$\Rightarrow \Rightarrow \int \frac{1}{2} \sqrt{t} dt$$

$$\Rightarrow \int \frac{1}{2} t^{1/2} dt$$

$$\Rightarrow \frac{4t^{3/2}}{6} + c$$

But $t = 1 - \tan^2 x$

$$\Rightarrow \frac{-2(1 - \tan^2 x)^{3/2}}{3} + c.$$

42. Question

Evaluate the following integrals:

$$\int \log x \frac{\sin \left\{1 + (\log x)^2\right\}}{x} dx$$

Answer

Assume $1 + (\log x)^2 = t$

$$d(1 + (\log x)^2) = dt$$

$$\Rightarrow \frac{2 \log x}{x} dx = dt$$

$$\Rightarrow \frac{\log x}{x} dx = \frac{dt}{2}$$

$$\Rightarrow \int \sin t \frac{dt}{2}$$

$$\Rightarrow \frac{1}{2} \int \sin t \, dt$$

$$\Rightarrow \frac{-1}{2} \cos t + c$$

$$\text{But } t = 1 + (\log x)^2$$

$$\Rightarrow \frac{-1}{2} \cos(1 + \log x^2) + c.$$

43. Question

Evaluate the following integrals:

$$\int \frac{1}{x^2} \cos^2 \left(\frac{1}{x} \right) dx$$

Answer

$$\text{Assume } \frac{1}{x} = t$$

$$\Rightarrow \frac{1}{x^2} dx = -dt$$

Substituting t and dt we get

$$\Rightarrow \int \cos^2 t \, dt$$

$$\Rightarrow \cos^2 x = \frac{1 + \cos 2x}{2}$$

∴ The given equation becomes,

$$\Rightarrow \int \frac{1 + \cos 2t}{2} dt$$

$$\text{We know } \int \cos ax \, dx = \frac{1}{a} \sin ax + c$$

$$\Rightarrow \frac{1}{2} \int dt - \frac{1}{2} \int \cos(2t) \, dt$$

$$\Rightarrow \frac{t}{2} - \frac{1}{4} \sin(2t) + c$$

$$\text{But } \frac{1}{x} = t$$

$$\Rightarrow \frac{1}{2x} - \frac{1}{4} \sin \left(\frac{2}{x} \right) + c.$$

44. Question

Evaluate the following integrals:

$$\int \sec^4 x \tan x \, dx$$

Answer

$$\text{Put } \tan x = t$$

$$d(\tan x) = dt$$

$$\sec^2 x \, dx = dt$$

$$\Rightarrow dx = \frac{dt}{\sec^2 x}$$

$$\text{We can write } \sec^4 x = \sec^2 x \cdot \sec^2 x$$

Now, the question becomes

$$\Rightarrow \int \sec^2 x \cdot \sec^2 x \cdot \tan x \frac{dt}{\sec^2 x}$$

$$\Rightarrow \int \sec^2 x \cdot \tan x \, dt$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\tan x = t$$

$$t^2 + 1 = \sec^2 x$$

$$\Rightarrow \int (t^2 + 1) t \, dt$$

$$\Rightarrow \int t^3 \, dt + \int t \cdot dt$$

$$\Rightarrow \frac{t^4}{4} + \frac{t^2}{2} + c$$

$$\text{But } t = \tan x$$

$$\Rightarrow \frac{\tan^4 x}{4} + \frac{\tan^2 x}{2} + c$$

45. Question

Evaluate the following integrals:

$$\int \frac{e^{\sqrt{x}} \cos(e^{\sqrt{x}})}{\sqrt{x}} \, dx$$

Answer

$$\text{Assume } e^{\sqrt{x}} = t$$

$$d(e^{\sqrt{x}}) = dt$$

$$\Rightarrow \frac{e^{\sqrt{x}}}{2\sqrt{x}} \, dx = dt$$

$$\Rightarrow \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx = 2dt$$

Substituting t and dt

$$\Rightarrow 2 \int \cos t \, dt$$

$$= 2 \sin t + c$$

$$\text{But } t = e^{\sqrt{x}}$$

$$\Rightarrow 2 \sin(e^{\sqrt{x}}) + c.$$

46. Question

Evaluate the following integrals:

$$\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) \, dx$$

Answer

$$\text{Assume } \frac{1}{x} = t$$

$$\Rightarrow \frac{1}{x^2} \, dx = dt$$

Substituting t and dt we get

$$\Rightarrow \int \cos^2 t \, dt$$

$$\Rightarrow \cos^2 x = \frac{1 + \cos 2x}{2}$$

$\therefore$ The given equation becomes,

$$\Rightarrow \int \frac{1 - \cos 2t}{2} dx$$

$$\text{We know } \int \cos ax \, dx = \frac{1}{a} \sin ax + c$$

$$\Rightarrow \frac{1}{2} \int dx - \frac{1}{2} \int \cos(2t) \, dt$$

$$\Rightarrow \frac{t}{2} - \frac{1}{4} \sin(t) + c$$

$$\text{But } \frac{1}{x} = t$$

$$\Rightarrow \frac{1}{2x} - \frac{1}{4} \sin\left(\frac{1}{x}\right) + c.$$

47. Question

Evaluate the following integrals:

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

Answer

Assume $\sqrt{x} = t$

$$d(\sqrt{x}) = dt$$

$$\Rightarrow \frac{1}{2\sqrt{x}} dx = dt$$

$$\Rightarrow \frac{1}{\sqrt{x}} dx = 2dt$$

Substituting t and dt

$$\Rightarrow 2 \int \sin t \, dt$$

$$= -2 \cos t + c$$

But $\sqrt{x} = t$

$$\Rightarrow 2 \cos(\sqrt{x}) + c.$$

48. Question

Evaluate the following integrals:

$$\int \frac{(x+1)e^x}{\sin^2(xe^x)} dx$$

Answer

Assume $xe^x = t$

$$d(xe^x) = dt$$

$$(e^x + xe^x) dx = dt$$

$$e^x(1+x) dx = dt$$

Substituting t and dt

$$\Rightarrow \int \frac{dt}{\sin^2 t}$$

$$\Rightarrow \int \csc^2 t \, dt$$

$$\Rightarrow -\cot t + c$$

$$\text{But } t = xe^x + 1$$

$$\Rightarrow -\cot(xe^x) + c.$$

49. Question

Evaluate the following integrals:

$$\int 5^{x+\tan^{-1}x} \left(\frac{x^2+2}{x^2+1} \right) dx$$

Answer

$$\text{Assume } x + \tan^{-1}x = t$$

$$d(x + \tan^{-1}x) = dt$$

$$\Rightarrow 1 + \frac{1}{x^2+1} = dt$$

$$\Rightarrow \frac{2+x^2}{x^2+1} = dt$$

Substituting t and dt

$$\Rightarrow \int 5^t dt$$

$$\Rightarrow \frac{5^t}{\log 5} + c$$

$$\text{But } t = x + \tan^{-1}x$$

$$\Rightarrow \frac{5^{x+\tan^{-1}x}}{\log 5} + c.$$

50. Question

Evaluate the following integrals:

$$\int \frac{e^{m \sin^{-1}x}}{\sqrt{1-x^2}} dx$$

Answer

$$\text{Assume } \sin^{-1}x = t$$

$$d(\sin^{-1}x) = dt$$

$$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$$

$\therefore$ Substituting t and dt in given equation we get

$$\Rightarrow \int e^{mt} dt$$

$$\Rightarrow \frac{e^{mt}}{m} + c$$

$$\text{But } t = \sin^{-1}x$$

$$\Rightarrow \frac{e^{m \sin^{-1}x}}{m} + c$$

51. Question

Evaluate the following integrals:

$$\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$$

Answer

Assume $\sqrt{x} = t$

$$d(\sqrt{x}) = dt$$

$$\Rightarrow \frac{1}{2\sqrt{x}} dx = dt$$

$$\Rightarrow \frac{1}{\sqrt{x}} dx = 2dt$$

Substituting t and dt

$$\Rightarrow 2 \int \cos t dt$$

$$= 2 \sin t + c$$

But $\sqrt{x} = t$

$$\Rightarrow 2 \sin(\sqrt{x}) + c.$$

52. Question

Evaluate the following integrals:

$$\int \frac{\sin(\tan^{-1} x)}{1+x^2} dx$$

Answer

Assume $\tan^{-1} x = t$

$$d(\tan^{-1} x) = dt$$

$$\Rightarrow \frac{1}{x^2 + 1} = dt$$

Substituting t and dt

$$\Rightarrow \int \sin t dt$$

$$= -\cos t + c$$

But $t = \tan^{-1} x$

$$\Rightarrow -\cos(\tan^{-1} x) + c.$$

53. Question

Evaluate the following integrals:

$$\int \frac{\sin(\log x)}{x} dx$$

Answer

Assume $\log x = t$

$$d(\log x) = dt$$

$$\Rightarrow \frac{1}{x} dx = dt$$

Substituting t and dt

$$\Rightarrow \int \sin t \, dt$$

$$= -\cos t + c$$

But $t = \log x$

$$\Rightarrow \cos(\log x) + c.$$

54. Question

Evaluate the following integrals:

$$\int \frac{e^{m \tan^{-1} x}}{1+x^2} dx$$

Answer

Assume $\tan^{-1} x = t$

$$d(\tan^{-1} x) = dt$$

$$\Rightarrow \frac{1}{x^2 + 1} = dt$$

Substituting t and dt

$$\Rightarrow \int e^{mt} dt$$

$$\Rightarrow \frac{e^{mt}}{m} + c$$

But $t = \tan^{-1} x$

$$\Rightarrow \frac{e^{m \tan^{-1} x}}{m} + c.$$

55. Question

Evaluate the following integrals:

$$\int \frac{x}{\sqrt{x^2 + a^2} + \sqrt{x^2 - a^2}} dx$$

Answer

Rationalize the given equation we get

$$\Rightarrow \int \frac{x}{\sqrt{x^2 + a^2} + \sqrt{x^2 - a^2}} \times \frac{\sqrt{x^2 + a^2} - \sqrt{x^2 - a^2}}{\sqrt{x^2 + a^2} - \sqrt{x^2 - a^2}} dx$$

$$\Rightarrow \int \frac{x(\sqrt{x^2 + a^2} - \sqrt{x^2 - a^2})}{2a^2} dx$$

Assume $x^2 = t$

$$2x \cdot dx = dt$$

$$\Rightarrow dx = \frac{dt}{2x}$$

Substituting t and dt

$$\Rightarrow \int \frac{(\sqrt{t + a^2} - \sqrt{t - a^2})}{4a^2} dt$$

$$\Rightarrow \frac{1}{4a^2} \int (\sqrt{t + a^2} - \sqrt{t - a^2}) dt$$

$$\Rightarrow \frac{1}{4a^2} \int (t + a^2)^{1/2} dt - \int (t - a^2)^{1/2} dt$$

$$\Rightarrow \frac{1}{4a^2} \left(\frac{2}{3} (t + a^2)^{3/2} - \frac{2}{3} (t - a^2)^{3/2} \right)$$

But $t = x^2$

$$\Rightarrow \frac{1}{4a^2} \left(\frac{2}{3} (x^2 + a^2)^{3/2} - \frac{2}{3} (x^2 - a^2)^{3/2} \right)$$

56. Question

Evaluate the following integrals:

$$\int \frac{x \tan^{-1} x^2}{1 + x^4} dx$$

Answer

Assume $\tan^{-1} x^2 = t$

$$d(\tan^{-1} x^2) = dt$$

$$\Rightarrow \frac{2x}{x^4 + 1} = dt$$

$$\Rightarrow \frac{x}{x^4 + 1} = \frac{dt}{2}$$

Substituting t and dt

$$\Rightarrow \frac{1}{2} \int t dt$$

$$\Rightarrow \frac{t^2}{4} + c$$

But $t = \tan^{-1} x^2$

$$\Rightarrow \frac{(\tan^{-1} x^2)^2}{4} + c.$$

57. Question

Evaluate the following integrals:

$$\int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}} dx$$

Answer

Assume $\sin^{-1} x = t$

$$d(\sin^{-1} x) = dt$$

$$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = dt$$

$\therefore$ Substituting t and dt in given equation we get

$$\Rightarrow \int t^3 dt$$

$$\Rightarrow \frac{t^4}{4} + c$$

But $t = \sin^{-1} x$

$$\Rightarrow \frac{(\sin^{-1} x)^4}{4} + c.$$

58. Question

Evaluate the following integrals:

$$\int \frac{\sin(2 + 3 \log x)}{x} dx$$

Answer

Assume $2 + 3 \log x = t$

$$d(2 + 3 \log x) = dt$$

$$\Rightarrow \frac{3}{x} dx = dt$$

$$\Rightarrow \frac{1}{x} dx = \frac{dt}{3}$$

Substituting t and dt

$$\Rightarrow \frac{1}{3} \int \sin t dt$$

$$= -\cos t + c$$

But $t = 2 + 3 \log x$

$$\Rightarrow \frac{-1}{3} \cos(2 + 3 \log x) + c.$$

59. Question

Evaluate the following integrals:

$$\int x e^{x^2} dx$$

Answer

Assume $x^2 = t$

$$\Rightarrow 2x \cdot dx = dt$$

$$\Rightarrow x \cdot dx = \frac{dt}{2}$$

Substituting t and dt

$$\Rightarrow \int e^t \cdot \frac{dt}{2}$$

$$\Rightarrow \frac{1}{2} e^t + c$$

But $x^2 = t$

$$\Rightarrow \frac{e^{x^2}}{2} + c.$$

60. Question

Evaluate the following integrals:

$$\int \frac{e^{2x}}{1 + e^x} dx$$

Answer

Assume $1 + e^x = t$

$$e^x = t - 1$$

$$d(1 + e^x) = dt$$

$$e^x dx = dt$$

$$dx = \frac{dt}{e^x}$$

Substitute t and dt we get

$$\Rightarrow \int e^{2x} \cdot \frac{dt}{e^x}$$

$$\Rightarrow \int e^x \cdot dt$$

$$\Rightarrow \int (t - 1) dt$$

$$\Rightarrow \int t \cdot dt - \int dt$$

$$\Rightarrow \frac{t^2}{2} - t + c$$

But $t = 1 + e^x$

$$\Rightarrow \frac{(1 + e^x)^2}{2} - (1 + e^x) + c$$

61. Question

Evaluate the following integrals:

$$\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} dx$$

Answer

Assume $\sqrt{x} = t$

$$d(\sqrt{x}) = dt$$

$$\Rightarrow \frac{1}{2\sqrt{x}} dx = dt$$

$$\Rightarrow \frac{1}{\sqrt{x}} dx = 2dt$$

Substituting t and dt

$$\Rightarrow 2 \int \sec^2 t \cdot dt$$

$$= 2 \tan t + c$$

But $\sqrt{x} = t$

$$\Rightarrow 2 \tan(\sqrt{x}) + c.$$

62. Question

Evaluate the following integrals:

$$\int \tan^3 2x \sec 2x \, dx$$

Answer

$$\tan^3 2x \cdot \sec 2x = \tan^2 2x \cdot \tan 2x \cdot \sec 2x \cdot dx$$

$$\tan^2 2x = \sec^2 2x - 1$$

$$\Rightarrow \tan^2 2x \cdot \tan 2x \cdot \sec 2x \cdot dx = (\sec^2 2x - 1) \cdot \tan 2x \cdot \sec 2x \cdot dx$$

$$\Rightarrow \sec^2 2x \tan 2x \cdot \sec 2x \cdot dx - \tan 2x \cdot \sec 2x \cdot dx$$

$$\therefore \int \sec^2 2x \cdot \tan 2x \cdot \sec 2x \, dx - \int \tan 2x \cdot \sec 2x \cdot dx$$

$$\Rightarrow \int \sec^2 2x \cdot \tan 2x \cdot \sec 2x \cdot dx - \frac{\sec 2x}{2} + c$$

Assume $\sec 2x = t$

$$d(\sec 2x) = dt$$

$$\sec 2x \cdot \tan 2x \cdot dx = dt$$

$$\Rightarrow \int t^2 \cdot dt - \frac{\sec 2x}{2} + c$$

$$\Rightarrow \frac{t^3}{3} - \frac{\sec 2x}{2} + c$$

But $t = \sec 2x$

$$\Rightarrow \frac{(\sec 2x)^3}{3} - \frac{\sec 2x}{2} + c.$$

63. Question

Evaluate the following integrals:

$$\int \frac{x + \sqrt{x+1}}{x+2} dx$$

Answer

The given equation can be written as

$$\Rightarrow \int \frac{x}{x+2} dx + \int \frac{\sqrt{x+1}}{x+2} dx$$

First integration be I1 and second be I2.

$\Rightarrow$ For I1

Add and subtract 2 from the numerator

$$\Rightarrow \int \frac{x+2-2}{x+2}$$

$$\Rightarrow \int \frac{x+2}{x+2} \cdot dx - \int \frac{2}{x+2} \cdot dx$$

$$\Rightarrow \int dx - 2 \int \frac{dx}{x+2}$$

$$\Rightarrow x - 2 \ln|x+2| + c1$$

$$\therefore I1 = x - 2 \ln|x+2| + c1$$

For I2

$$\Rightarrow \int \frac{\sqrt{x+1}}{x+2} dx$$

Assume $x+1 = t$

$$dt = dx$$

$$\Rightarrow \int \frac{\sqrt{t}}{t+1} dt$$

Substitute $u = \sqrt{t}$

$$dt = 2\sqrt{t} \cdot du$$

$$t = u^2$$

$$\Rightarrow 2 \int \frac{u^2}{u^2 + 1} du$$

Add and subtract 1 in the above equation:

$$\Rightarrow 2 \int \frac{u^2 + 1 - 1}{u^2 + 1} du$$

$$\Rightarrow 2 \int \frac{u^2 + 1}{u^2 + 1} du - \int \frac{1}{u^2 + 1} du$$

$$\Rightarrow 2 \int du - \int \frac{1}{u^2 + 1} du$$

$$\Rightarrow 2u - \tan^{-1}(u) + c_2$$

But $u = \sqrt{t}$

$$\therefore 2\sqrt{t} - \tan^{-1}(\sqrt{t}) + c_2$$

Also $t = x + 1$

$$\therefore 2\sqrt{x + 1} - \tan^{-1}(x + 1) + c_2$$

$$I = I_1 + I_2$$

$$\therefore I = x - 2\ln|x + 2| + c_1 + 2\sqrt{x + 1} - \tan^{-1}(x + 1) + c_2$$

$$I = x - 2\ln|x + 2| + 2\sqrt{x + 1} - \tan^{-1}(x + 1) + c.$$

64. Question

Evaluate the following integrals:

$$\int 5^{5^{5^x}} 5^{5^x} 5^x dx$$

Answer

Assume $5^{5^{5^x}} = t$

$$\Rightarrow d(5^{5^{5^x}}) = dt$$

$$\Rightarrow 5^{5^{5^x}} \cdot 5^{5^x} 5^x (\log 5^3) dx = dt$$

Substituting t and dt

$$\Rightarrow 5^{5^{5^x}} \cdot 5^{5^x} 5^x \cdot dx = \frac{dt}{(\log 5^3)}$$

$$\Rightarrow \int \frac{dt}{(\log 5^3)}$$

$$\Rightarrow \frac{1}{(\log 5^3)} \int dt + c$$

$$\Rightarrow \frac{t}{(\log 5^3)} + c$$

But $t = 5^{5^{5^x}}$

$$\Rightarrow \frac{5^{5^{5^x}}}{(\log 5^3)} + c$$

65. Question

Evaluate the following integrals:

$$\int \frac{1}{x\sqrt{x^4-1}} dx$$

Answer

Assume $x^2 = t$

$$2x \cdot dx = dt$$

$$\Rightarrow dx = \frac{dt}{2x}$$

Substituting t and dt

$$\Rightarrow \int \frac{dt}{2x} \times \frac{1}{x\sqrt{t^2-1}}$$

$$\Rightarrow \int \frac{dt}{2x^2} \times \frac{1}{\sqrt{t^2-1}}$$

$$\Rightarrow \frac{1}{2} \int \frac{dt}{t\sqrt{t^2-1}}$$

$$\Rightarrow \frac{1}{2} \sec^{-1} t + c$$

But $t = x^2$

$$\Rightarrow \frac{1}{2} \sec^{-1} x^2 + c$$

66. Question

Evaluate the following integrals:

$$\int \sqrt{e^x - 1} dx$$

Answer

Assume $e^x - 1 = t^2$

$$d(e^x - 1) = d(t^2)$$

$$e^x \cdot dx = 2t \cdot dt$$

$$\Rightarrow dx = \frac{2t}{e^x} dt$$

$$e^x = t^2 + 1$$

$$\Rightarrow dx = \frac{2t}{t^2 + 1} dt$$

Substituting t and dt

$$\Rightarrow \int \sqrt{t^2} \cdot \frac{2t}{t^2 + 1} dt$$

$$\Rightarrow \int t \cdot \frac{2t}{t^2 + 1} dt$$

$$\Rightarrow \int \frac{2t^2}{t^2 + 1} dt$$

$$\Rightarrow 2 \int \frac{t^2}{t^2 + 1} dt$$

Add and subtract 1 in numerator

$$\Rightarrow 2 \int \frac{t^2 + 1 - 1}{t^2 + 1} dt$$

$$\Rightarrow 2 \int \frac{t^2+1}{t^2+1} dt - 2 \int \frac{1}{t^2+1} dt$$

$$\Rightarrow 2 \int dt - 2 \int \frac{1}{t^2+1} dt$$

$$\Rightarrow \int \frac{1}{t^2+1} dt = \tan^{-1} t + c$$

$$\Rightarrow 2t - 2\tan^{-1}(t) + c$$

$$\text{But } t = (e^x - 1)^{1/2}$$

$$\Rightarrow 2(e^x - 1)^{1/2} - 2\tan^{-1}(e^x - 1)^{1/2} + c$$

67. Question

Evaluate the following integrals:

$$\int \frac{1}{(x+1)(x^2+2x+2)} dx$$

Answer

$$\text{We can write } x^2 + 2x + 1 + 1 = (x+1)^2 + 1$$

$$\Rightarrow \frac{1 \cdot dx}{(x+1)(x+1)^2 + 1}$$

$$\text{Assume } x+1 = \tan t$$

$$\Rightarrow dx = \sec^2 t \cdot dt$$

$$\Rightarrow \int \frac{\sec^2 t \cdot dt}{\tan t \cdot \tan^2 t + 1}$$

$$\Rightarrow \tan^2 t + 1 = \sec^2 t.$$

$$\Rightarrow \int \frac{dt}{\tan t}$$

$$\Rightarrow \frac{\cos t}{\sin t} dt$$

$$\Rightarrow \log|\sin t| + c$$

$$\Rightarrow \sin t = \frac{\tan t}{\sec^2 t}$$

$$\text{But } \tan t = x+1$$

$$\Rightarrow \sin t = \frac{x+1}{(1+x)^2 + 1}$$

The final answer is

$$\Rightarrow \log \sin \left| \frac{x+1}{x^2+2x+2} \right| + c$$

68. Question

Evaluate the following integrals:

$$\int \frac{x^5}{\sqrt{1+x^3}} dx$$

Answer

$$\text{Assume } x^3 + 1 = t^2$$

$$d(x^3 + 1) = d(t^2)$$

$$3x^2 \cdot dx = 2t \cdot dt$$

$$\Rightarrow dx = \frac{2t}{3x^2} dt$$

$$x^3 + 1 = t^2$$

$$\Rightarrow dx = \frac{2t}{3x^2} dt$$

Substituting t and dt

$$\Rightarrow \int \frac{x^5}{\sqrt{t^2}} \cdot \frac{2t}{3x^2} dt$$

$$\Rightarrow \int \frac{x^3}{t} \cdot \frac{2t}{3} dt$$

$$\Rightarrow \int \frac{2x^3}{3} dt$$

$$\Rightarrow x^3 = t^2 - 1$$

$$\Rightarrow \frac{2}{3} \int (t^2 - 1) \cdot dt$$

$$\Rightarrow \frac{2}{3} \int t^2 dt - \frac{2}{3} \int dt$$

$$\Rightarrow \frac{2}{3} \times \frac{t^3}{3} - \frac{2}{3} t + c$$

$$\Rightarrow \frac{2}{9} (x^3 + 1)^{3/2} - \frac{2}{3} (x^3 + 1)^{1/2} + c$$

69. Question

Evaluate the following integrals:

$$\int 4x^3 \sqrt{5 - x^2} dx$$

Answer

$$\text{Assume } 5 - x^2 = t^2$$

$$d(5 - x^2) = d(t^2)$$

$$- 2x \cdot dx = 2t \cdot dt$$

$$\Rightarrow x dx = - t \cdot dt$$

$$\Rightarrow dx = \frac{-t}{x} dt$$

Substituting t and dt

$$\Rightarrow \int 4x^3 \sqrt{t^2} \frac{-t}{x} dt$$

$$\Rightarrow 4 \int x^2 t^2$$

$$\Rightarrow x^2 = 5 - t^2$$

$$\Rightarrow 4 \int (5 - t^2) t^2 \cdot dt$$

$$\Rightarrow 20 \int t^2 dt - 4 \int t^4 dt$$

$$\Rightarrow 20 \times \frac{t^3}{3} - 4 \frac{t^5}{5} + c$$

$$\Rightarrow 20(5 - x^2)^{3/2} - \frac{4}{5} (5 - x^2)^{5/2} + c$$

70. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{x+x}} dx$$

Answer

$$x = t^2$$

$$d(x) = 2t.dt$$

$$dx = 2t.dt$$

Substituting t and dt we get

$$\Rightarrow \int \frac{2t.dt}{t^2+t}$$

$$\Rightarrow 2 \int \frac{t.dt}{t^2+t}$$

$$\Rightarrow 2 \int \frac{1}{1+t} dt$$

$$\Rightarrow 2(\ln|1+t|)$$

$$\text{But } t = \sqrt{x}$$

$$\Rightarrow 2(\ln|1+\sqrt{x}|) + c.$$

71. Question

Evaluate the following integrals:

$$\int \frac{1}{x^2(x^4+1)^{3/4}} dx$$

Answer

$$I = \int \frac{1}{x^2(x^4+1)^{3/4}} dx$$

$$\Rightarrow \int \frac{1}{x^5(1+\frac{1}{x^4})^{3/4}} dx$$

$$\text{Let } 1 + \frac{1}{x^4} = t$$

$$\Rightarrow -\frac{4}{x^5} dx = dt$$

$$\Rightarrow \frac{1}{x^5} dx = \frac{-dt}{4}$$

$$I = \frac{-1}{4} \int \frac{1}{t^{3/4}} dt$$

$$\Rightarrow \frac{-1}{4} \left(\frac{t^{1/4}}{\frac{1}{4}} \right) + c$$

$$\Rightarrow -t^{1/4} + c$$

$$\text{But } t = 1 + \frac{1}{x^4}$$

$$\Rightarrow -\left(1 + \frac{1}{x^4}\right)^{1/4} + c$$

72. Question

Evaluate the following integrals:

$$\int \frac{\sin^5 x}{\cos^4 x} dx$$

Answer

$$\sin^5 x = \sin^4 x \cdot \sin x$$

$$\text{Assume } \cos x = t$$

$$d(\cos x) = dt$$

$$-\sin x \cdot dx = dt$$

$$\Rightarrow dx = \frac{-dt}{\sin x}$$

Substitute t and dt we get

$$\Rightarrow \int \frac{\sin^4 x \cdot \sin x}{\cos^4 x} \times \frac{-dt}{\sin x}$$

$$\Rightarrow \int \frac{-dt(1 - \cos^2 x)^2}{\cos^4 x}$$

$$\Rightarrow \int \frac{-dt(1 - t^2)^2}{t^4}$$

$$\Rightarrow - \int \frac{1 + t^4 - 2t^2}{t^4} dt$$

$$\Rightarrow - \int \frac{1}{t^4} dt - \int \frac{t^4}{t^4} dt + 2 \int \frac{t^2}{t^4} dt$$

$$\Rightarrow - \int t^{-4} dt - \int dt + 2 \int t^{-2} dt$$

$$\Rightarrow \frac{t^{-3}}{3} - t - 2t^{-1} + c$$

$$\text{But } t = \cos x$$

$$\Rightarrow \frac{\cos^{-3} x}{3} - \cos x - 2 \cos^{-1} x + c$$

Exercise 19.10

1. Question

Evaluate the following integrals: $\int x^2 \sqrt{x+2} dx$

Answer

$$\text{Let } I = \int x^2 \sqrt{x+2} dx$$

$$\text{Substituting, } x+2 = t \Rightarrow dx = dt,$$

$$I = \int (t-2)^2 \sqrt{t} dt$$

$$\Rightarrow I = \int (t^2 - 4t + 4) \sqrt{t} dt$$

$$\Rightarrow I = \int \left(t^{\frac{5}{2}} - 4t^{\frac{3}{2}} + 4t^{\frac{1}{2}} \right) dt$$

$$\Rightarrow I = \frac{2}{7}t^{\frac{7}{2}} - \frac{8}{5}t^{\frac{5}{2}} + \frac{8}{2}t^{\frac{3}{2}} + c$$

$$\Rightarrow I = \frac{2}{7}(x+2)^{\frac{7}{2}} - \frac{8}{5}(x+2)^{\frac{5}{2}} + \frac{8}{2}(x+2)^{\frac{3}{2}} + c$$

$$\text{Therefore, } \int x^2 \sqrt{x+2} dx = \frac{2}{7}(x+2)^{\frac{7}{2}} - \frac{8}{5}(x+2)^{\frac{5}{2}} + \frac{8}{2}(x+2)^{\frac{3}{2}} + c$$

2. Question

Evaluate the following integrals: $\int \frac{x^2}{\sqrt{x-1}} dx$

Answer

$$\text{Let } I = \int \frac{x^2}{\sqrt{x-1}} dx$$

Substituting $x - 1 = t \Rightarrow dx = dt$,

$$\Rightarrow I = \int \frac{(t+1)^2}{\sqrt{t}} dt$$

$$\Rightarrow I = \int \frac{t^2 + 2t + 1}{\sqrt{t}} dt$$

$$\Rightarrow I = \int \left(t^{\frac{3}{2}} + 2t^{\frac{1}{2}} + t^{-\frac{1}{2}} \right) dt$$

$$\Rightarrow I = \frac{2}{5}t^{\frac{5}{2}} + 2t^{\frac{1}{2}} + \frac{4}{3}t^{\frac{3}{2}} + c$$

$$\Rightarrow I = \frac{\left(6t^{\frac{5}{2}} + 30t^{\frac{1}{2}} + 20t^{\frac{3}{2}} \right)}{15} + c$$

$$\Rightarrow I = \frac{2}{15}t^{\frac{1}{2}}(3t^2 + 15 + 10t) + c$$

$$\Rightarrow I = \frac{2}{15}(x-1)^{\frac{1}{2}}(3(x-1)^2 + 15 + 10(x-1)) + c$$

$$\Rightarrow I = \frac{2}{15}(x-1)^{\frac{1}{2}}(3(x^2 - 2x + 1)^2 + 15 + 10x - 10) + c$$

$$\Rightarrow I = \frac{2}{15}(x-1)^{\frac{1}{2}}(3x^2 + 4x + 8) + c$$

$$\text{Therefore, } \int \frac{x^2}{\sqrt{x-1}} dx = \frac{2}{15}(x-1)^{\frac{1}{2}}(3x^2 + 4x + 8) + c$$

3. Question

Evaluate the following integrals: $\int \frac{x^2}{\sqrt{3x+4}} dx$

Answer

$$\text{Let } I = \int \frac{x^2}{\sqrt{3x+4}} dx$$

Substituting $3x + 4 = t \Rightarrow 3dx = dt$,

$$\Rightarrow I = \int \frac{\left(\frac{t-4}{3}\right)^2}{3\sqrt{t}} dt$$

$$\Rightarrow I = \frac{1}{27} \int \frac{t^2 + 16 - 8t}{\sqrt{t}} dt$$

$$\Rightarrow I = \frac{1}{27} \int \left(t^{\frac{3}{2}} - 8t^{\frac{1}{2}} + 16t^{-\frac{1}{2}} \right) dt$$

$$\Rightarrow I = \frac{1}{27} \left[\frac{2}{5} t^{\frac{5}{2}} - \frac{16}{3} t^{\frac{3}{2}} + 32t^{\frac{1}{2}} \right] + c$$

$$\Rightarrow I = \frac{1}{27} \left[\frac{2}{5} (3x+4)^{\frac{5}{2}} - \frac{16}{3} (3x+4)^{\frac{3}{2}} + 32(3x+4)^{\frac{1}{2}} \right] + c$$

$$\Rightarrow I = \frac{2}{135} (3x+4)^{\frac{5}{2}} - \frac{16}{81} (3x+4)^{\frac{3}{2}} + \frac{32}{27} (3x+4)^{\frac{1}{2}} + c$$

Therefore, $\int \frac{x^2}{\sqrt{3x+4}} dx$
 $= \frac{2}{135} (3x+4)^{\frac{5}{2}} - \frac{16}{81} (3x+4)^{\frac{3}{2}} + \frac{32}{27} (3x+4)^{\frac{1}{2}} + c$

4. Question

Evaluate the following integrals: $\int \frac{2x-1}{(x-1)^2} dx$

Answer

$$\text{Let } I = \int \frac{2x-1}{(x-1)^2} dx$$

Substituting $x-1 = t \Rightarrow dx = dt$

$$\Rightarrow I = \int \frac{2(t+1)-1}{t^2} dt$$

$$\Rightarrow I = \int \frac{2t+1}{t^2} dt$$

$$\Rightarrow I = \int \left(\frac{2}{t} + \frac{1}{t^2} \right) dt$$

$$\Rightarrow I = 2 \log|t| + \frac{1}{t} + c$$

$$\Rightarrow I = 2 \log|x-1| + \frac{1}{x-1} + c$$

Therefore, $\int \frac{2x-1}{(x-1)^2} dx = 2 \log|x-1| + \frac{1}{x-1} + c$

5. Question

Evaluate the following integrals: $\int (2x^2+3)\sqrt{x+2} dx$

Answer

$$\text{Let } I = \int (2x^2+3)\sqrt{x+2} dx$$

Substituting $x+2 = t \Rightarrow dx = dt$

$$\Rightarrow I = \int [2(t-2)^2 + 3]\sqrt{t} dt$$

$$\Rightarrow I = \int [2t^2 - 8t + 8 + 3]\sqrt{t} dt$$

$$\Rightarrow I = \int \left[2t^{\frac{5}{2}} - 8t^{\frac{3}{2}} + 11t^{\frac{1}{2}} \right] dt$$

$$\Rightarrow I = \frac{4}{7}t^{\frac{7}{2}} - \frac{16}{5}t^{\frac{5}{2}} + \frac{22}{3}t^{\frac{3}{2}} + c$$

$$\Rightarrow I = \frac{4}{7}(x+2)^{\frac{7}{2}} - \frac{16}{5}(x+2)^{\frac{5}{2}} + \frac{22}{3}(x+2)^{\frac{3}{2}} + c$$

$$\therefore \int (2x^2 + 3)\sqrt{x+2} dx = \frac{4}{7}(x+2)^{\frac{7}{2}} - \frac{16}{5}(x+2)^{\frac{5}{2}} + \frac{22}{3}(x+2)^{\frac{3}{2}} + c$$

6. Question

Evaluate the following integrals: $\int \frac{x^2 + 3x + 1}{(x+1)^2} dx$

Answer

$$\text{Let } I = \int \frac{x^2 + 3x + 1}{(x+1)^2} dx$$

Substituting $x + 1 = t \Rightarrow dx = dt$

$$\Rightarrow I = \int \frac{(t-1)^2 + 3(t-1) + 1}{t^2} dt$$

$$\Rightarrow I = \int \frac{t^2 - 2t + 1 + 3t - 3 + 1}{t^2} dt$$

$$\Rightarrow I = \int \frac{t^2 + t - 1}{t^2} dt$$

$$\Rightarrow I = \int \left(1 + \frac{1}{t} - \frac{1}{t^2} \right) dt$$

$$\Rightarrow I = t + \log|t| - \frac{1}{t} + c$$

$$\Rightarrow I = (x+1) + \log|x+1| + \frac{1}{x+1} + c$$

$$\text{Therefore, } \int \frac{x^2 + 3x + 1}{(x+1)^2} dx = (x+1) + \log|x+1| + \frac{1}{x+1} + c$$

7. Question

Evaluate the following integrals: $\int \frac{x^2}{\sqrt{1-x}} dx$

Answer

$$\text{Let } I = \int \frac{x^2}{\sqrt{1-x}} dx$$

Substituting $1 - x = t \Rightarrow dx = -dt$,

$$\Rightarrow I = - \int \frac{(1-t)^2}{\sqrt{t}} dt$$

$$\Rightarrow I = - \int \frac{t^2 - 2t + 1}{\sqrt{t}} dt$$

$$\Rightarrow I = - \int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}} + t^{-\frac{1}{2}} \right) dt$$

$$\Rightarrow I = - \left[\frac{2}{5} t^{\frac{5}{2}} + 2t^{\frac{1}{2}} - \frac{4}{3} t^{\frac{3}{2}} \right] + c$$

$$\Rightarrow I = \frac{- \left(6t^{\frac{5}{2}} + 30t^{\frac{1}{2}} - 20t^{\frac{3}{2}} \right)}{15} + c$$

$$\Rightarrow I = \frac{-2}{15} t^{\frac{1}{2}} (3t^2 + 15 - 10t) + c$$

$$\Rightarrow I = \frac{-2}{15} (1-x)^{\frac{1}{2}} (3(1-x)^2 + 15 - 10(1-x)) + c$$

$$\Rightarrow I = \frac{2}{15} (1-x)^{\frac{1}{2}} (3(x^2 - 2x + 1)^2 + 15 + 10x - 10) + c$$

$$\Rightarrow I = \frac{2}{15} (1-x)^{\frac{1}{2}} (3x^2 + 4x + 8) + c$$

$$\text{Therefore, } \int \frac{x^2}{\sqrt{1-x}} dx = \frac{2}{15} (1-x)^{\frac{1}{2}} (3x^2 + 4x + 8) + c$$

8. Question

Evaluate the following integrals: $\int x(1-x)^{23} dx$

Answer

$$\text{Let } I = \int x(1-x)^{23} dx$$

Substituting $1-x = t \Rightarrow dx = -dt$

$$\Rightarrow I = - \int (1-t)t^{23} dt$$

$$\Rightarrow I = - \int (t^{23} - t^{24}) dt$$

$$\Rightarrow I = - \left[\frac{t^{24}}{24} - \frac{t^{25}}{25} \right] + c$$

$$\Rightarrow I = \frac{t^{25}}{25} - \frac{t^{24}}{24} + c$$

$$\Rightarrow I = \frac{(1-x)^{25}}{25} - \frac{(1-x)^{24}}{24} + c$$

$$\Rightarrow I = \frac{1}{600} (1-x)^{24} [24(1-x) - 25]$$

$$\Rightarrow I = -\frac{1}{600} (1-x)^{24} [1 + 24x] + c$$

9. Question

Evaluate the following integrals: $\int \frac{1}{\sqrt{x} + \sqrt[4]{x}} dx$

Answer

$$\text{Let } I = \int \frac{1}{\sqrt{x} + \sqrt[4]{x}} dx$$

$$\Rightarrow I = \int \frac{1}{\sqrt[4]{x}(\sqrt[4]{x} + 1)} dx$$

Multiplying and dividing by $\sqrt{x}$

$$\Rightarrow I = \int \frac{x^{\frac{1}{2}}}{x^{\frac{3}{4}}(\sqrt[4]{x} + 1)} dx$$

$$\text{Let, } \sqrt[4]{x} + 1 = t \Rightarrow \frac{1}{4}x^{-\frac{3}{4}}dx = dt$$

$$\text{So, } \Rightarrow I = 4 \int \frac{(t-1)^2}{t} dt$$

$$\Rightarrow I = 4 \int \frac{t^2 - 2t + 1}{t} dt$$

$$\Rightarrow I = 4 \int \left(t - 2 + \frac{1}{t} \right) dt$$

$$\Rightarrow I = 4 \left(\frac{t^2}{2} - 2t + \log|t| \right) + c$$

$$\Rightarrow I = 4 \left(\frac{(\sqrt[4]{x} + 1)^2}{2} - 2(\sqrt[4]{x} + 1) + \log|(\sqrt[4]{x} + 1)| \right) + c$$

$$\begin{aligned} \text{Therefore, } \int \frac{1}{\sqrt{x} + \sqrt[4]{x}} dx \\ = 4 \left(\frac{(\sqrt[4]{x} + 1)^2}{2} - 2(\sqrt[4]{x} + 1) + \log|(\sqrt[4]{x} + 1)| \right) + c \end{aligned}$$

10. Question

Evaluate the following integrals: $\int \frac{1}{x^{1/3}(x^{1/3} - 1)} dx$

Answer

$$\text{Let } I = \int \frac{1}{x^{\frac{1}{3}}(x^{\frac{1}{3}} - 1)} dx$$

Multiplying and dividing by $x^{\frac{1}{3}}$

$$\Rightarrow I = \int \frac{x^{\frac{1}{3}}}{x^{\frac{2}{3}}(x^{\frac{1}{3}} - 1)} dx$$

$$\text{Let, } x^{\frac{1}{3}} - 1 = t \Rightarrow \frac{1}{3}x^{-\frac{2}{3}}dx = dt$$

$$\text{So, } \Rightarrow I = 3 \int \frac{(t + 1)}{t} dt$$

$$\Rightarrow I = 3 \int \left(t + \frac{1}{t} \right) dt$$

$$\Rightarrow I = 3 \left(\frac{t^2}{2} + \log|t| \right) + c$$

$$\Rightarrow I = 3 \left(\frac{\left(x^{\frac{1}{3}} - 1 \right)^2}{2} + \log \left| \left(x^{\frac{1}{3}} - 1 \right) \right| \right) + c$$

$$\text{Therefore, } \int \frac{1}{\sqrt{x} + \sqrt[4]{x}} dx = 3 \left(\frac{\left(x^{\frac{1}{3}} - 1 \right)^2}{2} + \log \left| \left(x^{\frac{1}{3}} - 1 \right) \right| \right) + c$$

Exercise 19.11

1. Question

Evaluate the following integrals:

$$\int \tan^3 x \sec^2 x \, dx$$

Answer

$$\text{Let } I = \int \tan^3 x \sec^2 x \, dx$$

Let $\tan x = t$, then

$$\Rightarrow \sec^2 x \, dx = dt$$

$$\Rightarrow I = \int t^3 dt$$

$$\Rightarrow I = \frac{t^4}{4} + c$$

$$\Rightarrow I = \frac{\tan^4 x}{4} + c$$

$$\text{Therefore, } \int \tan^3 x \sec^2 x \, dx = \frac{\tan^4 x}{4} + c$$

2. Question

Evaluate the following integrals:

$$\int \tan x \sec^4 x \, dx$$

Answer

$$\text{Let } I = \int \tan x \sec^4 x \, dx$$

$$\Rightarrow I = \int \tan x \sec^2 x \sec^2 x \, dx$$

$$\Rightarrow I = \int \tan x (1 + \tan^2 x) \sec^2 x \, dx$$

$$\Rightarrow I = \int (\tan x + \tan^3 x) \sec^2 x \, dx$$

Let $\tan x = t$, then

$$\Rightarrow \sec^2 x \, dx = dt$$

$$\Rightarrow I = \int (t + t^3) dt$$

$$\Rightarrow I = \frac{t^2}{2} + \frac{t^4}{4} + c$$

$$\Rightarrow I = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + c$$

$$\text{Therefore, } \int \tan x \sec^4 x \, dx = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + c$$

3. Question

Evaluate the following integrals:

$$\int \tan^5 x \sec^4 x \, dx$$

Answer

$$\text{Let } I = \int \tan^5 x \sec^4 x \, dx$$

$$\Rightarrow I = \int \tan^5 x \sec^2 x \sec^2 x \, dx$$

$$\Rightarrow I = \int \tan^5 x (1 + \tan^2 x) \sec^2 x \, dx$$

$$\Rightarrow I = \int (\tan^5 x + \tan^7 x) \sec^2 x \, dx$$

Let $\tan x = t$, then

$$\Rightarrow \sec^2 x \, dx = dt$$

$$\Rightarrow I = \int (t^5 + t^7) dt$$

$$\Rightarrow I = \frac{t^6}{6} + \frac{t^8}{8} + c$$

$$\Rightarrow I = \frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + c$$

$$\text{Therefore, } \int \tan^5 x \sec^4 x \, dx = \frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + c$$

4. Question

Evaluate the following integrals:

$$\int \sec^6 x \tan x \, dx$$

Answer

$$\text{Let } I = \int \sec^6 x \tan x \, dx$$

$$\Rightarrow I = \int \sec^5 x (\sec x \tan x) dx$$

Substituting, $\sec x = t \Rightarrow \sec x \tan x dx = dt$

$$\Rightarrow I = \int t^5 dt$$

$$\Rightarrow I = \frac{t^6}{6} + c$$

$$\Rightarrow I = \frac{\sec^6 x}{6} + c$$

$$\text{Therefore, } \int \sec^5 x (\sec x \tan x) dx = \frac{\sec^6 x}{6} + c$$

5. Question

Evaluate the following integrals:

$$\int \tan^5 x dx$$

Answer

$$\text{Let } I = \int \tan^5 x dx$$

$$\Rightarrow I = \int \tan^2 x \tan^3 x dx$$

$$\Rightarrow I = \int (\sec^2 x - 1) \tan^3 x dx$$

$$\Rightarrow I = \int \tan^3 x \sec^2 x dx - \int \tan^3 x dx$$

$$\Rightarrow I = \int \tan^3 x \sec^2 x dx - \int (\sec^2 x - 1) \tan x dx$$

$$\Rightarrow I = \int \tan^3 x \sec^2 x dx - \int (\sec^2 x \tan x) dx + \int \tan x dx$$

Let $\tan x = t$, then

$$\Rightarrow \sec^2 x dx = dt$$

$$\Rightarrow I = \int t^3 dt - \int t dt + \int \tan x dx$$

$$\Rightarrow I = \frac{t^4}{4} - \frac{t^2}{2} + \log|\sec x| + c$$

$$\Rightarrow I = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log|\sec x| + c$$

$$\text{Therefore, } \int \tan^5 x dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log|\sec x| + c$$

6. Question

Evaluate the following integrals:

$$\int \sqrt{\tan x} \sec^4 x dx$$

Answer

$$\text{Let } I = \int \sqrt{\tan x} \sec^4 x \, dx$$

$$\Rightarrow I = \int \sqrt{\tan x} \sec^2 x \sec^2 x \, dx$$

$$\Rightarrow I = \int \sqrt{\tan x} (1 + \tan^2 x) \sec^2 x \, dx$$

$$\Rightarrow I = \int (\tan^{\frac{1}{2}} x + \tan^{\frac{5}{2}} x) \sec^2 x \, dx$$

Let $\tan x = t$, then

$$\Rightarrow \sec^2 x \, dx = dt$$

$$\Rightarrow I = \int \left(t^{\frac{1}{2}} + t^{\frac{5}{2}} \right) dt$$

$$\Rightarrow I = \frac{2}{3} t^{\frac{3}{2}} + \frac{2}{7} t^{\frac{7}{2}} + c$$

$$\Rightarrow I = \frac{2}{3} \tan^{\frac{3}{2}} x + \frac{2}{7} \tan^{\frac{7}{2}} x + c$$

$$\text{Therefore, } \int \sqrt{\tan x} \sec^4 x \, dx = \frac{2}{3} \tan^{\frac{3}{2}} x + \frac{2}{7} \tan^{\frac{7}{2}} x + c$$

7. Question

Evaluate the following integrals:

$$\int \sec^4 2x \, dx$$

Answer

$$\text{Let } I = \int \sec^4 2x \, dx$$

$$\Rightarrow I = \int \sec^2 2x \sec^2 2x \, dx$$

$$\Rightarrow I = \int (1 + \tan^2 2x) \sec^2 2x \, dx$$

$$\Rightarrow I = \int (\sec^2 2x + \tan^2 2x \sec^2 2x) \, dx$$

Let $\tan 2x = t$, then

$$\Rightarrow 2 \sec^2 2x \, dx = dt$$

$$\Rightarrow I = \frac{1}{2} \int (1 + t^2) dt$$

$$\Rightarrow I = \frac{1}{2} t + \frac{1}{2} \cdot \frac{1}{3} t^3 + c$$

$$\Rightarrow I = \frac{1}{2} \tan 2x + \frac{1}{6} \tan^3 2x + c$$

$$\text{Therefore, } \int \sec^4 2x \, dx = \frac{1}{2} \tan 2x + \frac{1}{6} \tan^3 2x + c$$

8. Question

Evaluate the following integrals:

$$\int \operatorname{cosec}^4 3x \, dx$$

Answer

$$\text{Let } I = \int \operatorname{cosec}^4 3x \, dx$$

$$\Rightarrow I = \int \operatorname{cosec}^2 3x \operatorname{cosec}^2 3x \, dx$$

$$\Rightarrow I = \int (1 + \cot^2 3x) \operatorname{cosec}^2 3x \, dx$$

$$\Rightarrow I = \int (\operatorname{cosec}^2 3x + \cot^2 3x \operatorname{cosec}^2 3x) \, dx$$

Let $\cot 3x = t$, then

$$\Rightarrow -3 \operatorname{cosec}^2 3x \, dx = dt$$

$$\Rightarrow I = -\frac{1}{3} \int (1 + t^2) \, dt$$

$$\Rightarrow I = -\frac{1}{3}t - \frac{1}{3} \cdot \frac{1}{3}t^3 + c$$

$$\Rightarrow I = -\frac{1}{3} \cot 3x - \frac{1}{9} \cot^3 3x + c$$

$$\text{Therefore, } \int \operatorname{cosec}^4 3x \, dx = -\frac{1}{3} \cot 3x - \frac{1}{9} \cot^3 3x + c$$

9. Question

Evaluate the following integrals:

$$\int \cot^n x \operatorname{cosec}^2 x \, dx, \, n \neq -1$$

Answer

$$\text{Let } I = \int \cot^n x \operatorname{cosec}^2 x \, dx$$

$$\text{Let } \cot x = t \Rightarrow -\operatorname{cosec}^2 x \, dx = dt$$

$$\Rightarrow I = -\int t^n \, dt$$

$$\Rightarrow I = -\frac{t^{n+1}}{n+1} + c$$

$$\Rightarrow I = -\frac{\cot^{n+1} x}{n+1} + c$$

$$\text{Therefore, } \int \cot^n x \operatorname{cosec}^2 x \, dx = -\frac{\cot^{n+1} x}{n+1} + c$$

10. Question

Evaluate the following integrals:

$$\int \cot^5 x \operatorname{cosec}^4 x \, dx$$

Answer

$$\text{Let } I = \int \cot^5 x \operatorname{cosec}^4 x \, dx$$

$$\Rightarrow I = \int \cot^5 x \operatorname{cosec}^2 x \operatorname{cosec}^2 x dx$$

$$\Rightarrow I = \int \cot^5 x (1 + \cot^2 x) \operatorname{cosec}^2 x dx$$

$$\Rightarrow I = \int (\cot^5 x + \cot^7 x) \operatorname{cosec}^2 x dx$$

Let $\cot x = t$, then

$$\Rightarrow -\operatorname{cosec}^2 x dx = dt$$

$$\Rightarrow I = -\int (t^5 + t^7) dt$$

$$\Rightarrow I = -\frac{t^6}{6} - \frac{t^8}{8} + c$$

$$\Rightarrow I = -\frac{\cot^6 x}{6} - \frac{\cot^8 x}{8} + c$$

$$\text{Therefore, } \int \cot^5 x \operatorname{cosec}^4 x dx = -\frac{\cot^6 x}{6} - \frac{\cot^8 x}{8} + c$$

11. Question

Evaluate the following integrals:

$$\int \cot^5 x dx$$

Answer

$$\text{Let } I = \int \cot^5 x dx$$

$$\Rightarrow I = \int \cot^2 x \cot^3 x dx$$

$$\Rightarrow I = \int (\operatorname{cosec}^2 x - 1) \cot^3 x dx$$

$$\Rightarrow I = \int \cot^3 x \operatorname{cosec}^2 x dx - \int \cot^3 x dx$$

$$\Rightarrow I = \int \cot^3 x \operatorname{cosec}^2 x dx - \int (\operatorname{cosec}^2 x - 1) \cot x dx$$

$$\Rightarrow I = \int \cot^3 x \operatorname{cosec}^2 x dx - \int (\operatorname{cosec}^2 x \cot x) dx + \int \cot x dx$$

Let $\cot x = t$, then

$$\Rightarrow -\operatorname{cosec}^2 x dx = dt$$

$$\Rightarrow I = -\int t^3 dt + \int t dt + \int \cot x dx$$

$$\Rightarrow I = -\frac{t^4}{4} + \frac{t^2}{2} + \log|\sin x| + c$$

$$\Rightarrow I = -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \log|\sin x| + c$$

$$\text{Therefore, } \int \cot^5 x dx = -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \log|\sin x| + c$$

12. Question

Evaluate the following integrals:

$$\int \cot^6 x \, dx$$

Answer

$$\text{Let } I = \int \cot^6 x \, dx$$

$$\Rightarrow I = \int \cot^2 x \cot^4 x \, dx$$

$$\Rightarrow I = \int (\operatorname{cosec}^2 x - 1) \cot^4 x \, dx$$

$$\Rightarrow I = \int \cot^4 x \operatorname{cosec}^2 x \, dx - \int \cot^4 x \, dx$$

$$\Rightarrow I = \int \cot^4 x \operatorname{cosec}^2 x \, dx - \int (\operatorname{cosec}^2 x - 1) \cot^2 x \, dx$$

$$\Rightarrow I = \int \cot^4 x \operatorname{cosec}^2 x \, dx - \int (\operatorname{cosec}^2 x \cot^2 x) \, dx + \int \cot^2 x \, dx$$

$$\Rightarrow I = \int \cot^4 x \operatorname{cosec}^2 x \, dx - \int (\operatorname{cosec}^2 x \cot^2 x) \, dx + \int (\operatorname{cosec}^2 x - 1) \, dx$$

Let $\cot x = t$, then

$$\Rightarrow -\operatorname{cosec}^2 x \, dx = dt$$

$$\Rightarrow I = -\int t^4 \, dt + \int t^2 \, dt - \int dt - \int dx$$

$$\Rightarrow I = -\frac{t^5}{5} + \frac{t^3}{3} - t - x + c$$

$$\Rightarrow I = -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} - \cot x - x + c$$

$$\text{Therefore, } \int \cot^6 x \, dx \Rightarrow I = -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} - \cot x - x + c$$

Exercise 19.12

1. Question

Evaluate the following integrals:

$$\int \sin^4 x \cos^3 x \, dx$$

Answer

$$\text{Let } \sin x = t$$

We know the Differentiation of $\sin x = \cos x$

$$dt = d(\sin x) = \cos x \, dx$$

$$\text{So, } dx = \frac{dt}{\cos x}$$

substitute all in above equation,

$$\int \sin^4 x \cos^3 x \, dx = \int t^4 \cos^3 x \frac{dt}{\cos x}$$

$$\begin{aligned}
&= \int t^4 \cos^2 x \, dt \\
&= \int t^4 (1 - \sin^2 x) \, dt \\
&= \int t^4 (1 - t^2) \, dt \\
&= \int (t^4 - t^6) \, dt
\end{aligned}$$

We know, basic integration formula, $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ for any $c \neq -1$

$$\text{Hence, } \int (t^4 - t^6) \, dt = \frac{t^5}{5} - \frac{t^7}{7} + c$$

Put back $t = \sin x$

$$\int \sin^4 x \cos^3 x \, dx = \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + c$$

2. Question

Evaluate the following integrals:

$$\int \sin^5 x \, dx$$

Answer

$$\begin{aligned}
\int \sin^5 x \, dx &= \int \sin^3 x \sin^2 x \, dx \\
&= \int \sin^3 x (1 - \cos^2 x) \, dx \quad \{ \text{since } \sin^2 x + \cos^2 x = 1 \} \\
&= \int (\sin^3 x - \sin^3 x \cos^2 x) \, dx \\
&= \int (\sin x (\sin^2 x) - \sin^3 x \cos^2 x) \, dx \\
&= \int (\sin x (1 - \cos^2 x) - \sin^3 x \cos^2 x) \, dx \quad \{ \text{since } \sin^2 x + \cos^2 x = 1 \} \\
&= \int (\sin x - \sin x \cos^2 x - \sin^3 x \cos^2 x) \, dx \\
&= \int \sin x \, dx - \int \sin x \cos^2 x \, dx - \int \sin^3 x \cos^2 x \, dx \quad (\text{separate the integrals})
\end{aligned}$$

We know, $d(\cos x) = -\sin x \, dx$

So put $\cos x = t$ and $dt = -\sin x \, dx$ in above integrals

$$\begin{aligned}
&= \int \sin x \, dx - \int \sin x \cos^2 x \, dx - \int \sin^3 x \cos^2 x \, dx \\
&= \int \sin x \, dx - \int t^2 (-dt) - \int (\sin^2 x \sin x) t^2 \, dx \\
&= \int \sin x \, dx - \int t^2 (-dt) - \int (1 - \cos^2 x) t^2 (-dt) \\
&= \int \sin x \, dx + \int t^2 \, dt + \int (1 - t^2) t^2 \, dt \\
&= \int \sin x \, dx + \int t^2 \, dt + \int (t^2 - t^4) \, dt \\
&= -\cos x + \frac{t^3}{3} + \frac{t^3}{3} - \frac{t^5}{5} + c \quad (\text{since } \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \text{ for any } c \neq -1)
\end{aligned}$$

Put back $t = \cos x$

$$\begin{aligned}
&= -\cos x + \frac{t^3}{3} + \frac{t^3}{3} - \frac{t^5}{5} + c \\
&= -\cos x + \frac{\cos^3 x}{3} + \frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} + c \\
&= -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + c = -[\cos x - \frac{2}{3} \cos^3 x + \frac{1}{5} \cos^5 x] + c
\end{aligned}$$

3. Question

Evaluate the following integrals:

$$\int \cos^5 x \, dx$$

Answer

$$\int \cos^5 x \, dx = \int \cos^3 x \cos^2 x \, dx$$

$$= \int \cos^3 x (1 - \sin^2 x) \, dx \quad \{ \text{since } \sin^2 x + \cos^2 x = 1 \}$$

$$= \int (\cos^3 x - \cos^3 x \sin^2 x) \, dx$$

$$= \int (\cos x (\cos^2 x) - \cos^3 x \sin^2 x) \, dx$$

$$= \int (\cos x (1 - \sin^2 x) - \cos^3 x \sin^2 x) \, dx \quad \{ \text{since } \sin^2 x + \cos^2 x = 1 \}$$

$$= \int (\cos x - \cos x \sin^2 x - \cos^3 x \sin^2 x) \, dx$$

$$= \int \cos x \, dx - \int \cos x \sin^2 x \, dx - \int \cos^3 x \sin^2 x \, dx \quad (\text{separate the integrals})$$

We know, $d(\sin x) = \cos x \, dx$

So put $\sin x = t$ and $dt = \cos x \, dx$ in above integrals

$$= \int \cos x \, dx - \int t^2 \, dt - \int \cos x \cos^2 x \sin^2 x \, dx$$

$$= \int \cos x \, dx - \int t^2 (dt) - \int (\cos^2 x \cos x) t^2 \, dx$$

$$= \int \cos x \, dx - \int t^2 (dt) - \int (1 - \sin^2 x) t^2 (dt)$$

$$= \int \cos x \, dx - \int t^2 \, dt - \int (1 - t^2) t^2 \, dt$$

$$= \int \cos x \, dx - \int t^2 \, dt - \int (t^2 - t^4) \, dt$$

$$= \sin x - \frac{t^3}{3} - \frac{t^3}{3} + \frac{t^5}{5} + c \quad (\text{since } \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \text{ for any } c \neq -1)$$

Put back $t = \sin x$

$$= \sin x - \frac{\sin^3 x}{3} - \frac{\sin^3 x}{3} + \frac{\cos^5 x}{5} + c$$

$$= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + c$$

4. Question

Evaluate the following integrals:

$$\int \sin^5 x \cos x \, dx$$

Answer

Let $\sin x = t$

Then $d(\sin x) = dt = \cos x \, dx$

Put $t = \sin x$ and $dt = \cos x \, dx$ in above equation

$$\int \sin^5 x \cos x \, dx = \int t^5 \, dt$$

$$= \frac{t^6}{6} + c \quad (\text{since } \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \text{ for any } c \neq -1)$$

$$= \frac{\sin^6 x}{6} + c$$

5. Question

Evaluate the following integrals:

$$\int \sin^3 x \cos^6 x \, dx$$

Answer

Since power of sin is odd, put $\cos x = t$

Then $dt = -\sin x \, dx$

Substitute these in above equation,

$$\begin{aligned} \int \sin^3 x \cos^6 x \, dx &= \int \sin x \sin^2 x t^6 \, dx \\ &= \int (1 - \cos^2 x) t^6 \sin x \, dx \\ &= \int (1 - t^2) t^6 \, dt \\ &= \int (t^6 - t^8) \, dt \\ &= \frac{t^7}{7} - \frac{t^9}{9} + c \quad (\text{since } \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \text{ for any } c \neq -1) \\ &= \frac{1}{7} \cos^7 x + \frac{1}{9} \cos^9 x + c \end{aligned}$$

6. Question

Evaluate the following integrals:

$$\int \cos^7 x \, dx$$

Answer

$$\begin{aligned} \int \cos^7 x \, dx &= \int \cos^6 x \cos x \, dx \\ &= \int (\cos^2 x)^3 \cos x \, dx \\ &= \int (1 - \sin^2 x)^3 \cos x \, dx \quad \{\text{since } \sin^2 x + \cos^2 x = 1\} \end{aligned}$$

$$\text{We know } (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

Here, $a = 1$ and $b = \sin^2 x$

$$\begin{aligned} \text{Hence, } \int (1 - \sin^2 x)^3 \cos x \, dx &= \int (1 - \sin^6 x - 3\sin^2 x + 3\sin^4 x) \cos x \, dx \\ &= \int (\cos x \, dx - \sin^6 x \cos x \, dx - 3\sin^2 x \cos x \, dx + 3\sin^4 x \cos x \, dx) \quad \{\text{take } \cos x \, dx \text{ inside brackets}\} \\ &= \int \cos x \, dx - \int \sin^6 x \cos x \, dx - 3 \int \sin^2 x \cos x \, dx + 3 \int \sin^4 x \cos x \, dx \quad (\text{separate the integrals}) \end{aligned}$$

Put $\sin x = t$ and $\cos x \, dx = dt$

$$\begin{aligned} &= \int \cos x \, dx - \int t^6 \, dt - 3 \int t^2 \, dt + 3 \int t^4 \, dt \\ &= \sin x - \frac{t^7}{7} - \frac{3t^3}{3} - \frac{3t^5}{5} + c \\ &= \sin x - \frac{t^7}{7} - t^3 - \frac{3t^5}{5} + c \end{aligned}$$

Put back $t = \sin x$

$$= \sin x - \sin^3 x + \frac{3}{5} \sin^5 x - \frac{1}{7} \sin^7 x + c$$

7. Question

Evaluate the following integrals:

$$\int x \cos^3 x^2 \sin x^2 \, dx$$

Answer

Let $\cos x^2 = t$

Then $d(\cos x^2) = dt$

Since $d(x^n) = nx^{n-1}$ and $d(\cos x) = -\sin x \, dx$

$dt = 2x (-\sin x^2) = -2x \sin x^2 \, dx$

$x \sin x^2 dx = -\frac{dt}{2}$

hence $\int x \cos^3 x^2 \sin x^2 \, dx = \int t^3 \times -\frac{dt}{2}$

$= -\frac{1}{2} \int t^3 dt$

$= -\frac{1}{2} \times \frac{t^4}{4} + c$

$= -\frac{1}{8} \cos^4 x^2 + c$

8. Question

Evaluate the following integrals:

$\int \sin^7 x \, dx$

Answer

$\int \sin^7 x \, dx = \int \sin^6 x \sin x \, dx$

$= \int (\sin^2 x)^3 \sin x \, dx$

$= \int (1 - \cos^2 x)^3 \sin x \, dx \quad \{ \text{since } \sin^2 x + \cos^2 x = 1 \}$

We know $(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$

Here, $a = 1$ and $b = \cos^2 x$

Hence, $\int (1 - \cos^2 x)^3 \sin x \, dx = \int (1 - \cos^6 x - 3\cos^4 x + 3\cos^2 x) \sin x \, dx$

$= \int (\sin x \, dx - \cos^6 x \sin x \, dx - 3\cos^4 x \sin x \, dx + 3\cos^2 x \sin x \, dx) \quad \{ \text{take } \sin x \, dx \text{ inside brackets} \}$

$= \int \sin x \, dx - \int \cos^6 x \sin x \, dx - 3 \int \cos^4 x \sin x \, dx + 3 \int \cos^2 x \sin x \, dx \quad (\text{separate the integrals})$

Put $\cos x = t$ and $-\sin x \, dx = dt$

$= \int \sin x \, dx - \int t^6 (-dt) - 3 \int t^4 (-dt) + 3 \int t^2 (-dt)$

$= -\cos x + \frac{t^7}{7} + \frac{3t^3}{3} - \frac{3t^5}{5} + c$

$= -\cos x + \frac{t^7}{7} + t^3 - \frac{3t^5}{5} + c$

Put back $t = \cos x$

$= -\cos x + \cos^3 x - \frac{3}{5} \cos^5 x + \frac{1}{7} \cos^7 x + c$

9. Question

Evaluate the following integrals:

$\int \sin^3 x \cos^5 x \, dx$

Answer

Let $\cos x = t$ then $dt = -\sin x \, dx$

$$dx = -\frac{dt}{\sin x}$$

Substitute all these in the above equation,

$$\begin{aligned}\int \sin^3 x \cos^5 x \, dx &= \int \sin^2 x \cos^4 x \left(-\frac{dt}{\sin x}\right) \\&= -\int \sin^2 x \cos^4 x \, dt \\&= -\int (1 - \cos^2 x) \cos^4 x \, dt \\&= -\int (1 - t^2) t^4 \, dt \\&= -\int t^4 \, dt - \int t^6 \, dt \\&= -\frac{t^5}{5} + \frac{t^7}{7} + c \quad \left(\text{since } \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \text{ for any } c \neq -1\right) \\&= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + c \\&= \frac{1}{7} \cos^7 x - \frac{1}{5} \cos^5 x + c\end{aligned}$$

10. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin^4 x \cos^2 x} \, dx$$

Answer

$$\int \frac{1}{\sin^4 x \cos^2 x} \, dx = \int \sin^{-4} x \cos^{-2} x \, dx$$

Adding the powers : $-4 + -2 = -6$

Since all are even nos, we will divide each by $\cos^6 x$ to convert into positive power

$$\begin{aligned}\text{So, } \int \frac{1}{\sin^4 x \cos^2 x} \, dx &= \int \frac{\frac{1}{\cos^6 x}}{\frac{\sin^4 x \cos^2 x}{\cos^6 x}} \, dx \\&= \int \frac{\sec^6 x}{\frac{\sin^4 x}{\cos^4 x}} \, dx = \int \frac{\sec^6 x}{\tan^4 x} \, dx \\&= \int \frac{\sec^4 x \sec^2 x}{\tan^4 x} \, dx = \int \frac{(\sec^2 x)^2 \sec^2 x}{\tan^4 x} \, dx \\&= \int \frac{(1 + \tan^2 x)^2 \sec^2 x}{\tan^4 x} \, dx \quad \{\text{since } \sec^2 x = 1 + \tan^2 x\} \\&= \int \frac{(1 + \tan^4 x + 2\tan^2 x)^2 \sec^2 x}{\tan^4 x} \, dx \quad (\text{apply } (a + b)^2 = a^2 + b^2 + 2ab)\end{aligned}$$

Let $\tan x = t$, so $dt = d(\tan x) = \sec^2 x \, dx$

$$\text{So, } dx = \frac{dt}{\sec^2 x}$$

Put t and dx in the above equation,

$$\begin{aligned}\int \frac{(1 + \tan^4 x + 2\tan^2 x)^2 \sec^2 x}{\tan^4 x} \, dx &= \int \frac{(1 + t^4 + 2t^2)^2}{t^4} \sec^2 x * \frac{dt}{\sec^2 x} \\&= \int \frac{(1 + t^4 + 2t^2)^2}{t^4} \, dt\end{aligned}$$

$$\begin{aligned}
&= \int (1 + t^{-4} + 2t^{-2}) dt \\
&= t - \frac{t^{-3}}{3} - 2t^{-1} + c \\
&= t - \frac{2}{t} - \frac{1}{3t^3} + c \\
&= \tan x - \frac{2}{\tan x} - \frac{1}{3\tan^3 x} + c \\
&= \tan x - 2\cot x - \frac{1}{3}\cot^3 x + c \quad \{1/\tan x = \cot x\}
\end{aligned}$$

11. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin^3 x \cos^5 x} dx$$

Answer

$$\int \frac{1}{\sin^3 x \cos^5 x} dx = \int \sin^{-3} x \cos^{-5} x dx$$

Adding the powers, $-3 + -5 = -8$

Since it is an even number, we will divide numerator and denominator by $\cos^8 x$

$$\begin{aligned}
\int \frac{1}{\sin^3 x \cos^5 x} dx &= \int \frac{\frac{1}{\cos^8 x}}{\frac{\sin^3 x \cos^5 x}{\cos^8 x}} dx \\
&= \int \frac{\sec^8 x}{\tan^3 x} dx = \int \frac{\sec^6 x \sec^2 x}{\tan^3 x} dx = \int \frac{(\sec^2 x)^3 \sec^2 x}{\tan^3 x} dx \\
&= \int \frac{(1 + \tan^2 x)^3 \sec^2 x}{\tan^3 x} dx
\end{aligned}$$

We know, $(a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

Here, $a = 1$ and $b = \tan^2 x$

$$\text{Hence, } \int \frac{(1 + \tan^2 x)^3 \sec^2 x}{\tan^3 x} dx = \int \frac{(1 + \tan^6 x + 3\tan^2 x + 3\tan^4 x) \sec^2 x}{\tan^3 x} dx$$

Let $\tan x = t$, then $dt = d(\tan x) = \sec^2 x dx$

Put these values in above equation:

$$\begin{aligned}
&= \int \frac{1 + t^6 + 3t^2 + 3t^4}{t^3} dt = \int (t^{-3} + t^3 + 3t^{-1} + 3t) dt \\
&= -\frac{t^{-2}}{2} + \frac{t^4}{4} + 3\log t + \frac{3t^2}{2} + c \quad \left(\text{since } \int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ for any } c \neq -1 \text{ and } \int t^{-1} dt = \log t \right) \\
&= -\frac{1}{2t^2} + \frac{1}{4}t^4 + 3\log t + \frac{3}{2}t^2 + c \\
&= -\frac{1}{2\tan^2 x} + \frac{1}{4}\tan^4 x + 3\log(\tan x) + \frac{3}{2}\tan^2 x + c
\end{aligned}$$

12. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin^3 x \cos x} dx$$

Answer

$$\int \frac{1}{\sin^3 x \cos x} dx = \int \sin^{-3} x \cos^{-1} x dx$$

Adding the powers , $-3 + -1 = -4$

Since it is an even number, we will divide numerator and denominator by $\cos x$

$$\begin{aligned}\int \frac{1}{\sin^3 x \cos x} dx &= \int \frac{\frac{1}{\cos^4 x}}{\frac{\sin^3 x \cos x}{\cos^4 x}} dx \\&= \int \frac{\sec^4 x}{\tan^3 x} dx = \int \frac{\sec^2 x \sec^2 x}{\tan^3 x} dx \\&= \int \frac{(1 + \tan^2 x) \sec^2 x}{\tan^3 x} dx\end{aligned}$$

Let $\tan x = t$, then $dt = d(\tan x) = \sec^2 x dx$

Put these values in the above equation:

$$\begin{aligned}&= \int \frac{1 + t^2}{t^3} dt = \int (t^{-3} + t^{-1}) dt \\&= -\frac{t^{-2}}{2} + \log t + c \quad (\text{since } \int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ for any } c \neq -1 \text{ and } \int t^{-1} dt = \log t) \\&= -\frac{1}{2t^2} + \log t + c \\&= -\frac{1}{2\tan^2 x} + \log(\tan x) + c\end{aligned}$$

13. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin x \cos^3 x} dx$$

Answer

We know, $\sin^2 x + \cos^2 x = 1$

$$\text{Therefore } \frac{1}{\sin x \cos^3 x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x}$$

Divide each term of numerator separately by $\sin x \cos^3 x$

$$\begin{aligned}&= \frac{\sin^2 x}{\sin x \cos^3 x} + \frac{\cos^2 x}{\sin x \cos^3 x} = \frac{\sin x}{\cos^3 x} + \frac{1}{\sin x \cos x} \\&= \frac{\sin x}{\cos x} * \left(\frac{1}{\cos^2 x}\right) + \frac{\frac{1}{\sin x \cos x}}{\frac{\cos^2 x}{\cos^2 x}} \quad (\text{divide second term each by } \cos^2 x) \\&= \tan x \sec^2 x + \frac{\sec^2 x}{\tan x}\end{aligned}$$

Therefore,

$$\begin{aligned}\int \frac{1}{\sin x \cos^3 x} dx &= \int \left(\tan x \sec^2 x + \frac{\sec^2 x}{\tan x} \right) dx \\&= \int \tan x \sec^2 x dx + \int \frac{\sec^2 x}{\tan x} dx\end{aligned}$$

Put $\tan x = t$, $dt = \sec^2 x dx$

$$= \int \tan x \sec^2 x \, dx + \int \frac{\sec^2 x}{\tan x} \, dx = \int t \, dt + \int \frac{1}{t} \, dt$$

$$= \frac{t^2}{2} + \log t + c = \frac{1}{2} \tan^2 x + \log(\tan x) + c$$

Exercise 19.13

1. Question

Evaluate the following integrals:

$$\int \frac{x^2}{(a^2 - x^2)^{3/2}} \, dx$$

Answer

$$\int \frac{x^2}{(a^2 - x^2)^{3/2}} \, dx$$

PUT $x = a \sin \theta$, so $dx = a \cos \theta \, d\theta$ and $\theta = \sin^{-1}(x/a)$

Above equation becomes,

$$= \int \frac{a^2 \sin^2 \theta}{(a^2 - a^2 \sin^2 \theta)^{3/2}} (a \cos \theta \, d\theta) = \int \frac{a^2 \sin^2 \theta}{(a^2)(a^2 - a^2 \sin^2 \theta)^{3/2}} (a \cos \theta \, d\theta) \text{ \{take } a^2 \text{ outside}\}}$$

$$= \int \frac{a^2 \sin^2 \theta}{(a^2)^{3/2} (a^2 - a^2 \sin^2 \theta)^{3/2}} (a \cos \theta \, d\theta) = \int \sin^2 \theta * \frac{\cos \theta}{\cos^3 \theta} \, d\theta$$

$$= \int \frac{\sin^2 \theta}{\cos^2 \theta} \, d\theta = \int \tan^2 \theta \, d\theta = \int (\sec^2 \theta - 1) \, d\theta \text{ (\sec^2 \theta - 1 = \tan^2 \theta)}$$

$$= \int \sec^2 \theta \, d\theta - \int \theta \, d\theta = \tan \theta + c - \theta$$

$$= \tan \theta - \theta + c$$

Put $\theta = \sin^{-1}(x/a)$

$$= \tan \theta * \sin^{-1} \left(\frac{x}{a} \right) - \sin^{-1} \left(\frac{x}{a} \right) + c$$

2. Question

Evaluate the following integrals:

$$\int \frac{x^7}{(a^2 - x^2)^5} \, dx$$

Answer

PUT $x = a \sin \theta$, so $dx = a \cos \theta \, d\theta$ and $\theta = \sin^{-1}(x/a)$

Above equation becomes,

$$\int \frac{x^7}{(a^2 - x^2)^5} \, dx = \int \frac{a^7 \sin^7 \theta}{(a^2 - a^2 \sin^2 \theta)^5} (a \cos \theta \, d\theta) = \int \frac{a^7 \sin^7 \theta}{(a^2)^5 (1 - \sin^2 \theta)^5} (a \cos \theta \, d\theta) \text{ \{take } a^2 \text{ outside}\}}$$

$$= \int \frac{a^7 \sin^7 \theta}{(a^2)^5 (1 - \sin^2 \theta)^5} (a \cos \theta \, d\theta) = \int \frac{a^7 \sin^7 \theta}{(a^{10} (1 - \sin^2 \theta)^5)} (a \cos \theta \, d\theta)$$

$$= \frac{1}{a^2} \int \frac{1}{\cos^2 \theta} \, d\theta = \frac{1}{a^2} \int \sec^2 \theta \, d\theta = \frac{1}{a^2} (\tan \theta + c)$$

Put $\theta = \sin^{-1}(x/a)$

$$= \frac{1}{a^2} \left(\tan \sin^{-1}\left(\frac{x}{a}\right) + c \right)$$

3. Question

Evaluate the following integrals:

$$\int \cos \left\{ 2 \cot^{-1} \sqrt{\frac{1+x}{1-x}} \right\} dx$$

Answer

Let $x = \cos 2t$ and $t = \cos^{-1} \frac{x}{2}$

$$= \sqrt{\frac{1+x}{1-x}} = \sqrt{\frac{1+\cos 2t}{1-\cos 2t}}$$

We know $1 + \cos 2t = 2\cos^2 t$ and $1 - \cos 2t = 2\sin^2 t$

$$\text{Hence, } \sqrt{\frac{1+\cos 2t}{1-\cos 2t}} = \sqrt{\frac{\cos^2 t}{\sin^2 t}} = \sqrt{\cot^2 t} = \cot t$$

$$\text{Therefore, } \int \cos \left\{ 2 \cot^{-1} \sqrt{\frac{1+x}{1-x}} \right\} dx = \int \cos \theta \, dx$$

Put $t = \cos^{-1} \frac{x}{2}$

$$= \int \cos \theta \, dx = \int \cos \frac{\cos^{-1} x}{2} dx = \int \frac{x}{2} dx = \frac{1}{2} \frac{x^2}{2} + c = \frac{x^2}{4} + c$$

4. Question

Evaluate the following integrals:

$$\int \frac{\sqrt{1+x^2}}{x^4} dx$$

Answer

let $x = \tan \theta$, so $dx = \sec^2 \theta \, d\theta$ and $\theta = \tan^{-1} x$

Putting above values,

$$= \int \frac{\sqrt{1+x^2}}{x^4} dx = \int \frac{\sqrt{1+\tan^2 \theta}}{\tan^4 \theta} \sec^2 \theta \, d\theta = \int \sec^2 \theta / \tan^2 \theta \, d\theta$$

$$= \int \operatorname{cosec}^2 \theta \, d\theta = -\cot \theta + c$$

Put $\theta = \tan^{-1} x$

$$= -\cot \theta + c = -\cot \tan^{-1} x + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{1}{(x^2 + 2x + 10)^2} dx$$

Answer

$$= x^2 + 2x + 10 = x^2 + 2x + 1 - 1 + 10 \text{ (add and subtract 1)}$$

$$= (x^2 + 1)^2 - 1 + 10 = (x^2 + 1)^2 + 9$$

$$= (x^2 + 1)^2 + 3^2$$

Put $x + 1 = t$ hence $dx = dt$ and $x = t - 1$

$$\int \frac{1}{(x^2 + 2x + 10)^2} dx = \int \frac{1}{(x^2 + 1)^2 + 3^2} dx$$

$$= \int \frac{1}{t^2 + 3^2} dt$$

$$\text{We have, } \int \frac{dt}{t^2 + a^2} = \frac{1}{a} \log\left(\frac{t-a}{t+a}\right) + c$$

Here $a = 3$

$$\text{Therefore, } \int \frac{1}{t^2 + 3^2} dt = \frac{1}{3} \log\left(\frac{t-3}{t+3}\right) + c$$

Put $t = x + 1$

$$= \frac{1}{3} \log\left(\frac{t-3}{t+3}\right) + c = \frac{1}{3} \log\left(\frac{x+1-3}{x+1+3}\right) + c = \frac{1}{3} \log\left(\frac{x-2}{x+4}\right) + c$$

Exercise 19.14

1. Question

Evaluate the following integrals:

$$\int \frac{1}{a^2 - b^2 x^2} dx$$

Answer

$$\text{Taking out } b^2, \frac{1}{b^2} \int \frac{1}{\left(\frac{a^2}{b^2}\right) - x^2} dx$$

$$= \frac{1}{b^2} \int \frac{1}{\left(\frac{a^2}{b^2}\right) - x^2} dx = \frac{1}{b^2} \int \frac{1}{\left(\frac{a}{b}\right)^2 - x^2} dx$$

$$= \frac{1}{b^2} \times \frac{1}{2\left(\frac{a}{b}\right)} \log\left[\frac{\frac{a}{b} + x}{\frac{a}{b} - x}\right] + c \{ \text{since } \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \frac{x+a}{x-a} + c \}$$

$$= \frac{1}{2ab} \log \frac{a+bx}{a-bx} + c$$

2. Question

Evaluate the following integrals:

$$\int \frac{1}{a^2 x^2 - b^2} dx$$

Answer

take out a^2

$$= \frac{1}{a^2} \int \frac{1}{x^2 - \frac{b^2}{a^2}} dx$$

$$= \frac{1}{a^2} \int \frac{1}{x^2 - \left(\frac{b}{a}\right)^2} dx = \frac{1}{a^2} * \frac{1}{2\left(\frac{b}{a}\right)} \log\left[\frac{x - \left(\frac{b}{a}\right)}{x + \frac{b}{a}}\right] + c \{ \text{since } \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \frac{x+a}{x-a} + c \}$$

$$= \frac{1}{2ab} \log \frac{ax-b}{ax+b} + c$$

3. Question

Evaluate the following integrals:

$$\int \frac{1}{a^2 x^2 + b^2} dx$$

Answer

take out a^2

$$= \frac{1}{a^2} \int \frac{1}{x^2 + \frac{b^2}{a^2}} dx$$

$$= \frac{1}{a^2} \int \frac{1}{x^2 + \left(\frac{b}{a}\right)^2} dx = \frac{1}{a^2} * \frac{1}{\left(\frac{b}{a}\right)} \tan^{-1} \left[\frac{x}{\frac{b}{a}} \right] + c \quad \{ \text{since } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{b}{a} \right) + c \}$$

$$= \frac{1}{ab} \tan^{-1} \left(\frac{ax}{b} \right) + c$$

4. Question

Evaluate the following integrals:

$$\int \frac{x^2 - 1}{x^2 + 4} dx$$

Answer

Add and subtract 4 in the numerator, we get

$$= \int \frac{x^2 + 4 - 4 - 1}{x^2 + 4} dx = \int \frac{(x^2 + 4) - 4 - 1}{x^2 + 4} dx$$

$$= \int \frac{(x^2 + 4) - 5}{x^2 + 4} dx = \int \frac{(x^2 + 4)}{x^2 + 4} dx - \int \frac{5}{x^2 + 4} dx \quad \{ \text{separate the numerator terms} \}$$

$$= \int dx - \int \frac{5}{x^2 + 4} dx = \int dx - 5 \int \frac{1}{x^2 + 4} dx$$

$$= \int dx - 5 \int \frac{1}{x^2 + 2^2} dx = x - 5 \times \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c \quad \{ \text{since } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{b}{a} \right) + c \}$$

$$= x - \frac{5}{2} \tan^{-1} \left(\frac{x}{2} \right) + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{1 + 4x^2}} dx$$

Answer

$$\text{Let } I = \int \frac{1}{\sqrt{1 + 4x^2}} dx = \int \frac{1}{\sqrt{1 + (2x)^2}} dx$$

Let $t = 2x$, then $dt = 2dx$ or $dx = dt/2$

$$\text{Therefore, } \int \frac{1}{\sqrt{1 + (2x)^2}} dx = \frac{1}{2} \int \frac{dt}{\sqrt{1 + t^2}}$$

$$= \frac{1}{2} \log[t + \sqrt{1 + t^2}] + c \quad \{ \text{since } \int \frac{1}{\sqrt{a^2 + x^2}} dx = \log[x + \sqrt{a^2 + x^2}] + c \}$$

$$= \frac{1}{2} \log[2x + \sqrt{1 + 4x^2}] + c$$

6. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{a^2 + b^2 x^2}} dx$$

Answer

Let $bx = t$ then $dt = bdx$ or $dx = \frac{dt}{b}$

$$\text{Hence, } \int \frac{1}{\sqrt{a^2 + b^2 x^2}} dx = \frac{1}{b} \int \frac{1}{\sqrt{a^2 + t^2}} dt$$

$$= \frac{1}{b} \log[t + \sqrt{a^2 + t^2}] + c \text{ \{since } \int \frac{1}{\sqrt{a^2 + x^2}} dx = \log[x + \sqrt{a^2 + x^2}] + c \}$$

Put $t = bx$

$$= \frac{1}{b} \log[bx + \sqrt{a^2 + b^2 x^2}] + c$$

7. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{a^2 - b^2 x^2}} dx$$

Answer

Let $bx = t$ then $dt = bdx$ or $dx = \frac{dt}{b}$

$$\text{Hence, } \int \frac{1}{\sqrt{a^2 - b^2 x^2}} dx = \frac{1}{b} \int \frac{1}{\sqrt{a^2 - t^2}} dt$$

$$= \frac{1}{b} \int \sin^{-1}\left(\frac{t}{a}\right) + c \text{ \{since } \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c \}$$

Put $t = bx$

$$= \frac{1}{b} \int \sin^{-1}\left(\frac{bx}{a}\right) + c$$

8. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{(2-x)^2 + 1}} dx$$

Answer

Let $(2-x) = t$, then $dt = -dx$, or $dx = -dt$

$$\text{Hence, } \int \frac{1}{\sqrt{(2-x)^2 + 1}} dx = \int \frac{1}{t^2 + 1} (-dt)$$

$$= -\int \frac{1}{t^2 + 1} dt = -\log(t + \sqrt{t^2 + 1}) + c \text{ \{since } \int \frac{1}{\sqrt{a^2 + x^2}} dx = \log[x + \sqrt{a^2 + x^2}] + c \}$$

Put $t = 2-x$

$$= -\log \int ((2-x) + \sqrt{(2-x)^2 + 1}) + c$$

9. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{(2-x)^2 - 1}} dx$$

Answer

Let $(2-x) = t$, then $dt = -dx$, or $dx = -dt$

$$\text{Hence, } \int \frac{1}{\sqrt{(2-x)^2 - 1}} dx = \int \frac{1}{t^2 - 1} (-dt)$$

$$= -\int \frac{1}{t^2 - 1} dt = -\log \int (t + \sqrt{t^2 - 1}) + c \text{ \{since } \int \frac{1}{\sqrt{x^2 + a^2}} dx = \log[x + \sqrt{x^2 + a^2}] + c \}}$$

Put $t = 2-x$

$$= -\log \int ((2-x) + \sqrt{(2-x)^2 - 1}) + c$$

10. Question

Evaluate the following integrals:

$$\int \frac{x^4 + 1}{x^2 + 1} dx$$

Answer

We will use basic formula : $(a + b)^2 = a^2 + b^2 + 2ab$

Or, $a^2 + b^2 = (a + b)^2 - 2ab$

Here, $x^4 + 1 = x^4 + 1^4$

$$= (x^2)^2 + (1^2)^2$$

Applying above formula, we get, $x^4 + 1 = (x^2 + 1)^2 - 2 \times 1 \times x^2$

$$= (x^2 + 1)^2 - 2x^2$$

$$\text{Hence, } \int \frac{x^4 + 1}{x^2 + 1} dx = \int \frac{(x^2 + 1)^2 - 2x^2}{x^2 + 1} dx$$

Separate the numerator terms,

$$\int \frac{(x^2 + 1)^2 - 2x^2}{x^2 + 1} dx = \int \frac{(x^2 + 1)^2}{x^2 + 1} dx - \int \frac{2x^2}{x^2 + 1} dx$$

$$= \int (x^2 + 1) dx - \int \frac{2x^2 + 2 - 2}{x^2 + 1} dx \text{ \{ add and subtract 2 to the second term \}}$$

$$= \int (x^2 + 1) dx - \int \frac{2(x^2 + 1)}{x^2 + 1} dx - 2 \int \frac{1}{(x^2 + 1)} dx \text{ \{ } 2x^2 + 2 - 2 = 2(x^2 + 1) - 2 \}}$$

$$= \int (x^2 + 1) dx - \int 2 dx - 2 \int \frac{1}{(x^2 + 1)} dx$$

$$= \frac{x^3}{3} + x - 2x + 2 \tan^{-1} x + c \text{ \{ since } \int \frac{1}{x^2 + 1} dx = \tan^{-1}(x) + c \}}$$

$$= \frac{x^3}{3} - x + 2\tan^{-1}x + c$$

Exercise 19.15

1. Question

Evaluate the following integrals:

$$\int \frac{1}{4x^2 + 12x + 5} dx$$

Answer

$$\text{let } I = \int \frac{1}{4x^2 + 12x + 5} dx$$

$$= \frac{1}{4} \int \frac{1}{x^2 + 3x + \frac{5}{4}} dx$$

$$= \frac{1}{4} \int \frac{1}{x^2 + 2x \times \frac{3}{2} + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{5}{4}} dx$$

$$= \frac{1}{4} \int \frac{1}{\left(x + \frac{3}{2}\right)^2 - 1} dx$$

$$\text{Let } \left(x + \frac{3}{2}\right) = t \dots (i)$$

$$\Rightarrow dx = dt$$

so,

$$I = \frac{1}{4} \int \frac{1}{t^2 - (1)^2} dt$$

$$I = \frac{1}{4} \times \frac{1}{2 \times 1} \log \left| \frac{t-1}{t+1} \right| + c$$

$$[\text{since, } \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x-a}{x+a} \right| + c]$$

$$I = \frac{1}{8} \log \left| \frac{\frac{x-\frac{3}{2}}{\frac{3}{2}} - 1}{\frac{x-\frac{3}{2}}{\frac{3}{2}} + 1} \right| + c \text{ [using (i)]}$$

$$I = \frac{1}{8} \log \left| \frac{2x-1}{2x+5} \right| + c$$

2. Question

Evaluate the following integrals:

$$\int \frac{1}{x^2 - 10x + 34} dx$$

Answer

$$\text{let } I = \int \frac{1}{x^2 - 10x + 34} dx$$

$$I = \int \frac{1}{x^2 - 10x + 34} dx$$

$$= \int \frac{1}{x^2 + 2x \times 5 + (5)^2 - (5)^2 + 34} dx$$

$$= \int \frac{1}{(x-5)^2 - 9} dx$$

Let $(x-5) = t$ (i)

$$\Rightarrow dx = dt$$

so,

$$I = \int \frac{1}{t^2 + (3)^2} dt$$

$$I = \frac{1}{3} \tan^{-1}\left(\frac{t}{3}\right) + c$$

$$[\text{since, } \int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c]$$

$$I = \frac{1}{3} \tan^{-1}\left(\frac{x-5}{3}\right) + c \text{ [using (i)]}$$

$$I = \frac{1}{3} \tan^{-1}\left(\frac{x-5}{3}\right) + c$$

3. Question

Evaluate the following integrals:

$$\int \frac{1}{1+x-x^2} dx$$

Answer

$$: \text{let } I = \int \frac{1}{1+x-x^2} dx = \int \frac{1}{-(x^2-x-1)} dx$$

$$= \int \frac{1}{-(x^2-x-1)} dx$$

$$= \int \frac{1}{-(x^2-x-\frac{1}{4}-1+\frac{1}{4})} dx$$

$$= \int \frac{1}{-\left(\left(x-\frac{1}{2}\right)^2 - \frac{5}{4}\right)} dx$$

$$= \int \frac{1}{\left(\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2\right)} dx$$

$$I = \frac{1}{2 \times \frac{\sqrt{5}}{2}} \log \left| \frac{\frac{\sqrt{5}}{2} + (x-\frac{1}{2})}{\frac{\sqrt{5}}{2} - (x-\frac{1}{2})} \right| + c$$

$$[\text{since, } \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x-a}{x+a} \right| + c]$$

$$I = \frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{5} + 2x - 1}{\sqrt{5} - 2x + 1} \right| + c$$

$$I = \frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{5} - 1 + 2x}{\sqrt{5} + 1 - 2x} \right| + c$$

4. Question

Evaluate the following integrals:

$$\int \frac{1}{2x^2 - x - 1} dx$$

Answer

$$\text{let } I = \int \frac{1}{2x^2 - x - 1} dx$$

$$= \frac{1}{2} \int \frac{1}{x^2 - \frac{x}{2} - \frac{1}{2}} dx$$

$$= \frac{1}{2} \int \frac{1}{x^2 + 2x \times \frac{1}{4} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - \frac{1}{2}} dx$$

$$= \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{4}\right)^2 - \frac{9}{16}} dx$$

$$\text{Let } \left(x - \frac{1}{4}\right) = t \dots\dots(i)$$

$$\Rightarrow dx = dt$$

so,

$$I = \frac{1}{2} \int \frac{1}{t^2 - \left(\frac{3}{4}\right)^2} dt$$

$$I = \frac{1}{2} \times \frac{1}{2 \times \frac{3}{4}} \log \left| \frac{t - \frac{3}{4}}{t + \frac{3}{4}} \right| + c$$

$$[\text{since, } \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x - a}{x + a} \right| + c]$$

$$I = \frac{1}{3} \log \left| \frac{x - \frac{1}{4} - \frac{3}{4}}{x - \frac{1}{4} + \frac{3}{4}} \right| + c \text{ [using (i)]}$$

$$I = \frac{1}{3} \log \left| \frac{x - 1}{2x + 1} \right| + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{1}{x^2 + 6x + 13} dx$$

Answer

We have,

$$x^2 + 6x + 13 = x^2 + 6x + 3^2 - 3^2 + 13$$

$$= (x + 3)^2 + 4$$

$$\text{Sol, } \int \frac{1}{x^2 + 6x + 13} dx = \int \frac{1}{(x+3)^2 + 2^2} dx$$

$$\text{Let } x+3 = t$$

Then $dx = dt$

$$\int \frac{1}{(t)^2 + 2^2} dt = \frac{1}{2} \tan^{-1} \frac{t}{2} + c$$

$$[\text{since, } \int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c]$$

$$\frac{1}{2} \tan^{-1} \frac{x+3}{2} + c$$

Exercise 19.16

1. Question

Evaluate the following integrals:

$$\int \frac{\sec^2 x}{1 - \tan^2 x} dx$$

Answer

$$\text{let } I = \int \frac{\sec^2 x}{1 - \tan^2 x} dx$$

$$\text{Let } \tan x = t \dots (i)$$

$$\Rightarrow \sec^2 x dx = dt$$

so,

$$I = \int \frac{dt}{(1)^2 - t^2}$$

$$I = \frac{1}{2 \times 1} \log \left| \frac{1+t}{1-t} \right| + c \quad [\text{since, } \int \frac{1}{a^2 - (x)^2} dx = \frac{1}{2 \times a} \log \left| \frac{a+x}{a-x} \right| + c]$$

$$I = \frac{1}{2} \log \left| \frac{1+\tan x}{1-\tan x} \right| + c \quad [\text{using (i)}]$$

2. Question

Evaluate the following integrals:

$$\int \frac{e^x}{1 + e^{2x}} dx$$

Answer

$$\text{: let } I = \int \frac{e^x}{1 + e^{2x}} dx$$

$$\text{Let } e^x = t \dots (i)$$

$$\Rightarrow e^x dx = dt$$

so,

$$I = \int \frac{dt}{(1)^2 + t^2}$$

$$I = \tan^{-1} t + c$$

$$[\text{since, } \int \frac{1}{1 + (x)^2} dx = \tan^{-1} x + c]$$

$$I = \tan^{-1}(e^x) + c \quad [\text{using (i)}]$$

3. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{\sin^2 x + 4 \sin x + 5} dx$$

Answer

$$\text{Let } I = \int \frac{\cos x}{\sin^2 x + 4 \sin x + 5} dx$$

$$\text{Let } \sin x = t \dots (i)$$

$$\Rightarrow \cos x dx = dt$$

$$\text{So, } I = \int \frac{dt}{t^2 + 4t + 5}$$

$$= \int \frac{dt}{t^2 + (2t)(2) + 2^2 - 2^2 + 5}$$

$$\int \frac{dt}{(t+2)^2 + 1}$$

$$\text{Again, let } t + 2 = u \dots (ii)$$

$$\Rightarrow dt = du$$

$$I = \int \frac{du}{u^2 + 1}$$

$$= \tan^{-1} u + c$$

$$[\text{since, } \int \frac{1}{1 + (x)^2} dx = \tan^{-1} x + c]$$

$$= \tan^{-1}(\sin x + 2) + c \text{ [using (i), (ii)]}$$

4. Question

Evaluate the following integrals:

$$\int \frac{e^x}{e^{2x} + 5e^x + 6} dx$$

Answer

$$\text{let } I = \int \frac{e^x}{e^{2x} + 5e^x + 6} dx$$

$$\text{Let } e^x = t \dots (i)$$

$$\Rightarrow e^x dx = dt$$

$$= \int \frac{1}{t^2 + 5t + 6} dt$$

$$= \int \frac{1}{t^2 + 2t \times \frac{5}{2} + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 6} dt$$

$$= \int \frac{1}{\left(t + \frac{5}{2}\right)^2 - \frac{1}{4}} dt$$

$$\text{Let } t + \frac{5}{2} = u \dots (i)$$

$$\Rightarrow dt = du$$

so,

$$I = \int \frac{1}{u^2 - \left(\frac{1}{2}\right)^2} du$$

$$I = \frac{1}{2 \times \frac{1}{2}} \log \left| \frac{u - \frac{1}{2}}{u + \frac{1}{2}} \right| + c$$

$$[\text{since, } \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x - a}{x + a} \right| + c]$$

$$I = \log \left| \frac{2u - 1}{2u + 1} \right| + c$$

$$I = \log \left| \frac{2\left(t + \frac{5}{2}\right) - 1}{2\left(t + \frac{5}{2}\right) + 1} \right| + c \text{ [using (i)]}$$

$$I = \log \left| \frac{e^x + 2}{e^x + 3} \right| + c \text{ [using (ii)]}$$

5. Question

Evaluate the following integrals:

$$\int \frac{e^{3x}}{4e^{6x} - 9} dx$$

Answer

$$\text{let } I = \int \frac{e^{3x}}{4e^{6x} - 9} dx$$

$$\text{Let } e^{3x} = t \dots (i)$$

$$\Rightarrow 3e^{3x} dx = dt$$

$$I = \frac{1}{3} \int \frac{1}{4t^2 - 9} dt$$

$$= \frac{1}{12} \int \frac{1}{t^2 - \frac{9}{4}} dt$$

$$I = \frac{1}{12} \int \frac{1}{t^2 - \left(\frac{3}{2}\right)^2} dt$$

$$I = \frac{1}{36} \log \left| \frac{t - \frac{3}{2}}{t + \frac{3}{2}} \right| + c$$

$$[\text{since, } \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x - a}{x + a} \right| + c]$$

$$I = \log \left| \frac{2t - 3}{2t + 3} \right| + c$$

$$I = \log \left| \frac{2e^{3x} - 3}{2e^{3x} + 3} \right| + c \text{ [using (i)]}$$

6. Question

Evaluate the following integrals:

$$\int \frac{1}{e^x + e^{-x}} dx$$

Answer

$$\text{let } I = \int \frac{1}{e^x + e^{-x}} dx$$

$$= \int \frac{1}{e^x + \frac{1}{e^x}} dx$$

$$= \int \frac{e^x}{(e^x)^2 + 1} dx$$

$$\text{Let } e^x = t \dots\dots(i)$$

$$\Rightarrow e^x dx = dt$$

$$I = \int \frac{1}{(t)^2 + 1} dt$$

$$I = \tan^{-1} t + c$$

$$[\text{since, } \int \frac{1}{1 + (x)^2} dx = \tan^{-1} x + c]$$

$$I = \tan^{-1}(e^x) + c \text{ [using (i)]}$$

7. Question

Evaluate the following integrals:

$$\int \frac{x}{x^4 + 2x^2 + 3} dx$$

Answer

$$\text{Let } I = \int \frac{x}{x^4 + 2x^2 + 3} dx$$

$$\text{Let } x^2 = t \dots\dots\dots(i)$$

$$\Rightarrow 2x dx = dt$$

$$I = \frac{1}{2} \int \frac{1}{t^2 + 2t + 3} dt$$

$$= \frac{1}{2} \int \frac{1}{t^2 + 2t + 1 - 1 + 3} dt$$

$$= \frac{1}{2} \int \frac{1}{(t + 1)^2 + 2} dt$$

$$\text{Put } t + 1 = u \dots\dots\dots(ii)$$

$$\Rightarrow dt = du$$

$$I = \frac{1}{2} \int \frac{1}{(u)^2 + (\sqrt{2})^2} du$$

$$I = \frac{1}{2\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + c$$

$$\left[\text{since, } \int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c \right]$$

$$I = \frac{1}{2\sqrt{2}} \tan^{-1} \frac{t+1}{\sqrt{2}} + c \text{ [using (i)]}$$

$$I = \frac{1}{2\sqrt{2}} \tan^{-1} \frac{x^2+1}{\sqrt{2}} + c \text{ [using (ii)]}$$

8. Question

Evaluate the following integrals:

$$\int \frac{3x^5}{1+x^{12}} dx$$

Answer

$$\text{let } I = \int \frac{3x^5}{1+x^{12}} dx$$

$$= \int \frac{3x^5}{1+(x^6)^2} dx$$

$$\text{Let } x^6 = t \dots (i)$$

$$\Rightarrow 6x^5 dx = dt$$

$$I = \frac{3}{6} \int \frac{1}{(t)^2 + 1} dt$$

$$I = \frac{1}{2} \tan^{-1} t + c$$

$$\left[\text{since, } \int \frac{1}{1+(x)^2} dx = \tan^{-1} x + c \right]$$

$$I = \frac{1}{2} \tan^{-1}(x^6) + c \text{ [using (i)]}$$

9. Question

Evaluate the following integrals:

$$\int \frac{x^2}{x^6 - a^6} dx$$

Answer

$$\text{let } I = \int \frac{x^2}{x^6 - a^6} dx$$

$$= \int \frac{x^2}{(x^3)^2 - (a^3)^2} dx$$

$$\text{Let } x^3 = t \dots (i)$$

$$\Rightarrow 3x^2 dx = dt$$

$$I = \frac{1}{3} \int \frac{1}{t^2 - (a^3)^2} dt$$

$$I = \frac{1}{3} \times \frac{1}{2 \times a^3} \log \left| \frac{t - a^3}{t + a^3} \right| + c$$

$$[\text{since, } \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x-a}{x+a} \right| + c]$$

$$I = \frac{1}{6a^3} \log \left| \frac{x^3 - a^3}{x^3 + a^3} \right| + c \text{ [using (i)]}$$

10. Question

Evaluate the following integrals:

$$\int \frac{x^2}{x^6 + a^6} dx$$

Answer

$$\text{let } I = \int \frac{x^2}{x^6 + a^6} dx$$

$$= \int \frac{x^2}{(x^3)^2 + (a^3)^2} dx$$

$$\text{Let } x^3 = t \dots (i)$$

$$\Rightarrow 3x^2 dx = dt$$

$$I = \frac{1}{3} \int \frac{1}{t^2 + (a^3)^2} dt$$

$$I = \frac{1}{3a^3} \tan^{-1} \frac{t}{a^3} + c$$

$$[\text{since, } \int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c]$$

$$I = \frac{1}{3a^3} \tan^{-1} \frac{x^3}{a^3} + c \text{ [using (i)]}$$

11. Question

Evaluate the following integrals:

$$\int \frac{1}{x(x^6 + 1)} dx$$

Answer

$$\text{let } I = \int \frac{1}{x(x^6 + 1)} dx$$

$$= \int \frac{x^5}{x^6(x^6 + 1)} dx$$

$$\text{Let } x^6 = t \dots (i)$$

$$\Rightarrow 6x^5 dx = dt$$

$$I = \frac{1}{6} \int \frac{1}{t(t+1)} dt$$

$$I = \frac{1}{6} \int \left(\frac{1}{t} - \frac{1}{t+1} \right) dt$$

$$I = \frac{1}{6} \left(\int \frac{1}{t} dt - \int \frac{1}{(t+1)} dt \right)$$

$$I = \frac{1}{6} (\log t - \log(t+1)) + c$$

$$I = \frac{1}{6} (\log x^6 - \log(x^6+1)) + c \text{ [using (i)]}$$

$$I = \frac{1}{6} \log \frac{x^6}{x^6+1} + c \text{ [log m - log n = } \log \frac{m}{n}]$$

12. Question

Evaluate the following integrals:

$$\int \frac{x}{x^4 - x^2 + 1} dx$$

Answer

$$\text{Let } I = \int \frac{x}{x^4 - x^2 + 1} dx$$

$$\text{Let } x^2 = t \text{(i)}$$

$$\Rightarrow 2x dx = dt$$

$$I = \frac{1}{2} \int \frac{1}{t^2 - t + 1} dt$$

$$= \frac{1}{2} \int \frac{1}{t^2 - 2t\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1} dt$$

$$= \frac{1}{2} \int \frac{1}{\left(t - \frac{1}{2}\right)^2 + \frac{3}{4}} dt$$

$$\text{Put } t - 1/2 = u \text{(ii)}$$

$$\Rightarrow dt = du$$

$$I = \frac{1}{2} \int \frac{1}{(u)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} du$$

$$I = \frac{1}{2 \frac{\sqrt{3}}{2}} \tan^{-1} \frac{u}{\frac{\sqrt{3}}{2}} + c$$

$$\text{[since, } \int \frac{1}{x^2 + (a)^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a}\right) + c]$$

$$I = \frac{1}{2 \frac{\sqrt{3}}{2}} \tan^{-1} \frac{t - \frac{1}{2}}{\frac{\sqrt{3}}{2}} + c \text{ [using (i)]}$$

$$I = \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x^2 - 1}{\sqrt{3}} + c \text{ [using (ii)]}$$

13. Question

Evaluate the following integrals:

$$\int \frac{x}{3x^4 - 18x^2 + 11} dx$$

Answer

$$\text{Let } I = \int \frac{x}{3x^4 - 18x^2 + 11} dx$$

Let $x^2 = t$ (i)

$$\Rightarrow 2x dx = dt$$

$$I = \frac{1}{6} \int \frac{1}{t^2 - 6t + \frac{11}{3}} dt$$

$$= \frac{1}{6} \int \frac{1}{t^2 - 2t(3) + (3)^2 - (3)^2 + 11} dt$$

$$= \frac{1}{6} \int \frac{1}{(t-3)^2 - \frac{16}{3}} dt$$

Put $t - 3 = u$ (ii)

$$\Rightarrow dt = du$$

$$I = \frac{1}{6} \int \frac{1}{(u)^2 - \left(\frac{4}{\sqrt{3}}\right)^2} du$$

$$I = \frac{1}{6} \times \frac{1}{2 \times \frac{4}{\sqrt{3}}} \log \left| \frac{u - \frac{4}{\sqrt{3}}}{u + \frac{4}{\sqrt{3}}} \right| + c$$

$$[\text{since, } \int \frac{1}{x^2 - (a)^2} dx = \frac{1}{2 \times a} \log \left| \frac{x-a}{x+a} \right| + c]$$

$$I = \frac{\sqrt{3}}{48} \log \left| \frac{t-3-\frac{4}{\sqrt{3}}}{t-3+\frac{4}{\sqrt{3}}} \right| + c \text{ [using (ii)]}$$

$$I = \frac{\sqrt{3}}{48} \log \left| \frac{x^2-3-\frac{4}{\sqrt{3}}}{x^2-3+\frac{4}{\sqrt{3}}} \right| + c \text{ [using (i)]}$$

14. Question

Evaluate the following integrals:

$$\int \frac{e^x}{(1+e^x)(2+e^x)} dx$$

Answer

To evaluate the following integral following steps:

Let $e^x = t$ (i)

$$\Rightarrow e^x dx = dt$$

Now

$$\int \frac{e^x}{(1+e^x)(2+e^x)} dx = \int \frac{1}{(1+t)(2+t)} dt$$

$$= \int \frac{1}{(1+t)} dt - \int \frac{1}{(2+t)} dt$$

$$= \log |(1+t)| - \log |(2+t)| + c$$

$$= \log \left| \frac{1+t}{2+t} \right| + c \text{ [log m - log n = log } \frac{m}{n}]$$

$$= \log \left| \frac{1+e^x}{2+e^x} \right| + c \text{ [using(i)]}$$

15. Question

Evaluate the following integrals:

$$\int \frac{1}{\cos x + \operatorname{cosec} x} dx$$

Answer

$$\text{let } I = \frac{1}{\cos x + \operatorname{cosec} x} dx$$

Multiply and divide by $\sin x$

$$I = \frac{\frac{1}{\sin x}}{\frac{\cos x}{\sin x} + \frac{\operatorname{cosec} x}{\sin x}} dx$$

$$= \frac{\operatorname{cosec} x}{\cot x + \operatorname{cosec}^2 x} dx$$

$$= \frac{\operatorname{cosec} x}{\cot x + 1 + \cot^2 x} dx$$

$$= \frac{\operatorname{cosec} x}{\cot^2 x + \cot x + 1} dx$$

Let $\cot x = t$

$$-\operatorname{cosec} x dx = dt$$

$$\text{So, } I = - \int \frac{dt}{t^2 + t + 1}$$

$$= \int \frac{dt}{t^2 + 2t \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1}$$

$$= \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}} + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2t + 1}{\sqrt{3}} + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2 \cot x + 1}{\sqrt{3}} + c$$

Exercise 19.17

1. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{2x - x^2}} dx$$

Answer

$$\begin{aligned}
 \text{let } I &= \int \frac{1}{\sqrt{2x-x^2}} dx \\
 &= \int \frac{1}{\sqrt{-(x^2-2x)}} dx \\
 &= \int \frac{1}{\sqrt{-(x^2-2x(1)+1^2-1^2)}} dx \\
 &= \int \frac{1}{\sqrt{-(x-1)^2+1}} dx \\
 &= \int \frac{1}{\sqrt{1-(x-1)^2}} dx
 \end{aligned}$$

$$\text{let } (x-1)=t$$

$$dx=dt$$

$$\begin{aligned}
 \text{so, } I &= \int \frac{1}{\sqrt{1-t^2}} dt \\
 &= \sin^{-1} t + c \quad [\text{since } \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c]
 \end{aligned}$$

$$I = \sin^{-1}(x-1) + c$$

2. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{8+3x-x^2}} dx$$

Answer

$$8+3x-x^2 \text{ can be written as } 8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

Therefore

$$\begin{aligned}
 &8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right) \\
 &= \frac{41}{4}-\left(x-\frac{3}{2}\right)^2 \\
 \int \frac{1}{\sqrt{8+3x-x^2}} dx &= \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}} dx
 \end{aligned}$$

$$\text{Let } x-\frac{3}{2}=t$$

$$dx=dt$$

$$\begin{aligned}
 \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}} dx &= \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-t^2}} dt \\
 &= \sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right) + c
 \end{aligned}$$

$$\left[\text{since } \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= \sin^{-1}\left(\frac{x - \frac{3}{2}}{\frac{\sqrt{41}}{2}}\right) + c$$

$$= \sin^{-1}\left(\frac{2x - 3}{\sqrt{41}}\right) + c$$

3. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{5 - 4x - 2x^2}} dx$$

Answer

$$\text{Let } I = \int \frac{1}{\sqrt{5 - 4x - 2x^2}} dx$$

$$= \int \frac{1}{\sqrt{-2\left[x^2 + 2x - \frac{5}{2}\right]}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left[x^2 + 2x + (1)^2 - (1)^2 - \frac{5}{2}\right]}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left[(x + 1)^2 - \frac{7}{2}\right]}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\frac{7}{2} - (x + 1)^2}} dx$$

$$\text{Let } (x + 1) = t$$

Differentiating both sides, we get,

$$dx = dt$$

$$\text{So, } I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\left(\frac{7}{2}\right)^2 - t^2\right)}} dt$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{t}{\sqrt{\frac{7}{2}}}\right) + c$$

$$\left[\text{since } \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$I = \frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{\frac{2}{7}} \times (x + 1)\right) + c$$

4. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{3x^2 + 5x + 7}} dx$$

Answer

$$\text{let } I = \int \frac{1}{\sqrt{3x^2 + 5x + 7}} dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{x^2 + \frac{5}{3}x + \frac{7}{3}}} dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{x^2 + 2x\left(\frac{5}{6}\right) + \left(\frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2 + \frac{7}{3}}} dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(x + \frac{5}{6}\right)^2 - \frac{59}{36}}} dx$$

$$\text{let } \left(x + \frac{5}{6}\right) = t$$

$$dx = dt$$

$$I = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{t^2 - \left(\frac{\sqrt{59}}{6}\right)^2}} dt$$

$$= \frac{1}{\sqrt{3}} \log \left| t + \sqrt{t^2 - \left(\frac{\sqrt{59}}{6}\right)^2} \right| + c \quad \left[\text{since } \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log \left| x + \sqrt{x^2 - a^2} \right| + c \right]$$

$$I = \frac{1}{\sqrt{3}} \log \left| x + \frac{5}{6} + \sqrt{\left(x + \frac{5}{6}\right)^2 - \left(\frac{\sqrt{59}}{6}\right)^2} \right| + c$$

$$I = \frac{1}{\sqrt{3}} \log \left| x + \frac{5}{6} + \sqrt{x^2 + \frac{5x}{3} + \frac{7}{3}} \right| + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{(x - \alpha)(\beta - x)}} dx, (\beta > \alpha)$$

Answer

$$\text{let } I = \int \frac{1}{\sqrt{(x - \alpha)(\beta - x)}} dx, (\text{as } \beta > \alpha)$$

$$= \int \frac{1}{\sqrt{-x^2 - x(\alpha + \beta) - \alpha\beta}} dx$$

$$= \int \frac{1}{\sqrt{-\left[x^2 + 2x\left(\frac{\alpha + \beta}{2}\right) + \left(\frac{\alpha + \beta}{2}\right)^2 - \left(\frac{\alpha + \beta}{2}\right)^2 + \alpha\beta\right]}} dx$$

$$= \int \frac{1}{\sqrt{-\left[\left(x - \frac{\alpha + \beta}{2}\right)^2 - \left(\frac{\alpha + \beta}{2}\right)^2\right]}} dx$$

$$= \int \frac{1}{\sqrt{\left[\left(\frac{\beta - \alpha}{2}\right)^2 - \left(x - \frac{\alpha + \beta}{2}\right)^2\right]}} dx \quad [\beta > \alpha]$$

Let $(x - (\alpha + \beta)/2) = t$

$dx = dt$

$$I = \int \frac{1}{\sqrt{\left(\frac{\beta - \alpha}{2}\right)^2 - t^2}} dt$$

$$= \sin^{-1} \left(\frac{t}{\frac{\beta - \alpha}{2}} \right) + c$$

$$I = \sin^{-1} \left(2 \frac{x - \frac{\alpha + \beta}{2}}{\beta - \alpha} \right) + c$$

$$I = \sin^{-1} \left(\frac{2x - \alpha - \beta}{\beta - \alpha} \right)$$

6. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{7 - 3x - 2x^2}} dx$$

Answer

$$\text{let } I = \int \frac{1}{\sqrt{7 - 3x - 2x^2}} dx$$

$$= \int \frac{1}{\sqrt{-2 \left[x^2 + \frac{3}{2}x - \frac{7}{2} \right]}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{- \left[x^2 + 2x \left(\frac{3}{4} \right) + \left(\frac{3}{4} \right)^2 - \left(\frac{3}{4} \right)^2 - \frac{7}{2} \right]}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{- \left[\left(x - \frac{3}{4} \right)^2 - \frac{65}{16} \right]}} dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{\sqrt{65}}{4} \right)^2 - \left(x + \frac{3}{4} \right)^2}} dx$$

$$\text{let } \left(x + \frac{3}{4} \right) = t$$

$$dx=dt$$

$$I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{\sqrt{65}}{4}\right)^2 - (t)^2}} dt$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{t}{\frac{\sqrt{65}}{4}} \right) + c$$

$$I = \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{4 \left(x + \frac{3}{4} \right)}{\sqrt{65}} \right) + c$$

$$I = \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{4x + 3}{\sqrt{65}} \right) + c$$

7. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{16 - 6x - x^2}} dx$$

Answer

$$\text{let } I = \int \frac{1}{\sqrt{16 - 6x - x^2}} dx$$

$$= \int \frac{1}{\sqrt{-(x^2 + 6x - 16)}} dx$$

$$= \int \frac{1}{\sqrt{-(x^2 + 2x(3) + (3)^2 - (3)^2 - 16)}} dx$$

$$= \int \frac{1}{\sqrt{-(x - 3)^2 - 25}} dx$$

$$= \int \frac{1}{\sqrt{25 - (x + 3)^2}} dx$$

$$\text{let } (x + 3) = t$$

$$dx=dt$$

$$I = \int \frac{1}{\sqrt{5^2 - t^2}} dt$$

$$= \sin^{-1} \left(\frac{t}{5} \right) + c$$

$$I = \sin^{-1} \left(\frac{x + 3}{5} \right) + c$$

8. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{7 - 6x - x^2}} dx$$

Answer

$7-6x-x^2$ can be written as $7-(x^2+6x+9-9)$

Therefore

$$7-(x^2+6x+9-9)$$

$$= 16 - (x^2 + 6x + 9)$$

$$= 16 - (x + 3)^2$$

$$= (4)^2 - (x + 3)^2$$

$$\int \frac{1}{\sqrt{7-6x-x^2}} dx = \int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx$$

Let $x+3=t$

$dx=dt$

$$\int \frac{1}{\sqrt{(4)^2 - (x+3)^2}} dx = \int \frac{1}{\sqrt{(4)^2 - (t)^2}} dt$$

$$= \sin^{-1}\left(\frac{t}{4}\right) + c$$

$$= \sin^{-1}\left(\frac{x+3}{4}\right) + c$$

9. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{5x^2 - 2x}} dx$$

Answer

$$\text{we have } \int \frac{dx}{\sqrt{5x^2 - 2x}} = \int \frac{dx}{\sqrt{5\left(x^2 - \frac{2x}{5}\right)}}$$

$$= \frac{1}{\sqrt{5}} \int \frac{dx}{\sqrt{\left(x - \frac{1}{5}\right)^2 - \left(\frac{1}{5}\right)^2}} \text{ completing the square}$$

Put $x - 1/5 = t$ then $dx = dt$

$$\text{Therefore } \int \frac{dx}{\sqrt{5x^2 - 2x}} = \frac{1}{\sqrt{5}} \int \frac{dx}{\sqrt{(t)^2 - \left(\frac{1}{5}\right)^2}}$$

$$= \frac{1}{\sqrt{5}} \log \left| t + \sqrt{t^2 - \left(\frac{1}{5}\right)^2} \right| + c$$

$$= \frac{1}{\sqrt{5}} \log \left| x - \frac{1}{5} + \sqrt{x^2 - \frac{2x}{5}} \right| + c$$

Exercise 19.18**1. Question**

Evaluate the following integrals:

$$\int \frac{x}{\sqrt{x^4 + a^4}} dx$$

Answer

$$\int \frac{x}{\sqrt{x^4 + a^4}} dx = \int \frac{x}{\sqrt{(x^2)^2 + (a^2)^2}} dx$$

Let $x^2 = t$, so $2x dx = dt$

Or, $x dx = dt/2$

$$\text{Hence, } \int \frac{x}{\sqrt{(x^2)^2 + (a^2)^2}} dx = \int \frac{1}{\sqrt{t^2 + (a^2)^2}} \frac{dt}{2} = \frac{1}{2} \int \frac{1}{\sqrt{t^2 + (a^2)^2}} dt$$

$$\text{Since, } \int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log[x + \sqrt{(x^2 + a^2)}] + c$$

$$\text{Hence, } \frac{1}{2} \int \frac{1}{\sqrt{t^2 + (a^2)^2}} dt = \frac{1}{2} \log(t + \sqrt{t^2 + (a^2)^2}) + c$$

Put $t = x^2$

$$= \frac{1}{2} \log(x^2 + \sqrt{(x^2)^2 + (a^2)^2}) + c$$

$$= \frac{1}{2} \log[x^2 + \sqrt{x^4 + a^4}] + c$$

2. Question

Evaluate the following integrals:

$$\int \frac{\sec^2 x}{\sqrt{4 + \tan^2 x}} dx$$

Answer

Let $\tan x = t$

Then $dt = \sec^2 x dx$

$$\text{Therefore, } \int \frac{\sec^2 x}{\sqrt{4 + \tan^2 x}} dx = \int \frac{dt}{\sqrt{2^2 + t^2}}$$

$$\text{Since, } \int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log[x + \sqrt{(x^2 + a^2)}] + c$$

$$\text{Hence, } \int \frac{dt}{\sqrt{2^2 + t^2}} = \log[t + \sqrt{t^2 + 2^2}] + c$$

$$= \log[\tan x + \sqrt{\tan^2 x + 4}] + c$$

3. Question

Evaluate the following integrals:

$$\int \frac{e^x}{\sqrt{16 - e^{2x}}} dx$$

Answer

Let $e^x = t$

Then we have, $e^x dx = dt$

$$\text{Therefore, } \int \frac{e^x}{\sqrt{16 - e^{2x}}} dx = \int \frac{dt}{\sqrt{4^2 - t^2}}$$

Since we have, $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Hence, $\int \frac{dt}{\sqrt{4^2-t^2}} = \sin^{-1}\left(\frac{e^x}{a}\right) + c$

4. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{\sqrt{4 + \sin^2 x}} dx$$

Answer

Let $\sin x = t$

Then $dt = \cos x \, dx$

Hence, $\int \frac{\cos x}{\sqrt{4 + \sin^2 x}} dx = \int \frac{dt}{\sqrt{2^2 + t^2}}$

Since we have, $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log[x + \sqrt{(x^2 + a^2)}] + c$

Therefore, $\int \frac{dt}{\sqrt{2^2 + t^2}} = \log[t + \sqrt{t^2 + 2^2}] + c$

$= \log[t + \sqrt{t^2 + 2^2}] + c = \log[\sin x + \sqrt{\sin^2 x + 4}] + c$

5. Question

Evaluate the following integrals:

$$\int \frac{\sin x}{\sqrt{4 \cos^2 x - 1}} dx$$

Answer

Let $2 \cos x = t$

Then $dt = -2 \sin x \, dx$

Or, $\sin x \, dx = -\frac{dt}{2}$

Therefore, $\int \frac{\sin x}{\sqrt{4 \cos^2 x - 1}} dx = \int -\frac{dt}{2\sqrt{(t^2 - 1^2)}}$

Since, $\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \log[x + \sqrt{(x^2 - a^2)}] + c$

Therefore, $\int -\frac{dt}{2\sqrt{(t^2 - 1^2)}} = -\frac{1}{2} \log[t + \sqrt{t^2 - 1}] + c$

$= -\frac{1}{2} \log[2 \cos x + \sqrt{4 \cos^2 x - 1}] + c$

6. Question

Evaluate the following integrals:

$$\int \frac{x}{\sqrt{4 - x^4}} dx$$

Answer

Let $x^2 = t$

$$2x \, dx = dt \text{ or } x \, dx = dt/2$$

$$\text{Hence, } \int \frac{x}{\sqrt{4-x^4}} = \int \frac{dt}{2(\sqrt{2^2-t^2})}$$

$$\text{Since we have, } \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\text{So, } \int \frac{dt}{2(\sqrt{2^2-t^2})} = \frac{1}{2} \sin^{-1}\left(\frac{t}{2}\right) + c$$

$$\text{Put } t = x^2$$

$$= \frac{1}{2} \sin^{-1}\left(\frac{t}{2}\right) + c = \frac{1}{2} \sin^{-1}\left(\frac{x^2}{2}\right) + c$$

7. Question

Evaluate the following integrals:

$$\int \frac{1}{x\sqrt{4-9(\log x)^2}} dx$$

Answer

$$\text{Put } 3\log x = t$$

$$\text{We have } d(\log x) = 1/x$$

$$\text{Hence, } d(3\log x) = dt = 3/x \, dx$$

$$\text{Or } 1/x \, dx = dt/3$$

$$\text{Hence, } \int \frac{1}{x\sqrt{4-9(\log x)^2}} dx = \int \frac{1}{3\sqrt{2^2-t^2}} dt$$

$$\text{Since we have, } \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\text{Hence, } \int \frac{1}{3\sqrt{2^2-t^2}} dt = \frac{1}{3} \sin^{-1}\left(\frac{t}{2}\right) + c$$

$$\text{Put } t = 3\log x$$

$$= \frac{1}{3} \sin^{-1}\left(\frac{t}{2}\right) + c = \frac{1}{3} \sin^{-1}\left(\frac{3\log x}{2}\right) + c$$

8. Question

Evaluate the following integrals:

$$\int \frac{\sin 8x}{\sqrt{9+\sin^4 4x}} dx$$

Answer

$$\text{Let } t = \sin^2 4x$$

$$dt = 2\sin 4x \cos 4x \times 4 \, dx$$

$$\text{we know } \sin 2x = 2\sin x \cos x$$

$$\text{therefore, } dt = 4 \sin 8x \, dx$$

$$\text{or, } \sin 8x \, dx = dt/4$$

$$\int \frac{\sin 8x}{\sqrt{9+\sin^4 x}} dx = \frac{1}{4} \int \frac{dt}{\sqrt{3^2+t^2}}$$

Since we have, $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log[x + \sqrt{(x^2 + a^2)}] + c$

$$= \frac{1}{4} \int \frac{dt}{\sqrt{3^2 + t^2}} = \frac{1}{4} \log[t + \sqrt{t^2 + 3^2}] + c$$

$$= \frac{1}{4} \log[\sin^2 4x + \sqrt{9 + \sin^4 4x}] + c$$

9. Question

Evaluate the following integrals:

$$\int \frac{\cos 2x}{\sqrt{\sin^2 2x + 8}} dx$$

Answer

Let $t = \sin 2x$

$$dt = 2 \cos 2x dx$$

$$\cos 2x dx = dt/2$$

$$\int \frac{\cos 2x}{\sqrt{\sin^2 2x + 8}} dx = \frac{1}{2} \int \frac{dt}{\sqrt{t^2 + (2\sqrt{2})^2}}$$

Since we have, $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log[x + \sqrt{(x^2 + a^2)}] + c$

$$= \frac{1}{2} \int \frac{dt}{\sqrt{t^2 + (2\sqrt{2})^2}} = \frac{1}{2} \log[t + \sqrt{t^2 + 8}] + c$$

$$= \frac{1}{2} \log[t + \sqrt{t^2 + 8}] + c = \frac{1}{2} \log[\sin 2x + \sqrt{\sin^2 2x + 8}] + c$$

10. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{\sqrt{\sin^4 x + 4 \sin^2 x - 2}} dx$$

Answer

Let $t = \sin^2 x$

$$dt = 2 \sin x \cos x dx$$

$$\text{we know } \sin 2x = 2 \sin x \cos x$$

$$\text{therefore, } dt = \sin 2x dx$$

$$\int \frac{\sin 2x}{\sqrt{\sin^4 x + 4 \sin^2 x - 2}} dx = \int \frac{dt}{\sqrt{t^2 + 4t - 2}}$$

Add and subtract 2^2 in denominator

$$= \int \frac{dt}{\sqrt{t^2 + 4t - 2}} = \int \frac{dt}{\sqrt{t^2 + 2 \times 2t + 2^2 - 2^2 - 2}}$$

Let $t + 2 = u$

$$dt = du$$

$$= \int \frac{dt}{\sqrt{(t + 2)^2 - 6}} = \int \frac{dt}{\sqrt{(u^2 - 6)}}$$

$$\begin{aligned}
\text{Since, } \int \frac{1}{\sqrt{(x^2-a^2)}} dx &= \log[x + \sqrt{(x^2-a^2)}] + c \\
&= \int dt/\sqrt{(u^2-6)} = \log[u + \sqrt{u^2-6}] + c \\
&= \log[t + 2 + \sqrt{(t+2)^2-6}] + c \\
&= \log[t + 2 + \sqrt{(t+2)^2-6}] + c = \log[\sin^2 x + 2 + \sqrt{(\sin^2 x + 2)^2-6}] + c
\end{aligned}$$

11. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{\sqrt{\cos^4 x - \sin^2 x + 2}} dx$$

Answer

Let $t = \cos^2 x$

$$dt = 2\cos x \sin x dx = -\sin 2x dx$$

$$\text{therefore, } \int \frac{\sin 2x}{\sqrt{\cos^4 x - \sin^2 x + 2}} dx = \int -\frac{dt}{\sqrt{t^2 - (1-t^2) + 2}}$$

$$\text{since, } [\sin^2 x = 1 - \cos^2 x]$$

$$\begin{aligned}
\int -\frac{dt}{\sqrt{t^2 - (1-t^2) + 2}} &= \int -\frac{dt}{\sqrt{t^2 + t + 1}} = \int -\frac{dt}{\sqrt{t^2 + t + \frac{1}{4} + \frac{3}{4}}} \\
&= \int -\frac{dt}{\sqrt{(t + \frac{1}{2})^2 + \frac{3}{4}}}
\end{aligned}$$

$$\begin{aligned}
\text{Since, } \int \frac{1}{\sqrt{(x^2-a^2)}} dx &= \log[x + \sqrt{(x^2-a^2)}] + c \\
&= \int -\frac{dt}{\sqrt{(t + \frac{1}{2})^2 + \frac{3}{4}}} = \log[t + \frac{1}{2} + \sqrt{(t + \frac{1}{2})^2 - (\frac{\sqrt{3}}{2})^2}] + c \\
&= \log[t + \frac{1}{2} + \sqrt{t^2 + t + 1}] + c = \log[\cos^2 x + \frac{1}{2} + \sqrt{\cos^4 x + \cos^2 x + 1}] + c
\end{aligned}$$

12. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{\sqrt{4 - \sin^2 x}} dx$$

Answer

Let $\sin x = t$

$$dt = \cos x dx$$

$$\text{therefore, } \int \frac{\cos x}{\sqrt{4 - \sin^2 x}} dx = \int \frac{dt}{\sqrt{2^2 - t^2}}$$

$$\text{Since we have, } \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$= \int \frac{dt}{\sqrt{2^2 - t^2}} = \sin^{-1}\left(\frac{t}{2}\right) + c = \sin^{-1}\left(\frac{\sin x}{2}\right) + c$$

13. Question

Evaluate the following integrals:

$$\int \frac{1}{x^{\frac{2}{3}} \sqrt{x^{\frac{2}{3}} - 4}} dx$$

Answer

$$\text{Let } x^{\frac{1}{3}} = t$$

$$\text{So, } dt = \frac{1}{3} x^{\frac{1}{3}-1} dx$$

$$= dt = \frac{1}{3} x^{\frac{1}{3}-1} dx = \frac{1}{3} x^{-\frac{2}{3}}$$

$$\text{Or, } \frac{dx}{x^{\frac{2}{3}}} = 3 dt$$

$$\int \frac{1}{x^{\frac{2}{3}} \sqrt{x^{\frac{2}{3}} - 4}} dx = 3 \int \frac{dt}{\sqrt{t^2 - 2^2}}$$

$$\text{Since, } \int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \log[x + \sqrt{(x^2 - a^2)}] + c$$

$$= 3 \int \frac{dt}{\sqrt{t^2 - 2^2}} = 3 \log[t + \sqrt{t^2 - 4}] + c$$

$$= 3 \log\left[x^{\frac{1}{3}} + \sqrt{(x^{\frac{1}{3}})^2 - 4}\right] + c = 3 \log\left[x^{\frac{1}{3}} + \sqrt{x^{\frac{2}{3}} - 4}\right] + c$$

14. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{(1-x^2)} \left\{ 9 + (\sin^{-1} x)^2 \right\}} dx$$

Answer

$$\text{Let } \sin^{-1} x = t$$

$$dt = \frac{1}{\sqrt{1-x^2}} dx$$

$$\text{Therefore, } \int \frac{1}{\sqrt{(1-x^2)} \left\{ 9 + (\sin^{-1} x)^2 \right\}} dx = \int \frac{1}{\sqrt{3^2 - t^2}} dt$$

$$\text{Since we have, } \int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log[x + \sqrt{(x^2 + a^2)}] + c$$

$$= \int \frac{1}{\sqrt{3^2 - t^2}} dt = \log[t + \sqrt{9 + t^2}] + c$$

$$= \log[t + \sqrt{9 + t^2}] + c = \log[\sin^{-1} x + \sqrt{9 + (\sin^{-1} x)^2}] + c$$

15. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{\sqrt{\sin^2 x - 2 \sin x - 3}} dx$$

Answer

Let $\sin x = t$

$\cos x \, dx = dt$

$$\int \frac{\cos x}{\sqrt{\sin^2 x - 2 \sin x - 3}} dx = \int \frac{dt}{\sqrt{t^2 - 2t - 3}}$$

Add and subtract 1^2 in denominator

$$= \int \frac{dt}{\sqrt{t^2 - 2t - 3}} = \int \frac{dt}{\sqrt{t^2 - 2t + 1^2 - 1^2 - 3}} = \int \frac{dt}{\sqrt{(t-1)^2 - 2^2}}$$

Let $t - 1 = u$

$dt = du$

$$= \int \frac{dt}{\sqrt{(t-1)^2 - 2^2}} = \int \frac{du}{\sqrt{u^2 - 2^2}}$$

Since, $\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \log[x + \sqrt{(x^2 - a^2)}] + c$

$$= \int \frac{du}{\sqrt{u^2 - 2^2}} = \log[u + \sqrt{u^2 - 4}] + c$$

Put $u = t - 1$

$$= \log[t - 1 + \sqrt{(t-1)^2 - 4}] + c$$

Put $t = \sin x$

$$= \log[t - 1 + \sqrt{(t-1)^2 - 4}] + c$$

$$= \log[\sin x - 1 + \sqrt{(\sin x - 1)^2 - 4}] + c$$

$$= \log[\sin x - 1 + \sqrt{\sin^2 x - 2 \sin x - 3}] + c$$

16. Question

Evaluate the following integrals:

$$\int \sqrt{\operatorname{cosec} x - 1} \, dx$$

Answer

$$\int \sqrt{\operatorname{cosec} x - 1} \, dx$$

Since $\operatorname{cosec} x = 1/\sin x$

$$\int \sqrt{\operatorname{cosec} x - 1} \, dx = \int \sqrt{\frac{1}{\sin x} - 1} \, dx = \int \sqrt{\frac{1 - \sin x}{\sin x}} \, dx$$

Multiply with $(1 + \sin x)$ both numerator and denominator

$$= \int \sqrt{\frac{1 - \sin x}{\sin x}} \, dx = \int \sqrt{\frac{1 - \sin x * (1 + \sin x)}{\sin x * (1 + \sin x)}} \, dx$$

Since $(a + b) \times (a - b) = a^2 - b^2$,

$$\begin{aligned}
&= \int \sqrt{\frac{1 - \sin x \times (1 + \sin x)}{\sin x \times (1 + \sin x)}} dx = \int \sqrt{\frac{1 - \sin^2 x}{\sin x + \sin^2 x}} dx \\
&= \int \sqrt{\frac{\cos^2 x}{\sin x + \sin^2 x}} dx \\
&= \int \frac{\cos x}{\sqrt{\sin x + \sin^2 x}} dx
\end{aligned}$$

Let $\sin x = t$

$dt = \cos x dx$

therefore, $\int \frac{\cos x}{\sqrt{\sin x + \sin^2 x}} dx = \int \frac{dt}{\sqrt{t^2 - t}}$

multiply and divide by 2 and add and subtract $(1/2)^2$ in denominator,

$$= \int \frac{dt}{\sqrt{t^2 - 2t\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} = \frac{\int dt}{\sqrt{\left(t + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

Let $t + 1/2 = u$

$dt = du$

$$= \frac{\int dt}{\sqrt{\left(t + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} = \int \frac{dt}{\sqrt{u^2 - \left(\frac{1}{2}\right)^2}}$$

Since, $\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \log[x + \sqrt{(x^2 - a^2)}] + c$

$$= \int \frac{dt}{\sqrt{u^2 - \left(\frac{1}{2}\right)^2}} = \log\left[u + \sqrt{\left(u^2 - \left(\frac{1}{2}\right)^2\right)}\right] + c$$

$$= \log\left[t + \frac{1}{2} + \sqrt{\left(\left(t + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right)}\right] + c$$

$$= \log\left[\sin x + \frac{1}{2} + \sqrt{\sin^2 x + \sin x}\right] + c$$

17. Question

Evaluate the following integrals:

$$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx$$

Answer

$$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx = \int (\sin x - \cos x) / \sqrt{((\sin x + \cos x)^2 - 1)} dx$$

Let $\sin x + \cos x = t$

$(\cos x - \sin x) = dt$

Therefore, $\int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2 - 1}} dx = \int -\frac{dt}{\sqrt{t^2 - 1}}$

$$\begin{aligned}
 \text{Since, } \int \frac{1}{\sqrt{(x^2 - a^2)}} dx &= \log[x + \sqrt{(x^2 - a^2)}] + c \\
 &= \int -\frac{dt}{\sqrt{t^2 - 1}} = -\log[t + \sqrt{t^2 - 1}] + c \\
 &= -\log[t + \sqrt{t^2 - 1}] + c = -\log[\sin x + \cos x + \sqrt{\sin 2x}] + c
 \end{aligned}$$

18. Question

Evaluate the following integrals:

$$\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx$$

Answer

$$= \int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}} dx = \int \frac{\sin x - \cos x}{\sqrt{8 - (\sin x + \cos x)^2 + 1}} dx$$

Let $\sin x + \cos x = t$

$(\cos x - \sin x) = dt$

$$\text{Therefore, } \int \frac{\sin x - \cos x}{\sqrt{8 - (\sin x + \cos x)^2 + 1}} dx = \int \frac{dt}{\sqrt{9 - t^2}}$$

$$\text{Since we have, } \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$= \int \frac{dt}{\sqrt{9 - t^2}} = \int \frac{dt}{\sqrt{3^2 - t^2}} = \sin^{-1}\left(\frac{t}{3}\right) + c$$

$$= \sin^{-1}\left(\frac{\sin x + \cos x}{3}\right) + c = \sin^{-1}\left(\frac{\sin x}{3} + \frac{\cos x}{3}\right) + c = \sin^{-1}\left(\frac{\sin x}{3}\right) + \sin^{-1}\left(\frac{\cos x}{3}\right) + c$$

$$= \frac{x}{3} + \sin^{-1}\left(\frac{\sin x}{3}\right) + c$$

Exercise 19.19

1. Question

Evaluate the integral:

$$\int \frac{x}{x^2 + 3x + 2} dx$$

Answer

$$I = \int \frac{x}{x^2 + 3x + 2} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also $2x$. So there is a chance that we can make substitution for $x^2 + 3x + 2$ and it can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx}(x^2 + 3x + 2) = 2x + 3$$

$$\therefore \text{ Let, } x = A(2x + 3) + B$$

$$\Rightarrow x = 2Ax + 3A + B$$

On comparing both sides -

We have,

$$2A = 1 \Rightarrow A = 1/2$$

$$3A + B = 0 \Rightarrow B = -3A = -3/2$$

Hence,

$$I = \int \frac{\frac{1}{2}(2x+3) - \frac{3}{2}}{x^2+3x+2} dx$$

$$\therefore I = \frac{1}{2} \int \frac{2x+3}{x^2+3x+2} dx - \frac{3}{2} \int \frac{1}{x^2+3x+2} dx$$

$$\text{Let, } I_1 = \frac{1}{2} \int \frac{2x+3}{x^2+3x+2} dx \text{ and } I_2 = \frac{3}{2} \int \frac{1}{x^2+3x+2} dx$$

$$\text{Now, } I = I_1 - I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \frac{1}{2} \int \frac{2x+3}{x^2+3x+2} dx$$

$$\text{Let } u = x^2 + 3x + 2 \Rightarrow du = (2x + 3)dx$$

$$\therefore I_1 \text{ reduces to } \frac{1}{2} \int \frac{du}{u}$$

Hence,

$$I_1 = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \log|u| + C \left\{ \because \int \frac{dx}{x} = \log|x| + C \right\}$$

On substituting value of u , we have:

$$I_1 = \frac{1}{2} \log|x^2 + 3x + 2| + C \dots \text{eqn 2}$$

As, $I_2 = \frac{3}{2} \int \frac{1}{x^2+3x+2} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = \frac{3}{2} \int \frac{1}{x^2+3x+2} dx$$

$$\Rightarrow I_2 = \frac{3}{2} \int \frac{1}{\{x^2+2(\frac{3}{2})x+(\frac{3}{2})^2\}+2-(\frac{3}{2})^2} dx$$

$$\text{Using: } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I_2 = \frac{3}{2} \int \frac{1}{\left(x+\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2} dx$$

$$I_2 \text{ matches with } \int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$\therefore I_2 = \frac{3}{2} \left\{ \frac{1}{2(\frac{1}{2})} \log \left| \frac{\left(x+\frac{3}{2}\right) - \frac{1}{2}}{\left(x+\frac{3}{2}\right) + \frac{1}{2}} \right| + C \right\}$$

$$\Rightarrow I_2 = \frac{3}{2} \log \left| \frac{2x+3-1}{2x+3+1} \right| + C$$

$$\Rightarrow I_2 = \frac{3}{2} \log \left| \frac{2x+2}{2x+4} \right| + C = \frac{3}{2} \log \left| \frac{x+1}{x+2} \right| + C \dots \text{eqn 3}$$

From eqn 1:

$$I = I_1 - I_2$$

Using eqn 2 and eqn 3:

$$I = \frac{1}{2} \log |x^2 + 3x + 2| + \frac{3}{2} \log \left| \frac{x+1}{x+2} \right| + C$$

2. Question

Evaluate the integral:

$$\int \frac{x+1}{x^2+x+3} dx$$

Answer

$$I = \int \frac{x+1}{x^2+x+3} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also $2x$. So there is a chance that we can make substitution for $x^2 + x + 3$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx}(x^2 + x + 1) = 2x + 1$$

$$\therefore \text{ Let, } x = A(2x + 1) + B$$

$$\Rightarrow x = 2Ax + A + B$$

On comparing both sides -

We have,

$$2A = 1 \Rightarrow A = 1/2$$

$$A + B = 0 \Rightarrow B = -A = -1/2$$

Hence,

$$I = \int \frac{\frac{1}{2}(2x+1) - \frac{1}{2}}{x^2+x+3} dx$$

$$\therefore I = \frac{1}{2} \int \frac{2x+1}{x^2+x+3} dx - \frac{1}{2} \int \frac{1}{x^2+x+3} dx$$

$$\text{Let, } I_1 = \frac{1}{2} \int \frac{2x+1}{x^2+x+3} dx \text{ and } I_2 = \frac{1}{2} \int \frac{1}{x^2+x+3} dx$$

$$\text{Now, } I = I_1 - I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As } I_1 = \frac{1}{2} \int \frac{2x+1}{x^2+x+3} dx$$

$$\text{Let } u = x^2 + x + 3 \Rightarrow du = (2x + 1)dx$$

$$\therefore I_1 \text{ reduces to } \frac{1}{2} \int \frac{du}{u}$$

Hence,

$$I_1 = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \log |u| + C \left\{ \because \int \frac{dx}{x} = \log |x| + C \right\}$$

On substituting the value of u , we have:

$$I_1 = \frac{1}{2} \log |x^2 + x + 3| + C \dots \text{eqn 2}$$

As, $I_2 = \frac{1}{2} \int \frac{1}{x^2+x+3} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$i) \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad ii) \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = \frac{1}{2} \int \frac{1}{x^2+x+3} dx$$

$$\Rightarrow I_2 = \frac{1}{2} \int \frac{1}{\left\{ x^2 + 2\left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)^2 \right\} + 3 - \left(\frac{1}{2}\right)^2} dx$$

$$\text{Using: } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I_2 = \frac{1}{2} \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2} dx$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I_2 = \frac{1}{2} \left\{ \frac{1}{\left(\frac{\sqrt{11}}{2}\right)} \tan^{-1} \left(\frac{x + \frac{1}{2}}{\frac{\sqrt{11}}{2}} \right) + C \right\}$$

$$\Rightarrow I_2 = \frac{1}{\sqrt{11}} \tan^{-1} \left(\frac{2x+1}{\sqrt{11}} \right) + C \dots \text{eqn 3}$$

From eqn 1:

$$I = I_1 - I_2$$

Using eqn 2 and eqn 3:

$$I = \frac{1}{2} \log|x^2 + x + 3| + \frac{1}{\sqrt{11}} \tan^{-1} \left(\frac{2x+1}{\sqrt{11}} \right) + C$$

3. Question

Evaluate the integral:

$$\int \frac{x-3}{x^2 + 2x - 4} dx$$

Answer

$$I = \int \frac{x-3}{x^2 + 2x - 4} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also $2x$. So there is a chance that we can make substitution for $x^2 + 2x - 4$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx} (x^2 + 2x - 4) = 2x + 2$$

$$\therefore \text{ Let, } x - 3 = A(2x + 2) + B$$

$$\Rightarrow x - 3 = 2Ax + 2A + B$$

On comparing both sides -

We have,

$$2A = 1 \Rightarrow A = 1/2$$

$$2A + B = -3 \Rightarrow B = -3 - 2A = -4$$

Hence,

$$I = \int \frac{\frac{1}{2}(2x+2) - 4}{x^2 + 2x - 4} dx$$

$$\therefore I = \frac{1}{2} \int \frac{2x+2}{x^2+2x-4} dx - 4 \int \frac{1}{x^2+2x-4} dx$$

$$\text{Let, } I_1 = \frac{1}{2} \int \frac{2x+2}{x^2+2x-4} dx \text{ and } I_2 = \int \frac{1}{x^2+2x-4} dx$$

$$\text{Now, } I = I_1 - 4I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \frac{1}{2} \int \frac{2x+2}{x^2+2x-4} dx$$

$$\text{Let } u = x^2 + 2x - 4 \Rightarrow du = (2x + 2)dx$$

$$\therefore I_1 \text{ reduces to } \frac{1}{2} \int \frac{du}{u}$$

Hence,

$$I_1 = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \log|u| + C \left\{ \because \int \frac{dx}{x} = \log|x| + C \right\}$$

On substituting value of u , we have:

$$I_1 = \frac{1}{2} \log|x^2 + 2x - 4| + C \dots \text{eqn 2}$$

As, $I_2 = \int \frac{1}{x^2+2x-4} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = \int \frac{1}{x^2+2x-4} dx$$

$$\Rightarrow I_2 = \int \frac{1}{\{x^2+2(1)x+(1)^2\}-4-(1)^2} dx$$

$$\text{Using: } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I_2 = \int \frac{1}{(x+1)^2 - (\sqrt{5})^2} dx$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$\therefore I_2 = \frac{1}{2\sqrt{5}} \log \left| \frac{x+1-\sqrt{5}}{x+1+\sqrt{5}} \right| + C \dots \text{eqn 3}$$

From eqn 1:

$$I = I_1 - 4I_2$$

Using eqn 2 and eqn 3:

$$I = \frac{1}{2} \log|x^2 + 2x - 4| - 4\left(\frac{1}{2\sqrt{5}} \log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right|\right) + C$$

$$I = \frac{1}{2} \log|x^2 + 2x - 4| - \frac{2}{\sqrt{5}} \log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right| + C$$

4. Question

Evaluate the integral:

$$\int \frac{2x-3}{x^2+6x+13} dx$$

Answer

$$I = \int \frac{2x-3}{x^2+6x+13} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also $2x$. So there is a chance that we can make a substitution for $x^2 + 6x + 13$ and I can be reduced to a fundamental integration.

$$\text{As } \frac{d}{dx}(x^2 + 6x + 13) = 2x + 6$$

$$\therefore \text{Let, } 2x - 3 = A(2x + 6) + B$$

$$\Rightarrow 2x - 3 = 2Ax + 6A + B$$

On comparing both sides -

We have,

$$2A = 2 \Rightarrow A = 1$$

$$6A + B = -3 \Rightarrow B = -3 - 6A = -9$$

Hence,

$$I = \int \frac{(2x+6)-9}{x^2+6x+13} dx$$

$$\therefore I = \int \frac{2x+6}{x^2+6x+13} dx - 9 \int \frac{1}{x^2+6x+13} dx$$

$$\text{Let, } I_1 = \int \frac{2x+6}{x^2+6x+13} dx \text{ and } I_2 = \int \frac{1}{x^2+6x+13} dx$$

$$\text{Now, } I = I_1 - 9I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \int \frac{2x+6}{x^2+6x+13} dx$$

$$\text{Let } u = x^2 + 6x + 13 \Rightarrow du = (2x + 6)dx$$

$$\therefore I_1 \text{ reduces to } \int \frac{du}{u}$$

Hence,

$$I_1 = \int \frac{du}{u} = \log|u| + C \left\{ \because \int \frac{dx}{x} = \log|x| + C \right\}$$

On substituting value of u , we have:

$$I_1 = \log|x^2 + 6x + 13| + C \dots \text{eqn 2}$$

As, $I_2 = \int \frac{1}{x^2+6x+13} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = \int \frac{1}{x^2 + 6x + 13} dx$$

$$\Rightarrow I_2 = \int \frac{1}{\{x^2 + 2(3)x + (3)^2\} + 13 - (3)^2} dx$$

$$\text{Using: } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I_2 = \int \frac{1}{(x+3)^2 + (2)^2} dx$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I_2 = \frac{1}{2} \tan^{-1} \left(\frac{x+3}{2} \right) + C \dots \text{eqn 3}$$

From eqn 1:

$$I = I_1 - 9I_2$$

Using eqn 2 and eqn 3:

$$I = \log|x^2 + 6x + 13| - 9 \frac{1}{2} \tan^{-1} \left(\frac{x+3}{2} \right) + C$$

$$I = \log|x^2 + 6x + 13| - \frac{9}{2} \tan^{-1} \left(\frac{x+3}{2} \right) + C$$

5. Question

Evaluate the integral:

$$\int \frac{x-1}{3x^2 - 4x + 3} dx$$

Answer

$$I = \int \frac{x-1}{3x^2 - 4x + 3} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also $2x$. So there is a chance that we can make substitution for $3x^2 - 4x + 3$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx} (3x^2 - 4x + 3) = 6x - 4$$

$$\therefore \text{Let, } x - 1 = A(6x - 4) + B$$

$$\Rightarrow x - 1 = 6Ax - 4A + B$$

On comparing both sides -

We have,

$$6A = 1 \Rightarrow A = 1/6$$

$$-4A + B = -1 \Rightarrow B = -1 + 4A = -2/6 = -1/3$$

Hence,

$$I = \int \frac{\frac{1}{6}(6x-4) - \frac{1}{3}}{3x^2-4x+3} dx$$

$$\therefore I = \frac{1}{6} \int \frac{6x-4}{3x^2-4x+3} dx - \frac{1}{3} \int \frac{1}{3x^2-4x+3} dx$$

$$\text{Let, } I_1 = \frac{1}{6} \int \frac{6x-4}{3x^2-4x+3} dx \text{ and } I_2 = \frac{1}{3} \int \frac{1}{3x^2-4x+3} dx$$

$$\text{Now, } I = I_1 - I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \frac{1}{6} \int \frac{6x-4}{3x^2-4x+3} dx$$

$$\text{Let } u = 3x^2 - 4x + 3 \Rightarrow du = (6x - 4)dx$$

$$\therefore I_1 \text{ reduces to } \frac{1}{6} \int \frac{du}{u}$$

Hence,

$$I_1 = \frac{1}{6} \int \frac{du}{u} = \frac{1}{6} \log|u| + C \{ \because \int \frac{dx}{x} = \log|x| + C \}$$

On substituting value of u , we have:

$$I_1 = \frac{1}{6} \log|3x^2 - 4x + 3| + C \dots \text{eqn 2}$$

As, $I_2 = \frac{1}{3} \int \frac{1}{3x^2-4x+3} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in the denominator.

$$\therefore I_2 = \frac{1}{9} \int \frac{1}{x^2 - \frac{4}{3}x + 1} dx \{ \text{on taking 3 common from denominator} \}$$

$$\Rightarrow I_2 = \frac{1}{9} \int \frac{1}{\left\{ x^2 - 2\left(\frac{2}{3}\right)x + \left(\frac{2}{3}\right)^2 \right\} + 1 - \left(\frac{2}{3}\right)^2} dx$$

$$\text{Using: } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I_2 = \frac{1}{9} \int \frac{1}{\left(x - \frac{2}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2} dx$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I_2 = \frac{1}{9} \cdot \frac{1}{\frac{\sqrt{5}}{3}} \tan^{-1} \left(\frac{x - \frac{2}{3}}{\frac{\sqrt{5}}{3}} \right) + C$$

$$\therefore I_2 = \frac{3}{9\sqrt{5}} \tan^{-1} \left(\frac{3x-2}{\sqrt{5}} \right) + C = \frac{1}{3\sqrt{5}} \tan^{-1} \left(\frac{3x-2}{\sqrt{5}} \right) + C \dots \text{eqn 3}$$

From eqn 1:

$$I = I_1 - I_2$$

Using eqn 2 and eqn 3:

$$I = \frac{1}{6} \log|3x^2 - 4x + 3| - \frac{1}{3\sqrt{5}} \tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right) + C$$

6. Question

Evaluate the integral:

$$\int \frac{2x}{2+x-x^2} dx$$

Answer

$$I = \int \frac{2x}{2+x-x^2} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also $2x$. So there is a chance that we can make substitution for $-x^2 + x + 2$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx}(-x^2 + x + 2) = -2x + 1$$

$$\therefore \text{ Let, } 2x = A(-2x + 1) + B$$

$$\Rightarrow 2x = -2Ax + A + B$$

On comparing both sides -

We have,

$$-2A = 2 \Rightarrow A = -1$$

$$A + B = 0 \Rightarrow B = -A = 1$$

Hence,

$$I = \int \frac{-(-2x+1)+1}{2+x-x^2} dx$$

$$\therefore I = - \int \frac{(-2x+1)}{2+x-x^2} dx + \int \frac{1}{2+x-x^2} dx$$

$$\text{Let, } I_1 = - \int \frac{(-2x+1)}{2+x-x^2} dx \text{ and } I_2 = \int \frac{1}{2+x-x^2} dx$$

$$\text{Now, } I = I_1 + I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = - \int \frac{(-2x+1)}{2+x-x^2} dx$$

$$\text{Let } u = 2 + x - x^2 \Rightarrow du = (-2x + 1)dx$$

$$\therefore I_1 \text{ reduces to } - \int \frac{du}{u}$$

Hence,

$$I_1 = - \int \frac{du}{u} = -\log|u| + C \left\{ \because \int \frac{dx}{x} = \log|x| + C \right\}$$

On substituting value of u , we have:

$$I_1 = -\log|2 + x - x^2| + C \dots \text{eqn 2}$$

As, $I_2 = \int \frac{1}{2+x-x^2} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$i) \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad ii) \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = - \int \frac{1}{x^2 - x - 2} dx$$

$$\Rightarrow I_2 = - \int \frac{1}{\left\{ x^2 - 2\left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)^2 \right\} - 2 - \left(\frac{1}{2}\right)^2} dx$$

$$\text{Using: } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I_2 = - \int \frac{1}{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} dx$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$\therefore I_2 = - \frac{1}{2\left(\frac{3}{2}\right)} \log \left| \frac{\left(x - \frac{1}{2}\right) - \frac{3}{2}}{\left(x - \frac{1}{2}\right) + \frac{3}{2}} \right| + C$$

$$\therefore I_2 = - \frac{1}{3} \log \left| \frac{2x-1-3}{2x-1+3} \right| + C = - \frac{1}{3} \log \left| \frac{2x-4}{2x+2} \right| + C = - \frac{1}{3} \log \left| \frac{x-2}{x+1} \right| + C \dots \text{eqn 3}$$

From eqn 1:

$$I = I_1 + I_2$$

Using eqn 2 and eqn 3:

$$\therefore I = - \log |2 + x - x^2| - \frac{1}{3} \log \left| \frac{x-2}{x+1} \right| + C$$

7. Question

Evaluate the integral:

$$\int \frac{1-3x}{3x^2 + 4x + 2} dx$$

Answer

$$I = \int \frac{1-3x}{3x^2 + 4x + 2} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also $2x$. So there is a chance that we can make substitution for $3x^2 + 4x + 2$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx} (3x^2 + 4x + 2) = 6x + 4$$

$$\therefore \text{Let, } 1-3x = A(6x + 4) + B$$

$$\Rightarrow 1-3x = 6Ax + 4A + B$$

On comparing both sides -

We have,

$$6A = -3 \Rightarrow A = -1/2$$

$$4A + B = 1 \Rightarrow B = -4A + 1 = 3$$

Hence,

$$I = \int \frac{\frac{1}{2}(6x+4)+3}{3x^2+4x+2} dx$$

$$\therefore I = -\frac{1}{2} \int \frac{6x+4}{3x^2+4x+2} dx + \int \frac{3}{3x^2+4x+2} dx$$

$$\text{Let, } I_1 = -\frac{1}{2} \int \frac{6x+4}{3x^2+4x+2} dx \text{ and } I_2 = \int \frac{3}{3x^2+4x+2} dx$$

$$\text{Now, } I = I_1 + I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As } I_1 = -\frac{1}{2} \int \frac{6x+4}{3x^2+4x+2} dx$$

$$\text{Let } u = 3x^2 + 4x + 2 \Rightarrow du = (6x + 4)dx$$

$$\therefore I_1 \text{ reduces to } -\frac{1}{2} \int \frac{du}{u}$$

Hence,

$$I_1 = -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \log|u| + C \quad \left\{ \because \int \frac{dx}{x} = \log|x| + C \right\}$$

On substituting the value of u , we have:

$$I_1 = -\frac{1}{2} \log|3x^2 + 4x + 2| + C \dots \text{eqn 2}$$

As, $I_2 = \int \frac{3}{3x^2+4x+2} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = \int \frac{3}{3(x^2 + \frac{4}{3}x + \frac{2}{3})} dx = \int \frac{1}{x^2 + \frac{4}{3}x + \frac{2}{3}} dx$$

$$\Rightarrow I_2 = \int \frac{1}{\{x^2 + 2(\frac{2}{3})x + (\frac{2}{3})^2\} + \frac{2}{3} - (\frac{2}{3})^2} dx$$

$$\text{Using: } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I_2 = \int \frac{1}{\left(x + \frac{2}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2} dx$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I_2 = \frac{1}{\frac{\sqrt{2}}{3}} \tan^{-1} \left(\frac{x + \frac{2}{3}}{\frac{\sqrt{2}}{3}} \right) + C$$

$$\therefore I_2 = \frac{3}{\sqrt{2}} \tan^{-1} \left(\frac{3x+2}{\sqrt{2}} \right) + C \dots \text{eqn 3}$$

From eqn 1:

$$I = I_1 + I_2$$

Using eqn 2 and eqn 3:

$$\therefore I = -\frac{1}{2} \log|3x^2 + 4x + 2| + \frac{3}{\sqrt{2}} \tan^{-1} \left(\frac{3x+2}{\sqrt{2}} \right) + C$$

8. Question

Evaluate the integral:

$$\int \frac{2x+5}{x^2-x-2} dx$$

Answer

$$I = \int \frac{2x+5}{x^2-x-2} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also $2x$. So there is a chance that we can make substitution for $x^2 - x - 2$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx}(x^2 - x - 2) = 2x - 1$$

$$\therefore \text{Let, } 2x + 5 = A(2x - 1) + B$$

$$\Rightarrow 2x + 5 = 2Ax - A + B$$

On comparing both sides -

We have,

$$2A = 2 \Rightarrow A = 1$$

$$-A + B = 5 \Rightarrow B = A + 5 = 6$$

Hence,

$$I = \int \frac{(2x-1)+6}{x^2-x-2} dx$$

$$\therefore I = \int \frac{(2x-1)}{x^2-x-2} dx + \int \frac{6}{x^2-x-2} dx$$

$$\text{Let, } I_1 = \int \frac{(2x-1)}{x^2-x-2} dx \text{ and } I_2 = \int \frac{6}{x^2-x-2} dx$$

$$\text{Now, } I = I_1 + I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \int \frac{(2x-1)}{x^2-x-2} dx$$

$$\text{Let } u = x^2 - x - 2 \Rightarrow du = (2x - 1)dx$$

$$\therefore I_1 \text{ reduces to } \int \frac{du}{u}$$

Hence,

$$I_1 = \int \frac{du}{u} = \log|u| + C \left\{ \because \int \frac{dx}{x} = \log|x| + C \right\}$$

On substituting value of u , we have:

$$I_1 = \log|x^2 - x - 2| + C \dots \text{eqn 2}$$

As, $I_2 = \int \frac{6}{x^2-x-2} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$i) \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad ii) \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = \int \frac{6}{x^2 - x - 2} dx$$

$$\Rightarrow I_2 = \int \frac{6}{\{x^2 - 2(\frac{1}{2})x + (\frac{1}{2})^2\} - 2 - (\frac{1}{2})^2} dx$$

$$\text{Using: } a^2 - 2ab + b^2 = (a - b)^2$$

We have:

$$I_2 = 6 \int \frac{1}{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} dx$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$\therefore I_2 = \frac{6}{2\left(\frac{3}{2}\right)} \log \left| \frac{\left(x - \frac{1}{2}\right) - \frac{3}{2}}{\left(x - \frac{1}{2}\right) + \frac{3}{2}} \right| + C$$

$$\therefore I_2 = \frac{6}{3} \log \left| \frac{2x-1-3}{2x-1+3} \right| + C = 2 \log \left| \frac{2x-4}{2x+2} \right| + C = 2 \log \left| \frac{x-2}{x+1} \right| + C \dots \text{eqn 3}$$

From eqn 1, we have:

$$I = I_1 + I_2$$

Using eqn 2 and 3, we get -

$$I = \log |x^2 - x - 2| + 2 \log \left| \frac{x-2}{x+1} \right| + C \dots \text{ans}$$

9. Question

Evaluate the integral:

$$\int \frac{ax^3 + bx}{x^4 + c^2} dx$$

Answer

$$I = \int \frac{ax^3 + bx}{x^4 + c^2} dx$$

As we can see that there is a term of x^3 in numerator and derivative of x^4 is also $4x^3$. So there is a chance that we can make substitution for $x^4 + c^2$ and I can be reduced to a fundamental integration but there is also a x term present. So it is better to break this integration.

$$I = \int \frac{ax^3}{x^4 + c^2} dx + \int \frac{bx}{x^4 + c^2} dx = I_1 + I_2 \dots \text{eqn 1}$$

$$I_1 = \int \frac{ax^3}{x^4 + c^2} dx = \frac{a}{4} \int \frac{4x^3}{x^4 + c^2} dx$$

$$\text{As, } \frac{d}{dx} (x^4 + c^2) = 4x^3$$

To make the substitution, I_1 can be rewritten as

$$I_1 = \frac{a}{4} \int \frac{4x^3}{x^4 + c^2} dx$$

$$\therefore \text{Let, } x^4 + c^2 = u$$

$$\Rightarrow du = 4x^3 dx$$

I_1 is reduced to simple integration after substituting u and du as:

$$I_1 = \frac{a}{4} \int \frac{du}{u} = \frac{a}{4} \log|u| + C$$

$$\therefore I_1 = \frac{a}{4} \log|x^4 + c^2| + C \dots \text{eqn 2}$$

As,

$$I_2 = \int \frac{bx}{x^4 + c^2} dx$$

$\therefore$ we have derivative of x^2 in numerator and term of x^2 in denominator. So we can apply method of substitution here also.

$$\text{As, } I_2 = \int \frac{bx}{(x^2)^2 + c^2} dx$$

$$\text{Let, } x^2 = v$$

$$\Rightarrow dv = 2x dx$$

$$\therefore I_2 = \frac{b}{2} \int \frac{2x}{(x^2)^2 + c^2} dx = \frac{b}{2} \int \frac{dv}{(v)^2 + c^2}$$

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I_2 = \frac{b}{2} \frac{1}{c} \tan^{-1} \left(\frac{v}{c} \right) + K = \frac{b}{2c} \tan^{-1} \left(\frac{v}{c} \right) + K$$

$$\Rightarrow I_2 = \frac{b}{2c} \tan^{-1} \left(\frac{x^2}{c} \right) + K \dots \text{eqn 3}$$

From eqn 1, we have:

$$I = I_1 + I_2$$

Using eqn 2 and 3, we get -

$$I = \frac{a}{4} \log|x^4 + c^2| + \frac{b}{2c} \tan^{-1} \left(\frac{x^2}{c} \right) + K \dots \text{ans}$$

10. Question

Evaluate the integral:

$$\int \frac{(3 \sin x - 2) \cos x}{5 - \cos^2 x - 4 \sin x} dx$$

Answer

$$I = \int \frac{(3 \sin x - 2) \cos x}{5 - \cos^2 x - 4 \sin x} dx = \int \frac{(3 \sin x - 2) \cos x}{5 - (1 - \sin^2 x) - 4 \sin x} dx$$

$$\Rightarrow I = \int \frac{(3 \sin x - 2) \cos x}{4 + \sin^2 x - 4 \sin x} dx$$

$$\text{Let, } \sin x = t \Rightarrow \cos x dx = dt$$

$$\therefore I = \int \frac{(3t - 2)}{t^2 - 4t + 4} dt$$

As we can see that there is a term of t in numerator and derivative of t^2 is also $2t$. So there is a chance that we can make substitution for $t^2 - 4t + 4$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dt}(t^2 - 4t - 4) = 2t - 4$$

$$\therefore \text{Let, } 3t - 2 = A(2t - 4) + B$$

$$\Rightarrow 3t - 2 = 2At - 4A + B$$

On comparing both sides -

We have,

$$2A = 3 \Rightarrow A = 3/2$$

$$-4A + B = -2 \Rightarrow B = 4A - 2 = 4$$

Hence,

$$I = \int \frac{(3t-2)}{t^2-4t+4} dt$$

$$\therefore I = \int \frac{\frac{3}{2}(2t-4)}{t^2-4t+4} dt + \int \frac{4}{t^2-4t+4} dt$$

$$\text{Let, } I_1 = \frac{3}{2} \int \frac{(2t-4)}{t^2-4t+4} dt \text{ and } I_2 = \int \frac{4}{t^2-4t+4} dt$$

$$\text{Now, } I = I_1 + I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \frac{3}{2} \int \frac{(2t-4)}{t^2-4t+4} dt$$

$$\text{Let } u = t^2 - 4t + 4 \Rightarrow du = (2t - 4)dx$$

$$\therefore I_1 \text{ reduces to } \frac{3}{2} \int \frac{du}{u}$$

Hence,

$$I_1 = \frac{3}{2} \int \frac{du}{u} = \log|u| + C \left\{ \because \int \frac{dx}{x} = \log|x| + C \right\}$$

On substituting value of u , we have:

$$I_1 = \frac{3}{2} \log|t^2 - 4t + 4| + C$$

$$I_1 = \frac{3}{2} \log|t - 2|^2 + C = 3 \log|t - 2| + C \dots \text{eqn 2}$$

$$\therefore I_2 = \int \frac{4}{t^2-4t+4} dt$$

$$\Rightarrow I_2 = \int \frac{4}{\{t^2-2(2)t+2^2\}} dx$$

$$\text{Using: } a^2 - 2ab + b^2 = (a - b)^2$$

We have:

$$I_2 = 4 \int \frac{1}{(t-2)^2} dx$$

$$\text{As, } \int \frac{1}{x^2} dx = -\frac{1}{x}$$

$$\therefore I_2 = \frac{-4}{t-2} = \frac{4}{2-t} + C \dots \text{eqn 3}$$

From eqn 1, we have:

$$I = I_1 + I_2$$

Using eqn 2 and 3, we get -

$$I = 3 \log|t - 2| + \frac{4}{2-t} + C$$

Putting value of t in I:

$$I = 3 \log|\sin x - 2| + \frac{4}{2-\sin x} + C \dots \text{ans}$$

11. Question

Evaluate the integral:

$$\int \frac{x+2}{2x^2+6x+5} dx$$

Answer

$$I = \int \frac{x+2}{2x^2+6x+5} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $2x^2 + 6x + 5$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx}(2x^2 + 6x + 5) = 4x + 6$$

$$\therefore \text{Let, } x + 2 = A(4x + 6) + B$$

$$\Rightarrow x + 2 = 4Ax + 6A + B$$

On comparing both sides -

We have,

$$4A = 1 \Rightarrow A = 1/4$$

$$6A + B = 2 \Rightarrow B = -6A + 2 = 1/2$$

Hence,

$$I = \int \frac{\frac{1}{4}(4x+6) + \frac{1}{2}}{2x^2+6x+5} dx$$

$$\therefore I = \int \frac{\frac{1}{4}(4x+6)}{2x^2+6x+5} dx + \int \frac{\frac{1}{2}}{2x^2+6x+5} dx$$

$$\text{Let, } I_1 = \frac{1}{4} \int \frac{(4x+6)}{2x^2+6x+5} dx \text{ and } I_2 = \frac{1}{2} \int \frac{1}{2x^2+6x+5} dx$$

$$\text{Now, } I = I_1 + I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \frac{1}{4} \int \frac{(4x+6)}{2x^2+6x+5} dx$$

$$\text{Let } u = 2x^2 + 6x + 5 \Rightarrow du = (4x + 6)dx$$

$$\therefore I_1 \text{ reduces to } \frac{1}{4} \int \frac{du}{u}$$

Hence,

$$I_1 = \frac{1}{4} \int \frac{du}{u} = \frac{1}{4} \log|u| + C \{ \because \int \frac{dx}{x} = \log|x| + C \}$$

On substituting value of u, we have:

$$I_1 = \frac{1}{4} \log|2x^2 + 6x + 5| + C \dots \text{eqn 2}$$

As, $I_2 = \frac{1}{2} \int \frac{1}{2x^2 + 6x + 5} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = \frac{1}{2} \int \frac{1}{2x^2 + 6x + 5} dx = \frac{1}{2} \int \frac{1}{2(x^2 + 3x + \frac{5}{2})} dx = \frac{1}{4} \int \frac{1}{x^2 + 3x + \frac{5}{2}} dx$$

$$\Rightarrow I_2 = \frac{1}{4} \int \frac{6}{\{x^2 + 2(\frac{3}{2})x + (\frac{3}{2})^2\} + \frac{5}{2} - (\frac{3}{2})^2} dx$$

$$\text{Using: } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I_2 = \frac{1}{4} \int \frac{1}{(x + \frac{3}{2})^2 + (\frac{1}{2})^2} dx$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I_2 = \frac{1}{4} \cdot \frac{1}{\frac{1}{2}} \tan^{-1} \left(\frac{x + \frac{3}{2}}{\frac{1}{2}} \right) + C$$

$$\therefore I_2 = \frac{1}{2} \tan^{-1}(2x + 3) + C \dots \text{eqn 3}$$

From eqn 1, we have:

$$I = I_1 + I_2$$

Using eqn 2 and 3, we get -

$$I = \frac{1}{4} \log|2x^2 + 6x + 5| + C + \frac{1}{2} \tan^{-1}(2x + 3) + C \dots \text{ans}$$

12. Question

Evaluate the integral:

$$\int \frac{5x - 2}{1 + 2x + 3x^2} dx$$

Answer

$$I = \int \frac{5x - 2}{3x^2 + 2x + 1} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also $2x$. So there is a chance that we can make substitution for $3x^2 + 2x + 1$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx}(3x^2 + 2x + 1) = 6x + 2$$

$$\therefore \text{Let, } 5x - 2 = A(6x + 2) + B$$

$$\Rightarrow 5x - 2 = 6Ax + 2A + B$$

On comparing both sides -

We have,

$$6A = 5 \Rightarrow A = 5/6$$

$$2A + B = -2 \Rightarrow B = -2A - 2 = -11/3$$

Hence,

$$I = \int \frac{\frac{5}{6}(6x+2) - \frac{11}{3}}{3x^2+2x+1} dx$$

$$\therefore I = \int \frac{\frac{5}{6}(6x+2)}{3x^2+2x+1} dx + \int \frac{-\frac{11}{3}}{3x^2+2x+1} dx$$

$$\text{Let, } I_1 = \frac{5}{6} \int \frac{(6x+2)}{3x^2+2x+1} dx \text{ and } I_2 = -\frac{11}{3} \int \frac{1}{3x^2+2x+1} dx$$

$$\text{Now, } I = I_1 + I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \frac{5}{6} \int \frac{(6x+2)}{3x^2+2x+1}$$

$$\text{Let } u = 3x^2 + 2x + 1 \Rightarrow du = (6x + 2)dx$$

$$\therefore I_1 \text{ reduces to } \frac{5}{6} \int \frac{du}{u}$$

Hence,

$$I_1 = \frac{5}{6} \int \frac{du}{u} = \frac{5}{6} \log|u| + C \left\{ \because \int \frac{dx}{x} = \log|x| + C \right\}$$

On substituting value of u , we have:

$$I_1 = \frac{5}{6} \log|3x^2 + 2x + 1| + C \dots \text{eqn 2}$$

As, $I_2 = -\frac{11}{3} \int \frac{1}{3x^2+2x+1} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = -\frac{11}{3} \int \frac{1}{3x^2+2x+1} dx = -\frac{11}{3} \int \frac{1}{3(x^2+\frac{2}{3}x+\frac{1}{3})} dx = -\frac{11}{9} \int \frac{1}{x^2+\frac{2}{3}x+\frac{1}{3}} dx$$

$$\Rightarrow I_2 = -\frac{11}{9} \int \frac{6}{\{x^2+2(\frac{1}{3})x+(\frac{1}{3})^2\} + \frac{1}{3} - (\frac{1}{3})^2} dx$$

$$\text{Using: } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I_2 = -\frac{11}{9} \int \frac{1}{\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2} dx$$

$$I_2 \text{ matches with the form } \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I_2 = -\frac{11}{9} \frac{1}{\frac{\sqrt{2}}{3}} \tan^{-1} \left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}} \right) + C$$

$$\therefore I_2 = -\frac{11}{3\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C \dots \text{eqn 3}$$

From eqn 1, we have:

$$I = I_1 + I_2$$

Using eqn 2 and 3, we get -

$$I = \frac{5}{6} \log|3x^2 + 2x + 1| - \frac{11}{3\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

13. Question

Evaluate the integral:

$$\int \frac{x+5}{3x^2+13x-10} dx$$

Answer

$$I = \int \frac{x+5}{3x^2+13x-10} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $3x^2 + 13x - 10$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx}(3x^2 + 13x - 10) = 6x + 13$$

$$\therefore \text{Let, } x + 5 = A(6x + 13) + B$$

$$\Rightarrow x + 5 = 6Ax + 13A + B$$

On comparing both sides -

We have,

$$6A = 1 \Rightarrow A = 1/6$$

$$13A + B = 5 \Rightarrow B = -13A + 5 = 17/6$$

Hence,

$$I = \int \frac{\frac{1}{6}(6x+13) + \frac{17}{6}}{3x^2+13x-10} dx$$

$$\therefore I = \int \frac{\frac{1}{6}(6x+13)}{3x^2+13x-10} dx + \int \frac{\frac{17}{6}}{3x^2+13x-10} dx$$

$$\text{Let, } I_1 = \frac{1}{6} \int \frac{(6x+13)}{3x^2+13x-10} dx \text{ and } I_2 = \frac{17}{6} \int \frac{1}{3x^2+13x-10} dx$$

$$\text{Now, } I = I_1 + I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \frac{1}{6} \int \frac{(6x+13)}{3x^2+13x-10} dx$$

$$\text{Let } u = 3x^2 + 13x - 10 \Rightarrow du = (6x + 13)dx$$

$$\therefore I_1 \text{ reduces to } \frac{1}{6} \int \frac{du}{u}$$

Hence,

$$I_1 = \frac{1}{6} \int \frac{du}{u} = \frac{1}{6} \log|u| + C \left\{ \because \int \frac{dx}{x} = \log|x| + C \right\}$$

On substituting value of u, we have:

$$I_1 = \frac{1}{6} \log|3x^2 + 13x - 10| + C \dots \text{eqn 2}$$

As, $I_2 = \frac{17}{6} \int \frac{1}{3x^2 + 13x - 10} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = \frac{17}{6} \int \frac{1}{3x^2 + 13x - 10} dx = \frac{17}{6} \int \frac{1}{3(x^2 + \frac{13}{3}x - \frac{10}{3})} dx = \frac{17}{18} \int \frac{1}{x^2 + \frac{13}{3}x - \frac{10}{3}} dx$$

$$\Rightarrow I_2 = \frac{17}{18} \int \frac{6}{\{x^2 + 2(\frac{13}{6})x + (\frac{13}{6})^2\} - \frac{10}{3} - (\frac{13}{6})^2} dx$$

$$\text{Using: } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I_2 = \frac{17}{18} \int \frac{1}{\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2} dx$$

$$I_2 \text{ matches with the form } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$\therefore I_2 = \frac{17}{18} \times \frac{1}{2 \times \frac{17}{6}} \log \left| \frac{\left(x + \frac{13}{6}\right) - \frac{17}{6}}{\left(x + \frac{13}{6}\right) + \frac{17}{6}} \right| + C$$

$$\therefore I_2 = \frac{1}{6} \log \left| \frac{6x + 13 - 17}{6x + 13 + 17} \right| + C = \frac{1}{6} \log \left| \frac{6x - 4}{6x + 30} \right| + C \dots \text{eqn 3}$$

From eqn 1, we have:

$$I = I_1 + I_2$$

Using eqn 2 and 3, we get -

$$I = \frac{1}{6} \log|3x^2 + 13x - 10| + \frac{1}{6} \log \left| \frac{6x - 4}{6x + 30} \right| + C$$

4. Question

Evaluate the integral:

$$\int \frac{(3 \sin x - 2) \cos x}{13 - \cos^2 x - 7 \sin x} dx$$

Answer

$$I = \int \frac{(3 \sin x - 2) \cos x}{13 - \cos^2 x - 7 \sin x} dx = \int \frac{(3 \sin x - 2) \cos x}{13 - (1 - \sin^2 x) - 7 \sin x} dx$$

$$\Rightarrow I = \int \frac{(3 \sin x - 2) \cos x}{12 + \sin^2 x - 7 \sin x} dx$$

Let, $\sin x = t \Rightarrow \cos x dx = dt$

$$\therefore I = \int \frac{(3t - 2)}{t^2 - 7t + 12} dt$$

As we can see that there is a term of t in numerator and derivative of t^2 is also $2t$. So there is a chance that we can make substitution for $t^2 - 7t + 12$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dt}(t^2 - 7t + 12) = 2t - 7$$

$$\therefore \text{Let, } 3t - 2 = A(2t - 7) + B$$

$$\Rightarrow 3t - 2 = 2At - 7A + B$$

On comparing both sides -

We have,

$$2A = 3 \Rightarrow A = 3/2$$

$$-7A + B = -2 \Rightarrow B = 7A - 2 = 17/2$$

Hence,

$$I = \int \frac{(3t-2)}{t^2-7t+12} dt$$

$$\therefore I = \int \frac{\frac{3}{2}(2t-7)}{t^2-7t+12} dt + \int \frac{\frac{17}{2}}{t^2-7t+12} dt$$

$$\text{Let, } I_1 = \frac{3}{2} \int \frac{(2t-7)}{t^2-7t+12} dt \text{ and } I_2 = \frac{17}{2} \int \frac{1}{t^2-7t+12} dt$$

$$\text{Now, } I = I_1 + I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \frac{3}{2} \int \frac{(2t-7)}{t^2-7t+12} dt$$

$$\text{Let } u = t^2 - 7t + 12 \Rightarrow du = (2t - 7)dx$$

$$\therefore I_1 \text{ reduces to } \frac{3}{2} \int \frac{du}{u}$$

Hence,

$$I_1 = \frac{3}{2} \int \frac{du}{u} = \log|u| + C \{ \because \int \frac{dx}{x} = \log|x| + C \}$$

On substituting value of u , we have:

$$I_1 = \frac{3}{2} \log|t^2 - 7t + 12| + C \dots \text{eqn 2}$$

As, $I_2 = \frac{17}{2} \int \frac{1}{t^2-7t+12} dt$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \text{ ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I_2 = \frac{17}{2} \int \frac{1}{t^2-7t+12} dt$$

$$\Rightarrow I_2 = \frac{17}{2} \int \frac{1}{\{t^2 - 2(\frac{7}{2})t + (\frac{7}{2})^2\} + 12 - (\frac{7}{2})^2} dx$$

$$\text{Using: } a^2 - 2ab + b^2 = (a - b)^2$$

We have:

$$I_2 = \frac{17}{2} \int \frac{1}{\left(t - \frac{7}{2}\right)^2 - \left(\frac{1}{2}\right)^2} dx$$

$$I_2 \text{ matches with the form } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$\therefore I_2 = \frac{17}{2} \frac{1}{2\left(\frac{1}{2}\right)} \log \left| \frac{\left(\frac{t-7}{2}\right)^{\frac{1}{2}}}{\left(\frac{t-7}{2}\right)^{\frac{1}{2}} + \frac{1}{2}} \right| + C$$

$$I_2 = \frac{17}{2} \log \left| \frac{2t-7-1}{2t-7+1} \right| + C = \frac{17}{2} \log \left| \frac{2t-8}{2t-6} \right| + C$$

$$I_2 = \frac{17}{2} \log \left| \frac{t-4}{t-3} \right| + C \dots \text{eqn 3}$$

From eqn 1, we have:

$$I = I_1 + I_2$$

Using eqn 2 and 3, we get -

$$I = \frac{3}{2} \log |t^2 - 7t + 12| + \frac{17}{2} \log \left| \frac{t-4}{t-3} \right| + C$$

Putting value of t in I:

$$I = \frac{3}{2} \log |\sin^2 x - 7 \sin x + 12| + \frac{17}{2} \log \left| \frac{4 - \sin x}{3 - \sin x} \right| + C \dots \text{ans}$$

5. Question

Evaluate the integral:

$$\int \frac{x+7}{3x^2+25x+28} dx$$

Answer

$$I = \int \frac{x+7}{3x^2+25x+28} dx$$

As we can see that there is a term of x in numerator and derivative of x^2 is also 2x. So there is a chance that we can make substitution for $3x^2 + 13x - 10$ and I can be reduced to a fundamental integration.

$$\text{As, } \frac{d}{dx} (3x^2 + 25x + 28) = 6x + 25$$

$$\therefore \text{Let, } x + 7 = A(6x + 25) + B$$

$$\Rightarrow x + 7 = 6Ax + 25A + B$$

On comparing both sides -

We have,

$$6A = 1 \Rightarrow A = 1/6$$

$$25A + B = 5 \Rightarrow B = -25A + 5 = 5/6$$

Hence,

$$I = \int \frac{\frac{1}{6}(6x+25) + \frac{5}{6}}{3x^2+25x+28} dx$$

$$\therefore I = \int \frac{\frac{1}{6}(6x+25)}{3x^2+25x+28} dx + \int \frac{\frac{5}{6}}{3x^2+25x+28} dx$$

$$\text{Let, } I_1 = \frac{1}{6} \int \frac{(6x+25)}{3x^2+25x+28} dx \text{ and } I_2 = \frac{5}{6} \int \frac{1}{3x^2+25x+28} dx$$

$$\text{Now, } I = I_1 + I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \frac{1}{6} \int \frac{(6x+25)}{3x^2+25x+28} dx$$

$$\text{Let } u = 3x^2 + 25x + 28 \Rightarrow du = (6x + 25)dx$$

$$\therefore I_1 \text{ reduces to } \frac{1}{6} \int \frac{du}{u}$$

Hence,

$$I_1 = \frac{1}{6} \int \frac{du}{u} = \frac{1}{6} \log|u| + C \quad \left\{ \because \int \frac{dx}{x} = \log|x| + C \right\}$$

On substituting value of u, we have:

$$I_1 = \frac{1}{6} \log|3x^2 + 25x + 28| + C \dots \text{eqn 2}$$

As, $I_2 = \frac{5}{6} \int \frac{1}{3x^2 + 25x + 28} dx$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = \frac{5}{6} \int \frac{1}{3x^2 + 25x + 28} dx = \frac{5}{6} \int \frac{1}{3 \left(x^2 + \frac{25}{3}x + \frac{28}{3} \right)} dx = \frac{5}{18} \int \frac{1}{x^2 + \frac{25}{3}x + \frac{28}{3}} dx$$

$$\Rightarrow I_2 = \frac{5}{18} \int \frac{1}{\left\{ x^2 + 2 \left(\frac{25}{6} \right) x + \left(\frac{25}{6} \right)^2 \right\} + \frac{28}{3} - \left(\frac{25}{6} \right)^2} dx$$

$$\text{Using: } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I_2 = \frac{5}{18} \int \frac{1}{\left(x + \frac{25}{6} \right)^2 - \left(\frac{17}{6} \right)^2} dx$$

$$I_2 \text{ matches with the form } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$\therefore I_2 = \frac{5}{18} \times \frac{1}{2 \times \frac{17}{6}} \log \left| \frac{\left(x + \frac{25}{6} \right) - \frac{17}{6}}{\left(x + \frac{25}{6} \right) + \frac{17}{6}} \right| + C$$

$$\therefore I_2 = \frac{5}{102} \log \left| \frac{6x+25-17}{6x+25+17} \right| + C = \frac{5}{102} \log \left| \frac{6x-8}{6x+42} \right| + C \dots \text{eqn 3}$$

From eqn 1, we have:

$$I = I_1 + I_2$$

Using eqn 2 and 3, we get -

$$I = \frac{1}{6} \log|3x^2 + 25x + 28| + \frac{5}{102} \log \left| \frac{6x-8}{6x+42} \right| + C$$

16. Question

Evaluate the integral:

$$\int \frac{x^3}{x^4 + x^2 + 1} dx$$

Answer

$$\text{Let, } I = \int \frac{x^3}{x^4 + x^2 + 1} dx$$

$$I = \int \frac{x^2 x}{(x^2)^2 + x^2 + 1} dx$$

If we assume x^2 to be an another variable, we can simplify the integral as derivative of x^2 i.e. x is present in numerator.

$$\text{Let, } x^2 = u$$

$$\Rightarrow 2x dx = du$$

$$\Rightarrow x dx = 1/2 du$$

$$\therefore I = \frac{1}{2} \int \frac{u}{u^2 + u + 1} du$$

$$\text{As, } \frac{d}{du}(u^2 + u + 1) = 2u + 1$$

$$\therefore \text{Let, } u = A(2u + 1) + B$$

$$\Rightarrow u = 2Au + A + B$$

On comparing both sides -

We have,

$$2A = 1 \Rightarrow A = 1/2$$

$$A + B = 0 \Rightarrow B = -A = -1/2$$

Hence,

$$I = \frac{1}{2} \int \frac{\frac{1}{2}(2u+1) - \frac{1}{2}}{u^2 + u + 1} du$$

$$\therefore I = \frac{1}{4} \int \frac{(2u+1)}{u^2 + u + 1} du + \frac{1}{2} \int \frac{-\frac{1}{2}}{u^2 + u + 1} du$$

$$\text{Let, } I_1 = \frac{1}{4} \int \frac{(2u+1)}{u^2 + u + 1} du \text{ and } I_2 = -\frac{1}{4} \int \frac{1}{u^2 + u + 1} du$$

$$\text{Now, } I = I_1 + I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \frac{1}{4} \int \frac{(2u+1)}{u^2 + u + 1} du$$

$$\text{Let } v = u^2 + u + 1 \Rightarrow dv = (2u + 1)du$$

$$\therefore I_1 \text{ reduces to } \frac{1}{4} \int \frac{dv}{v}$$

Hence,

$$I_1 = \frac{1}{4} \int \frac{dv}{v} = \frac{1}{4} \log|v| + C \left\{ \because \int \frac{dx}{x} = \log|x| + C \right\}$$

On substituting value of u , we have:

$$I_1 = \frac{1}{4} \log|u^2 + u + 1| + C \dots \text{eqn 2}$$

As, $I_2 = -\frac{1}{4} \int \frac{1}{u^2 + u + 1} du$ and we don't have any derivative of function present in denominator. $\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = -\frac{1}{4} \int \frac{1}{u^2 + u + 1} du$$

$$\Rightarrow I_2 = -\frac{1}{4} \int \frac{1}{\left\{u^2 + 2\left(\frac{1}{2}\right)u + \left(\frac{1}{2}\right)^2\right\} + 1 - \left(\frac{1}{2}\right)^2} du$$

$$\text{Using: } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I_2 = -\frac{1}{4} \int \frac{1}{\left(u + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} du$$

$$\therefore I_2 = -\frac{1}{4} \cdot \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \left(\frac{u + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + C$$

$$\therefore I_2 = -\frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2u+1}{\sqrt{3}} \right) + C \dots \text{eqn 3}$$

From eqn 1, we have:

$$I = I_1 + I_2$$

Using eqn 2 and 3, we get -

$$I = \frac{1}{4} \log|u^2 + u + 1| - \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2u+1}{\sqrt{3}} \right) + C$$

Putting value of u in I:

$$I = \frac{1}{4} \log|x^2 + x^2 + 1| - \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2x^2+1}{\sqrt{3}} \right) + C$$

$$I = \frac{1}{4} \log|x^4 + x^2 + 1| - \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2x^2+1}{\sqrt{3}} \right) + C$$

17. Question

Evaluate the integral:

$$\int \frac{x^3 - 3x}{x^4 + 2x^2 - 4}$$

Answer

$$\text{Let, } I = \int \frac{x^3 - 3x}{x^4 + 2x^2 - 4} dx$$

$$I = \int \frac{(x^2 - 3)x}{(x^2)^2 + 2x^2 - 4} dx$$

If we assume x^2 to be an another variable, we can simplify the integral as derivative of x^2 i.e. x is present in numerator.

$$\text{Let, } x^2 = u$$

$$\Rightarrow 2x dx = du$$

$$\Rightarrow x dx = 1/2 du$$

$$\therefore I = \frac{1}{2} \int \frac{u-3}{u^2 + 2u - 4} du$$

$$\text{As, } \frac{d}{du} (u^2 + 2u - 4) = 2u + 2$$

$$\therefore \text{Let, } u - 3 = A(2u + 2) + B$$

$$\Rightarrow u - 3 = 2Au + 2A + B$$

On comparing both sides -

We have,

$$2A = 1 \Rightarrow A = 1/2$$

$$2A + B = -3 \Rightarrow B = -3 - 2A = -4$$

Hence,

$$I = \int \frac{\frac{1}{2}(2u+2)-4}{u^2+2u-4} du$$

$$\therefore I = \frac{1}{2} \int \frac{2u+2}{u^2+2u-4} du - 4 \int \frac{1}{u^2+2u-4} du$$

$$\text{Let, } I_1 = \frac{1}{2} \int \frac{2u+2}{u^2+2u-4} du \text{ and } I_2 = \int \frac{1}{u^2+2u-4} du$$

$$\text{Now, } I = I_1 - 4I_2 \dots \text{eqn 1}$$

We will solve I_1 and I_2 individually.

$$\text{As, } I_1 = \frac{1}{2} \int \frac{2u+2}{u^2+2u-4} du$$

$$\text{Let } v = u^2 + 2u - 4 \Rightarrow dv = (2u + 2)du$$

$$\therefore I_1 \text{ reduces to } \frac{1}{2} \int \frac{dv}{v}$$

Hence,

$$I_1 = \frac{1}{2} \int \frac{dv}{v} = \log|u| + C \left\{ \because \int \frac{dx}{x} = \log|x| + C \right\}$$

On substituting value of u , we have:

$$I_1 = \frac{1}{2} \log|u^2 + 2u - 4| + C \dots \text{eqn 2}$$

$$\text{As, } I_2 = \int \frac{1}{u^2+2u-4} du \text{ and we don't have any derivative of function present in denominator.}$$

$\therefore$ we will use some special integrals to solve the problem.

As denominator doesn't have any square root term. So one of the following two integrals will solve the problem.

$$\text{i) } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \quad \text{ii) } \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Now we have to reduce I_2 such that it matches with any of above two forms.

We will make to create a complete square so that no individual term of x is seen in denominator.

$$\therefore I_2 = \int \frac{1}{u^2+2u-4} du$$

$$\Rightarrow I_2 = \int \frac{1}{\{u^2+2(1)u+(1)^2\}-4-(1)^2} du$$

$$\text{Using: } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I_2 = \int \frac{1}{(u+1)^2 - (\sqrt{5})^2} du$$

$$I_2 \text{ matches with } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$\therefore I_2 = \frac{1}{2\sqrt{5}} \log \left| \frac{u+1-\sqrt{5}}{u+1+\sqrt{5}} \right| + C \dots \text{eqn 3}$$

From eqn 1:

$$I = I_1 - 4I_2$$

Using eqn 2 and eqn 3:

$$I = \frac{1}{2} \log|u^2 + 2u - 4| - 4\left(\frac{1}{2\sqrt{5}} \log\left|\frac{u+1-\sqrt{5}}{u+1+\sqrt{5}}\right|\right) + C$$

$$I = \frac{1}{2} \log|u^2 + 2u - 4| - \frac{2}{\sqrt{5}} \log\left|\frac{u+1-\sqrt{5}}{u+1+\sqrt{5}}\right| + C$$

Putting value of u in I:

$$I = \frac{1}{2} \log|x^4 + 2x^2 - 4| - \frac{2}{\sqrt{5}} \log\left|\frac{x^2+1-\sqrt{5}}{x^2+1+\sqrt{5}}\right| + C$$

Exercise 19.20

1. Question

Evaluate the following integrals:

$$\int \frac{x^2 + x + 1}{x^2 - x} dx$$

Answer

$$\text{Given } I = \int \frac{x^2 + x + 1}{x^2 - x} dx$$

$$\text{Expressing the integral } \int \frac{P(x)}{ax^2 + bx + c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2 + bx + c} dx$$

$$\Rightarrow \int \frac{x^2 + x + 1}{(x-1)x} dx$$

$$\Rightarrow \int \left(\frac{2x+1}{(x-1)x} + 1 \right) dx$$

$$\Rightarrow \int \frac{2x+1}{(x-1)x} dx + \int 1 dx$$

$$\text{Consider } \int \frac{2x+1}{(x-1)x} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{2x+1}{(x-1)x} = \frac{A}{x-1} + \frac{B}{x}$$

$$\Rightarrow 2x + 1 = Ax + B(x-1)$$

$$\Rightarrow 2x + 1 = Ax + Bx - B$$

$$\Rightarrow 2x + 1 = (A+B)x - B$$

$$\therefore B = -1 \text{ and } A + B = 2$$

$$\therefore A = 2 + 1 = 3$$

$$\text{Thus, } \Rightarrow \frac{2x+1}{(x-1)x} = \frac{3}{x-1} - \frac{1}{x}$$

$$\Rightarrow \int \left(\frac{3}{x-1} - \frac{1}{x} \right) dx$$

$$\Rightarrow 3 \int \frac{1}{x-1} dx - \int \frac{1}{x} dx$$

Consider $\int \frac{1}{x-1} dx$

Substitute $u = x - 1 \rightarrow dx = du$.

$$\Rightarrow \int \frac{1}{x-1} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\therefore \int \frac{1}{u} du = \log|u| = \log|x-1|$$

Then,

$$\Rightarrow 3 \int \frac{1}{x-1} dx - \int \frac{1}{x} dx = 3(\log|x-1|) - \int \frac{1}{x} dx$$

$$= 3(\log|x-1|) - \log|x|$$

$$\therefore \int \frac{2x+1}{(x-1)x} dx = 3(\log|x-1|) - \log|x|$$

Then,

$$\Rightarrow \int \frac{2x+1}{(x-1)x} dx + \int 1 dx = 3(\log|x-1|) - \log|x| + \int 1 dx$$

We know that $\int 1 dx = x + c$

$$\Rightarrow \int \frac{2x+1}{(x-1)x} dx + \int 1 dx = 3(\log|x-1|) - \log|x| + x + c$$

$$\therefore I = \int \frac{x^2+x-1}{x^2-x} dx = -\log|x| + x + 3(\log|x-1|) + c$$

2. Question

Evaluate the following integrals:

$$\int \frac{x^2 + x - 1}{x^2 + x - 6} dx$$

Answer

$$\text{Consider } I = \int \frac{x^2+x-1}{x^2+x-6} dx$$

$$\text{Expressing the integral } \int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$$

$$\text{Let } x^2 + x - 1 = x^2 + x - 6 + 5$$

$$\Rightarrow \int \frac{x^2+x-1}{x^2+x-6} dx = \int \left(\frac{x^2+x-6}{x^2+x-6} + \frac{5}{x^2+x-6} \right) dx$$

$$= \int \left(\frac{5}{x^2+x-6} + 1 \right) dx$$

$$= 5 \int \left(\frac{1}{x^2+x-6} \right) dx + \int 1 dx$$

$$\text{Consider } \int \frac{1}{x^2+x-6} dx$$

Factorizing the denominator,

$$\Rightarrow \int \frac{1}{x^2 + x - 6} dx = \int \frac{1}{(x-2)(x+3)} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$$

$$\Rightarrow 1 = A(x+3) + B(x-2)$$

$$\Rightarrow 1 = Ax + 3A + Bx - 2B$$

$$\Rightarrow 1 = (A+B)x + (3A-2B)$$

$$\Rightarrow \text{Then } A+B=0 \dots (1)$$

$$\text{And } 3A-2B=1 \dots (2)$$

Solving (1) and (2),

$$2 \times (1) \rightarrow 2A + 2B = 0$$

$$1 \times (2) \rightarrow 3A - 2B = 1$$

$$5A = 1$$

$$\therefore A = 1/5$$

Substituting A value in (1),

$$\Rightarrow A+B=0$$

$$\Rightarrow 1/5 + B = 0$$

$$\therefore B = -1/5$$

$$\text{Thus, } \frac{1}{(x-2)(x+3)} = \frac{1}{5(x-2)} - \frac{1}{5(x+3)}$$

$$= \frac{1}{5} \int \frac{1}{x-2} dx - \frac{1}{5} \int \frac{1}{x+3} dx$$

$$\text{Let } x-2 = u \rightarrow dx = du$$

$$\text{And } x+3 = v \rightarrow dx = dv.$$

$$\Rightarrow \frac{1}{5} \int \frac{1}{u} du - \frac{1}{5} \int \frac{1}{v} dv$$

$$\text{We know that } \int \frac{1}{x} dx = \log|x| + c$$

$$\Rightarrow \frac{1}{5} \log|u| - \frac{1}{5} \log|v|$$

$$\Rightarrow \frac{1}{5} \log|x-2| - \frac{1}{5} \log|x+3|$$

$$\Rightarrow \frac{1}{5} (\log|x-2| - \log|x+3|)$$

Then,

$$\Rightarrow 5 \int \left(\frac{1}{x^2 + x - 6} \right) dx + \int 1 dx = 5 \left(\frac{1}{5} (\log|x-2| - \log|x+3|) \right) + \int 1 dx$$

$$\text{We know that } \int 1 dx = x + c$$

$$\Rightarrow (\log|x-2| - \log|x+3|) + x + c$$

$$\therefore I = \int \frac{x^2+x-1}{x^2+x-6} dx = -\log|x+3| + x + \log|x-2| + c$$

3. Question

Evaluate the following integrals:

$$\int \frac{(1-x^2)}{x(1-2x)} dx$$

Answer

$$\text{Given } I = \int \frac{1-x^2}{(1-2x)x} dx$$

$$\text{Rewriting, we get } \int \frac{x^2-1}{x(2x-1)} dx$$

$$\text{Expressing the integral } \int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$$

$$\Rightarrow \int \frac{x^2-1}{x(2x-1)} dx = \int \left(\frac{x-2}{2x(2x-1)} + \frac{1}{2} \right) dx$$

$$= \frac{1}{2} \int \frac{x-2}{x(2x-1)} dx + \frac{1}{2} \int 1 dx$$

$$\text{Consider } \int \frac{x-2}{x(2x-1)} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{x-2}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1}$$

$$\Rightarrow x-2 = A(2x-1) + Bx$$

$$\Rightarrow x-2 = 2Ax - A + Bx$$

$$\Rightarrow x-2 = (2A+B)x - A$$

$$\therefore A = 2 \text{ and } 2A + B = 1$$

$$\therefore B = 1 - 4 = -3$$

$$\text{Thus, } \Rightarrow \frac{x-2}{x(2x-1)} = \frac{2}{x} - \frac{3}{2x-1}$$

$$\Rightarrow \int \left(\frac{2}{x} - \frac{3}{2x-1} \right) dx$$

$$\Rightarrow 2 \int \frac{1}{x} dx - 3 \int \frac{1}{2x-1} dx$$

$$\text{Consider } \int \frac{1}{x} dx$$

$$\text{We know that } \int \frac{1}{x} dx = \log|x| + c$$

$$\Rightarrow \int \frac{1}{x} dx = \log|x|$$

$$\text{And consider } \int \frac{1}{2x-1} dx$$

$$\text{Let } u = 2x - 1 \rightarrow dx = 1/2 du$$

$$\Rightarrow \int \frac{1}{2x-1} dx = \frac{1}{2} \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{u} du = \frac{\log|u|}{2} = \frac{\log|2x-1|}{2}$$

Then,

$$\begin{aligned} \Rightarrow \int \frac{x-2}{x(2x-1)} dx &= 2 \int \frac{1}{x} dx - 3 \int \frac{1}{2x-1} dx \\ &= 2(\log|x|) - 3 \left(\frac{\log|2x-1|}{2} \right) \end{aligned}$$

Then,

$$\begin{aligned} \Rightarrow \int \frac{x^2-1}{x(2x-1)} dx &= \frac{1}{2} \int \frac{x-2}{x(2x-1)} dx + \frac{1}{2} \int 1 dx \\ &= \frac{1}{2} \left(2(\log|x|) - 3 \left(\frac{\log|2x-1|}{2} \right) \right) + \frac{1}{2} \int 1 dx \end{aligned}$$

We know that $\int 1 dx = x + c$

$$\Rightarrow \log|x| - \frac{3 \log|2x-1|}{4} + \frac{x}{2} + c$$

$$\therefore I = \int \frac{1-x^2}{(1-2x)x} dx = -\frac{3 \log|2x-1|}{4} + \log|x| + \frac{x}{2} + c$$

4. Question

Evaluate the following integrals:

$$\int \frac{x^2+1}{x^2-5x+6} dx$$

Answer

$$\text{Consider } I = \int \frac{x^2+1}{x^2-5x+6} dx$$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$$\begin{aligned} \Rightarrow \int \frac{x^2+1}{x^2-5x+6} dx &= \int \left(\frac{5x-5}{x^2-5x+6} + 1 \right) dx \\ &= 5 \int \frac{x-1}{x^2-5x+6} dx + \int 1 dx \end{aligned}$$

$$\text{Consider } \int \frac{x-1}{x^2-5x+6} dx$$

Let $x-1 = \frac{1}{2}(2x-5) + \frac{3}{2}$ and split,

$$\begin{aligned} \Rightarrow \int \left(\frac{2x-5}{2(x^2-5x+6)} + \frac{3}{2(x^2-5x+6)} \right) dx \\ \Rightarrow \frac{1}{2} \int \frac{2x-5}{(x^2-5x+6)} dx + \frac{3}{2} \int \frac{1}{x^2-5x+6} dx \end{aligned}$$

$$\text{Consider } \int \frac{2x-5}{(x^2-5x+6)} dx$$

$$\text{Let } u = x^2 - 5x + 6 \rightarrow dx = \frac{1}{2x-5} du$$

$$\Rightarrow \int \frac{2x-5}{(x^2-5x+6)} dx = \int \frac{2x-5}{u} \frac{1}{2x-5} du$$

$$= \int \frac{1}{u} du$$

$$\text{We know that } \int \frac{1}{x} dx = \log|x| + c$$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x^2 - 5x + 6|$$

$$\text{Now consider } \int \frac{1}{x^2-5x+6} dx$$

$$\Rightarrow \int \frac{1}{x^2-5x+6} dx = \int \frac{1}{(x-3)(x-2)} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$$

$$\Rightarrow 1 = A(x-2) + B(x-3)$$

$$\Rightarrow 1 = Ax - 2A + Bx - 3B$$

$$\Rightarrow 1 = (A+B)x - (2A+3B)$$

$$\Rightarrow A+B=0 \text{ and } 2A+3B=-1$$

Solving the two equations,

$$\Rightarrow 2A+2B=0$$

$$2A+3B=-1$$

$$-B=1$$

$$\therefore B=-1 \text{ and } A=1$$

$$\Rightarrow \int \frac{1}{(x-3)(x-2)} dx = \int \left(\frac{1}{x-3} - \frac{1}{x-2} \right) dx$$

$$= \int \frac{1}{x-3} dx - \int \frac{1}{x-2} dx$$

$$\text{Consider } \int \frac{1}{x-3} dx$$

$$\text{Let } u = x - 3 \rightarrow dx = du$$

$$\Rightarrow \int \frac{1}{x-3} dx = \int \frac{1}{u} du$$

$$\text{We know that } \int \frac{1}{x} dx = \log|x| + c$$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x-3|$$

$$\text{Similarly } \int \frac{1}{x-2} dx$$

$$\text{Let } u = x - 2 \rightarrow dx = du$$

$$\Rightarrow \int \frac{1}{x-2} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x-2|$$

Then,

$$\begin{aligned} \Rightarrow \int \frac{1}{x^2 - 5x + 6} dx &= \int \frac{1}{(x-3)(x-2)} dx = \int \frac{1}{x-3} dx - \int \frac{1}{x-2} dx \\ &= \log|x-3| - \log|x-2| \end{aligned}$$

Then,

$$\begin{aligned} \Rightarrow \int \frac{x-1}{x^2 - 5x + 6} dx &= \frac{1}{2} \int \frac{2x-5}{(x^2 - 5x + 6)} dx + \frac{3}{2} \int \frac{1}{x^2 - 5x + 6} dx \\ &= \frac{1}{2} (\log|x^2 - 5x + 6|) + \frac{3}{2} (\log|x-3| - \log|x-2|) \\ &= \frac{\log|x^2 - 5x + 6|}{2} + \frac{3\log|x-3|}{2} - \frac{3\log|x-2|}{2} \end{aligned}$$

Then,

$$\Rightarrow \int \frac{x^2 + 1}{x^2 - 5x + 6} dx = 5 \int \frac{x-1}{x^2 - 5x + 6} dx + \int 1 dx$$

We know that $\int 1 dx = x + c$

$$\begin{aligned} \Rightarrow 5 \int \frac{x-1}{x^2 - 5x + 6} dx + \int 1 dx \\ = \frac{5\log|x^2 - 5x + 6|}{2} + \frac{15\log|x-3|}{2} - \frac{15\log|x-2|}{2} + x + c \\ = \frac{5\log|x-2|\log|x-3|}{2} + \frac{15\log|x-3|}{2} - \frac{15\log|x-2|}{2} + x + c \\ = x - 5\log|x-2| + 10\log|x-3| + c \end{aligned}$$

$$\therefore I = \int \frac{x^2 + 1}{x^2 - 5x + 6} dx = x - 5\log|x-2| + 10\log|x-3| + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{x^2}{x^2 + 7x + 10} dx$$

Answer

$$\text{Given } I = \int \frac{x^2}{x^2 + 7x + 10} dx$$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$$\begin{aligned} \Rightarrow \int \frac{x^2}{x^2 + 7x + 10} dx &= \int \left(\frac{-7x - 10}{x^2 + 7x + 10} + 1 \right) dx \\ &= - \int \frac{7x + 10}{x^2 + 7x + 10} dx + \int 1 dx \end{aligned}$$

$$\text{Consider } \int \frac{7x+10}{x^2+7x+10} dx$$

Let $7x + 10 = \frac{7}{2}(2x + 7) - \frac{29}{2}$ and split,

$$\Rightarrow \int \frac{7x + 10}{x^2 + 7x + 10} dx = \int \left(\frac{7(2x + 7)}{2(x^2 + 7x + 10)} - \frac{29}{2(x^2 + 7x + 10)} \right) dx$$
$$= \frac{7}{2} \int \frac{2x + 7}{x^2 + 7x + 10} dx - \frac{29}{2} \int \frac{1}{x^2 + 7x + 10} dx$$

Consider $\int \frac{2x+7}{x^2+7x+10} dx$

Let $u = x^2 + 7x + 10 \rightarrow dx = \frac{1}{2x+7} du$

$$\Rightarrow \int \frac{2x + 7}{(x^2 + 7x + 10)} dx = \int \frac{2x + 7}{u} \frac{1}{2x + 7} du$$
$$= \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x^2 + 7x + 10|$$

Now consider $\int \frac{1}{x^2+7x+10} dx$

$$\Rightarrow \int \frac{1}{x^2 + 7x + 10} dx = \int \frac{1}{(x + 2)(x + 5)} dx$$

By partial fraction decomposition,

$$\Rightarrow \frac{1}{(x + 2)(x + 5)} = \frac{A}{x + 2} + \frac{B}{x + 5}$$

$$\Rightarrow 1 = A(x + 2) + B(x + 5)$$

$$\Rightarrow 1 = Ax + 2A + Bx + 5B$$

$$\Rightarrow 1 = (A + B)x + (2A + 5B)$$

$$\Rightarrow A + B = 0 \text{ and } 2A + 5B = 1$$

Solving the two equations,

$$\Rightarrow 2A + 2B = 0$$

$$2A + 5B = 1$$

$$-3B = -1$$

$$\therefore B = 1/3 \text{ and } A = -1/3$$

$$\Rightarrow \int \frac{1}{(x + 2)(x + 5)} dx = \int \left(\frac{-1}{3(x + 2)} + \frac{1}{3(x + 5)} \right) dx$$

$$= -\frac{1}{3} \int \frac{1}{x + 2} dx + \frac{1}{3} \int \frac{1}{x + 5} dx$$

Consider $\int \frac{1}{x+2} dx$

Let $u = x + 2 \rightarrow dx = du$

$$\Rightarrow \int \frac{1}{x + 2} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x + 2|$$

Similarly $\int \frac{1}{x+5} dx$

Let $u = x + 5 \rightarrow dx = du$

$$\Rightarrow \int \frac{1}{x+5} dx = \int \frac{1}{u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x + 5|$$

Then,

$$\begin{aligned} \Rightarrow \int \frac{1}{x^2 + 7x + 10} dx &= \int \frac{1}{(x+2)(x+5)} dx = -\frac{1}{3} \int \frac{1}{x+2} dx + \frac{1}{3} \int \frac{1}{x+5} dx \\ &= \frac{-\log|x+2|}{3} + \frac{\log|x+5|}{3} \end{aligned}$$

Then,

$$\begin{aligned} \Rightarrow \int \frac{7x+10}{x^2+7x+10} dx &= \frac{7}{2} \int \frac{2x+7}{x^2+7x+10} dx - \frac{29}{2} \int \frac{1}{x^2+7x+10} dx \\ &= \frac{7}{2} (\log|x^2+7x+10|) - \frac{29}{2} \left(\frac{-\log|x+2|}{3} + \frac{\log|x+5|}{3} \right) \\ &= \frac{7 \log|x^2+7x+10|}{2} + \frac{29 \log|x+2|}{6} - \frac{29 \log|x+5|}{6} \end{aligned}$$

Then,

$$\Rightarrow \int \frac{x^2}{x^2+7x+10} dx = - \int \frac{7x+10}{x^2+7x+10} dx + \int 1 dx$$

We know that $\int 1 dx = x + c$

$$\begin{aligned} \Rightarrow - \int \frac{7x+10}{x^2+7x+10} dx + \int 1 dx \\ &= \frac{-7 \log|x^2+7x+10|}{2} - \frac{29 \log|x+2|}{6} + \frac{29 \log|x+5|}{6} + x + c \\ &= \frac{-7 \log|x+2| \log|x+5|}{2} - \frac{29 \log|x+2|}{6} + \frac{29 \log|x+5|}{6} + x + c \\ &= -\frac{25 \log|x+2|}{3} + \frac{4 \log|x+5|}{3} + x + c \\ \therefore I &= \int \frac{x^2}{x^2+7x+10} dx = -\frac{25 \log|x+2|}{3} + \frac{4 \log|x+5|}{3} + x + c \end{aligned}$$

6. Question

Evaluate the following integrals:

$$\int \frac{x^2 + x + 1}{x^2 - x + 1} dx$$

Answer

$$\text{Given } I = \int \frac{x^2+x+1}{x^2-x+1} dx$$

$$\text{Expressing the integral } \int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$$

$$\Rightarrow \int \frac{x^2+x+1}{x^2-x+1} dx = \int \left(\frac{2x}{x^2-x+1} + 1 \right) dx$$

$$= 2 \int \left(\frac{x}{x^2-x+1} \right) dx + \int 1 dx$$

$$\text{Consider } \int \frac{x}{x^2-x+1} dx$$

$$\text{Let } x = \frac{1}{2} (2x - 1) + \frac{1}{2} \text{ and split,}$$

$$\Rightarrow \int \left(\frac{2x-1}{2(x^2-x+1)} + \frac{1}{2(x^2-x+1)} \right) dx$$

$$\Rightarrow \frac{1}{2} \int \frac{2x-1}{(x^2-x+1)} dx + \frac{1}{2} \int \frac{1}{(x^2-x+1)} dx$$

$$\text{Consider } \int \frac{2x-1}{(x^2-x+1)} dx$$

$$\text{Let } u = x^2 - x + 1 \rightarrow dx = du/2x - 1$$

$$\Rightarrow \int \frac{2x-1}{(x^2-x+1)} dx = \int \frac{2x-1}{u} \frac{du}{2x-1}$$

$$= \int \frac{1}{u} du$$

$$\text{We know that } \int \frac{1}{x} dx = \log|x| + c$$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x^2 - x + 1|$$

$$\text{Now consider } \int \frac{1}{(x^2-x+1)} dx$$

$$\Rightarrow \int \frac{1}{(x^2-x+1)} dx = \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} dx$$

$$\text{Let } u = \frac{2x-1}{\sqrt{3}} \rightarrow dx = \frac{\sqrt{3}}{2} du$$

$$\Rightarrow \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} dx = \int \frac{2\sqrt{3}}{3u^2 + 3} du$$

$$= \frac{2}{\sqrt{3}} \int \frac{1}{u^2 + 1} du$$

$$\text{We know that } \int \frac{1}{x^2+1} dx = \tan^{-1} x + c$$

$$\Rightarrow \frac{2}{\sqrt{3}} \int \frac{1}{u^2 + 1} du = \frac{2 \tan^{-1} u}{\sqrt{3}} = \frac{2 \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right)}{\sqrt{3}}$$

Then,

$$\Rightarrow \int \frac{x}{x^2 - x + 1} dx = \frac{1}{2} \int \frac{2x - 1}{(x^2 - x + 1)} dx + \frac{1}{2} \int \frac{1}{(x^2 - x + 1)} dx$$

$$= \frac{1}{2} (\log|x^2 - x + 1|) + \frac{1}{2} \left(\frac{2 \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right)}{\sqrt{3}} \right)$$

$$= \frac{\log|x^2 - x + 1|}{2} + \frac{\tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right)}{\sqrt{3}}$$

$$\text{Now } 2 \int \left(\frac{x}{x^2 - x + 1} \right) dx + \int 1 dx$$

$$\text{We know that } \int 1 dx = x + c$$

$$\Rightarrow 2 \int \left(\frac{x}{x^2 - x + 1} \right) dx + \int 1 dx = 2 \left(\frac{\log|x^2 - x + 1|}{2} + \frac{\tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right)}{\sqrt{3}} \right) + x + c$$

$$= (\log|x^2 - x + 1|) + \left(\frac{2 \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right)}{\sqrt{3}} \right) + x + c$$

$$\therefore I = \int \frac{x^2 + x + 1}{x^2 - x + 1} dx = (\log|x^2 - x + 1|) + \left(\frac{2 \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right)}{\sqrt{3}} \right) + x + c$$

7. Question

Evaluate the following integrals:

$$\int \frac{(x-1)^2}{x^2 + 2x + 2} dx$$

Answer

$$\text{Given } I = \int \frac{(x-1)^2}{x^2 + 2x + 2} dx$$

$$\text{Expressing the integral } \int \frac{P(x)}{ax^2 + bx + c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2 + bx + c} dx$$

$$\Rightarrow \int \frac{(x-1)^2}{x^2 + 2x + 2} dx = \int \left(\frac{-4x - 1}{x^2 + 2x + 2} + 1 \right) dx$$

$$= - \int \frac{4x + 1}{x^2 + 2x + 2} dx + \int 1 dx$$

$$\text{Consider } \int \frac{4x+1}{x^2+2x+2} dx$$

Let $4x + 1 = 2(2x + 2) - 3$ and split,

$$\Rightarrow \int \frac{4x + 1}{x^2 + 2x + 2} dx = \int \left(\frac{2(2x + 2)}{x^2 + 2x + 2} - \frac{3}{x^2 + 2x + 2} \right) dx$$

$$= 4 \int \frac{x + 1}{x^2 + 2x + 2} dx - 3 \int \frac{1}{x^2 + 2x + 2} dx$$

$$\text{Consider } \int \frac{x+1}{x^2+2x+2} dx$$

$$\text{Let } u = x^2 + 2x + 2 \rightarrow du = (2x + 2) dx$$

$$\Rightarrow \int \frac{x+1}{(x^2+2x+2)} dx = \int \frac{x+1}{u} \frac{1}{2x+2} du$$

$$= \int \frac{1}{2u} du$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{u} du = \frac{\log|u|}{2} = \frac{\log|x^2+2x+2|}{2}$$

Now consider $\int \frac{1}{x^2+2x+2} dx$

$$\Rightarrow \int \frac{1}{x^2+2x+2} dx = \int \frac{1}{(x+1)^2+1} dx$$

Let $u = x + 1 \rightarrow dx = du$

$$\Rightarrow \int \frac{1}{(x+1)^2+1} dx = \int \frac{1}{u^2+1} du$$

We know that $\int \frac{1}{x^2+1} dx = \tan^{-1} x + c$

$$\Rightarrow \int \frac{1}{u^2+1} du = \tan^{-1} u = \tan^{-1}(x+1)$$

Then,

$$\Rightarrow \int \frac{4x+1}{x^2+2x+2} dx = 4 \int \frac{x+1}{x^2+2x+2} dx - 3 \int \frac{1}{x^2+2x+2} dx$$

$$= 4 \left(\frac{\log|x^2+2x+2|}{2} \right) - 3(\tan^{-1}(x+1))$$

$$= 2 \log|x^2+2x+2| - 3 \tan^{-1}(x+1)$$

Then,

$$\Rightarrow \int \frac{(x-1)^2}{x^2+2x+2} dx = - \int \frac{4x+1}{x^2+2x+2} dx + \int 1 dx$$

We know that $\int 1 dx = x + c$

$$\Rightarrow - \int \frac{4x+1}{x^2+2x+2} dx + \int 1 dx = -2 \log|x^2+2x+2| + 3 \tan^{-1}(x+1) + x + c$$

$$\therefore I = \int \frac{(x-1)^2}{x^2+2x+2} dx = -2 \log|x^2+2x+2| + 3 \tan^{-1}(x+1) + x + c$$

8. Question

Evaluate the following integrals:

$$\int \frac{x^3+x^2+2x+1}{x^2-x+1} dx$$

Answer

$$\text{Given } I = \int \frac{x^3+x^2+2x+1}{x^2-x+1} dx$$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$$\Rightarrow \int \frac{x^3 + x^2 + 2x + 1}{x^2 - x + 1} dx = \int \frac{3x - 1}{x^2 - x + 1} + x + 2 dx$$

$$= \int \frac{3x - 1}{x^2 - x + 1} dx + \int x dx + 2 \int 1 dx$$

$$\text{Consider } \int \frac{3x-1}{x^2-x+1} dx$$

$$\text{Let } 3x - 1 = \frac{3}{2}(2x - 1) + \frac{1}{2} \text{ and split,}$$

$$\Rightarrow \int \frac{3x - 1}{x^2 - x + 1} dx = \int \left(\frac{3(2x - 1)}{2(x^2 - x + 1)} + \frac{1}{2(x^2 - x + 1)} \right) dx$$

$$= \frac{3}{2} \int \frac{(2x - 1)}{(x^2 - x + 1)} dx + \frac{1}{2} \int \frac{1}{(x^2 - x + 1)} dx$$

$$\text{Consider } \int \frac{(2x-1)}{(x^2-x+1)} dx$$

$$\text{Let } u = x^2 - x + 1 \rightarrow dx = \frac{1}{2x-1} du$$

$$\Rightarrow \int \frac{(2x - 1)}{(x^2 - x + 1)} dx = \int \frac{(2x - 1)}{u} \frac{1}{2x - 1} du$$

$$= \int \frac{1}{u} du$$

$$\text{We know that } \int \frac{1}{x} dx = \log|x| + c$$

$$\Rightarrow \int \frac{1}{u} du = \log|u| = \log|x^2 - x + 1|$$

$$\text{Consider } \int \frac{1}{(x^2-x+1)} dx$$

$$\Rightarrow \int \frac{1}{(x^2 - x + 1)} dx = \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} dx$$

$$\text{Let } u = \frac{2x-1}{\sqrt{3}} \rightarrow dx = \frac{\sqrt{3}}{2} du$$

$$\Rightarrow \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} dx = \int \frac{2\sqrt{3}}{3u^2 + 3} du$$

$$= \frac{2}{\sqrt{3}} \int \frac{1}{u^2 + 1} du$$

$$\text{We know that } \int \frac{1}{x^2+1} dx = \tan^{-1} x + c$$

$$\Rightarrow \frac{2}{\sqrt{3}} \int \frac{1}{u^2 + 1} du = \frac{2 \tan^{-1} u}{\sqrt{3}} = \frac{2 \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right)}{\sqrt{3}}$$

Then,

$$\Rightarrow \int \frac{3x - 1}{x^2 - x + 1} dx = \frac{3}{2} \int \frac{2x - 1}{(x^2 - x + 1)} dx + \frac{1}{2} \int \frac{1}{(x^2 - x + 1)} dx$$

$$= \frac{3}{2} (\log|x^2 - x + 1|) + \frac{1}{2} \left(\frac{2 \tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right)}{\sqrt{3}} \right)$$

$$= \frac{3 \log|x^2 - x + 1|}{2} + \frac{\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}$$

Then,

$$\Rightarrow \int \frac{x^3 + x^2 + 2x + 1}{x^2 - x + 1} dx = \int \frac{3x - 1}{x^2 - x + 1} dx + \int x dx + 2 \int 1 dx$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ and $\int 1 dx = x + c$

$$\begin{aligned} \Rightarrow \int \frac{3x - 1}{x^2 - x + 1} dx + \int x dx + 2 \int 1 dx \\ = \frac{3 \log|x^2 - x + 1|}{2} + \frac{\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}} + \frac{x^2}{2} + 2x + c \end{aligned}$$

$$= \frac{3 \log|x^2 - x + 1| + x^2 + 4x}{2} + \frac{\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}} + c$$

$$\therefore I = \int \frac{x^3 + x^2 + 2x + 1}{x^2 - x + 1} dx = \frac{3 \log|x^2 - x + 1| + x^2 + 4x}{2} + \frac{\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}} + c$$

9. Question

Evaluate the following integrals:

$$\int \frac{x^2(x^4 + 4)}{x^2 + 4} dx$$

Answer

$$\text{Given } I = \int \frac{x^2(x^4 + 4)}{x^2 + 4} dx$$

Expressing the integral $\int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$

$$\Rightarrow \int \frac{x^2(x^4 + 4)}{x^2 + 4} dx = \int \left(-\frac{80}{x^2 + 4} + x^4 - 4x^2 + 20 \right) dx$$

$$= -80 \int \frac{1}{x^2 + 4} dx + \int x^4 dx - 4 \int x^2 dx + 20 \int 1 dx$$

Consider $\int \frac{1}{x^2 + 4} dx$

Let $u = 1/2 x \rightarrow dx = 2du$

$$\Rightarrow \int \frac{1}{x^2 + 4} dx = \int \frac{2}{4u^2 + 4} du$$

$$= \frac{1}{2} \int \frac{1}{u^2 + 1} du$$

We know that $\int \frac{1}{x^2 + 1} dx = \tan^{-1} x + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{u^2 + 1} du = \frac{\tan^{-1} u}{2} = \frac{\tan^{-1}\left(\frac{x}{2}\right)}{2}$$

Then,

$$\Rightarrow \int \frac{x^2(x^4 + 4)}{x^2 + 4} dx = -80 \int \frac{1}{x^2 + 4} dx + \int x^4 dx - 4 \int x^2 dx + 20 \int 1 dx$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ and $\int 1 dx = x + c$

$$\Rightarrow -80 \left(\frac{\tan^{-1}\left(\frac{x}{2}\right)}{2} \right) + \frac{x^5}{5} - \frac{4x^3}{3} + 20x + c$$

$$\Rightarrow -40 \tan^{-1}\left(\frac{x}{2}\right) + \frac{x^5}{5} - \frac{4x^3}{3} + 20x + c$$

$$\therefore I = \int \frac{x^2(x^4 + 4)}{x^2 + 4} dx = -40 \tan^{-1}\left(\frac{x}{2}\right) + \frac{x^5}{5} - \frac{4x^3}{3} + 20x + c$$

10. Question

Evaluate the following integrals:

$$\int \frac{x^2}{x^2 + 6x + 12} dx$$

Answer

$$\text{Given } I = \int \frac{x^2}{x^2 + 6x + 12} dx$$

Expressing the integral $\int \frac{P(x)}{ax^2 + bx + c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2 + bx + c} dx$

$$\Rightarrow \int \frac{x^2}{x^2 + 6x + 12} dx = \int \left(\frac{-6x - 12}{x^2 + 6x + 12} + 1 \right) dx$$

$$= -6 \int \frac{x + 2}{x^2 + 6x + 12} dx + \int 1 dx$$

$$\text{Consider } \int \frac{x+2}{x^2+6x+12} dx$$

Let $x + 2 = 1/2(2x + 6) - 1$ and split,

$$\Rightarrow \int \frac{x + 2}{x^2 + 6x + 12} dx = \int \left(\frac{(2x + 6)}{2(x^2 + 6x + 12)} - \frac{1}{(x^2 + 6x + 12)} \right) dx$$

$$= \int \frac{x + 3}{x^2 + 6x + 12} dx - \int \frac{1}{x^2 + 6x + 12} dx$$

$$\text{Consider } \int \frac{x+3}{x^2+6x+12} dx$$

$$\text{Let } u = x^2 + 6x + 12 \rightarrow dx = \frac{1}{2x+6} du$$

$$\Rightarrow \int \frac{x + 3}{(x^2 + 6x + 12)} dx = \int \frac{x + 3}{u} \frac{1}{2x + 6} du$$

$$= \int \frac{1}{2u} du$$

$$\text{We know that } \int \frac{1}{x} dx = \log|x| + c$$

$$\Rightarrow \frac{1}{2} \int \frac{1}{u} du = \frac{\log|u|}{2} = \frac{\log|x^2 + 6x + 12|}{2}$$

$$\text{Now consider } \int \frac{1}{x^2+6x+12} dx$$

$$\Rightarrow \int \frac{1}{x^2 + 6x + 12} dx = \int \frac{1}{(x + 3)^2 + 3} dx$$

$$\text{Let } u = \frac{x+3}{\sqrt{3}} \rightarrow dx = \sqrt{3} du$$

$$\Rightarrow \int \frac{1}{(x+3)^2 + 3} dx = \frac{\sqrt{3}}{3u^2 + 3}$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{u^2 + 1} du$$

$$\text{We know that } \int \frac{1}{x^2 + 1} dx = \tan^{-1} x + c$$

$$\Rightarrow \frac{1}{\sqrt{3}} \int \frac{1}{u^2 + 1} du = \frac{\tan^{-1} u}{\sqrt{3}} = \frac{\tan^{-1}(\frac{x+3}{\sqrt{3}})}{\sqrt{3}}$$

Then,

$$\begin{aligned} \Rightarrow \int \frac{x+2}{x^2 + 6x + 12} dx &= \int \frac{x+3}{x^2 + 6x + 12} dx - \int \frac{1}{x^2 + 6x + 12} dx \\ &= \frac{\log|x^2 + 6x + 12|}{2} - \frac{\tan^{-1}(\frac{x+3}{\sqrt{3}})}{\sqrt{3}} \end{aligned}$$

Then,

$$\Rightarrow \int \frac{x^2}{x^2 + 6x + 12} dx = -6 \int \frac{x+2}{x^2 + 6x + 12} dx + \int 1 dx$$

$$\text{We know that } \int 1 dx = x + c$$

$$\begin{aligned} \Rightarrow -6 \int \frac{x+2}{x^2 + 6x + 12} dx + \int 1 dx \\ = -3 \log|x^2 + 6x + 12| + \frac{6 \tan^{-1}(\frac{x+3}{\sqrt{3}})}{\sqrt{3}} + x + c \end{aligned}$$

$$= -3 \log|x^2 + 6x + 12| + 2\sqrt{3} \tan^{-1}(\frac{x+3}{\sqrt{3}}) + x + c$$

$$\therefore I = \int \frac{x^2}{x^2 + 6x + 12} dx = -3 \log|x^2 + 6x + 12| + 2\sqrt{3} \tan^{-1}(\frac{x+3}{\sqrt{3}}) + x + c$$

Exercise 19.21

1. Question

Evaluate the following integrals:

$$\int \frac{x}{\sqrt{x^2 + 6x + 10}} dx$$

Answer

$$\text{Given } I = \int \frac{x}{\sqrt{x^2 + 6x + 10}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$$

$$\Rightarrow px + q = \lambda(2ax + b) + \mu$$

$$\Rightarrow x = \lambda(2x + 6) + \mu$$

$$\therefore \lambda = 1/2 \text{ and } \mu = -3$$

Let $x = 1/2(2x + 6) - 3$ and split,

$$\begin{aligned} \Rightarrow \int \frac{x}{\sqrt{x^2 + 6x + 10}} dx &= \int \left(\frac{2x + 6}{2\sqrt{x^2 + 6x + 10}} - \frac{3}{\sqrt{x^2 + 6x + 10}} \right) dx \\ &= \int \frac{x + 3}{\sqrt{x^2 + 6x + 10}} dx - 3 \int \frac{1}{\sqrt{x^2 + 6x + 10}} dx \end{aligned}$$

$$\text{Consider } \int \frac{x+3}{\sqrt{x^2+6x+10}} dx$$

$$\text{Let } u = x^2 + 6x + 10 \rightarrow dx = \frac{1}{2x+6} du$$

$$\begin{aligned} \Rightarrow \int \frac{x + 3}{\sqrt{x^2 + 6x + 10}} dx &= \int \frac{1}{2\sqrt{u}} du \\ &= \frac{1}{2} \int \frac{1}{\sqrt{u}} du \end{aligned}$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\begin{aligned} \Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u}} du &= \frac{1}{2} (2\sqrt{u}) \\ &= \sqrt{u} = \sqrt{x^2 + 6x + 10} \end{aligned}$$

$$\text{Consider } \int \frac{1}{\sqrt{x^2+6x+10}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 6x + 10}} dx = \int \frac{1}{\sqrt{(x + 3)^2 + 1}} dx$$

$$\text{Let } u = x + 3 \rightarrow dx = du$$

$$\Rightarrow \int \frac{1}{\sqrt{(x + 3)^2 + 1}} dx = \int \frac{1}{\sqrt{(u)^2 + 1}} du$$

$$\text{We know that } \int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + c$$

$$\begin{aligned} \Rightarrow \int \frac{1}{\sqrt{u^2 + 1}} du &= \sinh^{-1}(u) \\ &= \sinh^{-1}(x + 3) \end{aligned}$$

Then,

$$\begin{aligned} \Rightarrow \int \frac{x}{\sqrt{x^2 + 6x + 10}} dx &= \int \frac{x + 3}{\sqrt{x^2 + 6x + 10}} dx - 3 \int \frac{1}{\sqrt{x^2 + 6x + 10}} dx \\ &= \sqrt{x^2 + 6x + 10} - 3 \sinh^{-1}(x + 3) + c \\ \therefore I &= \int \frac{x}{\sqrt{x^2 + 6x + 10}} dx = \sqrt{x^2 + 6x + 10} - 3 \sinh^{-1}(x + 3) + c \end{aligned}$$

2. Question

Evaluate the following integrals:

$$\int \frac{2x + 1}{\sqrt{x^2 + 2x - 1}} dx$$

Answer

$$\text{Given } I = \int \frac{2x+1}{\sqrt{x^2+2x-1}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$$

$$\Rightarrow px + q = \lambda(2ax + b) + \mu$$

$$\Rightarrow 2x + 1 = \lambda(2x + 2) + \mu$$

$$\therefore \lambda = 1 \text{ and } \mu = -1$$

Let $2x + 1 = 2x + 2 - 1$ and split,

$$\begin{aligned} \Rightarrow \int \frac{2x+1}{\sqrt{x^2+2x-1}} dx &= \int \left(\frac{2x+2}{\sqrt{x^2+2x-1}} - \frac{1}{\sqrt{x^2+2x-1}} \right) dx \\ &= 2 \int \frac{x+1}{\sqrt{x^2+2x-1}} dx - \int \frac{1}{\sqrt{x^2+2x-1}} dx \end{aligned}$$

$$\text{Consider } \int \frac{x+1}{\sqrt{x^2+2x-1}} dx$$

$$\text{Let } u = x^2 + 2x - 1 \rightarrow dx = \frac{1}{2x+2} du$$

$$\begin{aligned} \Rightarrow \int \frac{x+1}{\sqrt{x^2+2x-1}} dx &= \int \frac{1}{2\sqrt{u}} du \\ &= \frac{1}{2} \int \frac{1}{\sqrt{u}} du \end{aligned}$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\begin{aligned} \Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u}} du &= \frac{1}{2} (2\sqrt{u}) \\ &= \sqrt{u} = \sqrt{x^2 + 2x - 1} \end{aligned}$$

$$\text{Consider } \int \frac{1}{\sqrt{x^2+2x-1}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x-1}} dx = \int \frac{1}{\sqrt{(x+1)^2 - 2}} dx$$

$$\text{Let } u = \frac{x+1}{\sqrt{2}} \rightarrow dx = \sqrt{2} du$$

$$\begin{aligned} \Rightarrow \int \frac{1}{\sqrt{(x+1)^2 - 2}} dx &= \int \frac{\sqrt{2}}{\sqrt{2u^2 - 2}} du \\ &= \int \frac{1}{\sqrt{u^2 - 1}} du \end{aligned}$$

$$\text{We know that } \int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + c$$

$$\begin{aligned} \Rightarrow \int \frac{1}{\sqrt{u^2 - 1}} du &= \cosh^{-1}(u) \\ &= \cosh^{-1} \left(\frac{x+1}{\sqrt{2}} \right) \end{aligned}$$

Then,

$$\Rightarrow \int \frac{2x+1}{\sqrt{x^2+2x-1}} dx = 2 \int \frac{x+1}{\sqrt{x^2+2x-1}} dx - \int \frac{1}{\sqrt{x^2+2x-1}} dx$$
$$= 2\sqrt{x^2+2x-1} - \cosh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$$\therefore I = \int \frac{2x+1}{\sqrt{x^2+2x-1}} dx = 2\sqrt{x^2+2x-1} - \cosh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

3. Question

Evaluate the following integrals:

$$\int \frac{x+1}{\sqrt{x+5x-x^2}} dx$$

Answer

$$\text{Given } I = \int \frac{x+1}{\sqrt{4+5x-x^2}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px+q = \lambda \left\{ \frac{d}{dx} (ax^2+bx+c) \right\} + \mu$$

$$\Rightarrow px+q = \lambda(2ax+b) + \mu$$

$$\Rightarrow x+1 = \lambda(-2x+5) + \mu$$

$$\therefore \lambda = -1/2 \text{ and } \mu = 7/2$$

$$\text{Let } x+1 = -1/2(-2x+5) + 7/2$$

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2+5x+4}} dx = \int \left(\frac{-2x+5}{2\sqrt{-x^2+5x+4}} + \frac{7}{2\sqrt{-x^2+5x+4}} \right) dx$$
$$= \frac{1}{2} \int \frac{-2x+5}{\sqrt{-x^2+5x+4}} dx + \frac{7}{2} \int \frac{1}{\sqrt{-x^2+5x+4}} dx$$

$$\text{Consider } \int \frac{-2x+5}{\sqrt{-x^2+5x+4}} dx$$

$$\text{Let } u = -x^2+5x+4 \rightarrow dx = \frac{1}{-2x+5} du$$

$$\Rightarrow \int \frac{-2x+5}{\sqrt{-x^2+5x+4}} dx = - \int \frac{1}{\sqrt{u}} du$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow - \int \frac{1}{\sqrt{u}} du = -(2\sqrt{u})$$

$$= -2\sqrt{x^2+6x+10}$$

$$\text{Consider } \int \frac{1}{\sqrt{-x^2+5x+4}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{-x^2+5x+4}} dx = \int \frac{1}{\sqrt{-\left(x-\frac{5}{2}\right)^2 + \frac{41}{4}}} dx$$

$$\text{Let } u = \frac{2x-5}{\sqrt{41}} \rightarrow dx = \frac{\sqrt{41}}{2} du$$

$$\Rightarrow \int \frac{1}{\sqrt{-\left(x-\frac{5}{2}\right)^2 + \frac{41}{4}}} dx = \int \frac{\sqrt{41}}{\sqrt{41-41u^2}} du$$

$$= \int \frac{1}{\sqrt{1-u^2}} du$$

$$\text{We know that } \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + c$$

$$\Rightarrow \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right)$$

Then,

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2+5x+4}} dx = \frac{1}{2} \int \frac{-2x+5}{\sqrt{-x^2+5x+4}} dx + \frac{7}{2} \int \frac{1}{\sqrt{-x^2+5x+4}} dx$$

$$= -\sqrt{-x^2+5x+4} + \frac{7}{2} \left(\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right) \right) + c$$

$$\therefore I = \int \frac{x+1}{\sqrt{-x^2+5x+4}} dx = -\sqrt{-x^2+5x+4} + \frac{7}{2} \left(\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right) \right) + c$$

4. Question

Evaluate the following integrals:

$$\int \frac{6x-5}{\sqrt{3x^2-5x+1}} dx$$

Answer

$$\text{Given } I = \int \frac{6x-5}{\sqrt{3x^2-5x+1}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px+q = \lambda \left\{ \frac{d}{dx}(ax^2+bx+c) \right\} + \mu$$

$$\Rightarrow px+q = \lambda(2ax+b) + \mu$$

$$\Rightarrow 6x-5 = \lambda(6x-5) + \mu$$

$$\therefore \lambda = 1 \text{ and } \mu = 0$$

$$\text{Let } u = 3x^2-5x+1 \rightarrow dx = \frac{1}{6x-5} du$$

$$\Rightarrow \int \frac{6x-5}{\sqrt{3x^2-5x+1}} dx = \int \frac{1}{\sqrt{u}} du$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow \int \frac{1}{\sqrt{u}} du = (2\sqrt{u}) + c$$

$$= 2\sqrt{3x^2-5x+1} + c$$

$$\therefore I = \int \frac{6x-5}{\sqrt{3x^2-5x+1}} dx = 2\sqrt{3x^2-5x+1} + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{3x+1}{\sqrt{5-2x-x^2}} dx$$

Answer

$$\text{Given } I = \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px+q = \lambda \left\{ \frac{d}{dx} (ax^2+bx+c) \right\} + \mu$$

$$\Rightarrow px+q = \lambda(2ax+b) + \mu$$

$$\Rightarrow 3x+1 = \lambda(-2x-2) + \mu$$

$$\therefore \lambda = -3/2 \text{ and } \mu = -2$$

$$\text{Let } 3x+1 = -(3/2)(-2x-2) - 2$$

$$\Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx = \int \left(\frac{-3(-2x-2)}{2\sqrt{-x^2-2x+5}} - \frac{2}{\sqrt{-x^2-2x+5}} \right) dx$$

$$= 3 \int \frac{x+1}{\sqrt{-x^2-2x+5}} dx - 2 \int \frac{1}{\sqrt{-x^2-2x+5}} dx$$

$$\text{Consider } \int \frac{x+1}{\sqrt{-x^2-2x+5}} dx$$

$$\text{Let } u = -x^2-2x+5 \rightarrow dx = \frac{1}{-2x-2} du$$

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2-2x+5}} dx = \int -\frac{1}{2\sqrt{u}} du$$

$$= -\frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow -\frac{1}{2} \int \frac{1}{\sqrt{u}} du = -(\sqrt{u})$$

$$= -\sqrt{-x^2-2x+5}$$

$$\text{Consider } \int \frac{1}{\sqrt{-x^2-2x+5}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{-x^2-2x+5}} dx = \int \frac{1}{\sqrt{6-(x+1)^2}} dx$$

$$\text{Let } u = \frac{x+1}{\sqrt{6}} \rightarrow dx = \sqrt{6} du$$

$$\Rightarrow \int \frac{1}{\sqrt{6-(x+1)^2}} dx = \int \frac{\sqrt{6}}{\sqrt{6-6u^2}} du$$

$$= \int \frac{1}{\sqrt{1-u^2}} du$$

We know that $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)$$

Then,

$$\Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx = 3 \int \frac{x+1}{\sqrt{-x^2-2x+5}} dx - 2 \int \frac{1}{\sqrt{-x^2-2x+5}} dx$$

$$= -3\sqrt{-x^2-2x+5} - 2\left(\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)\right) + c$$

$$\therefore I = \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx = -3\sqrt{-x^2-2x+5} - 2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + c$$

6. Question

Evaluate the following integrals:

$$\int \frac{x}{\sqrt{8+x-x^2}} dx$$

Answer

$$\text{Given } I = \int \frac{x}{\sqrt{-x^2+x+8}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px+q = \lambda \left\{ \frac{d}{dx}(ax^2+bx+c) \right\} + \mu$$

$$\Rightarrow px+q = \lambda(2ax+b) + \mu$$

$$\Rightarrow x = \lambda(-2x+1) + \mu$$

$$\therefore \lambda = -1/2 \text{ and } \mu = -1/2$$

Let $x = -1/2(-2x+1) - 1/2$ and split,

$$\Rightarrow \int \frac{x}{\sqrt{-x^2+x+8}} dx = \int \left(\frac{-(-2x+1)}{2\sqrt{-x^2+x+8}} - \frac{1}{2\sqrt{-x^2+x+8}} \right) dx$$

$$= \frac{1}{2} \int \frac{2x-1}{\sqrt{-x^2+x+8}} dx - \frac{1}{2} \int \frac{1}{\sqrt{-x^2+x+8}} dx$$

$$\text{Consider } \int \frac{2x-1}{\sqrt{-x^2+x+8}} dx$$

$$\text{Let } u = -x^2+x+8 \rightarrow dx = \frac{1}{-2x+1} du$$

$$\Rightarrow \int \frac{2x-1}{\sqrt{-x^2+x+8}} dx = \int -\frac{1}{\sqrt{u}} du$$

$$= - \int \frac{1}{\sqrt{u}} du$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow - \int \frac{1}{\sqrt{u}} du = -(2\sqrt{u})$$

$$= -2\sqrt{-x^2+x+8}$$

Consider $\int \frac{1}{\sqrt{-x^2+x+8}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{-x^2+x+8}} dx = \int \frac{1}{\sqrt{\frac{33}{4} - \left(x - \frac{1}{2}\right)^2}} dx$$

$$\text{Let } u = \frac{2x-1}{\sqrt{33}} \rightarrow dx = \frac{\sqrt{33}}{2} du$$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{33}{4} - \left(x - \frac{1}{2}\right)^2}} dx = \int \frac{\sqrt{33}}{\sqrt{33 - 33u^2}} du$$

$$= \int \frac{1}{\sqrt{1 - u^2}} du$$

We know that $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{1 - u^2}} du = \sin^{-1}(u)$$

$$= \sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right)$$

Then,

$$\Rightarrow \int \frac{x}{\sqrt{-x^2+x+8}} dx = \frac{1}{2} \int \frac{2x-1}{\sqrt{-x^2+x+8}} dx - \frac{1}{2} \int \frac{1}{\sqrt{-x^2+x+8}} dx$$

$$= -\sqrt{-x^2+x+8} - \frac{1}{2} \left(\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right) \right) + c$$

$$\therefore I = \int \frac{x}{\sqrt{-x^2+x+8}} dx = -\sqrt{-x^2+x+8} - \frac{\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right)}{2} + c$$

7. Question

Evaluate the following integrals:

$$\int \frac{x+2}{\sqrt{x^2+2x-1}} dx$$

Answer

$$\text{Given } I = \int \frac{x+2}{\sqrt{x^2+2x-1}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px+q = \lambda \left\{ \frac{d}{dx} (ax^2+bx+c) \right\} + \mu$$

$$\Rightarrow px+q = \lambda(2ax+b) + \mu$$

$$\Rightarrow x+2 = \lambda(2x+2) + \mu$$

$$\therefore \lambda = 1/2 \text{ and } \mu = 1$$

Let $x+2 = 1/2(2x+2) + 1$ and split,

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2+2x-1}} dx = \int \left(\frac{2x+2}{2\sqrt{x^2+2x-1}} + \frac{1}{\sqrt{x^2+2x-1}} \right) dx$$

$$= \int \frac{x+1}{\sqrt{x^2+2x-1}} dx + \int \frac{1}{\sqrt{x^2+2x-1}} dx$$

Consider $\int \frac{x+1}{\sqrt{x^2+2x-1}} dx$

Let $u = x^2 + 2x - 1 \rightarrow dx = \frac{1}{2x+2} du$

$$\Rightarrow \int \frac{x+1}{\sqrt{x^2+2x-1}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} (2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2+2x-1}$$

Consider $\int \frac{1}{\sqrt{x^2+2x-1}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x-1}} dx = \int \frac{1}{\sqrt{(x+1)^2-2}} dx$$

Let $u = \frac{x+1}{\sqrt{2}} \rightarrow dx = \sqrt{2} du$

$$\Rightarrow \int \frac{1}{\sqrt{(x+1)^2-2}} dx = \int \frac{\sqrt{2}}{\sqrt{2u^2-2}} du$$

$$= \int \frac{1}{\sqrt{u^2-1}} du$$

We know that $\int \frac{1}{\sqrt{x^2-1}} dx = \log(\sqrt{x^2-1} + x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2-1}} du = \log(\sqrt{u^2-1} + u)$$

$$= \log\left(\sqrt{\frac{(x+1)^2}{2}-1} + \frac{x+1}{\sqrt{2}}\right)$$

Then,

$$\Rightarrow \int \frac{x+1}{\sqrt{x^2+2x-1}} dx = \int \frac{x+1}{\sqrt{x^2+2x-1}} dx + \int \frac{1}{\sqrt{x^2+2x-1}} dx$$

$$= \sqrt{x^2+2x-1} + \log\left(\sqrt{\frac{(x+1)^2}{2}-1} + \frac{x+1}{\sqrt{2}}\right) + c$$

$$= \sqrt{x^2+2x-1} + \log(\sqrt{(x+1)^2-2} + x+1) + c$$

$$\therefore I = \int \frac{2x+1}{\sqrt{x^2+2x-1}} dx = \sqrt{x^2+2x-1} + \log(\sqrt{(x+1)^2-2} + x+1) + c$$

8. Question

Evaluate the following integrals:

$$\int \frac{x+2}{\sqrt{x^2-1}} dx$$

Answer

$$\text{Given } I = \int \frac{x+2}{\sqrt{x^2-1}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$$

$$\Rightarrow px + q = \lambda(2ax + b) + \mu$$

$$\Rightarrow x + 2 = \lambda(2x) + \mu$$

$$\therefore \lambda = 1/2 \text{ and } \mu = 2$$

Let $x + 2 = 1/2(2x) + 2$ and split,

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2-1}} dx = \int \left(\frac{2x}{2\sqrt{x^2-1}} + \frac{2}{\sqrt{x^2-1}} \right) dx$$

$$= \int \frac{x}{\sqrt{x^2-1}} dx + 2 \int \frac{1}{\sqrt{x^2-1}} dx$$

$$\text{Consider } \int \frac{x}{\sqrt{x^2-1}} dx$$

$$\text{Let } u = x^2 - 1 \rightarrow dx = \frac{1}{2x} du$$

$$\Rightarrow \int \frac{x}{\sqrt{x^2-1}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} (2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2-1}$$

$$\text{Consider } \int \frac{1}{\sqrt{x^2-1}} dx$$

$$\text{We know that } \int \frac{1}{\sqrt{x^2-1}} dx + c = \cosh^{-1} x + c$$

$$\Rightarrow \int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1}(x)$$

Then,

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2-1}} dx = \int \frac{x}{\sqrt{x^2-1}} dx + 2 \int \frac{1}{\sqrt{x^2-1}} dx$$

$$= \sqrt{x^2-1} + \cosh^{-1}(x) + c$$

$$\therefore I = \int \frac{x+2}{\sqrt{x^2-1}} dx = \sqrt{x^2-1} + \cosh^{-1}(x) + c$$

9. Question

Evaluate the following integrals:

Answer

$$\text{Given } I = \int \frac{x-1}{\sqrt{x^2+1}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$$

$$\Rightarrow px + q = \lambda(2ax + b) + \mu$$

$$\Rightarrow x - 1 = \lambda(2x) + \mu$$

$$\therefore \lambda = 1/2 \text{ and } \mu = -1$$

Let $x - 1 = 1/2(2x) - 1$ and split,

$$\Rightarrow \int \frac{x-1}{\sqrt{x^2+1}} dx = \int \left(\frac{2x}{2\sqrt{x^2+1}} - \frac{1}{\sqrt{x^2+1}} \right) dx$$

$$= \int \frac{x}{\sqrt{x^2+1}} dx - \int \frac{1}{\sqrt{x^2+1}} dx$$

$$\text{Consider } \int \frac{x}{\sqrt{x^2+1}} dx$$

$$\text{Let } u = x^2 + 1 \rightarrow dx = \frac{1}{2x} du$$

$$\Rightarrow \int \frac{x}{\sqrt{x^2+1}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} (2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2+1}$$

$$\text{Consider } \int \frac{1}{\sqrt{x^2+1}} dx$$

$$\text{We know that } \int \frac{1}{\sqrt{x^2+1}} dx + c = \sinh^{-1} x + c$$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1}(x)$$

Then,

$$\Rightarrow \int \frac{x-1}{\sqrt{x^2+1}} dx = \int \frac{x}{\sqrt{x^2+1}} dx - \int \frac{1}{\sqrt{x^2+1}} dx$$

$$= \sqrt{x^2+1} - \sinh^{-1}(x) + c$$

$$\therefore I = \int \frac{x-1}{\sqrt{x^2+1}} dx = \sqrt{x^2+1} - \sinh^{-1}(x) + c$$

10. Question

Evaluate the following integrals:

$$\int \frac{x}{\sqrt{x^2 + x + 1}} dx$$

Answer

$$\text{Given } I = \int \frac{x}{\sqrt{x^2 + x + 1}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$$

$$\Rightarrow px + q = \lambda(2ax + b) + \mu$$

$$\Rightarrow x = \lambda(2x + 1) + \mu$$

$$\therefore \lambda = 1/2 \text{ and } \mu = -1/2$$

$$\text{Let } x = 1/2(2x + 1) - 1/2 \text{ and split,}$$

$$\Rightarrow \int \frac{x}{\sqrt{x^2 + x + 1}} dx = \int \left(\frac{2x + 1}{2\sqrt{x^2 + x + 1}} - \frac{1}{2\sqrt{x^2 + x + 1}} \right) dx$$

$$= \frac{1}{2} \int \frac{2x + 1}{\sqrt{x^2 + x + 1}} dx - \frac{1}{2} \int \frac{1}{\sqrt{x^2 + x + 1}} dx$$

$$\text{Consider } \int \frac{2x+1}{\sqrt{x^2+x+1}} dx$$

$$\text{Let } u = x^2 + x + 1 \rightarrow dx = \frac{1}{2x+1} du$$

$$\Rightarrow \int \frac{2x + 1}{\sqrt{x^2 + x + 1}} dx = \int \frac{1}{\sqrt{u}} du$$

$$= \int \frac{1}{\sqrt{u}} du$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow \int \frac{1}{\sqrt{u}} du = (2\sqrt{u})$$

$$= 2\sqrt{u} = 2\sqrt{x^2 + x + 1}$$

$$\text{Consider } \int \frac{1}{\sqrt{x^2+x+1}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + x + 1}} dx = \int \frac{1}{\sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}} dx$$

$$\text{Let } u = \frac{2x+1}{\sqrt{3}} \rightarrow dx = \frac{\sqrt{3}}{2} du$$

$$\Rightarrow \int \frac{1}{\sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}} dx = \int \frac{\sqrt{3}}{\sqrt{3u^2 + 3}} du$$

$$= \int \frac{1}{\sqrt{u^2 + 1}} du$$

$$\text{We know that } \int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + c$$

$$\Rightarrow \int \frac{1}{\sqrt{u^2 + 1}} du = \sinh^{-1}(u)$$

$$= \sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

Then,

$$\Rightarrow \int \frac{x}{\sqrt{x^2 + x + 1}} dx = \frac{1}{2} \int \frac{2x+1}{\sqrt{x^2 + x + 1}} dx - \frac{1}{2} \int \frac{1}{\sqrt{x^2 + x + 1}} dx$$

$$= \sqrt{x^2 + x + 1} - \frac{\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)}{2} + c$$

$$\therefore I = \int \frac{x}{\sqrt{x^2 + x + 1}} dx = \sqrt{x^2 + x + 1} - \frac{\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)}{2} + c$$

11. Question

Evaluate the following integrals:

$$\int \frac{x+1}{\sqrt{x^2+1}} dx$$

Answer

$$\text{Given } I = \int \frac{x+1}{\sqrt{x^2+1}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$$

$$\Rightarrow px + q = \lambda(2ax + b) + \mu$$

$$\Rightarrow x + 1 = \lambda(2x) + \mu$$

$$\therefore \lambda = 1/2 \text{ and } \mu = 1$$

Let $x + 1 = 1/2(2x) + 1$ and split,

$$\Rightarrow \int \frac{x+1}{\sqrt{x^2+1}} dx = \int \left(\frac{2x}{2\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} \right) dx$$

$$= \int \frac{x}{\sqrt{x^2+1}} dx + \int \frac{1}{\sqrt{x^2+1}} dx$$

$$\text{Consider } \int \frac{x}{\sqrt{x^2+1}} dx$$

$$\text{Let } u = x^2 + 1 \rightarrow dx = \frac{1}{2x} du$$

$$\Rightarrow \int \frac{x}{\sqrt{x^2+1}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} (2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2 + 1}$$

$$\text{Consider } \int \frac{1}{\sqrt{x^2+1}} dx$$

$$\text{We know that } \int \frac{1}{\sqrt{x^2+1}} dx + c = \sinh^{-1} x + c$$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1}(x)$$

Then,

$$\Rightarrow \int \frac{x+1}{\sqrt{x^2+1}} dx = \int \frac{x}{\sqrt{x^2+1}} dx + \int \frac{1}{\sqrt{x^2+1}} dx$$

$$= \sqrt{x^2+1} + \sinh^{-1}(x) + c$$

$$\therefore I = \int \frac{x+1}{\sqrt{x^2+1}} dx = \sqrt{x^2+1} + \sinh^{-1}(x) + c$$

12. Question

Evaluate the following integrals:

$$\int \frac{2x+5}{\sqrt{x^2+2x+5}} dx$$

Answer

$$\text{Given } I = \int \frac{2x+5}{\sqrt{x^2+2x+5}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px+q = \lambda \left\{ \frac{d}{dx} (ax^2+bx+c) \right\} + \mu$$

$$\Rightarrow px+q = \lambda(2ax+b) + \mu$$

$$\Rightarrow 2x+5 = \lambda(2x+2) + \mu$$

$$\therefore \lambda = 1 \text{ and } \mu = 3$$

Let $2x+5 = 2x+2+3$ and split,

$$\Rightarrow \int \frac{2x+5}{\sqrt{x^2+2x+5}} dx = \int \left(\frac{2x+2}{\sqrt{x^2+2x+5}} + \frac{3}{\sqrt{x^2+2x+5}} \right) dx$$

$$= 2 \int \frac{x+1}{\sqrt{x^2+2x+5}} dx + 3 \int \frac{1}{\sqrt{x^2+2x+5}} dx$$

$$\text{Consider } \int \frac{x+1}{\sqrt{x^2+2x+5}} dx$$

$$\text{Let } u = x^2+2x+5 \rightarrow dx = \frac{1}{2x+2} du$$

$$\Rightarrow \int \frac{x+1}{\sqrt{x^2+2x+5}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} (2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2 + 2x + 5}$$

$$\text{Consider } \int \frac{1}{\sqrt{x^2 + 2x + 5}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 2x + 5}} dx = \int \frac{1}{\sqrt{(x+1)^2 + 4}} dx$$

$$\text{Let } u = \frac{x+1}{2} \rightarrow dx = 2du$$

$$\Rightarrow \int \frac{1}{\sqrt{(x+1)^2 + 4}} dx = \int \frac{2}{\sqrt{4u^2 + 4}} du$$

$$= \int \frac{1}{\sqrt{u^2 + 1}} du$$

$$\text{We know that } \int \frac{1}{\sqrt{x^2 + 1}} dx = \sinh^{-1} x + c$$

$$\Rightarrow \int \frac{1}{\sqrt{u^2 + 1}} du = \sinh^{-1}(u)$$

$$= \sinh^{-1}\left(\frac{x+1}{2}\right)$$

Then,

$$\Rightarrow \int \frac{2x+5}{\sqrt{x^2 + 2x + 5}} dx = 2 \int \frac{x+1}{\sqrt{x^2 + 2x + 5}} dx + 3 \int \frac{1}{\sqrt{x^2 + 2x + 5}} dx$$

$$= 2\sqrt{x^2 + 2x + 5} + 3\sinh^{-1}\left(\frac{x+1}{2}\right) + c$$

$$\therefore I = \int \frac{2x+5}{\sqrt{x^2 + 2x + 5}} dx = 2\sqrt{x^2 + 2x + 5} + 3\sinh^{-1}\left(\frac{x+1}{2}\right) + c$$

13. Question

Evaluate the following integrals:

$$\int \frac{3x+1}{\sqrt{5-2x-x^2}} dx$$

Answer

$$\text{Given } I = \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px+q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$$

$$\Rightarrow px + q = \lambda(2ax + b) + \mu$$

$$\Rightarrow 3x + 1 = \lambda(-2x - 2) + \mu$$

$$\therefore \lambda = -3/2 \text{ and } \mu = -2$$

$$\text{Let } 3x + 1 = - (3/2)(-2x - 2) - 2$$

$$\Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx = \int \left(\frac{-3(-2x-2)}{2\sqrt{-x^2-2x+5}} - \frac{2}{\sqrt{-x^2-2x+5}} \right) dx$$

$$= 3 \int \frac{x+1}{\sqrt{-x^2-2x+5}} dx - 2 \int \frac{1}{\sqrt{-x^2-2x+5}} dx$$

Consider $\int \frac{x+1}{\sqrt{-x^2-2x+5}} dx$

Let $u = -x^2 - 2x + 5 \rightarrow dx = \frac{1}{-2x-2} du$

$$\Rightarrow \int \frac{x+1}{\sqrt{-x^2-2x+5}} dx = \int -\frac{1}{2\sqrt{u}} du$$

$$= -\frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow -\frac{1}{2} \int \frac{1}{\sqrt{u}} du = -(\sqrt{u})$$

$$= -\sqrt{-x^2-2x+5}$$

Consider $\int \frac{1}{\sqrt{-x^2-2x+5}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{-x^2-2x+5}} dx = \int \frac{1}{\sqrt{6-(x+1)^2}} dx$$

Let $u = \frac{x+1}{\sqrt{6}} \rightarrow dx = \sqrt{6} du$

$$\Rightarrow \int \frac{1}{\sqrt{6-(x+1)^2}} dx = \int \frac{\sqrt{6}}{\sqrt{6-6u^2}} du$$

$$= \int \frac{1}{\sqrt{1-u^2}} du$$

We know that $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)$$

Then,

$$\Rightarrow \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx = 3 \int \frac{x+1}{\sqrt{-x^2-2x+5}} dx - 2 \int \frac{1}{\sqrt{-x^2-2x+5}} dx$$

$$= -3\sqrt{-x^2-2x+5} - 2 \left(\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) \right) + c$$

$$\therefore I = \int \frac{3x+1}{\sqrt{-x^2-2x+5}} dx = -3\sqrt{-x^2-2x+5} - 2 \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + c$$

14. Question

Evaluate the following integrals:

$$\int \sqrt{\frac{1-x}{1+x}} dx$$

Answer

Given $I = \int \frac{\sqrt{1-x}}{\sqrt{1+x}} dx$

Rationalizing the denominator,

$$\Rightarrow \int \sqrt{\frac{1-x}{1+x}} dx = \int \sqrt{\frac{1-x}{1+x} \times \frac{1-x}{1-x}} dx$$

$$= \int \frac{1-x}{\sqrt{1-x^2}} dx$$

Integral is of form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Writing numerator as $px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$

$$\Rightarrow px + q = \lambda(2ax + b) + \mu$$

$$\Rightarrow -x + 1 = \lambda(-2x) + \mu$$

$$\therefore \lambda = 1/2 \text{ and } \mu = 1$$

Let $-x + 1 = 1/2(-2x) + 1$ and split,

$$\Rightarrow \int \frac{1-x}{\sqrt{1-x^2}} dx = \int \left(\frac{-2x}{2\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \right) dx$$

$$= - \int \frac{x}{\sqrt{1-x^2}} dx + \int \frac{1}{\sqrt{1-x^2}} dx$$

Consider $\int \frac{x}{\sqrt{1-x^2}} dx$

Let $u = 1 - x^2 \rightarrow dx = \frac{-1}{2x} du$

$$\Rightarrow \int \frac{x}{\sqrt{1-x^2}} dx = \int \frac{-1}{2\sqrt{u}} du$$

$$= \frac{-1}{2} \int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} (2\sqrt{u})$$

$$= \sqrt{u} = -\sqrt{1-x^2}$$

Consider $\int \frac{1}{\sqrt{1-x^2}} dx$

We know that $\int \frac{1}{\sqrt{1-x^2}} dx + c = \sin^{-1} x + c$

$$\Rightarrow \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x)$$

Then,

$$\Rightarrow \int \frac{1-x}{\sqrt{1-x^2}} dx = - \int \frac{x}{\sqrt{1-x^2}} dx + \int \frac{1}{\sqrt{1-x^2}} dx$$

$$= \sqrt{1-x^2} + \sin^{-1}(x) + c$$

$$\therefore I = \int \sqrt{\frac{1-x}{1+x}} dx = \sqrt{1-x^2} + \sin^{-1}(x) + c$$

15. Question

Evaluate the following integrals:

$$\int \frac{2x+1}{\sqrt{x^2+4x+3}} dx$$

Answer

$$\text{Given } I = \int \frac{2x+1}{\sqrt{x^2+4x+3}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$$

$$\Rightarrow px + q = \lambda(2ax + b) + \mu$$

$$\Rightarrow 2x + 1 = \lambda(2x + 4) + \mu$$

$$\therefore \lambda = 1 \text{ and } \mu = -3$$

Let $2x + 1 = 2x + 4 - 3$ and split,

$$\begin{aligned} \Rightarrow \int \frac{2x+1}{\sqrt{x^2+4x+3}} dx &= \int \left(\frac{2x+4}{\sqrt{x^2+4x+3}} - \frac{3}{\sqrt{x^2+4x+3}} \right) dx \\ &= 2 \int \frac{x+2}{\sqrt{x^2+4x+3}} dx - 3 \int \frac{1}{\sqrt{x^2+4x+3}} dx \end{aligned}$$

$$\text{Consider } \int \frac{x+2}{\sqrt{x^2+4x+3}} dx$$

$$\text{Let } u = x^2 + 4x + 3 \rightarrow dx = \frac{1}{2x+4} du$$

$$\begin{aligned} \Rightarrow \int \frac{x+2}{\sqrt{x^2+4x+3}} dx &= \int \frac{1}{2\sqrt{u}} du \\ &= \frac{1}{2} \int \frac{1}{\sqrt{u}} du \end{aligned}$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} (2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2+4x+3}$$

$$\text{Consider } \int \frac{1}{\sqrt{x^2+4x+3}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+4x+3}} dx = \int \frac{1}{\sqrt{(x+2)^2-1}} dx$$

$$\text{Let } u = x + 2 \rightarrow dx = du$$

$$\Rightarrow \int \frac{1}{\sqrt{(x+2)^2-1}} dx = \int \frac{1}{\sqrt{u^2-1}} du$$

We know that $\int \frac{1}{\sqrt{x^2-1}} dx = \log(\sqrt{x^2-1} + x) + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2-1}} du = \log(\sqrt{u^2-1} + u)$$

$$= \log(\sqrt{(x+2)^2-1} + x+2)$$

Then,

$$\Rightarrow \int \frac{2x+1}{\sqrt{x^2+4x+3}} dx = 2 \int \frac{x+2}{\sqrt{x^2+4x+3}} dx - 3 \int \frac{1}{\sqrt{x^2+4x+3}} dx$$

$$= 2\sqrt{x^2+4x+3} - 3\log(\sqrt{(x+2)^2-1} + x+2) + c$$

$$= 2\sqrt{x^2+4x+3} - 3\log(\sqrt{x^2+4x+3} + x+2) + c$$

$$= 2\sqrt{(x+1)(x+3)} - 3\log(\sqrt{(x+1)(x+3)} + x+2) + c$$

$$\begin{aligned} \therefore I &= \int \frac{2x+1}{\sqrt{x^2+4x+3}} dx \\ &= 2\sqrt{(x+1)(x+3)} - 3\log(\sqrt{(x+1)(x+3)} + x+2) + c \end{aligned}$$

16. Question

Evaluate the following integrals:

$$\int \frac{2x+3}{\sqrt{x^2+4x+5}} dx$$

Answer

$$\text{Given } I = \int \frac{2x+3}{\sqrt{x^2+4x+5}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px+q = \lambda \left\{ \frac{d}{dx} (ax^2+bx+c) \right\} + \mu$$

$$\Rightarrow px+q = \lambda(2ax+b) + \mu$$

$$\Rightarrow 2x+3 = \lambda(2x+4) + \mu$$

$$\therefore \lambda = 1/2 \text{ and } \mu = -1$$

Let $2x+3 = 2x+4-1$ and split,

$$\Rightarrow \int \frac{2x+3}{\sqrt{x^2+4x+5}} dx = \int \left(\frac{2x+4}{\sqrt{x^2+4x+5}} - \frac{1}{\sqrt{x^2+4x+5}} \right) dx$$

$$= 2 \int \frac{x+2}{\sqrt{x^2+4x+5}} dx - \int \frac{1}{\sqrt{x^2+4x+5}} dx$$

$$\text{Consider } \int \frac{x+2}{\sqrt{x^2+4x+5}} dx$$

$$\text{Let } u = x^2+4x+5 \rightarrow dx = \frac{1}{2x+4} du$$

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2+4x+5}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} (2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2 + 4x + 5}$$

Consider $\int \frac{1}{\sqrt{x^2 + 4x + 5}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 4x + 5}} dx = \int \frac{1}{\sqrt{(x+2)^2 + 1}} dx$$

Let $u = x + 2 \rightarrow dx = du$

$$\Rightarrow \int \frac{1}{\sqrt{(x+2)^2 + 1}} dx = \int \frac{1}{\sqrt{u^2 + 1}} du$$

We know that $\int \frac{1}{\sqrt{x^2 + 1}} dx = \sinh^{-1} x + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2 + 1}} du = \sinh^{-1}(u)$$

$$= \sinh^{-1}(x + 2)$$

Then,

$$\Rightarrow \int \frac{2x + 3}{\sqrt{x^2 + 4x + 5}} dx = 2 \int \frac{x + 2}{\sqrt{x^2 + 4x + 5}} dx - \int \frac{1}{\sqrt{x^2 + 4x + 5}} dx$$

$$= 2\sqrt{x^2 + 4x + 5} - \sinh^{-1}(x + 2) + c$$

$$\therefore I = \int \frac{2x + 3}{\sqrt{x^2 + 4x + 5}} dx = 2\sqrt{x^2 + 4x + 5} - \sinh^{-1}(x + 2) + c$$

17. Question

Evaluate the following integrals:

$$\int \frac{5x + 3}{\sqrt{x^2 + 4x + 10}} dx$$

Answer

$$\text{Given } I = \int \frac{5x + 3}{\sqrt{x^2 + 4x + 10}} dx$$

Integral is of form $\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$

Writing numerator as $px + q = \lambda \left\{ \frac{d}{dx} (ax^2 + bx + c) \right\} + \mu$

$$\Rightarrow px + q = \lambda(2ax + b) + \mu$$

$$\Rightarrow 5x + 3 = \lambda(2x + 4) + \mu$$

$$\therefore \lambda = 5/2 \text{ and } \mu = -7$$

Let $5x + 3 = \frac{5}{2}(2x + 4) - 7$ and split,

$$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = \int \left(\frac{5(2x+4)}{2\sqrt{x^2+4x+10}} - \frac{7}{\sqrt{x^2+4x+10}} \right) dx$$

$$= 5 \int \frac{x+2}{\sqrt{x^2+4x+10}} dx - 7 \int \frac{1}{\sqrt{x^2+4x+10}} dx$$

Consider $\int \frac{x+2}{\sqrt{x^2+4x+10}} dx$

Let $u = x^2 + 4x + 10 \rightarrow dx = \frac{1}{2x+4} du$

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2+4x+10}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} (2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2+4x+10}$$

Consider $\int \frac{1}{\sqrt{x^2+4x+10}} dx$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+4x+10}} dx = \int \frac{1}{\sqrt{(x+2)^2+6}} dx$$

Let $u = \frac{x+2}{\sqrt{6}} \rightarrow dx = \sqrt{6} du$

$$\Rightarrow \int \frac{1}{\sqrt{(x+2)^2+6}} dx = \int \frac{\sqrt{6}}{\sqrt{6u^2+6}} du$$

$$= \int \frac{1}{\sqrt{u^2+1}} du$$

We know that $\int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + c$

$$\Rightarrow \int \frac{1}{\sqrt{u^2+1}} du = \sinh^{-1}(u)$$

$$= \sinh^{-1}\left(\frac{x+2}{\sqrt{6}}\right)$$

Then,

$$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = 5 \int \frac{x+2}{\sqrt{x^2+4x+10}} dx - 7 \int \frac{1}{\sqrt{x^2+4x+10}} dx$$

$$= 5\sqrt{x^2+4x+10} - 7\sinh^{-1}\left(\frac{x+2}{\sqrt{6}}\right) + c$$

$$\therefore I = \int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = 5\sqrt{x^2+4x+10} - 7\sinh^{-1}\left(\frac{x+2}{\sqrt{6}}\right) + c$$

18. Question

Evaluate the following integrals:

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}$$

Answer

$$\text{Given } I = \int \frac{x+2}{\sqrt{x^2+2x+3}} dx$$

$$\text{Integral is of form } \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$$

$$\text{Writing numerator as } px+q = \lambda \left\{ \frac{d}{dx} (ax^2+bx+c) \right\} + \mu$$

$$\Rightarrow px+q = \lambda(2ax+b) + \mu$$

$$\Rightarrow x+2 = \lambda(2x+2) + \mu$$

$$\therefore \lambda = 1/2 \text{ and } \mu = 1$$

Let $x+2 = 1/2(2x+2) + 1$ and split,

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \int \left(\frac{2x+2}{2\sqrt{x^2+2x+3}} + \frac{1}{\sqrt{x^2+2x+3}} \right) dx$$

$$= \int \frac{x+1}{\sqrt{x^2+2x+3}} dx + \int \frac{1}{\sqrt{x^2+2x+3}} dx$$

$$\text{Consider } \int \frac{x+1}{\sqrt{x^2+2x+3}} dx$$

$$\text{Let } u = x^2+2x+3 \rightarrow dx = \frac{1}{2x+2} du$$

$$\Rightarrow \int \frac{x+1}{\sqrt{x^2+2x+3}} dx = \int \frac{1}{2\sqrt{u}} du$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

$$\text{We know that } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} (2\sqrt{u})$$

$$= \sqrt{u} = \sqrt{x^2+2x+3}$$

$$\text{Consider } \int \frac{1}{\sqrt{x^2+2x+3}} dx$$

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x+3}} dx = \int \frac{1}{\sqrt{(x+1)^2+2}} dx$$

$$\text{Let } u = \frac{x+1}{\sqrt{2}} \rightarrow dx = \sqrt{2} du$$

$$\Rightarrow \int \frac{1}{\sqrt{(x+1)^2+2}} dx = \int \frac{\sqrt{2}}{\sqrt{2u^2+2}} du$$

$$= \int \frac{1}{\sqrt{u^2+1}} du$$

$$\text{We know that } \int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + c$$

$$\Rightarrow \int \frac{1}{\sqrt{u^2 + 1}} du = \sinh^{-1}(u)$$

$$= \sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

Then,

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2 + 2x + 3}} dx = \int \frac{x+1}{\sqrt{x^2 + 2x + 3}} dx + \int \frac{1}{\sqrt{x^2 + 2x + 3}} dx$$

$$= \sqrt{x^2 + 2x + 3} + \sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$$\therefore I = \int \frac{x+2}{\sqrt{x^2 + 2x + 3}} dx = \sqrt{x^2 + 2x + 3} + \sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

Exercise 19.22

1. Question

Evaluate the following integrals:

$$\int \frac{1}{4\cos^2 x + 9\sin^2 x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{4\cos^2 x + 9\sin^2 x} dx$$

Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$\Rightarrow I = \int \frac{1}{4\cos^2 x + 9\sin^2 x} dx = \int \frac{\sec^2 x}{4 + 9\tan^2 x} dx$$

Putting $\tan x = t$ and $\sec^2 x dx = dt$, we get

$$\Rightarrow I = \int \frac{dt}{4 + 9t^2} = \frac{1}{9} \int \frac{dt}{\frac{4}{9} + t^2}$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \frac{1}{9} \int \frac{dt}{\frac{4}{9} + t^2} = \frac{1}{9} \times \frac{1}{\frac{2}{3}} \tan^{-1}\left(\frac{t}{\frac{2}{3}}\right) + c$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{3t}{2}\right) + c$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{3\tan x}{2}\right) + c$$

$$\therefore I = \int \frac{1}{4\cos^2 x + 9\sin^2 x} dx = \frac{1}{6} \tan^{-1}\left(\frac{3\tan x}{2}\right) + c$$

2. Question

Evaluate the following integrals:

$$\int \frac{1}{4\sin^2 x + 5\cos^2 x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{4 \sin^2 x + 5 \cos^2 x} dx$$

Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$\Rightarrow I = \int \frac{1}{4 \sin^2 x + 5 \cos^2 x} dx = \int \frac{\sec^2 x}{4 \tan^2 x + 5} dx$$

Putting $\tan x = t$ and $\sec^2 x dx = dt$, we get

$$\Rightarrow I = \int \frac{dt}{4t^2 + 5} = \frac{1}{4} \int \frac{dt}{t^2 + \left(\frac{\sqrt{5}}{2}\right)^2}$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$\Rightarrow \frac{1}{4} \int \frac{dt}{t^2 + \left(\frac{\sqrt{5}}{2}\right)^2} = \frac{1}{4} \times \frac{1}{\frac{\sqrt{5}}{2}} \tan^{-1} \left(\frac{t}{\frac{\sqrt{5}}{2}} \right) + c$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{2t}{\sqrt{5}} \right) + c$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{2 \tan x}{\sqrt{5}} \right) + c$$

$$\therefore I = \int \frac{1}{4 \sin^2 x + 5 \cos^2 x} dx = \frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{2 \tan x}{\sqrt{5}} \right) + c$$

3. Question

Evaluate the following integrals:

$$\int \frac{2}{2 + \sin 2x} dx.$$

Answer

$$\text{Given } I = \int \frac{2}{2 + \sin 2x} dx$$

We know that $\sin 2x = 2 \sin x \cos x$

$$\Rightarrow \int \frac{2}{2 + \sin 2x} dx = \int \frac{2}{2 + 2 \sin x \cos x} dx$$

$$= \int \frac{1}{1 + \sin x \cos x} dx$$

Dividing the numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{1 + \sin x \cos x} dx = \int \frac{\sec^2 x}{\sec^2 x + \tan x} dx$$

Replacing $\sec^2 x$ in denominator by $1 + \tan^2 x$,

$$\Rightarrow \int \frac{\sec^2 x}{\sec^2 x + \tan x} dx = \int \frac{\sec^2 x}{1 + \tan^2 x + \tan x} dx$$

Putting $\tan x = t$ so that $\sec^2 x dx = dt$,

$$\Rightarrow \int \frac{\sec^2 x}{\tan^2 x + \tan x + 1} dx = \int \frac{dt}{t^2 + t + 1}$$

$$= \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$\Rightarrow \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \left(\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2t + 1}{\sqrt{3}} \right) + c$$

$$\therefore I = \int \frac{2}{2 + \sin 2x} dx = \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2t + 1}{\sqrt{3}} \right) + c$$

4. Question

Evaluate the following integrals:

$$\int \frac{\cos x}{\cos 3x} dx$$

Answer

$$\text{Given } I = \int \frac{\cos x}{\cos 3x} dx$$

$$\Rightarrow \int \frac{\cos x}{\cos 3x} dx = \int \frac{\cos x}{4 \cos^3 x - 3 \cos x} dx$$

$$= \int \frac{1}{4 \cos^2 x - 3} dx$$

Dividing numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{4 \cos^2 x - 3} dx = \int \frac{\sec^2 x}{4 - 3 \sec^2 x} dx$$

Replacing $\sec^2 x$ by $1 + \tan^2 x$ in denominator,

$$\Rightarrow \int \frac{\sec^2 x}{4 - 3 \sec^2 x} dx = \int \frac{\sec^2 x}{4 - 3 - 3 \tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{1 - 3 \tan^2 x} dx$$

Putting $\tan x = t$ and $\sec^2 x dx = dt$, we get

$$I = \int \frac{dt}{1 - 3t^2} = \frac{1}{3} \int \frac{1}{\frac{1}{3} - t^2} dt$$

We know that $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

$$\Rightarrow \frac{1}{3} \int \frac{1}{\frac{1}{3} - t^2} dt = \frac{1}{3} \times \frac{1}{2\sqrt{3}} \log \left| \frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t} \right| + c$$

$$= \frac{1}{6\sqrt{3}} \log \left| \frac{1 + \sqrt{3}t}{1 - \sqrt{3}t} \right| + c$$

$$= \frac{1}{6\sqrt{3}} \log \left| \frac{1 + \sqrt{3} \tan x}{1 - \sqrt{3} \tan x} \right| + c$$

$$\therefore I = \int \frac{\cos x}{\cos 3x} dx = \frac{1}{6\sqrt{3}} \log \left| \frac{1 + \sqrt{3} \tan x}{1 - \sqrt{3} \tan x} \right| + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{1}{1 + 3 \sin^2 x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{1 + 3 \sin^2 x} dx$$

Divide numerator and denominator by $\cos^2 x$,

$$\Rightarrow I = \int \frac{1}{1 + 3 \sin^2 x} dx = \int \frac{\sec^2 x}{\sec^2 x + 3 \tan^2 x} dx$$

Replacing $\sec^2 x$ in denominator by $1 + \tan^2 x$,

$$\begin{aligned} \Rightarrow \int \frac{\sec^2 x}{\sec^2 x + 3 \tan^2 x} dx &= \int \frac{\sec^2 x}{1 + \tan^2 x + 3 \tan^2 x} dx \\ &= \int \frac{\sec^2 x}{1 + 4 \tan^2 x} dx \end{aligned}$$

Putting $\tan x = t$ so that $\sec^2 x dx = dt$,

$$\begin{aligned} \Rightarrow \int \frac{\sec^2 x}{1 + 4 \tan^2 x} dx &= \int \frac{dt}{1 + 4t^2} \\ &= \frac{1}{4} \int \frac{1}{\frac{1}{4} + t^2} dt \end{aligned}$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$\begin{aligned} \Rightarrow \frac{1}{4} \int \frac{1}{\frac{1}{4} + t^2} dt &= \frac{1}{4} \times \frac{1}{2} \tan^{-1} \left(\frac{t}{\frac{1}{2}} \right) + c \\ &= \frac{1}{8} \tan^{-1} \left(\frac{\tan x}{\frac{1}{2}} \right) + c \end{aligned}$$

$$\therefore I = \int \frac{1}{1 + 3 \sin^2 x} dx = \frac{1}{8} \tan^{-1} \left(\frac{\tan x}{\frac{1}{2}} \right) + c$$

6. Question

Evaluate the following integrals:

$$\int \frac{1}{3 + 2 \cos^2 x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{3 + 2 \cos^2 x} dx$$

Divide numerator and denominator by $\cos^2 x$,

$$\Rightarrow I = \int \frac{1}{3 + 2 \cos^2 x} dx = \int \frac{\sec^2 x}{3 \sec^2 x + 2} dx$$

Replacing $\sec^2 x$ in denominator by $1 + \tan^2 x$,

$$\begin{aligned} \Rightarrow \int \frac{\sec^2 x}{3 \sec^2 x + 2} dx &= \int \frac{\sec^2 x}{3 + 3 \tan^2 x + 2} dx \\ &= \int \frac{\sec^2 x}{5 + 3 \tan^2 x} dx \end{aligned}$$

Putting $\tan x = t$ so that $\sec^2 x dx = dt$,

$$\begin{aligned} \Rightarrow \int \frac{\sec^2 x}{5 + 3 \tan^2 x} dx &= \int \frac{dt}{5 + 3t^2} \\ &= \frac{1}{3} \int \frac{1}{\frac{5}{3} + t^2} dt \end{aligned}$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$\begin{aligned} \Rightarrow \frac{1}{3} \int \frac{1}{\frac{5}{3} + t^2} dt &= \frac{1}{3} \times \sqrt{\frac{5}{3}} \tan^{-1} \left(\frac{t}{\sqrt{\frac{5}{3}}} \right) + c \\ &= \frac{\sqrt{5}}{3\sqrt{3}} \tan^{-1} \left(\frac{\sqrt{3} \tan x}{\sqrt{5}} \right) + c \\ \therefore I &= \int \frac{1}{3 + 2 \cos^2 x} dx = \frac{\sqrt{5}}{3\sqrt{3}} \tan^{-1} \left(\frac{\sqrt{3} \tan x}{\sqrt{5}} \right) + c \end{aligned}$$

7. Question

Evaluate the following integrals:

$$\int \frac{1}{(\sin x - 2 \cos x)(2 \sin x + \cos x)} dx$$

Answer

$$\begin{aligned} \text{Given } I &= \int \frac{1}{(\sin x - 2 \cos x)(2 \sin x + \cos x)} dx \\ \Rightarrow \int \frac{1}{(\sin x - 2 \cos x)(2 \sin x + \cos x)} dx \\ &= \int \frac{1}{2 \sin^2 x + \sin x \cos x - 4 \sin x \cos x - 2 \cos^2 x} dx \end{aligned}$$

Dividing the numerator and denominator by $\cos^2 x$,

$$\begin{aligned} \Rightarrow \int \frac{1}{2 \sin^2 x + \sin x \cos x - 4 \sin x \cos x - 2 \cos^2 x} dx \\ = \int \frac{\sec^2 x}{2 \tan^2 x - 3 \tan x - 2} dx \end{aligned}$$

Putting $\tan x = t$ so that $\sec^2 x dx = dt$.

$$\Rightarrow \int \frac{\sec^2 x}{2 \tan^2 x - 3 \tan x - 2} dx = \int \frac{dt}{2t^2 - 3t - 2}$$

$$= \frac{1}{2} \int \frac{1}{t^2 - \frac{3}{2} - 1} dt$$

$$= \frac{1}{2} \int \frac{1}{\left(t - \frac{3}{4}\right)^2 - \left(\frac{5}{4}\right)^2} dt$$

We know that $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{\left(t - \frac{3}{4}\right)^2 - \left(\frac{5}{4}\right)^2} dt = \frac{1}{2} \times \frac{1}{2\left(\frac{5}{4}\right)} \log \left| \frac{t - \frac{3}{4} - \frac{5}{4}}{t - \frac{3}{4} + \frac{5}{4}} \right| + c$$

$$= \frac{1}{5} \log \left| \frac{t-2}{t+\frac{1}{2}} \right| + c$$

$$= \frac{1}{5} \log \left| \frac{2 \tan x - 4}{2 \tan x + 1} \right| + c$$

$$\therefore I = \int \frac{1}{(\sin x - 2 \cos x)(2 \sin x + \cos x)} dx = \frac{1}{5} \log \left| \frac{2 \tan x - 4}{2 \tan x + 1} \right| + c$$

8. Question

Evaluate the following integrals:

$$\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$$

Answer

$$\text{Given } I = \int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$$

Dividing the numerator and denominator by $\cos^4 x$,

$$\Rightarrow \int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx = \int \frac{2 \tan x \sec^2 x}{\tan^4 x + 1} dx$$

Putting $\tan^2 x = t$ so that $2 \tan x \sec^2 x dx = dt$

$$\Rightarrow \int \frac{2 \tan x \sec^2 x}{\tan^4 x + 1} dx = \int \frac{dt}{t^2 + 1}$$

We know that $\int \frac{1}{1+x^2} dx = \tan^{-1}(x) + c$

$$\Rightarrow \int \frac{dt}{t^2 + 1} = \tan^{-1}(t) + c$$

$$= \tan^{-1}(\tan^2 x) + c$$

$$\therefore I = \int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx = \tan^{-1}(\tan^2 x) + c$$

9. Question

Evaluate the following integrals:

$$\int \frac{1}{\cos x (\sin x + 2 \cos x)} dx.$$

Answer

$$\text{Given } I = \int \frac{1}{\cos x (\sin x + 2 \cos x)} dx$$

$$\Rightarrow I = \int \frac{1}{\cos x (\sin x + 2 \cos x)} dx = \int \frac{1}{\cos x \sin x + 2 \cos^2 x} dx$$

Dividing the numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{\cos x \sin x + 2 \cos^2 x} dx = \int \frac{\sec^2 x}{\tan x + 2} dx$$

Putting $\tan x + 2 = t$ so that $\sec^2 x dx = dt$,

$$\Rightarrow \int \frac{\sec^2 x}{\tan x + 2} dx = \int \frac{dt}{t}$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{t} dt = \log|t| + c$$

$$= \log|\tan x + 2| + c$$

$$\therefore I = \int \frac{1}{\cos x (\sin x + 2 \cos x)} dx = \log|\tan x + 2| + c$$

10. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin^2 x + \sin 2x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{\sin^2 x + \sin 2x} dx$$

We know that $\sin 2x = 2 \sin x \cos x$

$$\Rightarrow I = \int \frac{1}{\sin^2 x + 2 \sin x \cos x} dx$$

Dividing numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{\sin^2 x + 2 \sin x \cos x} dx = \int \frac{\sec^2 x}{\tan^2 x + 2 \tan x} dx$$

Putting $\tan x = t$ so that $\sec^2 x dx = dt$,

$$\Rightarrow \int \frac{\sec^2 x}{\tan^2 x + 2 \tan x} dx = \int \frac{dt}{t^2 + 2t}$$

$$= \int \frac{1}{t^2 + 2t + 1^2 - 1^2} dt$$

$$= \int \frac{1}{(t+1)^2 - 1^2} dt$$

We know that $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

$$\Rightarrow \int \frac{1}{(t+1)^2 - 1^2} dt = \frac{1}{2} \log \left| \frac{t+1-1}{t+1+1} \right| + c$$

$$= \frac{1}{2} \log \left| \frac{t}{t+2} \right| + c$$

$$= \frac{1}{2} \log \left| \frac{\tan x}{\tan x + 2} \right| + c$$

$$\therefore I = \int \frac{1}{\sin^2 x + \sin 2x} dx = \frac{1}{2} \log \left| \frac{\tan x}{\tan x + 2} \right| + c$$

11. Question

Evaluate the following integrals:

$$\int \frac{1}{\cos 2x + 3 \sin^2 x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{\cos 2x + 3 \sin^2 x} dx$$

We know that $\cos 2x = 1 - 2 \sin^2 x$.

$$\Rightarrow \int \frac{1}{\cos 2x + 3 \sin^2 x} dx = \int \frac{1}{1 - 2 \sin^2 x + 3 \sin^2 x} dx$$

$$= \int \frac{1}{1 + \sin^2 x} dx$$

Dividing numerator and denominator by $\cos^2 x$,

$$\Rightarrow \int \frac{1}{1 + \sin^2 x} dx = \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x} dx$$

Replacing $\sec^2 x$ in denominator by $1 + \tan^2 x$,

$$\Rightarrow \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x} dx = \int \frac{\sec^2 x}{1 + 2 \tan^2 x} dx$$

Putting $\tan x = t$ so that $\sec^2 x dx = dt$,

$$\Rightarrow \int \frac{\sec^2 x}{1 + 2 \tan^2 x} dx = \int \frac{dt}{1 + 2t^2}$$

$$= \frac{1}{2} \int \frac{1}{\frac{1}{2} + t^2} dt$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$

$$\Rightarrow \frac{1}{2} \int \frac{1}{\frac{1}{2} + t^2} dt = \frac{1}{2} \times \frac{1}{\frac{1}{\sqrt{2}}} \tan^{-1} \left(\frac{t}{\frac{1}{\sqrt{2}}} \right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2} \tan x) + c$$

$$\therefore I = \int \frac{1}{\cos 2x + 3 \sin^2 x} dx = \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2} \tan x) + c$$

Exercise 19.23

1. Question

Evaluate the following integrals:

$$\int \frac{1}{5+4\cos x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{5+4\cos x} dx$$

$$\text{We know that } \cos x = \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$$

$$\Rightarrow \int \frac{1}{5+4\cos x} dx = \int \frac{1}{5+4\left(\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}\right)} dx$$

$$= \int \frac{1+\tan^2 \frac{x}{2}}{5\left(1+\tan^2 \frac{x}{2}\right)+4(1-\tan^2 \frac{x}{2})} dx$$

Replacing $1+\tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1+\tan^2 \frac{x}{2}}{5\left(1+\tan^2 \frac{x}{2}\right)+4(1-\tan^2 \frac{x}{2})} dx = \int \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2}+9} dx$$

Putting $\tan x/2 = t$ and $\sec^2(x/2)dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2}+9} dx = \int \frac{2dt}{t^2+9}$$

$$= 2 \int \frac{1}{t^2+9} dt$$

$$\text{We know that } \int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$\Rightarrow 2 \int \frac{1}{t^2+9} dt = 2 \left(\frac{1}{3} \right) \tan^{-1} \left(\frac{t}{3} \right) + c$$

$$= \frac{2}{3} \tan^{-1} \left(\frac{\tan x}{3} \right) + c$$

$$\therefore I = \int \frac{1}{5+4\cos x} dx = \frac{2}{3} \tan^{-1} \left(\frac{\tan x}{3} \right) + c$$

2. Question

Evaluate the following integrals:

$$\int \frac{1}{5-4\sin x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{5-4\sin x} dx$$

$$\text{We know that } \sin x = \frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}}$$

$$\Rightarrow \int \frac{1}{5-4\sin x} dx = \int \frac{1}{5-4\left(\frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}}\right)} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{5 \left(1 + \tan^2 \frac{x}{2}\right) - 4 \left(2 \tan \frac{x}{2}\right)} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{5 \left(1 + \tan^2 \frac{x}{2}\right) - 4 \left(2 \tan \frac{x}{2}\right)} dx = \int \frac{\sec^2 \frac{x}{2}}{5 + 5 \tan^2 \frac{x}{2} - 8 \tan \frac{x}{2}} dx$$

Putting $\tan x/2 = t$ and $\sec^2(x/2)dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{5 + 5 \tan^2 \frac{x}{2} - 8 \tan \frac{x}{2}} dx = \int \frac{2dt}{5 + 5t^2 - 8t}$$

$$= \frac{2}{5} \int \frac{1}{t^2 - \frac{8}{5}t + 1} dt$$

$$= \frac{2}{5} \int \frac{1}{\left(t - \frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} dt$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a}\right) + c$

$$\Rightarrow \frac{2}{5} \int \frac{1}{\left(t - \frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} dt = \frac{2}{5} \left(\frac{1}{\frac{3}{5}}\right) \tan^{-1} \left(\frac{t - \frac{4}{5}}{\frac{3}{5}}\right) + c$$

$$= \frac{2}{3} \tan^{-1} \left(\frac{5 \tan x - 4}{3}\right) + c$$

$$\therefore I = \int \frac{1}{5 - 4 \sin x} dx = \frac{2}{3} \tan^{-1} \left(\frac{5 \tan x - 4}{3}\right) + c$$

3. Question

Evaluate the following integrals:

$$\int \frac{1}{1 - 2 \sin x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{1 - 2 \sin x} dx$$

We know that $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\Rightarrow \int \frac{1}{1 - 2 \sin x} dx = \int \frac{1}{1 - 2 \left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right)} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{1 \left(1 + \tan^2 \frac{x}{2}\right) - 2 \left(2 \tan \frac{x}{2}\right)} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{1 \left(1 + \tan^2 \frac{x}{2}\right) - 2 \left(2 \tan \frac{x}{2}\right)} dx = \int \frac{\sec^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 4 \tan \frac{x}{2}} dx$$

Putting $\tan x/2 = t$ and $\sec^2(x/2)dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 4 \tan \frac{x}{2}} dx = \int \frac{2dt}{1 + t^2 - 4t}$$

$$= 2 \int \frac{1}{t^2 - 4t + 1} dt$$

$$= 2 \int \frac{1}{(t-2)^2 - (\sqrt{3})^2} dt$$

We know that $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

$$\Rightarrow 2 \int \frac{1}{(t-2)^2 - (\sqrt{3})^2} dt = 2 \left(\frac{1}{2\sqrt{3}} \right) \tan^{-1} \left(\frac{t-2-\sqrt{3}}{t+2+\sqrt{3}} \right) + c$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{\tan x - (2 + \sqrt{3})}{\tan x + (2 + \sqrt{3})} \right) + c$$

$$\therefore I = \int \frac{1}{1 - 2 \sin x} dx = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{\tan x - (2 + \sqrt{3})}{\tan x + (2 + \sqrt{3})} \right) + c$$

4. Question

Evaluate the following integrals:

$$\int \frac{1}{4 \cos x - 1} dx$$

Answer

$$\text{Given } I = \int \frac{1}{4 \cos x - 1} dx$$

We know that $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\Rightarrow \int \frac{1}{-1 + 4 \cos x} dx = \int \frac{1}{-1 + 4 \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{-1 \left(1 + \tan^2 \frac{x}{2} \right) + 4(1 - \tan^2 \frac{x}{2})} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{-1 \left(1 + \tan^2 \frac{x}{2} \right) + 4(1 - \tan^2 \frac{x}{2})} dx = \int \frac{\sec^2 \frac{x}{2}}{-5 \tan^2 \frac{x}{2} + 3} dx$$

Putting $\tan \frac{x}{2} = t$ and $\frac{1}{2} \sec^2 \left(\frac{x}{2} \right) dx = dt$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{-5 \tan^2 \frac{x}{2} + 3} dx = \int \frac{dt}{3 - 5t^2}$$

$$= \frac{1}{5} \int \frac{1}{\frac{3}{5} - t^2} dt$$

We know that $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

$$\Rightarrow \frac{1}{5} \int \frac{1}{\frac{3}{5} - t^2} dt = \frac{1}{5} \left(\frac{1}{\sqrt{\frac{3}{5}}} \right) \log \left| \frac{\sqrt{\frac{3}{5}} + t}{\sqrt{\frac{3}{5}} - t} \right| + c$$

$$= \frac{1}{\sqrt{15}} \log \left| \frac{\sqrt{3} + \sqrt{5} \tan \frac{x}{2}}{\sqrt{3} - \sqrt{5} \tan \frac{x}{2}} \right| + c$$

$$\therefore I = \int \frac{1}{4 \cos x - 1} dx = \frac{1}{\sqrt{15}} \log \left| \frac{\sqrt{3} + \sqrt{5} \tan \frac{x}{2}}{\sqrt{3} - \sqrt{5} \tan \frac{x}{2}} \right| + c$$

5. Question

Evaluate the following integrals:

$$\int \frac{1}{1 - \sin x + \cos x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{1 - \sin x + \cos x} dx$$

We know that $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ and $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\begin{aligned} \Rightarrow \int \frac{1}{1 - \sin x + \cos x} dx &= \int \frac{1}{1 - \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}} dx \\ &= \int \frac{1 + \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx \end{aligned}$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2 dx = 2dt$,

$$\begin{aligned} \Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx &= \int \frac{\sec^2 \frac{x}{2}}{2 - 2 \tan \frac{x}{2}} dx \\ &= \int \frac{2dt}{2 - 2t} \\ &= \int \frac{1}{1 - t} dt \end{aligned}$$

We know that $\int \frac{1}{x} dx = \log|x| + c$

$$\Rightarrow \int \frac{1}{1 - t} dt = -\log|1 - t| + c$$

$$= -\log \left| 1 - \tan \frac{x}{2} \right| + c$$

$$\therefore I = \int \frac{1}{1 - \sin x + \cos x} dx = -\log \left| 1 - \tan \frac{x}{2} \right| + c$$

6. Question

Evaluate the following integrals:

$$\int \frac{1}{3 + 2 \sin x + \cos x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{3 + 2 \sin x + \cos x} dx$$

$$\text{We know that } \sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \text{ and } \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\Rightarrow \int \frac{1}{3 + 2 \sin x + \cos x} dx = \int \frac{1}{3 + 2 \left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{3 + 3 \tan^2 \frac{x}{2} + 4 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2 dx = 2dt$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{3 + 3 \tan^2 \frac{x}{2} + 4 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx = \int \frac{\sec^2 \frac{x}{2}}{2 \tan^2 \frac{x}{2} + 4 \tan \frac{x}{2} + 4} dx$$

$$= \int \frac{2dt}{2t^2 + 4t + 4}$$

$$= \int \frac{1}{t^2 + 2t + 2} dt$$

$$= \int \frac{1}{(t+1)^2 + 1^2} dt$$

$$\text{We know that } \int \frac{1}{1+x^2} dx = \tan^{-1} x + c$$

$$\Rightarrow \int \frac{1}{(t+1)^2 + 1^2} dt = \tan^{-1}(t+1) + c$$

$$= \tan^{-1}\left(\tan \frac{x}{2} + 1\right) + c$$

$$\therefore I = \int \frac{1}{3 + 2 \sin x + \cos x} dx = \tan^{-1}\left(\tan \frac{x}{2} + 1\right) + c$$

7. Question

Evaluate the following integrals:

$$\int \frac{1}{13 + 3 \cos x + 4 \sin x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{13 + 3 \cos x + 4 \sin x} dx$$

$$\text{We know that } \sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \text{ and } \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\Rightarrow \int \frac{1}{13 + 4 \sin x + 3 \cos x} dx = \int \frac{1}{13 + 4 \left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + 3 \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{13 + 13 \tan^2 \frac{x}{2} + 8 \tan \frac{x}{2} + 3 - 3 \tan^2 \frac{x}{2}} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2 dx = 2dt$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{13 + 13 \tan^2 \frac{x}{2} + 8 \tan \frac{x}{2} + 3 - 3 \tan^2 \frac{x}{2}} dx = \int \frac{\sec^2 \frac{x}{2}}{10 \tan^2 \frac{x}{2} + 8 \tan \frac{x}{2} + 16} dx$$

$$= \int \frac{2dt}{10t^2 + 8t + 16}$$

$$= \frac{2}{10} \int \frac{1}{t^2 + \frac{4}{5}t + \frac{8}{5}} dt$$

$$= \frac{1}{5} \int \frac{1}{\left(t + \frac{2}{5}\right)^2 + \frac{6}{5}} dt$$

We know that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow \frac{1}{5} \int \frac{1}{\left(t + \frac{2}{5}\right)^2 + \frac{6}{5}} dt = \frac{1}{5} \left(\frac{1}{\frac{6}{5}}\right) \tan^{-1} \frac{t + \frac{2}{5}}{\frac{6}{5}} + c$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{5 \tan \frac{x}{2} + 2}{6} \right) + c$$

$$\therefore I = \int \frac{1}{13 + 3 \cos x + 4 \sin x} dx = \frac{1}{6} \tan^{-1} \left(\frac{5 \tan \frac{x}{2} + 2}{6} \right) + c$$

8. Question

Evaluate the following integrals:

$$\int \frac{1}{\cos x - \sin x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{\cos x - \sin x} dx$$

We know that $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ and $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\Rightarrow \int \frac{1}{-\sin x + \cos x} dx = \int \frac{1}{-\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{-2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2 dx = 2dt$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{-2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx = \int \frac{\sec^2 \frac{x}{2}}{-\tan^2 \frac{x}{2} - 2 \tan \frac{x}{2} + 1} dx$$

$$= - \int \frac{2dt}{t^2 + 2t - 1}$$

$$= -2 \int \frac{1}{(t+1)^2 - (\sqrt{2})^2} dt$$

$$= 2 \int \frac{1}{(\sqrt{2})^2 - (t+1)^2} dt$$

$$\text{We know that } \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

$$\Rightarrow 2 \int \frac{1}{(\sqrt{2})^2 - (t+1)^2} dt = \frac{2}{2\sqrt{2}} \log \left| \frac{\sqrt{2} + t + 1}{\sqrt{2} - t - 1} \right| + c$$

$$= \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \tan \frac{x}{2} + 1}{\sqrt{2} - \tan \frac{x}{2} - 1} \right| + c$$

$$\therefore I = \int \frac{1}{\cos x - \sin x} dx = \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \tan \frac{x}{2} + 1}{\sqrt{2} - \tan \frac{x}{2} - 1} \right| + c$$

9. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin x + \cos x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{\sin x + \cos x} dx$$

$$\text{We know that } \sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \text{ and } \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\Rightarrow \int \frac{1}{\sin x + \cos x} dx = \int \frac{1}{\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2 dx = 2dt$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx = \int \frac{\sec^2 \frac{x}{2}}{-\tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} + 1} dx$$

$$= - \int \frac{2dt}{t^2 - 2t - 1}$$

$$= -2 \int \frac{1}{(t-1)^2 - (\sqrt{2})^2} dt$$

$$= 2 \int \frac{1}{(\sqrt{2})^2 - (t-1)^2} dt$$

$$\text{We know that } \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

$$\Rightarrow 2 \int \frac{1}{(\sqrt{2})^2 - (t-1)^2} dt = \frac{2}{2\sqrt{2}} \log \left| \frac{\sqrt{2} + t - 1}{\sqrt{2} - t + 1} \right| + c$$

$$= \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \tan \frac{x}{2} - 1}{\sqrt{2} - \tan \frac{x}{2} + 1} \right| + c$$

$$\therefore I = \int \frac{1}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \tan \frac{x}{2} - 1}{\sqrt{2} - \tan \frac{x}{2} + 1} \right| + c$$

10. Question

Evaluate the following integrals:

$$\int \frac{1}{5 - 4 \cos x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{5 - 4 \cos x} dx$$

$$\text{We know that } \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\Rightarrow \int \frac{1}{5 - 4 \cos x} dx = \int \frac{1}{5 - 4 \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{5 \left(1 + \tan^2 \frac{x}{2} \right) - 4 \left(1 - \tan^2 \frac{x}{2} \right)} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{5 \left(1 + \tan^2 \frac{x}{2} \right) - 4 \left(1 - \tan^2 \frac{x}{2} \right)} dx = \int \frac{\sec^2 \frac{x}{2}}{9 \tan^2 \frac{x}{2} + 1} dx$$

Putting $\tan x/2 = t$ and $\sec^2(x/2)dx = 2dt$,

$$\Rightarrow \int \frac{\sec^2 \frac{x}{2}}{9 \tan^2 \frac{x}{2} + 1} dx = \int \frac{2dt}{9t^2 + 1}$$

$$= \frac{2}{9} \int \frac{1}{t^2 + \frac{1}{9}} dt$$

$$\text{We know that } \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$\Rightarrow \frac{2}{9} \int \frac{1}{t^2 + \frac{1}{9}} dt = \frac{2}{9} \left(\frac{1}{\frac{1}{3}} \right) \tan^{-1} \left(\frac{t}{\frac{1}{3}} \right) + c$$

$$= \frac{2}{3} \tan^{-1} (3 \tan x) + c$$

$$\therefore I = \int \frac{1}{5 - 4 \cos x} dx = \frac{2}{3} \tan^{-1} (3 \tan x) + c$$

11. Question

Evaluate the following integrals:

$$\int \frac{1}{2 + \sin x + \cos x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{2 + \sin x + \cos x} dx$$

$$\text{We know that } \sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \text{ and } \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\Rightarrow \int \frac{1}{2 + \sin x + \cos x} dx = \int \frac{1}{2 + \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{2 + 2 \tan^2 \frac{x}{2} - 2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2 dx = 2dt$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{2 + 2 \tan^2 \frac{x}{2} - 2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx = \int \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2} - 2 \tan \frac{x}{2} + 3} dx$$

$$= \int \frac{2dt}{t^2 - 2t + 3}$$

$$= 2 \int \frac{1}{(t+1)^2 + (\sqrt{2})^2} dt$$

$$\text{We know that } \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$\Rightarrow 2 \int \frac{1}{(t+1)^2 + (\sqrt{2})^2} dt = 2 \left(\frac{1}{\sqrt{2}} \right) \tan^{-1} \left(\frac{t+1}{\sqrt{2}} \right)$$

$$= \sqrt{2} \tan^{-1} \left(\frac{\tan \frac{x}{2} + 1}{\sqrt{2}} \right)$$

$$\therefore I = \int \frac{1}{2 + \sin x + \cos x} dx = \sqrt{2} \tan^{-1} \left(\frac{\tan \frac{x}{2} + 1}{\sqrt{2}} \right)$$

12. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin x + \sqrt{3} \cos x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{\sin x + \sqrt{3} \cos x} dx$$

$$\text{We know that } \sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \text{ and } \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\Rightarrow \int \frac{1}{\sin x + \sqrt{3} \cos x} dx = \int \frac{1}{\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \sqrt{3} \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2} + \sqrt{3} - \sqrt{3} \tan^2 \frac{x}{2}} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2 dx = 2dt$,

$$\begin{aligned}
&\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2} + \sqrt{3} - \sqrt{3} \tan^2 \frac{x}{2}} dx = \int \frac{\sec^2 \frac{x}{2}}{-\sqrt{3} \tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} + \sqrt{3}} dx \\
&= - \int \frac{2 dt}{\sqrt{3} t^2 - 2t - \sqrt{3}} \\
&= - \frac{2}{\sqrt{3}} \int \frac{1}{\left(t - \frac{1}{\sqrt{3}}\right)^2 - \left(\frac{2}{\sqrt{3}}\right)^2} dt \\
&= \frac{2}{\sqrt{3}} \int \frac{1}{\left(\frac{2}{\sqrt{3}}\right)^2 - \left(t - \frac{1}{\sqrt{3}}\right)^2} dt
\end{aligned}$$

We know that $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$

$$\begin{aligned}
&\Rightarrow \frac{2}{\sqrt{3}} \int \frac{1}{\left(\frac{2}{\sqrt{3}}\right)^2 - \left(t - \frac{1}{\sqrt{3}}\right)^2} dt = \frac{2}{\sqrt{3}} \left(\frac{1}{2 \left(\frac{2}{\sqrt{3}}\right)} \right) \log \left| \frac{\frac{2}{\sqrt{3}} + t - \frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}} - t + \frac{1}{\sqrt{3}}} \right| + c \\
&= \frac{1}{2} \log \left| \frac{\frac{2}{\sqrt{3}} + \tan \frac{x}{2} - \frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}} - \tan \frac{x}{2} + \frac{1}{\sqrt{3}}} \right| + c \\
&\therefore I = \int \frac{1}{\sin x + \sqrt{3} \cos x} dx = \frac{1}{2} \log \left| \frac{\frac{2}{\sqrt{3}} + \tan \frac{x}{2} - \frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}} - \tan \frac{x}{2} + \frac{1}{\sqrt{3}}} \right| + c
\end{aligned}$$

13. Question

Evaluate the following integrals:

$$\int \frac{1}{\sqrt{3} \sin x + \cos x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{\sqrt{3} \sin x + \cos x} dx$$

Let $\sqrt{3} = r \cos \theta$ and $1 = r \sin \theta$

$$r = \sqrt{3 + 1} = 2$$

And $\tan \theta = 1/\sqrt{3} \rightarrow \theta = \pi/6$

$$\begin{aligned}
&\Rightarrow \int \frac{1}{\sqrt{3} \sin x + \cos x} dx = \int \frac{1}{r \cos \theta \sin x + r \sin \theta \cos x} dx \\
&= \frac{1}{r} \int \frac{1}{\sin(x + \theta)} dx \\
&= \frac{1}{r} \int \operatorname{cosec}(x + \theta) dx
\end{aligned}$$

We know that $\int \operatorname{cosec} x dx = \log \left| \tan \frac{x}{2} \right| + c$

$$\begin{aligned}
&\Rightarrow \frac{1}{r} \int \operatorname{cosec}(x + \theta) dx = \frac{1}{2} \log \left| \tan \left(\frac{x}{2} + \frac{\theta}{2} \right) \right| + c \\
&= \frac{1}{2} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) \right| + c
\end{aligned}$$

$$\therefore I = \int \frac{1}{\sqrt{3}\sin x + \cos x} dx = \frac{1}{2} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) \right| + c$$

14. Question

Evaluate the following integrals:

$$\int \frac{1}{\sin x - \sqrt{3} \cos x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{\sin x - \sqrt{3} \cos x} dx$$

Let $1 = r \cos \theta$ and $\sqrt{3} = r \sin \theta$

$$r = \sqrt{3+1} = 2$$

And $\tan \theta = \sqrt{3} \rightarrow \theta = \pi/3$

$$\Rightarrow \int \frac{1}{\sin x - \sqrt{3} \cos x} dx = \int \frac{1}{r \cos \theta \sin x - r \sin \theta \cos x} dx$$

$$= \frac{1}{r} \int \frac{1}{\sin(x - \theta)} dx$$

$$= \frac{1}{r} \int \operatorname{cosec}(x - \theta) dx$$

We know that $\int \operatorname{cosec} x dx = \log \left| \tan \frac{x}{2} \right| + c$

$$\Rightarrow \frac{1}{r} \int \operatorname{cosec}(x - \theta) dx = \frac{1}{2} \log \left| \tan \left(\frac{x}{2} - \frac{\theta}{2} \right) \right| + c$$

$$= \frac{1}{2} \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{6} \right) \right| + c$$

$$\therefore I = \int \frac{1}{\sin x - \sqrt{3} \cos x} dx = \frac{1}{2} \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{6} \right) \right| + c$$

15. Question

Evaluate the following integrals:

$$\int \frac{1}{5 + 7 \cos x + \sin x} dx$$

Answer

$$\text{Given } I = \int \frac{1}{5 + 7 \cos x + \sin x} dx$$

We know that $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ and $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$$\Rightarrow \int \frac{1}{5 + \sin x + 7 \cos x} dx = \int \frac{1}{5 + \left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + 7 \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)} dx$$

$$= \int \frac{1 + \tan^2 \frac{x}{2}}{5 + 5 \tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} + 7 - 7 \tan^2 \frac{x}{2}} dx$$

Replacing $1 + \tan^2 x/2$ in numerator by $\sec^2 x/2$ and putting $\tan x/2 = t$ and $\sec^2 x/2 dx = 2dt$,

$$\Rightarrow \int \frac{1 + \tan^2 \frac{x}{2}}{5 + 5 \tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} + 7 - 7 \tan^2 \frac{x}{2}} dx = \int \frac{\sec^2 \frac{x}{2}}{-2 \tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} + 12} dx$$

$$= \int \frac{2dt}{-2t^2 + 2t + 12}$$

$$= - \int \frac{1}{t^2 - t - 6} dt$$

$$= - \int \frac{1}{\left(t - \frac{1}{2}\right)^2 - \frac{5^2}{2}} dt$$

We know that $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

$$\Rightarrow - \int \frac{1}{\left(t - \frac{1}{2}\right)^2 - \frac{5^2}{2}} dt = - \left(\frac{1}{2 \left(\frac{5}{2}\right)} \right) \log \left| \frac{t - \frac{1}{2} - \frac{5}{2}}{t - \frac{1}{2} + \frac{5}{2}} \right| + c$$

$$= \frac{-1}{5} \log \left| \frac{\tan \frac{x}{2} - 3}{\tan \frac{x}{2} + 2} \right| + c$$

$$\therefore I = \int \frac{1}{5 + 7 \cos x + \sin x} dx = \frac{-1}{5} \log \left| \frac{\tan \frac{x}{2} - 3}{\tan \frac{x}{2} + 2} \right| + c$$

Exercise 19.24

1. Question

Evaluate the integral

$$\int \frac{1}{1 - \cot x} dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos, polynomial, log etc and formula for some special functions.

$$\text{Let, } I = \int \frac{1}{1 - \cot x} dx$$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

$$\text{If } I \text{ has the form } \int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$$

Then substitute numerator as -

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + c) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{1}{1 - \cot x} dx = \int \frac{1}{\frac{\cos x}{\sin x}} dx = \int \frac{\sin x}{\sin x - \cos x} dx$$

As I matches with the form described above, So we will take the steps as described.

$$\therefore \sin x = A \frac{d}{dx}(\sin x - \cos x) + B(\sin x - \cos x) + C$$

$$\Rightarrow \sin x = A(\cos x + \sin x) + B(\sin x - \cos x) + C \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow \sin x = \sin x (B + A) + \cos x (A - B) + C$$

Comparing both sides we have:

$$C = 0$$

$$A - B = 0 \Rightarrow A = B$$

$$B + A = 1 \Rightarrow 2A = 1 \Rightarrow A = 1/2$$

$$\therefore A = B = 1/2$$

Thus I can be expressed as:

$$I = \int \frac{\frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(\sin x - \cos x)}{\sin x - \cos x} dx$$

$$I = \int \frac{\frac{1}{2}(\cos x + \sin x)}{\sin x - \cos x} dx + \int \frac{\frac{1}{2}(\sin x - \cos x)}{\sin x - \cos x} dx$$

$$\therefore \text{Let } I_1 = \frac{1}{2} \int \frac{(\cos x + \sin x)}{\sin x - \cos x} dx \text{ and } I_2 = \frac{1}{2} \int \frac{(\sin x - \cos x)}{\sin x - \cos x} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \frac{1}{2} \int \frac{(\cos x + \sin x)}{\sin x - \cos x} dx$$

$$\text{Let, } u = \sin x - \cos x \Rightarrow du = (\cos x + \sin x) dx$$

So, I_1 reduces to:

$$I_1 = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \log|u| + C_1$$

$$\therefore I_1 = \frac{1}{2} \log|\sin x - \cos x| + C_1 \dots \text{equation 2}$$

$$\text{As, } I_2 = \frac{1}{2} \int \frac{(\sin x - \cos x)}{\sin x - \cos x} dx = \frac{1}{2} \int dx$$

$$\therefore I_2 = \frac{x}{2} + C_2 \dots \text{equation 3}$$

From equation 1, 2 and 3 we have:

$$I = \frac{1}{2} \log|\sin x - \cos x| + C_1 + \frac{x}{2} + C_2$$

$$\therefore I = \frac{1}{2} \log|\sin x - \cos x| + \frac{x}{2} + C$$

2. Question

Evaluate the integral

$$\int \frac{1}{1 - \tan x} dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos, polynomial, log etc and formula for some

special functions.

$$\text{Let, } I = \int \frac{1}{1-\tan x} dx$$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

$$\text{If } I \text{ has the form } \int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$$

Then substitute numerator as -

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + c) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{1}{1-\tan x} dx = \int \frac{1}{1-\frac{\sin x}{\cos x}} dx = \int \frac{\cos x}{\cos x - \sin x} dx$$

As I matches with the form described above, So we will take the steps as described.

$$\therefore \cos x = A \frac{d}{dx} (\cos x - \sin x) + B(\cos x - \sin x) + C$$

$$\Rightarrow \cos x = A(-\sin x - \cos x) + B(\cos x - \sin x) + C \quad \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow \cos x = -\sin x (B + A) + \cos x (B - A) + C$$

Comparing both sides we have:

$$C = 0$$

$$B - A = 1 \Rightarrow A = B - 1$$

$$B + A = 0 \Rightarrow 2B - 1 = 0 \Rightarrow B = 1/2$$

$$\therefore A = B - 1 = -1/2$$

Thus I can be expressed as:

$$I = \int \frac{\frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(\cos x - \sin x)}{(\cos x - \sin x)} dx$$

$$I = \int \frac{\frac{1}{2}(\cos x + \sin x)}{(\cos x - \sin x)} dx + \int \frac{\frac{1}{2}(\cos x - \sin x)}{(\cos x - \sin x)} dx$$

$$\therefore \text{Let } I_1 = \frac{1}{2} \int \frac{(\cos x + \sin x)}{(\cos x - \sin x)} dx \text{ and } I_2 = \frac{1}{2} \int \frac{(\cos x - \sin x)}{(\cos x - \sin x)} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \frac{1}{2} \int \frac{(\cos x + \sin x)}{(\cos x - \sin x)} dx$$

$$\text{Let, } u = \cos x - \sin x \Rightarrow du = -(\cos x + \sin x) dx$$

So, I_1 reduces to:

$$I_1 = -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \log|u| + C_1$$

$$\therefore I_1 = -\frac{1}{2} \log|\cos x - \sin x| + C_1 \dots \text{equation 2}$$

$$\text{As, } I_2 = \frac{1}{2} \int \frac{(\cos x - \sin x)}{(\cos x - \sin x)} dx = \frac{1}{2} \int dx$$

$$\therefore I_2 = \frac{x}{2} + C_2 \dots \text{equation 3}$$

From equation 1, 2 and 3 we have:

$$I = -\frac{1}{2} \log|\cos x - \sin x| + C_1 + \frac{x}{2} + C_2$$

$$\therefore I = -\frac{1}{2} \log|\cos x - \sin x| + \frac{x}{2} + C$$

3. Question

Evaluate the integral

$$\int \frac{3 + 2 \cos x + 4 \sin x}{2 \sin x + \cos x + 3} dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some special functions.

$$\text{Let, } I = \int \frac{3+2 \cos x+4 \sin x}{2 \sin x+\cos x+3} dx$$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

$$\text{If } I \text{ has the form } \int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$$

Then substitute numerator as -

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + f) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{3+2 \cos x+4 \sin x}{2 \sin x+\cos x+3} dx$$

As I matches with the form described above, So we will take the steps as described.

$$\therefore 3 + 2 \cos x + 4 \sin x = A \frac{d}{dx} (2 \sin x + \cos x + 3) + B(2 \sin x + \cos x + 3) + C$$

$$\Rightarrow 3 + 2 \cos x + 4 \sin x = A(2 \cos x - \sin x) + B(2 \sin x + \cos x + 3) + C \quad \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow 3 + 2 \cos x + 4 \sin x = \sin x (2B - A) + \cos x (B + 2A) + 3B + C$$

Comparing both sides we have:

$$3B + C = 3$$

$$B + 2A = 2$$

$$2B - A = 4$$

On solving for A ,B and C we have:

$$A = 0, B = 2 \text{ and } C = -3$$

Thus I can be expressed as:

$$I = \int \frac{2(2 \sin x + \cos x + 3) - 3}{2 \sin x + \cos x + 3} dx$$

$$I = \int \frac{2(2 \sin x + \cos x + 3)}{2 \sin x + \cos x + 3} dx + \int \frac{-3}{2 \sin x + \cos x + 3} dx$$

$$\therefore \text{Let } I_1 = 2 \int \frac{(2 \sin x + \cos x + 3)}{2 \sin x + \cos x + 3} dx \text{ and } I_2 = -3 \int \frac{1}{2 \sin x + \cos x + 3} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = 2 \int \frac{(2 \sin x + \cos x + 3)}{2 \sin x + \cos x + 3} dx$$

So, I_1 reduces to:

$$I_1 = 2 \int dx = 2x + C_1 \dots \text{equation 2}$$

$$\text{As, } I_2 = -3 \int \frac{1}{2 \sin x + \cos x + 3} dx$$

To solve the integrals of the form $\int \frac{1}{a \sin x + b \cos x + c} dx$

To apply substitution method we take following procedure.

We substitute:

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \text{ and } \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\therefore I_2 = -3 \int \frac{1}{2 \sin x + \cos x + 3} dx$$

$$\Rightarrow I_2 = -3 \int \frac{1}{2 \left(\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + 3 \left(\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right) + 3} dx$$

$$\Rightarrow I_2 = -3 \int \frac{1 + \tan^2 \frac{x}{2}}{4 \tan^2 \frac{x}{2} + 1 - \tan^2 \frac{x}{2} + 3(1 + \tan^2 \frac{x}{2})} dx$$

$$\Rightarrow I_2 = -3 \int \frac{\sec^2 \frac{x}{2}}{2(2 \tan^2 \frac{x}{2} + 2 + 1 \tan^2 \frac{x}{2})} dx$$

$$\text{Let, } t = \tan \frac{x}{2} \Rightarrow dt = \frac{1}{2} \sec^2 \frac{x}{2} dx$$

$$\therefore I_2 = -3 \int \frac{1}{(2t + 2 + t^2)} dt$$

As, the denominator is polynomial without any square root term. So one of the special integral will be used to solve I_2 .

$$I_2 = -3 \int \frac{1}{(2t + 2 + t^2)} dt$$

$$\Rightarrow I_2 = -3 \int \frac{1}{(t^2 + 2(1)t + 1) + 1} dt$$

$$\therefore I_2 = -3 \int \frac{1}{(t+1)^2 + 1} dt \{ \because a^2 + 2ab + b^2 = (a+b)^2 \}$$

As, I_2 matches with the special integral form

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$I_2 = -3 \tan^{-1}(t + 1)$$

Putting value of t we have:

$$\therefore I_2 = -3 \tan^{-1} \left(\tan \frac{x}{2} + 1 \right) + C_2 \dots \text{equation 3}$$

From equation 1, 2 and 3:

$$I = 2x + C_1 - 3 \tan^{-1} \left(\tan \frac{x}{2} + 1 \right) + C_2$$

$$\therefore I = 2x - 3 \tan^{-1} \left(\tan \frac{x}{2} + 1 \right) + C \dots \text{ans}$$

4. Question

Evaluate the integral

$$\int \frac{1}{p + q \tan x} dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some special functions.

$$\text{Let, } I = \int \frac{1}{p+q \tan x} dx$$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

$$\text{If } I \text{ has the form } \int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$$

Then substitute numerator as -

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + c) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{1}{p+q \tan x} dx = \int \frac{1}{p+q \frac{\sin x}{\cos x}} dx = \int \frac{\cos x}{p \cos x + q \sin x} dx$$

As I matches with the form described above, So we will take the steps as described.

$$\therefore \cos x = A \frac{d}{dx} (p \cos x + q \sin x) + B(p \cos x + q \sin x) + C$$

$$\Rightarrow \cos x = A(-p \sin x + q \cos x) + B(p \cos x - q \sin x) + C \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow \cos x = -\sin x (Bq + Ap) + \cos x (Bp + Aq) + C$$

Comparing both sides we have:

$$C = 0$$

$$Bp + Aq = 1$$

$$Bq + Ap = 0$$

On solving above equations, we have:

$$A = \frac{q}{p^2+q^2} \quad B = \frac{p}{p^2+q^2} \quad \text{and } C = 0$$

Thus I can be expressed as:

$$I = \int \frac{\frac{q}{p^2+q^2} (-p \sin x + q \cos x) + \frac{p}{p^2+q^2} (p \cos x + q \sin x)}{(p \cos x + q \sin x)} dx$$

$$I = \int \frac{\frac{q}{p^2+q^2} (-p \sin x + q \cos x)}{(p \cos x + q \sin x)} dx + \int \frac{\frac{p}{p^2+q^2} (p \cos x + q \sin x)}{(p \cos x + q \sin x)} dx$$

$$\therefore \text{Let } I_1 = \frac{q}{p^2+q^2} \int \frac{(-p \sin x + q \cos x)}{(p \cos x + q \sin x)} dx \text{ and } I_2 = \frac{p}{p^2+q^2} \int \frac{(p \cos x + q \sin x)}{(p \cos x + q \sin x)} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \frac{q}{p^2+q^2} \int \frac{(-p \sin x + q \cos x)}{(p \cos x + q \sin x)} dx$$

$$\text{Let, } u = p \cos x + q \sin x \Rightarrow du = (-p \sin x + q \cos x) dx$$

So, I_1 reduces to:

$$I_1 = \frac{q}{p^2+q^2} \int \frac{du}{u} = \frac{q}{p^2+q^2} \log|u| + C_1$$

$$\therefore I_1 = \frac{q}{p^2+q^2} \log|(p \cos x + q \sin x)| + C_1 \dots \text{equation 2}$$

$$\text{As, } I_2 = \frac{p}{p^2+q^2} \int \frac{(p \cos x + q \sin x)}{(p \cos x + q \sin x)} dx = \frac{p}{p^2+q^2} \int dx$$

$$\therefore I_2 = \frac{px}{p^2+q^2} + C_2 \dots \text{equation 3}$$

From equation 1, 2 and 3 we have:

$$I = \frac{q}{p^2+q^2} \log|(p \cos x + q \sin x)| + C_1 + \frac{px}{p^2+q^2} + C_2$$

$$\therefore I = \frac{q}{p^2+q^2} \log|(p \cos x + q \sin x)| + \frac{px}{p^2+q^2} + C$$

5. Question

Evaluate the integral

$$\int \frac{5 \cos x + 6}{2 \cos x + \sin x + 3} dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos, polynomial, log etc and formula for some special functions.

$$\text{Let, } I = \int \frac{5 \cos x + 6}{2 \cos x + \sin x + 3} dx$$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

$$\text{If } I \text{ has the form } \int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$$

Then substitute numerator as -

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + f) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{5 \cos x + 6}{2 \cos x + \sin x + 3} dx$$

As I matches with the form described above, So we will take the steps as described.

$$\therefore 5 \cos x + 6 = A \frac{d}{dx} (2 \cos x + \sin x + 3) + B(2 \cos x + \sin x + 3) + C$$

$$\Rightarrow 5 \cos x + 6 = A(-2 \sin x + \cos x) + B(2 \cos x + \sin x + 3) + C \quad \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow 5 \cos x + 6 = \sin x (B - 2A) + \cos x (2B + A) + 3B + C$$

Comparing both sides we have:

$$3B + C = 6$$

$$2B + A = 5$$

$$B - 2A = 0$$

On solving for A, B and C we have:

$$A = 1, B = 2 \text{ and } C = 0$$

Thus I can be expressed as:

$$I = \int \frac{(-2 \sin x + \cos x) + 2(2 \cos x + \sin x + 3)}{2 \cos x + \sin x + 3} dx$$

$$I = \int \frac{(-2 \sin x + \cos x)}{2 \cos x + \sin x + 3} dx + \int \frac{2(2 \cos x + \sin x + 3)}{2 \cos x + \sin x + 3} dx$$

$$\therefore \text{Let } I_1 = \int \frac{(-2 \sin x + \cos x)}{2 \cos x + \sin x + 3} dx \text{ and } I_2 = \int \frac{2(2 \cos x + \sin x + 3)}{2 \cos x + \sin x + 3} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \int \frac{(-2 \sin x + \cos x)}{2 \cos x + \sin x + 3} dx$$

$$\text{Let, } 2 \cos x + \sin x + 3 = u$$

$$\Rightarrow (-2 \sin x + \cos x) dx = du$$

So, I_1 reduces to:

$$I_1 = \int \frac{du}{u} = \log|u| + C_1$$

$$\therefore I_1 = \log|2 \cos x + \sin x + 3| + C_1 \dots \text{equation 2}$$

$$\text{As, } I_2 = \int \frac{2(2 \cos x + \sin x + 3)}{2 \cos x + \sin x + 3} dx$$

$$\Rightarrow I_2 = 2 \int dx = 2x + C_2 \dots \text{equation 3}$$

From equation 1, 2 and 3 we have:

$$I = \log|2 \cos x + \sin x + 3| + C_1 + 2x + C_2$$

$$\therefore I = \log|2 \cos x + \sin x + 3| + 2x + C$$

6. Question

Evaluate the integral

$$\int \frac{2 \sin x + 3 \cos x}{3 \sin x + 4 \cos x} dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos, polynomial, log etc and formula for some special functions.

$$\text{Let, } I = \int \frac{2\sin x + 3\cos x}{4\cos x + 3\sin x} dx$$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

$$\text{If } I \text{ has the form } \int \frac{a\sin x + b\cos x + c}{d\sin x + e\cos x + f} dx$$

Then substitute numerator as -

$$a\sin x + b\cos x + c = A \frac{d}{dx} (d\sin x + e\cos x + f) + B(d\sin x + e\cos x + c) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{2\sin x + 3\cos x}{4\cos x + 3\sin x} dx$$

As I matches with the form described above, So we will take the steps as described.

$$\therefore 2\sin x + 3\cos x = A \frac{d}{dx} (3\sin x + 4\cos x) + B(4\cos x + 3\sin x) + C$$

$$\Rightarrow 2\sin x + 3\cos x = A(3\cos x - 4\sin x) + B(4\cos x + 3\sin x) + C \quad \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow 2\sin x + 3\cos x = \sin x (3B - 4A) + \cos x (3A + 4B) + C$$

Comparing both sides we have:

$$C = 0$$

$$3B - 4A = 2$$

$$4B + 3A = 3$$

On solving for A, B and C we have:

$$A = 1/25, B = 18/25 \text{ and } C = 0$$

Thus I can be expressed as:

$$I = \int \frac{\frac{1}{25}(3\cos x - 4\sin x) + \frac{18}{25}(4\cos x + 3\sin x)}{4\cos x + 3\sin x} dx$$

$$I = \int \frac{\frac{1}{25}(3\cos x - 4\sin x)}{4\cos x + 3\sin x} dx + \int \frac{\frac{18}{25}(4\cos x + 3\sin x)}{4\cos x + 3\sin x} dx$$

$$\therefore \text{Let } I_1 = \frac{1}{25} \int \frac{(3\cos x - 4\sin x)}{4\cos x + 3\sin x} dx \text{ and } I_2 = \frac{18}{25} \int \frac{(4\cos x + 3\sin x)}{4\cos x + 3\sin x} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \frac{1}{25} \int \frac{(3\cos x - 4\sin x)}{4\cos x + 3\sin x} dx$$

$$\text{Let, } 4\cos x + 3\sin x = u$$

$$\Rightarrow (-4\sin x + 3\cos x)dx = du$$

So, I_1 reduces to:

$$I_1 = \frac{1}{25} \int \frac{du}{u} = \frac{1}{25} \log|u| + C_1$$

$$\therefore I_1 = \frac{1}{25} \log|4\cos x + 3\sin x| + C_1 \dots \text{equation 2}$$

$$\text{As, } I_2 = \frac{18}{25} \int \frac{(4\cos x + 3\sin x)}{4\cos x + 3\sin x} dx$$

$$\Rightarrow I_2 = \frac{18}{25} \int dx = \frac{18x}{25} + C_2 \dots \text{equation 3}$$

From equation 1, 2 and 3 we have:

$$I = \frac{1}{25} \log|4 \cos x + 3 \sin x| + C_1 + \frac{18x}{25} + C_2$$

$$\therefore I = \frac{1}{25} \log|4 \cos x + 3 \sin x| + \frac{18x}{25} + C$$

7. Question

Evaluate the integral

$$\int \frac{1}{3 + 4 \cot x} dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some special functions.

$$\text{Let, } I = \int \frac{1}{3 + 4 \cot x} dx$$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

$$\text{If } I \text{ has the form } \int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$$

Then substitute numerator as -

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + c) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{1}{3 + 4 \cot x} dx = \int \frac{1 \frac{\cos x}{\sin x}}{3 + 4 \frac{\cos x}{\sin x}} dx = \int \frac{\sin x}{3 \sin x + 4 \cos x} dx$$

As I matches with the form described above, So we will take the steps as described.

$$\therefore \sin x = A \frac{d}{dx} (3 \sin x + 4 \cos x) + B(4 \cos x + 3 \sin x) + C$$

$$\Rightarrow \sin x = A(3 \cos x - 4 \sin x) + B(4 \cos x + 3 \sin x) + C \quad \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow \sin x = \sin x (3B - 4A) + \cos x (3A + 4B) + C$$

Comparing both sides we have:

$$C = 0$$

$$3B - 4A = 1$$

$$4B + 3A = 0$$

On solving for A ,B and C we have:

$$A = -4/25, B = 3/25 \text{ and } C = 0$$

Thus I can be expressed as:

$$I = \int \frac{\frac{-4}{25}(3 \cos x - 4 \sin x) + \frac{3}{25}(4 \cos x + 3 \sin x)}{4 \cos x + 3 \sin x} dx$$

$$I = \int \frac{-4(3\cos x - 4\sin x)}{4\cos x + 3\sin x} dx + \int \frac{\frac{3}{25}(4\cos x + 3\sin x)}{4\cos x + 3\sin x} dx$$

$$\therefore \text{Let } I_1 = -\frac{4}{25} \int \frac{(3\cos x - 4\sin x)}{4\cos x + 3\sin x} dx \text{ and } I_2 = \frac{3}{25} \int \frac{(4\cos x + 3\sin x)}{4\cos x + 3\sin x} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = -\frac{4}{25} \int \frac{(3\cos x - 4\sin x)}{4\cos x + 3\sin x} dx$$

$$\text{Let, } 4\cos x + 3\sin x = u$$

$$\Rightarrow (-4\sin x + 3\cos x)dx = du$$

So, I_1 reduces to:

$$I_1 = -\frac{4}{25} \int \frac{du}{u} = -\frac{4}{25} \log|u| + C_1$$

$$\therefore I_1 = -\frac{4}{25} \log|4\cos x + 3\sin x| + C_1 \dots \text{equation 2}$$

$$\text{As, } I_2 = \frac{3}{25} \int \frac{(4\cos x + 3\sin x)}{4\cos x + 3\sin x} dx$$

$$\Rightarrow I_2 = \frac{3}{25} \int dx = \frac{3x}{25} + C_2 \dots \text{equation 3}$$

From equation 1, 2 and 3 we have:

$$I = -\frac{4}{25} \log|4\cos x + 3\sin x| + C_1 + \frac{3x}{25} + C_2$$

$$\therefore I = -\frac{4}{25} \log|4\cos x + 3\sin x| + \frac{3x}{25} + C$$

8. Question

Evaluate the integral

$$\int \frac{2\tan x + 3}{3\tan x + 4} dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos, polynomial, log etc and formula for some special functions.

$$\text{Let, } I = \int \frac{2\tan x + 3}{3\tan x + 4} dx$$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

$$\text{If } I \text{ has the form } \int \frac{a\sin x + b\cos x + c}{d\sin x + e\cos x + f} dx$$

Then substitute numerator as -

$$a\sin x + b\cos x + c = A \frac{d}{dx} (d\sin x + e\cos x + f) + B(d\sin x + e\cos x + c) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{2\tan x + 3}{3\tan x + 4} dx = \int \frac{2 \frac{\sin x}{\cos x} + 3}{3 \frac{\sin x}{\cos x} + 4} = \int \frac{2\sin x + 3\cos x}{4\cos x + 3\sin x} dx$$

As I matches with the form described above, So we will take the steps as described.

$$\therefore 2 \sin x + 3 \cos x = A \frac{d}{dx} (3 \sin x + 4 \cos x) + B(4 \cos x + 3 \sin x) + C$$

$$\Rightarrow 2 \sin x + 3 \cos x = A(3 \cos x - 4 \sin x) + B(4 \cos x + 3 \sin x) + C \quad \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow 2 \sin x + 3 \cos x = \sin x (3B - 4A) + \cos x (3A + 4B) + C$$

Comparing both sides we have:

$$C = 0$$

$$3B - 4A = 2$$

$$4B + 3A = 3$$

On solving for A, B and C we have:

$$A = 1/25, B = 18/25 \text{ and } C = 0$$

Thus I can be expressed as:

$$I = \int \frac{\frac{1}{25}(3 \cos x - 4 \sin x) + \frac{18}{25}(4 \cos x + 3 \sin x)}{4 \cos x + 3 \sin x} dx$$

$$I = \int \frac{\frac{1}{25}(3 \cos x - 4 \sin x)}{4 \cos x + 3 \sin x} dx + \int \frac{\frac{18}{25}(4 \cos x + 3 \sin x)}{4 \cos x + 3 \sin x} dx$$

$$\therefore \text{Let } I_1 = \frac{1}{25} \int \frac{(3 \cos x - 4 \sin x)}{4 \cos x + 3 \sin x} dx \text{ and } I_2 = \frac{18}{25} \int \frac{(4 \cos x + 3 \sin x)}{4 \cos x + 3 \sin x} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \frac{1}{25} \int \frac{(3 \cos x - 4 \sin x)}{4 \cos x + 3 \sin x} dx$$

$$\text{Let, } 4 \cos x + 3 \sin x = u$$

$$\Rightarrow (-4 \sin x + 3 \cos x) dx = du$$

So, I_1 reduces to:

$$I_1 = \frac{1}{25} \int \frac{du}{u} = \frac{1}{25} \log|u| + C_1$$

$$\therefore I_1 = \frac{1}{25} \log|4 \cos x + 3 \sin x| + C_1 \dots \text{equation 2}$$

$$\text{As, } I_2 = \frac{18}{25} \int \frac{(4 \cos x + 3 \sin x)}{4 \cos x + 3 \sin x} dx$$

$$\Rightarrow I_2 = \frac{18}{25} \int dx = \frac{18x}{25} + C_2 \dots \text{equation 3}$$

From equation 1, 2 and 3 we have:

$$I = \frac{1}{25} \log|4 \cos x + 3 \sin x| + C_1 + \frac{18x}{25} + C_2$$

$$\therefore I = \frac{1}{25} \log|4 \cos x + 3 \sin x| + \frac{18x}{25} + C$$

9. Question

Evaluate the integral

$$\int \frac{1}{4 + 3 \tan x} dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos ,polynomial, log etc and formula for some special functions.

$$\text{Let, } I = \int \frac{1}{4+3\tan x} dx$$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

$$\text{If } I \text{ has the form } \int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$$

Then substitute numerator as -

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + c) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{1}{4+3\tan x} dx = \int \frac{1}{4+3\frac{\sin x}{\cos x}} dx = \int \frac{\cos x}{3 \sin x + 4 \cos x} dx$$

As I matches with the form described above, So we will take the steps as described.

$$\therefore \cos x = A \frac{d}{dx} (3 \sin x + 4 \cos x) + B(4 \cos x + 3 \sin x) + C$$

$$\Rightarrow \cos x = A(3 \cos x - 4 \sin x) + B(4 \cos x + 3 \sin x) + C \quad \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow \cos x = \sin x (3B - 4A) + \cos x (3A + 4B) + C$$

Comparing both sides we have:

$$C = 0$$

$$3B - 4A = 0$$

$$4B + 3A = 1$$

On solving for A, B and C we have:

$$A = 3/25, B = 4/25 \text{ and } C = 0$$

Thus I can be expressed as:

$$I = \int \frac{\frac{3}{25}(3 \cos x - 4 \sin x) + \frac{4}{25}(4 \cos x + 3 \sin x)}{4 \cos x + 3 \sin x} dx$$

$$I = \int \frac{\frac{3}{25}(3 \cos x - 4 \sin x)}{4 \cos x + 3 \sin x} dx + \int \frac{\frac{4}{25}(4 \cos x + 3 \sin x)}{4 \cos x + 3 \sin x} dx$$

$$\therefore \text{Let } I_1 = \frac{3}{25} \int \frac{(3 \cos x - 4 \sin x)}{4 \cos x + 3 \sin x} dx \text{ and } I_2 = \frac{4}{25} \int \frac{(4 \cos x + 3 \sin x)}{4 \cos x + 3 \sin x} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \frac{3}{25} \int \frac{(3 \cos x - 4 \sin x)}{4 \cos x + 3 \sin x} dx$$

$$\text{Let, } 4 \cos x + 3 \sin x = u$$

$$\Rightarrow (-4 \sin x + 3 \cos x) dx = du$$

So, I_1 reduces to:

$$I_1 = \frac{3}{25} \int \frac{du}{u} = \frac{3}{25} \log|u| + C_1$$

$$\therefore I_1 = \frac{3}{25} \log|4 \cos x + 3 \sin x| + C_1 \dots \text{equation 2}$$

$$\text{As, } I_2 = \frac{4}{25} \int \frac{(4 \cos x + 3 \sin x)}{4 \cos x + 3 \sin x} dx$$

$$\Rightarrow I_2 = \frac{4}{25} \int dx = \frac{4x}{25} + C_2 \dots \text{equation 3}$$

From equation 1, 2 and 3 we have:

$$I = \frac{3}{25} \log|4 \cos x + 3 \sin x| + C_1 + \frac{4x}{25} + C_2$$

$$\therefore I = \frac{3}{25} \log|4 \cos x + 3 \sin x| + \frac{4x}{25} + C$$

10. Question

Evaluate the integral

$$\int \frac{8 \cot x + 1}{3 \cot x + 2} dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos, polynomial, log etc and formula for some special functions.

$$\text{Let, } I = \int \frac{8 \cot x + 1}{3 \cot x + 2} dx$$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

$$\text{If } I \text{ has the form } \int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$$

Then substitute numerator as -

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + f) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{8 \cot x + 1}{3 \cot x + 2} dx = \int \frac{8 \frac{\cos x}{\sin x} + 1}{3 \frac{\cos x}{\sin x} + 2} = \int \frac{8 \cos x + \sin x}{3 \cos x + 2 \sin x} dx$$

As I matches with the form described above, So we will take the steps as described.

$$\therefore \sin x + 8 \cos x = A \frac{d}{dx} (3 \cos x + 2 \sin x) + B(3 \cos x + 2 \sin x) + C$$

$$\Rightarrow \sin x + 8 \cos x = A(-3 \sin x + 2 \cos x) + B(3 \cos x + 2 \sin x) + C \quad \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow \sin x + 8 \cos x = \sin x (2B - 3A) + \cos x (2A + 3B) + C$$

Comparing both sides we have:

$$C = 0$$

$$2B - 3A = 1$$

$$3B + 2A = 8$$

On solving for A, B and C we have:

$$A = 1, B = 2 \text{ and } C = 0$$

Thus I can be expressed as:

$$I = \int \frac{(-3 \sin x + 2 \cos x) + 2(3 \cos x + 2 \sin x)}{3 \cos x + 2 \sin x} dx$$

$$I = \int \frac{(-3 \sin x + 2 \cos x)}{3 \cos x + 2 \sin x} dx + \int \frac{2(3 \cos x + 2 \sin x)}{3 \cos x + 2 \sin x} dx$$

$$\therefore \text{Let } I_1 = \int \frac{(-3 \sin x + 2 \cos x)}{3 \cos x + 2 \sin x} dx \text{ and } I_2 = \int \frac{2(3 \cos x + 2 \sin x)}{3 \cos x + 2 \sin x} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \int \frac{(-3 \sin x + 2 \cos x)}{3 \cos x + 2 \sin x} dx$$

$$\text{Let, } 3 \cos x + 2 \sin x = u$$

$$\Rightarrow (-3 \sin x + 2 \cos x) dx = du$$

So, I_1 reduces to:

$$I_1 = \int \frac{du}{u} = \log|u| + C_1$$

$$\therefore I_1 = \log|3 \cos x + 2 \sin x| + C_1 \dots \text{equation 2}$$

$$\text{As, } I_2 = \int \frac{2(3 \cos x + 2 \sin x)}{3 \cos x + 2 \sin x} dx$$

$$\Rightarrow I_2 = 2 \int dx = 2x + C_2 \dots \text{equation 3}$$

From equation 1, 2 and 3 we have:

$$I = \frac{1}{25} \log|3 \cos x + 2 \sin x| + C_1 + 2x + C_2$$

$$\therefore I = \frac{1}{25} \log|3 \cos x + 2 \sin x| + 2x + C$$

11. Question

Evaluate the integral

$$\int \frac{4 \sin x + 5 \cos x}{5 \sin x + 4 \cos x} dx$$

Answer

Ideas required to solve the problems:

* Integration by substitution: A change in the variable of integration often reduces an integral to one of the fundamental integration. If derivative of a function is present in an integration or if chances of its presence after few modification is possible then we apply integration by substitution method.

* Knowledge of integration of fundamental functions like sin, cos, polynomial, log etc and formula for some special functions.

$$\text{Let, } I = \int \frac{4 \sin x + 5 \cos x}{5 \sin x + 4 \cos x} dx$$

To solve such integrals involving trigonometric terms in numerator and denominators. We use the basic substitution method and to apply this simply we follow the undermentioned procedure-

$$\text{If } I \text{ has the form } \int \frac{a \sin x + b \cos x + c}{d \sin x + e \cos x + f} dx$$

Then substitute numerator as -

$$a \sin x + b \cos x + c = A \frac{d}{dx} (d \sin x + e \cos x + f) + B(d \sin x + e \cos x + c) + C$$

Where A, B and C are constants

$$\text{We have, } I = \int \frac{4\sin x + 5\cos x}{5\sin x + 4\cos x} dx$$

As I matches with the form described above, So we will take the steps as described.

$$\therefore 4\sin x + 5\cos x = A \frac{d}{dx}(5\sin x + 4\cos x) + B(4\cos x + 5\sin x) + C$$

$$\Rightarrow 4\sin x + 5\cos x = A(5\cos x - 4\sin x) + B(4\cos x + 5\sin x) + C \quad \left\{ \because \frac{d}{dx} \cos x = -\sin x \right\}$$

$$\Rightarrow 4\sin x + 5\cos x = \sin x (5B - 4A) + \cos x (5A + 4B) + C$$

Comparing both sides we have:

$$C = 0$$

$$5B - 4A = 4$$

$$4B + 5A = 5$$

On solving for A, B and C we have:

$$A = 9/41, B = 40/41 \text{ and } C = 0$$

Thus I can be expressed as:

$$I = \int \frac{\frac{9}{41}(5\cos x - 4\sin x) + \frac{40}{41}(4\cos x + 5\sin x)}{4\cos x + 5\sin x} dx$$

$$I = \int \frac{\frac{9}{41}(5\cos x - 4\sin x)}{4\cos x + 5\sin x} dx + \int \frac{\frac{40}{41}(4\cos x + 5\sin x)}{4\cos x + 5\sin x} dx$$

$$\therefore \text{Let } I_1 = \frac{9}{41} \int \frac{(5\cos x - 4\sin x)}{4\cos x + 5\sin x} \text{ and } I_2 = \frac{40}{41} \int \frac{(4\cos x + 5\sin x)}{4\cos x + 5\sin x} dx$$

$$\Rightarrow I = I_1 + I_2 \dots \text{equation 1}$$

$$I_1 = \frac{9}{41} \int \frac{(5\cos x - 4\sin x)}{4\cos x + 5\sin x}$$

$$\text{Let, } 4\cos x + 5\sin x = u$$

$$\Rightarrow (-4\sin x + 5\cos x)dx = du$$

So, I_1 reduces to:

$$I_1 = \frac{9}{41} \int \frac{du}{u} = \frac{9}{41} \log|u| + C_1$$

$$\therefore I_1 = \frac{9}{41} \log|4\cos x + 5\sin x| + C_1 \dots \text{equation 2}$$

$$\text{As, } I_2 = \frac{40}{41} \int \frac{(4\cos x + 5\sin x)}{4\cos x + 5\sin x} dx$$

$$\Rightarrow I_2 = \frac{40}{41} \int dx = \frac{40x}{41} + C_2 \dots \text{equation 3}$$

From equation 1, 2 and 3 we have:

$$I = \frac{9}{41} \log|4\cos x + 5\sin x| + C_1 + \frac{40x}{41} + C_2$$

$$\therefore I = \frac{9}{41} \log|4\cos x + 5\sin x| + \frac{40x}{41} + C$$

Exercise 19.25

1. Question

Evaluate the following integrals:

$$\int x \cos x \, dx$$

Answer

$$\text{Let } I = \int x \cos x \, dx$$

$$\text{We know that, } \int UV = U \int V \, dx - \int \frac{d}{dx} U \int V \, dx$$

Using integration by parts,

$$I = x \int \cos x \, dx - \int \frac{d}{dx} x \int \cos x \, dx = \int x \cos x \, dx$$

$$\text{We have, } \int \sin x = -\cos x, \int \cos x = \sin x$$

$$= x \times \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + c$$

2. Question

Evaluate the following integrals:

$$\int \log(x+1) \, dx$$

Answer

$$\text{Let } I = \int \log(x+1) \, dx$$

That is,

$$I = \int 1 \times \log(x+1) \, dx$$

Using integration by parts,

$$I = \log(x+1) \int 1 \, dx - \int \frac{d}{dx} \log(x+1) \int 1 \, dx$$

$$\text{We know that, } \int 1 \, dx = x \text{ and } \int \log x = \frac{1}{x}$$

$$= \log(x+1) \times x - \int \frac{1}{x+1} \times x \, dx$$

$$\frac{x}{x+1} = 1 - \frac{1}{x+1}$$

$$= x \log(x+1) - \int \left(1 - \frac{1}{x+1}\right) dx$$

$$= x \log(x+1) - x + \log(x+1) + c$$

3. Question

Evaluate the following integrals:

$$\int x^3 \log x \, dx$$

Answer

$$\text{Let } I = \int x^3 \log x \, dx$$

Using integration by parts,

$$I = \log x \int x^3 \, dx - \int \frac{d}{dx} \log x \int x^3 \, dx$$

We have, $\int x^n dx = \frac{x^{n+1}}{n+1}$ and $\int \log x = \frac{1}{x}$

$$= \log x \times \frac{x^4}{4} - \int \frac{1}{x} \times \frac{x^4}{4}$$

$$= \log x \times \frac{x^4}{4} - \frac{1}{4} \int x^3 dx$$

$$= \frac{x^4}{4} \log x - \frac{1}{4} \times \frac{x^4}{4}$$

$$= \frac{x^4}{4} \log x - \frac{x^4}{16} + c$$

4. Question

Evaluate the following integrals:

$$\int x e^x dx$$

Answer

$$\text{Let } I = \int x e^x dx$$

Using integration by parts,

$$I = x \int e^x dx - \int \frac{d}{dx} x \int e^x dx$$

We know that , $\int e^x dx = e^x$ and $\frac{d}{dx} x = 1$

$$= x e^x - \int e^x dx$$

$$= x e^x - e^x + c$$

5. Question

Evaluate the following integrals:

$$\int x e^{2x} dx$$

Answer

$$\text{Let } I = \int x e^{2x} dx$$

Using integration by parts,

$$I = x \int e^{2x} dx - \int \frac{d}{dx} x \int e^{2x} dx$$

We know that , $\int e^{nx} dx = \frac{e^x}{n}$ and $\frac{d}{dx} x = 1$

$$= \frac{x e^{2x}}{2} - \int \frac{e^{2x}}{2} dx$$

$$= \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + c$$

$$I = \left(\frac{x}{2} - \frac{1}{4} \right) e^{2x} + c$$

6. Question

Evaluate the following integrals:

$$\int x^2 e^{-x} dx$$

Answer

$$\text{Let } I = \int x^2 e^{-x} dx$$

Using integration by parts,

$$= x^2 \int e^{-x} dx - \int \frac{d}{dx} x^2 \int e^{-x} dx$$

We know that, $\int e^{nx} dx = \frac{e^x}{n}$ and $\frac{d}{dx} x^n = nx^{n-1}$

$$= x^2 \times -e^{-x} - \int 2x \times -e^{-x} dx$$

Using integration by parts in second integral, $= -x^2 e^{-x} + 2 \left(x \int e^{-x} dx - \int \frac{d}{dx} x \int e^{-x} dx \right)$

$$= -x^2 e^{-x} + 2(-xe^{-x} + (-e^{-x})) + c$$

$$= -x^2 e^{-x} + 2(-xe^{-x} - e^{-x}) + c$$

$$I = -e^{-x}(x^2 + 2x + 2) + c$$

7. Question

Evaluate the following integrals:

$$\int x^2 \cos x dx$$

Answer

$$\text{Let } I = \int x^2 \cos x dx$$

Using integration by parts,

$$= x^2 \int \cos x dx - \int \frac{d}{dx} x^2 \int \cos x dx$$

We know that, $\int \cos x dx = \sin x$ and $\frac{d}{dx} x^n = nx^{n-1}$

$$= x^2 \sin x - \int 2x \sin x dx$$

$$= x^2 \sin x - 2 \int x \sin x dx$$

We know that, $\int \sin x dx = -\cos x$

$$= x^2 \sin x - 2 \left(x \int \sin x dx - \int \frac{d}{dx} x \int \sin x dx \right)$$

$$= x^2 \sin x - 2 \left(-x \cos x + \int \cos x dx \right)$$

$$= x^2 \sin x - 2(-x \cos x + \sin x) + c$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + c$$

8. Question

Evaluate the following integrals:

$$\int x^2 \cos 2x dx$$

Answer

$$\text{Let } I = \int x^2 \cos 2x \, dx$$

Using integration by parts,

$$= x^2 \int \cos 2x \, dx - \int \frac{d}{dx} x^2 \int \cos 2x \, dx$$

We know that,

$$\int \cos 2x \, dx = \sin 2x \text{ and } \frac{d}{dx} x^2 = 2x$$

$$\text{Then, } = \frac{x^2}{2} \sin 2x - \int 2x \frac{\sin 2x \, dx}{2}$$

$$= \frac{x^2}{2} \sin 2x - \int x \sin 2x \, dx$$

Using integration by parts in $\int x \sin 2x \, dx$

$$= \frac{x^2}{2} \sin 2x - \left(x \int \sin 2x \, dx - \int \frac{d}{dx} x \int \sin 2x \, dx \right)$$

$$= \frac{x^2}{2} \sin 2x - \left(\frac{-x}{2} \cos 2x + \frac{1}{2} \int \cos 2x \, dx \right)$$

$$= \frac{x^2}{2} \sin 2x - \left(\frac{-x}{2} \cos 2x + \frac{1}{4} \sin 2x \right) + c$$

$$= \frac{x^2}{2} \sin 2x + \frac{x}{2} \cos 2x - \frac{1}{4} \sin 2x + c$$

9. Question

Evaluate the following integrals:

$$\int x \sin 2x \, dx$$

Answer

$$\text{Let } I = \int x \sin 2x \, dx$$

Using integration by parts,

$$= x \int \sin 2x \, dx - \int \frac{d}{dx} x \int \sin 2x \, dx$$

$$\text{We know that, } \int \sin nx = \frac{-\cos nx}{n} \text{ and } \int \cos nx = \frac{\sin nx}{n}$$

$$= \frac{x}{2} - \cos 2x + \int \frac{\cos 2x \, dx}{2}$$

$$= -\frac{x}{2} \cos 2x + \frac{1}{2} \frac{\sin 2x}{2} + c$$

$$= -\frac{x}{2} \cos 2x + \frac{1}{4} \sin 2x + c$$

10. Question

Evaluate the following integrals:

$$\int \frac{\log(\log x)}{x} \, dx$$

Answer

$$\text{Let } I = \int \frac{\log(\log x)}{x} dx$$

$$\text{It can be written as, } = \int \left(\frac{1}{x}\right) (\log(\log x)) dx$$

Using integration by parts,

$$I = \log(\log x) \int \frac{1}{x} dx - \int \left(\frac{1}{x \log x} \int \frac{1}{x} dx\right) dx$$

$$\text{We know that, } \int \log x = \frac{1}{x} \text{ and } \frac{d}{dx} \frac{1}{x} = \log x$$

$$= \log x (\log x) \times \log x - \int \frac{1}{x \log x} \times \log x dx$$

$$= \log x (\log x) \times \log x - \int \frac{1}{x} dx$$

$$= \log x (\log x) \times \log x - \log x + c$$

$$= \log x (\log(\log x) - 1) + c$$

11. Question

Evaluate the following integrals:

$$\int x^2 \cos x dx$$

Answer

$$\text{Let } I = \int x^2 \cos x dx$$

Using integration by parts,

$$= x^2 \int \cos x dx - \int \frac{d}{dx} x^2 \int \cos x dx$$

We know that,

$$\int \cos nx = \frac{\sin nx}{n}$$

$$= x^2 \sin x - \int 2x \sin x dx$$

$$= x^2 \sin x - 2 \int x \sin x dx$$

Using integration by parts in second integral,

$$= x^2 \sin x - 2 \left(x \int \sin x dx - \int \frac{d}{dx} x \int \sin x dx \right)$$

$$= x^2 \sin x - 2 \left(-x \cos x + \int \cos x dx \right)$$

$$= x^2 \sin x - 2(-x \cos x + \sin x) + c$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + c$$

12. Question

Evaluate the following integrals:

$$\int x \operatorname{cosec}^2 x dx$$

Answer

$$\text{Let } I = \int x \operatorname{cosec}^2 x \, dx$$

Using integration by parts,

$$I = x \int \operatorname{cosec}^2 x \, dx - \int \frac{d}{dx} x \int \operatorname{cosec}^2 x \, dx$$

We know that, $\int \operatorname{cosec}^2 x \, dx = -\cot x$ and $\int \cot x \, dx = \log |\sin x|$

$$= x \times -\cot x - \int -\cot x \, dx$$

$$= -x \cot x + \log |\sin x| + c$$

13. Question

Evaluate the following integrals:

$$\int x \cos^2 x \, dx$$

Answer

$$\text{Let } I = \int x \cos^2 x \, dx$$

Using integration by parts,

$$I = x \int \cos^2 x \, dx - \int \frac{d}{dx} x \int \cos^2 x \, dx$$

We know that, $\cos^2 x = \frac{\cos 2x + 1}{2}$

$$= x \int \left[\frac{\cos 2x + 1}{2} \right] dx - \int \left[1 \int \left[\frac{\cos 2x + 1}{2} \right] dx \right] dx$$

We know that,

$$\int \cos nx = \frac{\sin nx}{n}$$

$$= \frac{x}{2} \left[\frac{\sin 2x}{2} + x \right] - \frac{1}{2} \int \left(x + \frac{\sin 2x}{2} \right) dx$$

$$= \frac{x}{4} \sin 2x + \frac{x^2}{2} - \frac{1}{2} \times \frac{x^2}{2} - \frac{1}{4} \left(-\frac{\cos 2x}{2} \right) + c$$

$$I = \frac{x}{4} \sin 2x + \frac{x^2}{4} + \frac{1}{8} \cos 2x + c$$

14. Question

Evaluate the following integrals:

$$\int x^n \log x \, dx$$

Answer

$$\text{Let } I = \int x^n \log x \, dx$$

Using integration by parts,

$$I = \log x \int x^n \, dx - \int \frac{d}{dx} \log x \int x^n \, dx$$

We know that,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} \text{ and } \frac{d}{dx} \log x = \frac{1}{x}$$

$$= \log x \frac{x^{n+1}}{n+1} - \int \frac{1}{x} \times \frac{x^{n+1}}{n+1} dx$$

$$= \log x \frac{x^{n+1}}{n+1} - \int \frac{x^n}{n+1} dx$$

$$= \log x \frac{x^{n+1}}{n+1} - \frac{1}{n+1} \left[\int x^n dx \right]$$

We know that,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$= \log x \frac{x^{n+1}}{n+1} - \frac{1}{(n+1)^2} x^{n+1} + c$$

15. Question

Evaluate the following integrals:

$$\int \frac{\log x}{x^n} dx$$

Answer

$$\text{Let } I = \int \frac{\log x}{x^n} dx = \int \log x \frac{1}{x^n} dx$$

Using integration by parts,

$$\int \log x \frac{1}{x^n} dx = \log x \int \frac{1}{x^n} dx - \int \frac{d}{dx} \log x \int \frac{1}{x^n} dx$$

We know that,

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$= \log x \left(\frac{x^{1-n}}{1-n} \right) - \int \frac{1}{x} \left(\frac{x^{1-n}}{1-n} \right) dx$$

$$= \log x \left(\frac{x^{1-n}}{1-n} \right) - \int \left(\frac{x^{-n}}{1-n} \right) dx$$

$$= \log x \left(\frac{x^{1-n}}{1-n} \right) - \left(\frac{1}{1-n} \right) \left(= \log x \left(\frac{x^{1-n}}{1-n} \right) - \right)$$

$$= \log x \left(\frac{x^{1-n}}{1-n} \right) - \left(\frac{x^{1-n}}{(1-n)^2} \right) + c$$

16. Question

Evaluate the following integrals:

$$\int x^2 \sin^2 x dx$$

Answer

$$\text{Let } I = \int x^2 \sin^2 x dx$$

We know that,

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$= \int x^2 \left(\frac{1 - \cos 2x}{2} \right) dx$$

Using integration by parts,

$$= \int \frac{x^2}{2} dx - \int \frac{x^2 \cos 2x}{2} dx$$

$$= \frac{x^3}{6} - \frac{1}{2} \left[\int x^2 \cos 2x dx \right]$$

Using integration by parts in second integral,

$$= \frac{x^3}{6} - \frac{1}{2} \left[x^2 \int \cos 2x dx - \int \frac{d}{dx} x^2 \int \cos 2x dx \right]$$

$$= \frac{x^3}{6} - \frac{1}{2} \left(x^2 \frac{\sin 2x}{2} \right) + \frac{1}{2} \times 2 \int x \frac{\sin 2x}{2} dx$$

Using integration by parts again,

$$= \frac{x^3}{6} - \frac{1}{2} \left(x^2 \frac{\sin 2x}{2} \right) + \frac{1}{2} \left[x \int \sin 2x dx - \int \frac{d}{dx} x \int \sin 2x dx \right]$$

$$= \frac{x^3}{6} - \frac{1}{2} \left(x^2 \frac{\sin 2x}{2} \right) + \frac{1}{2} \left(\frac{x}{2} - \cos 2x + \int \frac{\cos 2x dx}{2} \right)$$

$$= \frac{x^3}{6} - \frac{1}{2} \left(x^2 \frac{\sin 2x}{2} \right) + \frac{1}{2} \left(-\frac{x}{2} \cos 2x + \frac{1}{2} \frac{\sin 2x}{2} \right) + c$$

$$= \frac{x^3}{6} - \frac{1}{4} (x^2 \sin 2x) - \frac{1}{4} x \cos 2x + \frac{1}{8} \sin 2x + c$$

17. Question

Evaluate the following integrals:

$$\int 2x^3 e^{x^2} dx$$

Answer

$$\text{Let } I = \int 2x^3 e^{x^2} dx$$

$$\text{Put } x^2 = t$$

$$2x dx = dt$$

$$I = \int t e^t dt$$

Using integration by parts,

$$= t \int e^t dt - \int \frac{d}{dt} t \int e^t dt$$

$$\text{We have, } \int e^x dx = e^x$$

$$= t e^t - e^t + c$$

$$= e^t (t - 1) + c$$

Substitute value for t,

$$I = e^{x^2} (x^2 - 1) + c$$

18. Question

Evaluate the following integrals:

$$\int x^3 \cos x^2 dx$$

Answer

$$\text{Let } I = \int x^3 \cos x^2 dx$$

$$\text{Put } x^2 = t$$

$$2x dx = dt$$

$$I = \frac{1}{2} \int t \cos t dt$$

Using integration by parts,

$$I = \frac{1}{2} \left(t \int \cos t dt - \int \frac{d}{dt} t \int \cos t dt \right)$$

$$= \frac{1}{2} \left(t \times \sin t - \int \sin t dt \right)$$

$$= \frac{1}{2} (t \sin t + \cos t) + c$$

Substitute value for t,

$$= \frac{1}{2} (x^2 \sin x^2 + \cos x^2) + c$$

19. Question

Evaluate the following integrals:

$$\int x \sin x \cos x dx$$

Answer

$$\text{Let } I = \int x \sin x \cos x dx = \frac{1}{2} \int x \times 2 \sin x \cos x dx$$

We know that, $\sin 2x = 2 \sin x \cos x$

$$= \frac{1}{2} \int x \sin 2x$$

Using integration by parts,

$$= \frac{1}{2} \left(x \int \sin 2x dx - \int \frac{d}{dx} x \int \sin 2x dx \right)$$

We have,

$$\int \sin nx = \frac{-\cos nx}{n} \text{ and } \int \cos nx = \frac{\sin nx}{n}$$

$$= \frac{1}{2} \left(\frac{x}{2} - \cos 2x + \int \frac{\cos 2x dx}{2} \right)$$

$$= \frac{1}{2} \left(-\frac{x}{2} \cos 2x + \frac{1}{2} \frac{\sin 2x}{2} \right) + c$$

$$= -\frac{x}{4} \cos 2x + \frac{1}{8} \sin 2x + c$$

20. Question

Evaluate the following integrals:

$$\int \sin x \log (\cos x) dx$$

Answer

$$\text{Let } I = \int \sin x \log(\cos x) dx$$

$$\text{Put } \cos x = t$$

$$-\sin x dx = dt$$

$$I = \int -\log t dt$$

Using integration by parts,

$$= \int 1 \times -\log t dt$$

$$= -\left(\log t \int dt - \int \frac{d}{dt} \log t \int 1 dt\right)$$

$$= -\left(t \log t - \int \frac{1}{t} \times t dt\right)$$

$$= -\left(t \log t - \int dt\right)$$

$$= -(t \log t - t) + c$$

$$= t(1 - \log t) + c$$

Replace t by $\cos x$

$$I = \cos x(1 - \log(\cos x)) + c$$

21. Question

Evaluate the following integrals:

$$\int (\log x)^2 x dx$$

Answer

$$\text{Let } I = \int (\log x)^2 x dx$$

Using integration by parts,

$$= (\log x)^2 \int x dx - \int \frac{d}{dx} (\log x)^2 \int x dx$$

$$= (\log x)^2 \frac{x^2}{2} - \int \left(2(\log x) \left(\frac{1}{x}\right) \int x dx\right) dx$$

$$= \frac{x^2}{2} (\log x)^2 - 2 \int (\log x) \left(\frac{1}{x}\right) \left(\frac{x^2}{2}\right) dx$$

$$= \frac{x^2}{2} (\log x)^2 - \int x \log x dx$$

Using integration by integration by parts in second integral,

$$= \frac{x^2}{2} (\log x)^2 - \left[\log x \int x dx - \int \frac{d}{dx} \log x \int x dx \right]$$

$$\text{We know that, } \int x dx = \frac{x^2}{2} \text{ and } \frac{d}{dx} \log x = \frac{1}{x}$$

$$= \frac{x^2}{2} (\log x)^2 - \log x \frac{x^2}{2} - \int \frac{1}{x} \times \frac{x^2}{2}$$

$$= \frac{x^2}{2} (\log x)^2 - \log x \frac{x^2}{2} - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2}{2} (\log x)^2 - \log x \frac{x^2}{2} - \frac{1}{2} \frac{x^2}{2} + c$$

$$= \frac{x^2}{2} (\log x)^2 - \log x \frac{x^2}{2} - \frac{x^2}{4} + c$$

$$I = \frac{x^2}{2} \left[(\log x)^2 - \log x - \frac{1}{2} \right] + c$$

22. Question

Evaluate the following integrals:

$$\int e^{\sqrt{x}} \, dx$$

Answer

$$\text{Let } I = \int e^{\sqrt{x}} \, dx$$

$$\sqrt{x} = t; x = t^2$$

$$dx = 2t \, dt$$

$$I = 2 \int e^t t \, dt$$

Using integration by parts,

$$I = 2 \left(t \int e^t \, dt - \int \frac{d}{dt} t \int e^t \, dt \right)$$

$$= 2 \left(te^t - \int e^t \, dt \right)$$

$$= 2(te^t - e^t) + c$$

$$= 2e^t(t - 1) + c$$

Replace the value of t

$$= 2e^{\sqrt{x}}(\sqrt{x} - 1) + c$$

23. Question

Evaluate the following integrals:

$$\int \frac{\log(x+2)}{(x+2)^2} \, dx$$

Answer

$$\text{Let } I = \int \frac{\log(x+2)}{(x+2)^2} \, dx$$

$$\frac{1}{x+2} = t$$

$$\frac{-1}{(x+2)^2} \, dx = dt$$

$$I = - \int \log\left(\frac{1}{t}\right) dt$$

Using integration by parts,

$$= - \int \log t^{-1} dt$$

$$= - \int 1 \times \log t^{-1} dt$$

We know that, $\frac{d}{dt} \log t = \frac{1}{t}$ and $\int dt = t$

$$I = \log t \int dt - \int \left(\frac{d}{dt} \log t \int dt \right) dt$$

$$= \log t \int dt - \int \left(\frac{1}{t} \int dt \right) dt$$

$$= t \log t - \int \frac{1}{t} \times t dt$$

$$= t \log t - t + c$$

Replace the value of t,

$$= \frac{1}{x+2} (\log(x+2)^{-1} - 1) + c$$

$$= -\frac{1}{x+2} - \frac{\log(x+2)}{x+2} + c$$

24. Question

Evaluate the following integrals:

$$\int \frac{x + \sin x}{1 + \cos x} dx$$

Answer

$$\text{Let } I = \int \frac{x + \sin x}{1 + \cos x} dx$$

$1 + \cos x$ can be written as, $2 \cos^2 \frac{x}{2}$ and $\sin x$ can be written as $2 \sin \frac{x}{2} \cos \frac{x}{2}$

$$= \int \frac{x}{2 \cos^2 \frac{x}{2}} dx + \int \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} dx$$

$$= \frac{1}{2} \int x \sec^2 \frac{x}{2} + \int \tan \frac{x}{2} dx$$

Using integration by parts,

$$= \frac{1}{2} \left[x \int \sec^2 \frac{x}{2} - \int \frac{d}{dx} x \int \sec^2 \frac{x}{2} dx \right] + \int \tan \frac{x}{2} dx$$

$$= \frac{1}{2} \left[2x \tan \frac{x}{2} - 2 \int \tan \frac{x}{2} dx \right] + \int \tan \frac{x}{2} dx$$

$$= x \tan \frac{x}{2} - \int \tan \frac{x}{2} dx + \int \tan \frac{x}{2} dx$$

$$= x \tan \frac{x}{2} + c$$

25. Question

Evaluate the following integrals:

$$\int \log_{10} x \, dx$$

Answer

$$\text{Let } I = \int \log_{10} x \, dx$$

$$= \int \frac{\log x}{\log 10} \, dx$$

$$= \frac{1}{\log 10} \int 1 \times \log x \, dx$$

Using integration by parts,

$$= \frac{1}{\log 10} \left(\log x \int dx - \int \frac{d}{dx} \log x \int 1 \, dx \right)$$

$$\text{We know that } \frac{d}{dx} \log x = \frac{1}{x}$$

$$= \frac{1}{\log 10} \left(x \log x - \int \frac{1}{x} \times x \, dx \right)$$

$$= \frac{1}{\log 10} \left(x \log x - \int dx \right)$$

$$= \frac{1}{\log 10} (x \log x - x) + c$$

$$= \frac{x}{\log 10} (1 - \log x) + c$$

26. Question

Evaluate the following integrals:

$$\int \cos \sqrt{x} \, dx$$

Answer

$$\text{Let } I = \int \cos \sqrt{x} \, dx$$

$$\sqrt{x} = t; x = t^2$$

$$dx = 2t \, dt$$

$$= \int 2t \cos t \, dt$$

$$I = 2 \int t \cos t \, dt$$

Using integration by parts,

$$I = 2 \left(t \int \cos t \, dt - \int \frac{d}{dt} t \int \cos t \, dt \right)$$

$$= 2 \left(t \times \sin t - \int \sin t \, dt \right)$$

$$= 2(t \sin t + \cos t) + c$$

$$\text{Replace the value of } t, I = 2(\sqrt{x} \sin \sqrt{x} + \cos \sqrt{x}) + c$$

27. Question

Evaluate the following integrals:

$$\int \frac{x \cos^{-1} x}{\sqrt{1-x^2}} dx$$

Answer

$$\text{Let } I = \int \frac{x \cos^{-1} x}{\sqrt{1-x^2}} dx$$

$$\text{Let } t = \cos^{-1} x$$

$$dt = \frac{1}{\sqrt{1-x^2}} dx$$

Also,

$$\cos t = x$$

Thus,

$$I = - \int t \cos t dt$$

Now let us solve this by 'by parts' method

Using integration by parts,

$$I = -t \left(\int \cos t dt - \int \frac{d}{dt} t \int \cos t dt \right)$$

Let

$$U=t; du=dt$$

$$\int \cos t dt = v; \sin t = dv$$

Thus,

$$I = - \left[t \sin t - \int \sin t dt \right]$$

$$I = -[t \sin t + \cos t] + c$$

Substituting

$$t = \cos^{-1} x$$

$$I = -[\cos^{-1} x \sin t + x] + c$$

$$I = -[\cos^{-1} x \sqrt{1-x^2} + x] + c$$

28. Question

Evaluate the following integrals:

$$\int \frac{\log x}{(x+1)^2} dx$$

Answer

We know that integration by parts is given by:

$$\int UV = U \int V dv - \int \frac{d}{dx} U \int V dv$$

Choosing $\log x$ as first function and $\frac{1}{(x+1)^2}$ as second function we get,

$$\int \frac{\log x}{(x+1)^2} dx = \log x \int \left(\frac{1}{(x+1)^2} \right) dx - \int \left(\frac{d}{dx} (\log x) \right) \int \frac{1}{(x+1)^2} dx \, dx$$

$$\int \frac{\log x}{(x+1)^2} dx = \log x \left(-\frac{1}{x+1} \right) + \int \frac{1}{x} \left(\frac{1}{x+1} \right) dx$$

$$\int \frac{\log x}{(x+1)^2} dx = -\frac{\log x}{x+1} + \int \frac{(x+1) - (x)}{x(x+1)} dx$$

$$\int \frac{\log x}{(x+1)^2} dx = -\frac{\log x}{x+1} + \int \left(\frac{1}{x} - \frac{1}{x+1} \right) dx$$

$$\int \frac{\log x}{(x+1)^2} dx = -\frac{\log x}{x+1} + \log x - \log(x+1) + c$$

$$\int \frac{\log x}{(x+1)^2} dx = -\frac{\log x}{x+1} + \log \left(\frac{x}{x+1} \right) + c$$

29. Question

Evaluate the following integrals:

$$\int \operatorname{cosec}^3 x \, dx$$

Answer

$$\text{Let } I = \int \operatorname{cosec}^3 x \, dx$$

$$= \int \operatorname{cosec} x \times \operatorname{cosec}^2 x \, dx$$

Using integration by parts,

$$= \operatorname{cosec} x \int \operatorname{cosec}^2 x \, dx - \int \frac{d}{dx} \operatorname{cosec} x \int \operatorname{cosec}^2 x \, dx$$

We know that, $\int \operatorname{cosec}^2 x \, dx = -\cot x$ and $\frac{d}{dx} \operatorname{cosec} x = \operatorname{cosec} x \cot x$

$$= \operatorname{cosec} x \times -\cot x + \int \operatorname{cosec} x \cot x \times -\cot x \, dx$$

$$= -\operatorname{cosec} x \cot x + \int \operatorname{cosec} x \cot^2 x \, dx$$

Using integration by parts,

$$= -\operatorname{cosec} x \cot x + \int \operatorname{cosec} x (\operatorname{cosec}^2 x - 1) \, dx$$

$$= -\operatorname{cosec} x \cot x + \int \operatorname{cosec}^3 x \, dx - \int \operatorname{cosec} x \, dx$$

$$I = -\operatorname{cosec} x \cot x - I + \log \left| \tan \frac{x}{2} \right| + c_1$$

$$2I = -\operatorname{cosec} x \cot x + \log \left| \tan \frac{x}{2} \right| + c_1$$

$$I = -\frac{1}{2} \operatorname{cosec} x \cot x + \frac{1}{2} \log \left| \tan \frac{x}{2} \right| + c_1$$

30. Question

Evaluate the following integrals:

$$\int \sec^{-1} \sqrt{x} \, dx$$

Answer

$$\text{Let } I = \int \sec^{-1} \sqrt{x} \, dx$$

$$\sqrt{x} = t; x = t^2$$

$$dx = 2t \, dt$$

$$I = \int 2t \sec^{-1} t \, dt$$

Using integration by parts,

$$= 2 \left[\sec^{-1} t \int t \, dt - \int \frac{d}{dt} \sec^{-1} t \int t \, dt \right]$$

$$\text{We know that, } \frac{d}{dt} \sec^{-1} t = \frac{1}{t\sqrt{t^2-1}}$$

$$= 2 \left[\frac{t^2}{2} \sec^{-1} t - \int \frac{1}{t\sqrt{t^2-1}} \int t \, dt \right]$$

$$= 2 \left[\frac{t^2}{2} \sec^{-1} t - \int \frac{t^2}{2t\sqrt{t^2-1}} \, dt \right]$$

$$= t^2 \sec^{-1} t - \int \frac{t}{t\sqrt{t^2-1}} \, dt$$

$$= t^2 \sec^{-1} t - \frac{1}{2} \int \frac{2t}{\sqrt{t^2-1}} \, dt$$

$$= t^2 \sec^{-1} t - \frac{1}{2} \times 2\sqrt{t^2-1} + c$$

Substitute value for t,

$$I = x \sec^{-1} \sqrt{x} - \sqrt{x-1} + c$$

31. Question

Evaluate the following integrals:

$$\int \sin^{-1} \sqrt{x} \, dx$$

Answer

$$\text{Let } I = \int \sin^{-1} \sqrt{x} \, dx$$

$$\sqrt{x} = t; x = t^2$$

$$dx = 2t \, dt$$

$$= \int \sin^{-1} t \, 2t \, dt$$

Using integration by parts,

$$= \sin^{-1} t \int 2t \, dt - \int \frac{d}{dt} \sin^{-1} t \int 2t \, dt$$

$$\text{We know that, } \frac{d}{dt} \sin^{-1} t = \frac{t}{\sqrt{1-t^2}}$$

$$= t^2 \sin^{-1} t - 2 \int \frac{t^2}{\sqrt{1-t^2}} dt$$

$$\text{let us solve, } \int \frac{t^2}{\sqrt{1-t^2}} dt$$

$$= \int \frac{t^2 - 1 + 1}{\sqrt{1-t^2}} dt = \int \frac{t^2 - 1}{\sqrt{1-t^2}} dt + \int \frac{1}{\sqrt{1-t^2}} dt$$

$$\int \frac{1}{\sqrt{1-t^2}} dt = \sin^{-1} t$$

$$\int \frac{t^2 - 1}{\sqrt{1-t^2}} dt = \int -\sqrt{1-t^2} dt$$

$$t = \sin u; dt = \cos u du$$

$$\int -\sqrt{1-t^2} dt = \int -\cos^2 u du = - \int \left[\frac{1 + \cos 2u}{2} \right] du$$

$$= -\frac{u}{2} - \frac{\sin 2u}{4}$$

$$u = \sin^{-1} t \text{ and } t = \sqrt{x}$$

$$= -\frac{\sin^{-1} t}{2} - \frac{\sin(2\sin^{-1} t)}{4}$$

$$\text{There fore, } \int \sin^{-1} \sqrt{x} dx = x \sin^{-1} \sqrt{x} - \frac{\sin^{-1} \sqrt{x}}{2} - \frac{\sin(2\sin^{-1} t)}{4}$$

$$= x \sin^{-1} \sqrt{x} - \frac{\sin^{-1} \sqrt{x}}{2} - \frac{\sqrt{x(1-x)}}{2}$$

32. Question

Evaluate the following integrals:

$$\int x \tan^2 x dx$$

Answer

$$\text{Let } I = \int x \tan^2 x dx$$

$$= \int x (\sec^2 x - 1) dx$$

$$= \int x \sec^2 x dx - \int x dx$$

Using integration by parts,

$$= x \int \sec^2 x dx - \int \frac{d}{dx} x \int \sec^2 x dx - \frac{x^2}{2}$$

$$\text{We know that, } \int \sec^2 x dx = \tan x$$

$$= x \tan x - \int \tan x dx - \frac{x^2}{2}$$

$$= x \tan x - \log |\sec x| - \frac{x^2}{2} + c$$

33. Question

Evaluate the following integrals:

$$\int x \left(\frac{\sec 2x - 1}{\sec 2x + 1} \right) dx$$

Answer

Let $I = \int x \left(\frac{\sec 2x - 1}{\sec 2x + 1} \right) dx$ it can be written in terms of $\cos x$

$$= \int x \left(\frac{1 - \cos 2x}{1 + \cos 2x} \right) dx$$

$$= \int x \left(\frac{\sec^2 x}{\cos^2 x} \right) dx$$

$$= \int x \tan^2 x dx$$

$$= \int x (\sec^2 x - 1) dx$$

$$= \int x \sec^2 x - \int x dx$$

Using integration by parts,

$$= x \int \sec^2 x dx - \int \frac{d}{dx} x \int \sec^2 x dx - \frac{x^2}{2}$$

$$= x \tan x - \int \tan x dx - \frac{x^2}{2}$$

$$= x \tan x - \log |\sec x| - \frac{x^2}{2} + c$$

34. Question

Evaluate the following integrals:

$$\int (x + 1)e^x \log(xe^x) dx$$

Answer

$$\text{Let } I = \int (x + 1)e^x \log(xe^x) dx$$

$$xe^x = t$$

$$(1 \times e^x + xe^x) dx = dt$$

$$(x + 1)e^x dx = dt$$

$$I = \int \log t dt$$

$$= \int 1 \times \log t dt$$

Using integration by parts,

$$= \log t \int dt - \int \frac{d}{dt} \log t \int dt$$

$$= t \log t - \int \frac{1}{t} t dt$$

$$= t \log t - t + c$$

$$= t(\log t - 1) + c$$

Substitute value for t,

$$I = xe^x(\log xe^x - 1) + c$$

35. Question

Evaluate the following integrals:

$$\int \sin^{-1}(3x - 4x^3) dx$$

Answer

$$\text{Let } \int \sin^{-1}(3x - 4x^3) dx$$

$$x = \sin \theta$$

$$dx = \cos \theta d\theta$$

$$= \int \sin^{-1}(3 \sin \theta - 4 \sin^3 \theta) \cos \theta d\theta$$

$$\text{We know that } 3 \sin \theta - 4 \sin^3 \theta = \sin 3\theta$$

$$= \int \sin^{-1}(\sin 3\theta) \cos \theta d\theta$$

$$\text{We know that, } \int \sin^{-1}(\sin 3\theta) = 3\theta$$

$$= \int 3\theta \cos \theta d\theta$$

$$= 3 \int \theta \cos \theta d\theta$$

Using integration by parts,

$$= 3 \left(\theta \int \cos \theta d\theta - \int \frac{d}{d\theta} \theta \int \cos \theta d\theta \right)$$

$$= 3 \left(\theta \times \sin \theta - \int \sin \theta d\theta \right)$$

$$= 3(\theta \sin \theta + \cos \theta) + c$$

$$I = 3 \left[x \sin^{-1} x + \sqrt{1 - x^2} \right] + c$$

36. Question

Evaluate the following integrals:

$$\int \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$$

Answer

$$\text{Let } I = \int \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$$

$$x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$$

$$\sin^{-1} \left(\frac{2x}{1+x^2} \right) = \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx = \int 2\theta \sec^2\theta d\theta$$

Using integration by parts,

$$= 2\left(\theta \int \sec^2\theta d\theta - \int \frac{d}{d\theta}\theta \int \sec^2\theta d\theta\right)$$

$$= 2\left(\theta \tan \theta - \int \tan \theta d\theta\right)$$

We know that, $\int \tan \theta d\theta = \log|\cos \theta|$

$$= 2(\theta \tan \theta - \log|\cos \theta|) + c$$

$$= 2\left[x \tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + c$$

$$= 2x \tan^{-1}x + 2\log\left|(1+x^2)^{-\frac{1}{2}}\right| + c$$

$$= 2x \tan^{-1}x + 2\left[\frac{1}{2}\log(1+x)^2\right] + c$$

$$= 2x \tan^{-1}x + \log(1+x)^2 + c$$

37. Question

Evaluate the following integrals:

$$\int \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) dx$$

Answer

$$\text{Let } I = \int \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) dx$$

$$x = \tan \theta \Rightarrow dx = \sec^2\theta d\theta$$

We know that, $\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = \tan 3\theta$

$$I = \int \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right) \sec^2\theta d\theta$$

We know that, $\tan^{-1}(\tan 3\theta) = 3\theta$

$$= \int \tan^{-1}(\tan 3\theta) \sec^2\theta d\theta$$

$$= \int 3\theta \sec^2\theta d\theta$$

Using integration by parts,

$$= 3\left(\theta \int \sec^2\theta d\theta - \int \frac{d}{d\theta}\theta \int \sec^2\theta d\theta\right)$$

$$= 3\left(\theta \tan \theta - \int \tan \theta d\theta\right)$$

$$= 3(\theta \tan \theta - \log|\sec \theta|) + c$$

$$= 3\left[x \tan^{-1}x + \log\left|\sqrt{1+x^2}\right|\right] + c$$

$$= 3x \tan^{-1} x + \frac{3}{2} \log|1 + x^2| + c$$

38. Question

Evaluate the following integrals:

$$\int x^2 \sin^{-1} x \, dx$$

Answer

$$\text{Let } I = \int x^2 \sin^{-1} x \, dx$$

Using integration by parts,

$$I = \sin^{-1} x \int x^2 dx - \int \frac{d}{dx} \sin^{-1} x \int x^2 dx$$

$$= \frac{x^3}{3} \sin^{-1} x - \int \frac{x^3}{3\sqrt{1-x^2}} dx$$

$$I = \frac{x^3}{3} \sin^{-1} x - \int I_1 + C \text{----- (1)}$$

$$I_1 = - \int \frac{x^3}{3\sqrt{1-x^2}} dx$$

$$\text{Let } 1-x^2=t^2$$

$$-2x \, dx = 2t \, dt$$

$$-x \, dx = t \, dt$$

$$I_1 = - \int \frac{(1-t^2)t \, dt}{t}$$

$$I_1 = \int (t^2 - 1) dt$$

$$= \frac{t^3}{3} - t + c_2$$

$$= \frac{(1-x^2)^{\frac{3}{2}}}{3} - (1-x^2)^{\frac{1}{2}} + c_2$$

$$= \frac{x^3}{3} \sin^{-1} x - \frac{(1-x^2)^{\frac{3}{2}}}{9} + \frac{1}{3} (1-x^2)^{\frac{1}{2}} + c$$

39. Question

Evaluate the following integrals:

$$\int \frac{\sin^{-1} x}{x^2} dx$$

Answer

$$\text{Let } I = \int \frac{\sin^{-1} x}{x^2} dx$$

$$= \int \frac{1}{x^2} \sin^{-1} x \, dx$$

Using integration by parts,

$$I = \left[\sin^{-1}x \times \int \frac{1}{x^2} - \int \left(\frac{1}{\sqrt{1-x^2}} \right) \int \frac{1}{x^2} dx \right] dx$$

$$= \sin^{-1}x \left(\frac{-1}{x} \right) - \int \frac{1}{\sqrt{1-x^2}} \left(\frac{-1}{x} \right) dx$$

$$I = \frac{-1}{x} \sin^{-1}x + \int \frac{1}{x\sqrt{1-x^2}} dx$$

$$I = \frac{-1}{x} \sin^{-1}x + I_1 \text{-----(1)}$$

Where,

$$I_1 = \int \frac{1}{x\sqrt{1-x^2}}$$

$$1-x^2 = t^2$$

$$-2x dx = 2t dt$$

$$I_1 = \int \frac{t dt}{(1-t^2)\sqrt{t}}$$

$$= \frac{1}{2} \log \left| \frac{t-1}{t+1} \right|$$

$$= \frac{1}{2} \log \left| \frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1} \right| + c_1$$

$$I = \frac{-1}{x} \sin^{-1}x + \frac{1}{2} \log \left| \frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1} \right| + c$$

$$= \frac{-1}{x} \sin^{-1}x + \frac{1}{2} \log \left(\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1} \right) \left(\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}-1} \right) + c$$

$$= \frac{-1}{x} \sin^{-1}x + \frac{1}{2} \log \left(\frac{(\sqrt{1-x^2}-1)^2}{-x^2} \right) + c$$

$$= \frac{-1}{x} \sin^{-1}x + \log \left| \frac{1-\sqrt{1-x^2}}{x} \right| + c$$

40. Question

Evaluate the following integrals:

Answer

$$\text{Let } I = \int \frac{x^2 \tan^{-1}x}{1+x^2} dx$$

$$\tan^{-1}x = t; x = \tan t \int \frac{x^2 \tan^{-1}x}{1+x^2} dx$$

$$\frac{1}{1+x^2} dx = dt$$

$$I = \int t \tan^2 t dt$$

$$\text{We know that, } \tan^2 t = \sec^2 t - 1$$

$$= \int t(\sec^2 t - 1) dt$$

$$= \int t \sec^2 t dt - \int t dt$$

Using integration by parts,

$$= \left(t \int \sec^2 t dt - \int \frac{d}{dt} t \int \sec^2 t dt \right) - \frac{t^2}{2}$$

$$= \left(t \tan t - \int \tan t dt \right) - \frac{t^2}{2}$$

$$= (t \tan t - \log|\sec t|) - \frac{t^2}{2} + c$$

$$= \left[x \tan^{-1} x + \log|\sqrt{1+x^2}| \right] - \frac{\tan^2 x}{2} + c$$

$$= x \tan^{-1} x + \frac{1}{2} \log|1+x^2| - \frac{\tan^2 x}{2} + c$$

41. Question

Evaluate the following integrals:

$$\int \cos^{-1}(4x^3 - 3x) dx$$

Answer

$$\text{Let } I = \int \cos^{-1}(4x^3 - 3x) dx$$

$$x = \cos \theta \Rightarrow dx = -\sin \theta d\theta$$

$$I = - \int \cos^{-1}(4\cos^3 \theta - 3\cos \theta) \sin \theta d\theta$$

$$\text{We know that, } (4\cos^3 \theta - 3\cos \theta) = \cos 3\theta$$

$$= - \int \cos^{-1}(\cos 3\theta) \sin \theta d\theta$$

$$= - \int 3\theta \sin \theta d\theta$$

Using integration by parts,

$$= -3 \left[\theta \int \sin \theta d\theta - \int \frac{d}{d\theta} \theta \int \sin \theta d\theta \right]$$

$$= 3[-\theta \cos \theta + \int \cos \theta d\theta]$$

$$= 3\theta \cos \theta - 3\sin \theta + c$$

$$I = 3x \cos^{-1} x - 3\sqrt{1-x^2} + c$$

42. Question

Evaluate the following integrals:

$$\int \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) dx$$

Answer

$$\text{Let } I = \int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$$

$$\text{Let } x = \tan t$$

$$dx = \sec^2 t \, dt$$

$$I = \int \cos^{-1}\left(\frac{1 - \tan^2 t}{1 + \tan^2 t}\right) \sec^2 t \, dt$$

$$\text{We know that } \frac{1 - \tan^2 t}{1 + \tan^2 t} = \cos 2t$$

$$= \int \cos^{-1}(\cos 2t) \sec^2 t \, dt$$

$$= \int 2t \sec^2 t \, dt$$

Using integration by parts,

$$= 2\left[t \int \sec^2 t \, dt - \int \frac{d}{dt} t \int \sec^2 t \, dt\right]$$

$$= 2\left[t \tan t - \int \tan t \, dt\right]$$

$$= 2[t \tan t - \log \sec t] + c$$

$$= 2[x \tan^{-1} x - \log |\sqrt{1+x^2}|] + c$$

$$= 2x \tan^{-1} x - \log |1+x^2| + c$$

43. Question

Evaluate the following integrals:

$$\int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$$

Answer

$$\text{Let } I = \int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$$

$$x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$$

$$I = \int \tan^{-1}\left(\frac{2 \tan \theta}{1 - 2 \tan^2 \theta}\right) \sec^2 \theta d\theta$$

$$\text{We know that, } \frac{2 \tan \theta}{1 - 2 \tan^2 \theta} = \tan 2 \theta$$

$$= \int \tan^{-1}(\tan 2 \theta) \sec^2 \theta d\theta$$

$$\int 2\theta \sec^2 \theta d\theta$$

Using integration by parts,

$$= 2\left(\theta \int \sec^2 \theta d\theta - \int \frac{d}{d\theta} \theta \int \sec^2 \theta d\theta\right)$$

$$= 2\left(\theta \tan \theta - \int \tan \theta \, d\theta\right)$$

$$\begin{aligned}
&= 2(\theta \tan \theta - \log|\sec \theta|) + c \\
&= 2 \left[x \tan^{-1} x + \log \left| \sqrt{1+x^2} \right| \right] + c \\
&= 2x \tan^{-1} x + \log|1+x^2| + c
\end{aligned}$$

44. Question

Evaluate the following integrals:

$$\int (x+1) \log x \, dx$$

Answer

$$\text{Let } I = \int (x+1) \log x \, dx$$

Using integration by parts,

$$= \log x \int (x+1) dx - \int \frac{d}{dx} \log x \int (x+1) dx$$

$$\text{We know that, } \frac{d}{dx} \log x = \frac{1}{x}$$

$$= \log x \left(\frac{x^2}{2} + x \right) - \int \frac{1}{x} \left(\frac{x^2}{2} + x \right) dx$$

$$= \left(\frac{x^2}{2} + x \right) \log x - \int \frac{x}{2} dx - \int dx$$

$$= \left(\frac{x^2}{2} + x \right) \log x - \frac{x^2}{4} - x + c$$

$$= \left(\frac{x^2}{2} + x \right) \log x - \left(\frac{x^2}{4} + x \right) + c$$

45. Question

Evaluate the following integrals:

$$\int x^2 \tan^{-1} x \, dx$$

Answer

$$\text{Let } I = \int x^2 \tan^{-1} x \, dx$$

Using integration by parts,

Taking inverse function as first function and algebraic function as second function,

$$= \tan^{-1} x \int x^2 dx - \int \left(\frac{1}{1+x^2} \right) \int x^2 dx$$

$$= \tan^{-1} x \frac{x^3}{3} - \frac{1}{3} \int \frac{x^3}{1+x^2} dx$$

$$= \tan^{-1} x \frac{x^3}{3} - \frac{1}{3} \int x - \frac{x}{1+x^2} dx$$

$$= \tan^{-1} x \frac{x^3}{3} - \frac{1}{3} \times \frac{x^2}{2} + \int \frac{x}{1+x^2} dx$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \log|1+x^2| + c$$

46. Question

Evaluate the following integrals:

$$\int (e^{\log x} + \sin x) \cos x \, dx$$

Answer

$$\text{Let } I = \int (e^{\log x} + \sin x) \cos x \, dx$$

$$= \int (x + \sin x) \cos x \, dx$$

$$= \int x \cos x \, dx + \int \sin x \cos x \, dx$$

Using integration by parts,

$$= x \int \cos x \, dx - \int \frac{d}{dx} x \int \cos x \, dx + \frac{1}{2} \int \sin 2x \, dx$$

$$= x \times \sin x - \int \sin x \, dx + \frac{1}{2} \left(\frac{-\cos 2x}{2} \right) + c$$

$$= x \sin x + \cos x - \frac{1}{4} \cos 2x + c$$

$$= x \sin x + \cos x - \frac{1}{4} [1 - 2\sin^2 x] + c$$

$$I = x \sin x + \cos x - \frac{1}{4} + \frac{1}{2} \sin^2 x + c$$

$$I = x \sin x + \cos x + \frac{1}{2} \sin^2 x + c - \frac{1}{4}$$

$$I = x \sin x + \cos x + \frac{1}{2} \sin^2 x + k \text{ where, } k = c - \frac{1}{4}$$

47. Question

Evaluate the following integrals:

$$\int \frac{(x \tan^{-1} x)}{(1+x^2)^{3/2}} dx$$

Answer

$$\text{Let } I = \int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx$$

$$\tan^{-1} x = t$$

$$\frac{1}{1+x^2} dx = dt$$

$$I = \int \frac{t \tan t}{\sqrt{1+\tan^2 t}} dt$$

We know that, $\sqrt{1+\tan^2 t} = \sec t$

$$= \int \frac{t \tan t}{\sec t} dt$$

$$= \int t \frac{\sin t}{\cos t} \cos t dt$$

$$= \int t \sin t \, dt$$

Using integration by parts,

$$= t \int \sin t \, dt - \int \frac{d}{dt} t \int \sin t \, dt$$

$$= -t \cos t + \int \cos t \, dt$$

$$= -t \cos t + \sin t + c$$

Substitute value for t

$$I = \frac{\tan^{-1} x}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} + c$$

48. Question

Evaluate the following integrals:

$$\int \tan^{-1}(\sqrt{x}) \, dx$$

Answer

$$\text{Let } I = \int \tan^{-1}(\sqrt{x}) \, dx$$

$$x = t^2$$

$$dx = 2t \, dt$$

$$I = \int 2t \tan^{-1} t \, dt$$

Using integration by parts,

$$= 2 \left(\tan^{-1} t \int t \, dt - \int \frac{d}{dt} \tan^{-1} t \int t \, dt \right)$$

We know that,

$$\frac{d}{dt} \tan^{-1} t = \frac{1}{1+t^2}$$

$$= 2 \left[\frac{t^2}{2} \tan^{-1} t - \int \frac{t^2}{2(1+t^2)} \, dt \right]$$

$$= t^2 \tan^{-1} t - \int \frac{t^2 + 1 - 1}{1+t^2} \, dt$$

$$= t^2 \tan^{-1} t - \int \left(1 - \frac{1}{1+t^2} \right) \, dt$$

$$= t^2 \tan^{-1} t - t + \tan^{-1} t + c$$

$$= (t^2 + 1) \tan^{-1} t - t + c$$

$$= (x + 1) \tan^{-1} \sqrt{x} - \sqrt{x} + c$$

49. Question

Evaluate the following integrals:

$$\int x^3 \tan^{-1} x \, dx$$

Answer

$$\text{Let } I = \int x^3 \tan^{-1} x \, dx$$

Using integration by parts,

We know that,

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

$$= \tan^{-1} x \int x^3 dx - \int \left(\frac{1}{1+x^2} \right) \int x^3 dx$$

$$= \tan^{-1} x \frac{x^4}{4} - \frac{1}{4} \int \frac{x^4}{1+x^2} dx$$

$$\frac{1}{4} \int \frac{x^4}{1+x^2} dx = \frac{1}{4} \left[\int \frac{1}{1+x^2} dx + (x^2-1) dx \right] = \frac{1}{4} \left[\tan^{-1} x + \frac{x^3}{3} - x \right]$$

$$= \frac{x^4}{4} \tan^{-1} x - \frac{1}{4} \left[\tan^{-1} x + \frac{x^3}{3} - x \right] + c$$

50. Question

Evaluate the following integrals:

$$\int x \sin x \cos 2x \, dx$$

Answer

$$\text{Let } I = \int x \sin x \cos 2x \, dx = \frac{1}{2} \int x \times 2 \sin x \cos 2x \, dx$$

Using integration by parts,

$$= \frac{1}{2} \int x (\sin(x+2x) - \sin(2x-x)) dx$$

$$= \frac{1}{2} \int x (\sin 3x - \sin x) dx$$

Using integration by parts,

$$= \frac{1}{2} \left(x \int (\sin 3x - \sin x) dx - \int \frac{d}{dx} x \int (\sin 3x - \sin x) dx \right)$$

$$= \frac{1}{2} \left[x \left(-\frac{\cos 3x}{3} + \cos x \right) - \int - \left(\frac{\cos 3x}{3} + \cos x \right) dx \right]$$

$$I = \frac{1}{2} \left[-x \frac{\cos 3x}{3} + x \cos x + \frac{1}{9} \sin 3x - \sin x \right] + c$$

51. Question

Evaluate the following integrals:

$$\int (\tan^{-1} x^2) x \, dx$$

Answer

$$\text{Let } I = \int (\tan^{-1} x^2) x \, dx$$

$$x^2 = t$$

$$2x dx = dt$$

$$I = \frac{1}{2} \int (\tan^{-1} t) dt$$

Using integration by parts,

$$= \frac{1}{2} \left(\tan^{-1} t \int dt - \int \frac{d}{dt} \tan^{-1} t \int dt \right)$$

We know that,

$$\frac{d}{dt} \tan^{-1} t = \frac{1}{2(1+t^2)}$$

$$= \frac{1}{2} \left[t \tan^{-1} t - \int \frac{t}{(1+t^2)} dt \right]$$

$$= \frac{t}{2} \tan^{-1} t - \frac{1}{4} \int \frac{2t}{1+t^2} dt$$

$$= \frac{t}{2} \tan^{-1} t - \frac{1}{4} \log|1+t^2| + c$$

$$= \frac{x^2}{2} \tan^{-1} x^2 - \frac{1}{4} \log|1+x^4| + c$$

52. Question

Evaluate the following integrals:

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$$

Answer

$$\text{Let } I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$$

We are splitting this in to two functions

First we find the integral of:

$$\int \frac{x}{\sqrt{1-x^2}} dx$$

$$\text{Put } 1-x^2=t$$

$$-2x dx = dt$$

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$$

$$I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$$

Using integration by parts,

$$= (\sin^{-1} x) \times -\sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}} (-\sqrt{1-x^2}) dx$$

$$= (\sin^{-1} x) \times -\sqrt{1-x^2} - \int dx$$

$$= (\sin^{-1} x) \times -\sqrt{1-x^2} + x + c$$

$$= x - \sqrt{1-x^2} (\sin^{-1} x) + c$$

53. Question

Evaluate the following integrals:

$$\int \sin^3 \sqrt{x} \, dx$$

Answer

Let

$$\sqrt{x} = t$$

$$x = t^2$$

$$dx = 2t \, dt$$

$$I = 2 \int t \sin^3 t \, dt$$

$$= 2 \int t \left(\frac{3 \sin t - \sin 3t}{4} \right) dt$$

$$= \frac{1}{2} \int t(3 \sin t - \sin 3t) dt$$

Using integration by parts,

$$= \frac{1}{2} \left[t \left(-3 \cos t + \frac{1}{3} \cos 3t \right) - \int \left(-3 \cos t + \frac{\cos 3t}{3} \right) dt \right]$$

$$= \frac{1}{2} \left[\frac{-9t \cos t + t \cos 3t}{3} - \left\{ -3 \sin t + \frac{\sin 3t}{9} \right\} \right] + c$$

$$= \frac{1}{2} \left[\frac{-9 \cos t + t \cos 3t}{3} + \frac{27 \sin t - 3 \sin 3t}{9} \right] + c$$

$$= \frac{1}{18} [-27 \cos t + 3t \cos 3t + 27 \sin t - 3 \sin 3t] + c$$

$$I = \frac{1}{18} [3\sqrt{x} \cos 3\sqrt{x} + 27 \sin \sqrt{x} - 27\sqrt{x} \cos \sqrt{x} - 3 \sin 3\sqrt{x}] + c$$

54. Question

Evaluate the following integrals:

$$\int x \sin^3 x \, dx$$

Answer

$$\text{Let } I = \int x \sin^3 x \, dx$$

$$\text{We know that, } \sin^3 x = \frac{3 \sin x - \sin 3x}{4}$$

$$= \int x \left(\frac{3 \sin x - \sin 3x}{4} \right) dx$$

$$= \frac{1}{4} \int x(3 \sin x - \sin 3x) dx$$

Using integration by parts,

$$I = \frac{1}{4} \left[x \int (3 \sin x - \sin 3x) dx - \int 1 \int (3 \sin x - \sin 3x) dx \right]$$

$$= \frac{1}{4} \left[x \left(-3 \cos x + \frac{\cos 3x}{3} \right) - \int \left(-3 \cos x + \frac{\cos 3x}{3} \right) dx \right]$$

$$= \frac{1}{4} \left[-3x \cos x + \frac{x \cos 3x}{3} + 3 \sin x - \frac{\sin 3x}{9} \right] + c$$

$$I = \frac{1}{36} [3x \cos 3x - 27x \cos x + 27 \sin x - \sin 3x] + c$$

55. Question

Evaluate the following integrals:

$$\int \cos^3 \sqrt{x} \, dx$$

Answer

Let

$$\sqrt{x} = t$$

$$x = t^2$$

$$dx = 2t \, dt$$

$$\text{let } I = 2 \int t \cos^3 t \, dt$$

$$\text{we know that, } \cos^3 t \, dt = \frac{3 \cos t + \cos 3t}{4}$$

$$= 2 \int t \left(\frac{3 \cos t + \cos 3t}{4} \right) dt$$

$$= \frac{1}{2} \int t (3 \cos t - \cos 3t) dt$$

Using integration by parts,

$$= \frac{1}{2} \left[t \left(3 \sin t + \frac{1}{3} \sin 3t \right) + \int \left(3 \sin t + \frac{\sin 3t}{3} \right) dt \right]$$

$$= \frac{1}{2} \left[\frac{9t \sin t + t \sin 3t}{3} + \left\{ 3 \cos t + \frac{\cos 3t}{9} \right\} \right] + c$$

$$= \frac{1}{18} [27 t \sin t + 3t \sin 3t + 9 \cos t + \cos 3t] + c$$

$$I = \frac{1}{18} [27 \sqrt{x} \sin \sqrt{x} + 3 \sqrt{x} \sin 3\sqrt{x} + 9 \cos \sqrt{x} + \cos 3\sqrt{x}] + c$$

56. Question

Evaluate the following integrals:

$$\int x \cos^3 x \, dx$$

Answer

$$\text{Let } I = \int x \cos^3 x \, dx$$

$$\text{we know that, } \cos^3 t \, dt = \frac{3 \cos t + \cos 3t}{4}$$

$$= \int x \left(\frac{3 \cos x + \cos 3x}{4} \right) dx$$

$$= \frac{1}{4} \int x (3 \cos x + \cos 3x) dx$$

Using integration by parts,

$$I = \frac{1}{4} \left[x \int (3 \cos x + \cos 3x) dx - \int 1 \int (3 \cos x + \cos 3x) dx \right]$$

$$= \frac{1}{4} \left[x \left(3 \sin x + \frac{\sin 3x}{3} \right) - \int \left(3 \sin x + \frac{\sin 3x}{3} \right) dx \right]$$

$$= \frac{1}{4} \left[3x \sin x + \frac{x \sin 3x}{3} + 3 \cos x + \frac{\cos 3x}{9} \right] + c$$

$$I = \frac{3x \sin x}{4} + \frac{x \sin 3x}{12} + \frac{3 \cos x}{4} + \frac{\cos 3x}{36} + c$$

57. Question

Evaluate the following integrals:

$$\int \tan^{-1} \sqrt{\frac{1-x}{1+x}} dx$$

Answer

$$\text{Let } I = \int \tan^{-1} \sqrt{\frac{1-x}{1+x}} dx$$

$$x = \cos \theta ; dx = -\sin \theta d\theta$$

$$I = \int \tan^{-1} \left(\tan \frac{\theta}{2} \right) - \sin \theta d\theta$$

$$= -\frac{1}{2} \int \theta \sin \theta d\theta$$

Using integration by parts,

$$= -\frac{1}{2} \left[\theta \int \sin \theta d\theta - \int \frac{d}{d\theta} \theta \int \sin \theta d\theta \right]$$

$$= \frac{1}{2} [-\theta \cos \theta + \int \cos \theta d\theta]$$

$$= \frac{1}{2} [-\theta \cos \theta + \sin \theta] + c$$

$$I = \frac{1}{2} [-x \cos^{-1} x + \sqrt{1-x^2}] + c$$

58. Question

Evaluate the following integrals:

$$\int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$$

Answer

$$\text{Let } I = \int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$$

$$\text{Let } x = a \tan^2 \theta$$

$$dx = 2a \tan^2 \theta \sec^2 \theta$$

$$I = \int \left(\sin^{-1} \sqrt{\frac{a \tan^2 \theta}{a + a \tan^2 \theta}} \right) 2a \tan^2 \theta \sec^2 \theta d\theta$$

$$= \int \sin^{-1} (\sin \theta) 2a \tan^2 \theta \sec^2 \theta d\theta$$

$$= \int 2\theta a \tan^2 \theta \sec^2 \theta d\theta$$

$$= 2a \int \theta \tan^2 \theta \sec^2 \theta d\theta$$

Using integration by parts,

$$= 2a \left(\theta \int \tan^2 \theta \sec^2 \theta d\theta - \int 1 \int \tan^2 \theta \sec^2 \theta d\theta \right)$$

$$= 2a \left[\theta \frac{\tan^2 \theta}{2} - \int \frac{\tan^2 \theta}{2} d\theta \right]$$

$$= a\theta \tan^2 \theta - \frac{2a}{2} \int (\sec^2 \theta - 1) d\theta$$

$$= a\theta \tan^2 \theta - a \tan \theta + a\theta + c$$

$$= a \left(\tan^{-1} \sqrt{\frac{x}{a}} \right) \frac{x}{a} - a \sqrt{\frac{x}{a}} + a \tan^{-1} \sqrt{\frac{x}{a}} + c$$

$$= x \tan^{-1} \sqrt{\frac{x}{a}} - \sqrt{ax} + a \tan^{-1} \sqrt{\frac{x}{a}} + c$$

59. Question

Evaluate the following integrals:

$$\int \frac{x^3 \sin^{-1} x^2}{\sqrt{1-x^4}} dx$$

Answer

$$\text{Let } I = \int \frac{x^3 \sin^{-1} x^2}{\sqrt{1-x^4}} dx$$

$$\sin^{-1} x^2 = t$$

$$\frac{1}{\sqrt{1-x^4}} 2x dx = dt$$

$$I = \int \frac{x^2 \sin^{-1} x^2}{\sqrt{1-x^4}} x dx$$

$$= \int (\sin t) t \frac{dt}{2}$$

Using integration by parts,

$$= \frac{1}{2} \left[t \int \sin t dt - \int \frac{d}{dt} t \int \sin t dt \right]$$

$$= \frac{1}{2} [-t \cos t - \int -\cos t dt]$$

$$= \frac{1}{2} [-t \cos t + \sin t] + c$$

$$= \frac{1}{2} [x^2 - \sqrt{1-x^4} \sin^{-1} x^2] + c$$

60. Question

Evaluate the following integrals:

$$\int \frac{x^2 \sin^{-1} x}{(1-x^2)^{3/2}} dx$$

Answer

$$\text{Let } I = \int \frac{x^2 \sin^{-1} x}{(1-x^2)^{3/2}} dx$$

$$\sin^{-1} x = t$$

$$\frac{1}{\sqrt{1-x^2}} dx = dt$$

$$I = \int \frac{\sin^2 t \times t dt}{1 - \sin^2 t}$$

$$= \int \frac{t \sin^2 t}{\cos^2 t} dt$$

$$= \int t \tan^2 t dt$$

$$= \int t(\sec^2 t - 1) dt$$

Using integration by parts,

$$= \int t \sec^2 t dt - \int t dt$$

$$= t \int \sec^2 t dt - \int \frac{d}{dt} t \int \sec^2 t dt - \frac{t^2}{2}$$

We know that, $\int \sec^2 t dt = \tan t$

$$= t \tan t - \int \tan t dt - \frac{t^2}{2}$$

$$= t \tan t - \log|\sec t| - \frac{t^2}{2} + c$$

$$I = \frac{x}{\sqrt{1-x^2}} \sin^{-1} x + \log|1-x^2| - \frac{1}{2} (\sin^{-1} x)^2 + c$$

Exercise 19.26

1. Question

Evaluate the following integrals:

$$\int e^x (\cos x - \sin x) dx$$

Answer

$$\text{Let } I = \int e^x (\cos x - \sin x) dx$$

Using integration by parts,

$$= \int e^x \cos x dx - \int e^x \sin x dx$$

We know that, $\frac{d}{dx} \cos x = -\sin x$

$$\begin{aligned}
&= \cos x \int e^x - \int \frac{d}{dx} \cos x \int e^x - \int e^x \sin x \, dx \\
&= e^x \cos x + \int e^x \sin x \, dx - \int e^x \sin x \, dx \\
&= e^x \cos x + c
\end{aligned}$$

2. Question

Evaluate the following integrals:

$$\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3} \right) dx$$

Answer

$$\begin{aligned}
\text{Let } I &= \int e^x \left(\frac{1}{x^2} - \frac{2}{x^3} \right) dx \\
&= \int e^x x^{-2} dx - 2 \int e^x x^{-3} dx
\end{aligned}$$

Integrating by parts

$$= x^{-2} \int e^x dx - \int \frac{d}{dx} x^{-2} \int e^x dx - 2 \int e^x x^{-3} dx$$

We know that,

$$\begin{aligned}
\int x^n dx &= \frac{x^{n+1}}{n+1} \\
&= e^x x^{-2} + 2 \int e^x x^{-3} dx - 2 \int e^x x^{-3} dx \\
&= \frac{e^x}{x^2} + c
\end{aligned}$$

3. Question

Evaluate the following integrals:

$$\int e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx$$

Answer

$$\text{Let } I = \int e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx$$

We know that, $\sin^2 x + \cos^2 x = 1$ and $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$

$$\begin{aligned}
&= e^x \left(\frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right) \\
&= \frac{e^x \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right)^2}{2 \cos^2 \frac{x}{2}} \\
&= \frac{1}{2} e^x \left(\frac{\sin \frac{x}{2} + \cos \frac{x}{2}}{\cos \frac{x}{2}} \right)^2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} e^x \left[\tan \frac{x}{2} + 1 \right]^2 \\
&= \frac{1}{2} e^x \left[1 + \tan \frac{x}{2} \right]^2 \\
&= \frac{1}{2} e^x \left[1 + \tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} \right] \\
&= \frac{1}{2} e^x \left[\sec^2 \frac{x}{2} + 2 \tan \frac{x}{2} \right] \\
&= e^x \left[\frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right] \dots \dots (1)
\end{aligned}$$

Let $\tan \frac{x}{2} = f(x)$

$$f'(x) = \frac{1}{2} \sec^2 \frac{x}{2}$$

We know that,

$$\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + c$$

From equation(1), we obtain

$$\int e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx = e^x \tan \frac{x}{2} + c$$

4. Question

Evaluate the following integrals:

$$\int e^x (\cot x - \operatorname{cosec}^2 x) dx$$

Answer

$$\text{Let } I = \int e^x (\cot x - \operatorname{cosec}^2 x) dx$$

$$= \int e^x \cot x dx - \int e^x \operatorname{cosec}^2 x dx$$

Integrating by parts,

$$= \cot x \int e^x dx - \int \frac{d}{dx} \cot x \int e^x dx - \int e^x \operatorname{cosec}^2 x dx$$

$$= \cot x e^x + \int e^x \operatorname{cosec}^2 x dx - \int e^x \operatorname{cosec}^2 x dx$$

$$= e^x \cot x + c$$

5. Question

Evaluate the following integrals:

$$\int e^x \left(\frac{x-1}{2x^2} \right) dx$$

Answer

$$\int e^x \left(\frac{x-1}{2x^2} \right) dx$$

$$\text{Let } I = \int e^x \frac{1}{2x} dx - \int e^x \frac{1}{2x^2} dx$$

Integrating by parts,

$$\begin{aligned} &= \frac{e^x}{2x} - \int e^x \left(\frac{d}{dx} \left(\frac{1}{2x} \right) \right) dx - \int \frac{e^x}{2x^2} dx \\ &= \frac{e^x}{2x} + \int \frac{e^x}{2x^2} dx - \int \frac{e^x}{2x^2} dx \\ &= \frac{e^x}{2x} + c \end{aligned}$$

6. Question

Evaluate the following integrals:

$$\int e^x \sec x (1 + \tan x) dx$$

Answer

$$\text{Let } I = \int e^x \sec x (1 + \tan x) dx$$

$$= \int e^x \sec x dx + \int e^x \sec x \tan x dx$$

Integrating by parts,

$$\begin{aligned} &= e^x \sec x dx - \int e^x \frac{d}{dx} \sec x dx + \int e^x \sec x \tan x dx \\ &= e^x \sec x dx - \int e^x \sec x \tan x dx + \int e^x \sec x \tan x dx \\ &= e^x \sec x dx + c \end{aligned}$$

7. Question

Evaluate the following integrals:

$$\int e^x (\tan x - \log \cos x) dx$$

Answer

$$\text{Let } I = \int e^x (\tan x - \log \cos x) dx$$

$$I = \int e^x \tan x dx - \int e^x \log \cos x dx$$

Integrating by parts,

$$\begin{aligned} &= \int e^x \tan x dx - \{ e^x \log \cos x - \int e^x \left(\frac{d}{dx} \log \cos x \right) dx \} \\ &= \int e^x \tan x dx - e^x \log \cos x dx - \int e^x \tan x dx \\ &= -e^x \log \cos x dx + c \\ &= e^x \log \sec x + c \end{aligned}$$

8. Question

Evaluate the following integrals:

$$\int e^x [\sec x + \log (\sec x + \tan x)] dx$$

Answer

$$\text{Let } I = \int e^x [\sec x + \log (\sec x + \tan x)] dx$$

$$I = \int e^x \sec x dx + \int \log(\sec x + \tan x) dx$$

Integrating by parts

$$= \int e^x \sec x dx + e^x \log(\sec x + \tan x) - \int e^x \sec x dx$$

$$= e^x \log(\sec x + \tan x) + c$$

9. Question

Evaluate the following integrals:

$$\int e^x (\cot x + \log \sin x) dx$$

Answer

$$\text{Let } I = \int e^x (\cot x + \log \sin x) dx$$

$$= \int e^x \cot x dx + \int e^x \log \sin x dx$$

Integrating by parts

$$= \int e^x \log \sin x dx + \int e^x \cot x dx$$

$$= (\log \sin x) e^x - \int e^x \frac{d}{dx} \log \sin x dx + \int e^x \cot x dx + c$$

$$= (\log \sin x) e^x - \int e^x \cot x dx + \int e^x \cot x dx + c$$

$$= (\log \sin x) e^x + c$$

10. Question

Evaluate the following integrals:

$$\int e^x \frac{x-1}{(x+1)^3} dx$$

Answer

$$\text{Let } I = \int e^x \frac{x+1-2}{(x+1)^3} dx$$

$$= \int e^x \left\{ \frac{1}{(x+1)^2} + \frac{-2}{(x+1)^2} \right\} dx$$

$$= \int e^x \frac{1}{(x+1)^2} dx + \int e^x \frac{-2}{(x+1)^2} dx$$

Integrating by parts

$$= e^x \frac{1}{(x+1)^2} - \int e^x \frac{-2}{(x+1)^2} + \int e^x \frac{-2}{(x+1)^2}$$

$$= e^x \frac{1}{(x+1)^2} + c$$

11. Question

Evaluate the following integrals:

$$\int e^x \left(\frac{\sin 4x - 4}{1 - \cos 4x} \right) dx$$

Answer

$$\text{Let } I = \int e^x \left(\frac{\sin 4x - 4}{1 - \cos 4x} \right) dx$$

$$= \int e^x \left\{ \frac{2 \sin 2x \cos 2x}{2 \sin^2 x} - \frac{4}{2 \sin^2 x} \right\} dx$$

$$= \int e^x \{ \cot 2x - 2 \operatorname{cosec}^2 2x \} dx$$

$$= \int e^x \cot 2x dx - \int e^x 2 \operatorname{cosec}^2 2x dx$$

Integrating by parts,

$$= e^x \cot 2x - \int e^x \frac{d}{dx} \cot 2x dx - 2 \int e^x \operatorname{cosec}^2 2x dx$$

$$= e^x \cot 2x + 2 \int e^x \operatorname{cosec}^2 2x - 2 \int e^x \operatorname{cosec}^2 2x$$

$$= e^x \cot 2x + c$$

12. Question

Evaluate the following integrals:

$$\int \frac{2-x}{(1-x)^2} e^x dx$$

Answer

$$\text{Let } I = \int \frac{2-x}{(1-x)^2} e^x dx$$

$$= \int e^x \left\{ \frac{(1-x) + 1}{(1-x)^2} \right\} dx$$

$$= \int e^x \left\{ \frac{1}{1-x} + \frac{1}{(1-x)^2} \right\}$$

$$\frac{1}{1-x} = f(x) \quad \frac{1}{(1-x)^2} = f'(x)$$

We know that, $\int e^x \{f(x) + f'(x)\} = e^x f(x) + c$

$$= e^x \frac{1}{1-x} + c$$

13. Question

Evaluate the following integrals:

$$\int e^x \frac{1+x}{(2+x)^2} dx$$

Answer

$$\text{Let } I = \int \frac{1+x}{(2+x)^2} e^x dx$$

$$\begin{aligned}
&= \int e^x \left\{ \frac{(x+2) - 1}{(x+2)^2} \right\} dx \\
&= \int e^x \left\{ \frac{1}{x+2} - \frac{1}{(x+2)^2} \right\} \\
&= \int e^x \frac{1}{x+2} dx - \int e^x \frac{1}{(x+2)^2} dx
\end{aligned}$$

Using integration by parts,

$$\begin{aligned}
&= \frac{e^x}{x+2} + \int e^x \frac{1}{(x+2)^2} dx - \int e^x \frac{1}{(x+2)^2} dx \\
&= e^x \frac{1}{x+2} + c
\end{aligned}$$

14. Question

Evaluate the following integrals:

$$\int \frac{\sqrt{1 - \sin x}}{1 + \cos x} e^{-x/2} dx$$

Answer

$$\text{Let } I = \int \frac{\sqrt{1 - \sin x}}{1 + \cos x} e^{-x/2} dx$$

$$\text{put } \frac{x}{2} = t \Rightarrow x = 2t \Rightarrow dx = 2dt$$

$$\int \frac{\sqrt{1 - \sin x}}{1 + \cos x} e^{-x/2} dx = 2 \int \frac{\sqrt{1 - \sin 2t}}{1 + \cos 2t} e^{-t} dt$$

$$= 2 \int \frac{\sqrt{\sin^2 t + \cos^2 t - 2 \sin t \cos t}}{1 + \cos 2t} e^{-t} dt$$

$$= 2 \int \frac{\sqrt{(\cos t - \sin t)^2}}{2 \cos^2 t} e^{-t} dt$$

$$= \int (\sec t - \tan t \sec t) e^{-t} dt$$

$$= \int \sec t e^{-t} dt - \int \tan t \sec t e^{-t} dt$$

Integrating by parts

$$= e^{-t} \sec t + \int \tan t \sec t e^{-t} dt - \int \tan t \sec t e^{-t} dt$$

$$= e^{-t} \sec t + c$$

$$= e^{-\frac{x}{2}} \sec \frac{x}{2} + c$$

15. Question

Evaluate the following integrals:

$$\int e^x \left(\log x + \frac{1}{x} \right) dx$$

Answer

$$\text{Let } I = \int e^x \left(\log x + \frac{1}{x} \right) dx$$

We know that

$$\int e^x \{f(x) + f'(x)\} = e^x f(x) + c$$

Here,

$$f(x) = \log x; f'(x) = \frac{1}{x}$$

$$\int e^x \left(\log x + \frac{1}{x} \right) dx = e^x \log x + c$$

16. Question

Evaluate the following integrals:

$$\int e^x \left(\log x + \frac{1}{x^2} \right) dx$$

Answer

$$\text{Let } I = \int e^x \left(\log x + \frac{1}{x^2} \right) dx$$

$$= \int e^x \left(\log x + \frac{1}{x} - \frac{1}{x} + \frac{1}{x^2} \right) dx$$

$$= \int e^x \left(\log x - \frac{1}{x} \right) dx + \int e^x \left(\frac{1}{x} + \frac{1}{x^2} \right) dx$$

Using integration by parts,

$$= e^x \left(\log x - \frac{1}{x} \right) - \int e^x \frac{d}{dx} \left(\log x - \frac{1}{x} \right) dx + \int e^x \left(\frac{1}{x} + \frac{1}{x^2} \right) dx$$

$$= e^x \left(\log x - \frac{1}{x} \right) - \int e^x \left(\frac{1}{x} + \frac{1}{x^2} \right) dx + \int e^x \left(\frac{1}{x} + \frac{1}{x^2} \right) dx$$

$$= e^x \left(\log x - \frac{1}{x} \right) + c$$

17. Question

Evaluate the following integrals:

$$\int \frac{e^x}{x} \{x(\log x)^2 + 2 \log x\} dx$$

Answer

$$\text{Let } I = \int \frac{e^x}{x} \{x(\log x)^2 + 2 \log x\} dx$$

$$= \int e^x (\log x)^2 dx + 2 \int \frac{e^x}{x} \log x dx$$

Using integration by parts,

$$= e^x (\log x)^2 - \int e^x \frac{d}{dx} (\log x)^2 + 2 \int \frac{e^x}{x} \log x dx$$

$$= e^x (\log x)^2 - 2 \int \frac{e^x}{x} \log x dx + 2 \int \frac{e^x}{x} \log x dx$$

$$= e^x(\log x)^2 + c$$

18. Question

Evaluate the following integrals:

$$\int e^x \cdot \frac{\sqrt{1-x^2} \sin^{-1} x + 1}{\sqrt{1-x^2}} dx$$

Answer

$$\text{Let } I = \int e^x \frac{\sqrt{1-x^2} \sin^{-1} x + 1}{\sqrt{1-x^2}} dx$$

$$I = \int e^x \sin^{-1} x + \int e^x \frac{1}{\sqrt{1-x^2}} dx$$

Integrating by parts

$$= e^x \sin^{-1} x - \int e^x \left(\frac{d}{dx} (\sin^{-1} x) \right) dx + \int e^x \frac{1}{\sqrt{1-x^2}} dx$$

$$= e^x \sin^{-1} x - \int e^x \frac{1}{\sqrt{1-x^2}} dx + \int e^x \frac{1}{\sqrt{1-x^2}} dx$$

$$= e^x \sin^{-1} x + c$$

19. Question

Evaluate the following integrals:

$$\int e^{2x} (-\sin x + 2 \cos x) dx$$

Answer

$$\text{Let } I = \int e^{2x} (-\sin x + 2 \cos x) dx$$

$$I = \int e^{2x} - \sin x dx + 2 \int e^{2x} \cos x dx$$

Applying by parts in the second integral,

$$I = - \int e^{2x} \sin x dx + 2 \left\{ \frac{1}{2} e^{2x} \cos x + \int \frac{1}{2} e^{2x} \sin x dx \right\}$$

$$= - \int e^{2x} \sin x dx + e^{2x} \cos x + \int e^{2x} \sin x dx + c$$

$$= e^{2x} \cos x + c$$

20. Question

Evaluate the following integrals:

$$\int e^x \left(\tan^{-1} x + \frac{1}{1+x^2} \right) dx$$

Answer

$$\text{Let } I = \int e^x \left(\tan^{-1} x + \frac{1}{1+x^2} \right) dx$$

$$\text{here, } f(x) = \tan^{-1} x \text{ and } f'(x) = \frac{1}{1+x^2}$$

and we know that,

$$\int e^x \{f(x) + f'(x)\} = e^x f(x) + c$$

$$\int e^x \left(\tan^{-1} x + \frac{1}{1+x^2} \right) dx = e^x \tan^{-1} x + c$$

21. Question

Evaluate the following integrals:

$$\int e^x \left(\frac{\sin x \cos x - 1}{\sin^2 x} \right) dx$$

Answer

$$\text{Let } I = \int e^x \left(\frac{\sin x \cos x - 1}{\sin^2 x} \right) dx$$

$$= \int e^x (\cot x - \operatorname{cosec}^2 x) dx$$

$$= \int e^x (\cot x + -\operatorname{cosec}^2 x) dx$$

We know that, $\int e^x \{f(x) + f'(x)\} = e^x f(x) + c$

let $f(x) = \cot x$; $f'(x) = -\operatorname{cosec}^2 x$

$$\int e^x \left(\frac{\sin x \cos x - 1}{\sin^2 x} \right) dx = e^x \cot x + c$$

22. Question

Evaluate the following integrals:

$$\int \{ \tan (\log x) + \sec^2 (\log x) \} dx$$

Answer

$$\text{Let } I = \int [\tan(\log x) + \sec^2(\log x)] dx$$

$$\log x = z \Rightarrow x = e^z \Rightarrow dx = e^z dz$$

$$I = \int (\tan z + \sec^2 z) e^z dz$$

$$f(z) = \tan z; f'(z) = \sec^2 z$$

We know that, $\int e^x \{f(x) + f'(x)\} = e^x f(x) + c$

$$I = x \tan(\log x) + c$$

23. Question

Evaluate the following integrals:

$$\int e^x \frac{(x-4)}{(x-2)^3} dx$$

Answer

$$\text{Let } I = \int e^x \frac{(x-4)}{(x-2)^3} dx$$

$$= \int e^x \frac{(x-2) - 2}{(x-2)^3} dx$$

$$= \int e^x \left\{ \frac{1}{(x-2)^2} - \frac{2}{(x-2)^2} \right\} dx$$

$$\text{Let } f(x) = \frac{1}{(x-2)^2} \text{ and } f'(x) = \frac{2}{(x-2)^2}$$

$$\text{We know that, } \int e^x \{f(x) + f'(x)\} = e^x f(x) + c$$

$$I = \frac{e^x}{(x-2)^2} + c$$

24. Question

Evaluate the following integrals:

$$\int e^{2x} \left(\frac{1 - \sin 2x}{1 - \cos 2x} \right) dx$$

Answer

$$\text{Let } I = \int e^{2x} \left(\frac{1 - \sin 2x}{1 - \cos 2x} \right) dx$$

We have,

$$\cos 2x = 1 - 2 \sin^2 x$$

$$I = \int e^{2x} \left(\frac{1 - \sin 2x}{1 - (1 - 2 \sin^2 x)} \right) dx$$

$$= \int e^{2x} \left(\frac{1 - \sin 2x}{2 \sin^2 x} \right) dx$$

$$= \int e^{2x} \left(\frac{\operatorname{cosec}^2 x}{2} - \frac{2 \sin x \cos x}{2 \sin^2 x} \right) dx$$

$$= \int e^{2x} \left(\frac{\operatorname{cosec}^2 x}{2} - \frac{\cos x}{\sin x} \right) dx$$

$$= \int e^{2x} \left(\frac{\operatorname{cosec}^2 x}{2} - \cot x \right) dx$$

Using integration by parts,

$$= \frac{1}{2} \int e^{2x} \operatorname{cosec}^2 x dx - \int e^{2x} \cot x dx$$

That is,

$$I = I_1 + I_2$$

$$I_1 = \frac{1}{2} \int e^{2x} \operatorname{cosec}^2 x dx$$

$$I_2 = \int e^{2x} \cot x dx$$

Consider

$$I_1 = \frac{1}{2} \int e^{2x} \operatorname{cosec}^2 x dx$$

take e^{2x} as first function and $\operatorname{cosec}^2 x$ as second function

$$u = e^{2x}; du = 2e^{2x} dx$$

$$\int \operatorname{cosec}^2 x \, dx = \int dv$$

$$\text{Let } v = -\cot x$$

$$I_1 = \frac{1}{2} \left[e^{2x}(-\cot x) - \int (-\cot x) 2e^{2x} dx \right]$$

$$I_1 = \frac{1}{2} \left[e^{2x}(-\cot x) - 2 \int \cot x e^{2x} dx \right]$$

$$I_1 = \frac{1}{2} (e^{2x}(-\cot x)) + \int \cot x e^{2x} dx$$

Thus,

$$I = \frac{1}{2} (e^{2x}(-\cot x)) + \int \cot x e^{2x} dx - \int e^{2x} \cot x dx$$

$$I = \frac{1}{2} [e^{2x}(-\cot x)] + c$$

Exercise 19.27

1. Question

Evaluate the following integrals:

$$\int e^{ax} \cos bx \, dx$$

Answer

$$\text{Let } I = \int e^{ax} \cos bx \, dx$$

Integrating by parts,

$$I = e^{ax} \frac{\sin bx}{b} - a \int e^{ax} \frac{\sin bx}{b} \, dx$$

$$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx \, dx$$

$$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \left[-e^{ax} \frac{\cos bx}{b} - a \int e^{ax} \frac{\cos bx}{b} \, dx \right]$$

$$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx \, dx$$

$$I = \frac{e^{ax}}{b^2} [b \sin bx + a \cos bx] - \frac{a^2}{b^2} I + c$$

$$= \frac{e^{ax}}{a^2 + b^2} [b \cos bx + a \sin bx] + c$$

2. Question

Evaluate the following integrals:

$$\int e^{ax} \sin (bx + c) \, dx$$

Answer

$$\text{Let } I = \int e^{ax} \sin(bx + c) \, dx$$

$$= -e^{ax} \frac{\cos(bx + c)}{b} + \int a e^{ax} \frac{\cos(bx + c)}{b} \, dx$$

$$= -\frac{1}{b} e^{ax} \cos(bx + c) + \frac{a}{b} \int e^{ax} \cos(bx + c)$$

$$I = \frac{e^{ax}}{b^2} \{a \sin(bx + c) - b \cos(bx + c)\} - \frac{a^2}{b^2} I + c_1$$

$$I = \left\{ \frac{a^2 + b^2}{b^2} \right\} - \frac{e^{ax}}{b^2} \{a \sin(bx + c) - b \cos(bx + c)\} + c_1$$

$$= \frac{e^{ax}}{a^2 + b^2} \{a \sin(bx + c) - b \cos(bx + c)\}$$

3. Question

Evaluate the following integrals:

$$\int \cos(\log x) dx$$

Answer

$$\text{Let } I = \int \cos(\log x) dx$$

$$\text{Let } \log x = t$$

$$\frac{1}{x} dx = dt$$

$$dx = x dt$$

$$= \int e^t \cos t dt$$

$$\text{We know that, } \int \cos(\log x) dx = \frac{e^{ax}}{a^2 + b^2} \{a \sin(bx + c) - b \cos(bx + c)\}$$

$$\text{Hence, } a=1, b=1$$

$$\text{So, } I = \frac{e^t}{2} [\cos t + \sin t] + c$$

Hence,

$$\int \cos(\log x) dx = \frac{e^{\log x}}{2} \{\cos(\log x) + \sin(\log x)\} + c$$

$$I = \frac{x}{2} \{\cos(\log x) + \sin(\log x)\} + c$$

4. Question

Evaluate the following integrals:

$$\int e^{2x} \cos(3x + 4) dx$$

Answer

$$\text{Let } I = \int e^{2x} \cos(3x + 4) dx$$

Integrating by parts

$$I = e^{2x} \frac{\sin(3x + 4)}{3} - \int 2e^{2x} \frac{\sin(3x + 4)}{3} dx$$

$$= \frac{1}{3} e^{2x} \sin(3x + 4) - \frac{2}{3} \int e^{2x} \sin(3x + 4) dx$$

$$= \frac{1}{3} e^{2x} \sin(3x + 4) - \frac{2}{3} \left\{ -e^{2x} \frac{\cos(3x + 4)}{3} + \int 2e^{2x} \frac{\cos(3x + 4)}{3} dx \right\}$$

$$I = \frac{e^{2x}}{9} [2 \cos(3x + 4) + 3 \sin(3x + 4)] + c$$

Hence,

$$I = \frac{e^{2x}}{9} [2 \cos(3x + 4) + 3 \sin(3x + 4)] + c$$

5. Question

Evaluate the following integrals:

$$\int e^{2x} \sin x \cos x \, dx$$

Answer

$$\text{Let } I = \int e^{2x} \sin x \cos x \, dx$$

$$= \frac{1}{2} \int e^{2x} 2 \sin x \cos x \, dx$$

$$= \frac{1}{2} \int e^{2x} \sin 2x \, dx$$

We know that,

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} \{a \sin bx - b \cos bx\} + c$$

$$= \frac{e^{2x}}{8} \{2 \sin 2x - 2 \cos 2x\} + c$$

$$I = \frac{1}{2} \frac{e^{2x}}{8} \{2 \sin 2x - 2 \cos 2x\} + c$$

$$I = \frac{e^{2x}}{8} \{\sin 2x - \cos 2x\} + c$$

6. Question

Evaluate the following integrals:

$$e^{2x} \sin x \, dx$$

Answer

$$\text{Let } I = \int e^{2x} \sin x \, dx$$

Integrating by parts,

$$I = \sin x \int e^{2x} \, dx - \int \frac{d}{dx} \sin x \int e^{2x} \, dx$$

$$I = \sin x \frac{e^{2x}}{2} - \int \cos x \frac{e^{2x}}{2} \, dx$$

$$I = \sin x \frac{e^{2x}}{2} - \frac{1}{2} \int e^{2x} \cos x \, dx$$

Again integrating by parts,

$$I = \sin x \frac{e^{2x}}{2} - \frac{1}{2} \left\{ \cos x \int e^{2x} \, dx - \int \frac{d}{dx} \cos x \int e^{2x} \, dx \right\}$$

$$I = \sin x \frac{e^{2x}}{2} - \frac{1}{2} \left[\cos x \frac{e^{2x}}{2} - \int (-\sin x) \frac{e^{2x}}{2} \, dx \right]$$

$$I = \sin x \frac{e^{2x}}{2} - \frac{1}{2} \left[\cos x \frac{e^{2x}}{2} + \frac{1}{2} \int \sin x e^{2x} dx \right]$$

$$I = \sin x \frac{e^{2x}}{2} - \frac{1}{2} \cos x \frac{e^{2x}}{2} - \frac{1}{4} I$$

$$I + \frac{1}{4} I = \sin x \frac{e^{2x}}{2} - \frac{1}{2} \cos x \frac{e^{2x}}{2}$$

$$\frac{5}{4} I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4}$$

$$I = \frac{4}{5} \left[\frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} \right] + c$$

$$I = \frac{e^{2x}}{5} [2 \sin x - \cos x] + c$$

7. Question

Evaluate the following integrals:

$$\int e^{2x} \sin(3x + 1) dx$$

Answer

$$\text{Let } I = \int e^{2x} \sin(3x + 1) dx$$

Now Integrating by parts choosing $\sin(3x + 1)$ as first function and e^{2x} as second function we get,

$$I = \sin(3x + 1) \int e^{2x} dx - \int \left(\frac{d}{dx} \sin(3x + 1) \right) \int e^{2x} dx dx$$

$$I = \frac{e^{2x}}{2} \sin(3x + 1) - \int \frac{3e^{2x}}{2} \cos(3x + 1) dx$$

Now again integrating by parts by taking $\cos(3x + 1)$ as first function and e^{2x} as second function we get,

$$I = \frac{e^{2x}}{2} \sin(3x + 1) - [\cos(3x + 1) \int \frac{3e^{2x}}{2} dx - \int \frac{3}{2} \left(\frac{d}{dx} \cos(3x + 1) \right) \int e^{2x} dx dx]$$

$$I = \frac{e^{2x}}{2} \sin(3x + 1) - \frac{3}{4} e^{2x} \cos(3x + 1) - \frac{9}{4} \int e^{2x} \sin(3x + 1) dx$$

$$\int e^{2x} \sin(3x + 1) dx = I$$

Therefore,

$$I = \frac{e^{2x}}{2} \sin(3x + 1) - \frac{3}{4} e^{2x} \cos(3x + 1) - \frac{9}{4} I$$

$$I + \frac{9}{4} I = \frac{e^{2x}}{2} \sin(3x + 1) - \frac{3}{4} e^{2x} \cos(3x + 1)$$

$$\frac{13I}{4} = \frac{e^{2x}}{2} \sin(3x + 1) - \frac{3}{4} e^{2x} \cos(3x + 1)$$

$$I = \frac{e^{2x}}{13} \{2 \sin(3x + 1) - 3 \cos(3x + 1)\} + c$$

8. Question

Evaluate the following integrals:

$$\int e^x \sin^2 x \, dx$$

Answer

$$\text{Let } I = \int e^x \sin^2 x \, dx$$

$$I = \frac{1}{2} \int e^x 2 \sin^2 x \, dx$$

$$= \frac{1}{2} \int e^x (1 - \cos 2x) \, dx$$

Using integration by parts,

$$= \frac{1}{2} \int e^x \, dx - \frac{1}{2} \int e^x \cos 2x \, dx$$

We know that, $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} \{a \cos bx - b \sin bx\} + c$

$$I = \frac{1}{2} \left[e^x - \frac{e^x}{5} (\cos 2x + 2 \sin 2x) \right] + c$$

$$= \frac{e^x}{2} - \frac{e^x}{10} (\cos 2x + 2 \sin 2x) + c$$

9. Question

Evaluate the following integrals:

$$\int \frac{1}{x^3} \sin(\log x) \, dx$$

Answer

$$\text{Let } I = \int \frac{1}{x^3} \sin(\log x) \, dx$$

$$\text{let } \log x = t \Rightarrow \frac{1}{x} dx = dt \Rightarrow dx = e^x dt$$

We know that

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} \{a \sin bx - b \cos bx\} + c$$

$$\int e^{-2t} \sin t \, dt = \frac{e^{-2t}}{5} \{-2 \sin t - \cos t\} + c$$

$$I = \frac{x^{-2}}{5} \{-2 \sin(\log x) - \cos(\log x)\} + c$$

$$= \frac{-1}{5x^2} \{2 \sin(\log x) + \cos(\log x)\} + c$$

10. Question

Evaluate the following integrals:

$$\int e^{2x} \cos^2 x \, dx$$

Answer

$$\text{Let } I = \int e^{2x} \cos^2 x \, dx$$

$$= \frac{1}{2} \int e^{2x} 2 \cos^2 x \, dx$$

$$= \frac{1}{2} \int e^{2x}(1 + \cos 2x) dx$$

$$= \frac{1}{2} \int e^{2x} dx + \frac{1}{2} \int e^{2x} \cos 2x dx$$

We know that, $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} \{a \cos bx - b \sin bx\} + c$

$$I = \frac{1}{2} \left[\frac{e^{2x}}{2} - \frac{e^{2x}}{8} (2 \cos 2x + 2 \sin 2x) \right] + c$$

$$= \frac{e^{2x}}{4} + \frac{e^{2x}}{16} (2 \cos 2x + 2 \sin 2x) + c$$

$$= \frac{e^{2x}}{4} + \frac{e^{2x}}{8} (\cos 2x + \sin 2x) + c$$

11. Question

Evaluate the following integrals:

$$\int e^{-2x} \sin x dx$$

Answer

$$\text{Let } I = \int e^{-2x} \sin x dx$$

We know that, $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} \{a \sin bx - b \cos bx\} + c$

$$= \frac{e^{-2x}}{5} \{-2 \sin x - \cos x\} + c$$

12. Question

Evaluate the following integrals:

$$\int x^2 e^{x^3} \cos x^3 dx$$

Answer

$$\text{Let } I = \int x^2 e^{x^3} \cos x^3 dx$$

$$x^3 = t$$

$$3x^2 dx = dt$$

$$I = \frac{1}{3} \int e^t \cos t dt$$

We know that, $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} \{a \cos bx - b \sin bx\} + c$

$$I = \frac{1}{3} \left[\frac{e^t}{2} (\cos t + \sin t) \right] + c$$

$$I = \frac{1}{3} \left[\frac{e^{x^3}}{2} (\cos x^3 + \sin x^3) \right] + c$$

Exercise 19.28

1. Question

Evaluate the integral:

$$\int \sqrt{3+2x-x^2} \, dx$$

Answer

Key points to solve the problem:

- Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) \, dx$
- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} \, dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\text{Let, } I = \int \sqrt{3+2x-x^2} \, dx$$

$$\therefore I = \int \sqrt{3 - (x^2 - 2(1)x)} \, dx = \int \sqrt{3 - (x^2 - 2(1)x + 1) + 1} \, dx$$

$$\text{Using } a^2 - 2ab + b^2 = (a - b)^2$$

We have:

$$I = \int \sqrt{4 - (x-1)^2} \, dx = \int \sqrt{2^2 - (x-1)^2} \, dx$$

$$\text{As } I \text{ match with the form: } \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I = \frac{x-1}{2} \sqrt{4 - (x-1)^2} + \frac{4}{2} \sin^{-1} \left(\frac{x-1}{2} \right) + C$$

$$\Rightarrow I = \frac{1}{2} (x-1) \sqrt{3+2x-x^2} + 2 \sin^{-1} \left(\frac{x-1}{2} \right) + C$$

2. Question

Evaluate the integral:

$$\int \sqrt{x^2 + x + 1} \, dx$$

Answer

Key points to solve the problem:

- Such problems require the use of the method of substitution along with a method of integration by parts. By the method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) \, dx$
- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} \, dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

Let, $I = \int \sqrt{x^2 + x + 1} \, dx$

$$\therefore I = \int \sqrt{x^2 + 2\left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)^2 + 1 - \left(\frac{1}{2}\right)^2} \, dx$$

Using $a^2 + 2ab + b^2 = (a + b)^2$

We have:

$$I = \int \sqrt{\left(x + \frac{1}{2}\right)^2 + 1 - \frac{1}{4}} \, dx = \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \, dx$$

As I match with the form:

$$\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\therefore I = \frac{\left(x + \frac{1}{2}\right)}{2} \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2} \log \left| \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right| + C$$

$$\Rightarrow I = \frac{1}{4}(2x + 1)\sqrt{x^2 + x + 1} + \frac{3}{8} \log \left| \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right| + C$$

$$\Rightarrow I = \frac{1}{4}(2x + 1)\sqrt{x^2 + x + 1} + \frac{3}{8} \log \left| \left(x + \frac{1}{2}\right) + \sqrt{x^2 + x + 1} \right| + C$$

3. Question

Evaluate the integral:

$$\int \sqrt{x - x^2} \, dx$$

Answer

Key points to solve the problem:

- Such problems require the use of the method of substitution along with a method of integration by parts. By the method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)dx$
- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} \, dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

Let, $I = \int \sqrt{x - x^2} \, dx$

$$\therefore I = \int \sqrt{-\left(x^2 - 2\left(\frac{1}{2}\right)x\right)} \, dx = \int \sqrt{\frac{1}{4} - \left(x^2 - 2\left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)^2\right)} \, dx$$

Using $a^2 - 2ab + b^2 = (a - b)^2$

We have:

$$I = \int \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2} \, dx = \int \sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} \, dx$$

As I match with the form: $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$

$$\therefore I = \frac{x-\frac{1}{2}}{2} \sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} + \frac{\frac{1}{4}}{2} \sin^{-1} \left(\frac{x-\frac{1}{2}}{\frac{1}{2}} \right) + C$$

$$\Rightarrow I = \frac{1}{4} (2x - 1) \sqrt{x - x^2} + \frac{1}{8} \sin^{-1} (2x - 1) + C$$

4. Question

Evaluate the integral:

$$\int \sqrt{1+x-2x^2} dx$$

Answer

Key points to solve the problem:

- Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) dx$
- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + C$$

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$$

$$\text{Let, } I = \int \sqrt{1+x-2x^2} dx$$

$$\therefore I = \int \sqrt{1 - 2 \left(x^2 - 2 \left(\frac{1}{4} \right) x \right)} dx = \int \sqrt{1 - 2 \left(x^2 - 2 \left(\frac{1}{4} \right) x + \left(\frac{1}{4} \right)^2 \right) + 2 \left(\frac{1}{4} \right)^2} dx$$

$$\text{Using } a^2 - 2ab + b^2 = (a - b)^2$$

We have:

$$I = \int \sqrt{\frac{9}{8} - 2 \left(x - \frac{1}{4} \right)^2} dx = \int \sqrt{2} \sqrt{\left(\frac{3}{4} \right)^2 - \left(x - \frac{1}{4} \right)^2} dx$$

As I match with the form: $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$

$$\therefore I = \sqrt{2} \left\{ \frac{x-\frac{1}{4}}{2} \sqrt{\left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2} + \frac{\frac{9}{16}}{2} \sin^{-1} \left(\frac{x-\frac{1}{4}}{\frac{3}{4}} \right) \right\} + C$$

$$\Rightarrow I = \frac{1}{8} (4x - 1) \sqrt{2 \left\{ \left(\frac{3}{4} \right)^2 - \left(x - \frac{1}{4} \right)^2 \right\}} + \frac{9\sqrt{2}}{32} \sin^{-1} \left(\frac{4x-1}{3} \right) + C$$

$$\Rightarrow I = \frac{1}{8} (4x - 1) \sqrt{1 + x - 2x^2} + \frac{9\sqrt{2}}{32} \sin^{-1} \left(\frac{4x-1}{3} \right) + C$$

5. Question

Evaluate the integral:

$$\int \cos x \sqrt{4 - \sin^2 x} dx$$

Answer

Key points to solve the problem:

- Such problems require the use of the method of substitution along with a method of integration by parts. By the method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) dx$

- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + C$$

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$$

$$\text{Let, } I = \int \cos x \sqrt{4 - \sin^2 x} dx$$

$$\text{Let, } \sin x = t$$

Differentiating both sides:

$$\Rightarrow \cos x dx = dt$$

Substituting $\sin x$ with t , we have:

$$\therefore I = \int \sqrt{4 - t^2} dt = \int \sqrt{2^2 - t^2} dt$$

$$\text{As } I \text{ match with the form: } \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I = \frac{t}{2} \sqrt{4 - (t)^2} + \frac{4}{2} \sin^{-1} \left(\frac{t}{2} \right) + C$$

Putting the value of t i.e. $t = \sin x$

$$\Rightarrow I = \frac{1}{2} \sin x \sqrt{4 - \sin^2 x} + 2 \sin^{-1} \left(\frac{\sin x}{2} \right) + C$$

6. Question

Evaluate the integral:

$$\int e^x \sqrt{e^{2x} + 1} dx$$

Answer

Key points to solve the problem:

- Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) dx$

- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + C$$

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$$

$$\text{Let, } I = \int e^x \sqrt{e^{2x} + 1} dx$$

Let, $e^x = t$

Differentiating both sides:

$$\Rightarrow e^x dx = dt$$

Substituting e^x with t , we have:

We have:

$$I = \int \sqrt{t^2 + 1} dt = \int \sqrt{t^2 + 1^2} dt$$

As I match with the form:

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$$

$$\therefore I = \frac{t}{2} \sqrt{t^2 + 1} + \frac{1}{2} \log |t + \sqrt{t^2 + 1}|$$

$$\Rightarrow I = \frac{t}{2} \sqrt{t^2 + 1} + \frac{1}{2} \log |t + \sqrt{t^2 + 1}| + C$$

Putting the value of t back:

$$\Rightarrow I = \frac{e^x}{2} \sqrt{e^{2x} + 1} + \frac{1}{2} \log |e^x + \sqrt{e^{2x} + 1}| + C$$

7. Question

Evaluate the integral:

$$\int \sqrt{9 - x^2} dx$$

Answer

Key points to solve the problem:

- Such problems require the use of the method of substitution along with a method of integration by parts. By the method of integration by parts if we have $\int f(x)g(x)dx = f(x) \int g(x)dx - \int f'(x)(\int g(x)dx) dx$

- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + C$$

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$$

$$\text{Let, } I = \int \sqrt{9 - x^2} dx$$

$$\therefore I = \int \sqrt{9 - x^2} dx = \int \sqrt{3^2 - x^2} dx$$

$$\text{As I match with the form: } \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I = \frac{x}{2} \sqrt{9 - (x)^2} + \frac{9}{2} \sin^{-1} \left(\frac{x}{3} \right) + C$$

8. Question

Evaluate the integral:

$$\int \sqrt{16x^2 + 25} dx$$

Answer

Key points to solve the problem:

- Such problems require the use of the method of substitution along with a method of integration by parts. By the method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) dx$
- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\text{Let, } I = \int \sqrt{16x^2 + 25} dx$$

We have:

$$I = \int \sqrt{16x^2 + 25} dx = \int \sqrt{(4x)^2 + 5^2} dx$$

$$\Rightarrow I = \int 4 \sqrt{x^2 + \left(\frac{5}{4}\right)^2} dx$$

As I match with the form:

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\therefore I = 4 \left\{ \frac{x}{2} \sqrt{x^2 + \left(\frac{5}{4}\right)^2} + \frac{\frac{25}{16}}{2} \log \left| x + \sqrt{x^2 + \left(\frac{5}{4}\right)^2} \right| \right\}$$

$$\Rightarrow I = \frac{x}{2} \sqrt{16x^2 + 25} + \frac{25}{8} \log \left| x + \sqrt{x^2 + \frac{25}{16}} \right| + C$$

9. Question

Evaluate the integral:

$$\int \sqrt{4x^2 - 5} dx$$

Answer

Key points to solve the problem:

- Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) dx$
- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\text{Let, } I = \int \sqrt{4x^2 - 5} \, dx$$

We have:

$$I = \int \sqrt{4x^2 - 5} \, dx = \int 2 \sqrt{x^2 - \frac{5}{4}} \, dx$$

$$\Rightarrow I = 2 \int \sqrt{x^2 - \left(\frac{\sqrt{5}}{2}\right)^2} \, dx$$

As I match with the form:

$$\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\therefore I = 2 \left\{ \frac{x}{2} \sqrt{x^2 - \left(\frac{\sqrt{5}}{2}\right)^2} - \frac{\frac{5}{4}}{2} \log \left| x + \sqrt{x^2 - \left(\frac{\sqrt{5}}{2}\right)^2} \right| \right\}$$

$$\Rightarrow I = x \sqrt{x^2 - \frac{5}{4}} - \frac{5}{4} \log \left| x + \sqrt{x^2 - \frac{5}{4}} \right| + C$$

10. Question

Evaluate the integral:

$$\int \sqrt{2x^2 + 3x + 4} \, dx$$

Answer

Key points to solve the problem:

- Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx)dx$
- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} \, dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\text{Let, } I = \int \sqrt{(2x^2 + 3x + 4)} \, dx$$

$$\therefore I = \int \sqrt{2 \left\{ x^2 + 2 \left(\frac{3}{4} \right) x + \left(\frac{3}{4} \right)^2 + 2 - \left(\frac{3}{4} \right)^2 \right\}} \, dx$$

$$\text{Using } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I = \sqrt{2} \int \sqrt{\left(x + \frac{3}{4}\right)^2 + 2 - \frac{9}{16}} \, dx = \int \sqrt{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2} \, dx$$

As I match with the form:

$$\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\therefore I = \sqrt{2} \left\{ \frac{\left(x + \frac{3}{4}\right)}{2} \sqrt{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2} + \frac{\left(\frac{\sqrt{23}}{4}\right)^2}{2} \log \left| \left(x + \frac{3}{4}\right) + \sqrt{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2} \right| \right\} + C$$

$$\Rightarrow I = \frac{1}{8}(4x + 3) \sqrt{2 \left\{ \left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2 \right\}} + \frac{23\sqrt{2}}{32} \log \left| \left(x + \frac{3}{4}\right) + \sqrt{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2} \right| + C$$

$$\Rightarrow I = \frac{1}{8}(4x + 3) \sqrt{2x^2 + 3x + 4} + \frac{23\sqrt{2}}{32} \log \left| \left(x + \frac{3}{4}\right) + \sqrt{x^2 + \frac{3}{2}x + 2} \right| + C$$

11. Question

Evaluate the integral:

$$\int \sqrt{3 - 2x - 2x^2} \, dx$$

Answer

Key points to solve the problem:

- Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) dx$
- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} \, dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\text{Let, } I = \int \sqrt{3 - 2x - 2x^2} \, dx$$

$$\therefore I = \int \sqrt{3 - 2 \left(x^2 + 2 \left(\frac{1}{2} \right) x \right)} \, dx = \int \sqrt{3 - 2 \left(x^2 + 2 \left(\frac{1}{2} \right) x + \left(\frac{1}{2} \right)^2 \right) + 2 \left(\frac{1}{2} \right)^2} \, dx$$

$$\text{Using } a^2 + 2ab + b^2 = (a + b)^2$$

We have:

$$I = \int \sqrt{\frac{7}{4} - 2 \left(x + \frac{1}{2} \right)^2} \, dx = \int \sqrt{2} \sqrt{\left(\frac{\sqrt{7}}{2} \right)^2 - \left(x + \frac{1}{2} \right)^2} \, dx$$

$$\text{As } I \text{ match with the form: } \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I = \sqrt{2} \left\{ \frac{x + \frac{1}{2}}{2} \sqrt{\left(\frac{\sqrt{7}}{2} \right)^2 - \left(x + \frac{1}{2} \right)^2} + \frac{7}{2} \sin^{-1} \left(\frac{x + \frac{1}{2}}{\frac{\sqrt{7}}{2}} \right) \right\} + C$$

$$\Rightarrow I = \frac{1}{4}(2x + 1) \sqrt{2 \left\{ \left(\frac{\sqrt{7}}{2} \right)^2 - \left(x + \frac{1}{2} \right)^2 \right\}} + \frac{7\sqrt{2}}{8} \sin^{-1} \left(\frac{2x+1}{\sqrt{7}} \right) + C$$

$$\Rightarrow I = \frac{1}{4}(2x + 1) \sqrt{3 - 2x - 2x^2} + \frac{7\sqrt{2}}{8} \sin^{-1} \left(\frac{2x+1}{\sqrt{7}} \right) + C$$

12. Question

Evaluate the integral:

$$\int x\sqrt{x^4+1} \, dx$$

Answer

Key points to solve the problem:

- Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) dx$
- To solve the integrals of the form: $\int \sqrt{ax^2+bx+c} \, dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2-x^2} \, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2-a^2} \, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C$$

$$\int \sqrt{x^2+a^2} \, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C$$

$$\text{Let, } I = \int x\sqrt{x^4+1} \, dx = \int x\sqrt{(x^2)^2+1} \, dx$$

$$\text{Let, } x^2 = t$$

Differentiating both sides:

$$\Rightarrow 2x \, dx = dt \Rightarrow x \, dx = \frac{1}{2} dt$$

Substituting x^2 with t , we have:

We have:

$$I = \frac{1}{2} \int \sqrt{t^2+1} \, dt = \frac{1}{2} \int \sqrt{t^2+1^2} \, dt$$

As I match with the form:

$$\int \sqrt{x^2+a^2} \, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C$$

$$\therefore I = \frac{1}{2} \left\{ \frac{t}{2}\sqrt{t^2+1} + \frac{1}{2}\log|t+\sqrt{t^2+1}| \right\} + C$$

$$\Rightarrow I = \frac{t}{4}\sqrt{t^2+1} + \frac{1}{4}\log|t+\sqrt{t^2+1}| + C$$

Putting the value of t back:

$$\Rightarrow I = \frac{x^2}{4}\sqrt{(x^2)^2+1} + \frac{1}{4}\log|x^2+\sqrt{(x^2)^2+1}| + C$$

$$\Rightarrow I = \frac{x^2}{4}\sqrt{x^4+1} + \frac{1}{4}\log|x^2+\sqrt{x^4+1}| + C$$

13. Question

Evaluate the integral:

$$\int x^2\sqrt{a^6-x^6} \, dx$$

Answer

Key points to solve the problem:

- Such problems require the use of method of substitution along with method of integration by parts. By

method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) dx$

• To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\text{Let, } I = \int x^2 \sqrt{a^6 - x^6} dx = \int x^2 \sqrt{a^6 - (x^3)^2} dx$$

$$\text{Let, } x^3 = t$$

Differentiating both sides:

$$\Rightarrow 3x^2 dx = dt$$

$$\Rightarrow x^2 dx = \frac{1}{3} dt$$

Substituting x^3 with t , we have:

$$\therefore I = \frac{1}{3} \int \sqrt{(a^3)^2 - t^2} dt = \int \sqrt{(a^3)^2 - t^2} dt$$

$$\text{As } I \text{ match with the form: } \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I = \frac{1}{3} \left\{ \frac{t}{2} \sqrt{a^6 - (t)^2} + \frac{a^6}{2} \sin^{-1} \left(\frac{t}{a^3} \right) + C \right\}$$

Putting the value of t i.e. $t = x^3$

$$\Rightarrow I = \frac{x^3}{6} \sqrt{a^6 - x^6} + \frac{a^6}{6} \sin^{-1} \left(\frac{x^3}{a^3} \right) + C$$

14. Question

Evaluate the integral:

$$\int \frac{\sqrt{16 + (\log x)^2}}{x} dx$$

Answer

Key points to solve the problem:

• Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) dx$

• To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\text{Let, } I = \int \frac{1}{x} \sqrt{16 + (\log x)^2} dx$$

$$\text{Let, } \log x = t$$

Differentiating both sides:

$$\Rightarrow \frac{1}{x} dx = dt$$

Substituting $(\log x)$ with t , we have:

We have:

$$I = \int \sqrt{t^2 + 16} dt = \int \sqrt{t^2 + 4^2} dt$$

As I match with the form:

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$$

$$\therefore I = \left\{ \frac{t}{2} \sqrt{t^2 + 16} + \frac{16}{2} \log |t + \sqrt{t^2 + 16}| \right\} + C$$

Putting the value of t back:

$$\Rightarrow I = \frac{\log x}{2} \sqrt{(\log x)^2 + 16} + 8 \log |\log x + \sqrt{(\log x)^2 + 16}| + C$$

15. Question

Evaluate the integral:

$$\int \sqrt{2ax - x^2} dx$$

Answer

Key points to solve the problem:

- Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) dx$
- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + C$$

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$$

$$\text{Let, } I = \int \sqrt{2ax - x^2} dx$$

$$\therefore I = \int \sqrt{-(x^2 - 2(ax)x)} dx = \int \sqrt{a^2 - (x^2 - 2(a)x + (a)^2)} dx$$

$$\text{Using } a^2 - 2ab + b^2 = (a - b)^2$$

We have:

$$I = \int \sqrt{a^2 - (x - a)^2} dx = \int \sqrt{(a)^2 - (x - a)^2} dx$$

$$\text{As } I \text{ match with the form: } \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I = \frac{x-a}{2} \sqrt{(a)^2 - (x-a)^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x-a}{a} \right) + C$$

$$\Rightarrow I = \frac{1}{2}(x-a)\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$$

16. Question

Evaluate the integral:

$$\int \sqrt{3-x^2} \, dx$$

Answer

Key points to solve the problem:

- Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) dx$
- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} \, dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\text{Let, } I = \int \sqrt{3-x^2} \, dx$$

$$\therefore I = \int \sqrt{3-x^2} \, dx = \int \sqrt{(\sqrt{3})^2 - x^2} \, dx$$

$$\text{As } I \text{ match with the form: } \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\therefore I = \frac{x}{2}\sqrt{3-x^2} + \frac{3}{2}\sin^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

17. Question

Evaluate the integral:

$$\int \sqrt{x^2 - 2x} \, dx$$

Answer

Key points to solve the problem:

- Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) dx$
- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} \, dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log \left| x + \sqrt{x^2 - a^2} \right| + C$$

$$\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log \left| x + \sqrt{x^2 + a^2} \right| + C$$

$$\text{Let, } I = \int \sqrt{x^2 - 2x} \, dx$$

We have:

$$I = \int \sqrt{x^2 - 2x} dx = \int \sqrt{x^2 - 2(1)x + 1^2 - 1^2} dx$$

Using $a^2 - 2ab + b^2 = (a-b)^2$

$$I = \int \sqrt{(x-1)^2 - 1^2} dx$$

As I match with the form:

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + C$$

$$\therefore I = \frac{x-1}{2} \sqrt{(x-1)^2 - 1} - \frac{1}{2} \log |x-1 + \sqrt{(x-1)^2 - 1}| + C$$

$$\Rightarrow I = \frac{x-1}{2} \sqrt{x^2 - 2x} - \frac{1}{2} \log |x-1 + \sqrt{x^2 - 2x}| + C$$

18. Question

Evaluate the integral:

$$\int \sqrt{2x - x^2} dx$$

Answer

Key points to solve the problem:

- Such problems require the use of method of substitution along with method of integration by parts. By method of integration by parts if we have $\int f(x)g(x)dx = f(x)\int g(x)dx - \int f'(x)(\int g(x)dx) dx$
- To solve the integrals of the form: $\int \sqrt{ax^2 + bx + c} dx$ after applying substitution and integration by parts we have direct formulae as described below:

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + C$$

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$$

$$\text{Let, } I = \int \sqrt{2x - x^2} dx$$

$$\therefore I = \int \sqrt{-(x^2 - 2(1)x)} dx = \int \sqrt{1^2 - (x^2 - 2(1)x + (1)^2)} dx$$

Using $a^2 - 2ab + b^2 = (a - b)^2$

We have:

$$I = \int \sqrt{1^2 - (x-1)^2} dx = \int \sqrt{(1)^2 - (x-1)^2} dx$$

$$\text{As I match with the form: } \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\therefore I = \frac{x-1}{2} \sqrt{(1)^2 - (x-1)^2} + \frac{1^2}{2} \sin^{-1} \left(\frac{x-1}{1} \right) + C$$

$$\Rightarrow I = \frac{1}{2} (x-1) \sqrt{2x - x^2} + \frac{1}{2} \sin^{-1}(x-1) + C$$

Exercise 19.29

1. Question

Evaluate the following integrals -

$$\int (x+1)\sqrt{x^2-x+1} \, dx$$

Answer

$$\text{Let } I = \int (x+1)\sqrt{x^2-x+1} \, dx$$

$$\text{Let us assume } x+1 = \lambda \frac{d}{dx}(x^2-x+1) + \mu$$

$$\Rightarrow x+1 = \lambda \left[\frac{d}{dx}(x^2) - \frac{d}{dx}(x) + \frac{d}{dx}(1) \right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow x+1 = \lambda(2x^{2-1} - 1 + 0) + \mu$$

$$\Rightarrow x+1 = \lambda(2x-1) + \mu$$

$$\Rightarrow x+1 = 2\lambda x + \mu - \lambda$$

Comparing the coefficient of x on both sides, we get

$$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$$

Comparing the constant on both sides, we get

$$\mu - \lambda = 1$$

$$\Rightarrow \mu - \frac{1}{2} = 1$$

$$\therefore \mu = \frac{3}{2}$$

$$\text{Hence, we have } x+1 = \frac{1}{2}(2x-1) + \frac{3}{2}$$

Substituting this value in I , we can write the integral as

$$I = \int \left[\frac{1}{2}(2x-1) + \frac{3}{2} \right] \sqrt{x^2-x+1} \, dx$$

$$\Rightarrow I = \int \left[\frac{1}{2}(2x-1)\sqrt{x^2-x+1} + \frac{3}{2}\sqrt{x^2-x+1} \right] dx$$

$$\Rightarrow I = \int \frac{1}{2}(2x-1)\sqrt{x^2-x+1} \, dx + \int \frac{3}{2}\sqrt{x^2-x+1} \, dx$$

$$\Rightarrow I = \frac{1}{2} \int (2x-1)\sqrt{x^2-x+1} \, dx + \frac{3}{2} \int \sqrt{x^2-x+1} \, dx$$

$$\text{Let } I_1 = \frac{1}{2} \int (2x-1)\sqrt{x^2-x+1} \, dx$$

$$\text{Now, put } x^2-x+1 = t$$

$$\Rightarrow (2x-1)dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = \frac{1}{2} \int \sqrt{t} \, dt$$

$$\Rightarrow I_1 = \frac{1}{2} \int t^{\frac{1}{2}} \, dt$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{3} (x^2 - x + 1)^{\frac{3}{2}} + c$$

Let $I_2 = \frac{3}{2} \int \sqrt{x^2 - x + 1} dx$

We can write $x^2 - x + 1 = x^2 - 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + 1$$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = \frac{3}{2} \int \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx$$

Recall $\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln|x + \sqrt{x^2 + a^2}| + c$

$$\Rightarrow I_2 = \frac{3}{2} \left[\frac{\left(x - \frac{1}{2}\right)}{2} \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2} \ln \left| \left(x - \frac{1}{2}\right) + \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right| \right] + c$$

$$\Rightarrow I_2 = \frac{3}{2} \left[\frac{2x-1}{4} \sqrt{x^2 - x + 1} + \frac{3}{8} \ln \left| x - \frac{1}{2} + \sqrt{x^2 - x + 1} \right| \right] + c$$

$$\therefore I_2 = \frac{3}{8} (2x-1) \sqrt{x^2 - x + 1} + \frac{9}{16} \ln \left| x - \frac{1}{2} + \sqrt{x^2 - x + 1} \right| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{1}{3} (x^2 - x + 1)^{\frac{3}{2}} + \frac{3}{8} (2x-1) \sqrt{x^2 - x + 1} + \frac{9}{16} \ln \left| x - \frac{1}{2} + \sqrt{x^2 - x + 1} \right| + c$$

Thus, $\int (x+1)\sqrt{x^2-x+1}dx = \frac{1}{3}(x^2-x+1)^{\frac{3}{2}} + \frac{3}{8}(2x-1)\sqrt{x^2-x+1} + \frac{9}{16}\ln\left|x-\frac{1}{2}+\sqrt{x^2-x+1}\right| + c$

2. Question

Evaluate the following integrals -

$$\int (x+1)\sqrt{2x^2+3} dx$$

Answer

Let $I = \int (x+1)\sqrt{2x^2+3}dx$

Let us assume $x+1 = \lambda \frac{d}{dx}(2x^2+3) + \mu$

$$\Rightarrow x+1 = \lambda \left[\frac{d}{dx}(2x^2) + \frac{d}{dx}(3) \right] + \mu$$

$$\Rightarrow x+1 = \lambda \left[2 \frac{d}{dx}(x^2) + \frac{d}{dx}(3) \right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow x+1 = \lambda(2 \times 2x^{2-1} + 0) + \mu$$

$$\Rightarrow x+1 = \lambda(4x) + \mu$$

$$\Rightarrow x+1 = 4\lambda x + \mu$$

Comparing the coefficient of x on both sides, we get

$$4\lambda = 1 \Rightarrow \lambda = \frac{1}{4}$$

Comparing the constant on both sides, we get

$$\mu = 1$$

Hence, we have $x+1 = \frac{1}{4}(4x) + 1$

Substituting this value in I, we can write the integral as

$$I = \int \left[\frac{1}{4}(4x) + 1 \right] \sqrt{2x^2+3} dx$$

$$\Rightarrow I = \int \left[\frac{1}{2}(4x)\sqrt{2x^2+3} + \sqrt{2x^2+3} \right] dx$$

$$\Rightarrow I = \int \frac{1}{4}(4x)\sqrt{2x^2+3} dx + \int \sqrt{2x^2+3} dx$$

$$\Rightarrow I = \frac{1}{4} \int (4x)\sqrt{2x^2+3} dx + \int \sqrt{2x^2+3} dx$$

Let $I_1 = \frac{1}{4} \int (4x)\sqrt{2x^2+3} dx$

Now, put $2x^2+3 = t$

$$\Rightarrow (4x)dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = \frac{1}{4} \int \sqrt{t} dt$$

$$\Rightarrow I_1 = \frac{1}{4} \int t^{\frac{1}{2}} dt$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = \frac{1}{4} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{4} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{4} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{6} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{6} (2x^2 + 3)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = \int \sqrt{2x^2 + 3} dx$$

$$\text{We can write } 2x^2 + 3 = 2 \left(x^2 + \frac{3}{2} \right)$$

$$\Rightarrow 2x^2 + 3 = 2 \left[x^2 + \left(\sqrt{\frac{3}{2}} \right)^2 \right]$$

Hence, we can write I_2 as

$$I_2 = \int \sqrt{2 \left[x^2 + \left(\sqrt{\frac{3}{2}} \right)^2 \right]} dx$$

$$\Rightarrow I_2 = \sqrt{2} \int \sqrt{x^2 + \left(\sqrt{\frac{3}{2}} \right)^2} dx$$

$$\text{Recall } \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln |x + \sqrt{x^2 + a^2}| + c$$

$$\Rightarrow I_2 = \sqrt{2} \left[\frac{x}{2} \sqrt{x^2 + \left(\sqrt{\frac{3}{2}} \right)^2} + \frac{\left(\sqrt{\frac{3}{2}} \right)^2}{2} \ln \left| x + \sqrt{x^2 + \left(\sqrt{\frac{3}{2}} \right)^2} \right| \right] + c$$

$$\Rightarrow I_2 = \sqrt{2} \left[\frac{x}{2} \sqrt{x^2 + \frac{3}{2}} + \frac{3}{4} \ln \left| x + \sqrt{x^2 + \frac{3}{2}} \right| \right] + c$$

$$\Rightarrow I_2 = \sqrt{2} \left[\frac{x}{2\sqrt{2}} \sqrt{2x^2 + 3} + \frac{3}{2 \times 2} \ln \left| x + \sqrt{x^2 + \frac{3}{2}} \right| \right] + c$$

$$\therefore I_2 = \frac{x}{2}\sqrt{2x^2+3} + \frac{3}{2\sqrt{2}}\ln\left|x + \sqrt{x^2 + \frac{3}{2}}\right| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{1}{6}(2x^2+3)^{\frac{3}{2}} + \frac{x}{2}\sqrt{2x^2+3} + \frac{3}{2\sqrt{2}}\ln\left|x + \sqrt{x^2 + \frac{3}{2}}\right| + c$$

$$\text{Thus, } \int (x+1)\sqrt{2x^2+3}dx = \frac{1}{6}(2x^2+3)^{\frac{3}{2}} + \frac{x}{2}\sqrt{2x^2+3} + \frac{3}{2\sqrt{2}}\ln\left|x + \sqrt{x^2 + \frac{3}{2}}\right| + c$$

3. Question

Evaluate the following integrals -

$$\int (2x-5)\sqrt{2+3x-x^2} dx$$

Answer

$$\text{Let } I = \int (2x-5)\sqrt{2+3x-x^2} dx$$

$$\text{Let us assume } 2x-5 = \lambda \frac{d}{dx}(2+3x-x^2) + \mu$$

$$\Rightarrow 2x-5 = \lambda \left[\frac{d}{dx}(2) + \frac{d}{dx}(3x) - \frac{d}{dx}(x^2) \right] + \mu$$

$$\Rightarrow 2x-5 = \lambda \left[\frac{d}{dx}(2) + 3 \frac{d}{dx}(x) - \frac{d}{dx}(x^2) \right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow 2x-5 = \lambda(0+3-2x^{2-1}) + \mu$$

$$\Rightarrow 2x-5 = \lambda(3-2x) + \mu$$

$$\Rightarrow 2x-5 = -2\lambda x + 3\lambda + \mu$$

Comparing the coefficient of x on both sides, we get

$$-2\lambda = 2 \Rightarrow \lambda = -1$$

Comparing the constant on both sides, we get

$$3\lambda + \mu = -5$$

$$\Rightarrow 3(-1) + \mu = -5$$

$$\Rightarrow -3 + \mu = -5$$

$$\therefore \mu = -2$$

$$\text{Hence, we have } 2x-5 = -(3-2x) - 2$$

Substituting this value in I , we can write the integral as

$$I = \int [-(3-2x) - 2]\sqrt{2+3x-x^2} dx$$

$$\Rightarrow I = \int [-(3-2x)\sqrt{2+3x-x^2} - 2\sqrt{2+3x-x^2}] dx$$

$$\Rightarrow I = - \int (3-2x)\sqrt{2+3x-x^2} dx - \int 2\sqrt{2+3x-x^2} dx$$

$$\Rightarrow I = - \int (3 - 2x)\sqrt{2 + 3x - x^2} dx - 2 \int \sqrt{2 + 3x - x^2} dx$$

$$\text{Let } I_1 = - \int (3 - 2x)\sqrt{2 + 3x - x^2} dx$$

$$\text{Now, put } 2 + 3x - x^2 = t$$

$$\Rightarrow (3 - 2x)dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = - \int \sqrt{t} dt$$

$$\Rightarrow I_1 = - \int t^{\frac{1}{2}} dt$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = - \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$\Rightarrow I_1 = - \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = - \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = - \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = - \frac{2}{3} (2 + 3x - x^2)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = -2 \int \sqrt{2 + 3x - x^2} dx$$

$$\text{We can write } 2 + 3x - x^2 = -(x^2 - 3x - 2)$$

$$\Rightarrow 2 + 3x - x^2 = - \left[x^2 - 2(x) \left(\frac{3}{2} \right) + \left(\frac{3}{2} \right)^2 - \left(\frac{3}{2} \right)^2 - 2 \right]$$

$$\Rightarrow 2 + 3x - x^2 = - \left[\left(x - \frac{3}{2} \right)^2 - \frac{9}{4} - 2 \right]$$

$$\Rightarrow 2 + 3x - x^2 = - \left[\left(x - \frac{3}{2} \right)^2 - \frac{17}{4} \right]$$

$$\Rightarrow 2 + 3x - x^2 = \frac{17}{4} - \left(x - \frac{3}{2} \right)^2$$

$$\Rightarrow 2 + 3x - x^2 = \left(\frac{\sqrt{17}}{2} \right)^2 - \left(x - \frac{3}{2} \right)^2$$

Hence, we can write I_2 as

$$I_2 = -2 \int \sqrt{\left(\frac{\sqrt{17}}{2} \right)^2 - \left(x - \frac{3}{2} \right)^2} dx$$

$$\text{Recall } \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$$

$$\Rightarrow I_2 = -2 \left[\frac{\left(x - \frac{3}{2}\right)}{2} \sqrt{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2} + \frac{\left(\frac{\sqrt{17}}{2}\right)^2}{2} \sin^{-1} \left(\frac{x - \frac{3}{2}}{\frac{\sqrt{17}}{2}} \right) \right] + c$$

$$\Rightarrow I_2 = -2 \left[\frac{2x-3}{4} \sqrt{2+3x-x^2} + \frac{17}{8} \sin^{-1} \left(\frac{2x-3}{\sqrt{17}} \right) \right] + c$$

$$\therefore I_2 = -\frac{1}{2} (2x-3) \sqrt{2+3x-x^2} - \frac{17}{4} \sin^{-1} \left(\frac{2x-3}{\sqrt{17}} \right) + c$$

Substituting I_1 and I_2 in I , we get

$$I = -\frac{2}{3} (2+3x-x^2)^{\frac{3}{2}} - \frac{1}{2} (2x-3) \sqrt{2+3x-x^2} - \frac{17}{4} \sin^{-1} \left(\frac{2x-3}{\sqrt{17}} \right) + c$$

Thus, $\int (2x-5) \sqrt{2+3x-x^2} dx = -\frac{2}{3} (2+3x-x^2)^{\frac{3}{2}} - \frac{1}{2} (2x-3) \sqrt{2+3x-x^2} - \frac{17}{4} \sin^{-1} \left(\frac{2x-3}{\sqrt{17}} \right) + c$

4. Question

Evaluate the following integrals -

$$\int (x+2) \sqrt{x^2+x+1} dx$$

Answer

Let $I = \int (x+2) \sqrt{x^2+x+1} dx$

Let us assume $x+2 = \lambda \frac{d}{dx} (x^2+x+1) + \mu$

$$\Rightarrow x+2 = \lambda \left[\frac{d}{dx} (x^2) + \frac{d}{dx} (x) + \frac{d}{dx} (1) \right] + \mu$$

We know $\frac{d}{dx} (x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow x+2 = \lambda (2x^{2-1} + 1 + 0) + \mu$$

$$\Rightarrow x+2 = \lambda (2x+1) + \mu$$

$$\Rightarrow x+2 = 2\lambda x + \lambda + \mu$$

Comparing the coefficient of x on both sides, we get

$$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$$

Comparing the constant on both sides, we get

$$\lambda + \mu = 2$$

$$\Rightarrow \frac{1}{2} + \mu = 2$$

$$\therefore \mu = \frac{3}{2}$$

Hence, we have $x+2 = \frac{1}{2} (2x+1) + \frac{3}{2}$

Substituting this value in I , we can write the integral as

$$I = \int \left[\frac{1}{2} (2x+1) + \frac{3}{2} \right] \sqrt{x^2+x+1} dx$$

$$\Rightarrow I = \int \left[\frac{1}{2} (2x+1) \sqrt{x^2+x+1} + \frac{3}{2} \sqrt{x^2+x+1} \right] dx$$

$$\Rightarrow I = \int \frac{1}{2} (2x+1) \sqrt{x^2+x+1} dx + \int \frac{3}{2} \sqrt{x^2+x+1} dx$$

$$\Rightarrow I = \frac{1}{2} \int (2x+1) \sqrt{x^2+x+1} dx + \frac{3}{2} \int \sqrt{x^2+x+1} dx$$

$$\text{Let } I_1 = \frac{1}{2} \int (2x+1) \sqrt{x^2+x+1} dx$$

Now, put $x^2 + x + 1 = t$

$$\Rightarrow (2x+1)dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = \frac{1}{2} \int \sqrt{t} dt$$

$$\Rightarrow I_1 = \frac{1}{2} \int t^{\frac{1}{2}} dt$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{3} (x^2+x+1)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = \frac{3}{2} \int \sqrt{x^2+x+1} dx$$

$$\text{We can write } x^2+x+1 = x^2 + 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$$

$$\Rightarrow x^2+x+1 = \left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + 1$$

$$\Rightarrow x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = \frac{3}{2} \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx$$

$$\text{Recall } \int \sqrt{x^2+a^2} dx = \frac{x}{2} \sqrt{x^2+a^2} + \frac{a^2}{2} \ln|x+\sqrt{x^2+a^2}| + c$$

$$\Rightarrow I_2 = \frac{3}{2} \left[\frac{\left(x + \frac{1}{2}\right)}{2} \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2} \ln \left| \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right| \right] + c$$

$$\Rightarrow I_2 = \frac{3}{2} \left[\frac{2x+1}{4} \sqrt{x^2+x+1} + \frac{3}{8} \ln \left| x + \frac{1}{2} + \sqrt{x^2+x+1} \right| \right] + c$$

$$\therefore I_2 = \frac{3}{8} (2x+1) \sqrt{x^2+x+1} + \frac{9}{16} \ln \left| x + \frac{1}{2} + \sqrt{x^2+x+1} \right| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{1}{3} (x^2+x+1)^{\frac{3}{2}} + \frac{3}{8} (2x+1) \sqrt{x^2+x+1} + \frac{9}{16} \ln \left| x + \frac{1}{2} + \sqrt{x^2+x+1} \right| + c$$

Thus, $\int (x+2) \sqrt{x^2+x+1} dx = \frac{1}{3} (x^2+x+1)^{\frac{3}{2}} + \frac{3}{8} (2x+1) \sqrt{x^2+x+1} + \frac{9}{16} \ln \left| x + \frac{1}{2} + \sqrt{x^2+x+1} \right| + c$

5. Question

Evaluate the following integrals -

$$\int (4x+1) \sqrt{x^2-x-2} dx$$

Answer

Let $I = \int (4x+1) \sqrt{x^2-x-2} dx$

Let us assume $4x+1 = \lambda \frac{d}{dx} (x^2-x-2) + \mu$

$$\Rightarrow 4x+1 = \lambda \left[\frac{d}{dx} (x^2) - \frac{d}{dx} (x) - \frac{d}{dx} (2) \right] + \mu$$

We know $\frac{d}{dx} (x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow 4x+1 = \lambda (2x^{2-1} - 1 - 0) + \mu$$

$$\Rightarrow 4x+1 = \lambda (2x-1) + \mu$$

$$\Rightarrow 4x+1 = 2\lambda x + \mu - \lambda$$

Comparing the coefficient of x on both sides, we get

$$2\lambda = 4 \Rightarrow \lambda = \frac{4}{2} = 2$$

Comparing the constant on both sides, we get

$$\mu - \lambda = 1$$

$$\Rightarrow \mu - 2 = 1$$

$$\therefore \mu = 3$$

Hence, we have $4x+1 = 2(2x-1) + 3$

Substituting this value in I , we can write the integral as

$$I = \int [2(2x-1) + 3] \sqrt{x^2-x-2} dx$$

$$\Rightarrow I = \int \left[2(2x-1)\sqrt{x^2-x-2} + 3\sqrt{x^2-x-2} \right] dx$$

$$\Rightarrow I = \int 2(2x-1)\sqrt{x^2-x-2} dx + \int 3\sqrt{x^2-x-2} dx$$

$$\Rightarrow I = 2 \int (2x-1)\sqrt{x^2-x-2} dx + 3 \int \sqrt{x^2-x-2} dx$$

$$\text{Let } I_1 = 2 \int (2x-1)\sqrt{x^2-x-2} dx$$

$$\text{Now, put } x^2 - x - 2 = t$$

$$\Rightarrow (2x-1)dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = 2 \int \sqrt{t} dt$$

$$\Rightarrow I_1 = 2 \int t^{\frac{1}{2}} dt$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = 2 \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = 2 \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = 2 \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{4}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{4}{3} (x^2 - x - 2)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = 3 \int \sqrt{x^2-x-2} dx$$

$$\text{We can write } x^2 - x - 2 = x^2 - 2(x) \left(\frac{1}{2} \right) + \left(\frac{1}{2} \right)^2 - \left(\frac{1}{2} \right)^2 - 2$$

$$\Rightarrow x^2 - x - 2 = \left(x - \frac{1}{2} \right)^2 - \frac{1}{4} - 2$$

$$\Rightarrow x^2 - x - 2 = \left(x - \frac{1}{2} \right)^2 - \frac{9}{4}$$

$$\Rightarrow x^2 - x - 2 = \left(x - \frac{1}{2} \right)^2 - \left(\frac{3}{2} \right)^2$$

Hence, we can write I_2 as

$$I_2 = 3 \int \sqrt{\left(x - \frac{1}{2} \right)^2 - \left(\frac{3}{2} \right)^2} dx$$

$$\text{Recall } \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + c$$

$$\Rightarrow I_2 = 3 \left[\frac{\left(x - \frac{1}{2}\right)}{2} \sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} - \frac{\left(\frac{3}{2}\right)^2}{2} \ln \left| \left(x - \frac{1}{2}\right) + \sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} \right| \right] + c$$

$$\Rightarrow I_2 = 3 \left[\frac{2x-1}{4} \sqrt{x^2-x-2} - \frac{9}{8} \ln \left| x - \frac{1}{2} + \sqrt{x^2-x-2} \right| \right] + c$$

$$\therefore I_2 = \frac{3}{4} (2x-1) \sqrt{x^2-x-2} - \frac{27}{8} \ln \left| x - \frac{1}{2} + \sqrt{x^2-x-2} \right| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{4}{3} (x^2-x-2)^{\frac{3}{2}} + \frac{3}{4} (2x-1) \sqrt{x^2-x-2} - \frac{27}{8} \ln \left| x - \frac{1}{2} + \sqrt{x^2-x-2} \right| + c$$

Thus, $\int (4x+1) \sqrt{x^2-x-2} dx = \frac{4}{3} (x^2-x-2)^{\frac{3}{2}} + \frac{3}{4} (2x-1) \sqrt{x^2-x-2} - \frac{27}{8} \ln \left| x - \frac{1}{2} + \sqrt{x^2-x-2} \right| + c$

6. Question

Evaluate the following integrals -

$$\int (x-2) \sqrt{2x^2-6x+5} dx$$

Answer

Let $I = \int (x-2) \sqrt{2x^2-6x+5} dx$

Let us assume $x-2 = \lambda \frac{d}{dx} (2x^2-6x+5) + \mu$

$$\Rightarrow x-2 = \lambda \left[\frac{d}{dx} (2x^2) - \frac{d}{dx} (6x) - \frac{d}{dx} (5) \right] + \mu$$

$$\Rightarrow x-2 = \lambda \left[2 \frac{d}{dx} (x^2) - 6 \frac{d}{dx} (x) - \frac{d}{dx} (5) \right] + \mu$$

We know $\frac{d}{dx} (x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow x-2 = \lambda (2 \times 2x^{2-1} - 6 - 0) + \mu$$

$$\Rightarrow x-2 = \lambda (4x-6) + \mu$$

$$\Rightarrow x-2 = 4\lambda x + \mu - 6\lambda$$

Comparing the coefficient of x on both sides, we get

$$4\lambda = 1 \Rightarrow \lambda = \frac{1}{4}$$

Comparing the constant on both sides, we get

$$\mu - 6\lambda = -2$$

$$\Rightarrow \mu - 6 \left(\frac{1}{4} \right) = -2$$

$$\Rightarrow \mu - \frac{3}{2} = -2$$

$$\therefore \mu = -\frac{1}{2}$$

Hence, we have $x-2 = \frac{1}{4} (4x-6) - \frac{1}{2}$

Substituting this value in I, we can write the integral as

$$\begin{aligned}
 I &= \int \left[\frac{1}{4}(4x-6) - \frac{1}{2} \right] \sqrt{2x^2-6x+5} dx \\
 \Rightarrow I &= \int \left[\frac{1}{4}(4x-6)\sqrt{2x^2-6x+5} - \frac{1}{2}\sqrt{2x^2-6x+5} \right] dx \\
 \Rightarrow I &= \int \frac{1}{4}(4x-6)\sqrt{2x^2-6x+5} dx - \int \frac{1}{2}\sqrt{2x^2-6x+5} dx \\
 \Rightarrow I &= \frac{1}{4} \int (4x-6)\sqrt{2x^2-6x+5} dx - \frac{1}{2} \int \sqrt{2x^2-6x+5} dx
 \end{aligned}$$

$$\text{Let } I_1 = \frac{1}{4} \int (4x-6)\sqrt{2x^2-6x+5} dx$$

Now, put $2x^2 - 6x + 5 = t$

$$\Rightarrow (4x-6)dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = \frac{1}{4} \int \sqrt{t} dt$$

$$\Rightarrow I_1 = \frac{1}{4} \int t^{\frac{1}{2}} dt$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = \frac{1}{4} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{4} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{4} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{6} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{6} (2x^2 - 6x + 5)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = -\frac{1}{2} \int \sqrt{2x^2-6x+5} dx$$

$$\text{We can write } 2x^2 - 6x + 5 = 2 \left(x^2 - 3x + \frac{5}{2} \right)$$

$$\Rightarrow 2x^2 - 6x + 5 = 2 \left[x^2 - 2(x) \left(\frac{3}{2} \right) + \left(\frac{3}{2} \right)^2 - \left(\frac{3}{2} \right)^2 + \frac{5}{2} \right]$$

$$\Rightarrow 2x^2 - 6x + 5 = 2 \left[\left(x - \frac{3}{2} \right)^2 - \frac{9}{4} + \frac{5}{2} \right]$$

$$\Rightarrow 2x^2 - 6x + 5 = 2 \left[\left(x - \frac{3}{2} \right)^2 + \frac{1}{4} \right]$$

$$\Rightarrow 2x^2 - 6x + 5 = 2 \left[\left(x - \frac{3}{2} \right)^2 + \left(\frac{1}{2} \right)^2 \right]$$

Hence, we can write I_2 as

$$I_2 = -\frac{1}{2} \int \sqrt{2 \left[\left(x - \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \right]} dx$$

$$\Rightarrow I_2 = -\frac{\sqrt{2}}{2} \int \sqrt{\left(x - \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} dx$$

$$\Rightarrow I_2 = -\frac{1}{\sqrt{2}} \int \sqrt{\left(x - \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} dx$$

Recall $\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln|x + \sqrt{x^2 + a^2}| + c$

$$\Rightarrow I_2 = -\frac{1}{\sqrt{2}} \left[\frac{\left(x - \frac{3}{2}\right)}{2} \sqrt{\left(x - \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} + \frac{\left(\frac{1}{2}\right)^2}{2} \ln \left| \left(x - \frac{3}{2}\right) + \sqrt{\left(x - \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \right| \right] + c$$

$$\Rightarrow I_2 = -\frac{1}{\sqrt{2}} \left[\frac{2x-3}{4} \sqrt{x^2 - 3x + \frac{5}{2}} + \frac{1}{8} \ln \left| x - \frac{3}{2} + \sqrt{x^2 - 3x + \frac{5}{2}} \right| \right] + c$$

$$\Rightarrow I_2 = -\frac{1}{\sqrt{2}} \left[\frac{2x-3}{4\sqrt{2}} \sqrt{2x^2 - 6x + 5} + \frac{1}{8} \ln \left| x - \frac{3}{2} + \sqrt{x^2 - 3x + \frac{5}{2}} \right| \right] + c$$

$$\therefore I_2 = -\frac{1}{8} (2x-3) \sqrt{2x^2 - 6x + 5} - \frac{1}{8\sqrt{2}} \ln \left| x - \frac{3}{2} + \sqrt{x^2 - 3x + \frac{5}{2}} \right| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{1}{6} (2x^2 - 6x + 5)^{\frac{3}{2}} - \frac{1}{8} (2x-3) \sqrt{2x^2 - 6x + 5} - \frac{1}{8\sqrt{2}} \ln \left| x - \frac{3}{2} + \sqrt{x^2 - 3x + \frac{5}{2}} \right| + c$$

$$\text{Thus, } \int (x-2) \sqrt{2x^2 - 6x + 5} dx = \frac{1}{6} (2x^2 - 6x + 5)^{\frac{3}{2}} - \frac{1}{8} (2x-3) \sqrt{2x^2 - 6x + 5} - \frac{1}{8\sqrt{2}} \ln \left| x - \frac{3}{2} + \sqrt{x^2 - 3x + \frac{5}{2}} \right| + c$$

7. Question

Evaluate the following integrals -

$$\int (x+1) \sqrt{x^2 + x + 1} dx$$

Answer

$$\text{Let } I = \int (x+1) \sqrt{x^2 + x + 1} dx$$

$$\text{Let us assume } x+1 = \lambda \frac{d}{dx} (x^2 + x + 1) + \mu$$

$$\Rightarrow x + 1 = \lambda \left[\frac{d}{dx}(x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(1) \right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow x + 1 = \lambda(2x^{2-1} + 1 + 0) + \mu$$

$$\Rightarrow x + 1 = \lambda(2x + 1) + \mu$$

$$\Rightarrow x + 1 = 2\lambda x + \lambda + \mu$$

Comparing the coefficient of x on both sides, we get

$$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$$

Comparing the constant on both sides, we get

$$\lambda + \mu = 1$$

$$\Rightarrow \frac{1}{2} + \mu = 1$$

$$\therefore \mu = \frac{1}{2}$$

$$\text{Hence, we have } x + 1 = \frac{1}{2}(2x + 1) + \frac{1}{2}$$

Substituting this value in I , we can write the integral as

$$I = \int \left[\frac{1}{2}(2x + 1) + \frac{1}{2} \right] \sqrt{x^2 + x + 1} dx$$

$$\Rightarrow I = \int \left[\frac{1}{2}(2x + 1)\sqrt{x^2 + x + 1} + \frac{1}{2}\sqrt{x^2 + x + 1} \right] dx$$

$$\Rightarrow I = \int \frac{1}{2}(2x + 1)\sqrt{x^2 + x + 1} dx + \int \frac{1}{2}\sqrt{x^2 + x + 1} dx$$

$$\Rightarrow I = \frac{1}{2} \int (2x + 1)\sqrt{x^2 + x + 1} dx + \frac{1}{2} \int \sqrt{x^2 + x + 1} dx$$

$$\text{Let } I_1 = \frac{1}{2} \int (2x + 1)\sqrt{x^2 + x + 1} dx$$

Now, put $x^2 + x + 1 = t$

$$\Rightarrow (2x + 1)dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = \frac{1}{2} \int \sqrt{t} dt$$

$$\Rightarrow I_1 = \frac{1}{2} \int t^{\frac{1}{2}} dt$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{3} (x^2 + x + 1)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = \frac{1}{2} \int \sqrt{x^2 + x + 1} dx$$

$$\text{We can write } x^2 + x + 1 = x^2 + 2(x) \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$$

$$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 1$$

$$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Hence, we can write I_2 as

$$I_2 = \frac{1}{2} \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx$$

$$\text{Recall } \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln|x + \sqrt{x^2 + a^2}| + c$$

$$\Rightarrow I_2 = \frac{1}{2} \left[\frac{\left(x + \frac{1}{2}\right)}{2} \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2} \ln \left| \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right| \right] + c$$

$$\Rightarrow I_2 = \frac{1}{2} \left[\frac{2x+1}{4} \sqrt{x^2 + x + 1} + \frac{3}{8} \ln \left| x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right| \right] + c$$

$$\therefore I_2 = \frac{1}{8} (2x+1) \sqrt{x^2 + x + 1} + \frac{3}{16} \ln \left| x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{1}{3} (x^2 + x + 1)^{\frac{3}{2}} + \frac{1}{8} (2x+1) \sqrt{x^2 + x + 1} + \frac{3}{16} \ln \left| x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right| + c$$

$$\text{Thus, } \int (x+1) \sqrt{x^2 + x + 1} dx = \frac{1}{3} (x^2 + x + 1)^{\frac{3}{2}} + \frac{1}{8} (2x+1) \sqrt{x^2 + x + 1} + \frac{3}{16} \ln \left| x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right| + c$$

8. Question

Evaluate the following integrals -

$$\int (2x+3) \sqrt{x^2 + 4x + 3} dx$$

Answer

$$\text{Let } I = \int (2x + 3)\sqrt{x^2 + 4x + 3} dx$$

$$\text{Let us assume } 2x + 3 = \lambda \frac{d}{dx}(x^2 + 4x + 3) + \mu$$

$$\Rightarrow 2x + 3 = \lambda \left[\frac{d}{dx}(x^2) + \frac{d}{dx}(4x) + \frac{d}{dx}(3) \right] + \mu$$

$$\Rightarrow 2x + 3 = \lambda \left[\frac{d}{dx}(x^2) + 4 \frac{d}{dx}(x) + \frac{d}{dx}(3) \right] + \mu$$

$$\text{We know } \frac{d}{dx}(x^n) = nx^{n-1} \text{ and derivative of a constant is 0.}$$

$$\Rightarrow 2x + 3 = \lambda(2x^{2-1} + 4 + 0) + \mu$$

$$\Rightarrow 2x + 3 = \lambda(2x + 4) + \mu$$

$$\Rightarrow 2x + 3 = 2\lambda x + 4\lambda + \mu$$

Comparing the coefficient of x on both sides, we get

$$2\lambda = 2 \Rightarrow \lambda = 1$$

Comparing the constant on both sides, we get

$$4\lambda + \mu = 3$$

$$\Rightarrow 4(1) + \mu = 3$$

$$\Rightarrow 4 + \mu = 3$$

$$\therefore \mu = -1$$

$$\text{Hence, we have } 2x + 3 = (2x + 4) - 1$$

Substituting this value in I , we can write the integral as

$$I = \int [(2x + 4) - 1]\sqrt{x^2 + 4x + 3} dx$$

$$\Rightarrow I = \int \left[(2x + 4)\sqrt{x^2 + 4x + 3} - \sqrt{x^2 + 4x + 3} \right] dx$$

$$\Rightarrow I = \int (2x + 4)\sqrt{x^2 + 4x + 3} dx - \int \sqrt{x^2 + 4x + 3} dx$$

$$\text{Let } I_1 = \int (2x + 4)\sqrt{x^2 + 4x + 3} dx$$

$$\text{Now, put } x^2 + 4x + 3 = t$$

$$\Rightarrow (2x + 4)dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = \int \sqrt{t} dt$$

$$\Rightarrow I_1 = \int t^{\frac{1}{2}} dt$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$\Rightarrow I_1 = \frac{\frac{3}{2}t^2}{\frac{2}{2}} + c$$

$$\Rightarrow I_1 = \frac{2}{3}t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{2}{3}(x^2 + 4x + 3)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = - \int \sqrt{x^2 + 4x + 3} dx$$

$$\text{We can write } x^2 + 4x + 3 = x^2 + 2(x)(2) + 2^2 - 2^2 + 3$$

$$\Rightarrow x^2 + 4x + 3 = (x + 2)^2 - 4 + 3$$

$$\Rightarrow x^2 + 4x + 3 = (x + 2)^2 - 1$$

$$\Rightarrow x^2 + 4x + 3 = (x + 2)^2 - 1^2$$

Hence, we can write I_2 as

$$I_2 = - \int \sqrt{(x + 2)^2 - 1^2} dx$$

$$\text{Recall } \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + c$$

$$\Rightarrow I_2 = - \left[\frac{(x + 2)}{2} \sqrt{(x + 2)^2 - 1^2} - \frac{1^2}{2} \ln|(x + 2) + \sqrt{(x + 2)^2 - 1^2}| \right] + c$$

$$\Rightarrow I_2 = - \left[\frac{(x + 2)}{2} \sqrt{x^2 + 4x + 3} - \frac{1}{2} \ln|x + 2 + \sqrt{x^2 + 4x + 3}| \right] + c$$

$$\therefore I_2 = -\frac{1}{2}(x + 2)\sqrt{x^2 + 4x + 3} + \frac{1}{2} \ln|x + 2 + \sqrt{x^2 + 4x + 3}| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{2}{3}(x^2 + 4x + 3)^{\frac{3}{2}} - \frac{1}{2}(x + 2)\sqrt{x^2 + 4x + 3} + \frac{1}{2} \ln|x + 2 + \sqrt{x^2 + 4x + 3}| + c$$

$$\text{Thus, } \int (2x + 3)\sqrt{x^2 + 4x + 3} dx = \frac{2}{3}(x^2 + 4x + 3)^{\frac{3}{2}} - \frac{1}{2}(x + 2)\sqrt{x^2 + 4x + 3} + \frac{1}{2} \ln|x + 2 + \sqrt{x^2 + 4x + 3}| + c$$

9. Question

Evaluate the following integrals -

$$\int (2x - 4)\sqrt{x^2 - 4x + 3} dx$$

Answer

$$\text{Let } I = \int (2x - 5)\sqrt{x^2 - 4x + 3} dx$$

$$\text{Let us assume } 2x - 5 = \lambda \frac{d}{dx}(x^2 - 4x + 3) + \mu$$

$$\Rightarrow 2x - 5 = \lambda \left[\frac{d}{dx}(x^2) - \frac{d}{dx}(4x) + \frac{d}{dx}(3) \right] + \mu$$

$$\Rightarrow 2x - 5 = \lambda \left[\frac{d}{dx}(x^2) - 4 \frac{d}{dx}(x) + \frac{d}{dx}(3) \right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow 2x - 5 = \lambda(2x^{2-1} - 4 + 0) + \mu$$

$$\Rightarrow 2x - 5 = \lambda(2x - 4) + \mu$$

$$\Rightarrow 2x - 5 = 2\lambda x + \mu - 4\lambda$$

Comparing the coefficient of x on both sides, we get

$$2\lambda = 2 \Rightarrow \lambda = 1$$

Comparing the constant on both sides, we get

$$\mu - 4\lambda = -5$$

$$\Rightarrow \mu - 4(1) = -5$$

$$\Rightarrow \mu - 4 = -5$$

$$\therefore \mu = -1$$

Hence, we have $2x - 5 = (2x - 4) - 1$

Substituting this value in I, we can write the integral as

$$I = \int [(2x - 4) - 1]\sqrt{x^2 - 4x + 3} dx$$

$$\Rightarrow I = \int [(2x - 4)\sqrt{x^2 - 4x + 3} - \sqrt{x^2 - 4x + 3}] dx$$

$$\Rightarrow I = \int (2x - 4)\sqrt{x^2 - 4x + 3} dx - \int \sqrt{x^2 - 4x + 3} dx$$

$$\text{Let } I_1 = \int (2x - 4)\sqrt{x^2 - 4x + 3} dx$$

Now, put $x^2 - 4x + 3 = t$

$$\Rightarrow (2x - 4)dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = \int \sqrt{t} dt$$

$$\Rightarrow I_1 = \int t^{\frac{1}{2}} dt$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$\Rightarrow I_1 = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{2}{3} (x^2 - 4x + 3)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = - \int \sqrt{x^2 - 4x + 3} dx$$

We can write $x^2 - 4x + 3 = x^2 - 2(x)(2) + 2^2 - 2^2 + 3$

$$\Rightarrow x^2 - 4x + 3 = (x - 2)^2 - 4 + 3$$

$$\Rightarrow x^2 - 4x + 3 = (x - 2)^2 - 1$$

$$\Rightarrow x^2 - 4x + 3 = (x - 2)^2 - 1^2$$

Hence, we can write I_2 as

$$I_2 = - \int \sqrt{(x-2)^2 - 1^2} dx$$

$$\text{Recall } \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + c$$

$$\Rightarrow I_2 = - \left[\frac{(x-2)}{2} \sqrt{(x-2)^2 - 1^2} - \frac{1^2}{2} \ln|(x-2) + \sqrt{(x-2)^2 - 1^2}| \right] + c$$

$$\Rightarrow I_2 = - \left[\frac{(x-2)}{2} \sqrt{x^2 - 4x + 3} - \frac{1}{2} \ln|x - 2 + \sqrt{x^2 - 4x + 3}| \right] + c$$

$$\therefore I_2 = -\frac{1}{2}(x-2)\sqrt{x^2 - 4x + 3} + \frac{1}{2} \ln|x - 2 + \sqrt{x^2 - 4x + 3}| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{2}{3}(x^2 - 4x + 3)^{\frac{3}{2}} - \frac{1}{2}(x-2)\sqrt{x^2 - 4x + 3} + \frac{1}{2} \ln|x - 2 + \sqrt{x^2 - 4x + 3}| + c$$

$$\text{Thus, } \int (2x-5)\sqrt{x^2 - 4x + 3} dx = \frac{2}{3}(x^2 - 4x + 3)^{\frac{3}{2}} - \frac{1}{2}(x-2)\sqrt{x^2 - 4x + 3} + \frac{1}{2} \ln|x - 2 + \sqrt{x^2 - 4x + 3}| + c$$

10. Question

Evaluate the following integrals -

$$\int x\sqrt{x^2 + x} dx$$

Answer

$$\text{Let } I = \int x\sqrt{x^2 + x} dx$$

$$\text{Let us assume } x = \lambda \frac{d}{dx}(x^2 + x) + \mu$$

$$\Rightarrow x = \lambda \left[\frac{d}{dx}(x^2) + \frac{d}{dx}(x) \right] + \mu$$

$$\text{We know } \frac{d}{dx}(x^n) = nx^{n-1}$$

$$\Rightarrow x = \lambda(2x^{2-1} + 1) + \mu$$

$$\Rightarrow x = \lambda(2x + 1) + \mu$$

$$\Rightarrow x = 2\lambda x + \lambda + \mu$$

Comparing the coefficient of x on both sides, we get

$$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$$

Comparing the constant on both sides, we get

$$\lambda + \mu = 0$$

$$\Rightarrow \frac{1}{2} + \mu = 0$$

$$\therefore \mu = -\frac{1}{2}$$

$$\text{Hence, we have } x = \frac{1}{2}(2x + 1) - \frac{1}{2}$$

Substituting this value in I, we can write the integral as

$$I = \int \left[\frac{1}{2}(2x + 1) - \frac{1}{2} \right] \sqrt{x^2 + x} dx$$

$$\Rightarrow I = \int \left[\frac{1}{2}(2x + 1) \sqrt{x^2 + x} - \frac{1}{2} \sqrt{x^2 + x} \right] dx$$

$$\Rightarrow I = \int \frac{1}{2}(2x + 1) \sqrt{x^2 + x} dx - \int \frac{1}{2} \sqrt{x^2 + x} dx$$

$$\Rightarrow I = \frac{1}{2} \int (2x + 1) \sqrt{x^2 + x} dx - \frac{1}{2} \int \sqrt{x^2 + x} dx$$

$$\text{Let } I_1 = \frac{1}{2} \int (2x + 1) \sqrt{x^2 + x} dx$$

$$\text{Now, put } x^2 + x = t$$

$$\Rightarrow (2x + 1)dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = \frac{1}{2} \int \sqrt{t} dt$$

$$\Rightarrow I_1 = \frac{1}{2} \int t^{\frac{1}{2}} dt$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = \frac{1}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = \frac{1}{3} (x^2 + x)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = -\frac{1}{2} \int \sqrt{x^2 + x} dx$$

$$\text{We can write } x^2 + x = x^2 + 2(x) \left(\frac{1}{2} \right) + \left(\frac{1}{2} \right)^2 - \left(\frac{1}{2} \right)^2$$

$$\Rightarrow x^2 + x = \left(x + \frac{1}{2} \right)^2 - \left(\frac{1}{2} \right)^2$$

Hence, we can write I_2 as

$$I_2 = -\frac{1}{2} \int \sqrt{\left(x + \frac{1}{2} \right)^2 - \left(\frac{1}{2} \right)^2} dx$$

Recall $\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + c$

$$\Rightarrow I_2 = -\frac{1}{2} \left[\frac{\left(x + \frac{1}{2}\right)}{2} \sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} - \frac{\left(\frac{1}{2}\right)^2}{2} \ln \left| \left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right| \right] + c$$

$$\Rightarrow I_2 = -\frac{1}{2} \left[\frac{2x+1}{4} \sqrt{x^2+x} - \frac{1}{8} \ln \left| x + \frac{1}{2} + \sqrt{x^2+x} \right| \right] + c$$

$$\therefore I_2 = -\frac{1}{8} (2x+1) \sqrt{x^2+x} + \frac{1}{16} \ln \left| x + \frac{1}{2} + \sqrt{x^2+x} \right| + c$$

Substituting I_1 and I_2 in I , we get

$$I = \frac{1}{3} (x^2+x)^{\frac{3}{2}} - \frac{1}{8} (2x+1) \sqrt{x^2+x} + \frac{1}{16} \ln \left| x + \frac{1}{2} + \sqrt{x^2+x} \right| + c$$

$$\text{Thus, } \int x \sqrt{x^2+x} dx = \frac{1}{3} (x^2+x)^{\frac{3}{2}} - \frac{1}{8} (2x+1) \sqrt{x^2+x} + \frac{1}{16} \ln \left| x + \frac{1}{2} + \sqrt{x^2+x} \right| + c$$

11. Question

Evaluate the following integrals -

$$\int (x-3) \sqrt{x^2+3x-18} dx$$

Answer

$$\text{Let } I = \int (x-3) \sqrt{x^2+3x-18} dx$$

$$\text{Let us assume } x-3 = \lambda \frac{d}{dx} (x^2+3x-18) + \mu$$

$$\Rightarrow x-3 = \lambda \left[\frac{d}{dx} (x^2) + \frac{d}{dx} (3x) - \frac{d}{dx} (18) \right] + \mu$$

$$\Rightarrow x-3 = \lambda \left[\frac{d}{dx} (x^2) + 3 \frac{d}{dx} (x) - \frac{d}{dx} (18) \right] + \mu$$

We know $\frac{d}{dx} (x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow x-3 = \lambda (2x^{2-1} + 3 + 0) + \mu$$

$$\Rightarrow x-3 = \lambda (2x+3) + \mu$$

$$\Rightarrow x-3 = 2\lambda x + 3\lambda + \mu$$

Comparing the coefficient of x on both sides, we get

$$2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$$

Comparing the constant on both sides, we get

$$3\lambda + \mu = -3$$

$$\Rightarrow 3 \left(\frac{1}{2} \right) + \mu = -3$$

$$\Rightarrow \frac{3}{2} + \mu = -3$$

$$\therefore \mu = -\frac{9}{2}$$

$$\text{Hence, we have } x - 3 = \frac{1}{2}(2x + 3) - \frac{9}{2}$$

Substituting this value in I, we can write the integral as

$$\begin{aligned} I &= \int \left[\frac{1}{2}(2x + 3) - \frac{9}{2} \right] \sqrt{x^2 + 3x - 18} dx \\ \Rightarrow I &= \int \left[\frac{1}{2}(2x + 3) \sqrt{x^2 + 3x - 18} - \frac{9}{2} \sqrt{x^2 + 3x - 18} \right] dx \\ \Rightarrow I &= \int \frac{1}{2}(2x + 3) \sqrt{x^2 + 3x - 18} dx - \int \frac{9}{2} \sqrt{x^2 + 3x - 18} dx \\ \Rightarrow I &= \frac{1}{2} \int (2x + 3) \sqrt{x^2 + 3x - 18} dx - \frac{9}{2} \int \sqrt{x^2 + 3x - 18} dx \end{aligned}$$

$$\text{Let } I_1 = \frac{1}{2} \int (2x + 3) \sqrt{x^2 + 3x - 18} dx$$

$$\text{Now, put } x^2 + 3x - 18 = t$$

$$\Rightarrow (2x + 3)dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$\begin{aligned} I_1 &= \frac{1}{2} \int \sqrt{t} dt \\ \Rightarrow I_1 &= \frac{1}{2} \int t^{\frac{1}{2}} dt \\ \text{Recall } \int x^n dx &= \frac{x^{n+1}}{n+1} + c \\ \Rightarrow I_1 &= \frac{1}{2} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c \\ \Rightarrow I_1 &= \frac{1}{2} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c \\ \Rightarrow I_1 &= \frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c \\ \Rightarrow I_1 &= \frac{1}{3} t^{\frac{3}{2}} + \\ \therefore I_1 &= \frac{1}{3} (x^2 + 3x - 18)^{\frac{3}{2}} + c \end{aligned}$$

$$\text{Let } I_2 = -\frac{9}{2} \int \sqrt{x^2 + 3x - 18} dx$$

$$\text{We can write } x^2 + 3x - 18 = x^2 + 2(x) \left(\frac{3}{2} \right) + \left(\frac{3}{2} \right)^2 - \left(\frac{3}{2} \right)^2 - 18$$

$$\Rightarrow x^2 + 3x - 18 = \left(x + \frac{3}{2} \right)^2 - \frac{9}{4} - 18$$

$$\Rightarrow x^2 + 3x - 18 = \left(x + \frac{3}{2} \right)^2 - \frac{81}{4}$$

$$\Rightarrow x^2 + 3x - 18 = \left(x + \frac{3}{2} \right)^2 - \left(\frac{9}{2} \right)^2$$

Hence, we can write I_2 as

$$I_2 = -\frac{9}{2} \int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2} dx$$

$$\text{Recall } \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + c$$

$$\begin{aligned} \Rightarrow I_2 &= -\frac{9}{2} \left[\frac{\left(x + \frac{3}{2}\right)}{2} \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2} \right. \\ &\quad \left. - \frac{\left(\frac{9}{2}\right)^2}{2} \ln \left| \left(x + \frac{3}{2}\right) + \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2} \right| \right] + c \\ \Rightarrow I_2 &= -\frac{9}{2} \left[\frac{(2x+3)}{4} \sqrt{x^2 + 3x - 18} - \frac{81}{8} \ln \left| x + \frac{3}{2} + \sqrt{x^2 + 3x - 18} \right| \right] + c \\ \therefore I_2 &= -\frac{9}{8} (2x+3) \sqrt{x^2 + 3x - 18} + \frac{729}{16} \ln \left| x + \frac{3}{2} + \sqrt{x^2 + 3x - 18} \right| + c \end{aligned}$$

Substituting I_1 and I_2 in I , we get

$$\begin{aligned} I &= \frac{1}{3} (x^2 + 3x - 18)^{\frac{3}{2}} - \frac{9}{8} (2x+3) \sqrt{x^2 + 3x - 18} \\ &\quad + \frac{729}{16} \ln \left| x + \frac{3}{2} + \sqrt{x^2 + 3x - 18} \right| + c \end{aligned}$$

$$\text{Thus, } \int (x-3) \sqrt{x^2 + 3x - 18} dx = \frac{1}{3} (x^2 + 3x - 18)^{\frac{3}{2}} - \frac{9}{8} (2x+3) \sqrt{x^2 + 3x - 18} + \frac{729}{16} \ln \left| x + \frac{3}{2} + \sqrt{x^2 + 3x - 18} \right| + c$$

12. Question

Evaluate the following integrals -

$$\int (x+3) \sqrt{3-4x-x^2} dx$$

Answer

$$\text{Let } I = \int (x+3) \sqrt{3-4x-x^2} dx$$

$$\text{Let us assume } x+3 = \lambda \frac{d}{dx} (3-4x-x^2) + \mu$$

$$\Rightarrow x+3 = \lambda \left[\frac{d}{dx} (3) - \frac{d}{dx} (4x) - \frac{d}{dx} (x^2) \right] + \mu$$

$$\Rightarrow x+3 = \lambda \left[\frac{d}{dx} (3) - 4 \frac{d}{dx} (x) - \frac{d}{dx} (x^2) \right] + \mu$$

We know $\frac{d}{dx} (x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow x+3 = \lambda(0 - 4 - 2x^{2-1}) + \mu$$

$$\Rightarrow x+3 = \lambda(-4 - 2x) + \mu$$

$$\Rightarrow x+3 = -2\lambda x + \mu - 4\lambda$$

Comparing the coefficient of x on both sides, we get

$$-2\lambda = 1 \Rightarrow \lambda = -\frac{1}{2}$$

Comparing the constant on both sides, we get

$$\mu - 4\lambda = 3$$

$$\Rightarrow \mu - 4\left(-\frac{1}{2}\right) = 3$$

$$\Rightarrow \mu + 2 = 3$$

$$\therefore \mu = 1$$

$$\text{Hence, we have } x + 3 = -\frac{1}{2}(-4 - 2x) + 1$$

Substituting this value in I, we can write the integral as

$$I = \int \left[-\frac{1}{2}(-4 - 2x) + 1 \right] \sqrt{3 - 4x - x^2} dx$$

$$\Rightarrow I = \int \left[-\frac{1}{2}(-4 - 2x)\sqrt{3 - 4x - x^2} + \sqrt{3 - 4x - x^2} \right] dx$$

$$\Rightarrow I = -\int \frac{1}{2}(-4 - 2x)\sqrt{3 - 4x - x^2} dx + \int \sqrt{3 - 4x - x^2} dx$$

$$\Rightarrow I = -\frac{1}{2} \int (-4 - 2x)\sqrt{3 - 4x - x^2} dx + \int \sqrt{3 - 4x - x^2} dx$$

$$\text{Let } I_1 = -\frac{1}{2} \int (-4 - 2x)\sqrt{3 - 4x - x^2} dx$$

$$\text{Now, put } 3 - 4x - x^2 = t$$

$$\Rightarrow (-4 - 2x)dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = -\frac{1}{2} \int \sqrt{t} dt$$

$$\Rightarrow I_1 = -\frac{1}{2} \int t^{\frac{1}{2}} dt$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = -\frac{1}{2} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = -\frac{1}{2} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = -\frac{1}{2} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = -\frac{1}{3} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = -\frac{1}{3} (3 - 4x - x^2)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = \int \sqrt{3 - 4x - x^2} dx$$

$$\text{We can write } 3 - 4x - x^2 = -(x^2 + 4x - 3)$$

$$\Rightarrow 3 - 4x - x^2 = -[x^2 + 2(x)(2) + 2^2 - 2^2 - 3]$$

$$\Rightarrow 3 - 4x - x^2 = -[(x + 2)^2 - 4 - 3]$$

$$\Rightarrow 3 - 4x - x^2 = -[(x + 2)^2 - 7]$$

$$\Rightarrow 3 - 4x - x^2 = 7 - (x + 2)^2$$

$$\Rightarrow 3 - 4x - x^2 = (\sqrt{7})^2 - (x + 2)^2$$

Hence, we can write I_2 as

$$I_2 = \int \sqrt{(\sqrt{7})^2 - (x + 2)^2} dx$$

$$\text{Recall } \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$$

$$\Rightarrow I_2 = \frac{(x + 2)}{2} \sqrt{(\sqrt{7})^2 - (x + 2)^2} + \frac{(\sqrt{7})^2}{2} \sin^{-1} \left(\frac{x + 2}{\sqrt{7}} \right) + c$$

$$\therefore I_2 = \frac{1}{2} (x + 2) \sqrt{3 - 4x - x^2} + \frac{7}{2} \sin^{-1} \left(\frac{x + 2}{\sqrt{7}} \right) + c$$

Substituting I_1 and I_2 in I , we get

$$I = -\frac{1}{3} (3 - 4x - x^2)^{\frac{3}{2}} + \frac{1}{2} (x + 2) \sqrt{3 - 4x - x^2} + \frac{7}{2} \sin^{-1} \left(\frac{x + 2}{\sqrt{7}} \right) + c$$

$$\text{Thus, } \int (x + 3) \sqrt{3 - 4x - x^2} dx = -\frac{1}{3} (3 - 4x - x^2)^{\frac{3}{2}} + \frac{1}{2} (x + 2) \sqrt{3 - 4x - x^2} + \frac{7}{2} \sin^{-1} \left(\frac{x + 2}{\sqrt{7}} \right) + c$$

13. Question

Evaluate the following integrals -

$$\int (3x + 1) \sqrt{4 - 3x - 2x^2} dx$$

Answer

$$\text{Let } I = \int (3x + 1) \sqrt{4 - 3x - 2x^2} dx$$

$$\text{Let us assume } 3x + 1 = \lambda \frac{d}{dx} (4 - 3x - 2x^2) + \mu$$

$$\Rightarrow 3x + 1 = \lambda \left[\frac{d}{dx} (4) - \frac{d}{dx} (3x) - \frac{d}{dx} (2x^2) \right] + \mu$$

$$\Rightarrow 3x + 1 = \lambda \left[\frac{d}{dx} (4) - 3 \frac{d}{dx} (x) - 2 \frac{d}{dx} (x^2) \right] + \mu$$

$$\text{We know } \frac{d}{dx} (x^n) = nx^{n-1} \text{ and derivative of a constant is 0.}$$

$$\Rightarrow 3x + 1 = \lambda(0 - 3 - 2 \times 2x^{2-1}) + \mu$$

$$\Rightarrow 3x + 1 = \lambda(-3 - 4x) + \mu$$

$$\Rightarrow 3x + 1 = -4\lambda x + \mu - 3\lambda$$

Comparing the coefficient of x on both sides, we get

$$-4\lambda = 3 \Rightarrow \lambda = -\frac{3}{4}$$

Comparing the constant on both sides, we get

$$\mu - 3\lambda = 1$$

$$\Rightarrow \mu - 3 \left(-\frac{3}{4} \right) = 1$$

$$\Rightarrow \mu + \frac{9}{4} = 1$$

$$\therefore \mu = -\frac{5}{4}$$

$$\text{Hence, we have } 3x + 1 = -\frac{3}{4}(-3 - 4x) - \frac{5}{4}$$

Substituting this value in I, we can write the integral as

$$\begin{aligned} I &= \int \left[-\frac{3}{4}(-3 - 4x) - \frac{5}{4} \right] \sqrt{4 - 3x - 2x^2} dx \\ \Rightarrow I &= \int \left[-\frac{3}{4}(-3 - 4x)\sqrt{4 - 3x - 2x^2} - \frac{5}{4}\sqrt{4 - 3x - 2x^2} \right] dx \\ \Rightarrow I &= -\int \frac{3}{4}(-3 - 4x)\sqrt{4 - 3x - 2x^2} dx - \int \frac{5}{4}\sqrt{4 - 3x - 2x^2} dx \\ \Rightarrow I &= -\frac{3}{4} \int (-3 - 4x)\sqrt{4 - 3x - 2x^2} dx - \frac{5}{4} \int \sqrt{4 - 3x - 2x^2} dx \end{aligned}$$

$$\text{Let } I_1 = -\frac{3}{4} \int (-3 - 4x)\sqrt{4 - 3x - 2x^2} dx$$

$$\text{Now, put } 4 - 3x - 2x^2 = t$$

$$\Rightarrow (-3 - 4x)dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = -\frac{3}{4} \int \sqrt{t} dt$$

$$\Rightarrow I_1 = -\frac{3}{4} \int t^{\frac{1}{2}} dt$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = -\frac{3}{4} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = -\frac{3}{4} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = -\frac{3}{4} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = -\frac{1}{2} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = -\frac{1}{2} (4 - 3x - 2x^2)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = -\frac{5}{4} \int \sqrt{4 - 3x - 2x^2} dx$$

$$\text{We can write } 4 - 3x - 2x^2 = -(2x^2 + 3x - 4)$$

$$\Rightarrow 4 - 3x - 2x^2 = -2 \left[x^2 + \frac{3}{2}x - 2 \right]$$

$$\Rightarrow 4 - 3x - 2x^2 = -2 \left[x^2 + 2(x) \left(\frac{3}{4} \right) + \left(\frac{3}{4} \right)^2 - \left(\frac{3}{4} \right)^2 - 2 \right]$$

$$\Rightarrow 4 - 3x - 2x^2 = -2 \left[\left(x + \frac{3}{4} \right)^2 - \frac{9}{16} - 2 \right]$$

$$\Rightarrow 4 - 3x - 2x^2 = -2 \left[\left(x + \frac{3}{4} \right)^2 - \frac{41}{16} \right]$$

$$\Rightarrow 4 - 3x - 2x^2 = 2 \left[\frac{41}{16} - \left(x + \frac{3}{4} \right)^2 \right]$$

$$\Rightarrow 4 - 3x - 2x^2 = 2 \left[\left(\frac{\sqrt{41}}{4} \right)^2 - \left(x + \frac{3}{4} \right)^2 \right]$$

Hence, we can write I_2 as

$$I_2 = -\frac{5}{4} \int \sqrt{2 \left[\left(\frac{\sqrt{41}}{4} \right)^2 - \left(x + \frac{3}{4} \right)^2 \right]} dx$$

$$\Rightarrow I_2 = -\frac{5\sqrt{2}}{4} \int \sqrt{\left(\frac{\sqrt{41}}{4} \right)^2 - \left(x + \frac{3}{4} \right)^2} dx$$

Recall $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$

$$\Rightarrow I_2 = -\frac{5\sqrt{2}}{4} \left[\frac{\left(x + \frac{3}{4} \right)}{2} \sqrt{\left(\frac{\sqrt{41}}{4} \right)^2 - \left(x + \frac{3}{4} \right)^2} + \frac{\left(\frac{\sqrt{41}}{4} \right)^2}{2} \sin^{-1} \left(\frac{x + \frac{3}{4}}{\frac{\sqrt{41}}{4}} \right) \right] + c$$

$$\Rightarrow I_2 = -\frac{5\sqrt{2}}{4} \left[\frac{(4x+3)}{8} \sqrt{2 - \frac{3}{2}x - x^2} + \frac{41}{32} \sin^{-1} \left(\frac{4x+3}{\sqrt{41}} \right) \right] + c$$

$$\Rightarrow I_2 = -\frac{5\sqrt{2}}{32} (4x+3) \sqrt{2 - \frac{3}{2}x - x^2} - \frac{205\sqrt{2}}{128} \sin^{-1} \left(\frac{4x+3}{\sqrt{41}} \right) + c$$

$$\therefore I_2 = -\frac{5}{32} (4x+3) \sqrt{4 - 3x - 2x^2} - \frac{205\sqrt{2}}{128} \sin^{-1} \left(\frac{4x+3}{\sqrt{41}} \right) + c$$

Substituting I_1 and I_2 in I , we get

$$I = -\frac{1}{2} (4 - 3x - 2x^2)^{\frac{3}{2}} - \frac{5}{32} (4x+3) \sqrt{4 - 3x - 2x^2} - \frac{205\sqrt{2}}{128} \sin^{-1} \left(\frac{4x+3}{\sqrt{41}} \right) + c$$

$$\text{Thus, } \int (3x+1) \sqrt{4 - 3x - 2x^2} dx = -\frac{1}{2} (4 - 3x - 2x^2)^{\frac{3}{2}} - \frac{5}{32} (4x+3) \sqrt{4 - 3x - 2x^2} - \frac{205\sqrt{2}}{128} \sin^{-1} \left(\frac{4x+3}{\sqrt{41}} \right) + c$$

14. Question

Evaluate the following integrals -

$$\int (2x+5) \sqrt{10 - 4x - 3x^2} dx$$

Answer

$$\text{Let } I = \int (2x+5) \sqrt{10 - 4x - 3x^2} dx$$

Let us assume, $2x + 5 = \lambda \frac{d}{dx}(10 - 4x - 3x^2) + \mu$

$$\Rightarrow 2x + 5 = \lambda \left[\frac{d}{dx}(10) - \frac{d}{dx}(4x) - \frac{d}{dx}(3x^2) \right] + \mu$$

$$\Rightarrow 2x + 5 = \lambda \left[\frac{d}{dx}(10) - 4 \frac{d}{dx}(x) - 3 \frac{d}{dx}(x^2) \right] + \mu$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow 2x + 5 = \lambda(0 - 4 - 3 \times 2x^{2-1}) + \mu$$

$$\Rightarrow 2x + 5 = \lambda(-4 - 6x) + \mu$$

$$\Rightarrow 2x + 5 = -6\lambda x + \mu - 4\lambda$$

Comparing the coefficient of x on both sides, we get

$$-6\lambda = 2 \Rightarrow \lambda = -\frac{2}{6} = -\frac{1}{3}$$

Comparing the constant on both sides, we get

$$\mu - 4\lambda = 5$$

$$\Rightarrow \mu - 4\left(-\frac{1}{3}\right) = 5$$

$$\Rightarrow \mu + \frac{4}{3} = 5$$

$$\therefore \mu = \frac{11}{3}$$

$$\text{Hence, we have } 2x + 5 = -\frac{1}{3}(-4 - 6x) + \frac{11}{3}$$

Substituting this value in I , we can write the integral as

$$I = \int \left[-\frac{1}{3}(-4 - 6x) + \frac{11}{3} \right] \sqrt{10 - 4x - 3x^2} dx$$

$$\Rightarrow I = \int \left[-\frac{1}{3}(-4 - 6x)\sqrt{10 - 4x - 3x^2} + \frac{11}{3}\sqrt{10 - 4x - 3x^2} \right] dx$$

$$\Rightarrow I = -\int \frac{1}{3}(-4 - 6x)\sqrt{10 - 4x - 3x^2} dx + \int \frac{11}{3}\sqrt{10 - 4x - 3x^2} dx$$

$$\Rightarrow I = -\frac{1}{3} \int (-4 - 6x)\sqrt{10 - 4x - 3x^2} dx + \frac{11}{3} \int \sqrt{10 - 4x - 3x^2} dx$$

$$\text{Let } I_1 = -\frac{1}{3} \int (-4 - 6x)\sqrt{10 - 4x - 3x^2} dx$$

Now, put $10 - 4x - 3x^2 = t$

$$\Rightarrow (-4 - 6x)dx = dt \text{ (Differentiating both sides)}$$

Substituting this value in I_1 , we can write

$$I_1 = -\frac{1}{3} \int \sqrt{t} dt$$

$$\Rightarrow I_1 = -\frac{1}{3} \int t^{\frac{1}{2}} dt$$

$$\text{Recall } \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\Rightarrow I_1 = -\frac{1}{3} \left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + c$$

$$\Rightarrow I_1 = -\frac{1}{3} \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + c$$

$$\Rightarrow I_1 = -\frac{1}{3} \times \frac{2}{3} t^{\frac{3}{2}} + c$$

$$\Rightarrow I_1 = -\frac{2}{9} t^{\frac{3}{2}} + c$$

$$\therefore I_1 = -\frac{2}{9} (10 - 4x - 3x^2)^{\frac{3}{2}} + c$$

$$\text{Let } I_2 = \frac{11}{3} \int \sqrt{10 - 4x - 3x^2} dx$$

$$\text{We can write } 10 - 4x - 3x^2 = -(3x^2 + 4x - 10)$$

$$\Rightarrow 10 - 4x - 3x^2 = -3 \left[x^2 + \frac{4}{3}x - \frac{10}{3} \right]$$

$$\Rightarrow 10 - 4x - 3x^2 = -3 \left[x^2 + 2(x) \left(\frac{2}{3} \right) + \left(\frac{2}{3} \right)^2 - \left(\frac{2}{3} \right)^2 - \frac{10}{3} \right]$$

$$\Rightarrow 10 - 4x - 3x^2 = -3 \left[\left(x + \frac{2}{3} \right)^2 - \frac{4}{9} - \frac{10}{3} \right]$$

$$\Rightarrow 10 - 4x - 3x^2 = -3 \left[\left(x + \frac{2}{3} \right)^2 - \frac{34}{9} \right]$$

$$\Rightarrow 10 - 4x - 3x^2 = 3 \left[\frac{34}{9} - \left(x + \frac{2}{3} \right)^2 \right]$$

$$\Rightarrow 10 - 4x - 3x^2 = 3 \left[\left(\frac{\sqrt{34}}{3} \right)^2 - \left(x + \frac{2}{3} \right)^2 \right]$$

Hence, we can write I_2 as

$$I_2 = \frac{11}{3} \int \sqrt{3 \left[\left(\frac{\sqrt{34}}{3} \right)^2 - \left(x + \frac{2}{3} \right)^2 \right]} dx$$

$$\Rightarrow I_2 = \frac{11\sqrt{3}}{3} \int \sqrt{\left(\frac{\sqrt{34}}{3} \right)^2 - \left(x + \frac{2}{3} \right)^2} dx$$

$$\text{Recall } \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$$

$$\Rightarrow I_2 = \frac{11\sqrt{3}}{3} \left[\frac{\left(x + \frac{2}{3} \right)}{2} \sqrt{\left(\frac{\sqrt{34}}{3} \right)^2 - \left(x + \frac{2}{3} \right)^2} + \frac{\left(\frac{\sqrt{34}}{3} \right)^2}{2} \sin^{-1} \left(\frac{x + \frac{2}{3}}{\frac{\sqrt{34}}{3}} \right) \right] + c$$

$$\Rightarrow I_2 = \frac{11\sqrt{3}}{3} \left[\frac{(3x+2)}{6} \sqrt{\frac{10}{3} - \frac{4}{3}x - x^2} + \frac{34}{18} \sin^{-1} \left(\frac{3x+2}{\sqrt{34}} \right) \right] + c$$

$$\Rightarrow I_2 = -\frac{11\sqrt{3}}{18}(3x+2)\sqrt{\frac{10}{3}-\frac{4}{3}x-x^2}-\frac{374\sqrt{3}}{54}\sin^{-1}\left(\frac{3x+2}{\sqrt{34}}\right)+c$$

$$\therefore I_2 = -\frac{11}{18}(3x+2)\sqrt{10-4x-3x^2}-\frac{187\sqrt{3}}{27}\sin^{-1}\left(\frac{3x+2}{\sqrt{34}}\right)+c$$

Substituting I_1 and I_2 in I , we get

$$I = -\frac{2}{9}(10-4x-3x^2)^{\frac{3}{2}}-\frac{11}{18}(3x+2)\sqrt{10-4x-3x^2}-\frac{187\sqrt{3}}{27}\sin^{-1}\left(\frac{3x+2}{\sqrt{34}}\right)+c$$

Thus, $\int (2x+5)\sqrt{10-4x-3x^2}dx = -\frac{2}{9}(10-4x-3x^2)^{\frac{3}{2}}-\frac{11}{18}(3x+2)\sqrt{10-4x-3x^2}-\frac{187\sqrt{3}}{27}\sin^{-1}\left(\frac{3x+2}{\sqrt{34}}\right)+c$

Exercise 19.30

1. Question

Evaluate the following integral:

$$\int \frac{2x+1}{(x+1)(x-2)} dx$$

Answer

Here the denominator is already factored.

So let

$$\frac{2x+1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \dots \dots (i)$$

$$\Rightarrow \frac{2x+1}{(x+1)(x-2)} = \frac{A(x-2) + B(x+1)}{(x+1)(x-2)}$$

$$\Rightarrow 2x+1 = A(x-2) + B(x+1) \dots \dots (ii)$$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put $x = 2$ in the above equation, we get

$$\Rightarrow 2(2)+1 = A(2-2) + B(2+1)$$

$$\Rightarrow 3B = 5$$

$$\Rightarrow B = \frac{5}{3}$$

Now put $x = -1$ in equation (ii), we get

$$\Rightarrow 2(-1)+1 = A((-1)-2) + B((-1)+1)$$

$$\Rightarrow -3A = -1$$

$$\Rightarrow A = \frac{1}{3}$$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{A}{x+1} + \frac{B}{x-2} \right] dx$$

$$\Rightarrow \int \left[\frac{\frac{1}{3}}{x+1} + \frac{\frac{5}{3}}{x-2} \right] dx$$

Split up the integral,

$$\Rightarrow \frac{1}{3} \int \left[\frac{1}{x+1} \right] dx + \frac{5}{3} \int \left[\frac{1}{x-2} \right] dx$$

Let substitute $u = x + 1 \Rightarrow du = dx$ and $z = x - 2 \Rightarrow dz = dx$, so the above equation becomes,

$$\Rightarrow \frac{1}{3} \int \left[\frac{1}{u} \right] du + \frac{5}{3} \int \left[\frac{1}{z} \right] dz$$

On integrating we get

$$\Rightarrow \frac{1}{3} \log|u| + \frac{5}{3} \log|z| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{3} \log|x+1| + \frac{5}{3} \log|x-2| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{2x+1}{(x+1)(x-2)} dx = \frac{1}{3} \log|x+1| + \frac{5}{3} \log|x-2| + C$$

2. Question

Evaluate the following integral:

$$\int \frac{1}{x(x-2)(x-4)} dx$$

Answer

Here the denominator is already factored.

So let

$$\frac{1}{x(x-2)(x-4)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-4} \dots \dots (i)$$

$$\Rightarrow \frac{1}{x(x-2)(x-4)} = \frac{A(x-2)(x-4) + Bx(x-4) + Cx(x-2)}{x(x-2)(x-4)}$$

$$\Rightarrow 1 = A(x-2)(x-4) + Bx(x-4) + Cx(x-2) \dots \dots (ii)$$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put $x = 0$ in the above equation, we get

$$\Rightarrow 1 = A(0-2)(0-4) + B(0)(0-4) + C(0)(0-2)$$

$$\Rightarrow 1 = 8A + 0 + 0$$

$$\Rightarrow A = \frac{1}{8}$$

Now put $x = 2$ in equation (ii), we get

$$\Rightarrow 1 = A(2-2)(2-4) + B(2)(2-4) + C(2)(2-2)$$

$$\Rightarrow 1 = 0 - 4B + 0$$

$$\Rightarrow B = -\frac{1}{4}$$

Now put $x = 4$ in equation (ii), we get

$$\Rightarrow 1 = A(4 - 2)(4 - 4) + B(4)(4 - 4) + C(4)(4 - 2)$$

$$\Rightarrow 1 = 0 + 0 + 8C$$

$$\Rightarrow C = \frac{1}{8}$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-4} \right] dx$$

$$\Rightarrow \int \left[\frac{\frac{1}{8}}{x} + \frac{-\frac{1}{4}}{x-2} + \frac{\frac{1}{8}}{x-4} \right] dx$$

Split up the integral,

$$\Rightarrow \frac{1}{8} \int \left[\frac{1}{x} \right] dx - \frac{1}{4} \int \left[\frac{1}{x-2} \right] dx + \frac{1}{8} \int \left[\frac{1}{x-4} \right] dx$$

Let substitute $u = x - 4 \Rightarrow du = dx$ and $z = x - 2 \Rightarrow dz = dx$, so the above equation becomes,

$$\Rightarrow \frac{1}{8} \int \left[\frac{1}{x} \right] dx - \frac{1}{4} \int \left[\frac{1}{z} \right] dz + \frac{1}{8} \int \left[\frac{1}{u} \right] du$$

On integrating we get

$$\Rightarrow \frac{1}{8} \log|x| - \frac{1}{4} \log|z| + \frac{1}{8} \log|u| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{8} \log|x| - \frac{1}{4} \log|x-2| + \frac{1}{8} \log|x-4| + C$$

We will take $\frac{1}{8}$ common, we get

$$\Rightarrow \frac{1}{8} [\log|x| - 2 \log|x-2| + \log|x-4| + C]$$

Applying the logarithm rule we can rewrite the above equation as

$$\Rightarrow \frac{1}{8} \left[\log \left| \frac{x}{(x-2)^2} \right| + \log|x-4| + C \right]$$

$$\Rightarrow \frac{1}{8} \left[\log \left| \frac{x(x-4)}{(x-2)^2} \right| \right] + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{1}{x(x-2)(x-4)} dx = \frac{1}{8} \left[\log \left| \frac{x(x-4)}{(x-2)^2} \right| \right] + C$$

3. Question

Evaluate the following integral:

$$\int \frac{x^2 + x - 1}{x^2 + x - 6} dx$$

Answer

First we simplify numerator, we get

$$\begin{aligned} & \frac{x^2 + x - 1}{x^2 + x - 6} \\ &= \frac{x^2 + x - 6 + 5}{x^2 + x - 6} \\ &= \frac{x^2 + x - 6}{x^2 + x - 6} + \frac{5}{x^2 + x - 6} \\ &= 1 + \frac{5}{x^2 + x - 6} \end{aligned}$$

Now we will factorize denominator by splitting the middle term, we get

$$\begin{aligned} & 1 + \frac{5}{x^2 + x - 6} \\ &= 1 + \frac{5}{x^2 + 3x - 2x - 6} \\ &= 1 + \frac{5}{x(x + 3) - 2(x + 3)} \\ &= 1 + \frac{5}{(x + 3)(x - 2)} \end{aligned}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\begin{aligned} \frac{5}{(x + 3)(x - 2)} &= \frac{A}{x + 3} + \frac{B}{x - 2} \dots \dots (i) \\ \Rightarrow \frac{5}{(x + 3)(x - 2)} &= \frac{A(x - 2) + B(x + 3)}{(x + 3)(x - 2)} \\ \Rightarrow 5 &= A(x - 2) + B(x + 3) \dots \dots (ii) \end{aligned}$$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put $x = 2$ in the above equation, we get

$$\begin{aligned} \Rightarrow 5 &= A(2 - 2) + B(2 + 3) \\ \Rightarrow 5 &= 0 + 5B \\ \Rightarrow B &= 1 \end{aligned}$$

Now put $x = -3$ in equation (ii), we get

$$\begin{aligned} \Rightarrow 5 &= A((-3) - 2) + B((-3) + 3) \\ \Rightarrow 5 &= -5A \\ \Rightarrow A &= -1 \end{aligned}$$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[1 + \frac{A}{x + 3} + \frac{B}{x - 2} \right] dx$$

$$\Rightarrow \int \left[1 + \frac{-1}{x+3} + \frac{1}{x-2} \right] dx$$

Split up the integral,

$$\Rightarrow \int 1 dx - \int \left[\frac{1}{x+3} \right] dx + \int \left[\frac{1}{x-2} \right] dx$$

Let substitute $u = x + 3 \Rightarrow du = dx$ and $z = x - 2 \Rightarrow dz = dx$, so the above equation becomes,

$$\Rightarrow \int 1 dx - \int \left[\frac{1}{u} \right] du + \int \left[\frac{1}{z} \right] dz$$

On integrating we get

$$\Rightarrow x - \log|u| + \log|z| + C$$

Substituting back, we get

$$\Rightarrow x - \log|x+3| + \log|x-2| + C$$

Applying the logarithm rule, we can rewrite the above equation as

$$\Rightarrow x + \log \left| \frac{x-2}{x+3} \right| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{x^2 + x - 1}{x^2 + x - 6} dx = x + \log \left| \frac{x-2}{x+3} \right| + C$$

4. Question

Evaluate the following integral:

$$\int \frac{3 + 4x - x^2}{(x+2)(x-1)} dx$$

Answer

First we simplify numerator, we get

$$\begin{aligned} & \frac{3 + 4x - x^2}{(x+2)(x-1)} \\ &= \frac{-(x^2 - 4x - 3)}{x^2 + x - 2} \\ &= \frac{-(x^2 + x - 5x - 2 - 1)}{x^2 + x - 2} \\ &= \frac{-(x^2 + x - 2)}{x^2 + x - 2} + \frac{5x + 1}{x^2 + x - 2} \\ &= -1 + \frac{5x + 1}{(x+2)(x-1)} \end{aligned}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\begin{aligned} \frac{5x + 1}{(x+2)(x-1)} &= \frac{A}{x+2} + \frac{B}{x-1} \dots \dots (i) \\ \Rightarrow \frac{5x + 1}{(x+2)(x-1)} &= \frac{A(x-1) + B(x+2)}{(x+2)(x-1)} \end{aligned}$$

$$\Rightarrow 5x + 1 = A(x - 1) + B(x + 2) \dots (ii)$$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put $x = 1$ in the above equation, we get

$$\Rightarrow 5(1) + 1 = A(1 - 1) + B(1 + 2)$$

$$\Rightarrow 6 = 0 + 3B$$

$$\Rightarrow B = 2$$

Now put $x = -2$ in equation (ii), we get

$$\Rightarrow 5(-2) + 1 = A((-2) - 1) + B((-2) + 2)$$

$$\Rightarrow -9 = -3A + 0$$

$$\Rightarrow A = 3$$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[-1 + \frac{5x + 1}{(x + 2)(x - 1)} \right] dx$$

$$\Rightarrow \int \left[-1 + \frac{A}{x + 2} + \frac{B}{x - 1} \right] dx$$

$$\Rightarrow \int \left[-1 + \frac{3}{x + 2} + \frac{2}{x - 1} \right] dx$$

Split up the integral,

$$\Rightarrow - \int 1 dx + 3 \int \left[\frac{1}{x + 2} \right] dx + 2 \int \left[\frac{1}{x - 1} \right] dx$$

Let substitute $u = x + 2 \Rightarrow du = dx$ and $z = x - 1 \Rightarrow dz = dx$, so the above equation becomes,

$$\Rightarrow - \int 1 dx + 3 \int \left[\frac{1}{u} \right] du + 2 \int \left[\frac{1}{z} \right] dz$$

On integrating we get

$$\Rightarrow -x + 3 \log|u| + 2 \log|z| + C$$

Substituting back, we get

$$\Rightarrow -x + 3 \log|x + 2| + 2 \log|x - 1| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{3 + 4x - x^2}{(x + 2)(x - 1)} dx = -x + 3 \log|x + 2| + 2 \log|x - 1| + C$$

5. Question

Evaluate the following integral:

$$\int \frac{x^2 + 1}{x^2 - 1} dx$$

Answer

First we simplify numerator, we get

$$\begin{aligned}
& \frac{x^2 + 1}{x^2 - 1} \\
&= \frac{x^2 - 1 + 2}{x^2 - 1} \\
&= \frac{x^2 - 1}{x^2 - 1} + \frac{2}{x^2 - 1} \\
&= 1 + \frac{2}{(x-1)(x+1)}
\end{aligned}$$

Now the denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\begin{aligned}
\frac{2}{(x+1)(x-1)} &= \frac{A}{x+1} + \frac{B}{x-1} \dots \dots (i) \\
\Rightarrow \frac{2}{(x+2)(x-1)} &= \frac{A(x-1) + B(x+1)}{(x+2)(x-1)} \\
\Rightarrow 2 &= A(x-1) + B(x+1) \dots \dots (ii)
\end{aligned}$$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put $x = 1$ in the above equation, we get

$$\begin{aligned}
\Rightarrow 2 &= A(1-1) + B(1+1) \\
\Rightarrow 2 &= 0 + 2B \\
\Rightarrow B &= 1
\end{aligned}$$

Now put $x = -1$ in equation (ii), we get

$$\begin{aligned}
\Rightarrow 2 &= A((-1)-1) + B((-1)+1) \\
\Rightarrow 2 &= -2A + 0 \\
\Rightarrow A &= -1
\end{aligned}$$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\begin{aligned}
& \int \left[1 + \frac{2}{(x-1)(x+1)} \right] dx \\
& \Rightarrow \int \left[1 + \frac{A}{x+1} + \frac{B}{x-1} \right] dx \\
& \Rightarrow \int \left[1 + \frac{-1}{x+1} + \frac{1}{x-1} \right] dx
\end{aligned}$$

Split up the integral,

$$\Rightarrow \int 1 dx - \int \left[\frac{1}{x+1} \right] dx + \int \left[\frac{1}{x-1} \right] dx$$

Let substitute $u = x + 1 \Rightarrow du = dx$ and $z = x - 1 \Rightarrow dz = dx$, so the above equation becomes,

$$\Rightarrow \int 1 dx - \int \left[\frac{1}{u} \right] du + \int \left[\frac{1}{z} \right] dz$$

On integrating we get

$$\Rightarrow x - \log|u| + \log|z| + C$$

Substituting back, we get

$$\Rightarrow x - \log|x + 1| + \log|x - 1| + C$$

Applying the logarithm rule we get

$$\Rightarrow x + \log\left|\frac{x-1}{x+1}\right| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{x^2 + 1}{x^2 - 1} dx = x + \log\left|\frac{x-1}{x+1}\right| + C$$

6. Question

Evaluate the following integral:

$$\int \frac{x^2}{(x-1)(x-2)(x-3)} dx$$

Answer

Denominator is already factorized, so let

$$\frac{x^2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \dots\dots(i)$$

$$\begin{aligned} \Rightarrow \frac{x^2}{(x-1)(x-2)(x-3)} &= \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)} \end{aligned}$$

$$\Rightarrow x^2 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \dots\dots(ii)$$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put $x = 1$ in the above equation, we get

$$\Rightarrow 1^2 = A(1-2)(1-3) + B(1-1)(1-3) + C(1-1)(1-2)$$

$$\Rightarrow 1 = 2A + 0 + 0$$

$$\Rightarrow A = \frac{1}{2}$$

Now put $x = 2$ in equation (ii), we get

$$\Rightarrow 2^2 = A(2-2)(2-3) + B(2-1)(2-3) + C(2-1)(2-2)$$

$$\Rightarrow 4 = 0 - B + 0$$

$$\Rightarrow B = -4$$

Now put $x = 3$ in equation (ii), we get

$$\Rightarrow 3^2 = A(3-2)(3-3) + B(3-1)(3-3) + C(3-1)(3-2)$$

$$\Rightarrow 9 = 0 + 0 + 2C$$

$$\Rightarrow C = \frac{9}{2}$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \right] dx$$

$$\Rightarrow \int \left[\frac{\frac{1}{2}}{x-1} + \frac{-4}{x-2} + \frac{\frac{9}{2}}{x-3} \right] dx$$

Split up the integral,

$$\Rightarrow \frac{1}{2} \int \left[\frac{1}{x-1} \right] dx - 4 \int \left[\frac{1}{x-2} \right] dx + \frac{9}{2} \int \left[\frac{1}{x-3} \right] dx$$

Let substitute $u = x - 1 \Rightarrow du = dx$, $y = x - 2 \Rightarrow dy = dx$ and $z = x - 3 \Rightarrow dz = dx$, so the above equation becomes,

$$\Rightarrow \frac{1}{2} \int \left[\frac{1}{u} \right] du - 4 \int \left[\frac{1}{y} \right] dy + \frac{9}{2} \int \left[\frac{1}{z} \right] dz$$

On integrating we get

$$\Rightarrow \frac{1}{2} \log|u| - 4 \log|y| + \frac{9}{2} \log|z| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{2} \log|x-1| - 4 \log|x-2| + \frac{9}{2} \log|x-3| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{x^2}{(x-1)(x-2)(x-3)} dx = \frac{1}{2} \log|x-1| - 4 \log|x-2| + \frac{9}{2} \log|x-3| + C$$

7. Question

Evaluate the following integral:

$$\int \frac{5x}{(x+1)(x^2-4)} dx$$

Answer

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x-2)(x+2)}$$

The denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{5x}{(x+1)(x-2)(x+2)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+2} \dots \dots (i)$$

$$\Rightarrow \frac{5x}{(x+1)(x-2)(x+2)} = \frac{A(x-2)(x+2) + B(x+1)(x+2) + C(x+1)(x-2)}{(x+1)(x-2)(x+2)}$$

$$\Rightarrow 5x = A(x-2)(x+2) + B(x+1)(x+2) + C(x+1)(x-2) \dots \dots (ii)$$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put $x = -1$ in the above equation, we get

$$\Rightarrow 5(-1) = A((-1)-2)((-1)+2) + B((-1)+1)((-1)+2) + C((-1)+1)((-1)-2)$$

$$\Rightarrow -5 = -3A + 0 + 0$$

$$\Rightarrow A = \frac{5}{3}$$

Now put $x = -2$ in equation (ii), we get

$$\Rightarrow 5(-2) = A((-2) - 2)((-2) + 2) + B((-2) + 1)((-2) + 2) + C((-2) + 1)((-2) - 2)$$

$$\Rightarrow -10 = 0 + 0 + 4C$$

$$\Rightarrow C = -\frac{10}{4} = -\frac{5}{2}$$

Now put $x = 2$ in equation (ii), we get

$$\Rightarrow 5(2) = A((2) - 2)((2) + 2) + B((2) + 1)((2) + 2) + C((2) + 1)((2) - 2)$$

$$\Rightarrow 10 = 0 + 12B + 0$$

$$\Rightarrow B = \frac{10}{12} = \frac{5}{6}$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+2} \right] dx$$

$$\Rightarrow \int \left[\frac{\frac{5}{3}}{x+1} + \frac{-\frac{5}{2}}{x-2} + \frac{\frac{5}{6}}{x+2} \right] dx$$

Split up the integral,

$$\Rightarrow \frac{5}{3} \int \left[\frac{1}{x+1} \right] dx - \frac{5}{2} \int \left[\frac{1}{x-2} \right] dx + \frac{5}{6} \int \left[\frac{1}{x+2} \right] dx$$

Let substitute $u = x + 1 \Rightarrow du = dx$, $y = x - 2 \Rightarrow dy = dx$ and $z = x + 2 \Rightarrow dz = dx$, so the above equation becomes,

$$\Rightarrow \frac{5}{3} \int \left[\frac{1}{u} \right] du - \frac{5}{2} \int \left[\frac{1}{y} \right] dy + \frac{5}{6} \int \left[\frac{1}{z} \right] dz$$

On integrating we get

$$\Rightarrow \frac{5}{3} \log|u| - \frac{5}{2} \log|y| + \frac{5}{6} \log|z| + C$$

Substituting back, we get

$$\Rightarrow \frac{5}{3} \log|x+1| - \frac{5}{2} \log|x-2| + \frac{5}{6} \log|x+2| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{5x}{(x+1)(x^2-4)} dx = \frac{5}{3} \log|x+1| - \frac{5}{2} \log|x-2| + \frac{5}{6} \log|x+2| + C$$

8. Question

Evaluate the following integral:

$$\int \frac{x^2+1}{x(x^2-1)} dx$$

Answer

$$\frac{x^2 + 1}{x(x^2 - 1)} = \frac{x^2 + 1}{x(x-1)(x+1)}$$

The denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{x^2 + 1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \dots \dots (i)$$

$$\Rightarrow \frac{x^2 + 1}{x(x-1)(x+1)} = \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$$\Rightarrow x^2 + 1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1) \dots \dots (ii)$$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put x = 0 in the above equation, we get

$$\Rightarrow 0^2 + 1 = A(0-1)(0+1) + B(0)(0+1) + C(0)(0-1)$$

$$\Rightarrow 1 = -A + 0 + 0$$

$$\Rightarrow A = -1$$

Now put x = -1 in equation (ii), we get

$$\Rightarrow (-1)^2 + 1 = A((-1)-1)((-1)+1) + B(-1)((-1)+1) + C(-1)((-1)-1)$$

$$\Rightarrow 2 = 0 + 0 + C$$

$$\Rightarrow C = 1$$

Now put x = 1 in equation (ii), we get

$$\Rightarrow 1^2 + 1 = A(1-1)(1+1) + B(1)(1+1) + C(1)(1-1)$$

$$\Rightarrow 2 = 0 + 2B + 0$$

$$\Rightarrow B = 1$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{x^2 + 1}{x(x-1)(x+1)} \right] dx$$

$$\Rightarrow \int \left[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \right] dx$$

$$\Rightarrow \int \left[\frac{-1}{x} + \frac{1}{x-1} + \frac{1}{x+1} \right] dx$$

Split up the integral,

$$\Rightarrow - \int \left[\frac{1}{x} \right] dx + \int \left[\frac{1}{x-1} \right] dx + \int \left[\frac{1}{x+1} \right] dx$$

Let substitute u = x + 1 $\Rightarrow$ du = dx, y = x - 1 $\Rightarrow$ dy = dx, so the above equation becomes,

$$\Rightarrow - \int \left[\frac{1}{x} \right] dx + \int \left[\frac{1}{y} \right] dy + \int \left[\frac{1}{u} \right] du$$

On integrating we get

$$\Rightarrow -\log|x| + \log|y| + \log|u| + C$$

Substituting back, we get

$$\Rightarrow -\log|x| + \log|x-1| + \log|x+1| + C$$

Applying the rules of logarithm we get

$$\Rightarrow -\log|x| + \log|(x-1)(x+1)| + C$$

$$\Rightarrow \log \left| \frac{x^2 - 1}{x} \right| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{x^2 + 1}{x(x^2 - 1)} dx = \log \left| \frac{x^2 - 1}{x} \right| + C$$

9. Question

Evaluate the following integral:

$$\int \frac{2x - 3}{(x^2 - 1)(2x + 3)} dx$$

Answer

$$\frac{2x - 3}{(x^2 - 1)(2x + 3)} = \frac{2x - 3}{(x - 1)(x + 1)(2x + 3)}$$

The denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{2x - 3}{(x - 1)(x + 1)(2x + 3)} = \frac{A}{(x - 1)} + \frac{B}{x + 1} + \frac{C}{2x + 3} \dots \dots (i)$$

$$\begin{aligned} \Rightarrow \frac{2x - 3}{(x - 1)(x + 1)(2x + 3)} \\ = \frac{A(x + 1)(2x + 3) + B(x - 1)(2x + 3) + C(x - 1)(x + 1)}{(x - 1)(x + 1)(2x + 3)} \end{aligned}$$

$$\Rightarrow 2x - 3 = A(x + 1)(2x + 3) + B(x - 1)(2x + 3) + C(x - 1)(x + 1) \dots \dots (ii)$$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put $x = -1$ in the above equation, we get

$$\Rightarrow 2(-1) - 3 = A((-1) + 1)(2(-1) + 3) + B((-1) - 1)(2(-1) + 3) + C((-1) - 1)((-1) + 1)$$

$$\Rightarrow -5 = 0 - 2B + 0$$

$$\Rightarrow B = \frac{5}{2}$$

Now put $x = 1$ in equation (ii), we get

$$\Rightarrow 2(1) - 3 = A((1) + 1)(2(1) + 3) + B((1) - 1)(2(1) + 3) + C((1) - 1)((1) + 1)$$

$$\Rightarrow -1 = 10A + 0 + 0$$

$$\Rightarrow A = -\frac{1}{10}$$

Now put $x = -\frac{3}{2}$ in equation (ii), we get

$$\Rightarrow 2\left(-\frac{3}{2}\right) - 3$$

$$= A\left(\left(-\frac{3}{2}\right) + 1\right)\left(2\left(-\frac{3}{2}\right) + 3\right)$$

$$+ B\left(\left(-\frac{3}{2}\right) - 1\right)\left(2\left(-\frac{3}{2}\right) + 3\right) + C\left(\left(-\frac{3}{2}\right) - 1\right)\left(\left(-\frac{3}{2}\right) + 1\right)$$

$$\Rightarrow -6 = 0 + 0 + \frac{5}{4}C$$

$$\Rightarrow C = -\frac{24}{5}$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{2x-3}{(x-1)(x+1)(2x+3)} \right] dx$$

$$\Rightarrow \int \left[\frac{A}{(x-1)} + \frac{B}{x+1} + \frac{C}{2x+3} \right] dx$$

$$\Rightarrow \int \left[\frac{-\frac{1}{10}}{(x-1)} + \frac{\frac{5}{2}}{x+1} + \frac{-\frac{24}{5}}{2x+3} \right] dx$$

Split up the integral,

$$\Rightarrow -\frac{1}{10} \int \left[\frac{1}{x-1} \right] dx + \frac{5}{2} \int \left[\frac{1}{x+1} \right] dx - \frac{24}{5} \int \left[\frac{1}{2x+3} \right] dx$$

Let substitute

$$u = x + 1 \Rightarrow du = dx,$$

$$y = x - 1 \Rightarrow dy = dx \text{ and}$$

$$z = 2x + 3 \Rightarrow dz = 2dx \Rightarrow dx = \frac{dz}{2} \text{ so the above equation becomes,}$$

$$\Rightarrow -\frac{1}{10} \int \left[\frac{1}{y} \right] dy + \frac{5}{2} \int \left[\frac{1}{u} \right] du - \frac{24}{5} \int \left[\frac{1}{z} \right] \frac{dz}{2}$$

On integrating we get

$$\Rightarrow -\frac{1}{10} \log|y| + \frac{5}{2} \log|u| - \frac{12}{5} \log|z| + C$$

Substituting back, we get

$$\Rightarrow -\frac{1}{10} \log|x-1| + \frac{5}{2} \log|x+1| - \frac{12}{5} \log|2x+3| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{2x-3}{(x^2-1)(2x+3)} dx$$

$$= -\frac{1}{10} \log|x-1| + \frac{5}{2} \log|x+1| - \frac{12}{5} \log|2x+3| + C$$

10. Question

Evaluate the following integral:

$$\int \frac{x^3}{(x-1)(x-2)(x-3)} dx$$

Answer

First we simplify numerator, we will rewrite denominator as shown below

$$\frac{x^3}{(x-1)(x-2)(x-3)} = \frac{x^3}{x^3 - 6x^2 + 11x - 6}$$

Add and subtract numerator with $(-6x^2 + 11x - 6)$, we get

$$\frac{x^3 - 6x^2 + 11x - 6 + (6x^2 - 11x + 6)}{x^3 - 6x^2 + 11x - 6}$$

$$\Rightarrow = 1 + \frac{6x^2 - 11x + 6}{x^3 - 6x^2 + 11x - 6}$$

$$\Rightarrow = 1 + \frac{6x^2 - 11x + 6}{(x-1)(x-2)(x-3)}$$

The denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{6x^2 - 11x + 6}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{x-2} + \frac{C}{x-3} \dots\dots(i)$$

$$\begin{aligned} \Rightarrow \frac{6x^2 - 11x + 6}{(x-1)(x-2)(x-3)} \\ = \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)} \end{aligned}$$

$$\Rightarrow 6x^2 - 11x + 6 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \dots\dots(ii)$$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put $x = 1$ in the above equation, we get

$$\Rightarrow 6(1)^2 - 11(1) + 6 = A(1-2)(1-3) + B(1-1)(1-3) + C(1-1)(1-2)$$

$$\Rightarrow 1 = 2A + 0 + 0$$

$$\Rightarrow A = \frac{1}{2}$$

Now put $x = 2$ in equation (ii), we get

$$6(2)^2 - 11(2) + 6 = A(2-2)(2-3) + B(2-1)(2-3) + C(2-1)(2-2)$$

$$\Rightarrow 8 = 0 - B + 0$$

$$\Rightarrow B = -8$$

Now put $x = 3$ in equation (ii), we get

$$\Rightarrow 6(3)^2 - 11(3) + 6 = A(3-2)(3-3) + B(3-1)(3-3) + C(3-1)(3-2)$$

$$\Rightarrow 27 = 0 + 0 + 2C$$

$$\Rightarrow C = \frac{27}{2}$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[1 + \frac{6x^2 - 11x + 6}{(x-1)(x-2)(x-3)} \right] dx$$

$$\Rightarrow \int \left[1 + \frac{A}{(x-1)} + \frac{B}{x-2} + \frac{C}{x-3} \right] dx$$

$$\Rightarrow \int \left[1 + \frac{\frac{1}{2}}{(x-1)} + \frac{-8}{x-2} + \frac{\frac{27}{2}}{x-3} \right] dx$$

Split up the integral,

$$\Rightarrow \int 1 dx + \frac{1}{2} \int \left[\frac{1}{x-1} \right] dx - 8 \int \left[\frac{1}{x-2} \right] dx + \frac{27}{2} \int \left[\frac{1}{x-3} \right] dx$$

Let substitute

$$u = x - 1 \Rightarrow du = dx,$$

$$y = x - 2 \Rightarrow dy = dx \text{ and}$$

$$z = x - 3 \Rightarrow dz = dx, \text{ so the above equation becomes,}$$

$$\Rightarrow \int 1 dx + \frac{1}{2} \int \left[\frac{1}{u} \right] du - 8 \int \left[\frac{1}{y} \right] dy + \frac{27}{2} \int \left[\frac{1}{z} \right] dz$$

On integrating we get

$$\Rightarrow x + \frac{1}{2} \log|u| - 8 \log|y| + \frac{27}{2} \log|z| + C$$

Substituting back, we get

$$\Rightarrow x + \frac{1}{2} \log|x-1| - 8 \log|x-2| + \frac{27}{2} \log|x-3| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\begin{aligned} \int \frac{x^3}{(x-1)(x-2)(x-3)} dx \\ = x + \frac{1}{2} \log|x-1| - 8 \log|x-2| + \frac{27}{2} \log|x-3| + C \end{aligned}$$

11. Question

Evaluate the following integral:

$$\int \frac{\sin 2x}{(1 + \sin x)(2 + \sin x)} dx$$

Answer

The denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{\sin 2x}{(1 + \sin x)(2 + \sin x)} = \frac{A}{(1 + \sin x)} + \frac{B}{2 + \sin x} \dots \dots (i)$$

$$\Rightarrow \frac{\sin 2x}{(1 + \sin x)(2 + \sin x)} = \frac{A(2 + \sin x) + B(1 + \sin x)}{(1 + \sin x)(2 + \sin x)}$$

$$\Rightarrow \sin 2x = A(2 + \sin x) + B(1 + \sin x) = 2A + A \sin x + B + B \sin x$$

$$\Rightarrow 2 \sin x \cos x = \sin x (A + B) + (2A + B) \dots \dots (ii)$$

We need to solve for A and B.

We will equate similar terms, we get.

$$2A + B = 0 \Rightarrow B = -2A$$

$$\text{And } A + B = 2 \cos x$$

Substituting the value of B, we get

$$A - 2A = 2 \cos x \Rightarrow A = -2 \cos x$$

$$\text{Hence } B = -2A = -2(-2 \cos x)$$

$$\Rightarrow B = 4 \cos x$$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\begin{aligned} & \int \left[\frac{\sin 2x}{(1 + \sin x)(2 + \sin x)} \right] dx \\ & \Rightarrow \int \left[\frac{A}{(1 + \sin x)} + \frac{B}{2 + \sin x} \right] dx \\ & \Rightarrow \int \left[\frac{-2 \cos x}{(1 + \sin x)} + \frac{4 \cos x}{2 + \sin x} \right] dx \end{aligned}$$

Split up the integral,

$$\Rightarrow - \int \frac{2 \cos x}{(1 + \sin x)} dx + \int \frac{4 \cos x}{2 + \sin x} dx$$

Let substitute

$$u = \sin x \Rightarrow du = \cos x dx,$$

so the above equation becomes,

$$\Rightarrow -2 \int \frac{1}{(1 + u)} du + 4 \int \frac{1}{2 + u} du$$

Now substitute

$$v = 1 + u \Rightarrow dv = du$$

$$z = 2 + u \Rightarrow dz = du$$

So above equation becomes,

$$\Rightarrow -2 \int \frac{1}{(v)} dv + 4 \int \frac{1}{z} dz$$

On integrating we get

$$\Rightarrow -2 \log|v| + 4 \log|z| + C$$

Substituting back, we get

$$\Rightarrow 4 \log|2 + u| - 2 \log|1 + u| + C$$

$$\Rightarrow 4 \log|2 + \sin x| - 2 \log|1 + \sin x| + C$$

Applying logarithm rule, we get

$$\Rightarrow \log|(2 + \sin x)^4| - \log|(1 + \sin x)^2| + C$$

$$\Rightarrow \log \left| \frac{(2 + \sin x)^4}{(1 + \sin x)^2} \right| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{\sin 2x}{(1 + \sin x)(2 + \sin x)} dx = \log \left| \frac{(2 + \sin x)^4}{(1 + \sin x)^2} \right| + C$$

12. Question

Evaluate the following integral:

$$\int \frac{2x}{(x^2 + 1)(x^2 + 3)} dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{2x}{(x^2 + 1)(x^2 + 3)} = \frac{Ax + B}{(x^2 + 1)} + \frac{Cx + D}{x^2 + 3} \dots \dots (i)$$

$$\Rightarrow \frac{2x}{(x^2 + 1)(x^2 + 3)} = \frac{(Ax + B)(x^2 + 3) + (Cx + D)(x^2 + 1)}{(x^2 + 1)(x^2 + 3)}$$

$$\Rightarrow 2x = (Ax + B)(x^2 + 3) + (Cx + D)(x^2 + 1)$$

$$\Rightarrow 2x = Ax^3 + 3Ax + Bx^2 + 3B + Cx^3 + Cx + Dx^2 + D$$

$$\Rightarrow 2x = (A + C)x^3 + (B + D)x^2 + (3A + C)x + (3B + D) \dots \dots (ii)$$

By equating similar terms, we get

$$A + C = 0 \Rightarrow A = -C \dots \dots \dots (iii)$$

$$B + D = 0 \Rightarrow B = -D \dots \dots \dots (iv)$$

$$3A + C = 2$$

$$\Rightarrow 3(-C) + C = 2 \text{ (from equation (iii))}$$

$$\Rightarrow C = -1$$

So equation (iii) becomes $A = 1$

And also $3B + D = 0$ (from equation (ii))

$$\Rightarrow 3(-D) + D = 0 \text{ (from equation (iv))}$$

$$\Rightarrow D = 0$$

So equation (iv) becomes, $B = 0$

We put the values of A, B, C and D values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{2x}{(x^2 + 1)(x^2 + 3)} \right] dx$$

$$\Rightarrow \int \left[\frac{Ax + B}{(x^2 + 1)} + \frac{Cx + D}{x^2 + 3} \right] dx$$

$$\Rightarrow \int \left[\frac{(1)x + 0}{(x^2 + 1)} + \frac{(-1)x + 0}{x^2 + 3} \right] dx$$

Split up the integral,

$$\Rightarrow \int \frac{x}{(x^2 + 1)} dx - \int \left[\frac{x}{x^2 + 3} \right] dx$$

Let substitute

$$u = x^2 + 1 \Rightarrow du = 2x dx \Rightarrow dx = \frac{1}{2x} du$$

$$v = x^2 + 3 \Rightarrow dv = 2x dx \Rightarrow dx = \frac{1}{2x} dv$$

so the above equation becomes,

$$\Rightarrow \frac{1}{2} \int \frac{1}{(u)} du - \frac{1}{2} \int \left[\frac{1}{v} \right] dv$$

On integrating we get

$$\Rightarrow \frac{1}{2} \log|u| - \frac{1}{2} \log|v| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{2} \log|x^2 + 1| - \frac{1}{2} \log|x^2 + 3| + C$$

$$\Rightarrow \frac{1}{2} [\log|x^2 + 1| - \log|x^2 + 3|] + C$$

Applying the logarithm rule we get

$$\Rightarrow \frac{1}{2} \left[\log \left| \frac{(x^2 + 1)}{x^2 + 3} \right| \right] + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{2x}{(x^2 + 1)(x^2 + 3)} dx = \frac{1}{2} \left[\log \left| \frac{(x^2 + 1)}{x^2 + 3} \right| \right] + C$$

13. Question

Evaluate the following integral:

$$\int \frac{1}{x \log x (2 + \log x)} dx$$

Answer

Let substitute $u = \log x \Rightarrow du = \frac{1}{x} dx$, so the given equation becomes

$$\int \frac{1}{x \log x (2 + \log x)} dx = \int \frac{1}{u(2 + u)} du \dots (i)$$

Denominator is factorised, so let separate the fraction through partial fraction, hence let

$$\frac{1}{u(2 + u)} = \frac{A}{u} + \frac{B}{(2 + u)} \dots (ii)$$

$$\Rightarrow \frac{1}{u(2 + u)} = \frac{A(2 + u) + Bu}{u(2 + u)}$$

$$\Rightarrow 1 = A(2 + u) + Bu \dots (ii)$$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put $u = -2$ in above equation, we get

$$\Rightarrow 1 = A(2 + (-2)) + B(-2)$$

$$\Rightarrow 1 = -2B$$

$$\Rightarrow B = -\frac{1}{2}$$

Now put $u = 0$ in equation (ii), we get

$$\Rightarrow 1 = A(2 + 0) + B(0)$$

$$\Rightarrow 1 = 2A + 0$$

$$\Rightarrow A = \frac{1}{2}$$

We put the values of A and B values back into our partial fractions in equation (ii) and replace this as the integrand. We get

$$\begin{aligned} & \int \left[\frac{1}{u(2+u)} \right] du \\ & \Rightarrow \int \left[\frac{A}{u} + \frac{B}{(2+u)} \right] du \\ & \Rightarrow \int \left[\frac{\frac{1}{2}}{u} + \frac{-\frac{1}{2}}{(2+u)} \right] du \end{aligned}$$

Split up the integral,

$$\Rightarrow \frac{1}{2} \int \frac{1}{u} du - \frac{1}{2} \int \left[\frac{1}{2+u} \right] du$$

Let substitute

$z = 2 + u \Rightarrow dz = du$, so the above equation becomes,

$$\Rightarrow \frac{1}{2} \int \frac{1}{u} du - \frac{1}{2} \int \left[\frac{1}{z} \right] dz$$

On integrating we get

$$\Rightarrow \frac{1}{2} \log|u| - \frac{1}{2} \log|z| + C$$

Substituting back the value of z, we get

$$\Rightarrow \frac{1}{2} \log|u| - \frac{1}{2} \log|2+u| + C$$

Now substitute back the value of u, we get

$$\Rightarrow \frac{1}{2} [\log|\log x| - \log|2 + \log x|] + C$$

Applying the rules of logarithm we get

$$\Rightarrow \frac{1}{2} \log \left| \frac{\log x}{2 + \log x} \right| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{1}{x \log x (2 + \log x)} dx = \frac{1}{2} \log \left| \frac{\log x}{2 + \log x} \right| + C$$

14. Question

Evaluate the following integral:

$$\int \frac{x^2 + x + 1}{(x^2 + 1)(x + 2)} dx$$

Answer

Denominator is factorised, so let separate the fraction through partial fraction, hence let

$$\frac{x^2 + x + 1}{(x^2 + 1)(x + 2)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x + 2} \dots\dots (i)$$

$$\Rightarrow \frac{x^2 + x + 1}{(x^2 + 1)(x + 2)} = \frac{(Ax + B)(x + 2) + (Cx + D)(x^2 + 1)}{(x^2 + 1)(x + 2)}$$

$$\Rightarrow x^2 + x + 1 = (Ax + B)(x + 2) + (Cx + D)(x^2 + 1)$$

$$\Rightarrow x^2 + x + 1 = Ax^2 + 2Ax + Bx + 2B + Cx^3 + Cx + Dx^2 + D$$

$$\Rightarrow x^2 + x + 1 = Cx^3 + (A + D)x^2 + (2A + B + C)x + (2B + D) \dots\dots (ii)$$

We need to solve for A, B, C and D. We will equate the like terms we get,

$$C = 0 \dots\dots\dots (iii)$$

$$A + D = 1 \Rightarrow A = 1 - D \dots\dots\dots (iv)$$

$$2A + B + C = 1$$

$$\Rightarrow 2(1 - D) + B + 0 = 1 \text{ (from equation (iii) and (iv))}$$

$$\Rightarrow B = 2D - 1 \dots\dots\dots (v)$$

$$2B + D = 1$$

$$\Rightarrow 2(2D - 1) + D = 1 \text{ (from equation (v), we get)}$$

$$\Rightarrow 4D - 2 + D = 1$$

$$\Rightarrow 5D = 3$$

$$\Rightarrow D = \frac{3}{5} \dots\dots\dots (vi)$$

Equation (vi) in (v) and (iv), we get

$$B = 2\left(\frac{3}{5}\right) - 1 = \frac{1}{5}$$

$$A = 1 - \frac{3}{5} = \frac{2}{5}$$

We put the values of A, B, C, and D values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{x^2 + x + 1}{(x^2 + 1)(x + 2)} \right] dx$$

$$\Rightarrow \int \left[\frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x + 2} \right] dx$$

$$\Rightarrow \int \left[\frac{\left(\frac{2}{5}\right)x + \frac{1}{5}}{x^2 + 1} + \frac{(0)x + \frac{3}{5}}{x + 2} \right] dx$$

Split up the integral,

$$\Rightarrow \frac{1}{5} \int \frac{2x}{x^2 + 1} dx + \frac{1}{5} \int \frac{1}{x^2 + 1} dx + \frac{3}{5} \int \left[\frac{1}{x + 2} \right] dx$$

Let substitute

$$u = x^2 + 1 \Rightarrow du = 2x dx,$$

$$y = x + 2 \Rightarrow dy = dx, \text{ so the above equation becomes,}$$

$$\Rightarrow \frac{1}{5} \int \frac{1}{u} du + \frac{1}{5} \int \frac{1}{x^2 + 1} dx + \frac{3}{5} \int \left[\frac{1}{y} \right] dy$$

On integrating we get

$$\Rightarrow \frac{1}{5} \log|u| + \frac{1}{5} \tan^{-1} x + \frac{3}{5} \log|y| + C$$

(the standard integral of $\frac{1}{x^2 + 1} = \tan^{-1} x$)

Substituting back, we get

$$\Rightarrow \frac{1}{5} \log|x^2 + 1| + \frac{1}{5} \tan^{-1} x + \frac{3}{5} \log|x + 2| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{x^2 + x + 1}{(x^2 + 1)(x + 2)} dx = \frac{1}{5} \log|x^2 + 1| + \frac{1}{5} \tan^{-1} x + \frac{3}{5} \log|x + 2| + C$$

15. Question

Evaluate the following integral:

$$\int \frac{ax^2 + bx + c}{(x - a)(x - b)(x - c)} dx, \text{ where } a, b, c \text{ are distinct.}$$

Answer

Denominator is factorised, so let separate the fraction through partial fraction, hence let

$$\frac{ax^2 + bx + c}{(x - a)(x - b)(x - c)} = \frac{A}{(x - a)} + \frac{B}{x - b} + \frac{C}{x - c} \dots\dots (i)$$

$$\Rightarrow \frac{ax^2 + bx + c}{(x - a)(x - b)(x - c)} = \frac{A(x - b)(x - c) + B(x - a)(x - c) + C(x - a)(x - b)}{(x - a)(x - b)(x - c)}$$

$$\Rightarrow ax^2 + bx + c = A(x - b)(x - c) + B(x - a)(x - c) + C(x - a)(x - b) \dots\dots (ii)$$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put $x = a$ in the above equation, we get

$$\Rightarrow a(a)^2 + b(a) + c = A(a - b)(a - c) + B(a - a)(a - c) + C(a - a)(a - b)$$

$$\Rightarrow a^3 + ab + c = (a - b)(a - c)A + 0 + 0$$

$$\Rightarrow A = \frac{a^3 + ab + c}{(a - b)(a - c)}$$

Now put $x = b$ in equation (ii), we get

$$\Rightarrow a(b)^2 + b(b) + c = A(b - b)(b - c) + B(b - a)(b - c) + C(b - a)(b - b)$$

$$\Rightarrow ab^2 + b^2 + c = 0 + (b - a)(b - c)B + 0$$

$$\Rightarrow B = \frac{a^3 + ab + c}{(a - b)(a - c)}$$

Now put $x = c$ in equation (ii), we get

$$\begin{aligned} \Rightarrow a(c)^2 + b(c) + c \\ = A(c - b)(c - c) + B(c - a)(c - c) + C(c - a)(c - b) \end{aligned}$$

$$\Rightarrow ac^2 + bc + c = 0 + 0 + (c - a)(c - b)C$$

$$\Rightarrow C = \frac{ac^2 + bc + c}{(c - a)(c - b)}$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\begin{aligned} \int \left[\frac{ax^2 + bx + c}{(x - a)(x - b)(x - c)} \right] dx \\ \Rightarrow \int \left[\frac{A}{x - a} + \frac{B}{x - b} + \frac{C}{x - c} \right] dx \\ \Rightarrow \int \left[\frac{\frac{a^3 + ab + c}{(a - b)(a - c)}}{(x - a)} + \frac{\frac{a^3 + ab + c}{(a - b)(a - c)}}{x - b} + \frac{\frac{ac^2 + bc + c}{(c - a)(c - b)}}{x - c} \right] dx \end{aligned}$$

Split up the integral,

$$\begin{aligned} \Rightarrow \frac{a^3 + ab + c}{(a - b)(a - c)} \int \frac{1}{x - a} dx + \frac{a^3 + ab + c}{(a - b)(a - c)} \int \left[\frac{1}{x - b} \right] dx \\ + \frac{ac^2 + bc + c}{(c - a)(c - b)} \int \left[\frac{1}{x - c} \right] dx \end{aligned}$$

Let substitute

$$u = x - a \Rightarrow du = dx,$$

$$y = x - b \Rightarrow dy = dx \text{ and}$$

$$z = x - c \Rightarrow dz = dx, \text{ so the above equation becomes,}$$

$$\Rightarrow \frac{a^3 + ab + c}{(a - b)(a - c)} \int \frac{1}{u} du + \frac{a^3 + ab + c}{(a - b)(a - c)} \int \left[\frac{1}{y} \right] dy + \frac{ac^2 + bc + c}{(c - a)(c - b)} \int \left[\frac{1}{z} \right] dz$$

On integrating we get

$$\Rightarrow \frac{a^3 + ab + c}{(a - b)(a - c)} \log|u| + \frac{a^3 + ab + c}{(a - b)(a - c)} \log|y| + \frac{ac^2 + bc + c}{(c - a)(c - b)} \log|z| + C$$

Substituting back, we get

$$\begin{aligned} \Rightarrow \frac{a^3 + ab + c}{(a - b)(a - c)} \log|x - a| + \frac{a^3 + ab + c}{(a - b)(a - c)} \log|x - b| \\ + \frac{ac^2 + bc + c}{(c - a)(c - b)} \log|x - c| + C \end{aligned}$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{ax^2 + bx + c}{(x-a)(x-b)(x-c)} dx$$

$$= \frac{a^3 + ab + c}{(a-b)(a-c)} \log|x-a| + \frac{a^3 + ab + c}{(a-b)(a-c)} \log|x-b|$$

$$+ \frac{ac^2 + bc + c}{(c-a)(c-b)} \log|x-c| + C$$

16. Question

Evaluate the following integral:

$$\int \frac{x}{(x^2 + 1)(x - 1)} dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{x}{(x^2 + 1)(x - 1)} = \frac{Ax + B}{(x^2 + 1)} + \frac{Cx + D}{x - 1} \dots \dots (i)$$

$$\Rightarrow \frac{x}{(x^2 + 1)(x - 1)} = \frac{(Ax + B)(x - 1) + (Cx + D)(x^2 + 1)}{(x^2 + 1)(x - 1)}$$

$$\Rightarrow x = (Ax + B)(x - 1) + (Cx + D)(x^2 + 1)$$

$$\Rightarrow x = Ax^2 - Ax + Bx - B + Cx^2 + Cx + Dx^2 + D$$

$$\Rightarrow x = (C) x^2 + (A + D) x^2 + (B - A + C)x + (D - B) \dots \dots (ii)$$

By equating similar terms, we get

$$C = 0 \dots \dots \dots (iii)$$

$$A + D = 0 \Rightarrow A = -D \dots \dots \dots (iv)$$

$$B - A + C = 1$$

$$\Rightarrow B - (-D) + 0 = 2 \text{ (from equation(iii) and (iv))}$$

$$\Rightarrow B = 2 - D \dots \dots \dots (v)$$

$$D - B = 0 \Rightarrow D - (2 - D) = 0 \Rightarrow 2D = 2 \Rightarrow D = 1$$

So equation(iv) becomes $A = -1$

So equation (v) becomes, $B = 2 - 1 = 1$

We put the values of A, B, C, and D values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{x}{(x^2 + 1)(x - 1)} \right] dx$$

$$\Rightarrow \int \frac{Ax + B}{(x^2 + 1)} + \frac{Cx + D}{x - 1} dx$$

$$\Rightarrow \int \left[\frac{(-1)x + 1}{(x^2 + 1)} + \frac{(0)x + 1}{x - 1} \right] dx$$

Split up the integral,

$$\Rightarrow \int \frac{1}{(x^2 + 1)} dx - \int \frac{x}{(x^2 + 1)} dx + \int \left[\frac{1}{x - 1} \right] dx$$

Let substitute

$$u = x^2 + 1 \Rightarrow du = 2x dx \Rightarrow x dx = \frac{1}{2} du$$

$$v = x - 1 \Rightarrow dv = dx$$

so the above equation becomes,

$$\Rightarrow \int \frac{1}{(x^2 + 1)} dx - \frac{1}{2} \int \frac{1}{(u)} du + \int \left[\frac{1}{v} \right] dv$$

On integrating we get

$$\Rightarrow \tan^{-1} x - \frac{1}{2} \log|u| + \log|v| + C$$

(the standard integral of $\frac{1}{x^2 + 1} = \tan^{-1} x$)

Substituting back, we get

$$\Rightarrow \tan^{-1} x - \frac{1}{2} \log|x^2 + 1| + \log|x - 1| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{x}{(x^2 + 1)(x - 1)} dx = \tan^{-1} x - \frac{1}{2} \log|x^2 + 1| + \log|x - 1| + C$$

17. Question

Evaluate the following integral:

$$\int \frac{1}{(x - 1)(x + 1)(x + 2)} dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{1}{(x - 1)(x + 1)(x + 2)} = \frac{A}{(x - 1)} + \frac{B}{x + 1} + \frac{C}{x + 2} \dots \dots (i)$$

$$\Rightarrow \frac{1}{(x - 1)(x + 1)(x + 2)} = \frac{A(x + 1)(x + 2) + B(x - 1)(x + 2) + C(x - 1)(x + 1)}{(x - 1)(x + 1)(x + 2)}$$

$$\Rightarrow 1 = A(x + 1)(x + 2) + B(x - 1)(x + 2) + C(x - 1)(x + 1) \dots \dots (ii)$$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put $x = 1$ in the above equation, we get

$$\Rightarrow 1 = A(1 + 1)(1 + 2) + B(1 - 1)(1 + 2) + C(1 - 1)(1 + 1)$$

$$\Rightarrow 1 = 6A + 0 + 0$$

$$\Rightarrow A = \frac{1}{6}$$

Now put $x = -1$ in equation (ii), we get

$$\Rightarrow 1 = A(-1 + 1)(-1 + 2) + B(-1 - 1)(-1 + 2) + C(-1 - 1)(-1 + 1)$$

$$\Rightarrow 1 = 0 - 2B + 0$$

$$\Rightarrow B = -\frac{1}{2}$$

Now put $x = -2$ in equation (ii), we get

$$\Rightarrow 1 = A(-2 + 1)(-2 + 2) + B(-2 - 1)(-2 + 2) + C(-2 - 1)(-2 + 1)$$

$$\Rightarrow 1 = 0 + 0 + 3C$$

$$\Rightarrow C = \frac{1}{3}$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\begin{aligned} & \int \left[\frac{1}{(x-1)(x+1)(x+2)} \right] dx \\ & \Rightarrow \int \left[\frac{A}{(x-1)} + \frac{B}{x+1} + \frac{C}{x+2} \right] dx \\ & \Rightarrow \int \left[\frac{\frac{1}{6}}{(x-1)} + \frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{3}}{x+2} \right] dx \end{aligned}$$

Split up the integral,

$$\Rightarrow \frac{1}{6} \int \left[\frac{1}{(x-1)} \right] dx - \frac{1}{2} \int \left[\frac{1}{x+1} \right] dx + \frac{1}{3} \int \left[\frac{1}{x+2} \right] dx$$

Let substitute

$$u = x - 1 \Rightarrow du = dx,$$

$$y = x + 1 \Rightarrow dy = dx \text{ and}$$

$$z = x + 2 \Rightarrow dz = dx, \text{ so the above equation becomes,}$$

$$\Rightarrow \frac{1}{6} \int \left[\frac{1}{u} \right] du - \frac{1}{2} \int \left[\frac{1}{y} \right] dy + \frac{1}{3} \int \left[\frac{1}{z} \right] dz$$

On integrating we get

$$\Rightarrow \frac{1}{6} \log|u| - \frac{1}{2} \log|y| + \frac{1}{3} \log|z| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{6} \log|x-1| - \frac{1}{2} \log|x+1| + \frac{1}{3} \log|x+2| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\begin{aligned} & \int \frac{1}{(x-1)(x+1)(x+2)} dx \\ & = \frac{1}{6} \log|x-1| - \frac{1}{2} \log|x+1| + \frac{1}{3} \log|x+2| + C \end{aligned}$$

18. Question

Evaluate the following integral:

$$\int \frac{x^2}{(x^2+4)(x^2+9)} dx$$

Answer

Denominator is factorised, so let separate the fraction through partial fraction, hence let

$$\frac{x^2}{(x^2 + 4)(x^2 + 9)} = \frac{Ax + B}{(x^2 + 4)} + \frac{Cx + D}{x^2 + 9} \dots \dots (i)$$

$$\Rightarrow \frac{x^2}{(x^2 + 4)(x^2 + 9)} = \frac{(Ax + B)(x^2 + 9) + (Cx + D)(x^2 + 4)}{(x^2 + 4)(x^2 + 9)}$$

$$\Rightarrow x^2 = (Ax + B)(x^2 + 9) + (Cx + D)(x^2 + 4)$$

$$\Rightarrow x^2 = Ax^3 + 9Ax + Bx^2 + 9B + Cx^3 + 4Cx + Dx^2 + 4D$$

$$\Rightarrow x^2 = (A + C)x^3 + (B + D)x^2 + (9A + 4C)x + (9B + 4D) \dots \dots (ii)$$

By equating similar terms, we get

$$A + C = 0 \Rightarrow A = -C \dots \dots \dots (iii)$$

$$B + D = 1 \Rightarrow B = 1 - D \dots \dots \dots (iv)$$

$$9A + 4C = 0$$

$$\Rightarrow 9(-C) + 4C = 0 \text{ (from equation(iii))}$$

$$\Rightarrow C = 0 \dots \dots \dots (v)$$

$$9B + 4D = 0 \Rightarrow 9(1 - D) + 4D = 0 \Rightarrow 5D = 9 \Rightarrow D = \frac{9}{5}$$

$$\text{So equation(iv) becomes } B = 1 - \frac{9}{5} = -\frac{4}{5}$$

$$\text{So equation (iii) becomes, } A = 0$$

We put the values of A, B, C, and D values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \frac{x^2}{(x^2 + 4)(x^2 + 9)} dx$$

$$\Rightarrow \int \left[\frac{Ax + B}{(x^2 + 4)} + \frac{Cx + D}{x^2 + 9} \right] dx$$

$$\Rightarrow \int \left[\frac{(0)x - \frac{4}{5}}{(x^2 + 4)} + \frac{(0)x + \frac{9}{5}}{x^2 + 9} \right] dx$$

Split up the integral,

$$\Rightarrow -\frac{4}{5} \int \frac{1}{(x^2 + 4)} dx + \frac{9}{5} \int \frac{1}{(x^2 + 9)} dx$$

Let substitute

$$u = \frac{x}{2} \Rightarrow du = \frac{1}{2} dx \Rightarrow dx = 2du \text{ in first part}$$

$$v = \frac{x}{3} \Rightarrow dv = \frac{1}{3} dx \Rightarrow dx = 3dv \text{ in second part}$$

so the above equation becomes,

$$\Rightarrow \frac{9}{5} \int \frac{3}{((3v)^2 + 9)} dv - \frac{4}{5} \int \frac{2}{((2u)^2 + 4)} du$$

$$\Rightarrow \frac{9}{5} \int \frac{3}{(9v^2 + 9)} dv - \frac{4}{5} \int \frac{2}{(4u^2 + 4)} du$$

$$\Rightarrow \frac{3}{5} \int \frac{1}{v^2 + 1} dv - \frac{2}{5} \int \frac{1}{u^2 + 1} du$$

On integrating we get

$$\Rightarrow \frac{3}{5} \tan^{-1} v - \frac{2}{5} \tan^{-1} u + C$$

(the standard integral of $\frac{1}{x^2 + 1} = \tan^{-1} x$)

Substituting back, we get

$$\Rightarrow \frac{3}{5} \tan^{-1} \left(\frac{x}{3} \right) - \frac{2}{5} \tan^{-1} \left(\frac{x}{2} \right) + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{x^2}{(x^2 + 4)(x^2 + 9)} dx = \frac{3}{5} \tan^{-1} \left(\frac{x}{3} \right) - \frac{2}{5} \tan^{-1} \left(\frac{x}{2} \right) + C$$

19. Question

Evaluate the following integral:

$$\int \frac{5x^2 - 1}{x(x-1)(x+1)} dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{5x^2 - 1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \dots \dots (i)$$

$$\Rightarrow \frac{5x^2 - 1}{x(x-1)(x+1)} = \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$$\Rightarrow 5x^2 - 1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1) \dots \dots (ii)$$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put $x = 0$ in the above equation, we get

$$\Rightarrow 5(0)^2 - 1 = A(0-1)(0+1) + B(0)(0+1) + C(0)(0-1)$$

$$\Rightarrow A = 1$$

Now put $x = 1$ in equation (ii), we get

$$\Rightarrow 5(1)^2 - 1 = A(1-1)(1+1) + B(1)(1+1) + C(1)(1-1)$$

$$\Rightarrow 4 = 0 + 2B + 0$$

$$\Rightarrow B = 2$$

Now put $x = -1$ in equation (ii), we get

$$\Rightarrow 5(-1)^2 - 1 = A(-1-1)(-1+1) + B(-1)(-1+1) + C(-1)(-1-1)$$

$$\Rightarrow 4 = 0 + 0 + 2C$$

$$\Rightarrow C = 2$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{5x^2 - 1}{x(x-1)(x+1)} \right] dx$$

$$\Rightarrow \int \left[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \right] dx$$

$$\Rightarrow \int \left[\frac{1}{x} + \frac{2}{x-1} + \frac{2}{x+1} \right] dx$$

Split up the integral,

$$\Rightarrow \int \left[\frac{1}{x} \right] dx + 2 \int \left[\frac{1}{x-1} \right] dx + 2 \int \left[\frac{1}{x+1} \right] dx$$

Let substitute

$$u = x - 1 \Rightarrow du = dx,$$

$$y = x + 1 \Rightarrow dy = dx, \text{ so the above equation becomes,}$$

$$\Rightarrow \int \left[\frac{1}{x} \right] dx + 2 \int \left[\frac{1}{u} \right] du + 2 \int \left[\frac{1}{y} \right] dy$$

On integrating we get

$$\Rightarrow \log|x| + 2 \log|u| + 2 \log|y| + C$$

Substituting back, we get

$$\Rightarrow \log|x| + 2 \log|x-1| + 2 \log|x+1| + C$$

Applying logarithm rule, we get

$$\Rightarrow \log|x| + \log|(x-1)^2| + \log|(x+1)^2| + C$$

$$\Rightarrow \log|x(x^2-1)^2| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{5x^2 - 1}{x(x-1)(x+1)} dx = \log|x(x^2-1)^2| + C$$

20. Question

Evaluate the following integral:

$$\int \frac{x^2 + 6x - 8}{x^3 - 4x} dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{x^2 + 6x - 8}{x^3 - 4x}$$

$$= \frac{x^2 + 6x - 8}{x(x^2 - 4)}$$

$$\frac{x^2 + 6x - 8}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} \dots \dots (1)$$

$$\Rightarrow \frac{x^2 + 6x - 8}{x(x-2)(x+2)} = \frac{A(x-2)(x+2) + Bx(x+2) + Cx(x-2)}{x(x-2)(x+2)}$$

$$\Rightarrow x^2 + 6x - 8 = A(x - 2)(x + 2) + Bx(x + 2) + Cx(x - 2) \dots (ii)$$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put $x = 0$ in the above equation, we get

$$\Rightarrow 0^2 + 6(0) - 8 = A(0 - 2)(0 + 2) + B(0)(0 + 2) + C(0)(0 - 2)$$

$$\Rightarrow -8 = -4A + 0 + 0$$

$$\Rightarrow A = 2$$

Now put $x = 2$ in equation (ii), we get

$$\Rightarrow 2^2 + 6(2) - 8 = A(2 - 2)(2 + 2) + B(2)(2 + 2) + C(2)(2 - 2)$$

$$\Rightarrow 8 = 0 + 8B + 0$$

$$\Rightarrow B = 1$$

Now put $x = -2$ in equation (ii), we get

$$\Rightarrow (-2)^2 + 6(-2) - 8 = A((-2) - 2)((-2) + 2) + B(-2)((-2) + 2) + C(-2)((-2) - 2)$$

$$\Rightarrow -16 = 0 + 0 + 8C$$

$$\Rightarrow C = -2$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{x^2 + 6x - 8}{x(x - 2)(x + 2)} \right] dx$$

$$\Rightarrow \int \left[\frac{A}{x} + \frac{B}{x - 2} + \frac{C}{x + 2} \right] dx$$

$$\Rightarrow \int \left[\frac{2}{x} + \frac{1}{x - 2} + \frac{-2}{x + 2} \right] dx$$

Split up the integral,

$$\Rightarrow 2 \int \left[\frac{1}{x} \right] dx + \int \left[\frac{1}{x - 2} \right] dx - 2 \int \left[\frac{1}{x + 2} \right] dx$$

Let substitute

$$u = x - 2 \Rightarrow du = dx,$$

$$y = x + 2 \Rightarrow dy = dx, \text{ so the above equation becomes,}$$

$$\Rightarrow 2 \int \left[\frac{1}{x} \right] dx + \int \left[\frac{1}{u} \right] du - 2 \int \left[\frac{1}{y} \right] dy$$

On integrating we get

$$\Rightarrow 2 \log|x| + \log|u| - 2 \log|y| + C$$

Substituting back, we get

$$\Rightarrow \log|x| + \log|x - 2| - 2 \log|x + 2| + C$$

Applying logarithm rule, we get

$$\Rightarrow \log|x(x - 2)| - \log|(x + 2)^2| + C$$

$$\Rightarrow \log \left| \frac{x(x - 2)}{(x + 2)^2} \right| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{x^2 + 6x - 8}{x(x-2)(x+2)} dx = \log \left| \frac{x(x-2)}{(x+2)^2} \right| + C$$

21. Question

Evaluate the following integral:

$$\int \frac{x^2 + 1}{(2x+1)(x^2-1)} dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\begin{aligned} & \frac{x^2 + 1}{(2x+1)(x^2-1)} \\ &= \frac{x^2 + 1}{(2x+1)(x-1)(x+1)} \\ & \frac{x^2 + 1}{(2x+1)(x-1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{x+1} \dots\dots (i) \end{aligned}$$

$$\begin{aligned} \Rightarrow & \frac{x^2 + 1}{(2x+1)(x-1)(x+1)} \\ &= \frac{A(x-1)(x+1) + B(2x+1)(x+1) + C(2x+1)(x-1)}{(2x+1)(x-1)(x+1)} \end{aligned}$$

$$\Rightarrow x^2 + 1 = A(x-1)(x+1) + B(2x+1)(x+1) + C(2x+1)(x-1) \dots\dots (ii)$$

We need to solve for A, B and C. One way to do this is to pick values for x which will cancel each variable.

Put $x = 1$ in the above equation, we get

$$\Rightarrow 1^2 + 1 = A(1-1)(1+1) + B(2(1)+1)(1+1) + C(2(1)+1)(1-1)$$

$$\Rightarrow 2 = 0 + 6B + 0$$

$$\Rightarrow B = \frac{1}{3}$$

Now put $x = -\frac{1}{2}$ in equation (ii), we get

$$\begin{aligned} \Rightarrow & \left(-\frac{1}{2}\right)^2 + 1 \\ &= A\left(\left(-\frac{1}{2}\right)-1\right)\left(-\frac{1}{2}+1\right) + B\left(2\left(-\frac{1}{2}\right)+1\right)\left(-\frac{1}{2}+1\right) \\ &+ C\left(2\left(-\frac{1}{2}\right)+1\right)\left(-\frac{1}{2}-1\right) \end{aligned}$$

$$\Rightarrow \frac{5}{4} = -\frac{3}{4}A + 0 + 0$$

$$\Rightarrow A = -\frac{5}{3}$$

Now put $x = -1$ in equation (ii), we get

$$\Rightarrow (-1)^2 + 1 = A(-1-1)(-1+1) + B(2(-1)+1)(-1+1) + C(2(-1)+1)(-1-1)$$

$$\Rightarrow 2 = 0 + 0 + 2C$$

$$\Rightarrow C = 1$$

We put the values of A, B, and C values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \left[\frac{x^2 + 1}{(2x + 1)(x - 1)(x + 1)} \right] dx$$

$$\Rightarrow \int \left[\frac{A}{2x + 1} + \frac{B}{x - 1} + \frac{C}{x + 1} \right] dx$$

$$\Rightarrow \int \left[\frac{-\frac{5}{3}}{2x + 1} + \frac{\frac{1}{3}}{x - 1} + \frac{1}{x + 1} \right] dx$$

Split up the integral,

$$\Rightarrow -\frac{5}{3} \int \left[\frac{1}{2x + 1} \right] dx + \frac{1}{3} \int \left[\frac{1}{x - 1} \right] dx + \int \left[\frac{1}{x + 1} \right] dx$$

Let substitute

$$u = x - 1 \Rightarrow du = dx,$$

$$y = x + 1 \Rightarrow dy = dx \text{ and}$$

$$z = 2x + 1 \Rightarrow dz = 2dx \text{ so the above equation becomes,}$$

$$\Rightarrow -\frac{5}{3} \int \left[\frac{1}{z} \right] \frac{dz}{2} + \frac{1}{3} \int \left[\frac{1}{u} \right] du + \int \left[\frac{1}{y} \right] dy$$

On integrating we get

$$\Rightarrow -\frac{5}{6} \log|z| + \frac{1}{3} \log|u| + \log|y| + C$$

Substituting back, we get

$$\Rightarrow -\frac{5}{6} \log|2x + 1| + \frac{1}{3} \log|x - 1| + \log|x + 1| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\begin{aligned} \int \frac{x^2 + 1}{(2x + 1)(x^2 - 1)} dx \\ = -\frac{5}{6} \log|2x + 1| + \frac{1}{3} \log|x - 1| + \log|x + 1| + C \end{aligned}$$

22. Question

Evaluate the following integral:

$$\int \frac{1}{x \{ 6(\log x)^2 + 7 \log x + 2 \}} dx$$

Answer

Let substitute $u = \log x \Rightarrow du = \frac{1}{x} dx$, so the given equation becomes

$$\int \frac{1}{x \{ 6(\log x)^2 + 7 \log x + 2 \}} dx = \int \frac{1}{\{ 6u^2 + 7u + 2 \}} du \dots (i)$$

Factorizing the denominator, we get

$$\int \frac{1}{(2u + 1)(3u + 2)} du$$

The denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{1}{(2u + 1)(3u + 2)} = \frac{A}{2u + 1} + \frac{B}{(3u + 2)} \dots\dots(ii)$$

$$\Rightarrow \frac{1}{(2u + 1)(3u + 2)} = \frac{A(3u + 2) + B(2u + 1)}{(2u + 1)(3u + 2)}$$

$$\Rightarrow 1 = A(3u + 2) + B(2u + 1) \dots\dots(ii)$$

We need to solve for A and B. One way to do this is to pick values for x which will cancel each variable.

Put $u = -\frac{2}{3}$ in the above equation, we get

$$\Rightarrow 1 = A\left(3\left(-\frac{2}{3}\right) + 2\right) + B\left(2\left(-\frac{2}{3}\right) + 1\right)$$

$$\Rightarrow 1 = -\frac{1}{3}B$$

$$\Rightarrow B = -3$$

Now put $u = -\frac{1}{2}$ in equation (ii), we get

$$\Rightarrow 1 = A\left(3\left(-\frac{1}{2}\right) + 2\right) + B\left(2\left(-\frac{1}{2}\right) + 1\right)$$

$$\Rightarrow 1 = \frac{1}{2}A$$

$$\Rightarrow A = 2$$

We put the values of A and B values back into our partial fractions in equation (ii) and replace this as the integrand. We get

$$\int \left[\frac{1}{(2u + 1)(3u + 2)} \right] du$$

$$\Rightarrow \int \left[\frac{A}{2u + 1} + \frac{B}{(3u + 2)} \right] du$$

$$\Rightarrow \int \left[\frac{2}{2u + 1} + \frac{-3}{(3u + 2)} \right] du$$

Split up the integral,

$$\Rightarrow 2 \int \frac{1}{2u + 1} du - 3 \int \left[\frac{1}{3u + 2} \right] du$$

Let substitute

$z = 2u + 1 \Rightarrow dz = 2du$ and $y = 3u + 2 \Rightarrow dy = 3du$ so the above equation becomes,

$$\Rightarrow \int \frac{1}{z} dz - \int \left[\frac{1}{y} \right] dy$$

On integrating we get

$$\Rightarrow \log|z| - \log|y| + C$$

Substituting back the value of z, we get

$$\Rightarrow \log|2u + 1| - \log|3u + 2| + C$$

Now substitute back the value of u, we get

$$\Rightarrow \log|2(\log x) + 1| - \log|3(\log x) + 2| + C$$

Applying the rules of logarithm we get

$$\Rightarrow \log \left| \frac{2(\log x) + 1}{3(\log x) + 2} \right| + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{1}{x\{6(\log x)^2 + 7\log x + 2\}} dx = \log \left| \frac{2(\log x) + 1}{3(\log x) + 2} \right| + C + C$$

23. Question

Evaluate the following integral:

$$\int \frac{1}{x(x^n + 1)} dx$$

Answer

$$\frac{1}{x(x^n + 1)}$$

Multiply numerator and denominator by x^{n-1} , we get

$$\int \frac{1}{x(x^n + 1)} dx \Rightarrow \int \frac{x^{n-1}}{x(x^n + 1)x^{n-1}} dx \Rightarrow \int \frac{x^{n-1}}{x^n(x^n + 1)} dx$$

$$\text{Let } x^n = t \Rightarrow nx^{n-1} dx = dt$$

So the above equation becomes,

$$\int \frac{x^{n-1}}{x^n(x^n + 1)} dx \Rightarrow \frac{1}{n} \int \frac{1}{t(t + 1)} dt$$

The denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{1}{t(t + 1)} = \frac{A}{t} + \frac{B}{t + 1} \dots\dots (i)$$

$$\Rightarrow \frac{1}{t(t + 1)} = \frac{A(t + 1) + Bt}{t(t + 1)}$$

$$\Rightarrow 1 = A(t + 1) + Bt \dots\dots (ii)$$

Put $t = 0$ in above equations we get

$$1 = A(0 + 1) + B(0)$$

$$\Rightarrow A = 1$$

Now put $t = -1$ in equation (ii) we get

$$1 = A(-1 + 1) + B(-1)$$

$$\Rightarrow B = -1$$

We put the values of A and B values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \frac{x^{n-1}}{x^n(x^n + 1)} dx \Rightarrow \frac{1}{n} \int \frac{1}{t(t + 1)} dt$$

$$\Rightarrow \frac{1}{n} \int \left[\frac{A}{t} + \frac{B}{t + 1} \right] dt$$

$$\Rightarrow \frac{1}{n} \int \left[\frac{1}{t} + \frac{-1}{t + 1} \right] dt$$

Split up the integral,

$$\Rightarrow \frac{1}{n} \left[\int \frac{1}{t} dt - \int \frac{1}{t + 1} dt \right]$$

Let substitute

$u = t + 1 \Rightarrow du = dt$, so the above equation becomes,

$$\Rightarrow \frac{1}{n} \left[\int \frac{1}{t} dt - \int \frac{1}{u} du \right]$$

On integrating we get

$$\Rightarrow \frac{1}{n} [\log t - \log u] + C$$

Substituting back the values of u , we get

$$\Rightarrow \frac{1}{n} [\log |t| - \log |t + 1|] + C$$

Substituting back the values of t , we get

$$\Rightarrow \frac{1}{n} [\log |x^n| - \log |x^n + 1|] + C$$

Applying the logarithm rules, we get

$$\Rightarrow \frac{1}{n} \left[\log \left| \frac{x^n}{x^n + 1} \right| \right] + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{1}{x(x^n + 1)} dx = \frac{1}{n} \left[\log \left| \frac{x^n}{x^n + 1} \right| \right] + C$$

24. Question

Evaluate the following integral:

$$\int \frac{x}{(x^2 - a^2)(x^2 - b^2)} dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{x}{(x^2 - a^2)(x^2 - b^2)} = \frac{Ax + B}{(x^2 - a^2)} + \frac{Cx + D}{(x^2 - b^2)} \dots \dots (i)$$

$$\Rightarrow \frac{x}{(x^2 - a^2)(x^2 - b^2)} = \frac{(Ax + B)(x^2 - b^2) + (Cx + D)(x^2 - a^2)}{(x^2 - a^2)(x^2 - b^2)}$$

$$\Rightarrow x = (Ax + B)(x^2 - b^2) + (Cx + D)(x^2 - a^2)$$

$$\Rightarrow x = Ax^3 - Ab^2x + Bx^2 - b^2B + Cx^3 - a^2Cx + Dx^2 - a^2D$$

$$\Rightarrow x = (A + C)x^3 + (B + D)x^2 + (-Ab^2 - Ca^2)x + (-b^2B - a^2D) \dots \dots (ii)$$

By equating similar terms, we get

$$A + C = 0 \Rightarrow A = -C \dots \dots \dots (iii)$$

$$B + D = 0 \Rightarrow B = -D \dots \dots \dots (iv)$$

$$-Ab^2 - Ca^2 = 1$$

$$\Rightarrow -(-C)b^2 - Ca^2 = 1 \text{ (from equation(iii))}$$

$$\Rightarrow C = \frac{1}{b^2 - a^2} \dots \dots \dots (v)$$

$$-b^2B - a^2D = 0$$

$$\Rightarrow -b^2(-D) - a^2D = 0$$

$$\Rightarrow D = 0$$

So equation(iv) becomes $B = 0$

$$\text{So equation (iii) becomes, } A = -\frac{1}{b^2 - a^2}$$

We put the values of A, B, C, and D values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\begin{aligned} & \int \frac{x}{(x^2 - a^2)(x^2 - b^2)} dx \\ & \Rightarrow \int \left[\frac{Ax + B}{(x^2 - a^2)} + \frac{Cx + D}{(x^2 - b^2)} \right] dx \\ & \Rightarrow \int \left[\frac{\left(-\frac{1}{b^2 - a^2}\right)x + 0}{(x^2 - a^2)} + \frac{\left(\frac{1}{b^2 - a^2}\right)x + 0}{(x^2 - b^2)} \right] dx \end{aligned}$$

Split up the integral,

$$\Rightarrow -\frac{1}{b^2 - a^2} \int \frac{1}{(x^2 - a^2)} dx + \frac{1}{b^2 - a^2} \int \frac{1}{(x^2 - b^2)} dx$$

Let substitute

$$u = x^2 - a^2 \Rightarrow du = 2dx$$

$$v = x^2 - b^2 \Rightarrow dv = 2dx, \text{ so the above equation becomes,}$$

$$\begin{aligned} & \Rightarrow -\frac{1}{b^2 - a^2} \int \frac{\frac{1}{u} du}{2} + \frac{1}{b^2 - a^2} \int \frac{\frac{1}{v} dv}{2} \\ & \Rightarrow -\frac{1}{2(b^2 - a^2)} \int \frac{1}{u} du + \frac{1}{2(b^2 - a^2)} \int \frac{1}{v} dv \end{aligned}$$

On integrating we get

$$\Rightarrow -\frac{1}{2(b^2 - a^2)} \log|u| + \frac{1}{2(b^2 - a^2)} \log|v| + C$$

Substituting back, we get

$$\Rightarrow \frac{1}{2(b^2 - a^2)} [\log|x^2 - b^2| - \log|x^2 - a^2|] + C$$

Applying the logarithm rule we get

$$\Rightarrow \frac{1}{2(b^2 - a^2)} \left[\log \left| \frac{x^2 - b^2}{x^2 - a^2} \right| \right] + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{x}{(x^2 - a^2)(x^2 - b^2)} dx = \frac{1}{2(b^2 - a^2)} \left[\log \left| \frac{x^2 - b^2}{x^2 - a^2} \right| \right] + C$$

25. Question

Evaluate the following integral:

$$\int \frac{x^2 + 1}{(x^2 + 4)(x^2 + 25)} dx$$

Answer

Denominator is factorized, so let separate the fraction through partial fraction, hence let

$$\frac{x^2 + 1}{(x^2 + 4)(x^2 + 25)} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{x^2 + 25} \dots \dots (i)$$

$$\Rightarrow \frac{x^2 + 1}{(x^2 + 4)(x^2 + 25)} = \frac{(Ax + B)(x^2 + 25) + (Cx + D)(x^2 + 4)}{(x^2 + 4)(x^2 + 25)}$$

$$\Rightarrow x^2 + 1 = (Ax + B)(x^2 + 25) + (Cx + D)(x^2 + 4)$$

$$\Rightarrow x^2 + 1 = Ax^3 + 25Ax + Bx^2 + 25B + Cx^3 + 4Cx + Dx^2 + 4D$$

$$\Rightarrow x^2 + 1 = (A + C)x^3 + (B + D)x^2 + (25A + 4C)x + (25B + 4D) \dots \dots (ii)$$

By equating similar terms, we get

$$A + C = 0 \Rightarrow A = -C \dots \dots \dots (iii)$$

$$B + D = 1 \Rightarrow B = 1 - D \dots \dots \dots (iv)$$

$$25A + 4C = 0$$

$$\Rightarrow 25(-C) + 4C = 0 \text{ (from equation(iii))}$$

$$\Rightarrow C = 0 \dots \dots \dots (v)$$

$$25B + 4D = 1 \Rightarrow 25(1 - D) + 4D = 1 \Rightarrow 21D = 24 \Rightarrow D = \frac{24}{21} = \frac{8}{7}$$

$$\text{So equation(iv) becomes } B = 1 - \frac{8}{7} = -\frac{1}{7}$$

So equation (iii) becomes, $A = 0$

We put the values of A, B, C, and D values back into our partial fractions in equation (i) and replace this as the integrand. We get

$$\int \frac{x^2 + 1}{(x^2 + 4)(x^2 + 25)} dx$$

$$\Rightarrow \int \left[\frac{Ax + B}{(x^2 + 4)} + \frac{Cx + D}{x^2 + 25} \right] dx$$

$$\Rightarrow \int \left[\frac{(0)x - \frac{1}{7}}{(x^2 + 4)} + \frac{(0)x + \frac{8}{7}}{x^2 + 25} \right] dx$$

Split up the integral,

$$\Rightarrow -\frac{1}{7} \int \frac{1}{(x^2 + 4)} dx + \frac{8}{7} \int \frac{1}{(x^2 + 25)} dx$$

Let substitute

$$u = \frac{x}{2} \Rightarrow du = \frac{1}{2} dx \Rightarrow dx = 2du \text{ in first part}$$

$$v = \frac{x}{5} \Rightarrow dv = \frac{1}{5} dx \Rightarrow dx = 5dv \text{ in second part}$$

so the above equation becomes,

$$\Rightarrow \frac{8}{7} \int \frac{5}{((5v)^2 + 25)} dv - \frac{1}{7} \int \frac{2}{((2u)^2 + 4)} du$$

$$\Rightarrow \frac{8}{7} \int \frac{5}{(25v^2 + 25)} dv - \frac{1}{7} \int \frac{2}{(4u^2 + 4)} du$$

$$\Rightarrow \frac{8}{35} \int \frac{1}{v^2 + 1} dv - \frac{1}{14} \int \frac{1}{u^2 + 1} du$$

On integrating we get

$$\Rightarrow \frac{8}{35} \tan^{-1} v - \frac{1}{14} \tan^{-1} u + C$$

(the standard integral of $\frac{1}{x^2 + 1} = \tan^{-1} x$)

Substituting back, we get

$$\Rightarrow \frac{8}{35} \tan^{-1} \left(\frac{x}{5} \right) - \frac{1}{14} \tan^{-1} \left(\frac{x}{2} \right) + C$$

Note: the absolute value signs account for the domain of the natural log function ($x > 0$).

Hence,

$$\int \frac{x^2 + 1}{(x^2 + 4)(x^2 + 25)} dx = \frac{8}{35} \tan^{-1} \left(\frac{x}{5} \right) - \frac{1}{14} \tan^{-1} \left(\frac{x}{2} \right) + C$$

26. Question

Evaluate the following integral:

$$\int \frac{x^3 + x + 1}{x^2 - 1}$$

Answer

Let

$$I = \int \frac{x^3 + x + 1}{x^2 - 1} dx = \int \left(x + \frac{2x + 1}{x^2 - 1} \right) dx$$

Now,

$$\text{Let } \frac{2x+1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$2x + 1 = A(x - 1) + B(x + 1)$$

$$\text{Put } x = 1$$

$$2 + 1 = A \times 0 + B \times 2$$

$$3 = 2B$$

$$B = \frac{3}{2}$$

$$\text{Put } x = -1$$

$$-2 + 1 = -2A + B \times 0$$

$$-1 = -2A$$

$$A = \frac{1}{2}$$

$$I = \int x dx + \frac{1}{2} \int \frac{dx}{x+1} + \frac{3}{2} \int \frac{dx}{x-1}$$

$$\int \frac{dx}{x} = \log|x| \text{ and } \int x dx = \frac{x^2}{2}$$

Therefore,

$$I = \frac{x^2}{2} + \frac{1}{2} \log|x+1| + \frac{3}{2} \log|x-1| + c$$

27. Question

Evaluate the following integral:

$$\int \frac{3x-2}{(x+1)^2(x+3)}$$

Answer

$$I = \int \frac{3x-2}{(x+1)^2(x+3)} dx$$

$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$$

$$3x - 2 = A(x+1)(x+3) + B(x+3) + C(x+1)^2$$

$$\text{Put } x = -1$$

$$-3 - 2 = A \times 0 + B \times (-1+3) + C \times 0$$

$$-5 = 2B$$

$$B = -\frac{5}{2}$$

$$\text{Put } x = -3$$

$$-9 - 2 = C \times (-2)(-2)$$

$$-11 = 4C$$

$$C = -\frac{11}{4}$$

Equating coefficients of constants

$$-2 = 3A + 3B + C$$

$$-2 = 3A + 3 \times \frac{-5}{2} - \frac{11}{4}$$

$$A = \frac{11}{4}$$

Thus,

$$I = \frac{11}{4} \int \frac{dx}{x+1} - \frac{5}{2} \int \frac{dx}{(x+1)^2} - \frac{11}{4} \int \frac{dx}{x+3}$$

$$I = \frac{11}{4} \log|x+1| - \frac{5}{2(x+1)} - \frac{11}{4} \log|x+3| + C$$

28. Question

Evaluate the following integral:

$$\int \frac{2x+1}{(x+2)(x-3)^2}$$

Answer

$$I = \int \frac{2x+1}{(x+2)(x-3)^2} dx$$

$$\frac{2x+1}{(x+2)(x-3)^2} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

$$2x+1 = A(x-3)^2 + B(x+2)(x-3) + C(x+2)$$

$$2x+1 = Ax^2 - 3Ax + 9A + Bx^2 - 5Bx - 6B + Cx + 2C$$

$$\text{Put } x = 3$$

$$7 = 5C$$

$$C = \frac{7}{5}$$

$$\text{Put } x = -2$$

$$-3 = 0A$$

$$-11 = 4C$$

$$C = -\frac{11}{4}$$

Equating coefficients of constants

$$-2 = 3A + 3B + C$$

$$-2 = 3A + 3 \times \frac{-5}{2} - \frac{11}{4}$$

$$A = \frac{11}{4}$$

Thus,

$$I = \frac{11}{4} \int \frac{dx}{x+1} - \frac{5}{2} \int \frac{dx}{(x+1)^2} - \frac{11}{4} \int \frac{dx}{x+3}$$

$$I = \frac{11}{4} \log|x + 1| - \frac{5}{2(x + 1)} - \frac{11}{4} \log|x + 3| + C$$

29. Question

Evaluate the following integral:

$$\int \frac{x^2 + 1}{(x - 2)^2 (x + 3)} dx$$

Answer

$$I = \int \frac{x^2 + 2}{(x - 2)^2 (x + 3)} dx$$

$$\frac{x^2 + 2}{(x - 2)^2 (x + 3)} = \frac{A}{x - 2} + \frac{B}{(x - 2)^2} + \frac{C}{x + 3}$$

$$x^2 + 1 = A(x - 2)(x + 3) + B(x + 3) + C(x - 2)^2$$

Put $x = 2$

$$4 + 1 = B \times 5$$

$$5 = 5B$$

$$B = \frac{5}{5} = 1$$

Put $x = -3$

$$10 = C \times 25$$

$$C = \frac{10}{25} = \frac{2}{5}$$

Equating coefficients of constants

$$1 = -6A + 3B + 4C$$

$$1 = -6A + 3 + \frac{8}{5}$$

$$A = \frac{3}{5}$$

Thus,

$$I = \frac{3}{5} \int \frac{dx}{x - 2} - \int \frac{dx}{(x - 2)^2} - \frac{2}{5} \int \frac{dx}{x + 3}$$

$$I = \frac{3}{5} \log|x - 2| - \frac{1}{(x - 2)} + \frac{2}{5} \log|x + 3| + C$$

30. Question

Evaluate the following integral:

$$\int \frac{x}{(x - 1)^2 (x + 2)} dx$$

Answer

$$I = \int \frac{x}{(x - 1)^2 (x + 2)} dx$$

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$$\text{Put } x = -2$$

$$-2 = 9C$$

$$C = -\frac{2}{9}$$

$$\text{Put } x = 1$$

$$1 = 3B$$

$$B = \frac{1}{3}$$

Equating coefficients of constants

$$0 = -2A + 2B + C$$

$$0 = -2A + 2 * \frac{1}{3} - \frac{2}{9}$$

$$A = \frac{2}{9}$$

Thus,

$$I = \frac{2}{9} \int \frac{dx}{x-1} + \frac{1}{3} \int \frac{dx}{(x-1)^2} - \frac{2}{9} \int \frac{dx}{x+2}$$

$$I = \frac{2}{9} \log|x-1| + \frac{1}{3} \left(\frac{-1}{(x-1)} \right) - \frac{2}{9} \log|x+2| + C$$

$$= \frac{2}{9} \log \left| \frac{x-1}{x+2} \right| - \frac{1}{3(x-1)} + C$$

31. Question

Evaluate the following integral:

$$\int \frac{x^2}{(x-1)(x+1)^2} dx$$

Answer

$$I = \int \frac{x^2}{(x-1)(x+1)^2} dx$$

$$\frac{x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$$\text{Put } x = 1$$

$$1 = 4A$$

$$A = \frac{1}{4}$$

$$\text{Put } x = -1$$

$$1 = -2C$$

$$C = -\frac{1}{2}$$

Equating coefficients of x^2

$$1 = A + B$$

$$1 = \frac{1}{4} + B$$

$$B = \frac{3}{4}$$

Thus,

$$I = \frac{1}{4} \int \frac{dx}{x-1} + \frac{3}{4} \int \frac{dx}{x+1} - \frac{1}{2} \int \frac{dx}{(x+1)^2}$$

$$I = \frac{1}{4} \log|x-1| + \frac{3}{4} \log|x+1| + \frac{1}{2(x+1)} + C$$

32. Question

Evaluate the following integral:

$$\int \frac{x^2 + x - 1}{(x+1)^2(x+2)} dx$$

Answer

$$I = \int \frac{x^2 + x - 1}{(x+1)^2(x+2)} dx$$

$$\frac{x^2 + x - 1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

$$x^2 + x - 1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$$

Put $x = -2$

$$1 = C$$

$$C = 1$$

Put $x = -1$

$$-1 = B$$

$$B = -1$$

Equating coefficients of constants

$$-1 = 2A + 2B + C$$

$$-1 = 2A - 2 + 1$$

$$A = 0$$

Thus,

$$I = 0 \times \int \frac{dx}{x+1} + (-1) \int \frac{dx}{(x+1)^2} + \int \frac{dx}{x+2}$$

$$I = -\left(\frac{-1}{(x+1)}\right) + \log|x+2| + C$$

$$= \left(\frac{1}{(x+1)} \right) + \log|x+2| + C$$

33. Question

Evaluate the following integral:

$$\int \frac{2x^2 + 7x - 3}{x^2(2x+1)} dx$$

Answer

$$I = \int \frac{2x^2 + 7x - 3}{x^2(2x+1)} dx$$

$$\frac{2x^2 + 7x - 3}{x^2(2x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x+1}$$

$$2x^2 + 7x - 3 = Ax(2x+1) + B(2x+1) + Cx^2$$

Equating constants

$$-3 = B$$

Equating coefficients of x

$$7 = A + 2B$$

$$7 = A - 6$$

$$A = 13$$

Equating coefficients of x^2

$$2 = 2A + C$$

$$2 = 26 + C$$

$$C = -24$$

Thus,

$$I = \int \frac{13dx}{x} - \int \frac{3dx}{x^2} - 24 \int \frac{dx}{2x+1}$$

$$I = 13 \log|x| + \frac{3}{x} - 12 \log|2x+1| + C$$

34. Question

Evaluate the following integral:

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} dx$$

Answer

$$I = \int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} = \int \frac{5x^2 + 20x + 6}{x(x+1)^2}$$

$$\frac{5x^2 + 20x + 6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$5x^2 + 20x + 6 = A(x+1)^2 + Bx(x+1) + Cx$$

Equating constants

$$6 = A$$

Equating coefficients of x^2

$$5 = A + B$$

$$B = -1$$

Equating coefficients of x

$$20 = 2A + B + C$$

$$20 = 12 - 1 + C$$

$$C = 9$$

$$I = \int \frac{6dx}{x} - \int \frac{dx}{x+1} + 9 \int \frac{dx}{(x+1)^2}$$

$$I = 6 \log|x| - \log|x+1| - \frac{9}{x+1} + C$$

35. Question

Evaluate the following integral:

$$\int \frac{18}{(x+2)(x^2+4)} dx$$

Answer

$$I = \int \frac{18}{(x+2)(x^2+4)}$$

$$\frac{18}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$$

$$18 = A(x^2+4) + (Bx+C)(x+2)$$

Equating constants

$$18 = 4A + 2C$$

Equating coefficients of x

$$0 = 2B + C$$

Equating coefficients of x^2

$$0 = A + B$$

Solving, we get

$$A = \frac{9}{4}, \quad B = -\frac{9}{4}, \quad C = \frac{9}{2}$$

Thus,

$$I = \frac{9}{4} \int \frac{dx}{x+2} + \left(-\frac{9}{4}\right) \int \frac{xdx}{x^2+4} + \frac{9}{2} \int \frac{dx}{x^2+4}$$

$$I = \frac{9}{4} \log|x+2| - \frac{9}{8} \log|x^2+4| + \frac{9}{4} \tan^{-1}\left(\frac{x}{2}\right) + C$$

36. Question

Evaluate the following integral:

$$\int \frac{5}{(x^2+1)(x+2)} dx$$

Answer

$$I = \int \frac{5}{(x^2+1)(x+2)}$$

$$\frac{5}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2}$$

$$5 = (Ax+B)(x+2) + C(x^2+1)$$

Equating constants

$$5 = 2B + C$$

Equating coefficients of x

$$0 = 2A + B$$

Equating coefficients of x^2

$$0 = A + C$$

Solving, we get

$$A = -1, B = 2, C = 1$$

Thus

$$I = \int \frac{-x+2}{x^2+1} dx + \int \frac{dx}{x+2}$$

$$= \int \frac{-x dx}{x^2+1} + 2 \int \frac{dx}{x^2+1} + \int \frac{dx}{x+2}$$

$$I = -\frac{1}{2} \log|x^2+1| + 2 \tan^{-1}x + \log|x+2| + C$$

37. Question

Evaluate the following integral:

$$\int \frac{x}{(x+1)(x^2+1)} dx$$

Answer

$$I = \int \frac{x}{(x+1)(x^2+1)}$$

$$\frac{x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$x = A(x^2+1) + (Bx+C)(x+1)$$

Equating constants

$$0 = A + C$$

Equating coefficients of x

$$1 = B + C$$

Equating coefficients of x^2

$$0 = A + B$$

Solving, we get

$$A = -\frac{1}{2} \quad B = \frac{1}{2} \quad C = \frac{1}{2}$$

Thus

$$I = -\frac{1}{2} \int \frac{dx}{x+1} + \frac{1}{2} \int \frac{xdx}{x^2+1} + \frac{1}{2} \int \frac{dx}{x^2+1}$$

$$I = -\frac{1}{2} \log|x+1| + \frac{1}{4} \log|x^2+1| + \frac{1}{2} \tan^{-1}x + C$$

38. Question

Evaluate the following integral:

$$\int \frac{1}{1+x+x^2+x^3} dx$$

Answer

$$I = \int \frac{1}{1+x+x^2+x^3} = \int \frac{dx}{(x^2+1)(x+1)}$$

$$\frac{1}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

$$1 = (Ax+B)(x+1) + C(x^2+1)$$

Equating constants

$$1 = B + C$$

Equating coefficients of x

$$0 = A + B$$

Equating coefficients of x^2

$$0 = A + C$$

Solving, we get

$$A = -\frac{1}{2} \quad B = \frac{1}{2} \quad C = \frac{1}{2}$$

Thus

$$I = -\frac{1}{2} \int \frac{xdx}{x^2+1} + \frac{1}{2} \int \frac{dx}{x^2+1} + \frac{1}{2} \int \frac{dx}{x+1}$$

$$I = -\frac{1}{4} \log|x^2+1| + \frac{1}{2} \tan^{-1}x + \frac{1}{2} \log|x+1| + C$$

39. Question

Evaluate the following integral:

$$\int \frac{1}{(x+1)^2(x^2+1)} dx$$

Answer

$$I = \frac{1}{(x+1)^2(x^2+1)}$$

$$\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$$

$$1 = A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2$$

$$= Ax^3 + Ax^2 + Ax + A + Bx^2 + B + Cx^3 + 2Cx^2 + Cx + Dx^2 + 2Dx + D$$

$$= (A+C)x^3 + (A+B+2C+D)x^2 + (A+C+2D)x + (A+B+D)$$

Equating constants

$$1 = A + B + D$$

Equating coefficients of x^3

$$0 = A + C$$

Equating coefficients of x^2

$$0 = A + B + 2C + D$$

Equating coefficients of x

$$0 = A + C + 2D$$

Solving we get

$$A = \frac{1}{2} \quad B = \frac{1}{2} \quad C = -\frac{1}{2} \quad D = 0$$

Thus,

$$I = \frac{1}{2} \int \frac{dx}{x+1} + \frac{1}{2} \int \frac{dx}{(x+1)^2} - \frac{1}{2} \int \frac{dx}{x^2+1}$$

$$I = \frac{1}{2} \log|x+1| - \frac{1}{2(x+1)} - \frac{1}{4} \log|x^2+1| + C$$

40. Question

Evaluate the following integral:

$$\int \frac{2x}{x^3-1} dx$$

Answer

$$I = \int \frac{2x}{x^3-1} dx = \int \frac{2x}{(x-1)(x^2+x+1)} dx$$

$$\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$2x = A(x^2+x+1) + (Bx+C)(x-1)$$

$$= (A+B)x^2 + (A-B+C)x + (A-C)$$

Equating constants,

$$A - C = 0$$

Equating coefficients of x

$$2 = A - B + C$$

Equating coefficients of x^2

$$0 = A + B$$

On solving,

We get

$$A = \frac{2}{3} \quad B = -\frac{2}{3} \quad C = \frac{2}{3}$$

$$\begin{aligned} I &= \frac{2}{3} \int \frac{dx}{x-1} - \frac{2}{3} \int \frac{(x-1)dx}{x^2+x+1} \\ &= \frac{2}{3} \int \frac{dx}{x-1} - \frac{2}{3} \cdot \frac{1}{2} \int \frac{(2x-2)dx}{x^2+x+1} \\ &= \frac{2}{3} \int \frac{dx}{x-1} - \frac{1}{3} \int \frac{(2x+1)dx}{x^2+x+1} + \int \frac{dx}{x^2+x+1} \\ &= \frac{2}{3} \int \frac{dx}{x-1} - \frac{1}{3} \int \frac{(2x+1)dx}{x^2+x+1} + \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \frac{2}{3} \log|x-1| - \frac{1}{3} \log|x^2+x+1| + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C \end{aligned}$$

41. Question

Evaluate the following integral:

$$\int \frac{1}{(x^2+1)(x^2+4)} dx$$

Answer

$$I = \int \frac{1}{(x^2+1)(x^2+4)} dx$$

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

$$1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

$$= (A+C)x^3 + (B+D)x^2 + (4A+C)x + 4B+D$$

Equating similar terms

$$A + C = 0$$

$$B + D = 0$$

$$4A + C = 0$$

$$4B + D = 1$$

$$\text{We get, } A = 0 \quad B = \frac{1}{3} \quad C = 0 \quad D = -\frac{1}{3}$$

Thus,

$$\begin{aligned} I &= \int \frac{\frac{1}{3} dx}{x^2+1} - \int \frac{\frac{1}{3} dx}{x^2+4} \\ &= \frac{1}{3} \tan^{-1}x - \frac{1}{6} \tan^{-1}\frac{x}{2} + C \end{aligned}$$

42. Question

Evaluate the following integral:

$$\int \frac{x^2}{(x^2+1)(3x^2+4)} dx$$

Answer

$$I = \int \frac{x^2}{(x^2+1)(3x^2+4)} dx$$

$$\frac{x^2}{(x^2+1)(3x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{3x^2+4}$$

$$\begin{aligned} x^2 &= (Ax+B)(3x^2+4) + (Cx+D)(x^2+1) \\ &= (3A+C)x^3 + (3B+D)x^2 + (4A+C)x + 4B+D \end{aligned}$$

Equating similar terms

$$3A+C=0$$

$$3B+D=1$$

$$4A+C=0$$

$$4B+D=0$$

Solving we get,

$$A=0, B=-1, C=0, D=4$$

Thus,

$$I = \int \frac{-dx}{x^2+1} - \int \frac{4dx}{3x^2+4}$$

$$I = -\tan^{-1}x + \frac{4}{3} \int \frac{dx}{x^2 + \left(\frac{2}{\sqrt{3}}\right)^2}$$

$$I = -\tan^{-1}x + \frac{4}{3} \cdot \frac{\sqrt{3}}{2} \tan^{-1} \frac{\sqrt{3}x}{2} + C$$

$$I = \frac{2}{\sqrt{3}} \tan^{-1} \frac{\sqrt{3}x}{2} - \tan^{-1}x + C$$

43. Question

Evaluate the following integral:

$$\int \frac{3x+5}{x^3-x^2-x+1} dx$$

Answer

$$I = \int \frac{3x+5}{x^3-x^2-x+1} dx = \int \frac{3x+5}{(x-1)^2(x+1)} dx$$

$$\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$3x+5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$$\text{Put } x = 1$$

$$8 = 2B$$

$$B = 4$$

$$\text{Put } x = -1$$

$$-3 + 5 = 4C$$

$$2 = 4C$$

$$C = \frac{1}{2}$$

$$\text{Put } x = 0$$

$$5 = -A + B + C$$

$$A = \frac{1}{2}$$

$$\int \frac{3x + 5}{(x-1)^2(x+1)} dx = \frac{1}{2} \int \frac{dx}{x-1} + 4 \int \frac{dx}{(x-1)^2} + \frac{1}{2} \int \frac{dx}{x+1}$$

$$= -\frac{1}{2} \ln|x-1| - \frac{4}{(x-1)} + \frac{1}{2} \ln|x+1| + C$$

$$= \frac{1}{2} \ln \left| \frac{x+1}{x-1} \right| - \frac{4}{(x-1)} + C$$

44. Question

Evaluate the following integral:

$$\int \frac{x^3 - 1}{x^3 + x} dx$$

Answer

$$I = \int \frac{x^3 - 1}{x^3 + x} dx = \int 1 - \frac{x+1}{x^3 + x} dx$$

$$= \int 1 dx - \int \frac{x+1}{x^3 + x} dx$$

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$X + 1 = A(x^2 + 1) + (Bx + C)(x)$$

Equating constants

$$A = 1$$

Equating coefficients of x

$$1 = C$$

Equating coefficients of x^2

$$0 = A + B$$

$$B = -1$$

$$I = -\int \frac{dx}{x} - \int \frac{-x+1}{x^2+1} + \int dx$$

$$\begin{aligned}
 I &= -\int \frac{dx}{x} + \int \frac{xdx}{x^2 + 1} - \int \frac{dx}{x^2 + 1} + \int dx \\
 &= -\log|x| + \frac{1}{2}\log|x^2 + 1| - \tan^{-1}x + x + c \\
 I &= x - \log|x| + \frac{1}{2}\log|x^2 + 1| - \tan^{-1}x + c
 \end{aligned}$$

45. Question

Evaluate the following integral:

$$\int \frac{x^2 + x + 1}{(x+1)^2(x+2)} dx$$

Answer

$$\begin{aligned}
 I &= \int \frac{x^2 + x + 1}{(x+1)^2(x+2)} dx \\
 \frac{x^2 + x + 1}{(x+1)^2(x+2)} &= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} \\
 x^2 + x + 1 &= A(x+1)(x+2) + B(x+2) + C(x+1)^2 \\
 \text{Put } x &= -2 \\
 3 &= C \\
 C &= 3 \\
 \text{Put } x &= -1 \\
 1 &= B \\
 B &= 1 \\
 \text{Equating coefficients of constants} \\
 1 &= 2A + 2B + C \\
 1 &= 2A + 2 + 3 \\
 A &= -2
 \end{aligned}$$

Thus,

$$\begin{aligned}
 I &= 2 \int \frac{dx}{x+1} + (1) \int \frac{dx}{(x+1)^2} + 3 \int \frac{dx}{x+2} \\
 I &= -2 \ln|x+1| - \left(\frac{1}{x+1}\right) + 3 \ln|x+2| + C
 \end{aligned}$$

46. Question

Evaluate the following integral:

$$\int \frac{1}{x(x^4+1)} dx$$

Answer

Let

$$I = \int \frac{1}{x(x^4 + 1)} dx$$

$$\frac{1}{x(x^4 + 1)} = \frac{A}{x} + \frac{Bx^3 + Cx^2 + Dx + E}{x^4 + 1}$$

$$1 = A(x^4 + 1) + (Bx^3 + Cx^2 + Dx + E)(x)$$

Equating constants

$$A = 1$$

Equating coefficients of x^4

$$0 = A + B$$

$$0 = 1 + B$$

$$B = -1$$

Equating coefficients of x^2

$$D = 0$$

Equating coefficients of x

$$E = 0$$

Thus,

$$I = \int \frac{dx}{x} + \int -\frac{x^2 dx}{x^4 + 1}$$

$$= \log|x| - \frac{1}{4} \log|x^4 + 1| + C$$

$$= \frac{4}{4} \log|x| - \frac{1}{4} \log|x^4 + 1| + C$$

$$= \frac{1}{4} \log|x^4| - \frac{1}{4} \log|x^4 + 1| + C$$

$$\frac{1}{4} \log \left| \frac{x^4}{x^4 + 1} \right| + C$$

47. Question

Evaluate the following integral:

$$\int \frac{1}{x(x^3 + 8)} dx$$

Answer

Consider the integral,

$$I = \int \frac{1}{x(x^3 + 8)} dx$$

Rewriting the above integral, we have

$$I = \int \frac{x^2}{x^3(x^3 + 8)} dx$$

$$I = \frac{1}{3} \int \frac{3x^2}{x^3(x^3 + 8)} dx$$

Substitute $x^3 = t$

$$3x^2 dx = dt$$

$$I = \frac{1}{3} \int \frac{dt}{t(t+8)}$$

$$\frac{1}{t(t+8)} = \frac{A}{t} + \frac{B}{t+8}$$

$$1 = A(t+8) + Bt$$

Equating constants

$$1 = 8A$$

$$A = \frac{1}{8}$$

Equating coefficients of t

$$0 = A + B$$

$$B = -\frac{1}{8}$$

$$I = \frac{1}{3} \int \frac{dt}{t(t+8)}$$

$$= \frac{1}{3} \int \left(\frac{\frac{1}{8}}{t} - \frac{\frac{1}{8}}{t+8} \right) dt$$

$$= \frac{1}{3} \times \frac{1}{8} \int \frac{dt}{t} - \frac{1}{3} \times \frac{1}{8} \int \frac{dt}{t+8}$$

$$= \frac{1}{24} \log t - \frac{1}{24} \log|t+8| + C$$

$$= \frac{1}{24} \log x^3 - \frac{1}{24} \log|x^3+8| + C$$

$$= \frac{3}{24} \log x - \frac{1}{24} \log|x^3+8| + C$$

$$= \frac{1}{8} \log x - \frac{1}{24} \log|x^3+8| + C$$

48. Question

Evaluate the following integral:

$$\int \frac{3}{(1-x)(1+x^2)} dx$$

Answer

$$I = \int \frac{3}{(1-x)(1+x^2)} dx$$

$$\frac{3}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}$$

$$3 = A(1+x^2) + (Bx+C)(1-x)$$

Equating similar terms

$$A - B = 0$$

$$B - C = 0$$

$$A + C = 3$$

Solving

$$A = \frac{3}{2}, B = \frac{3}{2}, C = \frac{3}{2}$$

Thus,

$$\begin{aligned} I &= \frac{3}{2} \int \frac{dx}{1-x} + \frac{3}{2} \int \frac{x dx}{1+x^2} + \frac{3}{2} \int \frac{dx}{1+x^2} \\ &= -\frac{3}{2} \log|1-x| + \frac{3}{2} \log|1+x^2| + \frac{3}{2} \tan^{-1}x + C \end{aligned}$$

$$I = \frac{3}{4} \left[\log \left| \frac{1+x^2}{(1-x)^2} \right| + 2 \tan^{-1}x \right] + C$$

49. Question

Evaluate the following integral:

$$\int \frac{\cos x}{(1 - \sin x)^3 (2 + \sin x)} dx$$

Answer

Let

$$\sin x = t$$

$$\cos x \, dx = dt$$

$$I = \int \frac{\cos x}{(1 - \sin x)^3 (2 + \sin x)} dx$$

$$= \int \frac{dt}{(1-t)^3 (2+t)}$$

$$\frac{1}{(1-t)^3 (2+t)} = \frac{A}{1-t} + \frac{B}{(1-t)^2} + \frac{C}{(1-t)^3} + \frac{D}{2+t}$$

$$1 = A(1-t)^2(2+t) + B(1-t)(2+t) + C(2+t) + D(1-t)^3$$

$$\text{Put } t = 1$$

$$1 = 3C$$

$$C = \frac{1}{3}$$

$$\text{Put } t = -2$$

$$1 = 27D$$

$$D = \frac{1}{27}$$

$$A = -\frac{1}{27} \quad B = \frac{1}{9}$$

$$\begin{aligned} \int \frac{dt}{(1-t)^3(2+t)} &= -\frac{1}{27} \int \frac{1}{1-t} dt + \frac{1}{9} \int \frac{dt}{(1-t)^2} + \frac{1}{3} \int \frac{dt}{(1-t)^3} + \frac{1}{27} \int \frac{dt}{2+t} \\ &= -\frac{1}{27} \log|1-t| + \frac{1}{9(1-t)} + \frac{1}{6(1-t)^2} + \frac{1}{27} \log|2+t| + C \end{aligned}$$

Put $t = \sin x$

$$= -\frac{1}{27} \log|1 - \sin x| + \frac{1}{9(1 - \sin x)} + \frac{1}{6(1 - \sin x)^2} + \frac{1}{27} \log|2 + \sin x| + C$$

50. Question

Evaluate the following integral:

$$\int \frac{2x^2 + 1}{x^2(x^2 + 4)} dx$$

Answer

$$I = \int \frac{2x^2 + 1}{x^2(x^2 + 4)} dx$$

Put $x^2 = t$

$2x dx = dt$

$$\frac{2t + 1}{t(t + 4)} = \frac{A}{t} + \frac{B}{t + 4}$$

$$2t + 1 = A(t + 4) + Bt$$

Equating constants

$$1 = 4A$$

$$A = \frac{1}{4}$$

Equating coefficients of t

$$2 = A + B$$

$$B = 2 - \frac{1}{4} = \frac{7}{4}$$

$$\frac{2x^2 + 1}{x^2(x^2 + 4)} = \frac{1}{4x^2} + \frac{7}{4(x^2 + 4)}$$

Thus we have

$$\begin{aligned} \int \frac{2x^2 + 1}{x^2(x^2 + 4)} dx &= \frac{1}{4} \int \frac{dx}{x^2} + \frac{7}{4} \int \frac{dx}{x^2 + 4} \\ &= -\frac{1}{4x} + \frac{7}{8} \tan^{-1}\left(\frac{x}{2}\right) + C \end{aligned}$$

51. Question

Evaluate the following integral:

$$\int \frac{\cos x}{(1 - \sin x)(2 - \sin x)} dx$$

Answer

We have,

$$I = \int \frac{\cos x}{(1 - \sin x)(2 - \sin x)} dx$$

Let $1 - \sin x = t$

$$\Rightarrow -\cos x dx = dt$$

$$\therefore I = - \int \frac{dt}{t(1 + t)}$$

$$\Rightarrow I = - \int \frac{(1 + t) - t}{t(1 + t)} dt$$

$$\Rightarrow I = - \int \left(\frac{1}{t} - \frac{1}{1 + t} \right) dt$$

$$\Rightarrow I = - (\ln t - \ln(1 + t)) + c$$

$$\Rightarrow I = \ln(1 + t) - \ln t + c$$

$$\Rightarrow I = \frac{\ln(1 + t)}{\ln t} + c$$

$$\Rightarrow I = \frac{\ln(2 - \sin x)}{\ln(1 - \sin x)} + c$$

$$\text{Therefore, } \int \frac{\cos x}{(1 - \sin x)(2 - \sin x)} dx = \frac{\ln(2 - \sin x)}{\ln(1 - \sin x)} + c$$

52. Question

Evaluate the following integral:

$$\int \frac{2x + 1}{(x - 2)(x - 3)} dx$$

Answer

$$\text{Let, } I = \int \frac{2x + 1}{(x - 2)(x - 3)} dx$$

$$\text{Now, let } \frac{2x + 1}{(x - 2)(x - 3)} = \frac{A}{x - 2} + \frac{B}{x - 3}$$

$$\Rightarrow 2x + 1 = A(x - 3) + B(x - 2)$$

$$\Rightarrow 2x + 1 = (A + B)x - 3A - 2B$$

Equating similar terms, we get,

$$A + B = 2 \text{ and } 3A + 2B = -1$$

$$\text{So, } A = -5, B = 7$$

$$\therefore I = -5 \int \frac{dx}{x - 2} + 7 \int \frac{dx}{x - 3}$$

$$\Rightarrow I = -5 \log |x - 2| + 7 \log |x - 3| + c$$

$$\Rightarrow I = \log |x - 2|^{-5} + \log |x - 3|^7 + c$$

$$\Rightarrow I = \log \left| \frac{(x-3)^7}{(x-2)^5} \right| + c$$

$$\text{Hence, } \int \frac{2x+1}{(x-2)(x-3)} dx = \log \left| \frac{(x-3)^7}{(x-2)^5} \right| + c$$

53. Question

Evaluate the following integral:

$$\int \frac{1}{(x^2+1)(x^2+2)} dx$$

Answer

$$\text{Let, } I = \int \frac{1}{(x^2+1)(x^2+2)} dx$$

$$\text{Let, } x^2 = y$$

$$\text{Then, } \frac{1}{(y+1)(y+2)} = \frac{A}{y+1} + \frac{B}{y+2}$$

$$\Rightarrow 1 = A(y+2) + B(y+1)$$

$$\Rightarrow 1 = (A+B)y + 2A + B$$

On equating similar terms, we get,

$$A + B = 0, \text{ and } 2A + B = 1$$

$$\text{We get, } A = 1, B = -1$$

$$\therefore I = \int \frac{dx}{x^2+1} - \int \frac{dx}{x^2+2}$$

$$\Rightarrow I = \tan^{-1} x - \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + c$$

$$\text{So, } \int \frac{1}{(x^2+1)(x^2+2)} dx = \tan^{-1} x - \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + c$$

54. Question

Evaluate the following integral:

$$\int \frac{1}{x(x^4-1)} dx$$

Answer

$$\text{Let, } I = \int \frac{1}{x(x^4-1)} dx$$

$$\text{Let, } \frac{1}{x(x^4-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} + \frac{D}{x^2+1}$$

$$\Rightarrow 1 = A(x+1)(x-1)(x^2+1) + Bx(x-1)(x^2+1) + Cx(x+1)(x^2+1) + Dx(x+1)(x-1)$$

$$\text{For, } x=0, A = -1$$

$$\text{For, } x=1, C = \frac{1}{4}$$

$$\text{For, } x = -1, B = \frac{1}{4}$$

$$\text{For, } x = 2, D = \frac{1}{4}$$

$$\therefore I = -\int \frac{dx}{x} + \frac{1}{4} \int \frac{dx}{x+1} + \frac{1}{4} \int \frac{dx}{x-1} + \frac{1}{4} \int \frac{dx}{x^2+1}$$

$$\Rightarrow I = -\ln|x| + \frac{1}{4} \ln|(x+1)| + \frac{1}{4} \ln|x-1| + \frac{1}{4} \tan^{-1} x + c$$

$$\Rightarrow I = -\ln|x| + \frac{1}{4} (\ln|x^2-1|) + \frac{1}{4} \tan^{-1} x + c$$

$$\Rightarrow I = -\frac{1}{4} \ln|x^4| + \frac{1}{4} \ln(x^2-1) + \frac{1}{4} \tan^{-1} x + c$$

$$\Rightarrow I = \frac{1}{4} \ln \left| \frac{x^2-1}{x^4} \right| + \frac{1}{4} \tan^{-1} x + c$$

$$\text{Thus, } \int \frac{1}{x(x^4-1)} dx = \frac{1}{4} \ln \left| \frac{x^2-1}{x^4} \right| + c$$

55. Question

Evaluate the following integral:

$$\int \frac{1}{x^4-1} dx$$

Answer

$$\text{Let, } I = \int \frac{1}{(x^4-1)} dx$$

$$\text{Let, } \frac{1}{(x^4-1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x^2+1}$$

$$\Rightarrow 1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + C(x+1)(x-1)$$

$$\text{For, } x = 1, B = \frac{1}{4}$$

$$\text{For, } x = -1, A = \frac{1}{4}$$

$$\text{For, } x = 0, A = -\frac{1}{2}$$

$$\therefore I = -\frac{1}{4} \int \frac{dx}{x+1} + \frac{1}{4} \int \frac{dx}{x-1} - \frac{1}{2} \int \frac{dx}{x^2+1}$$

$$\Rightarrow I = -\frac{1}{4} \ln|(x+1)| + \frac{1}{4} \ln|x-1| - \frac{1}{2} \tan^{-1} x + c$$

$$\Rightarrow I = \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \tan^{-1} x + c$$

$$\text{So, } \int \frac{1}{(x^4-1)} dx = \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \tan^{-1} x + c$$

56. Question

Evaluate the following integral:

$$\int \frac{2x}{(x^2 + 1)(x^2 + 2)^2} dx$$

Answer

$$\text{Let, } I = \int \frac{2x}{(x^2 + 1)(x^2 + 2)^2} dx$$

$$\text{Let } x^2 + 2 = t \Rightarrow 2x dx = dt$$

$$\therefore I = \int \frac{dt}{(t-1)t^2}$$

$$\text{Now, let, } \frac{1}{(t-1)t^2} = \frac{A}{t-1} + \frac{B}{t} + \frac{C}{t^2}$$

$$\Rightarrow 1 = At^2 + Bt(t-1) + C(t-1)$$

$$\text{For } t=1, A=1$$

$$\text{For } t=0, C=-1$$

$$\text{For } t=-1, B=-1$$

$$\therefore I = \int \frac{dt}{t-1} - \int \frac{dt}{t} - \int \frac{dt}{t^2}$$

$$\Rightarrow I = \log|t-1| - \log|t| + \frac{1}{t} + c$$

$$\text{So, } \int \frac{2x}{(x^2 + 1)(x^2 + 2)^2} dx = \log|t-1| - \log|t| + \frac{1}{t} + c$$

57. Question

Evaluate the following integral:

$$\int \frac{x^2}{(x-1)(x^2 + 1)} dx$$

Answer

$$\text{Let, } I = \int \frac{x^2}{(x-1)(x^2 + 1)} dx$$

$$\text{Let } \frac{x^2}{(x-1)(x^2 + 1)} = \frac{A}{x-1} + \frac{B}{x^2 + 1}$$

$$\Rightarrow x^2 = A(x^2 + 1) + B(x-1)$$

$$\text{For, } x=1, A = \frac{1}{2}$$

$$\text{For, } x=0, B = \frac{1}{2}$$

$$\therefore I = \frac{1}{2} \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{dx}{x^2 + 1}$$

$$\Rightarrow I = \frac{1}{2} \log|x-1| + \frac{1}{2} \tan^{-1} x + c$$

Hence, $\int \frac{x^2}{(x-1)(x^2+1)} dx = \frac{1}{2} \log|x-1| + \frac{1}{2} \tan^{-1} x + c$

58. Question

Evaluate the following integral:

$$\int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$$

Answer

Let, $I = \int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$

Let $x^2 = y$

Thus, $\frac{x^2}{(x^2+a^2)(x^2+b^2)} = \frac{y}{(y+a^2)(y+b^2)}$

Now, let $\frac{y}{(y+a^2)(y+b^2)} = \frac{A}{y+a^2} + \frac{B}{y+b^2}$

$$\Rightarrow y = A(y+b^2) + B(y+a^2)$$

$$\Rightarrow y = y(A+B) + (Ab^2 + Ba^2)$$

Equating the coefficients, we get,

$$A+B=1, \text{ and } Ab^2 + Ba^2 = 0$$

On solving we get, $A = -\frac{a^2}{b^2-a^2}, \quad B = \frac{b^2}{b^2-a^2}$

$$\therefore I = -\frac{a^2}{b^2-a^2} \int \frac{dx}{x^2+a^2} + \frac{b^2}{b^2-a^2} \int \frac{dx}{x^2+b^2}$$

$$\Rightarrow I = \frac{b}{b^2-a^2} \tan^{-1}\left(\frac{x}{b}\right) - \frac{a}{b^2-a^2} \tan^{-1}\left(\frac{x}{a}\right) + c$$

Thus, $\int \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = \frac{b}{b^2-a^2} \tan^{-1}\left(\frac{x}{b}\right) - \frac{a}{b^2-a^2} \tan^{-1}\left(\frac{x}{a}\right) + c$

59. Question

Evaluate the following integral:

$$\int \frac{1}{\cos x (5-4 \sin x)} dx$$

Answer

Let, $I = \int \frac{dx}{\cos x (5-4 \sin x)}$

Multiplying and dividing by $\cos x$

Let, $I = \int \frac{\cos x dx}{\cos^2 x (5-4 \sin x)}$

$$\Rightarrow I = \int \frac{\cos x dx}{(1-\sin^2 x)(5-4 \sin x)}$$

Let, $\sin x = t, \cos x dx = dt$

$$\therefore I = \int \frac{dt}{(1-t^2)(5-4t)}$$

$$\text{Now, let } \frac{1}{(1-t^2)(5-4t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{5-4t}$$

$$\Rightarrow 1 = A(1+t)(5-4t) + B(1-t)(5-4t) + C(1-t^2)$$

$$\text{For } t = 1, A = \frac{1}{2}$$

$$\text{For } t = -1, B = \frac{1}{18}$$

$$\text{For } t = \frac{5}{4}, C = -\frac{16}{9}$$

$$\therefore I = \frac{1}{2} \int \frac{dt}{1-t} + \frac{1}{18} \int \frac{dt}{1+t} - \frac{16}{9} \int \frac{dt}{5-4t}$$

$$\Rightarrow I = -\frac{1}{2} \log|1-t| + \frac{1}{18} \log|1+t| + \frac{4}{9} \log|5-4t| + c$$

$$\text{So, } I = -\frac{1}{2} \log|1-\sin x| + \frac{1}{18} \log|1+\sin x| + \frac{4}{9} \log|5-4\sin x| + c$$

60. Question

Evaluate the following integral:

$$\int \frac{1}{\sin x (3 + 2 \cos x)} dx$$

Answer

$$\text{Let, } I = \int \frac{1}{\sin x (3 + 2 \cos x)} dx$$

Multiplying and dividing by $\sin x$

$$\therefore I = \int \frac{\sin x}{\sin^2 x (3 + 2 \cos x)} dx$$

$$\therefore I = \int \frac{\sin x}{(1 - \cos^2 x)(3 + 2 \cos x)} dx$$

Let $\cos x = t$, $-\sin x dx = dt$

$$\text{So, } I = \int \frac{dt}{(t^2 - 1)(3 + 2t)}$$

$$\text{Now, let } \frac{1}{(t^2 - 1)(3 + 2t)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{C}{3+2t}$$

$$\Rightarrow 1 = A(t+1)(3+2t) + B(t-1)(3+2t) + C(t^2-1)$$

$$\text{For, } t = 1, A = \frac{1}{10}$$

$$\text{For, } t = -1, B = -\frac{1}{2}$$

$$\text{For, } t = -\frac{3}{2}, C = \frac{4}{5}$$

$$\therefore I = \frac{1}{10} \int \frac{dt}{t-1} - \frac{1}{2} \int \frac{dt}{t+1} + \frac{4}{5} \int \frac{dt}{3+2t}$$

$$\Rightarrow I = \frac{1}{10} \log|t-1| - \frac{1}{2} \log|t+1| + \frac{2}{5} \log|3+2t| + c$$

61. Question

Evaluate the following integral:

$$\int \frac{1}{\sin x + \sin 2x} dx$$

Answer

$$\text{Let, } I = \int \frac{1}{\sin x + \sin 2x} dx$$

$$\Rightarrow I = \int \frac{1}{\sin x + 2 \sin x \cos x} dx$$

Multiplying and dividing by $\sin x$

$$\Rightarrow I = \int \frac{\sin x}{\sin^2 x + 2 \sin^2 x \cdot \cos x} dx$$

$$\Rightarrow I = \int \frac{\sin x}{1 - \cos^2 x + 2(1 - \cos^2 x) \cos x} dx$$

Let $\cos x = t$, $-\sin x dx = dt$

$$\therefore I = \int \frac{dt}{(t^2 - 1) + 2(t^2 - 1)t}$$

$$\Rightarrow I = \int \frac{dt}{(t^2 - 1)(1 + 2t)}$$

$$\text{Let, } \frac{1}{(t^2 - 1)(1 + 2t)} = \frac{A}{t-1} + \frac{B}{1+t} + \frac{C}{1+2t}$$

$$\Rightarrow 1 = A(1+t)(1+2t) + B(t-1)(1+2t) + C(t^2-1)$$

$$\text{For } t = 1, A = \frac{1}{6}$$

$$\text{For } t = -1, B = \frac{1}{2}$$

$$\text{For } t = -\frac{1}{2}, C = -\frac{4}{3}$$

$$\text{So, } I = \frac{1}{6} \int \frac{dt}{t-1} + \frac{1}{2} \int \frac{dt}{t+1} - \frac{4}{3} \int \frac{dt}{1+2t}$$

$$\Rightarrow I = \frac{1}{6} \log|t-1| + \frac{1}{2} \log|1+t| - \frac{2}{3} \log|1+2t| + c$$

$$\text{So, } I = \frac{1}{6} \log|\cos x - 1| + \frac{1}{2} \log|1 + \cos x| - \frac{2}{3} \log|1 + 2 \cos x| + c$$

62. Question

Evaluate the following integral:

$$\int \frac{x+1}{x(1+xe^x)} dx$$

Answer

$$\text{Let, } I = \int \frac{x+1}{x(1+xe^x)} dx$$

$$\Rightarrow, \quad I = \int \frac{(x+1)(1+xe^x - xe^x)}{x(1+xe^x)} dx$$

$$\Rightarrow, \quad I = \int \frac{(x+1)(1+xe^x)}{x(1+xe^x)} dx - \int \frac{(x+1)(xe^x)}{x(1+xe^x)} dx$$

$$\Rightarrow, \quad I = \int \frac{(x+1)}{x} dx - \int \frac{(x+1)(e^x)}{(1+xe^x)} dx$$

$$\Rightarrow, \quad I = \log|xe^x| - \log|1+xe^x| + c$$

$$\Rightarrow, \quad I = \log \left| \frac{xe^x}{1+xe^x} \right| + c$$

$$\text{Hence, } \int \frac{x+1}{x(1+xe^x)} dx = \log \left| \frac{xe^x}{1+xe^x} \right| + c$$

63. Question

Evaluate the following integral:

$$\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} dx$$

Answer

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \frac{x^4 + 3x^2 + 2}{x^4 + 7x^2 + 12}$$

$$= \frac{(x^4 + 7x^2 + 12) - 4x^2 - 10}{x^4 + 7x^2 + 12}$$

$$= 1 - \frac{4x^2 + 10}{x^4 + 7x^2 + 12}$$

$$\text{Now, } \frac{4x^2 + 10}{x^4 + 7x^2 + 12} = \frac{4x^2 + 10}{(x^2+3)(x^2+4)}$$

$$\text{Let, } \frac{4x^2 + 10}{(x^2+3)(x^2+4)} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{x^2+4}$$

$$\Rightarrow 4x^2 + 10 = (Ax+B)(x^2+4) + (Cx+D)(x^2+3)$$

$$\text{For, } x=0, 10 = 4B + 3D \dots (i)$$

$$\text{For, } x=1, 14 = 5A + 5B + 4C + 4D \dots (ii)$$

$$\text{For, } x=-1, 14 = -5A + 5B - 4C + 4D \dots (iii)$$

$$\text{Also, by comparing coefficient of } x^3 \text{ we get, } 0=A+C \text{ (iv)}$$

On solving, (i), (ii), (iii), (iv) we get,

$$A=0, B=-2, C=0, D=6$$

$$\text{So, } \frac{(x^2 + 1)(x^2 + 2)}{(x^2 + 3)(x^2 + 4)} = 1 + \frac{2}{x^2 + 3} - \frac{6}{x^2 + 4}$$

$$\therefore \int \frac{(x^2 + 1)(x^2 + 2)}{(x^2 + 3)(x^2 + 4)} dx = \int \left(1 + \frac{2}{x^2 + 3} - \frac{6}{x^2 + 4} \right) dx$$

$$= x + \frac{2}{\sqrt{3}} \tan^{-1} x - 3 \tan^{-1} \frac{x}{2} + c$$

$$\text{Therefore, } \int \frac{(x^2 + 1)(x^2 + 2)}{(x^2 + 3)(x^2 + 4)} dx = x + \frac{2}{\sqrt{3}} \tan^{-1} x - 3 \tan^{-1} \frac{x}{2} + c$$

64. Question

Evaluate the following integral:

$$\int \frac{4x^4 + 3}{(x^2 + 2)(x^2 + 3)(x^2 + 4)} dx$$

Answer

$$\text{Let } I = \int \frac{4x^4 + 3}{(x^2 + 2)(x^2 + 3)(x^2 + 4)} dx$$

$$\text{Let } x^2 = y$$

$$\therefore \frac{4x^4 + 3}{(x^2 + 2)(x^2 + 3)(x^2 + 4)} = \frac{4y^2 + 3}{(y + 2)(y + 3)(y + 4)}$$

$$\text{Let, } \frac{4y^2 + 3}{(y + 2)(y + 3)(y + 4)} = \frac{A}{y + 2} + \frac{B}{y + 3} + \frac{C}{y + 4}$$

$$\Rightarrow 4y^2 + 3 = A(y + 3)(y + 4) + B(y + 2)(y + 4) + C(y + 2)(y + 3)$$

$$\text{For } y = -2, A = \frac{19}{2}$$

$$\text{For } y = -3, B = -39$$

$$\text{For } y = -4, C = \frac{67}{2}$$

$$\text{Thus, } I = \frac{19}{2} \int \frac{dx}{x^2 + 2} - 39 \int \frac{dx}{x^2 + 3} + \frac{67}{2} \int \frac{dx}{x^2 + 4}$$

$$\Rightarrow I = \frac{19}{2\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) - \frac{39}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + \frac{67}{4} \tan^{-1} \left(\frac{x}{2} \right) + c$$

65. Question

Evaluate the following integral:

$$\int \frac{x^4}{(x - 1)(x^2 + 1)} dx$$

Answer

$$\frac{x^4}{(x - 1)(x^2 + 1)} = \frac{x^4}{x^3 - x^2 + x - 1}$$

$$= \frac{x(x^3 - x^2 + x - 1) + 1(x^3 - x^2 + x - 1) + 1}{x^3 - x^2 + x - 1}$$

$$= x + 1 + \frac{1}{(x-1)(x^2 + 1)}$$

$$\text{Now, let } \frac{1}{(x-1)(x^2 + 1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 1}$$

$$\Rightarrow 1 = A(x^2 + 1) + (Bx + C)(x-1)$$

$$\text{For, } x = 1, A = \frac{1}{2}$$

$$\text{For, } x = 0, C = A - 1 = -\frac{1}{2}$$

$$\text{For, } x = -1, B = -\frac{1}{2}$$

$$\therefore \int \frac{x^4}{(x-1)(x^2 + 1)} dx = \int x dx + \int dx + \frac{1}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{x+1}{x^2 + 1} dx$$

$$= \frac{x^2}{2} + x + \frac{1}{2} \log|x-1| - \frac{1}{4} \log(x^2 + 1) - \frac{1}{2} \tan^{-1} x + c$$

66. Question

Evaluate the following integral:

$$\int \frac{x^2}{x^4 - x^2 - 12} dx$$

Answer

$$\frac{x^2}{x^4 - x^2 - 12} = \frac{x^2}{(x^2 - 4)(x^2 + 3)}$$

$$\text{Let, } \frac{x^2}{(x^2 - 4)(x^2 + 3)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x^2 + 3}$$

$$\Rightarrow x^2 = A(x+2)(x^2 + 3) + B(x-2)(x^2 + 3) + C(x-2)(x+2)$$

$$\text{For, } x = 2, A = \frac{1}{7}$$

$$\text{For, } x = -2, B = -\frac{1}{7}$$

$$\text{For, } x = 0, C = \frac{3}{7}$$

$$\therefore \int \frac{x^2}{x^4 - x^2 - 12} dx = \frac{1}{7} \int \frac{dx}{x-2} - \frac{1}{7} \int \frac{dx}{x+2} + \frac{3}{7} \int \frac{dx}{x^2 + 3}$$

$$= \frac{1}{7} \log|x-2| - \frac{1}{7} \log|x+2| + \frac{3}{7\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + c$$

67. Question

Evaluate the following integral:

$$\int \frac{x^2}{1-x^4} dx$$

Answer

$$\text{Let, } I = \int \frac{x^2}{1-x^4} dx$$

$$\text{Let, } \frac{x^2}{1-x^4} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{1+x^2}$$

$$\Rightarrow x^2 = A(1+x)(x^2+1) + B(1-x)(x^2+1) + c(x+1)(1-x)$$

$$\text{For, } x = 1, A = \frac{1}{4}$$

$$\text{For, } x = -1, B = \frac{1}{4}$$

$$\text{For, } x = 0, C = -\frac{1}{2}$$

$$\therefore I = \frac{1}{4} \int \frac{dx}{1-x} + \frac{1}{4} \int \frac{dx}{1+x} - \frac{1}{2} \int \frac{dx}{1+x^2}$$

$$\Rightarrow I = -\frac{1}{4} \log|1-x| + \frac{1}{4} \log|1+x| - \frac{1}{2} \tan^{-1} x + c$$

$$\Rightarrow I = \frac{1}{4} \log \left| \frac{1+x}{1-x} \right| - \frac{1}{2} \tan^{-1} x + c$$

$$\text{Hence, } \int \frac{x^2}{1-x^4} dx = \frac{1}{4} \log \left| \frac{1+x}{1-x} \right| - \frac{1}{2} \tan^{-1} x + c$$

68. Question

Evaluate the following integral:

$$\int \frac{x^2}{x^4 + x^2 - 2} dx$$

Answer

$$\text{Let, } I = \int \frac{x^2}{x^4 + x^2 - 2} dx$$

$$\text{Let, } \frac{x^2}{x^4 + x^2 - 2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x^2+2}$$

$$\Rightarrow x^2 = A(x-1)(x^2+2) + B(x+1)(x^2+2) + C(x^2-1)$$

$$\text{For, } x = 1, A = \frac{1}{6}$$

$$\text{For, } x = -1, B = -\frac{1}{6}$$

$$\text{For, } x = 0, C = -\frac{2}{3}$$

$$\therefore I = \frac{1}{6} \int \frac{dx}{x+1} - \frac{1}{6} \int \frac{dx}{x-1} - \frac{2}{3} \int \frac{dx}{x^2+2}$$

$$\Rightarrow I = \frac{1}{6} \log|x+1| - \frac{1}{6} \log|x-1| - \frac{2}{3\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + c$$

69. Question

Evaluate the following integral:

$$\int \frac{(x^2 + 1)(x^2 + 4)}{(x^2 + 3)(x^2 - 5)} dx$$

Answer

$$\frac{(x^2 + 1)(x^2 + 4)}{(x^2 + 3)(x^2 - 5)} = \frac{x^4 + 5x^2 + 4}{x^4 - 2x^2 - 15}$$

$$= \frac{(x^4 - 2x^2 - 15) + 7x^2 + 19}{x^4 - 2x^2 - 15}$$

$$= 1 + \frac{7x^2 + 19}{x^4 - 2x^2 - 15}$$

$$\text{Now, } \frac{7x^2 + 19}{x^4 - 2x^2 - 15} = \frac{7x^2 + 19}{(x^2 + 3)(x^2 - 5)}$$

$$\text{Let, } \frac{7x^2 + 19}{x^4 - 2x^2 - 15} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{x^2 - 5}$$

$$\Rightarrow 7x^2 + 19 = (Ax + B)(x^2 - 5) + (Cx + D)(x^2 + 3)$$

$$\text{For, } x=0, 19 = -5B + 3D \dots (i)$$

$$\text{For, } x=1, 26 = -4A - 4B + 4C + 4D \dots (ii)$$

$$\text{For, } x=-1, 14 = 4A - 4B - 4C + 4D \dots (iii)$$

$$\text{Also, by comparing coefficient of } x^3 \text{ we get, } 0=A + C \text{ (iv)}$$

On solving, (i), (ii), (iii), (iv) we get,

$$A = 0, B = -\frac{11}{8}, C = 0, D = \frac{69}{8}$$

$$\text{So, } \frac{(x^2 + 1)(x^2 + 4)}{(x^2 + 3)(x^2 - 5)} = 1 - \frac{11}{8} \frac{1}{x^2 + 3} + \frac{69}{8} \frac{1}{x^2 - 5}$$

$$\therefore \int \frac{(x^2 + 1)(x^2 + 4)}{(x^2 + 3)(x^2 - 5)} dx = \int \left(1 - \frac{11}{8} \frac{1}{x^2 + 3} + \frac{69}{8} \frac{1}{x^2 - 5} \right) dx$$

$$= x - \frac{11}{8\sqrt{3}} \tan^{-1} x + \frac{69}{16\sqrt{5}} \log \left| \frac{x - \sqrt{5}}{x + \sqrt{5}} \right| + c$$

$$\text{Thus, } I = x - \frac{11}{8\sqrt{3}} \tan^{-1} x + \frac{69}{16\sqrt{5}} \log \left| \frac{x - \sqrt{5}}{x + \sqrt{5}} \right| + c$$

Exercise 19.31

1. Question

Evaluate the following integral:

$$\int \frac{x^2 + 1}{x^4 + x^2 + 1} dx$$

Answer

re-writing the given equation as

$$\int \frac{1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx$$

Let $x - \frac{1}{x}$ as t

$$\left(1 + \frac{1}{x^2}\right) = dt$$

$$\int \frac{1}{t^2 + 3} dt$$

Using identity $\int \frac{1}{x^2+1} dx = \arctan(x)$

$$\frac{1}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right) + c$$

Substituting t as $x - \frac{1}{x}$

$$\frac{1}{\sqrt{3}} \arctan\left(\frac{\left(x - \frac{1}{x}\right)}{\sqrt{3}}\right) + c$$

2. Question

Evaluate the following integral:

$$\int \sqrt{\cot \theta} d\theta$$

Answer

let $\cot \theta$ as x^2

$$-\operatorname{cosec}^2 \theta d\theta = 2x dx$$

$$d\theta = -\frac{2x}{1 + \cot^2 \theta} dx$$

$$d\theta = -\frac{2x}{1 + x^4} dx$$

$$\int -\frac{2x^2}{1 + x^4} dx$$

re-writing the given equation as

$$\int \frac{1 + \frac{1}{x^2} + 1 - \frac{1}{x^2}}{\frac{1}{x^2} + x^2} dx$$

$$-\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 2} dx - \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 2} dx$$

Let $x - \frac{1}{x} = t$ and $x + \frac{1}{x} = z$

So $\left(1 + \frac{1}{x^2}\right) dx = dt$ and $\left(1 - \frac{1}{x^2}\right) dx = dz$

$$-\int \frac{dt}{(t)^2 + 2} - \int \frac{dz}{(z)^2 - 2}$$

Using identity $\int \frac{1}{x^2+1} dx = \arctan(x)$ and $\int \frac{dz}{(z)^2-1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$

$$-\frac{1}{2} \arctan\left(\frac{t}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}} \log \left| \frac{z - \sqrt{2}}{z + \sqrt{2}} \right| + c$$

Substituting t as $x - \frac{1}{x}$ and z as $x + \frac{1}{x}$

$$-\frac{1}{2} \arctan\left(\frac{x - \frac{1}{x}}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}} \log \left| \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} \right| + c$$

3. Question

Evaluate the following integral:

$$\int \frac{x^2 + 9}{x^4 + 81} dx$$

Answer

re-writing the given equation as

$$\int \frac{1 + \frac{9}{x^2}}{x^2 + \frac{81}{x^2}} dx$$

$$\int \frac{1 + \frac{9}{x^2}}{\left(x - \frac{9}{x}\right)^2 + 18} dx$$

Let $x - \frac{9}{x} = t$

$$\left(1 + \frac{9}{x^2}\right) dx = dt$$

$$\int \frac{dt}{t^2 + 18}$$

Using identity $\int \frac{1}{x^2+1} dx = \arctan(x)$

$$\frac{1}{3\sqrt{2}} \arctan\left(\frac{t}{3\sqrt{2}}\right) + c$$

Substituting t as $x - \frac{9}{x}$

$$\frac{1}{3\sqrt{2}} \arctan\left(\frac{x - \frac{9}{x}}{3\sqrt{2}}\right) + c$$

4. Question

Evaluate the following integral:

$$\int \frac{1}{x^4 + x^2 + 1} dx$$

Answer

re-writing the given equation as

$$\int \frac{\frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$\frac{1}{2} \int \frac{1 + \frac{1}{x^2} + \frac{1}{x^2} - 1}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$\frac{1}{2} \left[\int \frac{1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx + \int \frac{-1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx \right]$$

$$\frac{1}{2} \left[\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx + \int \frac{-1 + \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 1} dx \right]$$

$$\text{Let } x - \frac{1}{x} = t \text{ and } x + \frac{1}{x} = z$$

$$\left(1 + \frac{1}{x^2}\right) dx = dt \text{ and } \left(1 - \frac{1}{x^2}\right) dx = dz$$

$$\frac{1}{2} \left[\int \frac{dt}{(t)^2 + 3} - \int \frac{dz}{(z)^2 - 1} \right]$$

$$\text{Using identity } \int \frac{1}{x^2 + 1} dx = \arctan(x) \text{ and } \int \frac{dz}{(z)^2 - 1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$$

$$\frac{1}{2} \left[\frac{1}{\sqrt{3}} \left(\arctan \left(\frac{t}{\sqrt{3}} \right) - \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| \right) \right]$$

$$\text{Substituting } t \text{ as } x - \frac{1}{x} \text{ and } z \text{ as } x + \frac{1}{x}$$

$$\frac{1}{2} \left[\frac{1}{\sqrt{3}} \left(\arctan \left(\frac{x - \frac{1}{x}}{\sqrt{3}} \right) - \frac{1}{2} \log \left| \frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1} \right| \right) \right]$$

5. Question

Evaluate the following integral:

$$\int \frac{x^2 - 3x + 1}{x^4 + x^2 + 1} dx$$

Answer

re-writing the given equation as

$$\int \frac{1 - \frac{3}{x} + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx - \int \frac{3x}{x^4 + x^2 + 1} dx$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$$\left(1 + \frac{1}{x^2}\right) dx = dt \text{ and } 2x dx = dz$$

$$\int \frac{dt}{(t)^2 + 3} - \frac{3}{2} \int \frac{dz}{z^2 + z + 1}$$

$$\int \frac{dt}{(t)^2 + 3} - \frac{3}{2} \int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

Using identity $\int \frac{1}{x^2+1} dx = \arctan(x)$

$$\frac{1}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right) - \sqrt{3} \arctan\left(\frac{2z+1}{\sqrt{3}}\right) + c$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$$\frac{1}{\sqrt{3}} \arctan\left(\frac{x - \frac{1}{x}}{\sqrt{3}}\right) - \sqrt{3} \arctan\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + c$$

6. Question

Evaluate the following integral:

$$\int \frac{x^2 + 1}{x^4 - x^2 + 1} dx$$

Answer

re-writing the given equation as

$$\int \frac{1 + \frac{1}{x^2}}{x^2 - 1 + \frac{1}{x^2}} dx$$

$$\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 1} dx$$

Substituting t as $x - \frac{1}{x}$

$$\left(1 + \frac{1}{x^2}\right) dx = dt$$

$$\int \frac{dt}{t^2 + 1}$$

Using identity $\int \frac{1}{x^2+1} dx = \arctan(x)$

$$\arctan t + c$$

Substituting t as $x - \frac{1}{x}$

$$\arctan\left(x - \frac{1}{x}\right) + c$$

7. Question

Evaluate the following integral:

$$\int \frac{x^2 - 1}{x^4 + 1} dx$$

Answer

re-writing the given equation as

$$\int \frac{1 - \frac{1}{x^2}}{x^2 - \frac{1}{x^2}} dx$$

$$\int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 2} dx$$

$$\text{Assume } t = x + \frac{1}{x}$$

$$dt = \left(1 - \frac{1}{x^2}\right) dx$$

$$\int \frac{dt}{t^2 - 2}$$

$$\text{Using identity } \int \frac{dz}{(z)^2 - 1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$$

$$\frac{1}{2\sqrt{2}} \log \frac{t - \sqrt{2}}{t + \sqrt{2}} + c$$

$$\text{Substituting } t \text{ as } x + \frac{1}{x}$$

$$\frac{1}{2\sqrt{2}} \log \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} + c$$

8. Question

Evaluate the following integral:

$$\int \frac{x^2 + 1}{x^4 + 7x^2 + 1} dx$$

Answer

re-writing the given equation as

$$\int \frac{1 + \frac{1}{x^2}}{x^2 + 7 + \frac{1}{x^2}} dx$$

$$\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 9} dx$$

$$\text{Assume } t = x - \frac{1}{x}$$

$$dt = \left(1 + \frac{1}{x^2}\right) dx$$

$$\int \frac{dt}{(t)^2 + 9}$$

Using identity $\int \frac{1}{x^2+1} dx = \arctan(x)$

$$\frac{1}{3} \arctan\left(\frac{t}{3}\right) + c$$

Substituting t as $x - \frac{1}{x}$

$$\frac{1}{3} \arctan\left(\frac{x - \frac{1}{x}}{3}\right) + c$$

9. Question

Evaluate the following integral:

$$\int \frac{(x-1)^2}{x^4 + x^2 + 1} dx$$

Answer

re-writing the given equation as

$$\int \frac{x^2 - 2x + 1}{x^4 + x^2 + 1} dx$$

$$\int \frac{1 - \frac{2}{x} + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 3} dx - \int \frac{2x}{x^4 + x^2 + 1} dx$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$$\left(1 + \frac{1}{x^2}\right) dx = dt \text{ and } 2x dx = dz$$

$$\int \frac{dt}{(t)^2 + 3} - \frac{3}{2} \int \frac{dz}{z^2 + z + 1}$$

$$\int \frac{dt}{(t)^2 + 3} - \int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

Using identity $\int \frac{1}{x^2+1} dx = \arctan(x)$

$$\frac{1}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}} \arctan\left(\frac{2z+1}{\sqrt{3}}\right) + c$$

Substituting t as $x - \frac{1}{x}$ and z as x^2

$$\frac{1}{\sqrt{3}} \arctan\left(\frac{x - \frac{1}{x}}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}} \arctan\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + c$$

10. Question

Evaluate the following integral:

$$\int \frac{1}{x^4 + 3x^2 + 1} dx$$

Answer

re-writing the given equation as

$$\int \frac{\frac{1}{x^2}}{x^2 + 3 + \frac{1}{x^2}} dx$$

$$\frac{1}{2} \int \frac{\left(1 + \frac{1}{x^2}\right) - \left(1 - \frac{1}{x^2}\right)}{x^2 + 3 + \frac{1}{x^2}} dx$$

$$\frac{1}{2} \left[\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 5} dx - \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 + 1} dx \right]$$

Assume $t = x - \frac{1}{x}$ and $z = x + \frac{1}{x}$

$$dt = \left(1 + \frac{1}{x^2}\right) dx \text{ and } dz = \left(1 - \frac{1}{x^2}\right) dx$$

$$\frac{1}{2} \left[\int \frac{dt}{(t)^2 + 5} - \int \frac{dz}{(z)^2 + 1} \right]$$

Using identity $\int \frac{1}{x^2+1} dx = \arctan(x)$

$$\frac{1}{2\sqrt{5}} \arctan\left(\frac{t}{\sqrt{5}}\right) - \frac{1}{2} \arctan(z) + c$$

Substituting t as $x - \frac{1}{x}$ and z as $x + \frac{1}{x}$

$$\frac{1}{2\sqrt{5}} \arctan\left(\frac{x - \frac{1}{x}}{\sqrt{5}}\right) - \frac{1}{2} \arctan\left(x + \frac{1}{x}\right) + c$$

11. Question

Evaluate the following integral:

$$\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$$

Answer

Re-writing the given equation as

Multiplying $\sec^4 x$ in both numerator and denominator

$$\int \frac{\sec^4 x}{\tan^4 x + \tan^2 x + 1} dx$$

$$= \int \frac{(\tan^2 x + 1)\sec^2 x}{\tan^4 x + \tan^2 x + 1} dx$$

Assume $\tan x = t$

$$\sec^2 x dx = dt$$

$$= \int \frac{(t^2 + 1)dt}{t^4 + t^2 + 1}$$

$$= \int \frac{1 + \frac{1}{t^2}}{t^2 + 1 + \frac{1}{t^2}} dt$$

$$= \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 3} dt$$

$$\text{Assume } z = t - \frac{1}{t}$$

$$\Rightarrow dz = 1 + \frac{1}{t^2}$$

$$= \int \frac{dz}{z^2 + 3}$$

$$\text{Using identity } \int \frac{1}{x^2 + 1} dx = \arctan(x)$$

$$= \frac{1}{\sqrt{3}} \arctan\left(\frac{z}{\sqrt{3}}\right) + c$$

$$= \frac{1}{\sqrt{3}} \arctan\left(\frac{t - \frac{1}{t}}{\sqrt{3}}\right) + c$$

$$= \frac{1}{\sqrt{3}} \arctan\left(\frac{\tan x - \frac{1}{\tan x}}{\sqrt{3}}\right) + c$$

Exercise 19.32

1. Question

Evaluate the following integral:

$$\int \frac{1}{(x-1)\sqrt{x+2}} dx$$

Answer

$$\text{assume } x+2=t^2$$

$$dx=2tdt$$

$$\int \frac{2dt}{(t^2-3)}$$

$$\text{Using identity } \int \frac{dz}{(z)^2-1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$$

$$\frac{1}{\sqrt{3}} \log \left| \frac{t - \sqrt{3}}{t + \sqrt{3}} \right| + c$$

$$\frac{1}{\sqrt{3}} \log \left| \frac{\sqrt{x+2} - \sqrt{3}}{\sqrt{x+2} + \sqrt{3}} \right| + c$$

2. Question

Evaluate the following integral:

$$\int \frac{1}{(x-1)\sqrt{2x+3}} dx$$

Answer

assume $2x+3=t^2$

$dx=tdt$

$$\int \frac{dt}{\frac{t^2-3}{2}-1}$$

$$\int \frac{2dt}{(t^2-5)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$

$$\frac{1}{\sqrt{5}} \log \left| \frac{t-\sqrt{5}}{t+\sqrt{5}} \right| + c$$

$$\frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{(2x+3)} - \sqrt{5}}{\sqrt{2x+3} + \sqrt{5}} \right| + c$$

3. Question

Evaluate the following integral:

$$\int \frac{x+1}{(x-1)\sqrt{x+2}} dx$$

Answer

re-writing the given equation as

$$\int \frac{(x-1)+2}{(x-1)\sqrt{x+2}} dx$$

Now splitting the integral in two parts

$$\int \frac{(x-1)}{(x-1)\sqrt{x+2}} dx + \int \frac{2}{(x-1)\sqrt{x+2}} dx$$

For the first part using identity $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$2\sqrt{x+2}$$

For the second part

assume $x+2=t^2$

$dx=2tdt$

$$\int \frac{4dt}{(t^2-3)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$

$$\frac{2}{\sqrt{3}} \log \left| \frac{t-\sqrt{3}}{t+\sqrt{3}} \right| + c$$

$$\frac{2}{\sqrt{3}} \log \left| \frac{\sqrt{(x+2)} - \sqrt{3}}{\sqrt{x+2} + \sqrt{3}} \right| + c$$

Hence integral is

$$2\sqrt{x+2} + \frac{2}{\sqrt{3}} \log \left| \frac{\sqrt{(x+2)} - \sqrt{3}}{\sqrt{x+2} + \sqrt{3}} \right| + c$$

4. Question

Evaluate the following integral:

$$\int \frac{x^2}{(x-1)\sqrt{x+2}} dx$$

Answer

re-writing the given equation as

$$\int \frac{(x^2-1)+1}{(x-1)\sqrt{x+2}} dx$$

$$\int \frac{(x^2-1)}{(x-1)\sqrt{x+2}} dx + \int \frac{1}{(x-1)\sqrt{x+2}} dx$$

$$\int \frac{(x+1)}{\sqrt{x+2}} dx + \int \frac{1}{(x-1)\sqrt{x+2}} dx$$

$$\int \frac{(1)}{\sqrt{x+2}} dx + \int \sqrt{x+2} dx + \int \frac{1}{(x-1)\sqrt{x+2}} dx$$

For the first- and second-part using identity $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$\frac{2}{3} (x+2)^{\frac{3}{2}} + 2\sqrt{x+2}$$

For the second part

assume $x+2=t^2$

$$dx=2t dt$$

$$\int \frac{4dt}{(t^2-3)}$$

Using identity $\int \frac{dz}{(z^2-1)} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$

$$\frac{2}{\sqrt{3}} \log \left| \frac{t-\sqrt{3}}{t+\sqrt{3}} \right| + c$$

$$\frac{2}{\sqrt{3}} \log \left| \frac{\sqrt{(x+2)} - \sqrt{3}}{\sqrt{x+2} + \sqrt{3}} \right| + c$$

Hence integral is

$$\frac{2}{3} (x+2)^{\frac{3}{2}} + 2\sqrt{x+2} + \frac{2}{\sqrt{3}} \log \left| \frac{\sqrt{(x+2)} - \sqrt{3}}{\sqrt{x+2} + \sqrt{3}} \right| + c$$

5. Question

Evaluate the following integral:

$$\int \frac{x}{(x-3)\sqrt{x+1}} dx$$

Answer

re-writing the given equation as

$$\int \frac{(x-3) + 3}{(x-3)\sqrt{x+1}} dx$$

$$\int \frac{(x-3)}{(x-3)\sqrt{x+1}} dx + \int \frac{3}{(x-3)\sqrt{x+1}} dx$$

For the first part using identity $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$2\sqrt{x+1} + c$$

For the second part

assume $x+1=t^2$

$$dx=2t dt$$

$$\int \frac{2dt}{(t^2-4)}$$

Using identity $\int \frac{dz}{(z)^2-1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$

$$\frac{1}{2} \log \left| \frac{t-2}{t+2} \right| + c$$

$$\frac{1}{2} \log \left| \frac{\sqrt{(x+2)} - 2}{\sqrt{x+2} + 2} \right| + c$$

Hence integral is

$$\frac{1}{2} \log \left| \frac{\sqrt{(x+2)} - 2}{\sqrt{x+2} + 2} \right| + c + 2\sqrt{x+1}$$

6. Question

Evaluate the following integral:

$$\int \frac{1}{(x^2+1)\sqrt{x}} dx$$

Answer

let $x=t^2$

$$dx=2t dt$$

$$\int \frac{2dt}{t^4+1}$$

Dividing by t^2 in both numerator and denominator

$$\int \frac{\left[\left(1 + \frac{1}{t^2}\right) - \left(1 - \frac{1}{t^2}\right) \right] dt}{t^2 + \frac{1}{t^2}}$$

$$\int \frac{\left[\left(1 + \frac{1}{t^2} \right) \right] dt}{\left(t - \frac{1}{t} \right)^2 + 2} - \int \frac{\left(1 - \frac{1}{t^2} \right) dt}{\left(t + \frac{1}{t} \right)^2 - 2}$$

$$\text{Let } t - \frac{1}{t} = z \text{ and } t + \frac{1}{t} = y$$

$$\left(1 + \frac{1}{t^2} \right) dt = dz \text{ and } \left(1 - \frac{1}{t^2} \right) dt = dy$$

$$\int \frac{dz}{z^2 + 2} - \int \frac{dy}{y^2 - 2}$$

$$\text{Using identity } \int \frac{1}{x^2 + 1} dx = \arctan(x) \text{ and } \int \frac{dz}{(z)^2 - 1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$$

$$\frac{1}{\sqrt{2}} \arctan \left(\frac{z}{\sqrt{2}} \right) - \frac{1}{2\sqrt{2}} \log \left| \frac{y - \sqrt{2}}{y + \sqrt{2}} \right| + c$$

$$\text{Substituting } t - \frac{1}{t} = z \text{ and } t + \frac{1}{t} = y$$

$$\frac{1}{\sqrt{2}} \arctan \left(\frac{t - \frac{1}{t}}{\sqrt{2}} \right) - \frac{1}{2\sqrt{2}} \log \left| \frac{t + \frac{1}{t} - \sqrt{2}}{t + \frac{1}{t} + \sqrt{2}} \right| + c$$

$$\frac{1}{\sqrt{2}} \arctan \left(\frac{\sqrt{x} - \frac{1}{\sqrt{x}}}{\sqrt{2}} \right) - \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{x} + \frac{1}{\sqrt{x}} - \sqrt{2}}{\sqrt{x} + \frac{1}{\sqrt{x}} + \sqrt{2}} \right| + c$$

7. Question

Evaluate the following integral:

$$\int \frac{x}{(x^2 + 2x + 2)\sqrt{x+1}} dx$$

Answer

$$\text{assume } x+1=t^2$$

$$dx=2t dt$$

$$\int \frac{2(t^2 - 1)dt}{t^4 + 1}$$

Dividing by t^2 in both numerator and denominator

$$\int \frac{2\left(1 - \frac{1}{t^2}\right)dt}{t^2 + \frac{1}{t^2}}$$

$$\int \frac{2\left(1 - \frac{1}{t^2}\right)dt}{\left(t + \frac{1}{t}\right)^2 - 2}$$

$$\text{Let } \left(t + \frac{1}{t}\right) = z$$

$$\left(1 - \frac{1}{t^2}\right)dt = dz$$

$$\int \frac{2dz}{z^2 - 2}$$

Using identity $\int \frac{dz}{(z)^2 - 1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$

$$\frac{1}{\sqrt{2}} \log \left| \frac{z - \sqrt{2}}{z + \sqrt{2}} \right| + c$$

Substituting $\left(t + \frac{1}{t}\right) = z$

$$\frac{1}{\sqrt{2}} \log \left| \frac{t + \frac{1}{t} - \sqrt{2}}{t + \frac{1}{t} + \sqrt{2}} \right| + c$$

Substituting $t = \sqrt{x+1}$

$$\frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{x+1} + \frac{1}{\sqrt{x+1}} - \sqrt{2}}{\sqrt{x+1} + \frac{1}{\sqrt{x+1}} + \sqrt{2}} \right| + c$$

8. Question

Evaluate the following integral:

$$\int \frac{1}{(x-1)\sqrt{x^2+1}} dx$$

Answer

assume $x - 1 = \frac{1}{t}$

$$dx = -\frac{1}{t^2} dt$$

$$-\int \frac{dt}{\sqrt{2t^2 + 2t + 1}}$$

$$-\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{\left(t + \frac{1}{2}\right)^2 + \frac{1}{4}}}$$

Using identity $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log(x + \sqrt{x^2 + a^2}) + c$

$$-\frac{1}{\sqrt{2}} \log \left(t + \frac{1}{2} + \sqrt{\left(t + \frac{1}{2}\right)^2 + \frac{1}{4}} \right) + c$$

Substituting $t = \frac{1}{x-1}$

$$-\frac{1}{\sqrt{2}} \log \left(\frac{1}{x-1} + \frac{1}{2} + \sqrt{\left(\frac{1}{x-1} + \frac{1}{2}\right)^2 + \frac{1}{4}} \right) + c$$

9. Question

Evaluate the following integral:

$$\int \frac{1}{(x+1)\sqrt{x^2+x+1}} dx$$

Answer

$$\text{assume } x+1 = \frac{1}{t}$$

$$dx = -\frac{1}{t^2} dt$$

$$-\int \frac{dt}{\sqrt{1+t-t^2}}$$

$$-\int \frac{dt}{\sqrt{\frac{5}{4} - \left(t - \frac{1}{2}\right)^2}}$$

$$\text{Using identity } \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + c$$

$$-\arcsin\left(\frac{\left(t - \frac{1}{2}\right)}{\frac{\sqrt{5}}{2}}\right) + c$$

$$\text{Substituting } t = \frac{1}{x+1}$$

$$-\arcsin\left(\frac{\left(\frac{1}{x+1} - \frac{1}{2}\right)}{\frac{\sqrt{5}}{2}}\right) + c$$

10. Question

Evaluate the following integral:

$$\int \frac{1}{(x^2-1)\sqrt{x^2+1}} dx$$

Answer

$$\text{assume } x = \frac{1}{t}$$

$$dx = -\frac{1}{t^2} dt$$

$$-\int \frac{tdt}{(1-t^2)(\sqrt{1+t^2})}$$

$$\text{Let } 1+t^2=u^2$$

$$tdt=udu$$

$$\int \frac{udu}{(u^2-2)u}$$

$$\int \frac{du}{(u^2-2)}$$

$$\text{Using identity } \int \frac{dz}{(z^2-1)} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$$

$$\frac{1}{2\sqrt{2}} \log \left| \frac{u - \sqrt{2}}{u + \sqrt{2}} \right| + c$$

Substituting $u = \sqrt{1+t^2}$

$$\frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{1+t^2} - \sqrt{2}}{\sqrt{1+t^2} + \sqrt{2}} \right| + c$$

Substituting $t = \frac{1}{x}$

$$\frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{1 + \frac{1}{x^2}} - \sqrt{2}}{\sqrt{1 + \frac{1}{x^2}} + \sqrt{2}} \right| + c$$

11. Question

Evaluate the following integral:

$$\int \frac{x}{(x^2 + 4)\sqrt{x^2 + 1}} dx$$

Answer

assume $x^2 + 1 = u^2$

$$x dx = u du$$

$$\int \frac{u du}{(u^2 + 3)u}$$

$$\int \frac{du}{(u^2 + 3)}$$

Using identity $\int \frac{1}{x^2 + 1} dx = \arctan(x)$

$$\frac{1}{\sqrt{3}} \arctan \left(\frac{u}{\sqrt{3}} \right) + c$$

Substituting $u = \sqrt{1 + x^2}$

$$\frac{1}{\sqrt{3}} \arctan \left(\frac{\sqrt{1 + x^2}}{\sqrt{3}} \right) + c$$

12. Question

Evaluate the following integral:

$$\int \frac{1}{(1 + x^2)\sqrt{1 - x^2}} dx$$

Answer

assume $x = \frac{1}{t}$

$$dx = -\frac{1}{t^2} dt$$

$$-\int \frac{t dt}{(t^2 + 1)(\sqrt{t^2 - 1})}$$

$$\text{Let } t^2 - 1 = u^2$$

$$t dt = u du$$

$$-\int \frac{u du}{(u^2 + 2)u}$$

$$-\int \frac{du}{(u^2 + 2)}$$

$$\text{Using identity } \int \frac{1}{x^2+1} dx = \arctan(x)$$

$$-\frac{1}{\sqrt{2}} \arctan\left(\frac{u}{\sqrt{2}}\right) + c$$

$$\text{Substituting } u = \sqrt{t^2 - 1}$$

$$-\frac{1}{\sqrt{2}} \arctan\left(\frac{\sqrt{t^2 - 1}}{\sqrt{2}}\right) + c$$

$$\text{Substituting } t = \frac{1}{x}$$

$$-\frac{1}{\sqrt{2}} \arctan\left(\frac{\sqrt{\frac{1}{x^2} - 1}}{\sqrt{2}}\right) + c$$

13. Question

Evaluate the following integral:

$$\int \frac{1}{(2x^2 + 3)\sqrt{x^2 - 4}} dx$$

Answer

$$\text{assume } x = \frac{1}{t}$$

$$dx = -\frac{1}{t^2} dt$$

$$-\int \frac{t dt}{(3t^2 + 2)(\sqrt{1 - 4t^2})}$$

$$\text{Assume } 1 - 4t^2 = u^2$$

$$-4t dt = u du$$

$$-\frac{1}{4} \int \frac{u du}{\left(\frac{11 - 3u^2}{4}\right)u}$$

$$-\frac{1}{3} \int \frac{du}{\left(\frac{11}{3} - u^2\right)}$$

$$\text{Using identity } \int \frac{dz}{(z)^2 - 1} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$$

$$\frac{1}{2\sqrt{33}} \log \left| \frac{u - \sqrt{\frac{11}{3}}}{u + \sqrt{\frac{11}{3}}} \right| + c$$

Substituting $u = \sqrt{1 - 4t^2}$

$$\frac{1}{2\sqrt{33}} \log \left| \frac{\sqrt{1 - 4t^2} - \sqrt{\frac{11}{3}}}{\sqrt{1 - 4t^2} + \sqrt{\frac{11}{3}}} \right| + c$$

Substituting $t = \frac{1}{x}$

$$\frac{1}{2\sqrt{33}} \log \left| \frac{\sqrt{1 - \frac{4}{x^2}} - \sqrt{\frac{11}{3}}}{\sqrt{1 - \frac{4}{x^2}} + \sqrt{\frac{11}{3}}} \right| + c$$

14. Question

Evaluate the following integral:

$$\int \frac{x}{(x^2 + 4)\sqrt{x^2 + 9}} dx$$

Answer

assume $x^2 + 9 = u^2$

$x dx = u du$

$$\int \frac{u du}{(u^2 - 5)u}$$

$$\int \frac{du}{(u^2 - 5)}$$

Using identity $\int \frac{dz}{(z^2 - 1)} = \frac{1}{2} \log \left| \frac{z-1}{z+1} \right| + c$

$$\frac{1}{2\sqrt{5}} \log \left| \frac{u - \sqrt{5}}{u + \sqrt{5}} \right| + c$$

Substituting $u = \sqrt{9 + x^2}$

$$\frac{1}{2\sqrt{5}} \log \left| \frac{\sqrt{9 + x^2} - \sqrt{5}}{\sqrt{9 + x^2} + \sqrt{5}} \right| + c$$

Very short answer

16. Question

Write a value of $\int \frac{1}{1 + 2e^x} dx$

Answer

Take e^x out from the denominator.

$$y = \int \frac{1}{e^x(e^{-x} + 2)} dx$$

$$y = \int \frac{e^{-x}}{(e^{-x} + 2)} dx$$

Let, $e^{-x} + 2 = t$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = -e^{-x}$$

$$\Rightarrow -dt = e^{-x} dx$$

$$y = \int \frac{-dt}{t}$$

Use formula $\int \frac{1}{t} dt = \ln t$

$$Y = -\ln t + c$$

Again, put $e^{-x} + 2 = t$

$$Y = -\ln(e^{-x} + 2) + c$$

Note: Don't forget to replace t with the function of x at the end of solution. Always put constant c with indefinite integral.

17. Question

Write a value of $\int \frac{(\tan^{-1} x)^3}{1+x^2} dx$

Answer

Let, $\tan^{-1} x = t$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = \frac{1}{1+x^2}$$

$$\Rightarrow dt = \frac{dx}{1+x^2}$$

$$y = \int t^3 dt$$

Use formula $\int t^n dt = \frac{t^{n+1}}{n+1}$

$$y = \frac{t^4}{4} + c$$

Again, put $t = \tan^{-1} x$

$$y = \frac{(\tan^{-1} x)^4}{4} + c$$

18. Question

Write a value of $\int \frac{\sec^2 x}{(5 + \tan x)^4} dx$

Answer

Let, $\tan x = t$

Differentiating both side with respect to x

$$\frac{dt}{dx} = (\sec x)^2 \Rightarrow dt = \sec^2 x \, dx$$

$$y = \int \frac{dt}{(5+t)^4}$$

$$\text{Use formula } \int \frac{1}{(a+t)^n} dt = \frac{(a+t)^{-n+1}}{-n+1}$$

$$y = \frac{(5+t)^{-3}}{-3} + c$$

Again, put $t = \tan x$

$$y = -\frac{1}{3(5 + \tan x)^3} + c$$

19. Question

Write a value of $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx$

Answer

We know that

$$1 + \sin 2x = \sin^2 x + \cos^2 x + 2 \sin x \cos x = (\sin x + \cos x)^2$$

$$y = \int \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}} dx$$

$$y = \int \frac{(\sin x + \cos x)}{(\sin x + \cos x)} dx$$

$$y = \int dx$$

Use formula $\int c \, dx = cx$, where c is constant

$$y = x + c$$

20. Question

Write a value of $\int \log_e x \, dx$

Answer

$$y = \int 1 \times \log_e x \, dx$$

By using integration by parts

Let, $\log_e x$ as Ist function and 1 as IInd function

$$\text{Use formula } \int I \times II \, dx = I \int II \, dx - \int \left(\frac{d}{dx} I \right) \left(\int II \, dx \right) dx$$

$$y = \log_e x \int dx - \int \left(\frac{d}{dx} \log_e x \right) \left(\int dx \right) dx$$

$$y = (\log_e x)x - \int \left(\frac{1}{x} \right) (x) dx$$

$$y = x \log_e x - \int dx$$

$$y = x \log_e x - x + c$$

21. Question

Write a value of $\int a^x e^x dx$

Answer

We know that a and e are constant so, $a^x e^x = (ae)^x$

$$y = \int (ae)^x dx$$

Use formula $\int c^x = \frac{c^x}{\log c}$ where c is constant

$$y = \frac{(ae)^x}{\log(ae)} + c$$

$$y = \frac{a^x e^x}{\log a + 1} + c$$

22. Question

Write a value of $\int e^{2x^2 + \ln x} dx$

Answer

We know that $e^{a+b} = e^a e^b$

$$y = \int e^{2x^2} e^{\ln x} dx$$

$$y = \int e^{2x^2} x dx$$

Let, $x^2 = t$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = 2x$$

$$\Rightarrow \frac{1}{2} dt = x dx$$

$$y = \int \frac{1}{2} e^{2t} dt$$

Use formula $\int e^{a+bt} = \frac{e^{a+bt}}{b}$

$$y = \frac{1}{2} \frac{e^{2t}}{2} + c$$

Again, put $t = x^2$

$$y = \frac{e^{2x^2}}{4} + c$$

23. Question

Write a value of $\int (e^{x \log_e a} + e^{a \log_e x}) dx$

Answer

We know that by using property of logarithm

$$e^{x \log_e a} = e^{\log_e a^x} = a^x \text{ and } e^{a \log_e x} = e^{\log_e x^a} = x^a$$

$$y = \int a^x + x^a dx$$

$$y = \int a^x dx + \int x^a dx$$

$$\text{Use formula } \int a^x dx = \frac{a^x}{\log a} \text{ and } \int x^a dx = \frac{x^{a+1}}{a+1}$$

$$y = \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + c$$

24. Question

$$\text{Write a value of } \int \frac{\cos x}{\sin x \log \sin x} dx$$

Answer

$$\text{Let } \log(\sin x) = t$$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = \frac{\cos x}{\sin x} \Rightarrow dt = \frac{\cos x}{\sin x} dx$$

$$y = \int \frac{1}{t} dt$$

$$\text{Use formula } \int \frac{1}{t} dt = \log t$$

$$y = \log t + c$$

$$\text{Again, put } t = \log(\sin x)$$

$$y = \log(\log(\sin x)) + c$$

25. Question

$$\text{Write a value of } \int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x} dx$$

Answer

$$\text{We know that } \cos^2 x = 1 - \sin^2 x$$

$$(a^2 \sin^2 x + b^2 \cos^2 x) = a^2 \sin^2 x + b^2 (1 - \sin^2 x)$$

$$= (a^2 - b^2) \sin^2 x + b^2$$

$$y = \int \frac{\sin 2x}{(a^2 - b^2)(\sin x)^2 + b^2} dx$$

$$\text{Let, } \sin^2 x = t$$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = 2 \sin x \cos x$$

$$= \sin 2x$$

$$\Rightarrow dt = \sin 2x dx$$

$$y = \int \frac{dt}{(a^2 - b^2)t + b^2}$$

$$\text{Use formula } \int \frac{1}{ct+d} dt = \frac{\log(ct+d)}{c}$$

$$y = \frac{\log[(a^2 - b^2)t + b^2]}{(a^2 - b^2)} + c$$

Again, put $t = \sin^2 x$

$$y = \frac{\log[(a^2 - b^2)(\sin x)^2 + b^2]}{(a^2 - b^2)} + c$$

26. Question

Write a value of $\int \frac{a^x}{3 + a^x} dx$

Answer

Let, $3 + a^x = t$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = a^x \log a$$

$$\Rightarrow \frac{dt}{\log a} = a^x dx$$

$$y = \int \frac{1}{(\log a)t} dt$$

Use formula $\int \frac{1}{t} dt = \log t$

$$y = \frac{\log t}{\log a} + c$$

Again, put $t = 3 + a^x$

$$y = \frac{\log(3 + a^x)}{\log a} + c$$

27. Question

Write a value of $\int \frac{1 + \log x}{3 + x \log x} dx$

Answer

Let, $x(\log x) = t$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = x \frac{1}{x} + \log x = 1 + \log x$$

$$\Rightarrow dt = (1 + \log x) dx$$

$$y = \int \frac{1}{3 + t} dt$$

Use formula $\int \frac{1}{a+t} dt = \log(a+t)$

$$y = \log(3 + t) + c$$

Again, put $t = x(\log x)$

$$y = \log(3 + x(\log x)) + c$$

28. Question

Write a value of $\int \frac{\sin x}{\cos^3 x} dx$

Answer

Let, $\cos x = t$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = -\sin x$$

$$\Rightarrow -dt = \sin x dx$$

$$y = \int \frac{-1}{t^3} dt$$

$$\text{Use formula } \int \frac{1}{t^n} dt = \frac{t^{-n+1}}{-n+1}$$

$$y = -\frac{t^{-2}}{-2} + c$$

Again, put $t = \cos x$

$$y = \frac{1}{2(\cos x)^2} + c$$

29. Question

Write a value of $\int \frac{\sin x - \cos x}{\sqrt{1 + \sin 2x}} dx$

Answer

We know that

$$1 + \sin 2x = \sin^2 x + \cos^2 x + 2\sin x \cos x$$

$$= (\sin x + \cos x)^2$$

$$y = \int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2}} dx$$

$$y = \int \frac{(\sin x - \cos x)}{(\sin x + \cos x)} dx$$

Let, $\sin x + \cos x = t$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = \cos x - \sin x$$

$$\Rightarrow -dt = (\sin x - \cos x) dx$$

$$y = \int \frac{-1}{t} dt$$

$$\text{Use formula } \int \frac{1}{t} = \log t$$

$$y = -\log t + c$$

Again, put $t = \sin x + \cos x$

$$y = -\log(\sin x + \cos x) + c$$

30. Question

Write a value of $\int \frac{1}{x(\log x)^n} dx$

Answer

Let, $\log x = t$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = \frac{1}{x}$$

$$\Rightarrow dt = \frac{1}{x} dx$$

$$y = \int \frac{1}{t^n} dt$$

Use formula $\int \frac{1}{t^n} dt = \frac{t^{-n+1}}{-n+1}$

$$y = \frac{t^{-n+1}}{-n+1} + c$$

Again, put $t = \log x$

$$y = \frac{(\log x)^{-n+1}}{-n+1} + c$$

31. Question

Write a value of $\int e^{ax} \sin bx dx$

Answer

we know $\int f(x)g(x) = f(x) \int g(x) - f'(x) \int g(x)$

Let $\int e^{ax} \sin bx dx = i$

Given that $\int e^{ax} \sin bx dx$

$$i = \sin bx \int e^{ax} - \int b \cos bx \int e^{ax}$$

$$i = \sin bx \frac{e^{ax}}{a} - \int b \cos bx \frac{e^{ax}}{a}$$

$$i = \sin bx \frac{e^{ax}}{a} - \frac{1}{a} \left[b \cos bx \frac{e^{ax}}{a} - \frac{b^2}{a} \int e^{ax} \sin bx dx \right]$$

$$i = \sin bx \frac{e^{ax}}{a} - \frac{b}{a^2} \cos bx e^{ax} + \frac{b^2}{a^2} i$$

$$i \left(1 - \frac{b^2}{a^2} \right) = \frac{a \sin bx e^{ax} - b \cos bx e^{ax}}{a^2}$$

$$i = \frac{a \sin bx e^{ax} - b \cos bx e^{ax}}{a^2} \left(\frac{a^2}{a^2 - b^2} \right)$$

$$\int e^{ax} \sin bx dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 - b^2}$$

32. Question

Write a value of $\int e^{ax} \cos bx \, dx$.

Answer

we know $\int f(x)g(x) = f(x) \int g(x) - f'(x) \int g(x)$

Let $\int e^{ax} \cos bx \, dx = i$

Given that $\int e^{ax} \cos bx \, dx$

$$i = \cos bx \int e^{ax} - \int -b \sin bx \int e^{ax}$$

$$i = \cos bx \frac{e^{ax}}{a} + \int b \sin bx \frac{e^{ax}}{a}$$

$$i = \cos bx \frac{e^{ax}}{a} + \frac{1}{a} \left[b \sin bx \frac{e^{ax}}{a} - \frac{b^2}{a} \int e^{ax} \cos bx \, dx \right]$$

$$i = \cos bx \frac{e^{ax}}{a} + \frac{b}{a^2} \sin bx e^{ax} - \frac{b^2}{a^2} i$$

$$i \left(1 + \frac{b^2}{a^2} \right) = \frac{a \cos bx e^{ax} + b \sin bx e^{ax}}{a^2}$$

$$i = \frac{a \cos bx e^{ax} + b \sin bx e^{ax}}{a^2} \left(\frac{a^2}{a^2 + b^2} \right)$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}(a \sin bx + b \cos bx)}{a^2 + b^2}$$

33. Question

Write a value of $\int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx$.

Answer

given $\int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx$

$$= \int \frac{e^x}{x} dx - \int \frac{e^x}{x^2} dx$$

$$= \int \frac{e^x}{x} dx - \left[\frac{e^x}{x^2} - \int -\frac{e^x}{x} \right] + c$$

$$= -\frac{e^x}{x^2} + c$$

34. Question

Write a value of $\int e^{ax} |af(x) + f'(x)| \, dx$.

Answer

given $\int e^{ax} |af(x) + f'(x)| \, dx$

$$= a \int e^{ax} f(x) \, dx + \int e^{ax} f'(x) \, dx$$

$$= a \left[f(x) \frac{e^{ax}}{a} - \int f'(x) \frac{e^{ax}}{a} dx \right] + \int e^{ax} f'(x) dx$$

$$= f(x) e^{ax} + c$$

35. Question

Write a value of $\int \sqrt{4-x^2} dx$.

Answer

we know that $\int \sqrt{a^2 - x^2} dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$

Given $\int \sqrt{4-x^2}$

$$= \int \sqrt{2^2 - x^2}$$

$$= \frac{x\sqrt{2^2 - x^2}}{2} + \frac{2^2}{2} \sin^{-1}\left(\frac{x}{2}\right)$$

$$= \frac{x\sqrt{4-x^2}}{2} + \frac{x^2}{2} \sin^{-1}\left(\frac{x}{2}\right) + c$$

36. Question

Write a value of $\int \sqrt{9+x^2} dx$.

Answer

we know that $\int \sqrt{x^2 + a^2} dx = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \log|x + \sqrt{x^2 + a^2}| + c$

Given $\int x^2 + 9$

$$= \int x^2 + 3^2$$

$$= \frac{x\sqrt{x^2 + 3^2}}{2} + \frac{3^2}{2} \log|x + \sqrt{x^2 + 3^2}|$$

$$= \frac{x\sqrt{x^2 + 9}}{2} + \frac{9}{2} \log|x + \sqrt{x^2 + 9}| + c$$

37. Question

Write a value of $\int \sqrt{x^2 - 9} dx$

Answer

we know that $\int \sqrt{x^2 - a^2} dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log|x + \sqrt{x^2 - a^2}| + c$

Given $\int \sqrt{x^2 - 9} dx$

$$= \int \sqrt{x^2 - 3^2} dx$$

$$= \frac{x\sqrt{x^2 - 3^2}}{2} - \frac{3^2}{2} \log|x + \sqrt{x^2 - 3^2}|$$

$$= \frac{x\sqrt{x^2 - 9}}{2} - \frac{9}{2} \log|x + \sqrt{x^2 - 9}| + c$$

38. Question

Evaluate: $\int \frac{x^2}{1+x^3}$

Answer

let $1+x^3 = t$

Differentiating on both sides we get,

$$3x^2 dx = dt$$

$$x^2 dx = \frac{1}{3} dt$$

substituting it in $\int \frac{x^2}{1+x^3} dx$ we get,

$$= \int \frac{1}{3t} dt$$

$$= \frac{1}{3} \log t + c$$

$$= \frac{1}{3} \log(1+x^3) + c$$

39. Question

Evaluate: $\int \frac{x^2 + 4x}{x^3 + 6x^2 + 5} dx$

Answer

let $x^3 + 6x^2 + 5 = t$

Differentiating on both sides we get,

$$(3x^2 + 12x) dx = dt$$

$$3(x^2 + 4x) dx = dt$$

$$(x^2 + 4x) dx = \frac{1}{3} dt$$

Substituting it in $\int \frac{x^2+4x}{x^3+6x^2+5} dx$ we get,

$$= \int \frac{1}{3t} dt$$

$$= \frac{1}{3 \log(x^3 + 6x^2 + 5)} + c$$

40. Question

Evaluate: $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} dx$

Answer

let $\sqrt{x} = t$

Differentiating on both sides we get,

$$\frac{1}{2\sqrt{x}} dx = dt$$

$$\frac{1}{\sqrt{x}} dx = 2dt$$

substituting it in $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} dx$ we get,

$$= \int 2\sec^2 t dt$$

$$= 2 \tan t + c$$

$$= 2 \tan \sqrt{x} + c$$

41. Question

Evaluate: $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$.

Answer

$$\text{let } \sqrt{x} = t$$

Differentiating on both sides we get,

$$\frac{1}{2\sqrt{x}} dx = dt$$

$$\frac{1}{\sqrt{x}} dx = 2dt$$

substituting it in $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$ we get,

$$= \int 2 \sin t dt$$

$$= -2 \cos t + c$$

$$= -2 \cos \sqrt{x} + c$$

42. Question

Evaluate: $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$.

Answer

$$\text{let } \sqrt{x} = t$$

Differentiating on both sides we get,

$$\frac{1}{2\sqrt{x}} dx = dt$$

$$\frac{1}{\sqrt{x}} dx = 2dt$$

substituting it in $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$ we get,

$$= \int 2 \cos t dt$$

$$= 2 \sin t + c$$

$$= 2 \sin \sqrt{x} + c$$

43. Question

Evaluate: $\int \frac{(1 + \log x)^2}{x} dx.$

Answer

let $1 + \log x = t$

Differentiating on both sides we get,

$$\frac{1}{x} dx = dt$$

Substituting it in $\int \frac{(1 + \log x)^2}{x}$ we get,

$$= \int t^2 dt$$

$$= \frac{t^3}{3} + c$$

$$= \frac{(1 + \log x)^3}{3} + c$$

44. Question

Evaluate: $\int \sec^2 (7 - 4x) dx.$

Answer

let $7 - 4x = t$

Differentiating on both sides we get,

$$-4 dx = dt$$

$$dx = -\frac{1}{4} dt$$

substituting it in $\int \sec^2 (7 - 4x) dx$ we get,

$$= \int -\frac{1}{4} \sec^2 t dt$$

$$= \tan t + c$$

$$= \tan (7 - 4x) + c$$

45. Question

Evaluate: $\int \frac{\log x^x}{x} dx.$

Answer

given $\int \frac{\log x^x}{x} dx$

$$= \int \frac{x \log x}{x} dx$$

$$= \int \log x$$

$$= x \log x - x + c$$

1. Question

Write a value of $\int \frac{1 + \cot x}{x + \log \sin x} dx$.

Answer

let $x + \log \sin x = t$

Differentiating it on both sides we get,

$$(1 + \cot x) dx = dt$$

Given that $\int \frac{1 + \cot x}{x + \log \sin x} dx$

Substituting t in above equation we get,

$$= \int \frac{dt}{t}$$

$$= \log t + c$$

$$= \log(x + \log \sin x) + c$$

2. Question

Write a value of $\int e^{3 \log x} x^4 dx$.

Answer

Consider $\int e^{3 \log x} x^4$

$$e^{3 \log x} = e^{\log x^3}$$

$$= x^3$$

$$\int e^{3 \log x} x^4 = \int x^3 x^4 dx$$

$$= \int x^7 dx$$

$$= \frac{x^8}{8} + c$$

3. Question

Write a value of $\int x^2 \sin x^3 dx$.

Answer

let $x^3 = t$

Differentiating on both sides we get,

$$3 x^2 dx = dt$$

$$x^2 dx = \frac{1}{3} dt$$

substituting above equation in $\int x^2 \sin x^3 dx$ we get,

$$= \int \frac{1}{3} \sin t dt$$

$$= -\frac{1}{3} \cos t + c$$

$$= -\frac{1}{3} \cos x^3 + c$$

4. Question

Write a value of $\int \tan^3 x \sec^2 x \, dx$.

Answer

let $\tan x = t$

Differentiating on both sides we get,

$$\sec^2 x \, dx = dt$$

Substituting above equation in $\int \tan^3 x \sec^2 x \, dx$ we get,

$$= \int t^3 \, dt$$

$$= \frac{t^4}{4} + c$$

$$= \frac{\tan^4 x}{4} + c$$

5. Question

Write a value of $\int e^x (\sin x + \cos x) \, dx$.

Answer

we know $\int e^x (f(x) + f'(x)) \, dx = e^x f(x) + c$

Given, $\int e^x (\sin x + \cos x) \, dx$

Here $f(x) = \sin x$ and $f'(x) = \cos x$

Therefore $\int e^x (\sin x + \cos x) \, dx = e^x \sin x + c$

6. Question

Write a value of $\int \tan^6 x \sec^2 x \, dx$.

Answer

let $\tan x = t$

Differentiating on both sides we get,

$$\sec^2 x \, dx = dt$$

Substituting above equation in $\int \tan^6 x \sec^2 x \, dx$ we get,

$$= \int t^6 \, dt$$

$$= \frac{t^7}{7} + c$$

$$= \frac{\tan^7 x}{7} + c$$

7. Question

Write a value of $\int \frac{\cos x}{3 + 2 \sin x} \, dx$.

Answer

let $3+2\sin x=t$

Differentiating on both sides we get,

$$2\cos x \, dx = dt$$

$$\cos x \, dx = \frac{1}{2} dt$$

Substituting above equation in $\int \frac{\cos x}{3+2\sin x} dx$ we get,

$$\int \frac{1}{2t} dt$$

$$= \frac{1}{2} \log t + c$$

$$= \frac{1}{2} \log(3 + 2\sin x) + c$$

8. Question

Write a value of $\int e^x \sec x (1 + \tan x) dx$.

Answer

given,

$$\int e^x \sec x (1 + \tan x) dx = \int e^x (\sec x + \sec x \tan x) dx$$

$$= e^x \sec x + c$$

$$\therefore \int e^x (f(x) + f'(x)) dx = e^x f(x) + c$$

9. Question

Write a value of $\int \frac{\log x^n}{x} dx$.

Answer

let $\log x^n = t$

Differentiating on both sides we get,

$$\frac{1}{x^n} n x^{n-1} dx = dt$$

$$\frac{n}{x} dx = dt$$

$$\frac{1}{x} dx = \frac{1}{n} dt$$

Substituting above equations in $\int \frac{\log x^n}{x} dx$ we get,

$$\int \frac{1}{n} t dt$$

$$= \frac{1}{n} \frac{t^2}{2} + c$$

$$= \frac{(\log x^n)^2}{2n} + c$$

10. Question

Write a value of $\int \frac{(\log x)^n}{x} dx$.

Answer

let $\log x = t$

Differentiating on both sides we get,

$$\frac{1}{x} dx = dt$$

Substituting above equations in $\int \frac{(\log x)^n}{x} dx$ we get,

$$\begin{aligned} & \int t^n dt \\ &= \frac{t^{n+1}}{n+1} + c \\ &= \frac{(\log x)^{n+1}}{n+1} + c \end{aligned}$$

11. Question

Write a value of $\int e^{\log \sin x} \cos x dx$.

Answer

given $\int e^{\log \sin x} \cos x dx$

$$= \int \sin x \cos x dx \quad (\because e^{\log x} = x)$$

Let $\sin x = t$

Differentiating on both sides we get,

$$\cos x dx = dt$$

Substituting above equations in given equation we get,

$$= \int t dt$$

$$= \frac{t^2}{2} + c$$

$$= \frac{\sin^2 x}{2} + c$$

12. Question

Write a value of $\int \sin^3 x \cos x dx$.

Answer

let $\sin x = t$

Differentiating on both sides we get,

$$\cos x dx = dt$$

Substituting above equation in $\int \sin^3 x \cos x dx$ we get,

$$= \int t^3 dt$$

$$= \frac{t^4}{4} + c$$

$$= \frac{\sin^4 x}{4} + c$$

13. Question

Write a value of $\int \cos^4 x \sin x \, dx$.

Answer

let $\cos x = t$

Differentiating on both sides we get,

$$-\sin x \, dx = dt$$

Substituting above equation in $\int \cos^4 x \sin x \, dx$ we get,

$$= \int -t^4 \, dt$$

$$= -\frac{t^5}{5} + c$$

$$= -\frac{\cos^5 x}{5} + c$$

14. Question

Write a value of $\int \tan x \sec^3 x \, dx$.

Answer

given $\int \tan x \sec^3 x \, dx$

$$= \int (\tan x \sec x) \sec^2 x \, dx$$

Let $\sec x = t$

Differentiating on both sides we get,

$$\tan x \sec x \, dx = dt$$

Substituting above equation in $\int \tan x \sec^3 x \, dx$ we get,

$$= \int t^2 \, dt$$

$$= \frac{t^3}{3} + c$$

$$= \frac{\sec^3 x}{3} + c$$

15. Question

Write a value of $\int \frac{1}{1+e^x} \, dx$.

Answer

given $\int \frac{1}{1+e^x} \, dx$

$$= \int \left(1 - \frac{e^x}{1+e^x} \right) dx$$

Let $1+e^x = t$

Differentiating on both sides we get,

$$e^x dx = dt$$

Substituting above equation in given equation we get,

$$= \int \left(1 - \frac{1}{t}\right) dt$$

$$= t - \log t + c$$

$$= 1 + e^x - \log(1 + e^x) + c$$

46. Question

Evaluate: $\int 2^x dx$.

Answer

Given, $\int 2^x dx$.

$$= \frac{2^x}{\log 2} + c \text{ [since, } \int a^x dx = \frac{a^x}{\log a} \text{]}$$

47. Question

Evaluate: $\int \frac{1 - \sin x}{\cos^2 x} dx$.

Answer

Given, $\int \frac{1 - \sin x}{\cos^2 x} dx$.

$$= \int \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} dx$$

$$= \int \sec^2 x - \tan x \cdot \sec x dx \text{ [since, } \cos x = \frac{1}{\sec x} \text{]}$$

$$= \tan x - \sec x + c$$

48. Question

Evaluate: $\int \frac{x^3 - 1}{x^2} dx$.

Answer

Given, $\int \frac{x^3 - 1}{x^2} dx$.

$$= \int \frac{x^3}{x^2} - \frac{1}{x^2} dx$$

$$= \int x - \frac{1}{x^2} dx$$

$$\text{[since, } \int x^n dx = \frac{x^{n+1}}{n+1} \text{]}$$

$$= \frac{x^2}{2} - \frac{x^{-2+1}}{-2+1} + c$$

$$= \frac{x^2}{2} - \frac{x^{-1}}{-1} + c$$

$$= \frac{x^2}{2} + \frac{1}{x} + c$$

49. Question

Evaluate: $\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$.

Answer

Given, $\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$.

$$= \int \frac{x^2(x - 1) + x - 1}{x - 1} dx$$

$$= \int \frac{(x - 1)[x^2 + 1]}{x - 1} dx$$

$$= \int (x^2 + 1) dx \text{ [since, } \int x^n dx = \frac{x^{n+1}}{n+1} \text{]}$$

$$= \frac{x^3}{3} + x + c$$

50. Question

Evaluate: $\int \frac{e^{\tan^{-1} x}}{1 + x^2} dx$.

Answer

Given, $\int \frac{e^{\tan^{-1} x}}{1 + x^2} dx$.

Let $\tan^{-1} x = t$

$$\frac{d}{dx}(\tan^{-1} x) = dt$$

$$\frac{1}{1 + x^2} dx = dt$$

Now, $\int \frac{e^{\tan^{-1} x}}{1 + x^2} dx$.

$$= \int e^t dt$$

$$= e^t + c$$

$$= e^{\tan^{-1} x} + c$$

51. Question

Evaluate: $\int \frac{1}{\sqrt{1 - x^2}} dx$.

Answer

Given,

$$\int \frac{1}{\sqrt{1-x^2}} dx.$$

$$=\sin^{-1}x + c$$

(It is a standard formula).

52. Question

Evaluate: $\int \sec x (\sec x + \tan x) dx$.

Answer

Given, $\int \sec x (\sec x + \tan x) dx$

$$=\int (\sec^2 x + \sec x \cdot \tan x) dx$$

$$= \tan x + \sec x + c$$

53. Question

Evaluate: $\int \frac{1}{x^2+16} dx$.

Answer

Given, $\int \frac{1}{x^2+16} dx$.

We know that, $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$

By comparison, $a=4$

$$= \frac{1}{4} \tan^{-1} \frac{x}{4} + c$$

54. Question

Evaluate: $\int (1-x)\sqrt{x} dx$.

Answer

Given, $\int (1-x)\sqrt{x} dx$

$$= \int (\sqrt{x} - x\sqrt{x}) dx$$

$$= \int (x^{\frac{1}{2}} - x \cdot x^{\frac{1}{2}}) dx$$

$$= \int x^{\frac{1}{2}} - x^{\frac{3}{2}} dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c \text{ [since, } \int x^n dx = \frac{x^{n+1}}{n+1} \text{]}$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c$$

$$= \frac{2}{3} x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}} + c$$

55. Question

Evaluate: $\int \frac{x + \cos 6x}{3x^2 + \sin 6x} dx$.

Answer

Given,

$$\int \frac{x + \cos 6x}{3x^2 + \sin 6x} dx.$$

$$\text{Let } 3x^2 + \sin 6x = t$$

$$\Rightarrow \frac{d}{dx}(3x^2 + \sin 6x) = dt$$

$$\Rightarrow 6x + \cos 6x \cdot 6 = dt$$

$$\Rightarrow x + \cos 6x = \frac{dt}{6}$$

Substituting the values,

$$= \int \frac{1}{6t} dt$$

$$= \frac{1}{6} \log t + c$$

$$= \frac{1}{6} \log(3x^2 + \sin 6x) + c$$

56. Question

If $\int \left(\frac{x-1}{x^2} \right) e^x dx = f(x)e^x + C$, then write the value of $f(x)$.

⌋

Answer

Consider, $\int \frac{x-1}{x^2} e^x dx$

$$\int \left(\frac{x}{x^2} - \frac{1}{x^2} \right) e^x dx$$

$$= \int \left(\frac{1}{x} - \frac{1}{x^2} \right) e^x dx$$

It is clearly of the form,

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$$

By comparison, $f(x) = \frac{1}{x}$; $f'(x) = -\frac{1}{x^2}$

$$= e^x \frac{1}{x} + c$$

Therefore, the value of $f(x) = \frac{1}{x}$

57. Question

If $\int e^x (\tan x + 1) \sec x dx = e^x f(x) + C$, then write the value $f(x)$.

Answer

Given, $\int e^x(\tan x + 1)\sec x \, dx$

It is clearly of the form,

$$\int e^x[f(x) + f'(x)]dx = e^x f(x) + c$$

By comparison, $f(x)=1+\tan x$; $f'(x)=\sec x$

$$= e^x (1+\tan x) + C$$

Therefore, the value of $f(x)=1+\tan x$

58. Question

Evaluate: $\int \frac{2}{1-\cos 2x} dx$

Answer

Given, $\int \frac{2}{1-\cos 2x} dx$

We Know that, $\cos 2x=1-2\sin^2 x$

$$\Rightarrow 1-\cos 2x=2\sin^2 x$$

Substitute this in the given,

$$= \int \frac{2}{2\sin^2 x} dx$$

$$= \int \frac{1}{\sin^2 x} dx$$

$$= \int \operatorname{cosec}^2 x \, dx$$

$$= -\cot x + c$$

59. Question

Write the anti-derivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$.

Answer

Anti-derivative is nothing but integration

Therefore its Anti-derivative can be found by integrating the above given equation.

$$= \int 3\sqrt{x} + \frac{1}{\sqrt{x}} dx$$

$$= \int 3x^{\frac{1}{2}} + x^{-\frac{1}{2}} dx$$

$$= 3 \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c \text{ [since, } \int x^n dx = \frac{x^{n+1}}{n+1} \text{]}$$

$$= 3 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$$

$$= 2(x^{\frac{3}{2}} + x^{\frac{1}{2}}) + c$$

60. Question

Evaluate: $\int \cos^{-1}(\sin x) dx$

Answer

Given, $\int \cos^{-1}(\sin x) dx$

Let us consider, $\int \cos^{-1} dx$

We know that, $\int f(x).g(x) dx = f(x) \int g(x) dx - \int [f'(x) \int g(x)] dx$

By comparison, $f(x) = \cos^{-1} x$; $g(x) = 1$

$$= \cos^{-1} x \int 1 dx - \int -\frac{1}{\sqrt{1-x^2}} \cdot x dx$$

$$= x \cos^{-1} x - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} (-2x) dx$$

$$= x \cos^{-1} x - \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} (-2x) dx$$

$$= x \cos^{-1} x - \frac{1}{2} \frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c \quad (\text{since, } \int [f(x)^n \cdot f'(x)] dx = \frac{f(x)^{n+1}}{n+1})$$

$$= x \cos^{-1} x - (1-x^2)^{1/2} + c$$

$$= x \cos^{-1} x - \sqrt{1-x^2} + c$$

$$\text{Therefore, } \int \cos^{-1} x dx = x \cos^{-1} x - \sqrt{1-x^2} + c$$

Replace 'x' with ' $\sin x$ ' :-

$$\therefore \int \cos^{-1}(\sin x) dx = \sin x \cdot \cos^{-1}(\sin x) - \sqrt{1-(\sin x)^2} + c$$

$$= \sin x \cdot \cos^{-1} x (\sin x) - \sqrt{\cos^2 x} + c$$

$$= \sin x \cdot \cos^{-1} x (\sin x) - \cos x + c$$

61. Question

Evaluate: $\int \frac{1}{\sin^2 x \cos^2 x} dx$

Answer

Given, $\int \frac{1}{\sin^2 x \cos^2 x} dx$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx \quad [\text{since, } \sin^2 x + \cos^2 x = 1]$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x) dx$$

$$= \tan x - \cot x + c$$

62. Question

Evaluate: $\int \frac{1}{x(1+\log x)} dx$

Answer

Given, $\int \frac{1}{x(1+\log x)} dx$

Let $1+\log x=t$

$$\Rightarrow \frac{d}{dx}(1+\log x) = dt$$

$$\Rightarrow \frac{1}{x} dx = dt$$

$$= \int \frac{1}{t} dt$$

$$= \log t + c$$

$$= \log(1+\log x) + c$$

MCQ

18. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{x+3}{(x+4)^2} e^x dx =$

A. $\frac{e^x}{x+4} + C$

B. $\frac{e^x}{x+3} + C$

C. $\frac{1}{(x+4)^2} + C$

D. $\frac{e^x}{(x+4)^2} + C$

Answer

$$\int \frac{x+3}{(x+4)^2} e^x dx$$

$$= \int \frac{x+4}{(x+4)^2} e^x dx - \int \frac{1}{(x+4)^2} e^x dx$$

$$= \int e^x \left(\frac{1}{x+4} dx - \frac{1}{(x+4)^2} dx \right)$$

$$\left[\because f(x) = \frac{1}{x+4}; f'(x) = -\frac{1}{(x+4)^2} \right]$$

$$= e^x \left(\frac{1}{x+4} \right) + c$$

$$\because \{ \int e^x f(x) + f'(x) dx = e^x f(x) \}$$

18. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{x+3}{(x+4)^2} e^x dx =$

A. $\frac{e^x}{x+4} + C$

B. $\frac{e^x}{x+3} + C$

C. $\frac{1}{(x+4)^2} + C$

D. $\frac{e^x}{(x+4)^2} + C$

Answer

$$\int \frac{x+3}{(x+4)^2} e^x dx$$

$$= \int \frac{x+4}{(x+4)^2} e^x dx - \int \frac{1}{(x+4)^2} e^x dx$$

$$= \int e^x \left(\frac{1}{x+4} dx - \frac{1}{(x+4)^2} dx \right)$$

$$\left[\because f(x) = \frac{1}{x+4} ; f'(x) = -\frac{1}{(x+4)^2} \right]$$

$$= e^x \left(\frac{1}{x+4} \right) + c$$

$$\because \{ \int e^x f(x) + f'(x) dx = e^x f(x) \}$$

19. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{\sin x}{3+4\cos^2 x} dx$

A. $\log(3+4\cos^2 x) + C$

B. $\frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{\cos x}{\sqrt{3}} \right) + C$

C. $-\frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2\cos x}{\sqrt{3}} \right) + C$

D. $\frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2\cos x}{\sqrt{3}} \right) + C$

Answer

$$\int \frac{\sin x}{3+4(\cos x)^2} dx$$

$\Rightarrow \cos x = t$ then ;

$\Rightarrow -\sin(x)dx = dt$

$$= - \int \frac{dt}{3+4t^2} \left(\int \frac{dt}{a+bt^2} = \frac{1}{\sqrt{ab}} \tan^{-1} \sqrt{\frac{b}{a}} \right)$$

$$= - \frac{1}{2\sqrt{3}} \tan^{-1} \sqrt{\frac{4}{3}} t \text{ put } (\cos x = t)$$

$$\Rightarrow - \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2 \cos x}{\sqrt{3}} \right) + C$$

19. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{\sin x}{3+4\cos^2 x} dx$

A. $\log(3+4\cos^2 x) + C$

B. $\frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{\cos x}{\sqrt{3}} \right) + C$

C. $-\frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2 \cos x}{\sqrt{3}} \right) + C$

D. $\frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2 \cos x}{\sqrt{3}} \right) + C$

Answer

$$\int \frac{\sin x}{3+4(\cos x)^2} dx$$

$\Rightarrow \cos x = t$ then ;

$\Rightarrow -\sin(x)dx = dt$

$$= - \int \frac{dt}{3+4t^2} \left(\int \frac{dt}{a+bt^2} = \frac{1}{\sqrt{ab}} \tan^{-1} \sqrt{\frac{b}{a}} \right)$$

$$= - \frac{1}{2\sqrt{3}} \tan^{-1} \sqrt{\frac{4}{3}} t \text{ put } (\cos x = t)$$

$$\Rightarrow - \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2 \cos x}{\sqrt{3}} \right) + C$$

20. Question

Mark the correct alternative in each of the following:

Evaluate $\int e^x \left(\frac{1-\sin x}{1-\cos x} \right) dx$

A. $-e^x \tan \frac{x}{2} + C$

B. $-e^x \cot \frac{x}{2} + C$

C. $-\frac{1}{2}e^x \tan \frac{x}{2} + C$

D. $-\frac{1}{2}e^x \cot \frac{x}{2} + C$

Answer

$$\begin{aligned} \text{Given, } & \int e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx \\ &= - \int e^x \left(\frac{\sin x}{1 - \cos x} - \frac{1}{1 - \cos x} \right) dx \quad \{ \int e^x [f(x) + f'(x)] = e^x f(x) \} \\ \Rightarrow f(x) &= \frac{\sin x}{1 - \cos x}; f'(x) = - \frac{1}{1 - \cos x} \\ &= -e^x \left(\frac{\sin x}{1 - \cos x} \right) \\ \therefore \left[\frac{\sin x}{1 - \cos x} = \cot \frac{x}{2} \right] \\ &= -e^x \cot \frac{x}{2} + c \end{aligned}$$

20. Question

Mark the correct alternative in each of the following:

Evaluate $\int e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx$

A. $-e^x \tan \frac{x}{2} + C$

B. $-e^x \cot \frac{x}{2} + C$

C. $-\frac{1}{2}e^x \tan \frac{x}{2} + C$

D. $-\frac{1}{2}e^x \cot \frac{x}{2} + C$

Answer

$$\begin{aligned} \text{Given, } & \int e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx \\ &= - \int e^x \left(\frac{\sin x}{1 - \cos x} - \frac{1}{1 - \cos x} \right) dx \quad \{ \int e^x [f(x) + f'(x)] = e^x f(x) \} \\ \Rightarrow f(x) &= \frac{\sin x}{1 - \cos x}; f'(x) = - \frac{1}{1 - \cos x} \\ &= -e^x \left(\frac{\sin x}{1 - \cos x} \right) \\ \therefore \left[\frac{\sin x}{1 - \cos x} = \cot \frac{x}{2} \right] \end{aligned}$$

$$= -e^x \cot \frac{x}{2} + c$$

21. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{2}{(e^x + e^{-x})^2} dx$

A. $\frac{-e^{-x}}{e^x + e^{-x}} + C$

B. $-\frac{1}{e^x + e^{-x}} + C$

C. $\frac{-1}{(e^x + 1)^2} + C$

D. $\frac{1}{e^x - e^{-x}} + C$

Answer

Given $\int \frac{2}{(e^x + e^{-x})^2} dx$

$$= \int \frac{2e^{2x}}{(e^{2x} + 1)^2} dx$$

if $t = e^{2x} + 1$

; then $\frac{dt}{dx} = 2e^{2x}$

$$\Rightarrow \int \frac{dt}{t^2} = -\frac{1}{t} + c$$

$$\Rightarrow -\frac{1}{e^{2x} + 1} + c$$

$$= \frac{-e^{-x}}{e^x + e^{-x}} + C$$

21. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{2}{(e^x + e^{-x})^2} dx$

A. $\frac{-e^{-x}}{e^x + e^{-x}} + C$

B. $-\frac{1}{e^x + e^{-x}} + C$

$$C. \frac{-1}{(e^x + 1)^2} + C$$

$$D. \frac{1}{e^x - e^{-x}} + C$$

Answer

$$\text{Given } \int \frac{2}{(e^x + e^{-x})^2} dx$$

$$= \int \frac{2e^{2x}}{(e^{2x} + 1)^2} dx$$

$$\text{if } t = e^{2x} + 1$$

$$\text{; then } \frac{dt}{dx} = 2e^{2x}$$

$$\Rightarrow \int \frac{dt}{t^2} = -\frac{1}{t} + c$$

$$\Rightarrow -\frac{1}{e^{2x} + 1} + c$$

$$= \frac{-e^{-x}}{e^x + e^{-x}} + C$$

22. Question

Mark the correct alternative in each of the following:

$$\text{Evaluate } \int \frac{e^x (1+x)}{\cos^2(xe^x)} dx =$$

$$A. 2 \log_e \cos(xe^x) + C$$

$$B. \sec(xe^x) + C$$

$$C. \tan(xe^x) + C$$

$$D. \tan(x + e^x) + C$$

Answer

$$\text{let } (t) = xe^x;$$

$$\frac{dt}{dx} = e^x(1+x)$$

$$\Rightarrow \int \frac{dt}{(\cos t)^2} = \int (\sec t)^2 dt$$

$$= \tan t$$

$$(\text{put } (t) = xe^x)$$

$$= \tan(xe^x) + c$$

22. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{e^x(1+x)}{\cos^2(xe^x)} dx =$

A. $2 \log_e \cos(xe^x) + C$

B. $\sec(xe^x) + C$

C. $\tan(xe^x) + C$

D. $\tan(x + e^x) + C$

Answer

let $t = xe^x$;

$$\frac{dt}{dx} = e^x(1+x)$$

$$\Rightarrow \int \frac{dt}{(\cos t)^2} = \int (\sec t)^2 dt$$

$$= \tan t$$

(put $t = xe^x$)

$$= \tan(xe^x) + c$$

23. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{\sin^2 x}{\cos^4 x} dx =$

A. $\frac{1}{3} \tan^2 x + C$

B. $\frac{1}{2} \tan^2 x + C$

C. $\frac{1}{3} \tan^3 x + C$

D. none of these

Answer

$$I = \int (\tan x)^2 (\sec x)^2 dx$$

$$\Rightarrow \tan x = t \left[\frac{dt}{dx} = (\sec x)^2 \right]$$

$$\Rightarrow \int t^2 dt = \frac{t^3}{3} + c$$

$$\Rightarrow I = \frac{1}{3} (\tan x)^3 + c$$

23. Question

Mark the correct alternative in each of the following:

Evaluate $\int \frac{\sin^2 x}{\cos^4 x} dx =$

A. $\frac{1}{3} \tan^2 x + C$

B. $\frac{1}{2} \tan^2 x + C$

C. $\frac{1}{3} \tan^3 x + C$

D. none of these

Answer

$$I = \int (\tan x)^2 (\sec x)^2 dx$$

$$\Rightarrow \tan x = t \left[\frac{dt}{dx} = (\sec x)^2 \right]$$

$$\Rightarrow \int t^2 dt = \frac{t^3}{3} + c$$

$$\Rightarrow I = \frac{1}{3} (\tan x)^3 + c$$

24. Question

Mark the correct alternative in each of the following:

The primitive of the function $f(x) = \left(1 - \frac{1}{x^2}\right) a^{x + \frac{1}{x}}$, $a > 0$ is

A. $\frac{a^{x + \frac{1}{x}}}{\log_e a}$

B. $\log_e a \cdot a^{x + \frac{1}{x}}$

C. $\frac{a^{x + \frac{1}{x}}}{x} \log_e a$

D. $\frac{a^{x + \frac{1}{x}}}{x \log_e a}$

Answer

$$I = \int \left(1 - \frac{1}{x^2}\right) a^{x + \frac{1}{x}} dx$$

$$\Rightarrow \text{let } x + \frac{1}{x} = t;$$

$$1 - \frac{1}{x^2} = \frac{dt}{dx}$$

$$= \int a^t dt$$

$$\Rightarrow I = \frac{a^t}{\log_e a} \left(\text{put } t = x + \frac{1}{x} \right)$$

$$\Rightarrow I = \frac{a^{x+\frac{1}{x}}}{\log_e a} + C$$

24. Question

Mark the correct alternative in each of the following:

The primitive of the function $f(x) = \left(1 - \frac{1}{x^2}\right) a^{x+\frac{1}{x}}$, $a > 0$ is

A. $\frac{a^{x+\frac{1}{x}}}{\log_e a}$

B. $\log_e a \cdot a^{x+\frac{1}{x}}$

C. $\frac{a^{x+\frac{1}{x}}}{x} \log_e a$

D. $x \frac{a^{x+\frac{1}{x}}}{\log_e a}$

Answer

$$I = \int \left(1 - \frac{1}{x^2}\right) a^{x+\frac{1}{x}} dx$$

$$\Rightarrow \text{let } x + \frac{1}{x} = t;$$

$$1 - \frac{1}{x^2} = \frac{dt}{dx}$$

$$= \int a^t dt$$

$$\Rightarrow I = \frac{a^t}{\log_e a} \left(\text{put } t = x + \frac{1}{x} \right)$$

$$\Rightarrow I = \frac{a^{x+\frac{1}{x}}}{\log_e a} + C$$

25. Question

Mark the correct alternative in each of the following:

The value of $\int \frac{1}{x + x \log x} dx$ is

A. $1 + \log x$

B. $x + \log x$

C. $x \log(1 + \log x)$

D. $\log(1 + \log x)$

Answer

$$I = \int \frac{1}{x(1 + \log_e x)} dx$$

$$\Rightarrow \text{let } (1 + \log_e x) = t \left[\frac{dt}{dx} = \frac{1}{x} \right]$$

$$\Rightarrow \int \frac{1}{t} dt = \log_e t$$

$$\Rightarrow I = \log(1 + \log x) + C$$

25. Question

Mark the correct alternative in each of the following:

The value of $\int \frac{1}{x + x \log x} dx$ is

A. $1 + \log x$

B. $x + \log x$

C. $x \log(1 + \log x)$

D. $\log(1 + \log x)$

Answer

$$I = \int \frac{1}{x(1 + \log_e x)} dx$$

$$\Rightarrow \text{let } (1 + \log_e x) = t \left[\frac{dt}{dx} = \frac{1}{x} \right]$$

$$\Rightarrow \int \frac{1}{t} dt = \log_e t$$

$$\Rightarrow I = \log(1 + \log x) + C$$

26. Question

Mark the correct alternative in each of the following:

$\int \sqrt{\frac{x}{1-x}} dx$ is equal to

A. $\sin^{-1} \sqrt{x} + C$

B. $\sin^{-1} \left(\sqrt{x} - \sqrt{x(1-x)} \right) + C$

C. $\sin^{-1} \left\{ \sqrt{x(1-x)} \right\} + C$

D. $\sin^{-1} \sqrt{x} - \sqrt{x(1-x)} + C$

Answer

$$\text{let } x = (\sin t)^2; (dx = 2 \sin t \cos t dt)$$

$$I = \int \sqrt{\frac{(\sin t)^2}{1 - (\sin t)^2}} \times 2 \sin t \cos t dt$$

$$I = \int (\sin t)^2 dt$$

$$I = \int (1 - \cos 2t) dt$$

$$I = \int 1 dt - \int \cos 2t dt$$

$$I = t - \frac{\sin 2t}{2} + c \quad [t = \sin^{-1} \sqrt{x}] (\cos t = \sqrt{1-x})$$

$$I = \sin^{-1}(\sqrt{x}) - (\sqrt{x}\sqrt{1-x}) + c$$

26. Question

Mark the correct alternative in each of the following:

$$\int \sqrt{\frac{x}{1-x}} dx \text{ is equal to}$$

A. $\sin^{-1} \sqrt{x} + C$

B. $\sin^{-1}(\sqrt{x} - \sqrt{x(1-x)}) + C$

C. $\sin^{-1} \{ \sqrt{x(1-x)} \} + C$

D. $\sin^{-1} \sqrt{x} - \sqrt{x(1-x)} + C$

Answer

$$\text{let } x = (\sin t)^2; (dx = 2 \sin t \cos t dt)$$

$$I = \int \sqrt{\frac{(\sin t)^2}{1 - (\sin t)^2}} \times 2 \sin t \cos t dt$$

$$I = \int (\sin t)^2 dt$$

$$I = \int (1 - \cos 2t) dt$$

$$I = \int 1 dt - \int \cos 2t dt$$

$$I = t - \frac{\sin 2t}{2} + c \quad [t = \sin^{-1} \sqrt{x}] (\cos t = \sqrt{1-x})$$

$$I = \sin^{-1}(\sqrt{x}) - (\sqrt{x}\sqrt{1-x}) + c$$

27. Question

Mark the correct alternative in each of the following:

$$\int e^x \{f(x) + f'(x)\} dx =$$

A. $e^x f(x) + C$

B. $e^x + f(x) + C$

C. $2e^x f(x) + C$

D. $e^x - f(x) + C$

Answer

$$\text{let } I = \int e^x (f(x) + f'(x)) dx$$

Open the brackets, we get

$$I = \{ \int e^x f(x) dx + \int e^x f'(x) dx \}$$

$$= U + \int e^x f'(x) dx$$

$$U = \int e^x f(x) dx$$

To solve U using integration by parts

$$U = f(x) \int e^x dx - \int [f'(x) \int e^x]$$

$$= f(x) e^x - \int f'(x) e^x$$

$$= U + \int e^x f'(x) dx$$

$$I = e^x f(x) + \int f'(x) e^x dx - \int e^x f'(x) dx$$

$$I = e^x f(x) + C$$

27. Question

Mark the correct alternative in each of the following:

$$\int e^x \{f(x) + f'(x)\} dx =$$

A. $e^x f(x) + C$

B. $e^x + f(x) + C$

C. $2e^x f(x) + C$

D. $e^x - f(x) + C$

Answer

$$\text{let } I = \int e^x (f(x) + f'(x)) dx$$

Open the brackets, we get

$$I = \left\{ \int e^x f(x) dx + \int e^x f'(x) dx \right\}$$

$$= U + \int e^x f'(x) dx$$

$$U = \int e^x f(x) dx$$

To solve U using integration by parts

$$U = f(x) \int e^x dx - \int [f'(x) \int e^x]$$

$$= f(x) e^x - \int f'(x) e^x$$

$$= U + \int e^x f'(x) dx$$

$$I = e^x f(x) + \int f'(x) e^x dx - \int e^x f'(x) dx$$

$$I = e^x f(x) + C$$

28. Question

Mark the correct alternative in each of the following:

$$\text{The value of } \int \frac{\sin x + \cos x}{\sqrt{1 - \sin 2x}} dx \text{ is equal to}$$

A. $\sqrt{\sin 2x} + C$

B. $\sqrt{\cos 2x} + C$

C. $\pm (\sin x - \cos x) + C$

D. $\pm \log (\sin x - \cos x) + C$

Answer

$$I = \int \frac{\sin x + \cos x}{\sin x - \cos x} dx \quad (\sqrt{1 - \sin 2x} = \pm\{\sin x - \cos x\})$$

$$\text{Let } t = \sin x - \cos x \quad \left(\frac{dt}{dx} = \sin x + \cos x \right)$$

$$I = \int \frac{dt}{t}$$

$$I = \pm \log(\sin x - \cos x) + c$$

28. Question

Mark the correct alternative in each of the following:

The value of $\int \frac{\sin x + \cos x}{\sqrt{1 - \sin 2x}} dx$ is equal to

A. $\sqrt{\sin 2x} + C$

B. $\sqrt{\cos 2x} + C$

C. $\pm (\sin x - \cos x) + C$

D. $\pm \log (\sin x - \cos x) + C$

Answer

$$I = \int \frac{\sin x + \cos x}{\sin x - \cos x} dx \quad (\sqrt{1 - \sin 2x} = \pm\{\sin x - \cos x\})$$

$$\text{Let } t = \sin x - \cos x \quad \left(\frac{dt}{dx} = \sin x + \cos x \right)$$

$$I = \int \frac{dt}{t}$$

$$I = \pm \log(\sin x - \cos x) + c$$

29. Question

Mark the correct alternative in each of the following:

If $\int x \sin x \, dx = -x \cos x + \alpha$, then α is equal to

A. $\sin x + C$

B. $\cos x + C$

C. C

D. none of these

Answer

using integration by parts

$$I = \int x \sin x \, dx$$

$$= x \int \sin x \, dx - \int \frac{dx}{dx} (x) \int \sin x$$

$$I = x \cos x + \int \cos x \, dx$$

$$(\because \int \sin x = -\cos x)$$

$$= x \cos x + \sin x + c$$

29. Question

Mark the correct alternative in each of the following:

If $\int x \sin x \, dx = -x \cos x + \alpha$, then α is equal to

- A. $\sin x + C$
- B. $\cos x + C$
- C. C
- D. none of these

Answer

using integration by parts

$$I = \int x \sin x \, dx$$

$$= x \int \sin x \, dx - \int \frac{dx}{dx} (x) \int \sin x$$

$$I = x \cos x + \int \cos x \, dx$$

$$(\because \int \sin x = -\cos x)$$

$$= x \cos x + \sin x + c$$

30. Question

Mark the correct alternative in each of the following:

$$\int \frac{\cos 2x - 1}{\cos 2x + 1} dx =$$

- A. $\tan x - x + C$
- B. $x + \tan x + C$
- C. $x - \tan x + C$
- D. $-x - \cot x + C$

Answer

$$I = \int \frac{1 - 2(\sin x)^2 - 1}{2(\cos x)^2 - 1 + 1}$$

$$I = - \int \frac{(\sin x)^2}{(\cos x)^2} dx$$

$$I = - \int (\tan x)^2 dx$$

$$I = - \int (-1 + (\sec x)^2) dx$$

$$= (x - \tan x) + c$$

30. Question

Mark the correct alternative in each of the following:

$$\int \frac{\cos 2x - 1}{\cos 2x + 1} dx =$$

- A. $\tan x - x + C$
- B. $x + \tan x + C$
- C. $x - \tan x + C$
- D. $-x - \cot x + C$

Answer

$$I = \int \frac{1-2(\sin x)^2-1}{2(\cos x)^2-1+1}$$

$$I = - \int \frac{(\sin x)^2}{(\cos x)^2} dx$$

$$I = - \int (\tan x)^2 dx$$

$$I = - \int (-1 + (\sec x)^2) dx$$

$$= (x - \tan x) + c$$

31. Question

Mark the correct alternative in each of the following:

$$\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} dx \text{ is equal to}$$

A. $2(\sin x + x \cos \theta) + C$

B. $2(\sin x - x \cos \theta) + C$

C. $2(\sin x + 2x \cos \theta) + C$

D. $2(\sin x - 2x \cos \theta) + C$

Answer

$$I = \int \frac{\{2(\cos x)^2 - 1\} - \{2(\cos \theta)^2 - 1\}}{\cos x - \cos \theta} dx$$

$$I = 2 \int \frac{(\cos x)^2 - (\cos \theta)^2}{\cos x - \cos \theta} dx$$

$$I = 2 \int \frac{(\cos x - \cos \theta)(\cos x + \cos \theta)}{\cos x - \cos \theta} dx$$

$$I = 2 \int (\cos x + \cos \theta) dx$$

$$I = 2(\sin x + x \cos \theta) + c$$

31. Question

Mark the correct alternative in each of the following:

$$\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} dx \text{ is equal to}$$

A. $2(\sin x + x \cos \theta) + C$

B. $2(\sin x - x \cos \theta) + C$

C. $2(\sin x + 2x \cos \theta) + C$

D. $2(\sin x - 2x \cos \theta) + C$

Answer

$$I = \int \frac{\{2(\cos x)^2 - 1\} - \{2(\cos \theta)^2 - 1\}}{\cos x - \cos \theta} dx$$

$$I = 2 \int \frac{(\cos x)^2 - (\cos \theta)^2}{\cos x - \cos \theta} dx$$

$$I = 2 \int \frac{(\cos x - \cos \theta)(\cos x + \cos \theta)}{\cos x - \cos \theta} dx$$

$$I = 2 \int (\cos x + \cos \theta) dx$$

$$I = 2(\sin x + x \cos \theta) + c$$

32. Question

Mark the correct alternative in each of the following:

$$\int \frac{x^9}{(4x^2 + 1)^6} dx \text{ is equal to}$$

A. $\frac{1}{5x} \left(4 + \frac{1}{x^2} \right)^{-5} + C$

B. $\frac{1}{5} \left(4 + \frac{1}{x^2} \right)^{-5} + C$

C. $\frac{1}{10x} \left(\frac{1}{x^2} + 4 \right)^{-5} + C$

D. $\frac{1}{10} \left(\frac{1}{x^2} + 4 \right)^{-5} + C$

Answer

$$I = \int \frac{x^9}{(4x^2 + 1)^6} dx$$

$$I = \int \frac{x^9}{x^{12} \left(4 + \frac{1}{x^2} \right)^6} dx$$

$$I = \int \frac{1}{x^3 \left(4 + \frac{1}{x^2} \right)^6} dx$$

$$\text{Let } \left(4 + \frac{1}{x^2} \right) = t ; \frac{-2}{x^3} dx = dt$$

$$I = \int \frac{dt}{-2t^6}$$

$$I = \frac{1}{10} \left[\frac{1}{t^5} \right]$$

$$I = \frac{1}{10} \left(\left[4 + \frac{1}{x^2} \right]^{-5} \right) + c$$

32. Question

Mark the correct alternative in each of the following:

$$\int \frac{x^9}{(4x^2 + 1)^6} dx \text{ is equal to}$$

A. $\frac{1}{5x} \left(4 + \frac{1}{x^2} \right)^{-5} + C$

B. $\frac{1}{5} \left(4 + \frac{1}{x^2} \right)^{-5} + C$

$$C. \frac{1}{10x} \left(\frac{1}{x^2} + 4 \right)^{-5} + C$$

$$D. \frac{1}{10} \left(\frac{1}{x^2} + 4 \right)^{-5} + C$$

Answer

$$I = \int \frac{x^9}{(4x^2+1)^6} dx$$

$$I = \int \frac{x^9}{x^{12} \left(4 + \frac{1}{x^2} \right)^6} dx$$

$$I = \int \frac{1}{x^3 \left(4 + \frac{1}{x^2} \right)^6} dx$$

$$\text{Let } \left(4 + \frac{1}{x^2} \right) = t ; \frac{-2}{x^3} dx = dt$$

$$I = \int \frac{dt}{-2t^6}$$

$$I = \frac{1}{10} \left[\frac{1}{t^5} \right]$$

$$I = \frac{1}{10} \left(\left[4 + \frac{1}{x^2} \right]^{-5} \right) + c$$

33. Question

Mark the correct alternative in each of the following:

$$\int \frac{x^3}{\sqrt{1+x^2}} dx = a(1+x^2)^{3/2} + b\sqrt{1+x^2} + C, \text{ then}$$

$$A. a = \frac{1}{3}, b = 1$$

$$B. a = -\frac{1}{3}, b = 1$$

$$C. a = -\frac{1}{3}, b = -1$$

$$D. a = \frac{1}{3}, b = -1$$

Answer

$$\text{let } (\sqrt{1+x^2}) = t$$

$$\frac{x}{\sqrt{1+x^2}} dx = dt;$$

$$I = \int \frac{x^3}{\sqrt{1+x^2}} dx = \int x^2 dt = \int (t^2 - 1) dt$$

$$I = \frac{t^3}{3} - t [\text{put}(t) = \sqrt{1+x^2}]$$

$$I = \frac{(1+x^2)^{\frac{3}{2}}}{3} - \sqrt{1+x^2} + C$$

$$[a = \frac{1}{3}]; [b = -1]$$

33. Question

Mark the correct alternative in each of the following:

$$\int \frac{x^3}{\sqrt{1+x^2}} dx = a(1+x^2)^{3/2} + b\sqrt{1+x^2} + C, \text{ then}$$

A. $a = \frac{1}{3}, b = 1$

B. $a = -\frac{1}{3}, b = 1$

C. $a = -\frac{1}{3}, b = -1$

D. $a = \frac{1}{3}, b = -1$

Answer

$$\text{let } (\sqrt{1+x^2}) = t$$

$$\frac{x}{\sqrt{1+x^2}} dx = dt;$$

$$I = \int \frac{x^3}{\sqrt{1+x^2}} dx = \int x^2 dt = \int (t^2 - 1) dt$$

$$I = \frac{t^3}{3} - t [put(t) = \sqrt{1+x^2}]$$

$$I = \frac{(1+x^2)^{\frac{3}{2}}}{3} - \sqrt{1+x^2} + C$$

$$[a = \frac{1}{3}]; [b = -1]$$

34. Question

Mark the correct alternative in each of the following:

$$\int \frac{x^3}{x+1} dx$$

A. $x + \frac{x^2}{2} + \frac{x^3}{3} - \log|1-x| + C$

B. $x + \frac{x^2}{2} - \frac{x^3}{3} - \log|1-x| + C$

C. $x - \frac{x^2}{2} - \frac{x^3}{3} - \log|1+x| + C$

$$D. x - \frac{x^2}{2} + \frac{x^3}{3} - \log|1+x| + C$$

Answer

$$\begin{aligned} &= \int \frac{x^3+1}{x+1} dx - \int \frac{1}{x+1} dx \\ &= \int \frac{(x+1)(x^2-x+1)}{x+1} dx - \int \frac{1}{x+1} dx \\ &= \int (x^2-x+1) dx - \int \frac{1}{x+1} dx \\ &= \frac{x^3}{3} - \frac{x^2}{2} + x - \log(1+x) + c \end{aligned}$$

34. Question

Mark the correct alternative in each of the following:

$$\int \frac{x^3}{x+1} dx$$

$$A. x + \frac{x^2}{2} + \frac{x^3}{3} - \log|1-x| + C$$

$$B. x + \frac{x^2}{2} - \frac{x^3}{3} - \log|1-x| + C$$

$$C. x - \frac{x^2}{2} - \frac{x^3}{3} - \log|1+x| + C$$

$$D. x - \frac{x^2}{2} + \frac{x^3}{3} - \log|1+x| + C$$

Answer

$$\begin{aligned} &= \int \frac{x^3+1}{x+1} dx - \int \frac{1}{x+1} dx \\ &= \int \frac{(x+1)(x^2-x+1)}{x+1} dx - \int \frac{1}{x+1} dx \\ &= \int (x^2-x+1) dx - \int \frac{1}{x+1} dx \\ &= \frac{x^3}{3} - \frac{x^2}{2} + x - \log(1+x) + c \end{aligned}$$

35. Question

Mark the correct alternative in each of the following:

$$\text{If } \int \frac{1}{(x+2)(x^2+1)} dx = a \log|1+x^2| + b \tan^{-1}$$

$$x + \frac{1}{5} \log|x+2| + C, \text{ then}$$

A. $a = -\frac{1}{10}, b = -\frac{2}{5}$

B. $a = \frac{1}{10}, b = -\frac{2}{5}$

C. $a = -\frac{1}{10}, b = \frac{2}{5}$

D. $a = \frac{1}{10}, b = \frac{2}{5}$

Answer

$$U = \int \frac{1}{(x+2)(x^2+1)} dx$$

$$U = \int \frac{A}{x+2} dx + \int \frac{Bx+C}{x^2+1} dx$$

$$\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} \text{ (compare coefficient of } x^2, \text{ and } x \text{ both side)}$$

$$\left[A = \frac{1}{5}; B = -\frac{1}{5}; C = \frac{2}{5} \right] \text{ put the value of A,B,C in U}$$

$$U = \int \frac{\frac{1}{5}}{x+2} dx + \int \frac{-\frac{1}{5}x + \frac{2}{5}}{x^2+1} dx$$

$$U = \frac{1}{5} \left[\int \frac{1}{x+2} dx + \int \frac{-x}{x^2+1} dx + \int \frac{2}{x^2+1} dx \right]$$

$$U = \frac{1}{5} \left[\log(X+2) - \frac{1}{2} \log(x^2+1) + 2 \tan^{-1} X \right] + C$$

35. Question

Mark the correct alternative in each of the following:

If $\int \frac{1}{(x+2)(x^2+1)} dx = a \log |1+x^2| + b \tan^{-1} x + \frac{1}{5} \log |x+2| + C$, then

$x + \frac{1}{5} \log |x+2| + C$, then

A. $a = -\frac{1}{10}, b = -\frac{2}{5}$

B. $a = \frac{1}{10}, b = -\frac{2}{5}$

C. $a = -\frac{1}{10}, b = \frac{2}{5}$

D. $a = \frac{1}{10}, b = \frac{2}{5}$

Answer

$$U = \int \frac{1}{(x+2)(x^2+1)} dx$$

$$U = \int \frac{A}{x+2} dx + \int \frac{Bx+C}{x^2+1} dx$$

$$\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} \text{ (compare coefficient of } x^2, \text{ and } x \text{ both side)}$$

$$\left[A = \frac{1}{5}; B = -\frac{1}{5}; C = \frac{2}{5} \right] \text{ put the value of A,B,C in U}$$

$$U = \int \frac{\frac{1}{5}}{x+2} dx + \int \frac{-\frac{1}{5}x + \frac{2}{5}}{x^2+1} dx$$

$$U = \frac{1}{5} \left[\int \frac{1}{x+2} dx + \int \frac{-x}{x^2+1} dx + \int \frac{2}{x^2+1} dx \right]$$

$$U = \frac{1}{5} \left[\log(x+2) - \frac{1}{2} \log(x^2+1) + 2 \tan^{-1} x \right] + C$$

Revision exercise

106. Question

$$\int \frac{1}{x\sqrt{1+x^2}} dx$$

Answer

$$\text{Let } x = \sin^{\frac{2}{3}} t$$

Differentiate both side with respect to t

$$\frac{dx}{dt} = \frac{2}{3} \sin^{-\frac{1}{3}} t \cos t \Rightarrow dx = \frac{2}{3} \sin^{-\frac{1}{3}} t \cos t dt$$

$$y = \int \frac{1}{\sin^{\frac{2}{3}} t \sqrt{1 + \sin^2 t}} \cdot \frac{2}{3} \sin^{-\frac{1}{3}} t \cos t dt$$

$$y = \frac{2}{3} \int \operatorname{cosec} t dt$$

$$y = \frac{2}{3} \ln(\operatorname{cosec} t - \cot t) + c$$

$$\text{Again, put } t = \sin^{-1} x^{\frac{3}{2}}$$

$$y = \frac{2}{3} \ln(\operatorname{cosec} \sin^{-1} x^{\frac{3}{2}} - \cot \sin^{-1} x^{\frac{3}{2}}) + c$$

$$y = \frac{2}{3} \ln \left(x^{-\frac{3}{2}} - \frac{\sqrt{1-x^3}}{x^{\frac{3}{2}}} \right) + c$$

$$y = -\ln x + \frac{2}{3} \ln(1 - \sqrt{1-x^3}) + c$$

107. Question

$$\text{Evaluate } \int \frac{\sin x + \cos x}{\sin^4 x + \cos^4 x} dx$$

Answer

$$\int \frac{(\sin x + \cos x)}{\sin^4 x + \cos^4 x} dx$$

$$= \int \frac{(\sin x + \cos x)}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x} dx$$

$$= \int \frac{(\sin x + \cos x)}{1 - 2\sin^2 x \cos^2 x} dx$$

$$= \int \frac{2(\sin x + \cos x)}{2 - 4\sin^2 x \cos^2 x} dx$$

$$= \int \frac{2(\sin x + \cos x)}{2 - \sin^2 2x} dx$$

Let $\sin x - \cos x = t$,

$$(\cos x + \sin x) dx = dt$$

$$= \int \frac{2}{2 - (1 - t^2)^2} dt$$

$$= \int \frac{2}{(\sqrt{2} - 1 + t^2)(\sqrt{2} + 1 - t^2)} dt$$

$$= \frac{1}{\sqrt{2}} \int \left(\frac{1}{(\sqrt{2} + 1 + t^2)} - \frac{1}{(\sqrt{2} - 1 - t^2)} \right) dt$$

$$= \frac{1}{\sqrt{2}} \int \left(\frac{1}{(\sqrt{2} + 1 + t^2)} \right) dt - \frac{1}{\sqrt{2}} \int \left(\frac{1}{(\sqrt{2} - 1 - t^2)} \right) dt$$

$$= \frac{1}{\sqrt{2}} \int \left(\frac{1}{((\sqrt{2} + 1))^2 + t^2} \right) dt - \frac{1}{\sqrt{2}} \int \left(\frac{1}{((\sqrt{2} - 1))^2 - t^2} \right) dt$$

$$= \frac{1}{\sqrt{2}} \left[\frac{1}{2\sqrt{2} + 1} \log \left| \frac{t - \sqrt{\sqrt{2} + 1}}{t + \sqrt{\sqrt{2} + 1}} \right| \right] - \frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{\sqrt{2} - 1}} \tan^{-1} \left(\frac{t}{\sqrt{\sqrt{2} - 1}} \right) \right] + c$$

108. Question

Evaluate $\int x^2 \tan^{-1} x \, dx$

Answer

$$\int x^2 \tan^{-1} x \, dx$$

Here we will use integration by parts,

$$\int u \, dv = uv - \int v \, du$$

Choose u in these order LIATE (L-LOGS, I-INVERSE, A-ALGEBRAIC, T-TRIG, E-EXPONENTIAL)

So here, $u = \tan^{-1} x$

$$= \tan^{-1} x \int x^2 dx - \frac{1}{3} \int x^3 (d(\tan^{-1} x)) / \dots\dots\dots (dx + c$$

$$\int x^2 dx = \left(\frac{x^3}{3} \right) + c$$

$$= \left(\frac{x^3}{3} \right) \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1 + x^2} dx$$

Putting $1 + x^2 = t$,

$$2x dx = dt,$$

$$\begin{aligned}
 x \, dx &= \frac{dt}{2} \\
 &= \left(\frac{x^3}{3}\right) \tan^{-1} x - \frac{1}{3} \int \frac{xx^2}{1+x^2} dx \\
 &= \left(\frac{x^3}{3}\right) \tan^{-1} x - \frac{1}{3} \int \frac{(t-1)}{t} \frac{dt}{2} \\
 &= \left(\frac{x^3}{3}\right) \tan^{-1} x - \frac{1}{6} \int \frac{(t-1)}{t} dt \\
 &= \left(\frac{x^3}{3}\right) \tan^{-1} x - \frac{1}{6} \left[\int 1 \, dt - \int \frac{1}{t} dt \right] \\
 &= \left(\frac{x^3}{3}\right) \tan^{-1} x - \frac{1}{6} [-\log t + t] + c
 \end{aligned}$$

Resubstituting t

$$= \left(\frac{x^3}{3}\right) \tan^{-1} x - \frac{1}{6} [-\log(1+x^2) + (1+x^2)] + c$$

109. Question

Evaluate $\int \tan^{-1} \sqrt{x} \, dx$

Answer

$$\int \tan^{-1} \sqrt{x} \, dx$$

$$\int u \cdot dv = uv - \int v \, du$$

Choose u in these order

LIATE(L-LOGS,I-INVERSE,A-ALGEBRAIC,T-TRIG,E-EXPONENTIAL)

Here $u = \tan^{-1} \sqrt{x}$ and $v = 1$.

$$\therefore \int \tan^{-1} \sqrt{x} \, dx$$

$$\therefore x \tan^{-1} \sqrt{x} - \int x \cdot \frac{d(\tan^{-1} \sqrt{x})}{dx}$$

$$= x \tan^{-1} \sqrt{x} - \frac{1}{2} \int \frac{\sqrt{x}}{1+x} dx$$

Put $\sqrt{x} = t$;

$$\frac{1}{2\sqrt{x}} dx = dt;$$

$$dx = 2t \, dt$$

and $x = t^2$

$$= x \tan^{-1} \sqrt{x} - \int \frac{t^2}{1+t^2} dt$$

$$= x \tan^{-1} \sqrt{x} - \left[\int \frac{1+t^2}{1+t^2} dt - \int \frac{1}{1+t^2} dt \right]$$

$$= x \tan^{-1} \sqrt{x} - [\sqrt{x} - \tan^{-1} \sqrt{x}] + c$$

110. Question

Evaluate $\int \sin^{-1} \sqrt{x} \, dx$

Answer

$$\int \sin^{-1} \sqrt{x} \, dx$$

$$\int u \, dv = uv - \int v \, du$$

Choose u in these order LIATE(L-LOGS,I-INVERSE,A-ALGEBRAIC,T-TRIG,E-EXPONENTIAL)

$$u = \sin^{-1} \sqrt{x} \quad v = 1$$

$$\therefore \int \sin^{-1} \sqrt{x} = x \cdot \sin^{-1} \sqrt{x} - \frac{1}{2} \int \frac{\sqrt{x}}{\sqrt{1-x}} \, dx$$

$$\text{Put } \sqrt{x} = t;$$

$$dx = 2t \, dt$$

$$= x \cdot \sin^{-1} \sqrt{x} - \int \frac{t^2}{\sqrt{1-t^2}} \, dt$$

$$\text{Now put } t = \sin u;$$

$$dt = \cos u \, du;$$

$$\sqrt{1-t^2} = \sqrt{1-\sin^2 u}$$

$$= \cos u$$

$$= x \cdot \sin^{-1} \sqrt{x} - \int \frac{\sin^2 u \cos u \, du}{\sqrt{1-\sin^2 u}}$$

$$= x \cdot \sin^{-1} \sqrt{x} - \int \frac{\sin^2 u \cos u \, du}{\cos u}$$

$$= x \cdot \sin^{-1} \sqrt{x} - \int \sin^2 u \, du \dots (\text{Here we can substitute } \sin^2 u = (1 - \cos 2u)/2)$$

$$= x \cdot \sin^{-1} \sqrt{x} - \int \frac{1 - \cos 2u}{2} \, du$$

$$= x \cdot \sin^{-1} \sqrt{x} - \left[\int \frac{1 - \cos 2u}{2} \, du \right]$$

$$= x \cdot \sin^{-1} \sqrt{x} - \left[\frac{u}{2} - \frac{1}{4} \sin 2u \right] + c$$

$$\text{Put } u = \sin^{-1} \sqrt{x},$$

$$I = x \cdot \sin^{-1} \sqrt{x} - \left[\frac{\sin^{-1} \sqrt{x}}{2} - \frac{\sqrt{x} \sqrt{(1-x)}}{2} \right] + c$$

111. Question

Evaluate $\int \sec^{-1} \sqrt{x} \, dx$

Answer

$$\int \sec^{-1} \sqrt{x} \, dx$$

$$\int u \, dv = uv - \int v \, du$$

Choose u in these order LIATE(L-LOGS,I-INVERSE,A-ALGEBRAIC,T-TRIG,E-EXPONENTIAL)

Here $u = \sec^{-1} \sqrt{x}$ and $v=1$.

$$\begin{aligned} \int \sec^{-1} \sqrt{x} dx &= x \sec^{-1} x - \int \frac{x dx}{2x\sqrt{x-1}} \\ &= x \sec^{-1} x - \int \frac{dx}{2\sqrt{x-1}} \end{aligned}$$

Put $x-1=t$ $dx=dt$

$$\begin{aligned} &= x \sec^{-1} x - \int \frac{dt}{2\sqrt{t}} \\ &= x \sec^{-1} x - \frac{2}{2}(\sqrt{t}) + c \\ &= x \sec^{-1} x - (\sqrt{x-1}) + c \end{aligned}$$

112. Question

Evaluate $\int \tan^{-1} \sqrt{\frac{1-x}{1+x}} dx$

Answer

Put $x=\cos 2t$; $dx=-2\sin 2t$

$$\begin{aligned} &= \int \tan^{-1} \sqrt{\frac{1-x}{1+x}} dx = \int \tan^{-1} \sqrt{\frac{1-\cos 2t}{1+\cos 2t}} (-2\sin 2t) dt \\ &= \int \tan^{-1} \sqrt{\frac{1-\cos 2t}{1+\cos 2t}} (-2\sin 2t) dt \\ &= -2 \int \tan^{-1} \tan t \sin 2t dt \\ &= -2 \int t \sin 2t dt \\ &= -2 \left[-\frac{t \cos 2t}{2} + \frac{1}{2} \int \cos 2t dt \right] \\ &= t \cos 2t - \frac{\sin 2t}{2} + c \\ &= \frac{x \cos^{-1} x}{2} - \frac{\sqrt{1-x^2}}{2} + c \end{aligned}$$

113. Question

Evaluate $\int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$

Answer

$$\int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$$

Put $x=a \tan^2 t$; $dx=2a \cdot \tan t \cdot \sec^2 t dt$

$$\begin{aligned}
&= \int \sin^{-1} \sqrt{\frac{x}{a+x}} dx = \int \sin^{-1} \sqrt{\frac{a \tan^2 t}{a + a \tan^2 t}} 2a \tan t \sec^2 t dt = \int t \cdot 2a \tan t \sec^2 t dt \\
&= 2a \int t \tan t \sec^2 t dt \\
&= 2a \left[\frac{t(\tan^2 t)}{2} - \int \frac{\tan^2 t}{2} dt \right] + c \\
&= 2a \left[\frac{t(\tan^2 t)}{2} - \frac{\tan t}{2} + \frac{t}{2} \right] + c \\
&= a[t(\tan^2 t) - \tan t + t] + c \\
&= x \tan^{-1} \sqrt{\frac{x}{a}} - \sqrt{ax} + a \tan^{-1} \sqrt{\frac{x}{a}} + c.
\end{aligned}$$

114. Question

Evaluate $\int \sin^{-1}(3x - 4x^3) dx$

Answer

Put $x = \sin t$; $dx = \cos t dt$

$$\begin{aligned}
\int \sin^{-1}(3x - 4x^3) dx &= \int \sin^{-1}(3\sin t - 4\sin^3 t) \cos t dt \dots \dots (3\sin t - 4\sin^3 t) = \sin 3t. \\
&= \int \sin^{-1}(\sin 3t) \cos t dt = \int 3t \cos t dt \\
&= 3 \int t \cos t dt
\end{aligned}$$

By by parts,

$$\begin{aligned}
&= 3[t \sin t - \int \sin t dt] + c \\
&= 3[t \sin t + \cos t] + c \\
&= 3x \sin^{-1} x + 3\sqrt{1-x^2} + c.
\end{aligned}$$

115. Question

Evaluate $\int (\sin^{-1} x)^3 dx$

Answer

$$\int (\sin^{-1} x)^3 dx$$

Put $x = \sin t$;

$dx = \cos t dt$

$$\begin{aligned}
\int (\sin^{-1} x)^3 dx &= \int (\sin^{-1}(\sin t))^3 \cos t dt \\
\int t^3 \cos t dt &= [t^3 \sin t - 3 \int t^2 \sin t dt] = [t^3 \sin t - 3[-t^2 \cos t + 2 \int t \cos t dt]] \\
&= \left[t^3 \sin t + 3t^2 \cos t - 6 \int t \cos t dt \right] = [t^3 \sin t + 3t^2 \cos t - 6[t \sin t + \cos t]] + c
\end{aligned}$$

$$= [t^3 \sin t + 3t^2 \cos t - 6t \cos t - 6 \cos t] + c$$

$$= [(\sin^{-1} x)^3 x + 3(\sin^{-1} x)^2 \sqrt{1-x^2} - 6x \sin^{-1} x - 6\sqrt{1-x^2}] + c$$

116. Question

Evaluate $\int \cos^{-1}(1-2x^2) dx$

Answer

Put $x = \sin t$

$$; dx = \cos t dt;$$

$$\int \cos^{-1}(1-2x^2) dx = \int \cos^{-1}(1-2\sin^2 t) \cos t dt = \int \cos^{-1}(1-\sin^2 t - \sin^2 t) \cos t dt$$

$$\int \cos^{-1}(\cos^2 t - \sin^2 t) \cos t dt = \int \cos^{-1}(\cos 2t) \cos t dt$$

$$2 \int t \cos t dt = 2[ts \sin t + \cos t] + c$$

$$Ans = 2x \sin^{-1} x + 2\sqrt{1-x^2} + c$$

117. Question

Evaluate $\int \frac{x \sin^{-1} x}{(1-x^2)^{3/2}} dx$

Answer

$$\int \frac{x \sin^{-1} x}{(1-x^2)\sqrt{1-x^2}} dx$$

we can put $\sin^{-1} x = t; dx/(1-x^2)^{1/2} = dt; (1-x^2) = \cos^2 t$ and $x = \sin t$.

$$\int \frac{t \sin t}{\cos^2 t} dt = \int t \tan t \sec t dt$$

By by parts,

$$\int t \tan t \sec t dt = t \sec t - \int \sec t dt \dots\dots$$

$$\therefore \int \sec t \tan t dt = \int \frac{\sin t}{\cos^2 t} dt$$

$$= t \sec t - \log(\tan t + \sec t) + C'$$

Put $\cos t = u;$

$$-\sin t dt = du$$

$$= \sin^{-1} x \sec(\sin^{-1} x) - \log(\tan(\sin^{-1} x) + \sec(\sin^{-1} x)) + c' \int -u^{-2} du$$

$$= -(-u^{-1}) + c$$

$$= \sec t + C$$

118. Question

Evaluate $\int e^{2x} \left(\frac{1 + \sin 2x}{1 + \cos 2x} \right) dx$

Answer

Put $2x=t$ $dx=dt/2$

$$\begin{aligned}\frac{1}{2} \int e^t \left(\frac{1 + \sin t}{1 + \cos t} \right) dt &= \frac{1}{2} \int \left(e^t \tan \frac{t}{2} + \frac{1}{2} e^t \sec^2 \frac{t}{2} \right) dt \\&= \frac{1}{2} \int \left(e^t \tan \frac{t}{2} \right) dt + \frac{1}{4} \int e^t \sec^2 \frac{t}{2} dt \\&= \frac{1}{2} \int \left(e^t \tan \frac{t}{2} \right) dt + \frac{1}{4} \left[2e^t \tan \frac{t}{2} - \int 2e^t \tan \frac{t}{2} \right] = e^t \frac{\tan \frac{t}{2}}{2} + c\end{aligned}$$

119. Question

Evaluate $\int \frac{\sqrt{1 - \sin x}}{1 + \cos x} e^{-x/2} dx$

Answer

$$\begin{aligned}&= \int e^{-\frac{x}{2}} \frac{\sqrt{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}}}{2 \cos^2 \frac{x}{2}} dx \\&= \int e^{-\frac{x}{2}} \frac{(\sin \frac{x}{2} - \cos \frac{x}{2})}{2 \cos^2 \frac{x}{2}} dx \\&= \int e^{-\frac{x}{2}} \left(\frac{\sin \frac{x}{2}}{2 \cos^2 \frac{x}{2}} - \frac{\cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right) dx \\&= \int \left[\frac{1}{2} \tan \frac{x}{2} \sec \frac{x}{2} e^{-\frac{x}{2}} - \frac{1}{2} \sec \frac{x}{2} e^{-\frac{x}{2}} \right] dx \\&= \frac{1}{2} \int \tan \frac{x}{2} \sec \frac{x}{2} e^{-\frac{x}{2}} dx - \frac{1}{2} \int \sec \frac{x}{2} e^{-\frac{x}{2}} dx \\&= \frac{1}{2} \int \tan \frac{x}{2} \sec \frac{x}{2} e^{-\frac{x}{2}} dx - \frac{1}{2} \left[\sec \frac{x}{2} \int e^{-\frac{x}{2}} dx - \int \frac{d}{dx} (\sec \frac{x}{2}) \int (e^{-\frac{x}{2}} dx) dx \right] \\&= \frac{1}{2} \int \tan \frac{x}{2} \sec \frac{x}{2} e^{-\frac{x}{2}} dx + e^{-\frac{x}{2}} \sec \frac{x}{2} + \frac{1}{2} \int \frac{1}{2} \tan \frac{x}{2} \sec \frac{x}{2} \left(\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}} \right) dx \\&= \sec \frac{x}{2} (e^{-\frac{x}{2}}) + c\end{aligned}$$

120. Question

Evaluate $\int e^x \frac{(1-x)^2}{(1+x^2)^2} dx$

Answer

$$\begin{aligned}&= \int e^x \frac{(1+x^2-2x)}{(1+x^2)^2} dx \\&= \int e^x \frac{dx}{1+x^2} - \int \frac{2xe^x dx}{(1+x^2)^2} \\&= \int e^x \left[\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2} \right] dx \dots (\int e^x (f(x) + f'(x)) = e^x f(x) + c) \\&= e^x \frac{1}{1+x^2} + c\end{aligned}$$

121. Question

Evaluate $\int \frac{e^{m \tan^{-1} x}}{(1+x^2)^{3/2}} dx$

Answer

$$= e^m \int \frac{\tan^{-1} x}{(1+x^2)\sqrt{1+x^2}} dx$$

Put $\tan^{-1} x = t, dx/(1+x^2) = dt, 1+x^2 = \sec^2 x;$

$$= e^m \int \frac{t dt}{\sec t} = e^m \int t \cos t dt$$

$$= e^m \left[t \sin t - \int \sin t dt \right]$$

$$= e^m [t \sin t + \cos t] + c$$

$$= e^m \left[\frac{x \tan^{-1} x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} \right] + c$$

122. Question

Evaluate $\int \frac{x^2}{(x-1)^3(x+1)} dx$

Answer

$$= \int \frac{x^2}{(x-1)^3(x+1)} dx$$

By using partial differentiation,

$$= \frac{x^2}{(x-1)^3(x+1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

$$x^2 = A(x-1)^3 + B(x-1)^2(x+1) + C(x-1)^1(x+1) + D(x+1)$$

By substituting the x^2 coefficients and other coefficients we can get,

$$A = -1/8; B = 1/8; C = 3/4; D = 1/2;$$

$$= \int \frac{-dx}{8(x+1)} + \int \frac{dx}{8(x-1)} + \int \frac{3dx}{4(x-1)^2} + \int \frac{dx}{2(x-1)^3}$$

$$= -\frac{1}{8} \log(1+x) + \frac{1}{8} \log(x-1) - \frac{3}{4(x-1)} - \frac{1}{4} \left(\frac{1}{1-x^2} \right) + c$$

123. Question

Evaluate $\int \frac{x}{x^3-1} dx$

Answer

$$= \int \frac{x}{(x^3-1)} dx = \int \frac{x}{(x-1)(x^2+x+1)} dx$$

$$= \int \left(\frac{1}{3(x-1)} - \frac{x-1}{3(x^2+x+1)} \right)$$

$$\begin{aligned}
&= \frac{1}{3} \int \frac{1}{x-1} dx - \frac{1}{3} \int \frac{x-1}{x^2+x+1} dx \\
&= \frac{1}{3} \log(x-1) - \frac{1}{3} \left[\int \frac{(2x+1)}{2(x^2+x+1)} dx - \int \frac{3}{2((x^2+x+1))} dx \right] \\
&= \frac{1}{3} \log(x-1) - \frac{1}{3} [I_1 + I_2]
\end{aligned}$$

$$I_1 = \frac{1}{2} \int \frac{(2x+1)}{(x^2+x+1)} dx$$

put $x^2+x+1=t$;

$$(2x+1)dx=dt$$

$$I_1 = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \log t + c = \frac{1}{2} \log(x^2+x+1) + c$$

$$\text{Now, } I_2 = \frac{3}{2} \int \frac{dx}{x^2+x+1} = \frac{3}{2} \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

put $(2x+1)/\sqrt{3} = u$;

$$2dx/\sqrt{3}=du;$$

$$dx=\sqrt{3}du/2$$

$$= \frac{3}{2} \cdot \frac{2}{\sqrt{3}} \int \frac{du}{u^2+1} = \frac{3}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1} u + c = \sqrt{3} \tan^{-1} \frac{2x+1}{\sqrt{3}} + c$$

$$\text{So, answer is } = \frac{1}{3} \log(x-1) - \frac{1}{3} \left[\frac{1}{2} \log(x^2+x+1) - \sqrt{3} \tan^{-1} \frac{2x+1}{\sqrt{3}} \right] + c$$

124. Question

$$\text{Evaluate } \int \frac{1}{1+x+x^2+x^3} dx$$

Answer

$$= \int \frac{dx}{1+x+x^2+x^3} = \int \frac{dx}{(1+x)(1+x^2)}$$

We can write the integral as follows,

$$\begin{aligned}
&= \int \left[\frac{dx}{2(x+1)} \right] - \int \left[\frac{x-1}{2(x^2+1)} dx \right] = \frac{1}{2} \log(x+1) - \frac{1}{2} \left[\int \frac{xdx}{x^2+1} - \int \frac{dx}{x^2+1} \right] \\
&= \frac{1}{2} \log(x+1) - \frac{1}{2} \left[\log \frac{(x^2+1)}{2} - \tan^{-1} x \right] + c
\end{aligned}$$

125. Question

$$\text{Evaluate } \int \frac{1}{(x^2+2)(x^2+5)} dx$$

Answer

$$\int \frac{dx}{(x^2+5)(x^2+2)}$$

$$\text{By partial fractions, } \frac{1}{(x^2+5)(x^2+2)} = \frac{A}{x^2+5} + \frac{B}{x^2+2}$$

Solving these two equations, $2A+5B=1$ and $A+B=0$

We get $A=-1/3$ and $B=1/3$

$$= -\frac{1}{3} \int \frac{dx}{(x^2+5)} + \frac{1}{3} \int \frac{dx}{(x^2+2)} = -\frac{1}{3} \cdot \frac{1}{\sqrt{5}} \tan^{-1} \frac{x}{\sqrt{5}} + \frac{1}{3} \cdot \frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + c$$

126. Question

$$\int \frac{x^2 - 2}{x^5 - x} dx$$

Answer

By partial fractions,

$$= \frac{x^2 - 2}{x^2 - 5} = \frac{x^2 - 2}{(x-1)x(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x+1} + \frac{D}{x^2+1}$$

So by solving, $A=-\frac{1}{2}$; $B=2$; $C=-\frac{1}{2}$; $D = -3/2$

$$= \int -\frac{dx}{4(x-1)} + \int \frac{2}{x} dx - \frac{1}{4} \int \frac{dx}{x+1} - \frac{3}{2} \int \frac{xdx}{x^2+1}$$

$$= -\frac{1}{4} \log(x-1) + 2 \log x - \frac{1}{4} \log(x+1) - \frac{3}{4} \log(x^2+1) + c$$

127. Question

Evaluate $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} dx$

Answer

Let, $x = \sin^2 t$

Differentiating both side with respect to t

$$\frac{dx}{dt} = 2 \sin t \cos t \Rightarrow dx = 2 \sin t \cos t dt$$

$$y = \int \sqrt{\frac{1-\sin t}{1+\sin t}} 2 \sin t \cos t dt$$

$$y = \int \sqrt{\frac{(1-\sin t)}{(1+\sin t)} \times \frac{(1-\sin t)}{(1-\sin t)}} 2 \sin t \cos t dt$$

$$y = 2 \int (1-\sin t) \sin t dt$$

$$y = 2 \int \sin t - \frac{1-\cos 2t}{2} dt$$

$$y = 2 \left(-\cos t - \frac{t}{2} + \frac{\sin 2t}{4} \right) + c$$

Again, put $t = \sin \sqrt{x}$

$$y = 2 \left(-\cos \sin \sqrt{x} - \frac{\sin \sqrt{x}}{2} + \frac{\sin(2 \sin \sqrt{x})}{4} \right) + c$$

$$y = 2 \left(-\sqrt{1-x} - \frac{\sin\sqrt{x}}{2} + \frac{1}{2}\sqrt{x-x^2} \right) + c$$

128. Question

$$\int \frac{x^2 + x + 1}{(x+1)^2(x+2)} dx$$

Answer

$$= \int \frac{x^2 + x + 1}{(x+1)^2(x+2)} dx$$

by partial fraction,

$$\frac{x^2 + x + 1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

So we get these three equations ,

$$2A+2B+C=1$$

$$3A+B+2C=1$$

$$A+C=1$$

So the values are A=-2;C=3;B=1

$$\begin{aligned} \int \frac{x^2 + x + 1}{(x+1)^2(x+2)} dx &= \int \left(-\frac{2dx}{x+1} \right) + \int \frac{dx}{(x+1)^2} + \int \frac{3dx}{x+2} \\ &= -2 \log(x+1) + 3 \log(x+2) - \frac{1}{x+1} + c \end{aligned}$$

129. Question

$$\int \frac{\sin 4x - 2}{1 - \cos 4x} e^{2x} dx$$

Answer

Put $2x=t$;

$$2dx=dt; dx=dt/2$$

$$\begin{aligned} &= - \int \frac{\sin 4x - 2}{\cos 4x - 1} dx = - \frac{1}{2} \int \frac{e^t(\sin 2t - 2)}{\cos 2t - 1} dt = \frac{1}{4} \int \frac{e^t(2\sin t \cos t - 2)}{\cos^2 t} dt \\ &= \frac{2}{4} \int e^t \cot t dt - \frac{2}{4} \int e^t \operatorname{cosec}^2 t dt = \frac{1}{2} \left[\int e^t \cot t dt - \int e^t \operatorname{cosec}^2 t dt \right] \\ &= \frac{1}{2} \left[e^t \cot t + \int e^t \operatorname{cosec}^2 t dt - \int e^t \operatorname{cosec}^2 t dt \right] \\ &= \frac{1}{2} \left[\frac{e^{2x} \cot 2x}{2} \right] + c \end{aligned}$$

130. Question

$$\text{Evaluate } \int \frac{\left\{ \cot x + \cot^3 x \right\} x}{1 + \cot^3 x} dx$$

Answer

$$= \int \frac{\cot x(1 + \cot^2 x)}{1 + \cot^3 x} dx = \int \frac{\cot x \operatorname{cosec}^2 x}{1 + \cot^3 x} dx$$

Put $\cot x = t$, $-\operatorname{cosec}^2 x dx = dt$;

$$= - \int \frac{t dt}{t^3 + 1} = - \int \frac{t dt}{(t + 1)(t^2 - t + 1)}$$

By partial fractions it's a remembering thing

That if you see the above integral just apply the below return result,

$$\begin{aligned} &= - \int \left[\frac{(t + 1)}{3(t^2 - t + 1)} - \frac{1}{3(t + 1)} \right] dt \\ &= \frac{1}{3} \log(t + 1) - \frac{1}{3} \int \left[\frac{2t - 1}{2(t^2 - t + 1)} + \frac{3}{2(t^2 - t + 1)} \right] dt \\ &= \frac{1}{3} \log(t + 1) - \frac{1}{6} \log(t^2 - t + 1) - \frac{1}{2} \int \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \frac{3}{4}} \\ &= \frac{1}{3} \log(t + 1) - \frac{1}{6} \log(t^2 - t + 1) - \frac{1}{2} \left[\frac{2}{\sqrt{3}} \tan^{-1} \frac{(2t - 1)}{\sqrt{3}} \right] + c \\ &= \frac{1}{3} \log(\cot x + 1) - \frac{1}{6} \log(\cot^2 x - \cot x + 1) - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2\cot x - 1}{\sqrt{3}} \right) + c \end{aligned}$$

16. Question

Evaluate $\int \frac{1}{e^x + 1} dx$

Answer

$$\int \frac{1}{e^x + 1} dx$$

We can write above integral as

$$\begin{aligned} &\Rightarrow \int \frac{1 + e^x - e^x}{e^x + 1} dx \\ &\Rightarrow \underbrace{\int \frac{1 + e^x}{e^x + 1} dx}_{(1)} + \underbrace{\int \frac{-e^x}{e^x + 1} dx}_{(2)} \end{aligned}$$

Considering first integral:

$$\int \frac{1 + e^x}{1 + e^x} dx$$

Since the numerator and denominator are exactly same, our integrand simplifies to 1 and integrand becomes:

$$\Rightarrow \int dx$$

$$\Rightarrow x$$

$$\therefore \int \frac{1 + e^x}{1 + e^x} dx = x \dots (3)$$

Considering second integral:

$$\int \frac{-e^x}{e^x + 1} dx$$

Let $u = 1 + e^x$, $du = e^x dx$

Apply u - substitution:

$$\int \frac{1}{u} (-du) = -\ln|u|$$

Replacing the value of u we get,

$$\int \frac{-e^x}{e^x + 1} dx = -\ln|1 + e^x| + C \dots (4)$$

From (3) and (4) we get,

$$\Rightarrow \int \frac{1 + e^x}{e^x + 1} dx + \int \frac{-e^x}{e^x + 1} dx = x - \ln|1 + e^x| + C$$

$$\therefore \int \frac{1}{e^x + 1} dx = x - \ln|1 + e^x| + C$$

17. Question

Evaluate $\int \frac{e^x - 1}{e^x + 1} dx$

Answer

$$\int \frac{e^x - 1}{e^x + 1} dx$$

We can write above integrand as:

$$\int \left(\frac{e^x}{e^x + 1} - \frac{1}{e^x + 1} \right) dx$$

$$\Rightarrow \underbrace{\int \frac{e^x}{e^x + 1} dx}_{(A)} - \underbrace{\int \frac{1}{e^x + 1} dx}_{(B)}$$

Considering integrand (A)

$$A = \int \frac{e^x}{e^x + 1} dx$$

Put $e^x + 1 = t$

Differentiating w.r.t x we get,

$$e^x dx = dt$$

Substituting values we get

$$A = \int \frac{e^x}{e^x + 1} dx = \int \frac{dt}{t} dx = \ln|t| + C$$

Substituting the value of t we get,

$$A = \ln|e^x + 1| + C$$

$$\therefore A = \int \frac{e^x}{e^x + 1} dx = \ln|e^x + 1| + C \dots (i)$$

Considering integrand (B)

$$B = \int \frac{1}{e^x + 1} dx$$

We can write above integral as

$$\Rightarrow \int \frac{1 + e^x - e^x}{e^x + 1} dx$$

$$\underbrace{\hspace{1cm}} \Rightarrow \int \frac{1+e^x}{e^x+1} dx + \int \frac{-e^x}{e^x+1} dx$$

(1) (2)

Considering first integral:

$$\int \frac{1 + e^x}{1 + e^x} dx$$

Since the numerator and denominator are exactly same, our integrand simplifies to 1 and integrand becomes:

$$\Rightarrow \int dx$$

$$\Rightarrow x$$

$$\therefore \int \frac{1+e^x}{1+e^x} dx = x \dots (3)$$

Considering second integral:

$$\int \frac{-e^x}{e^x + 1} dx$$

$$\text{Let } u = 1 + e^x, du = e^x dx$$

Apply u - substitution:

$$\int \frac{1}{u} (-du) = -\ln|u|$$

Replacing the value of u we get,

$$\int \frac{-e^x}{e^x+1} dx = -\ln|1 + e^x| + C \dots (4)$$

From (3) and (4) we get,

$$\Rightarrow \int \frac{1 + e^x}{e^x + 1} dx + \int \frac{-e^x}{e^x + 1} dx = x - \ln|1 + e^x| + C$$

$$\therefore B = \int \frac{1}{e^x+1} dx = x - \ln|1 + e^x| + C \dots (ii)$$

From (i) and (ii) we get,

$$\int \frac{e^x}{e^x + 1} dx - \int \frac{1}{e^x + 1} dx = (\ln|e^x + 1| - (x - \ln|1 + e^x|)) + C$$

$$= 2 \ln|e^x + 1| - x + C$$

$$\therefore \int \frac{e^x - 1}{e^x + 1} dx = 2 \ln|e^x + 1| - x + C$$

18. Question

$$\text{Evaluate } \int \frac{1}{e^x + e^{-x}} dx$$

Answer

$$\int \frac{1}{e^x + e^{-x}} dx$$

We can write above integral as:

$$= \int \frac{1}{e^x + \frac{1}{e^x}} dx$$

$$= \int \frac{e^x}{e^{2x} + 1} dx \quad \text{--(1)}$$

Let $e^x = t$

Differentiating w.r.t x we get,

$$e^x dx = dt$$

$\therefore$ integral (1) becomes,

$$= \int \frac{1}{t^2 + 1} dt$$

$$= \tan^{-1}(t) + C \quad \left(\because \int \frac{1}{x^2 + 1} dx = \tan^{-1}(x) \right)$$

Putting value of t we get,

$$= \tan^{-1}(e^x) + C$$

$$\therefore \int \frac{1}{e^x + e^{-x}} dx = \tan^{-1}(e^x) + C$$

19. Question

Evaluate $\int \frac{\cos^7 x}{\sin x} dx$

Answer

$$\int \frac{\cos^7 x}{\sin x} dx$$

We can write above integral as:

$$\int \frac{(\cos^2 x)^3 \cdot \cos x}{\sin x} dx \quad \text{--(1)}$$

Put $\sin x = t$

Differentiating w.r.t x we get,

$$\cos x \cdot dx = dt$$

$\therefore$ integral (1) becomes,

$$= \int \frac{(\cos^2 x)^3}{t} dt$$

$$= \int \frac{(1 - \sin^2 x)^3}{t} dt \quad \text{--} \quad (\because \sin^2(x) + \cos^2(x) = 1)$$

$$= \int \frac{(1 - t^2)^3}{t} dt$$

$$= \int \frac{(1)^3 - (t^2)^3 - 3(1)(t^2)(1 - t^2)}{t} dt = \int \frac{1 - t^6 - 3t^2 + 3t^4}{t} dt$$

$$= \int \frac{1}{t} dt - \int \frac{t^6}{t} dt - \int \frac{3t^2}{t} dt + \int \frac{3t^4}{t} dt$$

$$= \log|t| - \frac{t^6}{6} - \frac{3t^2}{2} + \frac{3t^4}{4} + C$$

Putting value of $t = \sin(x)$ we get,

$$= \log|\sin x| - \frac{\sin^6 x}{6} - \frac{3 \sin^2 x}{2} + \frac{3 \sin^4 x}{4} + C$$

$$\therefore \int \frac{\cos^7 x}{\sin x} dx = \log|\sin x| - \frac{\sin^6 x}{6} - \frac{3 \sin^2 x}{2} + \frac{3 \sin^4 x}{4} + C$$

20. Question

Evaluate $\int \sin x \sin 2x \sin 3x dx$

Answer

$$\int \sin x \sin 2x \sin 3x dx$$

We can write above integral as:

$$= \frac{1}{2} \int (2 \sin x \sin 2x) \sin 3x dx \quad \text{--(1)}$$

We know that,

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

Now, considering A as x and B as 2x we get,

$$= 2 \sin x \sin 2x = \cos(x-2x) - \cos(x+2x)$$

$$= 2 \sin x \sin 2x = \cos(-x) - \cos(3x)$$

$$= 2 \sin x \sin 2x = \cos(x) - \cos(3x) \quad [\because \cos(-x) = \cos(x)]$$

$\therefore$ integral (1) becomes,

$$= \frac{1}{2} \int (\cos x - \cos 3x) \sin 3x dx$$

$$= \frac{1}{2} \int (\cos x \sin 3x - \cos 3x \sin 3x) dx$$

$$= \frac{1}{2} \left[\int (\cos x \sin 3x) dx - \int (\cos 3x \sin 3x) dx \right]$$

$$= \frac{1}{4} \left[\int 2(\cos x \sin 3x) dx - \int 2(\cos 3x \sin 3x) dx \right]$$

Considering $\int 2(\cos x \sin 3x) dx$

We know,

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

Now, considering A as 3x and B as x we get,

$$2 \sin 3x \cos x = \sin(4x) + \sin(2x)$$

$$\therefore \int 2(\cos x \sin 3x) dx = \int \sin 4x + \sin 2x dx \quad \text{--(2)}$$

Again, Considering $\int 2(\cos 3x \sin 3x) dx$

We know,

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

Now, considering A as 3x and B as 3x we get,

$$2 \sin 3x \cdot \cos 3x = \sin(6x) + \sin(0)$$

$$= \sin(6x)$$

$$\therefore \int 2(\cos 3x \cdot \sin 3x) dx = \int \sin 6x dx \quad \text{--(3)}$$

$\therefore$ integral becomes,

$$= \frac{1}{4} \left[\int 2(\cos x \cdot \sin 3x) dx - \int 2(\cos 3x \cdot \sin 3x) dx \right]$$

$$= \frac{1}{4} \left[\int (\sin 4x + \sin 2x) dx - \int \sin 6x dx \right] \text{ [From (2) and (3)]}$$

$$= \frac{1}{4} \left[\int \sin 4x dx + \int \sin 2x dx - \int \sin 6x dx \right]$$

$$= \frac{1}{4} \left[\frac{-\cos 4x}{4} + \left(\frac{-\cos 2x}{2} \right) - \left(\frac{-\cos 6x}{6} \right) \right] + C$$

$$\left[\because \int \sin(ax + b) dx = -\frac{\cos(ax + b)}{a} + C \right]$$

$$= \frac{1}{4} \left[\frac{\cos 6x}{6} - \frac{\cos 4x}{4} - \frac{\cos 2x}{2} \right] + C$$

$$\therefore \int \sin x \sin 2x \sin 3x dx = \frac{1}{4} \left[\frac{\cos 6x}{6} - \frac{\cos 4x}{4} - \frac{\cos 2x}{2} \right] + C$$

21. Question

Evaluate $\int \cos x \cos 2x \cos 3x dx$

Answer

$$\int \cos x \cos 2x \cos 3x dx$$

We can write above integral as:

$$= \frac{1}{2} \int (2 \cos x \cos 2x) \cos 3x dx \quad \text{--(1)}$$

We know that,

$$2 \cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$$

Now, considering A as x and B as 2x we get,

$$= 2 \cos x \cdot \cos 2x = \cos(x+2x) + \cos(x-2x)$$

$$= 2 \cos x \cdot \cos 2x = \cos(3x) + \cos(-x)$$

$$= 2 \cos x \cdot \cos 2x = \cos(3x) + \cos(x) \quad [\because \cos(-x) = \cos(x)]$$

$\therefore$ integral (1) becomes,

$$= \frac{1}{2} \int (\cos 3x + \cos x) \cos 3x dx$$

$$= \frac{1}{2} \int (\cos 3x \cdot \cos 3x + \cos x \cdot \cos 3x) dx$$

$$= \frac{1}{2} \left[\int (\cos^2 3x) dx + \int (\cos x \cdot \cos 3x) dx \right]$$

$$= \frac{1}{4} \left[\int 2(\cos^2 3x) + \int 2(\cos x \cdot \cos 3x) dx \right]$$

Considering $\int 2(\cos x \cdot \cos 3x) dx$

We know,

$$2 \cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$$

Now, considering A as x and B as 3x we get,

$$2 \cos x \cdot \cos 3x = \cos(4x) + \cos(-2x)$$

$$2 \cos x \cdot \cos 3x = \cos(4x) + \cos(2x) \quad [\because \cos(-x) = \cos(x)]$$

$$\therefore \int 2(\cos x \cdot \cos 3x) dx = \int (\cos 4x + \cos 2x) dx \quad \text{--(2)}$$

Considering $\int 2\cos^2 3x$

We know,

$$\cos 2A = 2\cos^2 A - 1$$

$$2\cos^2 A = 1 + \cos 2A$$

Now, considering A as 3x we get,

$$\int 2\cos^2 3x = \int 1 + \cos 2(3x) = \int 1 + \cos(6x)$$

$$\therefore \int 2(\cos^2 3x) dx = \int 1 + \cos 6x dx \quad \text{--(3)}$$

$\therefore$ integral becomes,

$$\begin{aligned} &= \frac{1}{4} \left[\int 2(\cos^2 3x) + \int 2(\cos x \cdot \cos 3x) dx \right] \\ &= \frac{1}{4} \left[\int (1 + \cos 6x) dx + \int (\cos 4x + \cos 2x) dx \right] \quad [\text{From (2) and (3)}] \\ &= \frac{1}{4} \left[\int (1 + \cos 6x) dx + \int \cos 4x dx + \int \cos 2x dx \right] \\ &= \frac{1}{4} \left[x + \frac{\sin 6x}{6} \right] + \frac{1}{4} \left[\frac{\sin 4x}{4} \right] + \frac{1}{4} \left[\frac{\sin 2x}{2} \right] + C \\ &= \frac{1}{4} \left[x + \frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2} \right] + C \\ \therefore \int \cos x \cos 2x \cos 3x dx &= \frac{1}{4} \left[x + \frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2} \right] + C \end{aligned}$$

22. Question

$$\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Answer

$$\int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

We can write above integral as

$$= \int \frac{\sin x + \cos x}{\sqrt{1 - 1 + \sin 2x}} dx \quad [\text{Adding and subtracting 1 in denominator}]$$

$$= \int \frac{\sin x + \cos x}{\sqrt{1 - (1 - \sin 2x)}} dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{1 - (\sin^2 x + \cos^2 x - 2 \sin x \cos x)}} dx \quad \because \sin^2 x + \cos^2 x = 1 \text{ and}$$

$$\sin 2x = 2 \sin x \cos x$$

$$= \int \frac{(\sin x + \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}} dx \quad \because \sin^2 x + \cos^2 x - 2 \sin x \cos x = (\sin x - \cos x)^2$$

Put $\sin x - \cos x = t$

Differentiating w.r.t x we get,

$$(\cos x + \sin x)dx = dt$$

Putting values we get,

$$= \int \frac{(\sin x + \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}} dx = \int \frac{dt}{\sqrt{1 - t^2}}$$

$$= \int \frac{dt}{\sqrt{1 - t^2}} = \sin^{-1} t + C$$

Putting value of t we get,

$$\therefore \int \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx = \sin^{-1}(\sin x - \cos x) + C$$

23. Question

$$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx$$

Answer

$$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx$$

We can write above integral as

$$= \int \frac{\sin x - \cos x}{\sqrt{1 + \sin 2x - 1}} dx \text{ [Adding and subtracting 1 in denominator]}$$

$$= \int \frac{\sin x - \cos x}{\sqrt{(1 + \sin 2x) - 1}} dx$$

$$= \int \frac{\sin x - \cos x}{\sqrt{(\sin^2 x + \cos^2 x + 2 \sin x \cos x) - 1}} dx \because \sin^2 x + \cos^2 x = 1 \text{ and}$$

$$\sin 2x = 2 \sin x \cos x$$

$$= \int \frac{(\sin x - \cos x)}{\sqrt{(\sin x + \cos x)^2 - 1}} dx \because \sin^2 x + \cos^2 x + 2 \sin x \cos x = (\sin x + \cos x)^2$$

Taking minus (-) common from numerator we get,

$$= - \int \frac{(-\sin x + \cos x)}{\sqrt{(\sin x + \cos x)^2 - 1}} dx$$

Put $\sin x + \cos x = t$

Differentiating w.r.t x we get,

$$(\cos x - \sin x)dx = dt$$

Putting values we get,

$$= - \int \frac{(\cos x - \sin x)}{\sqrt{(\sin x + \cos x)^2 - 1}} dx = - \int \frac{dt}{\sqrt{t^2 - 1}}$$

We know that,

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

Here $x = t$ and $a = 1$

$$\therefore - \int \frac{dt}{\sqrt{t^2 - 1}} = -\log|t + \sqrt{t^2 - 1}| + C$$

Putting value of t we get,

$$\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx = -\log|\sin x + \cos x + \sqrt{(\sin x + \cos x)^2 - 1}| + C$$

$\therefore$ from (1) we get,

$$\therefore \int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx = -\log|\sin x + \cos x + \sqrt{\sin 2x}| + C$$

24. Question

Evaluate $\int \frac{1}{\sin(x-a)\sin(x-b)} dx$

Answer

Let $I = \int \frac{1}{\sin(x-a)\sin(x-b)} dx$

Multiply and divide $\frac{1}{\sin(a-b)}$ in R.H.S we get,

$$I = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\sin(x-a)\sin(x-b)} dx$$

We can write above integral as:

$$\begin{aligned} &= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b+x-x)}{\sin(x-a)\sin(x-b)} dx \\ &= \frac{1}{\sin(a-b)} \int \frac{\sin[(x-b)-(x-a)]}{\sin(x-a)\sin(x-b)} dx \\ &= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)} \right] dx \end{aligned}$$

$$[\because \sin(A+B) = \sin A \cos B + \cos A \sin B]$$

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)\cos(x-a)}{\sin(x-a)\sin(x-b)} - \frac{\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)} \right] dx$$

By simplifying we get,

$$\begin{aligned} &= \frac{1}{\sin(a-b)} \int \left[\frac{\cos(x-a)}{\sin(x-a)} - \frac{\cos(x-b)}{\sin(x-b)} \right] dx \\ &= \frac{1}{\sin(a-b)} \int [\cot(x-a) - \cot(x-b)] dx \\ &= \frac{1}{\sin(a-b)} [\log|\sin(x-a)| - \log|\sin(x-b)|] + C \end{aligned}$$

$$[\because \int \cot x dx = \log|\sin x| + C]$$

$$= \frac{1}{\sin(a-b)} \left[\log \left| \frac{\sin(x-a)}{\sin(x-b)} \right| \right] + C$$

$$\therefore I = \int \frac{1}{\sin(x-a)\sin(x-b)} dx = \frac{1}{\sin(a-b)} \left[\log \left| \frac{\sin(x-a)}{\sin(x-b)} \right| \right] + C$$

25. Question

Evaluate $\int \frac{1}{\cos(x-a)\cos(x-b)} dx$

Answer

$$\text{Let } I = \int \frac{1}{\cos(x-a)\cos(x-b)} dx$$

Multiply and divide $\frac{1}{\sin(a-b)}$ in R.H.S we get,

$$I = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)} dx$$

We can write above integral as:

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b+x-x)}{\cos(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)} \right] dx$$

$$[\because \sin(A+B) = \sin A \cdot \cos B - \cos A \cdot \sin B]$$

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)\cos(x-a)}{\cos(x-a)\cos(x-b)} - \frac{\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)} \right] dx$$

By simplifying we get,

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x-b)}{\cos(x-b)} - \frac{\sin(x-a)}{\cos(x-a)} \right] dx$$

$$= \frac{1}{\sin(a-b)} \int [\tan(x-b) - \tan(x-a)] dx$$

$$= \frac{1}{\sin(a-b)} [-\log|\cos(x-b)| + \log|\cos(x-a)|]$$

$$[\because \int \tan x dx = -\log|\cos x| + C]$$

$$= \frac{1}{\sin(a-b)} [\log|\cos(x-a)| - \log|\cos(x-b)|]$$

$$= \frac{1}{\sin(a-b)} \left[\log \left| \frac{\cos(x-a)}{\cos(x-b)} \right| \right] + C$$

$$\therefore I = \int \frac{1}{\cos(x-a)\cos(x-b)} dx = \frac{1}{\sin(a-b)} \left[\log \left| \frac{\cos(x-a)}{\cos(x-b)} \right| \right] + C$$

26. Question

Evaluate $\int \frac{\sin x}{\sqrt{1+\sin x}} dx$

Answer

$$\int \frac{\sin x}{\sqrt{1+\sin x}} dx$$

We can write above integral as:

$$= \int \frac{1+\sin x-1}{\sqrt{1+\sin x}} dx \text{ (Adding and subtracting 1 in numerator)}$$

$$= \int \frac{1 + \sin x}{\sqrt{1 + \sin x}} dx - \int \frac{1}{\sqrt{1 + \sin x}} dx$$

$$= \int \sqrt{1 + \sin x} dx - \int \frac{1}{\sqrt{1 + \sin x}} dx$$

Consider

$$\sqrt{1 + \sin x} = \sqrt{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}} = \sqrt{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2}$$

($\because \sin^2 x + \cos^2 x = 1$ and $\sin 2x = 2 \sin x \cos x$)

$$\therefore \sqrt{1 + \sin x} = \sin \frac{x}{2} + \cos \frac{x}{2} \quad \dots (1)$$

$$\therefore \int \sqrt{1 + \sin x} dx - \int \frac{1}{\sqrt{1 + \sin x}} dx$$

$$= \int \left(\sin \frac{x}{2} + \cos \frac{x}{2}\right) dx - \int \frac{1}{\sin \frac{x}{2} + \cos \frac{x}{2}} dx$$

[From (1)]

Considering,

$$\int \left(\sin \frac{x}{2} + \cos \frac{x}{2}\right) dx - \int \frac{1}{\sin \frac{x}{2} + \cos \frac{x}{2}} dx$$

$$= -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} - \int \frac{1}{\frac{2 \tan \frac{x}{4}}{1 + \tan^2 \frac{x}{4}} + \frac{1 - \tan^2 \frac{x}{4}}{1 + \tan^2 \frac{x}{4}}} dx$$

$$\because \sin \frac{x}{2} = \frac{2 \tan \frac{x}{4}}{1 + \tan^2 \frac{x}{4}} \text{ and } \cos \frac{x}{2} = \frac{1 - \tan^2 \frac{x}{4}}{1 + \tan^2 \frac{x}{4}}$$

$$= -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} - \int \frac{1 + \tan^2 \frac{x}{4}}{\left(2 \tan \frac{x}{4} + 1 - \tan^2 \frac{x}{4}\right) + (1 - 1)} dx$$

(Adding and subtracting 1 in denominator)

$$= -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} + \int \frac{1 + \tan^2 \frac{x}{4}}{-\left[\left(-2 \tan \frac{x}{4} + 1 + \tan^2 \frac{x}{4}\right) - 2\right]} dx$$

$$= -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} - \int \frac{\sec^2 \frac{x}{4}}{\left(\tan \frac{x}{4} - 1\right)^2 - 2} dx \quad \dots (2)$$

$$\because -2 \tan \frac{x}{4} + 1 + \tan^2 \frac{x}{4} = \left(\tan \frac{x}{4} - 1\right)^2$$

$$\text{Put } \tan \frac{x}{4} - 1 = u$$

$$\sec^2 \frac{x}{4} dx = 4 du$$

Putting values in (2) we get,

$$= -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} - 4 \int \frac{du}{(u)^2 - (\sqrt{2})^2}$$

$$\text{We know } \int \frac{du}{(x)^2 - (a)^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$= -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} - 4 \frac{1}{2\sqrt{2}} \log \left| \frac{u - \sqrt{2}}{u + \sqrt{2}} \right| + C$$

Substituting value of u we get,

$$= -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} - \sqrt{2} \log \left| \frac{\tan \frac{x}{4} - 1 - \sqrt{2}}{\tan \frac{x}{4} - 1 + \sqrt{2}} \right| + C$$

$$\therefore \int \frac{\sin x}{\sqrt{1 + \sin x}} dx = -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} - \sqrt{2} \log \left| \frac{\tan \frac{x}{4} - 1 - \sqrt{2}}{\tan \frac{x}{4} - 1 + \sqrt{2}} \right| + C$$

27. Question

$$\text{Evaluate } \int \frac{\sin x}{\cos 2x} dx$$

Answer

$$\text{Let } I = \int \frac{\sin x}{\cos 2x} dx$$

$$\text{We know } \cos 2x = 2\cos^2 x - 1$$

Putting values in I we get,

$$I = \int \frac{\sin x}{\cos 2x} dx = \int \frac{\sin x}{2\cos^2 x - 1} dx$$

$$\text{Put } \cos x = t$$

Differentiating w.r.t to x we get,

$$\sin x dx = -dt$$

Putting values in integral we get,

$$I = - \int \frac{dt}{2t^2 - 1} = - \int \frac{dt}{(\sqrt{2}t)^2 - (1)^2}$$

$$\text{Again put } \sqrt{2} \times t = u$$

Differentiating w.r.t to t we get,

$$dt = \frac{du}{\sqrt{2}}$$

Putting values in integral we get,

$$I = \frac{1}{\sqrt{2}} \int \frac{du}{(1)^2 - (u)^2}$$

$$\text{We know } \int \frac{dx}{(1)^2 - (x)^2} = \sin^{-1} x + C$$

$$I = \frac{1}{\sqrt{2}} \sin^{-1} u + C$$

Substituting value of u we get,

$$I = \frac{1}{\sqrt{2}} \sin^{-1} \sqrt{2} t + C$$

Substituting value of t we get,

$$I = \frac{1}{\sqrt{2}} \sin^{-1}(\sqrt{2} \cos x) + C$$

$$\therefore I = \int \frac{\sin x}{\cos 2x} dx = \frac{1}{\sqrt{2}} \sin^{-1}(\sqrt{2} \cos x) + C$$

28. Question

Evaluate $\int \tan^3 x \, dx$

Answer

$$\int \tan^3 x \, dx$$

We can write above integral as:

$$\int \tan^3 x \, dx = \int (\tan^2 x)(\tan x) \, dx \text{ ----(Splitting } \tan^3 x)$$

$$= \int (\sec^2 x - 1)(\tan x) \, dx \text{ (Using } \tan^2 x = \sec^2 x - 1)$$

$$= \underbrace{\int \sec^2 x (\tan x) \, dx}_{(1)} - \underbrace{\int (\tan x) \, dx}_{(2)}$$

Considering integral (1)

Let $u = \tan x$

$$du = \sec^2 x \, dx$$

Substituting values we get,

$$\int \sec^2 x (\tan x) \, dx = \int u \, du = \frac{u^2}{2} + C$$

Substituting value of u we get,

$$\int \sec^2 x (\tan x) \, dx = \frac{\tan^2 x}{2} + C$$

$\therefore$ integral becomes,

$$\begin{aligned} \int \sec^2 x (\tan x) \, dx - \int (\tan x) \, dx &= \frac{\tan^2 x}{2} - \int (\tan x) \, dx \\ &= \frac{\tan^2 x}{2} - (-\log|\cos x|) + C \text{ [}\because \int \tan x \, dx = -\log|\cos x| + C \text{]} \end{aligned}$$

$$\therefore \int \tan^3 x \, dx = \frac{\tan^2 x}{2} + \log|\cos x| + C$$

29. Question

$$\int \tan^4 x \, dx$$

Answer

$$\int \tan^4 x \, dx$$

We can write above integral as:

$$\int \tan^4 x \, dx = \int (\tan^2 x)(\tan^2 x) \, dx \text{ ----(Splitting } \tan^4 x)$$

$$= \int (\sec^2 x - 1) \tan^2 x \, dx \text{ (Using } \tan^2 x = \sec^2 x - 1 \text{)}$$

$$= \underbrace{\int \sec^2 x (\tan^2 x) \, dx}_{(1)} - \underbrace{\int (\tan^2 x) \, dx}_{(2)}$$

Considering integral (1)

Let $u = \tan x$

$$du = \sec^2 x \, dx$$

Substituting values we get,

$$\int \sec^2 x (\tan^2 x) \, dx = \int u^2 \, du = \frac{u^3}{3} + C$$

Substituting value of u we get,

$$\int \sec^2 x (\tan^2 x) \, dx = \frac{\tan^3 x}{3} + C$$

Considering integral (2)

$$\int (\tan^2 x) \, dx = \int (\sec^2 x - 1) \, dx$$

$$= \int (\sec^2 x) \, dx - \int 1 \, dx$$

$$= \tan x - x + C$$

$\therefore$ integral becomes,

$$\int \sec^2 x (\tan^2 x) \, dx - \int (\tan^2 x) \, dx = \frac{\tan^3 x}{3} + C - (\tan x - x + C)$$

$$= \frac{\tan^3 x}{3} - \tan x + x + C \text{ [}\because C+C \text{ is a constant]}$$

$$\therefore \int \tan^4 x \, dx = \frac{\tan^3 x}{3} - \tan x + x + C$$

30. Question

$$\int \tan^5 x \, dx$$

Answer

$$\int \tan^5 x \, dx$$

We can write above integral as:

$$\int \tan^5 x \, dx = \int (\tan^3 x) (\tan^2 x) \, dx \text{ ----(Splitting } \tan^5 x \text{)}$$

$$= \int \tan^3 x (\sec^2 x - 1) \, dx \text{ (Using } \tan^2 x = \sec^2 x - 1 \text{)}$$

$$= \int \sec^2 x (\tan^3 x) \, dx - \int (\tan^3 x) \, dx$$

$$= \int \sec^2 x (\tan^3 x) \, dx - \int (\tan^2 x)(\tan x) \, dx \text{ ----(Splitting } \tan^3 x \text{)}$$

$$= \int \sec^2 x (\tan^3 x) \, dx - \int (\sec^2 x - 1)(\tan x) \, dx$$

(Using $\tan^2 x = \sec^2 x - 1$)

$$= \underbrace{\int \sec^2 x (\tan^3 x) dx}_{(1)} - \underbrace{\int \sec^2 x (\tan x) dx}_{(2)} - \underbrace{\int (\tan x) dx}_{(3)}$$

Considering integral (1)

Let $u = \tan x$

$$du = \sec^2 x dx$$

Substituting values we get,

$$\int \sec^2 x (\tan^3 x) dx = \int u^3 du = \frac{u^4}{4} + C$$

Substituting value of u we get,

$$\int \sec^2 x (\tan^3 x) dx = \frac{\tan^4 x}{4} + C$$

Considering integral (2)

Let $t = \tan x$

$$dt = \sec^2 x dx$$

Substituting values we get,

$$\int \sec^2 x (\tan x) dx = \int t dt = \frac{t^2}{2} + C$$

Substituting value of t we get,

$$\int \sec^2 x (\tan x) dx = \frac{\tan^2 x}{2} + C$$

Considering integral (3)

$$\int (\tan x) dx = -\log|\cos x| \quad [\because \int \tan x dx = -\log|\cos x| + C]$$

$\therefore$ integral becomes,

$$\begin{aligned} & \int \sec^2 x (\tan^3 x) dx - \int \sec^2 x (\tan x) dx - \int (\tan x) dx \\ &= \frac{\tan^4 x}{4} + C - \left(\frac{\tan^2 x}{2} + C \right) - (-\log|\cos x|) \\ &= \left(\frac{\tan^4 x}{4} \right) + \left(\frac{\tan^2 x}{2} \right) + (\log|\cos x|) + C \quad [\because C+C+C \text{ is a constant}] \\ \therefore \int \tan^5 x dx &= \left(\frac{\tan^4 x}{4} \right) + \left(\frac{\tan^2 x}{2} \right) + (\log|\cos x|) + C \end{aligned}$$

86. Question

Evaluate $\int \sqrt{a^2 - x^2} dx$

Answer

Let, $x = a \sin t$

Differentiate both side with respect to t

$$\frac{dx}{dt} = a \cos t \Rightarrow dx = a \cos t \, dt$$

$$y = \int \sqrt{a^2 - (a \sin t)^2} \, a \cos t \, dt$$

$$y = \int (a \cos t)(a \cos t) \, dt$$

$$y = \int a^2 \cos^2 t \, dt$$

$$y = \int a^2 \left(\frac{1 + \cos 2t}{2} \right) \, dt$$

$$y = \frac{a^2}{2} \int 1 + \cos 2t \, dt$$

$$y = \frac{a^2}{2} \left(t + \frac{\sin 2t}{2} \right) + c$$

$$\text{Again, put } t = \sin^{-1} \frac{x}{a}$$

$$y = \frac{a^2}{2} \left(\sin^{-1} \frac{x}{a} + \frac{\sin \left(2 \sin^{-1} \frac{x}{a} \right)}{2} \right) + c$$

$$y = \frac{a^2}{2} \left(\sin^{-1} \frac{x}{a} + \frac{2 \times \frac{x}{a} \times \sqrt{1 - \frac{x^2}{a^2}}}{2} \right) + c$$

$$y = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + c$$

87. Question

$$\text{Evaluate } \int \sqrt{3x^2 + 4x + 1} \, dx$$

Answer

Make perfect square of quadratic equation

$$3x^2 + 4x + 1 = 3 \left(x^2 + \frac{4}{3}x + \frac{1}{3} \right)$$

$$= 3 \left(x^2 + 2 \left(\frac{2}{3} \right) (x) + \left(\frac{2}{3} \right)^2 - \frac{1}{9} \right)$$

$$= 3 \left[\left(x + \frac{2}{3} \right)^2 - \frac{1}{9} \right]$$

$$y = \int \sqrt{3 \left[\left(x + \frac{2}{3} \right)^2 - \frac{1}{9} \right]} \, dx$$

$$y = \sqrt{3} \int \sqrt{\left[\left(x + \frac{2}{3} \right)^2 - \frac{1}{9} \right]} \, dx$$

$$\text{Using formula, } \int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln(x + \sqrt{x^2 - a^2})$$

$$y = \sqrt{3} \frac{\left(x + \frac{2}{3} \right)}{2} \sqrt{\left(x + \frac{2}{3} \right)^2 - \frac{1}{9}} - \frac{\sqrt{3}}{18} \ln \left(\left(x + \frac{2}{3} \right) + \sqrt{\left(x + \frac{2}{3} \right)^2 - \frac{1}{9}} \right) + c$$

$$y = \frac{3x+2}{6}\sqrt{3x^2+4x+1} - \frac{\sqrt{3}}{18}\ln\left(\left(x+\frac{2}{3}\right) + \sqrt{x^2+\frac{4x}{3}+\frac{1}{3}}\right) + c$$

88. Question

Evaluate $\int \sqrt{1+2x-3x^2} \, dx$

Answer

Make perfect square of quadratic equation

$$1 + 2x - 3x^2 = 3 \left[-\left(x^2 - \frac{2}{3}x - \frac{1}{3}\right) \right]$$

$$= 3 \left[\frac{4}{9} - \left(x^2 - 2\left(\frac{1}{3}\right)(x) + \left(\frac{1}{3}\right)^2\right) \right]$$

$$= 3 \left[\left(\frac{2}{3}\right)^2 - \left(x - \frac{1}{3}\right)^2 \right]$$

$$y = \sqrt{3} \int \left[\left(\frac{2}{3}\right)^2 - \left(x - \frac{1}{3}\right)^2 \right] dx$$

Using formula, $\int \sqrt{a^2 - x^2} \, dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2}$

$$y = \sqrt{3} \left(\frac{\left(\frac{2}{3}\right)^2}{2} \sin^{-1} \frac{\left(x - \frac{1}{3}\right)}{\left(\frac{2}{3}\right)} + \frac{\left(x - \frac{1}{3}\right)}{2} \sqrt{\left(\frac{2}{3}\right)^2 - \left(x - \frac{1}{3}\right)^2} \right) + c$$

$$y = \frac{2\sqrt{3}}{9} \sin^{-1} \frac{(3x-1)}{2} + \frac{(3x-1)}{6} \sqrt{1+2x-3x^2} + c$$

89. Question

Evaluate $\int x\sqrt{1+x-x^2} \, dx$

Answer

Make perfect square of quadratic equation

$$1 + x - x^2 = \frac{5}{4} - \left(x^2 - 2\left(\frac{1}{2}\right)(x) + \left(\frac{1}{2}\right)^2\right)$$

$$= \left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2$$

$$y = \int x \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2} \, dx$$

$$\text{Let, } x - \frac{1}{2} = t \Rightarrow x = t + \frac{1}{2}$$

Differentiate both side with respect to t

$$\frac{dx}{dt} = 1 \Rightarrow dx = dt$$

$$y = \int \left(t + \frac{1}{2}\right) \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2} \, dt$$

$$y = \int t \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2} + \frac{1}{2} \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2} dt$$

$$A = \int t \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2} dt$$

$$\text{Let, } t^2 = z$$

Differentiate both side with respect to z

$$2t \frac{dt}{dz} = 1 \Rightarrow t dt = \frac{1}{2} dz$$

$$A = \frac{1}{2} \int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - z} dz$$

$$A = \frac{1}{4} \int \sqrt{5 - 4z} dz$$

$$A = \frac{-1}{24} (5 - 4z)^{\frac{3}{2}} + c_1$$

$$\text{Put } z = t^2 \text{ and } t = x - \frac{1}{2}$$

$$A = \frac{-1}{24} \left(5 - 4 \left(x - \frac{1}{2} \right)^2 \right)^{\frac{3}{2}} + c_1$$

$$A = \frac{-1}{3} (1 + x - x^2)^{\frac{3}{2}} + c_1$$

$$B = \int \frac{1}{2} \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2} dt$$

$$B = \frac{1}{2} \left(\frac{\left(\frac{\sqrt{5}}{2}\right)^2}{2} \sin^{-1} \frac{t}{\left(\frac{\sqrt{5}}{2}\right)} + \frac{t}{2} \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2} \right) + c_2$$

$$B = \frac{5}{16} \sin^{-1} \left(\frac{2t}{\sqrt{5}} \right) + \frac{t}{8} \sqrt{5 - 4t^2} + c_2$$

$$\text{Put } t = x - \frac{1}{2}$$

$$B = \frac{5}{16} \sin^{-1} \left(\frac{2x-1}{\sqrt{5}} \right) + \frac{\left(x - \frac{1}{2}\right)}{8} \sqrt{5 - 4\left(x - \frac{1}{2}\right)^2} + c_2$$

$$B = \frac{5}{16} \sin^{-1} \left(\frac{2x-1}{\sqrt{5}} \right) + \frac{(2x-1)}{8} \sqrt{1+x-x^2} + c_2$$

The final answer is $y = A + B$

$$y = \frac{-1}{3} (1 + x - x^2)^{\frac{3}{2}} + \frac{5}{16} \sin^{-1} \left(\frac{2x-1}{\sqrt{5}} \right) + \frac{(2x-1)}{8} \sqrt{1+x-x^2} + c$$

$$y = \frac{1}{24} (8x^2 - 2x - 11) \sqrt{1+x-x^2} + \frac{5}{16} \sin^{-1} \left(\frac{2x-1}{\sqrt{5}} \right) + c$$

90. Question

Evaluate $\int (2x+3)\sqrt{4x^2+5x+6} \, dx$

Answer

Make perfect square of quadratic equation

$$4x^2 + 5x + 6 = 4 \left[\left(x + \frac{5}{8} \right)^2 + \frac{71}{64} \right]$$

$$y = 2 \int (2x+3) \sqrt{\left[\left(x + \frac{5}{8} \right)^2 + \left(\frac{\sqrt{71}}{8} \right)^2 \right]} \, dx$$

$$\text{Let, } x + \frac{5}{8} = t \Rightarrow x = t - \frac{5}{8}$$

Differentiate both side with respect to t

$$\frac{dx}{dt} = 1 \Rightarrow dx = dt$$

$$y = 2 \int \left(2t + \frac{7}{4} \right) \sqrt{\left[t^2 + \left(\frac{\sqrt{71}}{8} \right)^2 \right]} \, dt$$

$$A = 4 \int t \sqrt{\left(\frac{\sqrt{71}}{8} \right)^2 + t^2} \, dt$$

$$\text{Let, } t^2 = z$$

Differentiate both side with respect to z

$$2t \frac{dt}{dz} = 1 \Rightarrow t dt = \frac{1}{2} dz$$

$$A = 2 \int \sqrt{\left(\frac{\sqrt{71}}{8} \right)^2 + z} \, dz$$

$$A = \frac{1}{4} \int \sqrt{71 + 64z} \, dz$$

$$A = \frac{1}{384} (71 + 64z)^{\frac{3}{2}} + c_1$$

$$\text{Put } z = t^2 \text{ and } t = x + \frac{5}{8}$$

$$A = \frac{1}{384} \left(71 + 64 \left(x + \frac{5}{8} \right)^2 \right)^{\frac{3}{2}} + c_1$$

$$A = \frac{1}{6} (4x^2 + 5x + 6)^{\frac{3}{2}} + c_1$$

$$B = \int \frac{7}{2} \sqrt{\left(\frac{\sqrt{71}}{8} \right)^2 + t^2} \, dt$$

$$B = \frac{7}{2} \left(\frac{t}{2} \sqrt{\left(\frac{\sqrt{71}}{8} \right)^2 + t^2} + \frac{\left(\frac{\sqrt{71}}{8} \right)^2}{2} \ln \left(t + \sqrt{\left(\frac{\sqrt{71}}{8} \right)^2 + t^2} \right) \right) + c_2$$

Put $t = x + \frac{5}{8}$

$$B = \frac{7}{2} \left(\frac{\left(x + \frac{5}{8}\right)}{2} \sqrt{\left(\frac{\sqrt{71}}{8}\right)^2 + \left(x + \frac{5}{8}\right)^2} \right) +$$

$$\frac{7 \left(\frac{\sqrt{71}}{8}\right)^2}{4} \ln \left(\left(x + \frac{5}{8}\right) + \sqrt{\left(\frac{\sqrt{71}}{8}\right)^2 + \left(x + \frac{5}{8}\right)^2} \right) + c_2$$

$$B = \frac{7}{2} \left(\frac{(8x + 5)}{32} \sqrt{4x^2 + 5x + 6} \right) +$$

$$\frac{497}{256} \ln \left(\left(x + \frac{5}{8}\right) + \sqrt{\left(\frac{\sqrt{71}}{8}\right)^2 + \left(x + \frac{5}{8}\right)^2} \right) + c_2$$

The final answer is $y = A + B$

$$y = \frac{1}{6} (4x^2 + 5x + 6)^{\frac{3}{2}} + \frac{7}{2} \left(\frac{(8x + 5)}{32} \sqrt{4x^2 + 5x + 6} \right) +$$

$$\frac{497}{256} \ln \left(\left(x + \frac{5}{8}\right) + \sqrt{x^2 + \frac{5}{4}x + \frac{3}{2}} \right) + c$$

$$y = \frac{1}{192} (128x^2 + 328x + 297) \sqrt{4x^2 + 5x + 6} +$$

$$\frac{497}{256} \ln \left(\left(x + \frac{5}{8}\right) + \sqrt{x^2 + \frac{5}{4}x + \frac{3}{2}} \right) + c$$

91. Question

Evaluate $\int (1 + x^2) \cos 2x \, dx$

Answer

$$y = \int \cos 2x + x^2 \cos 2x \, dx$$

$$A = \int \cos 2x \, dx$$

$$A = \frac{\sin 2x}{2} + c_1$$

$$B = \int x^2 \cos 2x \, dx$$

Use the method of integration by parts

$$B = x^2 \int \cos 2x \, dx - \int \frac{d}{dx}(x^2) \left(\int \cos 2x \, dx \right) dx$$

$$B = x^2 \frac{\sin 2x}{2} - \int x \sin 2x \, dx$$

$$B = x^2 \frac{\sin 2x}{2} - (x \int \sin 2x \, dx - \int \frac{d}{dx}(x) \left(\int \sin 2x \, dx \right))$$

$$B = x^2 \frac{\sin 2x}{2} + x \frac{\cos 2x}{2} - \frac{\sin 2x}{4} + c_2$$

The final answer is $y = A + B$

$$y = \frac{\sin 2x}{2} + x^2 \frac{\sin 2x}{2} + x \frac{\cos 2x}{2} - \frac{\sin 2x}{4} + c$$

$$y = \frac{(1+x^2)}{2} \sin 2x + \frac{x}{2} \cos 2x - \frac{1}{4} \sin 2x + c$$

92. Question

Evaluate $\int \log_{10} x \, dx$

Answer

Use the method of integration by parts

$$y = \int 1 \times \log_{10} x \, dx$$

$$y = \log_{10} x \int dx - \int \frac{d}{dx} \log_{10} x \left(\int dx \right) dx$$

$$y = x \log_{10} x - \int x \frac{1}{x \log_e 10} dx$$

$$y = x \log_{10} x - \frac{x}{\log_e 10} + c$$

$$y = x(\log_e x - 1) \log_{10} e + c$$

93. Question

Evaluate $\int \frac{\log(\log x)}{x} dx$

Answer

Let, $\log x = t$

Differentiating both side with respect to t

$$\frac{1}{x} \frac{dx}{dt} = 1 \Rightarrow \frac{dx}{x} = dt$$

Note:- Always use direct formula for $\int \log x \, dx$

$$y = \int \log t \, dt$$

$$y = t \log t - t + c$$

Again, put $t = \log x$

$$y = (\log x) \log(\log x) - \log x + c$$

94. Question

Evaluate $\int x \sec^2 2x \, dx$

Answer

Use method of integration by parts

$$y = x \int \sec^2 2x \, dx - \int \frac{d}{dx} x \left(\int \sec^2 2x \, dx \right) dx$$

$$y = x \frac{\tan 2x}{2} - \int \frac{\tan 2x}{2} dx$$

Use formula $\int \tan x \, dx = \log \sec x$

$$y = \frac{x}{2} \tan 2x - \frac{\log(\sec 2x)}{4} + c$$

95. Question

Evaluate $\int x \sin^3 x \, dx$

Answer

We know that $\sin^3 x = \frac{3 \sin x - \sin 3x}{4}$

$$y = \int x \left(\frac{3 \sin x - \sin 3x}{4} \right) dx$$

$$y = \frac{3}{4} \int x \sin x \, dx - \frac{1}{4} \int x \sin 3x \, dx$$

Use method of integration by parts

$$\begin{aligned} y &= \frac{3}{4} \left(x \int \sin x \, dx - \int \frac{d}{dx} x \left(\int \sin x \, dx \right) dx \right) \\ &\quad - \frac{1}{4} \left(x \int \sin 3x \, dx - \int \frac{d}{dx} x \left(\int \sin 3x \, dx \right) dx \right) \\ &= \frac{3}{4} \left(-x \cos x + \int \cos x \, dx \right) - \frac{1}{4} \left(-x \frac{\cos 3x}{3} + \int \frac{\cos 3x}{3} dx \right) \end{aligned}$$

$$y = \frac{3}{4} (-x \cos x + \sin x) - \frac{1}{4} \left(-x \frac{\cos 3x}{3} + \frac{\sin 3x}{9} \right) + c$$

$$y = \frac{1}{4} \left(-3x \cos x + 3 \sin x + \frac{x}{3} \cos 3x - \frac{\sin 3x}{9} \right) + c$$

96. Question

Evaluate $\int (x+1)^2 e^x \, dx$

Answer

$$y = \int (x^2 + 2x + 1) e^x \, dx$$

$$y = \int (x^2 + 2x) e^x \, dx + \int e^x \, dx$$

We know that $\int (f(x) + f'(x)) e^x \, dx = f(x) e^x$

Here, $f(x) = x^2$ then $f'(x) = 2x$

$$y = x^2 e^x + e^x + c$$

$$y = (x^2 + 1) e^x + c$$

97. Question

Evaluate $\int \log \left(x + \sqrt{x^2 + a^2} \right) dx$

Answer

Use method of integration by parts

$$y = \log(x + \sqrt{x^2 + a^2}) \int dx - \int \frac{d}{dx} \log(x + \sqrt{x^2 + a^2}) \left(\int dx \right) dx$$

$$y = x \log(x + \sqrt{x^2 + a^2}) - \int \frac{1 + \frac{2x}{2\sqrt{x^2 + a^2}}}{x + \sqrt{x^2 + a^2}} x dx$$

$$y = x \log(x + \sqrt{x^2 + a^2}) - \int \frac{x}{\sqrt{x^2 + a^2}} dx$$

Let, $x^2 + a^2 = t$

Differentiating both side with respect to t

$$2x \frac{dx}{dt} = 1 \Rightarrow x dx = \frac{dt}{2}$$

$$y = x \log(x + \sqrt{x^2 + a^2}) - \frac{1}{2} \int \frac{1}{\sqrt{t}} dt$$

$$y = x \log(x + \sqrt{x^2 + a^2}) - \sqrt{t} + c$$

Again, put $t = x^2 + a^2$

$$y = x \log(x + \sqrt{x^2 + a^2}) - \sqrt{x^2 + a^2} + c$$

98. Question

Evaluate $\int \frac{\log x}{x^3} dx$

Answer

Use method of integration by parts

$$y = \log x \int \frac{1}{x^3} dx - \int \frac{d}{dx} \log x \left(\int \frac{1}{x^3} dx \right) dx$$

$$y = -\log x \frac{1}{2x^2} + \int \frac{1}{2x^3} dx$$

$$y = -\frac{1}{2x^2} \log x - \frac{1}{4x^2} + c$$

$$y = -\frac{1}{4x^2} (2 \log x + 1) + c$$

99. Question

Evaluate $\int \frac{\log(1-x)}{x^2} dx$

Answer

Use method of integration by parts

$$y = \log(1-x) \int \frac{1}{x^2} dx - \int \frac{d}{dx} \log(1-x) \left(\int \frac{1}{x^2} dx \right) dx$$

$$y = -\log(1-x) \frac{1}{x} - \int \frac{1}{(1-x)x} dx$$

$$y = -\frac{1}{x} \log(1-x) - \int \frac{x + (1-x)}{(1-x)x} dx$$

$$y = -\frac{1}{x} \log(1-x) - \int \frac{1}{(1-x)} + \frac{1}{x} dx$$

$$y = -\frac{1}{x} \log(1-x) + \log(1-x) - \log x + c$$

$$y = \left(1 - \frac{1}{x}\right) \log(1-x) - \log x + c$$

100. Question

Evaluate $\int x^3 (\log x)^2 dx$

Answer

Use method of integration by parts

$$y = \log^2 x \int x^3 dx - \int \frac{d}{dx} \log^2 x \left(\int x^3 dx \right) dx$$

$$y = \log^2 x \frac{x^4}{4} - \int \frac{2 \log x}{x} \frac{x^4}{4} dx$$

$$y = \frac{x^4}{4} \log^2 x - \frac{1}{2} (\log x \int x^3 dx - \int \frac{d}{dx} \log x \left(\int x^3 dx \right) dx)$$

$$y = \frac{x^4}{4} \log^2 x - \frac{1}{2} \left(\log x \frac{x^4}{4} - \int \frac{1}{x} \frac{x^4}{4} dx \right)$$

$$y = \frac{x^4}{4} \log^2 x - \frac{x^4}{8} \log x + \frac{x^4}{32} + c$$

101. Question

Evaluate $\int \frac{1}{x \sqrt{1+x^n}} dx$

Answer

Let, $\sqrt{1+x^n} = t$

Differentiate both side with respect to t

$$\frac{nx^{n-1} dx}{2\sqrt{1+x^n}} = 1 \Rightarrow \frac{dx}{x\sqrt{1+x^n}} = \frac{2dt}{n(t^2-1)}$$

$$y = \int \frac{2}{n(t^2-1)} dt$$

Use formula $\int \frac{1}{t^2-a^2} dt = \frac{1}{2a} \ln \left(\frac{t-a}{t+a} \right)$

$$y = \frac{1}{n} \ln \left(\frac{t-1}{t+1} \right) + c$$

Again put $t = \sqrt{1+x^n}$

$$y = \frac{1}{n} \ln \left(\frac{\sqrt{1+x^n}-1}{\sqrt{1+x^n}+1} \right) + c$$

102. Question

Evaluate $\int \frac{x^2}{\sqrt{1-x}} dx$

Answer

Let, $x = \sin^2 t$

Differentiate both side with respect to t

$$\frac{dx}{dt} = 2 \sin t \cos t \quad dt \Rightarrow dx = 2 \sin t \cos t \, dt$$

$$y = \int \frac{\sin^4 t}{\cos t} 2 \sin t \cos t \, dt$$

$$y = 2 \int \sin^5 t \, dt$$

$$y = 2 \int (1 - \cos^2 t)^2 \sin t \, dt$$

Let, $\cos t = z$

Differentiate both side with respect to z

$$-\sin t \frac{dt}{dz} = 1 \Rightarrow \sin t \, dt = -dz$$

$$y = -2 \int (1 - z^2)^2 dz$$

$$y = -2 \int 1 + z^4 - 2z^2 \, dz$$

$$y = -2 \left(z + \frac{z^5}{5} - 2 \frac{z^3}{3} \right) + c$$

Again put $z = \cos t$ and $t = \sin^{-1} \sqrt{x}$

$$y = -2 \left(\cos(\sin^{-1} \sqrt{x}) + \frac{\cos^5(\sin^{-1} \sqrt{x})}{5} - 2 \frac{\cos^3(\sin^{-1} \sqrt{x})}{3} \right) + c$$

$$y = -2 \left(\sqrt{1-x} + \frac{(1-x)^2 \sqrt{1-x}}{5} - \frac{2(1-x) \sqrt{1-x}}{3} \right) + c$$

$$y = \frac{-2}{15} \sqrt{1-x} (3x^2 + 4x + 8) + c$$

103. Question

Evaluate $\int \frac{x^5}{\sqrt{1+x^3}} \, dx$

Answer

Let, $1 + x^3 = t$

Differentiate both side with respect to t

$$3x^2 \frac{dx}{dt} = 1 \Rightarrow x^2 \, dx = \frac{dt}{3}$$

$$y = \frac{1}{3} \int \frac{(t-1)}{\sqrt{t}} \, dt$$

$$y = \frac{1}{3} \int \sqrt{t} - \frac{1}{\sqrt{t}} \, dt$$

$$y = \frac{1}{3} \left(\frac{2}{3} t^{\frac{3}{2}} - 2\sqrt{t} \right) + c$$

Again, put $t = 1 + x^3$

$$y = \frac{1}{3} \left(\frac{2}{3} (1 + x^3)^{\frac{3}{2}} - 2\sqrt{1 + x^3} \right) + c$$

$$y = \frac{2}{9} \sqrt{1 + x^3} (x^3 - 2) + c$$

104. Question

Evaluate $\int \frac{1+x^2}{\sqrt{1+x^2}} dx$

Answer

$$y = \int \sqrt{1+x^2} dx$$

Use formula $\sqrt{a^2 + x^2} = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln(x + \sqrt{x^2 + a^2})$

$$y = \frac{x}{2} \sqrt{x^2 + 1} + \frac{1}{2} \ln(x + \sqrt{x^2 + 1}) + c$$

105. Question

Evaluate $\int x \sqrt{\frac{1-x}{1+x}} dx$

Answer

Let, $x = \sin t$

Differentiate both side with respect to t

$$\frac{dx}{dt} = \cos t \Rightarrow dx = \cos t dt$$

$$y = \int \sin t \sqrt{\frac{1-\sin t}{1+\sin t}} \cos t dt$$

$$y = \int \sin t \sqrt{\frac{(1-\sin t)(1-\sin t)}{(1+\sin t)(1-\sin t)}} \cos t dt$$

$$y = \int \sin t (1 - \sin t) dt$$

$$y = \int \sin t dt - \int \sin^2 t dt$$

$$y = -\cos t - \int \frac{1 - \cos 2t}{2} dt$$

$$y = -\cos t - \left(\frac{t}{2} - \frac{\sin 2t}{4} \right) + c$$

Again put $t = \sin^{-1} x$

$$y = -\cos(\sin^{-1} x) - \left(\frac{(\sin^{-1} x)}{2} - \frac{\sin 2(\sin^{-1} x)}{4} \right) + c$$

$$y = -\sqrt{1-x^2} - \frac{\sin^{-1} x}{2} + \frac{x\sqrt{1-x^2}}{2} + c$$

$$y = \left(\frac{x}{2} - 1\right)\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x + c$$

66. Question

Evaluate $\int \frac{1}{\sin x(2+3\cos x)} dx$

Answer

To solve this type of solution, we are going to substitute the value of $\sin x$ and $\cos x$ in terms of $\tan(x/2)$

$$\sin x = \frac{2 \left[\tan \frac{x}{2} \right]}{1 + \tan^2 \frac{x}{2}}$$

$$\cos x = \frac{\left(1 - \frac{\tan^2 x}{2}\right)}{1 + \frac{\tan^2 x}{2}}$$

$$I = \int \frac{1}{\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \left(2 + 3 \cdot \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}\right)} dx$$

$$I = \int \frac{\sec^2 \frac{x}{2}}{2 \tan \frac{x}{2} (2 + 2 \tan^2 \frac{x}{2} + 3 - 3 \tan^2 \frac{x}{2})} dx$$

In this type of equations, we apply substitution method so that equation may be solve in simple way

Let $\tan\left(\frac{x}{2}\right) = t$

$$\frac{1}{2} \cdot \sec^2 \frac{x}{2} dx = dt$$

Put these terms in above equation, we get $I = \int \frac{dt}{t(5-t^2)}$

$$I = \int \frac{t^{-3} dt}{(5t^{-2} - 1)}$$

Let us now again apply the substitution method in above equation

Let $t^{-2} = k$

$$-2 \cdot t^{-3} dt = dk$$

Substitute these terms in above equation gives-

$$I = -\frac{1}{10} \int \frac{dk}{k}$$

$$I = \frac{1}{10k^2} = \frac{1}{10} \cdot \left(\frac{5-t^2}{t^2}\right)^2$$

$$= \frac{1}{10} \cdot \left(\frac{5}{t^2} - 1\right)^2$$

Now put the value of t , $t = \tan(x/2)$ in above equation gives us the finally solution

$$I = \frac{1}{10} \cdot \left(\frac{5}{\tan^2 \frac{x}{2}} - 1\right)^2$$

67. Question

Evaluate $\int \frac{1}{\sin x + \sin 2x} dx$

Answer

To solve this type of solution, we are going to substitute the value of $\sin x$ and $\cos x$ in terms of $\tan(x/2)$

$$\sin x = \frac{2 \left[\tan \frac{x}{2} \right]}{1 + \tan^2 \frac{x}{2}}$$

$$\cos x = \frac{\left(1 - \frac{\tan^2 x}{2} \right)}{1 + \frac{\tan^2 x}{2}}$$

$$I = \int \frac{1}{\frac{2 \tan x/2}{1 + \tan^2 \frac{x}{2}} \left(1 + 2 \cdot \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)} dx$$

$$I = \int \frac{\sec^2 \frac{x}{2}}{2 \tan \frac{x}{2} (3 - \tan^2 \frac{x}{2})} dx$$

In this type of equations we apply substitution method so that equation may be solve in simple way

Let $\tan \left(\frac{x}{2} \right) = t$

$$\frac{1}{2} \cdot \sec^2 \frac{x}{2} dx = dt$$

Put these terms in above equation, we get $I = \int \frac{dt}{t(3-t^2)}$

$$I = \int \frac{t^{-3} dt}{(3t^{-2} - 1)}$$

Let us now again apply the substitution method in above equation

Let $t^{-2} = k$

$$-2 \cdot t^{-3} dt = dk$$

Substitute these terms in above equation gives-

$$I = -\frac{1}{6} \int \frac{dk}{k}$$

$$I = \frac{1}{6k^2}$$

$$= \frac{1}{6} \cdot \left(\frac{3 - t^2}{t^2} \right)^2$$

$$= \frac{1}{6} \cdot \left(\frac{3}{t^2} - 1 \right)^2$$

Now put the value of t , $t = \tan(x/2)$ in above equation gives us the finally solution

$$I = \frac{1}{6} \cdot \left(\frac{3}{\tan^2 \frac{x}{2}} - 1 \right)^2$$

68. Question

Evaluate $\int \frac{1}{\sin^4 x + \cos^4 x} dx$

Answer

Consider $\int \frac{1}{\sin^4 x + \cos^4 x} dx$,

Divide num and denominator by $\cos^4 x$ to get,

$$\int \frac{1}{\sin^4 x + \cos^4 x} dx = \int \frac{\frac{1}{\cos^4 x}}{\frac{\sin^4 x}{\cos^4 x} + \frac{\cos^4 x}{\cos^4 x}} dx$$

$$= \int \frac{\sec^4 x}{\tan^4 x + 1} dx$$

$$= \int \frac{\sec^2 x \cdot \sec^2 x}{\tan^4 x + 1} dx$$

$$= \int \frac{\sec^2 x (1 + \tan^2 x)}{\tan^4 x + 1} dx$$

Let $\tan x = t$

$$\sec^2 x dx = dt$$

$$= \int \frac{(1 + t^2)}{t^4 + 1} dt$$

Now divide both numerator and denominator by $\frac{1}{t^2}$ to get,

$$= \int \frac{\left(\frac{1}{t^2} + 1\right)}{\left(t^2 + \frac{1}{t^2}\right) + 2 - 2} dt$$

$$= \int \frac{\left(\frac{1}{t^2} + 1\right)}{\left(1 - \frac{1}{t}\right)^2 + 2} dt$$

Let $1 - \frac{1}{t} = u$

$$\left(1 + \frac{1}{t^2}\right) dt = du$$

$$= \int \frac{du}{u^2 + 2}$$

$$= \int \frac{du}{u^2 + (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{1 - \frac{1}{t}}{\sqrt{2}} \right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{1 - \frac{1}{\tan x}}{\sqrt{2}} \right) + c$$

69. Question

Evaluate $\int \frac{1}{5 - 4 \sin x} dx$

Answer

in this integral we are going to put the value of $\sin(x)$ in terms of $\tan(x/2)$ -

$$I = \int \frac{2dt}{5 + 5t^2 - 8t}$$

$$I = \frac{2}{5} \int \frac{1}{\left(t - \frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} dt$$

By applying the formula of $1/(x^2+a^2)$ in above equation yields the integral-

$$I = \frac{2}{5} \cdot \frac{1}{\frac{3}{5}} \cdot \tan^{-1} \frac{\left(t - \frac{4}{5}\right)}{\left(\frac{3}{5}\right)}$$

$$I = \frac{2}{3} \cdot \tan^{-1} \frac{5t - 4}{3}$$

By putting the value of t in above equation ,

$$I = \frac{2}{3} \cdot \tan^{-1} \left(\frac{5}{3} \tan \frac{x}{2} - \frac{4}{3} \right)$$

70. Question

Evaluate $\int \sec^4 x \, dx$

Answer

above equation can be solve by using one formula that is $(1 + \tan^2 x = \sec^2 x)$

$$I = \int \sec^4 x \, dx$$

$$= \int \sec^2 x \sec^2 x \, dx$$

$$= \int \sec^2 x (1 + \tan^2 x) \, dx$$

$$= \int \sec^2 x \, dx + \int \sec^2 x \tan^2 x \, dx$$

Put $\tan x = t$ in above equation so that $\sec^2 x dx = dt$

$$I = \tan x + \int t^2 dt = \tan x + \frac{t^3}{3}$$

$$= \tan x + \frac{\tan^3 x}{3}$$

71. Question

Evaluate $\int \operatorname{cosec}^4 2x \, dx$

Answer

above equation can we solve by the formula of $(1 + \cot^2 x = \operatorname{cosec}^2 x)$

$$I = \int \operatorname{cosec}^4 2x \, dx$$

$$= \int \operatorname{cosec}^2 2x (1 + \cot^2 2x) \, dx$$

$$= \int \operatorname{cosec}^2 2x \, dx + \int \operatorname{cosec}^2 2x \cot^2 2x \, dx$$

Let us consider that $\cot 2x = t$ then $-2 \cdot \operatorname{cosec}^2 2x dx = dt$

$$I = -\frac{\cot(2x)}{2} - \frac{1}{2} \cdot (t^2 dt)$$

$$I = -\frac{\cot(2x)}{2} - \frac{1}{6} \cdot (\cot 2x)^3$$

72. Question

Evaluate $\int \frac{1 + \sin x}{\sin x (1 + \cos x)} dx$

Answer

first divide nominator by denominator -

$$\begin{aligned} I &= \int \frac{1}{\sin x (1 + \cos x)} dx + \int \frac{1}{1 + \cos x} dx \\ &= \int \frac{1}{\sin x (1 + \cos x)} dx + \int \frac{1}{1 + 2\cos^2 x - 1} dx \end{aligned}$$

: To solve this type of solution, we are going to substitute the value of $\sin x$ and $\cos x$ in terms of $\tan(x/2)$

$$\sin x = \frac{2 \left[\tan \frac{x}{2} \right]}{1 + \tan^2 \frac{x}{2}}$$

$$\cos x = \frac{\left(1 - \frac{\tan^2 x}{2} \right)}{1 + \frac{\tan^2 x}{2}}$$

$$I = \int \frac{1}{\frac{2 \tan x/2}{1 + \tan^2 \frac{x}{2}} \left(1 + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)} dx$$

$$I = \int \frac{\sec^2 x/2}{2 \tan x/2 (1 + \tan^2 \frac{x}{2} + 1 - \tan^2 \frac{x}{2})} dx$$

In this type of equations we apply substitution method so that equation may be solve in simple way

Let $\tan(x/2) = t$

$$1/2 \cdot \sec^2(x/2) dx = dt$$

Put these terms in above equation, we get $I = \int \frac{dt}{2t}$

Substitute these terms in above equation gives-

$$I = \frac{1}{2} \int \frac{dt}{t}$$

$$I = \frac{-1}{2t^2}$$

Now put the value of t , $t = \tan(x/2)$ in above equation gives us the finally solution

$$I = \frac{-1}{2} \cdot \left(\frac{1}{\tan^2 \frac{x}{2}} \right)$$

73. Question

Evaluate $\int \frac{1}{2 + \cos x} dx$

Answer

To solve this type of solution ,we are going to substitute the value of sinx and cosx in terms of tan(x/2)

$$\sin x = \frac{2 \left[\tan \frac{x}{2} \right]}{1 + \tan^2 \frac{x}{2}}$$

$$\cos x = \frac{\left(1 - \frac{\tan^2 x}{2} \right)}{1 + \frac{\tan^2 x}{2}}$$

$$I = \int \frac{1}{\left(2 + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right)} dx$$

$$I = \int \frac{\sec^2 \frac{x}{2}}{(2 + 2 \tan^2 \frac{x}{2} + 1 - \tan^2 \frac{x}{2})} dx$$

In this type of equations we apply substitution method so that equation may be solve in simple way

Let $\tan(x/2)=t$

$$1/2 \cdot \sec^2(x/2) dx = dt$$

Put these terms in above equation,we get $I = 2 \int \frac{dt}{(3+t^2)}$

$$I = \frac{2.1}{(\sqrt{3})} \tan^{-1} \frac{t}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} \cdot \tan^{-1} \left(\frac{x}{2\sqrt{3}} \right)$$

74. Question

Evaluate $\int \sqrt{\frac{a+x}{x}} dx$

Answer

to solve this integral we have to apply substitution method for which we are going to put $x=a \cdot \tan^2 k$

This means $dx = 2 \cdot a \cdot \tan k \cdot \sec^2 k \cdot dk$, then I will be,

$$I = \int \sqrt{\frac{a \sec^2 k}{a \tan^2 k}} \cdot 2a \cdot \tan k \cdot \sec^2 k \cdot dk = 2a \cdot \operatorname{cosec} k \cdot \tan k \cdot \sec^2 k \cdot dk$$

In this above integral let $\tan k = t$ then $\sec^2 k dk = dt$, put in above equation-

$$I = 2a \int \sqrt{(t^2 + 1)} \cdot dt$$

Apply the formula of $\sqrt{x^2 + a^2} = x/2 \cdot \sqrt{a^2 + x^2} + a^2/2 \ln|x + \sqrt{a^2 + x^2}|$

$$I = 2a \left[\frac{t}{2} \cdot \sqrt{1 + t^2} + \frac{1}{2} \cdot \ln |t + \sqrt{1 + t^2}| \right]$$

Now put the value of t in above integral $t = \tan k$, then finally integral will be-

$$I = 2a \left[\frac{\tan k}{2} \cdot \sqrt{1 + \tan^2 k} + \frac{1}{2} \cdot \ln |\tan k + \sqrt{1 + \tan^2 k}| \right]$$

Now put the value of k in terms of x that is $\tan^2 k = x/a$ in above integral -

$$I = 2a \left[\frac{1}{2} \sqrt{\frac{x}{a}} \cdot \sqrt{1 + \frac{x}{a}} + \frac{1}{2} \cdot \ln \left| \frac{1}{2} \sqrt{\frac{x}{a}} + \sqrt{1 + \frac{x}{a}} \right| \right]$$

75. Question

Evaluate $\int \frac{6x+5}{\sqrt{6+x-2x^2}} dx$

Answer

$$y = 6 \int \frac{x + \frac{5}{6}}{\sqrt{6+x-2x^2}} dx$$

$$y = \frac{6}{-4} \int \frac{-4\left(x + \frac{5}{6}\right)}{\sqrt{6+x-2x^2}} dx$$

$$y = -\frac{3}{2} \int \frac{-4x - \frac{10}{3} + 1 - 1}{\sqrt{6+x-2x^2}} dx$$

$$y = -\frac{3}{2} \int \frac{-4x + 1}{\sqrt{6+x-2x^2}} dx - \frac{3}{2} \int \frac{-\frac{10}{3} - 1}{\sqrt{6+x-2x^2}} dx$$

$$A = -\frac{3}{2} \int \frac{-4x + 1}{\sqrt{6+x-2x^2}} dx$$

Let, $6 + x - 2x^2 = t$

Differentiating both side with respect to t

$$(1 - 4x) \frac{dx}{dt} = 1 \Rightarrow (1 - 4x) dx = dt$$

$$A = -\frac{3}{2} \int \frac{1}{\sqrt{t}} dt$$

$$A = -\frac{3}{2} 2\sqrt{t} + c_1$$

Again, put $t = 6 + x - 2x^2$

$$A = -3\sqrt{6+x-2x^2} + c_1$$

$$B = -\frac{3}{2} \int \frac{-\frac{10}{3} - 1}{\sqrt{6+x-2x^2}} dx$$

$$B = \frac{13}{2} \int \frac{1}{\sqrt{6+x-2x^2}} dx$$

Make perfect square of quadratic equation

$$6 + x - 2x^2 = 2 \left(\left(\frac{7}{4} \right)^2 - \left(x - \frac{1}{4} \right)^2 \right)$$

$$B = \frac{13}{2\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{7}{4} \right)^2 - \left(x - \frac{1}{4} \right)^2}} dx$$

Use formula $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}$

$$B = \frac{13}{2\sqrt{2}} \sin^{-1} \frac{\left(x - \frac{1}{4} \right)}{\left(\frac{7}{4} \right)} + c_2$$

$$B = \frac{13}{2\sqrt{2}} \sin^{-1} \frac{4x-1}{7} + c_2$$

The final solution of the question is $y = A + B$

$$y = -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}} \sin^{-1} \left(\frac{4x-1}{7} \right) + C$$

76. Question

Evaluate $\int \frac{\sin^5 x}{\cos^4 x} dx$

Answer

to solve this type of integration we have to let $\cos x$ either $\sin x = t$ then manipulate them

Let $\cos x = t$ then $-\sin x dx = dt$

Also apply the formula of $(\sin^2 t + \cos^2 t = 1)$

$$\begin{aligned} I &= \int \frac{\sin^5 x}{\cos^4 x} dx = - \int \frac{(1-t^2)^2}{t^4} dt \\ &= - \int \frac{1+t^4-2t^2}{t^4} dt \\ &= - \left[\int t^{-4} dt + \int 1 dt - \int \frac{2}{t^2} dt \right] \\ I &= \frac{t^{-3}}{3} - t - \frac{2}{t} \end{aligned}$$

Now put the value of t in above integral

$$I = \frac{1}{3\cos^3 x} - \cos x - \frac{2}{\cos x}$$

77. Question

Evaluate $\int \frac{\cos^5 x}{\sin x} dx$

Answer

to solve this type of integration we have to let $\cos x$ either $\sin x = t$ then manipulate them

Let $\sin x = t$ then $\cos x dx = dt$

Also apply the formula of $(\sin^2 t + \cos^2 t = 1)$

$$\begin{aligned} I &= \int \frac{\cos^5 x}{\sin x} dx \\ &= \int \frac{(1-t^2)^2}{t} dt = \int \frac{1+t^4-2t^2}{t} dt = \int \frac{1}{t} dt + \int t^3 dt - \int 2t dt \\ I &= -\frac{1}{t^2} + \frac{t^4}{4} - t^2 \end{aligned}$$

Now put the value of t in above integral

$$I = \frac{-1}{\sin^2 x} + (\sin^4 x)/4 - \sin^2 x$$

78. Question

Evaluate $\int \frac{\sin^6 x}{\cos x} dx$

Answer

$$y = \int \left(\frac{\sin^4 x (1 - \cos^2 x)}{\cos x} \right) dx$$

$$y = \int \left(\frac{\sin^4 x}{\cos x} - \frac{\sin^4 x \cos^2 x}{\cos x} \right) dx$$

$$y = \int \left(\frac{\sin^2 x (1 - \cos^2 x)}{\cos x} - \sin^4 x \cos x \right) dx$$

$$y = \int \left(\frac{\sin^2 x}{\cos x} - \frac{\sin^2 x \cos^2 x}{\cos x} - \sin^4 x \cos x \right) dx$$

$$y = \int \left(\frac{\sin^2 x}{\cos x} - \sin^2 x \cos x - \sin^4 x \cos x \right) dx$$

$$y = \int \left(\frac{1 - \cos^2 x}{\cos x} \right) dx - \int (\sin^2 x \cos x + \sin^4 x \cos x) dx$$

Let, $\sin x = t$

Differentiating both side with respect to t

$$\cos x \frac{dx}{dt} = 1 \Rightarrow \cos x dx = dt$$

$$y = \int \left(\frac{1}{\cos x} - \cos x \right) dx - \int t^2 + t^4 dt$$

$$y = \ln(\sec x + \tan x) - \sin x - \frac{t^3}{3} - \frac{t^5}{5} + c$$

Again put $t = \sin x$

$$y = \ln(\sec x + \tan x) - \sin x - \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$$

$$y = \frac{1}{2} \ln \left(\frac{1 + \sin x}{1 - \sin x} \right) - \sin x - \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$$

79. Question

Evaluate $\int \frac{\sin^2 x}{\cos^6 x} dx$

Answer

dividing by $\cos^6 x$ yields-

$$I = \int \tan^2 x \cdot \sec^4 x dx$$

Let us consider $\tan x = t$

Then $\sec^2 x dx = dt$, put in above equation-

$$I = \int t^2 (1 + t^2) dt = \int (t^2 + t^4) dt = \int t^2 dt + \int t^4 dt = \frac{t^3}{3} + \frac{t^5}{5}$$

Now repute the value of t, which is $t = \tan x$

$$I = \frac{(\tan^3 x)}{3} + \frac{\tan^5 x}{5}$$

80. Question

Evaluate $\int \sec^6 x \, dx$

Answer

in this integral we will use the formula $1 + \tan^2 x = \sec^2 x$,

$$I = \int \sec^2 x \sec^4 x \, dx$$

$$= \int \sec^2 x (1 + \tan^2 x)^2 \, dx$$

Now put $\tan x = t$ which means $\sec^2 x dx = dt$,

$$I = \int (1 + t^2)^2 \, dt$$

$$= \int (1 + t^4 + 2t^2) \, dt$$

Now put the value of t , which is $t = \tan x$ in above integral-

$$I = \tan x + \frac{\tan^5 x}{5} + 2 \cdot \frac{\tan^3 x}{3}$$

81. Question

Evaluate $\int \tan^5 x \sec^3 x \, dx$

Answer

in this integral we will use the formula $1 + \tan^2 x = \sec^2 x$,

Then equation will be transform in below form-

$$I = \int \tan^5 x \sec^2 x \sec x \, dx$$

$$= \int \sec x \tan^5 x \sec^2 x \, dx$$

Now put $\tan x = t$ which means $\sec^2 x dx = dt$,

$$I = \int t^5 \cdot \sqrt{1 + t^2} \, dt$$

In this above integral put $1 + t^2 = k^2$

that is mean $t dt = k dk$

$$I = \int (k^4 + 1 - 2k) k^2 \, dk$$

$$= \int (k^6 + k^2 - 2k^3) \, dk$$

$$= \frac{k^7}{7} + \frac{k^3}{3} - \frac{k^4}{2}$$

Now put the value of $k = (1 + t^2) = \sec^2 x$ in above equation-

$$I = \frac{\sec^{14} x}{7} + \frac{\sec^6 x}{3} - \frac{\sec^8 x}{2}$$

82. Question

Evaluate $\int \tan^3 x \sec^4 x \, dx$

Answer

in this integral we will use the formula $1 + \tan^2 x = \sec^2 x$,

Then equation will be transform in below form-

$$I = \int \tan^3 x \sec^2 x \sec^2 x \, dx$$

$$= \int \tan^3 x (1 + \tan^2 x) \sec^2 x \, dx$$

Now put $\tan x = t$ which means $\sec^2 x dx = dt$,

$$I = \int t^3 (1 + t^2) dt = \int (t^4 + t^5) dt$$

$$I = \frac{t^5}{5} + \frac{t^6}{6}$$

Now put the value of t , which is $t = \tan x$ in above integral-

$$I = \frac{\tan^5 x}{5} + \frac{\tan^6 x}{6}$$

83. Question

Evaluate $\int \frac{1}{\sec x + \operatorname{cosec} x} dx$

Answer

$$y = \int \frac{\sin x \cos x}{\sin x + \cos x} dx$$

$$y = \frac{1}{2} \int \frac{1 + 2 \sin x \cos x - 1}{\sin x + \cos x} dx$$

Use $1 = \sin^2 x + \cos^2 x$

$$y = \frac{1}{2} \int \frac{\sin^2 x + \cos^2 x + 2 \sin x \cos x}{\sin x + \cos x} dx - \frac{1}{2} \int \frac{1}{\sin x + \cos x} dx$$

$$\text{Use } \sin x + \cos x = \sqrt{2} \left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right)$$

$$= \sqrt{2} \left(\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} \right)$$

$$= \sqrt{2} \sin \left(x + \frac{\pi}{4} \right)$$

$$y = \frac{1}{2} \int \frac{(\sin x + \cos x)^2}{\sin x + \cos x} dx - \frac{1}{2} \int \frac{1}{\sqrt{2} \sin \left(x + \frac{\pi}{4} \right)} dx$$

$$y = \frac{1}{2} \int \sin x + \cos x \, dx - \frac{1}{2\sqrt{2}} \int \operatorname{cosec} \left(x + \frac{\pi}{4} \right) dx$$

$$y = \frac{1}{2} (-\cos x + \sin x) - \frac{1}{2\sqrt{2}} \ln \left(\tan \left(\frac{x}{2} + \frac{\pi}{8} \right) \right) + c$$

84. Question

Evaluate $\int \sqrt{a^2 + x^2} \, dx$

Answer

in these type of problems we put the value of $x = a \tan k$

That is mean that $dx = a \sec^2 k \, dk$

$$I = \int \sqrt{a^2 + a^2 \tan^2 k} \, a \sec^2 k \, dk$$

$$= \int a \cdot \sec k \cdot a \cdot \sec^2 k \, dk$$

$$= \int a^2 \sec^3 k \, dk$$

By upper solve questions we can find out the value of integration of $\sec^3 x$, which is equal to

$$i = \int \sec^3 x \, dx = \frac{1 + \sec x \cdot \tan x}{2}$$

Put the value of integration of $\sec^3 x$ in above equation we get our finally integral which is -

$$I = a^2 \cdot \frac{1 + \sec k \cdot \tan k}{2}$$

Now put the value of k which is $\tan^{-1}(x/a)$ in above equation-

$$I = a^2 \cdot \left(\frac{1 + \frac{x}{a} \cdot \sec(\tan^{-1} \frac{x}{a})}{2} \right)$$

85. Question

Evaluate $\int \sqrt{x^2 - a^2} \, dx$

Answer

Consider $\int \sqrt{x^2 - a^2} \, dx$,

Let $I = \int \sqrt{x^2 - a^2} \, dx$ and $II = \int 1 \, dx$

As $\int I \cdot II \, dx = I \cdot \int II \, dx - \int [d/dx(I) \cdot \int II \, dx]$

So,

$$= \int \sqrt{x^2 - a^2} \, dx - \int \frac{d}{dx}(\sqrt{x^2 - a^2}) \cdot \int 1 \, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{1}{2\sqrt{x^2 - a^2}} \cdot 2x \cdot x \, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}} \, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}} \, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2}{\sqrt{x^2 - a^2}} \, dx - \int \frac{a^2}{\sqrt{x^2 - a^2}} \, dx$$

$$I = x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2} \, dx - \int \frac{a^2}{\sqrt{x^2 - a^2}} \, dx$$

$$I = x\sqrt{x^2 - a^2} - I - \int \frac{a^2}{\sqrt{x^2 - a^2}} \, dx$$

$$2I = x\sqrt{x^2 - a^2} - \int \frac{a^2}{\sqrt{x^2 - a^2}} \, dx$$

$$2I = x\sqrt{x^2 - a^2} - a^2 \log|x + \sqrt{x^2 - a^2}| + c$$

$$I = \frac{1}{2} \left(x\sqrt{x^2 - a^2} - a^2 \log|x + \sqrt{x^2 - a^2}| + c \right)$$

46. Question

Evaluate $\int \frac{1}{1-x-4x^2} dx$

Answer

Given, $\int \frac{1}{(1-x-4x^2)} dx$

$$= -\int \frac{1}{4x^2 + x - 1} dx$$

$$= -\int \frac{1}{4x^2 + x + \frac{1}{16} - \frac{17}{16}} dx$$

$$= -\int \frac{1}{\left(2x + \frac{1}{4}\right)^2 - \frac{17}{16}} dx$$

$$= -\int \frac{1}{\left(2x + \frac{1}{4}\right)^2 - \left(\frac{\sqrt{17}}{4}\right)^2} dx$$

It is clearly of the form, $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \frac{x-a}{x+a} + c$

Where $x = 2x + \frac{1}{4}$; $a = \frac{\sqrt{17}}{4}$

$$= -\frac{1}{2\left(\frac{\sqrt{17}}{4}\right)} \log \frac{2x + \frac{1}{4} - \frac{\sqrt{17}}{4}}{2x + \frac{1}{4} + \frac{\sqrt{17}}{4}} + c$$

$$= -\frac{2}{\sqrt{17}} \log \frac{2x + \frac{1}{4} - \frac{\sqrt{17}}{4}}{2x + \frac{1}{4} + \frac{\sqrt{17}}{4}} + c$$

47. Question

Evaluate $\int \frac{1}{3x^2 + 13x - 10} dx$

Answer

Given, $\int \frac{1}{3x^2 + 13x - 10} dx$

Now, $3x^2 + 13x - 10$

$$= 3x^2 + 15x - 2x - 10$$

$$= 3x(x+5) - 2(x-5)$$

$$= (x-5)(3x-2)$$

$$\frac{1}{3x^2 + 13x - 10} \cong \frac{A}{x+5} + \frac{B}{3x-2}$$

$$1 \cong A(3x-2) + B(x+5)$$

Equating 'x' coeff: -

$$0 = 3A + B$$

$$B = -3A$$

Equating constant:-

$$1 = -2A + 5B$$

$$1 = -2A + 5(-3A)$$

$$1 = -2A - 15A$$

$$1 = -17A$$

$$A = -\frac{1}{17}$$

$$B = -3\left(-\frac{1}{17}\right)$$

$$B = \frac{3}{17}$$

$$\frac{1}{3x^2 + 13x - 10} \cong -\frac{1}{17(x+5)} + \frac{3}{17(3x-2)}$$

$$\int \frac{1}{3x^2 + 13x - 10} dx = \int -\frac{1}{17(x+5)} + \frac{3}{17(3x-2)} dx$$

$$= -\frac{1}{17} \int \frac{1}{x+5} dx + \frac{3}{17} \int \frac{1}{3x-2} dx$$

$$= -\frac{1}{17} \log(x+5) + \frac{3}{17} \log(3x-2) + c$$

48. Question

Evaluate $\int \frac{\sin x}{\cos^2 x - 2 \cos x - 3} dx$

Answer

Given, $\int \frac{\sin x}{\cos^2 x - 2 \cos x - 3} dx$

Let $\cos x = t$

$-\sin x dx = dt$

$$= \int \frac{dt}{t^2 - 2t - 3}$$

Now, $t^2 - 2t - 3$

$$= t^2 - 3t + t - 3$$

$$= t(t-3) + t-3$$

$$= (t-3)(t+1)$$

$$\frac{1}{t^2 - 2t - 3} \cong \frac{A}{t-3} + \frac{B}{t+1}$$

$$1 \cong A(t-1) + B(t-3)$$

Equating 't' coeff:-

$$0 = A + B$$

$$A = -B$$

Equating constant:-

$$1 = -A - 3B$$

$$1 = -(-B) - 3B$$

$$1 = -2B$$

$$B = \frac{-1}{2}$$

$$A = -\left(\frac{-1}{2}\right)$$

$$A = \frac{1}{2}$$

$$\frac{1}{t^2 - 2t - 3} \cong \frac{1}{2(t-3)} + \frac{-1}{2(t+1)}$$

$$\int \frac{1}{t^2 - 2t - 3} dt = \frac{1}{2} \int \frac{1}{t-3} dt - \frac{1}{2} \int \frac{1}{t+1} dt$$

$$= \frac{1}{2} \log(t-3) - \frac{1}{2} \log(t+1) + c$$

$$= \frac{1}{2} [\log(\cos x - 3) - \log(\cos x + 1)] + c$$

49. Question

Evaluate $\int \sqrt{\operatorname{cosec} x - 1} dx$

Answer

Given, $\int \sqrt{\operatorname{cosec} x - 1} dx$

$$= \int \sqrt{\frac{1}{\sin x} - 1} dx$$

$$= \int \sqrt{\frac{1 - \sin x}{\sin x}} dx$$

Rationalising the denominator:-

$$= \int \sqrt{\frac{(1 - \sin x)(1 + \sin x)}{(\sin x)(1 + \sin x)}} dx$$

$$= \int \sqrt{\frac{(1 - \sin^2 x)}{\sin x(1 + \sin x)}} dx$$

$$= \int \sqrt{\frac{\cos^2 x}{\sin x(1 + \sin x)}} dx$$

$$= \int \frac{\cos x}{\sqrt{\sin x(1 + \sin x)}} dx$$

Let $\sin x = t$

$$\cos x dx = dt$$

$$= \int \frac{dt}{\sqrt{t(t+1)}}$$

$$= \int \frac{dt}{\sqrt{t^2 + t}}$$

$$= \int \frac{dt}{\sqrt{t^2 + t - \frac{1}{4} + \frac{1}{4}}}$$

$$= \int \frac{dt}{\sqrt{\left(t + \frac{1}{2}\right)^2 - \frac{1}{4}}}$$

$$= \int \frac{dt}{\sqrt{\left(t + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

Clearly, it is of the form $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cos^{-1}\left(\frac{x}{a}\right)$

Where $x = t + \frac{1}{2}$; $a = \frac{1}{2}$

$$= \cos^{-1}\left(\frac{t + \frac{1}{2}}{\frac{1}{2}}\right) + c$$

$$= \cos^{-1}\left[2\left(\sin x + \frac{1}{2}\right)\right] + c$$

50. Question

Evaluate $\int \frac{1}{\sqrt{3 - 2x - x^2}} dx$

Answer

Given, $\int \frac{1}{\sqrt{3 - 2x - x^2}} dx$

$$= \int \frac{1}{\sqrt{4 - 1 - 2x - x^2}} dx$$

$$= \int \frac{1}{\sqrt{4 - (x^2 + 2x + 1)}} dx$$

$$= \int \frac{1}{\sqrt{4 - (x + 1)^2}} dx$$

$$= \int \frac{1}{\sqrt{(2)^2 - (x + 1)^2}} dx$$

It is clearly of the form, $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Where, $a=2$; $x=x+1$

$$= \sin^{-1}\left(\frac{x + 1}{2}\right) + c$$

51. Question

Evaluate $\int \frac{x+1}{x^2 + 4x + 5} dx$

Answer

Given, $\int \frac{x+1}{x^2 + 4x + 5} dx$

Consider, $x+1 \cong A \frac{dy}{dx}(x^2 + 4x + 5) + B$

$$x+1 \cong A(2x+4)+B$$

Equating 'x' coeff:-

$$1=2A$$

$$A = \frac{1}{2}$$

equating constant:-

$$1=4A+B$$

$$1 = 4\left(\frac{1}{2}\right) + B$$

$$1=2+B$$

$$B=-1$$

$$x+1 \cong \frac{1}{2} (2x+4)-1$$

$$\text{Now, } \int \frac{x+1}{x^2+4x+5} dx$$

$$= \int \frac{\frac{1}{2}(2x+4) - 1}{x^2+4x+5} dx$$

$$= \frac{1}{2} \int \frac{2x+4}{x^2+4x+5} dx - \int \frac{1}{x^2+4x+5} dx$$

$$[\text{Since, } \int \frac{f'(x)}{f(x)} dx = \log[f(x)] + c]$$

$$= \frac{1}{2} \log(x^2+4x+5) - \int \frac{1}{x^2+4x+4+1} dx$$

$$= \frac{1}{2} \log(x^2+4x+5) - \int \frac{1}{(x+2)^2+(1)^2} dx$$

$$= \frac{1}{2} \log(x^2+4x+5) - \frac{1}{1} \tan^{-1}\left(\frac{x+2}{1}\right) dx$$

$$[\text{Since, } \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c]$$

$$= \frac{1}{2} \log(x^2+4x+5) - \tan^{-1}(x+2) + c$$

52. Question

$$\text{Evaluate } \int \frac{5x+7}{\sqrt{(x-5)(x-4)}} dx$$

Answer

$$\text{Given, } \int \frac{5x+7}{\sqrt{(x-5)(x-4)}} dx$$

$$= \int \frac{5x+7}{\sqrt{x^2-9x+20}} dx$$

$$\text{Now, } 5x+7 \cong A \frac{dy}{dx}(x^2-9x+20) + B$$

$$5x+7 \cong A(2x-9)+B$$

Equating 'x' coeff:-

$$5=2A$$

$$A=\frac{5}{2}$$

Equating constant:-

$$7=-9A+B \quad 7=-9\left(\frac{5}{2}\right)+B$$

$$B=7+\frac{45}{2}$$

$$B=\frac{59}{2}$$

$$5x+7 \cong \frac{5}{2}(2x-9) + \frac{59}{2}$$

$$= \int \frac{5x-7}{\sqrt{x^2-9x+20}} dx$$

$$= \int \frac{\frac{5}{2}(2x-9) + \frac{59}{2}}{\sqrt{x^2-9x+20}} dx$$

$$= \frac{5}{2} \int \frac{2x-9}{\sqrt{x^2-9x+20}} dx + \frac{59}{2} \int \frac{1}{\sqrt{x^2-9x+20}} dx$$

$$[\text{Since, } \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c]$$

$$= \frac{5}{2} \cdot 2(\sqrt{x^2-9x+20}) + \frac{59}{2} \int \frac{1}{\sqrt{\left(x+\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

$$= 5\sqrt{x^2-9x+20} + \frac{59}{2} \cdot \frac{1}{2\left(\frac{1}{2}\right)} \cdot \cosh^{-1}\left[\frac{x+\frac{9}{2}}{\frac{1}{2}}\right] + c \quad [\text{since, } \int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1}\left[\frac{x}{a}\right] + c]$$

$$= 5\sqrt{x^2-9x+20} + \frac{59}{2} \cosh^{-1} 2\left[x+\frac{9}{2}\right] + c$$

53. Question

Evaluate $\int \sqrt{\frac{1+x}{x}} dx$

Answer

Given, $\int \sqrt{\frac{1+x}{x}} dx$

Let $\sqrt{x+1} = u$

$$\Rightarrow u^2 = x+1$$

$$\Rightarrow u^2 - 1 = x$$

$$\frac{1}{2\sqrt{x+1}} dx = du$$

$$2 du = dx$$

$$\begin{aligned}
\int \sqrt{\frac{1+x}{x}} dx &= \int \frac{u}{u^2-1} 2u du \\
&= 2 \int \frac{u^2}{u^2-1} du \\
&= 2 \int \frac{u^2-1+1}{u^2-1} du \\
&= 2 \left[\int \frac{u^2-1}{u^2-1} du + \int \frac{1}{u^2-1} du \right] \\
&= 2 \left[\int 1 du + \int \frac{1}{u^2-1} du \right]
\end{aligned}$$

As we know,

$$\begin{aligned}
\int \frac{1}{x^2-a^2} dx &= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c \\
&= 2 \left[u + \frac{1}{2} \log \left| \frac{u-1}{u+1} \right| \right] + c
\end{aligned}$$

Now substitute back the value of u.

$$= 2\sqrt{x+1} + \frac{1}{2} \log \left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + c$$

54. Question

Evaluate $\int \sqrt{\frac{1-x}{x}} dx$

Answer

Given, $\sqrt{\frac{1-x}{x}} dx$

Let, $\sqrt{x} = t$

$$\frac{d}{dx}(\sqrt{x}) = dt$$

$$\frac{1}{2\sqrt{x}} dx = dt$$

$$dx = 2t dt$$

Now, $\int \frac{\sqrt{1-t^2}}{t} 2t dt$

$$= 2 \int \sqrt{1-t^2} dt$$

Consider, $t = \sin k$

$$dt = \cos k dk$$

$$= 2 \int \sqrt{1-\sin^2 k} \cdot \cos k dk$$

$$= 2 \int \sqrt{\cos^2 k} \cdot \cos k dk$$

$$= 2 \int \cos^2 k dk$$

$$= \int 2 \cos^2 k \, dk$$

$$= \int \cos 2k - 1 \, dk \text{ [since, } \cos 2x = 2\cos^2 x - 1]$$

$$= \frac{\sin 2k}{2} - k + c$$

$$= \frac{2 \sin k \cos k}{2} - k + c$$

$$= t \cos(\sin^{-1} t) - 2\sin^{-1} t + 2c$$

$$= \sqrt{x} \cos(\sin^{-1} \sqrt{x}) - 2 \sin^{-1} \sqrt{x} + 2c$$

55. Question

Evaluate $\int \frac{\sqrt{a} - \sqrt{x}}{1 - \sqrt{ax}} \, dx$

Answer

Given, $\int \frac{\sqrt{a} - \sqrt{x}}{1 - \sqrt{ax}} \, dx$

Let $1 - \sqrt{ax} = t$

$$-\frac{1}{2\sqrt{ax}} a \, dx = dt$$

$$dx = -\frac{2\sqrt{ax}}{a} \, dt$$

Now,

$$\sqrt{ax} = 1 + t$$

$$ax = (1 + t)^2$$

$$x = \frac{(1 + t)^2}{a}$$

$$= \int \frac{\sqrt{a} - \sqrt{\frac{(1 + t)^2}{a}}}{t} \times \frac{-2\sqrt{a}(1 + t)}{a} \, dt$$

$$= \int \frac{\sqrt{a} - \left(\frac{1 + t}{\sqrt{a}}\right)}{t} \times \frac{-2\sqrt{a}(1 + t)}{a} \, dt$$

$$= \int \frac{a - 1 - t}{t} \times \frac{-2\sqrt{a}(1 + t)}{a\sqrt{a}} \, dt$$

$$= \int \frac{(a - 1 - t)}{t} \times \frac{-2(1 + t)}{a} \, dt$$

$$= 2 \int \frac{(a - 1 - t)}{t} \times \frac{(-1 - t)}{a} \, dt$$

$$= 2 \int \frac{(-a - at + 1 + t + t + t^2)}{at} \, dt$$

$$= 2 \int \frac{(-a - at + 1 + 2t + t^2)}{at} \, dt$$

$$= 2 \int \left(-\frac{1}{t} - 1 + \frac{1}{at} + \frac{2}{a} + \frac{t}{a} \right) dt$$

$$= 2 \left[-\log t - t + \frac{1}{a} \log t + \frac{2}{a} t + \frac{t^2}{2a} \right] + c$$

$$= \left[-2 \log t - 2t + \frac{2}{a} \log t + \frac{4}{a} t + \frac{t^2}{a} \right] + c$$

Put back the value of t to get,

$$= \left[-2 \log(1 - \sqrt{ax}) - 2(1 - \sqrt{ax}) + \frac{2}{a} \log(1 - \sqrt{ax}) + \frac{4}{a} (1 - \sqrt{ax}) + \frac{(1 - \sqrt{ax})^2}{a} \right] + c$$

56. Question

Evaluate $\int \frac{1}{(\sin x - 2 \cos x)(2 \sin x + \cos x)} dx$

Answer

$$\begin{aligned} \text{Given, } & \int \frac{1}{(\sin x - 2 \cos x)(2 \sin x + \cos x)} dx \\ &= \int \frac{1}{2 \sin^2 x + \sin x \cos x - 4 \cos x \sin x - 2 \cos^2 x} dx \\ &= \int \frac{1}{2 \sin^2 x - 3 \cos x \sin x - 2 \cos^2 x} dx \\ &= \int \frac{1}{\cos^2 x [2 \tan^2 x - 3 \tan x - 2]} dx \\ &= \int \frac{\sec^2 x}{2 \tan^2 x - 3 \tan x - 2} dx \end{aligned}$$

Let $\tan x = t$

$$\frac{d}{dx}(\tan x) = dt$$

$$\sec^2 x \, dx = dt$$

$$\text{Now, } \int \frac{dt}{2t^2 - 3t - 2}$$

$$= \int \frac{dt}{(2t + 1)(t - 2)}$$

$$\text{Now, } \frac{1}{(2t+1)(t-2)} \cong \frac{A}{2t+1} + \frac{B}{t-2}$$

$$1 \cong A(t-2) + B(2t+1)$$

Equating 't' coeff: -

$$0 = A + 2B$$

$$A = -2B$$

Equating constant: -

$$1 = -2A + B$$

$$1 = -2(-2B) + B$$

$$1 = 5B$$

$$B = \frac{1}{5}$$

$$A = \frac{-2}{5}$$

$$\frac{1}{(2t+1)(t-2)} = \frac{-2}{5(2t+1)} + \frac{1}{5(t-2)}$$

$$\text{Now, } \int \frac{1}{(2t+1)(t-2)} dt = \frac{-2}{5} \int \frac{1}{2t+1} dt + \frac{1}{5} \int \frac{1}{t-2} dt$$

$$= \frac{-2}{5} \log(2t+1) + \frac{1}{5} \log(t-2) + c$$

$$= \frac{-2}{5} \log(2\tan x + 1) + \frac{1}{5} \log(\tan x - 2) + c$$

57. Question

$$\text{Evaluate } \int \frac{1}{4\sin^2 x + 4\sin x \cos x + 5\cos^2 x} dx$$

Answer

$$\text{Given, } \int \frac{1}{4\sin^2 x + 4\sin x \cos x + 5\cos^2 x} dx$$

$$= \int \frac{1}{\cos^2 x [4\tan^2 x + 4\tan x + 5]} dx$$

$$= \int \frac{\sec^2 x}{4\tan^2 x + 4\tan x + 5} dx$$

Let $\tan x = t$

$$\frac{d}{dx}(\tan x) = dt$$

$$\sec^2 x dx = dt$$

$$= \int \frac{dt}{4t^2 + 4t + 5}$$

$$= \int \frac{dt}{4t^2 + 4t + 1 + 4}$$

$$= \int \frac{dt}{(2t+1)^2 + (2)^2}$$

$$= \frac{1}{2} \tan^{-1} \left[\frac{2t+1}{2} \right] + c$$

$$= \frac{1}{2} \tan^{-1} \left[\frac{2\tan x + 1}{2} \right] + c$$

58. Question

$$\text{Evaluate } \int \frac{1}{a + b \tan x} dx$$

Answer

$$\text{Given, } \int \frac{1}{a + b \tan x} dx$$

Consider, $a=b=1$

$$= \int \frac{1}{1 + \tan x} dx$$

$$= \int \frac{1}{1 + \frac{\sin x}{\cos x}} dx$$

$$= \int \frac{\cos x}{\cos x + \sin x} dx$$

$$\text{Now, } \cos x = A (\cos x + \sin x) + B \frac{d}{dx}(\cos x + \sin x)$$

$$= A (\cos x + \sin x) + B (-\sin x + \cos x)$$

Equating 'cosx' coeff:- Equating 'sinx' coeff:-

$$1 = A + B \quad 0 = A - B$$

$$A = B$$

$$1 = A + A$$

$$2A = 1$$

$$A = 1/2 \quad B = 1/2$$

$$\cos x = \frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(-\sin x + \cos x)$$

$$= \int \frac{\cos x}{\cos x + \sin x} dx$$

$$= \int \frac{\frac{1}{2}(\cos x + \sin x) + \frac{1}{2}(-\sin x + \cos x)}{\cos x + \sin x} dx$$

$$= \int \frac{\frac{1}{2}(\cos x + \sin x)}{\cos x + \sin x} dx + \int \frac{\frac{1}{2}(-\sin x + \cos x)}{\cos x + \sin x} dx$$

$$= \frac{1}{2} \int 1 dx + \frac{1}{2} \int \frac{-\sin x + \cos x}{\cos x + \sin x} dx$$

$$[\text{since, } \int \frac{f'(x)}{f(x)} dx = \log[f(x)] + c]$$

$$= \frac{1}{2}(x) + \frac{1}{2} \log(\cos x + \sin x) + c$$

59. Question

$$\text{Evaluate } \int \frac{1}{\sin^2 x + \sin 2x} dx$$

Answer

$$\text{Given, } \int \frac{1}{\sin^2 x + \sin 2x} dx$$

$$= \int \frac{1}{\sin^2 x + 2 \sin x \cos x} dx$$

$$= \int \frac{1}{\sin^2 x (1 + 2 \cot x)} dx$$

$$= \int \frac{\operatorname{cosec}^2 x}{1 + \cot x} dx$$

$$\text{Let } \cot x = t$$

$$\frac{d}{dx}(\cot x) = dt$$

$$-\operatorname{cosec}^2 x dx = dt$$

$$\begin{aligned}\text{Now, } & - \int \frac{dt}{1+t} \\ &= -\log(1+t) + c \\ &= -\log(1+\cot x) + c\end{aligned}$$

60. Question

Evaluate $\int \frac{\sin x + 2 \cos x}{2 \sin x + \cos x} dx$

Answer

Given, $\int \frac{\sin x + 2 \cos x}{2 \sin x + \cos x} dx$

$$\sin x + 2 \cos x = A(2 \sin x + \cos x) + B \frac{d}{dx}(2 \sin x - \cos x)$$

$$= A(2 \sin x + \cos x) + B(2 \cos x - \sin x)$$

Equating 'sin x' coeff: -

$$1 = 2A - B$$

$$B = 2A - 1$$

Equating 'cos x' coeff:-

$$2 = A + 2B$$

$$2 = A + 2(2A - 1)$$

$$2 = A + 4A - 2$$

$$4 = 5A$$

$$A = \frac{4}{5}$$

$$B = 2\left(\frac{4}{5}\right) - 1$$

$$B = \frac{8}{5} - 1$$

$$B = \frac{3}{5}$$

$$\text{Now, } \sin x + 2 \cos x = \frac{4}{5}(2 \sin x + \cos x) + \frac{3}{5}(2 \cos x - \sin x)$$

$$= \int \frac{\frac{4}{5}(2 \sin x + \cos x) + \frac{3}{5}(2 \cos x - \sin x)}{2 \sin x + \cos x} dx$$

$$= \frac{4}{5} \int 1 dx + \frac{3}{5} \int \frac{2 \cos x - \sin x}{2 \sin x + \cos x} dx$$

$$= \frac{4}{5}(x) + \frac{3}{5} \log(2 \sin x + \cos x) + c$$

61. Question

Evaluate $\int \frac{x^3}{\sqrt{x^8 + 4}} dx$

Answer

Given, $\int \frac{x^3}{\sqrt{x^8+4}} dx$

Put, $x^4=t$

$4x^3 dx = dt$

$x^3 dx = \frac{1}{4} dt$

$$= \int \frac{x^3}{\sqrt{(x^4)^2 + 4}} dx$$

$$= \int \frac{\frac{1}{4} dt}{\sqrt{t^2 + 4}}$$

$$= \frac{1}{4} \int \frac{1}{\sqrt{t^2 + 2^2}} dt$$

$$= \frac{1}{4} \sinh^{-1} \left[\frac{t}{2} \right] + c$$

$$= \frac{1}{4} \sinh^{-1} \left[\frac{x^4}{2} \right] + c$$

62. Question

Evaluate $\int \frac{1}{2-3\cos 2x} dx$

Answer

Given, $\int \frac{1}{2-3\cos 2x} dx$

Put $\tan x = t$

$$\frac{d}{dx} (\tan x) = dt$$

$\sec^2 x dx = dt$

$$dx = \frac{dt}{1+t^2}$$

and $\cos 2x = \frac{1-t^2}{1+t^2}$

Now, $\int \frac{1}{2-3\left[\frac{1-t^2}{1+t^2}\right]} \cdot \frac{dt}{1+t^2}$

$$= \int \frac{1+t^2}{2(1+t^2) - 3(1-t^2)} \frac{dt}{1+t^2}$$

$$= \int \frac{1}{2+2t^2-3+3t^2} dt$$

$$= \int \frac{1}{5t^2-1} dt$$

$$= \frac{1}{5} \int \frac{1}{t^2 - \frac{1}{5}} dt$$

$$= \frac{1}{5} \int \frac{1}{t^2 - \left(\frac{1}{\sqrt{5}}\right)^2} dt \text{ [since, } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c]$$

$$= \frac{1}{5} \cdot \frac{1}{2\left(\frac{1}{\sqrt{5}}\right)} \log \left| \frac{t - \frac{1}{\sqrt{5}}}{t + \frac{1}{\sqrt{5}}} \right| + c$$

$$= \frac{1}{2\sqrt{5}} \log \left| \frac{\tan x - \frac{1}{\sqrt{5}}}{\tan x + \frac{1}{\sqrt{5}}} \right| + c$$

63. Question

Evaluate $\int \frac{\cos x}{\frac{1}{4} - \cos^2 x} dx$

Answer

Given, $\int \frac{\cos x}{\frac{1}{4} - \cos^2 x} dx$

$$= \int \frac{\cos x}{\frac{1}{4} - (1 - \sin^2 x)} dx$$

Let $\sin x = t$

$\cos x dx = dt$

$$= \int \frac{dt}{\frac{1}{4} - (1 - t^2)}$$

$$= \int \frac{dt}{\frac{1 - 4 + 4t^2}{4}}$$

$$= \int \frac{4 dt}{4t^2 - 3}$$

$$= 4 \int \frac{1}{(2t)^2 - (\sqrt{3})^2} dt$$

[since, $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$]

$$= 4 \cdot \frac{1}{2\sqrt{3}} \log \left| \frac{2t - \sqrt{3}}{2t + \sqrt{3}} \right| + c$$

$$= \frac{2}{\sqrt{3}} \log \left| \frac{2 \sin x - \sqrt{3}}{2 \sin x + \sqrt{3}} \right| + c$$

64. Question

Evaluate $\int \frac{1}{1 + 2 \cos x} dx$

Answer

Given, $\int \frac{1}{1 + 2 \cos x} dx$

Put $\tan \frac{x}{2} = t$

$dx = \frac{2}{1+t^2} dt$ and $\cos x = \frac{1-t^2}{1+t^2}$

$$\begin{aligned}
&= \int \frac{1}{1+2\left[\frac{1-t^2}{1+t^2}\right]} \cdot \frac{2}{1+t^2} dt \\
&= \int \frac{1+t^2}{1+t^2+2-2t^2} \cdot \frac{2}{1+t^2} dt \\
&= \int \frac{2}{3-t^2} dt \\
&= \int \frac{2}{(\sqrt{3})^2 - (t)^2} dt \text{ [since, } \int \frac{1}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c] \\
&= \frac{1}{2a} \log \left| \frac{\sqrt{3}+t}{\sqrt{3}-t} \right| + c \\
&= \frac{1}{2a} \log \left| \frac{\sqrt{3} + \tan \frac{x}{2}}{\sqrt{3} - \tan \frac{x}{2}} \right| + c
\end{aligned}$$

65. Question

Evaluate $\int \frac{1}{1-2\sin x} dx$

Answer

Given, $\int \frac{1}{1-2\sin x} dx$

Let $\tan \frac{x}{2} = t$

$$dx = \frac{2}{1+t^2} dt \text{ and } \sin x = \frac{2t}{1+t^2}$$

$$\begin{aligned}
&= \int \frac{1}{1-2\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2}{1+t^2} dt \\
&= \int \frac{1+t^2}{1+t^2-4t} \cdot \frac{2}{1+t^2} dt \\
&= \int \frac{2}{t^2-4t+1} dt \\
&= \int \frac{2}{t^2-4t+4-3} dt \\
&= \int \frac{2}{(t-2)^2 - (\sqrt{3})^2} dt \\
&= \frac{2}{2\sqrt{3}} \log \left| \frac{t-2-\sqrt{3}}{t-2+\sqrt{3}} \right| + c \text{ [since, } \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c] \\
&= \frac{1}{\sqrt{3}} \log \left| \frac{\tan \frac{x}{2} - 2 - \sqrt{3}}{\tan \frac{x}{2} - 2 + \sqrt{3}} \right| + c
\end{aligned}$$

31. Question

Evaluate $\int \cot^4 x dx$

Answer

In this question, first of all we expand $\cot^4 x$ as

$$\cot^4 x = (\operatorname{cosec}^2 x - 1)^2$$

$$= \operatorname{cosec}^4 x - 2\operatorname{cosec}^2 x + 1 \dots (1)$$

Now, write $\operatorname{cosec}^4 x$ as

$$\operatorname{cosec}^4 x = \operatorname{cosec}^2 x \operatorname{cosec}^2 x$$

$$= \operatorname{cosec}^2 x (1 + \cot^2 x)$$

$$= \operatorname{cosec}^2 x + \operatorname{cosec}^2 x \cot^2 x$$

Putting the value of $\operatorname{cosec}^4 x$ in eq(1)

$$\cot^4 x = \operatorname{cosec}^2 x + \operatorname{cosec}^2 x \cot^2 x - 2\operatorname{cosec}^2 x + 1$$

$$= \operatorname{cosec}^2 x \cot^2 x - \operatorname{cosec}^2 x + 1$$

$$y = \int \cot^4 x \, dx$$

$$= \int \operatorname{cosec}^2 x \cot^2 x \, dx + \int -\operatorname{cosec}^2 x + 1 \, dx$$

$$A = \int \operatorname{cosec}^2 x \cot^2 x \, dx$$

Let, $\cot x = t$

Differentiating both side with respect to x

$$\frac{dt}{dx} = -\operatorname{cosec}^2 x$$

$$\Rightarrow -dt = \operatorname{cosec}^2 x \, dx$$

$$A = \int -t^2 \, dt$$

$$\text{Using formula } \int t^n \, dt = \frac{t^{n+1}}{n+1}$$

$$A = -\frac{t^3}{3} + c_1$$

Again, put $t = \cot x$

$$A = -\frac{\cot^3 x}{3} + c_1$$

$$\text{Now, } B = \int -\operatorname{cosec}^2 x + 1 \, dx$$

Using formula $\int \operatorname{cosec}^2 x \, dx = -\cot x$ and $\int c \, dx = cx$

$$B = \cot x + x + c_2$$

Now, the complete solution is

$$y = A + B$$

$$y = -\frac{\cot^3 x}{3} + \cot x + x + c$$

32. Question

Evaluate $\int \cot^5 x \, dx$

Answer

$$y = \int \frac{\cos^5 x}{\sin^5 x} \, dx$$

$$y = \int \frac{\cos^4 x \cos x}{\sin^5 x} dx$$

$$y = \int \frac{(1 - \sin^2 x)^2 \cos x}{\sin^5 x} dx$$

Let, $\sin x = t$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = \cos x \Rightarrow dt = \cos x dx$$

$$y = \int \frac{(1 - t^2)^2}{t^5} dt$$

$$y = \int \frac{1 - 2t^2 + t^4}{t^5} dt$$

$$y = \int t^{-5} - 2t^{-3} + \frac{1}{t} dt$$

Using formula $\int t^n dt = \frac{t^{n+1}}{n+1}$ and $\int \frac{1}{t} dt = \ln t$

$$y = \frac{t^{-4}}{-4} - 2 \frac{t^{-2}}{-2} + \ln t + c$$

Again, put $t = \sin x$

$$y = -\frac{\sin^{-4} x}{4} + \sin^{-2} x + \ln t + c$$

33. Question

Evaluate $\int \frac{x^2}{(x-1)^3} dx$

Answer

$$y = \int \frac{(x-1+1)^2}{(x-1)^3} dx$$

$$y = \int \frac{(x-1)^2 + 2(x-1) + 1}{(x-1)^3} dx$$

$$y = \int \frac{1}{(x-1)} + 2 \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} dx$$

Using formula $\int \frac{1}{x} dx = \ln x$ and $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$y = \ln(x-1) + 2 \frac{(x-1)^{-1}}{-1} + \frac{(x-1)^{-2}}{-2} + c$$

$$y = \ln(x-1) - 2(x-1)^{-1} - \frac{(x-1)^{-2}}{2} + c$$

34. Question

Evaluate $\int x\sqrt{2x+3} dx$

Answer

In this question we write $x\sqrt{2x+3}$ as

$$\begin{aligned}
 x\sqrt{2x+3} &= \frac{2x\sqrt{2x+3}}{2} \\
 &= \frac{(2x+3-3)\sqrt{2x+3}}{2} \\
 &= \frac{(2x+3)\sqrt{2x+3} - 3\sqrt{2x+3}}{2} \\
 &= \frac{(2x+3)^{\frac{3}{2}} - 3\sqrt{2x+3}}{2}
 \end{aligned}$$

$$y = \int x\sqrt{2x+3} \, dx$$

$$y = \int \frac{(2x+3)^{\frac{3}{2}} - 3\sqrt{2x+3}}{2} \, dx$$

Using formula $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}$

$$y = \frac{(2x+3)^{\frac{5}{2}}}{2 \times 2 \times \frac{5}{2}} - \frac{3(2x+3)^{\frac{3}{2}}}{2 \times 2 \times \frac{3}{2}} + c$$

$$y = \frac{(2x+3)^{\frac{5}{2}}}{10} - \frac{(2x+3)^{\frac{3}{2}}}{2} + c$$

35. Question

Evaluate $\int \frac{x^3}{(1+x^2)^2} \, dx$

Answer

Let, $x = \tan t$

Differentiating both side with respect to t

$$\frac{dx}{dt} = \sec^2 t \Rightarrow dx = \sec^2 t \, dt$$

$$y = \int \frac{\tan^3 t}{\sec^4 t} \sec^2 t \, dt$$

$$y = \int \frac{\sin^3 t}{\cos t} \, dt$$

$$y = \int \frac{(1 - \cos^2 t) \sin t}{\cos t} \, dt$$

Again, let $\cos t = z$

Differentiating both side with respect to t

$$\frac{dz}{dt} = -\sin t \Rightarrow -dz = \sin t \, dt$$

$$y = - \int \frac{(1 - z^2)}{z} \, dz$$

$$y = - \int \frac{1}{z} - z \, dz$$

Using formula $\int \frac{1}{z} dz = \ln z$ and $\int z^n dz = \frac{z^{n+1}}{n+1}$

$$y = -\ln z + \frac{z^2}{2} + c$$

Again, put $z = \cos t = \cos(\tan^{-1}x)$

$$y = -\ln \cos(\tan^{-1} x) + \frac{\cos^2(\tan^{-1} x)}{2} + c$$

36. Question

Evaluate $\int x \sin^5 x^2 \cos x^2 dx$

Answer

Let, $\sin x^2 = t$

Differentiating both sides with respect to x

$$\frac{dt}{dx} = \cos x^2 \times 2x \Rightarrow \frac{dt}{2} = x \cos x^2 dx$$

$$y = \int \frac{t^5}{2} dt$$

Using formula $\int t^n dt = \frac{t^{n+1}}{n+1}$

$$y = \frac{t^6}{2 \times 6} + c$$

Again, put $t = \sin x^2$

$$y = \frac{\sin^6 x^2}{12} + c$$

37. Question

Evaluate $\int \sin^3 x \cos^4 x dx$

Answer

$$y = \int (1 - \cos^2 x) \cos^4 x \sin x dx$$

Let, $\cos x = t$

Differentiating both side with respect to x

$$\frac{dt}{dx} = -\sin x \Rightarrow -dt = \sin x dx$$

$$y = \int -(1 - t^2)t^4 dt$$

$$y = -\int t^4 - t^6 dt$$

Using formula $\int t^n dt = \frac{t^{n+1}}{n+1}$

$$y = -\left(\frac{t^5}{5} - \frac{t^7}{7}\right) + c$$

Again, put $t = \cos x$

$$y = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + c$$

38. Question

Evaluate $\int \sin^5 x \, dx$

Answer

$$y = \int (1 - \cos^2 x)^2 \sin x \, dx$$

Let, $\cos x = t$

Differentiating both side with respect to x

$$\frac{dt}{dx} = -\sin x \Rightarrow -dt = \sin x \, dx$$

$$y = -\int (1 - t^2)^2 \, dt$$

$$y = -\int 1 + t^4 - 2t^2 \, dt$$

Using formula $\int t^n \, dt = \frac{t^{n+1}}{n+1}$ and $\int c \, dt = ct$

$$y = -\left(t + \frac{t^5}{5} - 2\frac{t^3}{3}\right) + c$$

Again, put $t = \cos x$

$$y = -\left(\cos x + \frac{\cos^5 x}{5} - 2\frac{\cos^3 x}{3}\right) + c$$

39. Question

Evaluate $\int \cos^5 x \, dx$.

Answer

$$y = \int (1 - \sin^2 x)^2 \cos x \, dx$$

Let, $\sin x = t$

Differentiating both side with respect to x

$$\frac{dt}{dx} = \cos x \Rightarrow dt = \cos x \, dx$$

$$y = \int (1 - t^2)^2 \, dt$$

$$y = \int 1 + t^4 - 2t^2 \, dt$$

Using formula $\int t^n \, dt = \frac{t^{n+1}}{n+1}$ and $\int c \, dt = ct$

$$y = \left(t + \frac{t^5}{5} - 2\frac{t^3}{3}\right) + c$$

Again, put $t = \sin x$

$$y = \left(\sin x + \frac{\sin^5 x}{5} - 2\frac{\sin^3 x}{3}\right) + c$$

40. Question

Evaluate $\int \sqrt{\sin x} \cos^3 x \, dx$

Answer

$$y = \int \sqrt{\sin x} (1 - \sin^2 x) \cos x \, dx$$

Let, $\sin x = t$

Differentiating both side with respect to x

$$\frac{dt}{dx} = \cos x \Rightarrow dt = \cos x \, dx$$

$$y = \int \sqrt{t}(1 - t^2) \, dt$$

$$y = \int t^{\frac{1}{2}} - t^{\frac{5}{2}} \, dt$$

Using formula $\int t^n \, dt = \frac{t^{n+1}}{n+1}$

$$y = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \frac{t^{\frac{7}{2}}}{\frac{7}{2}} + c$$

Again, put $t = \sin x$

$$y = \frac{\sin^{\frac{3}{2}} x}{\frac{3}{2}} - \frac{\sin^{\frac{7}{2}} x}{\frac{7}{2}} + c$$

41. Question

Evaluate $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} \, dx$

Answer

$$y = \int \frac{\sin 2x}{(\sin^2 x)^2 + (1 - \sin^2 x)^2} \, dx$$

Let, $\sin^2 x = t$

Differentiating both side with respect to x

$$\frac{dt}{dx} = 2 \sin x \cos x \Rightarrow dt = \sin 2x \, dx$$

$$y = \int \frac{dt}{t^2 + (1-t)^2}$$

$$y = \int \frac{dt}{2t^2 - 2t + 1}$$

Try to make perfect square in denominator

$$y = \int \frac{dt}{2t^2 - 2t + \frac{1}{2} + \frac{1}{2}}$$

$$y = \int \frac{dt}{(\sqrt{2}t)^2 - 2(\sqrt{2}t)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}$$

$$y = \int \frac{dt}{\left(\sqrt{2}t - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

Using formula $\int \frac{dt}{t^2+a^2} = \frac{1}{a} \tan^{-1} \frac{t}{a}$

$$y = \frac{1}{\sqrt{2} \times \frac{1}{\sqrt{2}}} \tan^{-1} \frac{\left(\sqrt{2}t - \frac{1}{\sqrt{2}}\right)}{\frac{1}{\sqrt{2}}} + c$$

$$y = \sqrt{2} \tan^{-1} \left(\sqrt{2}t - \frac{1}{\sqrt{2}} \right) + c$$

Again, put $t = \sin^2 x$

$$y = \sqrt{2} \tan^{-1} \left(\sqrt{2} \sin^2 x - \frac{1}{\sqrt{2}} \right) + c$$

42. Question

Evaluate $\int \frac{1}{\sqrt{x^2 - a^2}} dx$

Answer

Let, $x = a \sec t$

Differentiating both side with respect to t

$$\frac{dx}{dt} = a \sec t \tan t \Rightarrow dx = a \sec t \tan t dt$$

$$y = \int \frac{a \sec t \tan t}{\sqrt{a^2 \sec^2 t - a^2}} dt$$

$$y = \int \frac{\sec t \tan t}{\tan t} dt$$

$$y = \int \sec t dt$$

Using formula $\int \sec t dt = \ln(\tan t + \sec t)$

$$y = \ln(\tan t + \sec t) + c_1$$

Again, put $t = \sec^{-1} \frac{x}{a}$

$$y = \ln \left(\tan \sec^{-1} \frac{x}{a} + \sec \sec^{-1} \frac{x}{a} \right) + c_1$$

$$y = \ln \left(\sqrt{\left(\frac{x}{a}\right)^2 - 1} + \frac{x}{a} \right) + c_1$$

$$y = \ln(x + \sqrt{x^2 - a^2}) - \ln a + c_1$$

$$y = \ln(x + \sqrt{x^2 - a^2}) + c$$

43. Question

Evaluate $\int \frac{1}{\sqrt{x^2 + a^2}} dx$

Answer

Let, $x = a \tan t$

Differentiating both side with respect to t

$$\frac{dx}{dt} = a \sec^2 t \Rightarrow dx = a \sec^2 t \, dt$$

$$y = \int \frac{a \sec^2 t}{\sqrt{a^2 \tan^2 t + a^2}} \, dt$$

$$y = \int \frac{\sec^2 t}{\sec t} \, dt$$

$$y = \int \sec t \, dt$$

Tip: This is very important formula. It is use directly in the question. So, learn it by heart.

Using formula $\int \sec t \, dt = \ln(\tan t + \sec t)$

$$y = \ln(\tan t + \sec t) + c_1$$

$$\text{Again, put } t = \tan^{-1} \frac{x}{a}$$

$$y = \ln \left(\tan \tan^{-1} \frac{x}{a} + \sec \tan^{-1} \frac{x}{a} \right) + c_1$$

$$y = \ln \left(\sqrt{\left(\frac{x}{a}\right)^2 + 1} + \frac{x}{a} \right) + c_1$$

$$y = \ln(x + \sqrt{x^2 + a^2}) - \ln a + c_1$$

$$y = \ln(x + \sqrt{x^2 + a^2}) + c$$

44. Question

$$\text{Evaluate } \int \frac{1}{4x^2 + 4x + 5} \, dx$$

Answer

In this question we can try to make perfect square in denominator

$$y = \int \frac{1}{(2x)^2 + 2(2x)(1) + 1 + 4} \, dx$$

$$y = \int \frac{1}{(2x+1)^2 + (2)^2} \, dx$$

$$\text{Using formula } \int \frac{dt}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$y = \frac{1}{2 \times 2} \tan^{-1} \frac{(2x+1)}{2} + c$$

$$y = \frac{1}{4} \tan^{-1} \frac{(2x+1)}{2} + c$$

45. Question

$$\text{Evaluate } \int \frac{1}{x^2 + 4x - 5} \, dx$$

Answer

In this question we can try to make perfect square in denominator

$$y = \int \frac{1}{x^2 + 2(x)(2) + 4 - (3)^2} dx$$

$$y = \int \frac{1}{(x+2)^2 - (3)^2} dx$$

Using formula $\int \frac{dt}{x^2 - a^2} = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + c$

$$y = \frac{1}{2 \times 3} \log\left(\frac{x+2-3}{x+2+3}\right) + c$$

$$y = \frac{1}{6} \log\left(\frac{x-1}{x+5}\right) + c$$

1. Question

Evaluate $\int \frac{1}{\sqrt{x} + \sqrt{x+1}} dx$

Answer

Rationalising denominator

We get, $\int \frac{\sqrt{x}-\sqrt{x+1}}{x-(x+1)} dx$

It becomes $\int \frac{\sqrt{x}-\sqrt{x+1}}{-1} dx$

$$= -\int \sqrt{x} dx - \int \sqrt{x+1} dx$$

$$= -\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

1. Question

Evaluate $\int \frac{1}{\sqrt{x} + \sqrt{x+1}} dx$

Answer

Rationalising denominator

We get, $\int \frac{\sqrt{x}-\sqrt{x+1}}{x-(x+1)} dx$

It becomes $\int \frac{\sqrt{x}-\sqrt{x+1}}{-1} dx$

$$= -\int \sqrt{x} dx - \int \sqrt{x+1} dx$$

$$= -\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

2. Question

Evaluate $\int \frac{1-x^4}{1-x} dx$

Answer

Factorising the equation

$$= \int \frac{(1-x^2)(1+x^2)}{1-x} dx$$

$$= \int \frac{(1-x)(1+x)(1+x^2)}{1-x} dx$$

On cancelling we get

$$= \int (1+x)(1+x^2) dx$$

$$= \int (1+x+x^2+x^3) dx$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + c$$

2. Question

Evaluate $\int \frac{1-x^4}{1-x} dx$

Answer

Factorising the equation

$$= \int \frac{(1-x^2)(1+x^2)}{1-x} dx$$

$$= \int \frac{(1-x)(1+x)(1+x^2)}{1-x} dx$$

On cancelling we get

$$= \int (1+x)(1+x^2) dx$$

$$= \int (1+x+x^2+x^3) dx$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + c$$

3. Question

Evaluate $\int \frac{x+2}{(x+1)^3} dx$

Answer

On simplifying we get,

$$\int \frac{(x+1)+1}{(x+1)^3} dx$$

$$= \int \frac{1}{(x+1)^2} dx + \int \frac{1}{(x+1)^3} dx$$

On solving we get

$$= -\frac{1}{x+1} - \frac{1}{2(x+1)^2} + c$$

3. Question

Evaluate $\int \frac{x+2}{(x+1)^3} dx$

Answer

On simplifying we get,

$$\int \frac{(x+1)+1}{(x+1)^2} dx$$
$$= \int \frac{1}{(x+1)^2} dx + \int \frac{1}{(x+1)^3} dx$$

On solving we get

$$= -\frac{1}{x+1} - \frac{1}{2(x+1)^2} + c$$

4. Question

Evaluate $\int \frac{8x+13}{\sqrt{4x+7}} dx$

Answer

On simplifying we get,

$$= \int \frac{4x+7}{\sqrt{4x+7}} dx + \int \frac{4x+7}{\sqrt{4x+7}} dx - \int \frac{1}{\sqrt{4x+7}} dx$$
$$= 2 \int \frac{4x+7}{\sqrt{4x+7}} dx - \int \frac{1}{\sqrt{4x+7}} dx$$
$$= 2 \int \sqrt{4x+7} dx - \int \frac{1}{\sqrt{4x+7}} dx$$
$$= 2 x \frac{(4x+7)^{3/2}}{\frac{3}{2}} x \times \frac{1}{4} - \frac{(4x+7)^{\frac{1}{2}}}{\frac{1}{2}} x \times \frac{1}{4} + c$$
$$= \frac{(4x+7)^{3/2}}{3} - \frac{(4x+7)^{\frac{1}{2}}}{2} + c$$

4. Question

Evaluate $\int \frac{8x+13}{\sqrt{4x+7}} dx$

Answer

On simplifying we get,

$$= \int \frac{4x+7}{\sqrt{4x+7}} dx + \int \frac{4x+7}{\sqrt{4x+7}} dx - \int \frac{1}{\sqrt{4x+7}} dx$$
$$= 2 \int \frac{4x+7}{\sqrt{4x+7}} dx - \int \frac{1}{\sqrt{4x+7}} dx$$
$$= 2 \int \sqrt{4x+7} dx - \int \frac{1}{\sqrt{4x+7}} dx$$
$$= 2 x \frac{(4x+7)^{3/2}}{\frac{3}{2}} x \times \frac{1}{4} - \frac{(4x+7)^{\frac{1}{2}}}{\frac{1}{2}} x \times \frac{1}{4} + c$$
$$= \frac{(4x+7)^{3/2}}{3} - \frac{(4x+7)^{\frac{1}{2}}}{2} + c$$

5. Question

Evaluate $\int \frac{1+x+x^2}{x^2(1+x)} dx$

Answer

On simplifying we get

$$\begin{aligned} & \int \frac{1+x}{x^2(1+x)} dx + \int \frac{x^2}{x^2(1+x)} dx \\ &= \int \frac{1}{x^2} dx + \int \frac{1}{1+x} dx \\ &= -x^{-1} + \ln(1+x) + c \end{aligned}$$

5. Question

Evaluate $\int \frac{1+x+x^2}{x^2(1+x)} dx$

Answer

On simplifying we get

$$\begin{aligned} & \int \frac{1+x}{x^2(1+x)} dx + \int \frac{x^2}{x^2(1+x)} dx \\ &= \int \frac{1}{x^2} dx + \int \frac{1}{1+x} dx \\ &= -x^{-1} + \ln(1+x) + c \end{aligned}$$

6. Question

Evaluate $\int \frac{(2^x + 3^x)^2}{6^x} dx$

Answer

On squaring numerator we get

$$\begin{aligned} &= \int \frac{2^{2x} + 2 \cdot 2^x \cdot 3^x + 3^{2x}}{2^x \cdot 3^x} dx \\ &= \int \left(\frac{2}{3} \right)^x + 2 + \left(\frac{3}{2} \right)^x dx \end{aligned}$$

Formula for $\int a^x dx = \frac{a^x}{\ln(a)}$

Solving above equation we get,

$$= \frac{\left(\frac{2}{3} \right)^x}{\ln\left(\frac{2}{3} \right)} + 2x + \frac{\left(\frac{3}{2} \right)^x}{\ln\left(\frac{3}{2} \right)} + c$$

6. Question

Evaluate $\int \frac{(2^x + 3^x)^2}{6^x} dx$

Answer

On squaring numerator we get

$$= \int \frac{2^{2x} + 2 \cdot 2^x \cdot 3^x + 3^{2x}}{2^x \cdot 3^x} dx$$

$$= \int \left(\frac{2}{3}\right)^x + 2 + \left(\frac{3}{2}\right)^x dx$$

$$\text{Formula for } \int a^x dx = \frac{a^x}{\ln(a)}$$

Solving above equation we get,

$$= \frac{\left(\frac{2}{3}\right)^x}{\ln\left(\frac{2}{3}\right)} + 2x + \frac{\left(\frac{3}{2}\right)^x}{\ln\left(\frac{3}{2}\right)} + c$$

7. Question

$$\text{Evaluate } \int \frac{\sin x}{1 + \sin x} dx$$

Answer

Multiplying numerator and denominator with $1 - \sin x$

$$\text{We get } \int \frac{\sin x(1 - \sin x)}{1 - \sin^2 x} dx$$

$$= \int \frac{\sin x(1 - \sin x)}{\cos^2 x} dx$$

$$= \int \frac{\sin x - \sin^2 x}{\cos^2 x} dx$$

$$= \int \frac{\sin x}{\cos^2 x} dx - \int \frac{\sin^2 x}{\cos^2 x} dx$$

$$\text{Taking } \int \frac{\sin x}{\cos^2 x} dx = A \text{ and } \int \frac{\sin^2 x}{\cos^2 x} dx = B$$

Solving for A

$$\text{Taking } \cos x = t$$

On differentiating both sides we get

$$-\sin x \, dx = dt$$

Putting values in A we get our equation as

$$= \int \frac{-dt}{t^2}$$

$$= t^{-1} + c$$

Substituting value of t,

$$= \sec x + c$$

Solving for B

$$\int \frac{1 - \cos^2 x}{\cos^2 x} dx$$

$$= \int \sec^2 x - \int 1 \, dx$$

$$= \tan x - x + c$$

Final answer is A+B

$$= \sec x + \tan x - x + c$$

7. Question

Evaluate $\int \frac{\sin x}{1 + \sin x} dx$

Answer

Multiplying numerator and denominator with $1 - \sin x$

We get $\int \frac{\sin x(1 - \sin x)}{1 - \sin^2 x} dx$

$$= \int \frac{\sin x(1 - \sin x)}{\cos^2 x} dx$$

$$= \int \frac{\sin x - \sin^2 x}{\cos^2 x} dx$$

$$= \int \frac{\sin x}{\cos^2 x} dx - \int \frac{\sin^2 x}{\cos^2 x} dx$$

Taking $\int \frac{\sin x}{\cos^2 x} dx = A$ and $\int \frac{\sin^2 x}{\cos^2 x} dx = B$

Solving for A

Taking $\cos x = t$

On differentiating both sides we get

$$-\sin x \, dx = dt$$

Putting values in A we get our equation as

$$= \int \frac{-dt}{t^2}$$

$$= t^{-1} + c$$

Substituting value of t,

$$= \sec x + c$$

Solving for B

$$\int \frac{1 - \cos^2 x}{\cos^2 x} dx$$

$$= \int \sec^2 x - \int 1 \, dx$$

$$= \tan x - x + c$$

Final answer is A+B

$$= \sec x + \tan x - x + c$$

8. Question

Evaluate $\int \frac{x^4 + x^2 - 1}{x^2 + 1} dx$

Answer

On simplifying we get

$$\int \frac{x^2(x^2 + 1)}{(x^2 + 1)} - \frac{1}{(x^2 + 1)} dx$$

$$= \int x^2 dx - \int \frac{1}{x^2 + 1} dx$$

$$= \frac{x^3}{3} - \tan^{-1} x + c$$

8. Question

Evaluate $\int \frac{x^4 + x^2 - 1}{x^2 + 1} dx$

Answer

On simplifying we get

$$\int \frac{x^2(x^2 + 1)}{(x^2 + 1)} - \frac{1}{(x^2 + 1)} dx$$

$$= \int x^2 dx - \int \frac{1}{x^2 + 1} dx$$

$$= \frac{x^3}{3} - \tan^{-1} x + c$$

9. Question

Evaluate $\int \sec^2 x \cos^2 2x dx$

Answer

$$\int \sec^2 x (\cos^2 x - \sin^2 x)^2 dx$$

Opening the square

$$= \int \frac{\cos^4 x - 2 \cos^2 x \sin^2 x + \sin^4 x}{\cos^2 x} dx$$

$$= \int \left(\cos^2 x - 2 \sin^2 x + \frac{\sin^2 x \sin^2 x}{\cos^2 x} \right) dx$$

$$= \int \left(\cos^2 x - 2 \sin^2 x + \frac{(1 - \cos^2 x)(1 - \cos^2 x)}{\cos^2 x} \right) dx$$

On multiplying $(1 - \cos^2 x)(1 - \cos^2 x)$ equation reduces to

$$= \int (\cos^2 x - 2 \sin^2 x + \sec^2 x - 2 + \cos^2 x) dx$$

$$= \int (2 \cos^2 x - 2 \sin^2 x + \sec^2 x - 2) dx$$

$$= \int (2(\cos^2 x - \sin^2 x) + \sec^2 x - 2) dx$$

$$= \int (2 \cos 2x + \sec^2 x - 2) dx$$

On solving this we get our answer i.e

$$= \frac{2 \sin 2x}{2} + \tan x - 2x + c$$

$$= \sin 2x + \tan x - 2x + c$$

9. Question

Evaluate $\int \sec^2 x \cos^2 2x \, dx$

Answer

$$\int \sec^2 x (\cos^2 x - \sin^2 x)^2 \, dx$$

Opening the square

$$\begin{aligned} &= \int \frac{\cos^4 x - 2 \cos^2 x \sin^2 x + \sin^4 x}{\cos^2 x} \, dx \\ &= \int \left(\cos^2 x - 2 \sin^2 x + \frac{\sin^2 x \sin^2 x}{\cos^2 x} \right) \, dx \\ &= \int \left(\cos^2 x - 2 \sin^2 x + \frac{(1 - \cos^2 x)(1 - \cos^2 x)}{\cos^2 x} \right) \, dx \end{aligned}$$

On multiplying $(1 - \cos^2 x)(1 - \cos^2 x)$ equation reduces to

$$= \int (\cos^2 x - 2 \sin^2 x + \sec^2 x - 2 + \cos^2 x) \, dx$$

$$= \int (2 \cos^2 x - 2 \sin^2 x + \sec^2 x - 2) \, dx$$

$$= \int (2(\cos^2 x - \sin^2 x) + \sec^2 x - 2) \, dx$$

$$= \int (2 \cos 2x + \sec^2 x - 2) \, dx$$

On solving this we get our answer i.e

$$= \frac{2 \sin 2x}{2} + \tan x - 2x + c$$

$$= \sin 2x + \tan x - 2x + c$$

10. Question

Evaluate $\int \operatorname{cosec}^2 x \cos^2 2x \, dx$

Answer

$$\int \operatorname{cosec}^2 x (\cos^2 x - \sin^2 x)^2 \, dx$$

Opening the square

$$\begin{aligned} &= \int \frac{\cos^4 x - 2 \cos^2 x \sin^2 x + \sin^4 x}{\sin^2 x} \, dx \\ &= \int \left(\frac{\cos^2 x \cos^2 x}{\sin^2 x} - 2 \cos^2 x + \sin^2 x \right) \, dx \\ &= \int \left(\frac{(1 - \sin^2 x)(1 - \sin^2 x)}{\sin^2 x} - 2 \cos^2 x + \sin^2 x \right) \, dx \end{aligned}$$

On multiplying $(1 - \sin^2 x)(1 - \sin^2 x)$ equation reduces to

$$= \int (\operatorname{cosec}^2 x - 2 + \sin^2 x - 2 \cos^2 x + \sin^2 x) \, dx$$

$$= \int (\operatorname{cosec}^2 x - 2 + 2 \sin^2 x - 2 \cos^2 x) \, dx$$

$$= \int (-2(\cos^2 x - \sin^2 x) + \operatorname{cosec}^2 x - 2) \, dx$$

$$= \int (-2 \cos 2x + \operatorname{cosec}^2 x - 2) \, dx$$

On solving this we get our answer i.e

$$= \frac{-2 \sin 2x}{2} - \cot x - 2x + c$$

$$= -\sin 2x - \cot x - 2x + c$$

10. Question

Evaluate $\int \operatorname{cosec}^2 x \cos^2 2x \, dx$

Answer

$$\int \operatorname{cosec}^2 x (\cos^2 x - \sin^2 x)^2 dx$$

Opening the square

$$\begin{aligned} &= \int \frac{\cos^4 x - 2 \cos^2 x \sin^2 x + \sin^4 x}{\sin^2 x} dx \\ &= \int \left(\frac{\cos^2 x \cos^2 x}{\sin^2 x} - 2 \cos^2 x + \sin^2 x \right) dx \\ &= \int \left(\frac{(1 - \sin^2 x)(1 - \sin^2 x)}{\sin^2 x} - 2 \cos^2 x + \sin^2 x \right) dx \end{aligned}$$

On multiplying $(1 - \sin^2 x)(1 - \sin^2 x)$ equation reduces to

$$= \int (\operatorname{cosec}^2 x - 2 + \sin^2 x - 2 \cos^2 x + \sin^2 x) dx$$

$$= \int (\operatorname{cosec}^2 x - 2 + 2 \sin^2 x - 2 \cos^2 x) dx$$

$$= \int (-2(\cos^2 x - \sin^2 x) + \operatorname{cosec}^2 x - 2) dx$$

$$= \int (-2 \cos 2x + \operatorname{cosec}^2 x - 2) dx$$

On solving this we get our answer i.e

$$= \frac{-2 \sin 2x}{2} - \cot x - 2x + c$$

$$= -\sin 2x - \cot x - 2x + c$$

11. Question

Evaluate $\int \sin^4 2x \, dx$

Answer

Replacing $2x$ by t

We get $dx = dt/2$ by differentiating both sides

Our equation has become

$$\begin{aligned} &\frac{1}{2} \int \sin^4 t \, dt \\ &= \frac{1}{2} \int \sin^2 t \cdot \sin^2 t \, dt = \frac{1}{2} \int \sin^2 t (1 - \cos^2 t) \, dt \\ &= \frac{1}{2} \int \sin^2 t \, dt - \frac{1}{2} \int \sin^2 t \cos^2 t \, dt \end{aligned}$$

simplifying $\sin^2 t \cos^2 t$

on multiplying and dividing by 4 we get $\sin^2 t \cos^2 t$ as $\sin^2 2t$

$$= \frac{1}{2} \int \frac{1 - \cos 2t}{2} dt - \frac{1}{2} \int \frac{\sin^2 2t}{4}$$

$$\begin{aligned}
&= \frac{1}{2} \int \frac{1 - \cos 2t}{2} dt - \frac{1}{2} \int \frac{1 - \cos 4t}{4.2} \\
&= \frac{1}{4} \int 1 - \cos 2t dt - \frac{1}{16} \int 1 - \cos 4t dt \\
&= \frac{t}{4} - \frac{\sin 2t}{8} - \frac{t}{8} + \frac{\sin 4t}{64} + c
\end{aligned}$$

Hence our final answer is

$$= \frac{t}{8} - \frac{\sin 2t}{8} + \frac{\sin 4t}{64} + c$$

11. Question

Evaluate $\int \sin^4 2x dx$

Answer

Replacing $2x$ by t

We get $dx = dt/2$ by differentiating both sides

Our equation has become

$$\begin{aligned}
&\frac{1}{2} \int \sin^4 t dt \\
&= \frac{1}{2} \int \sin^2 t \cdot \sin^2 t dt = \frac{1}{2} \int \sin^2 t \cdot (1 - \cos^2 t) dt \\
&= \frac{1}{2} \int \sin^2 t dt - \frac{1}{2} \int \sin^2 t \cdot \cos^2 t dt
\end{aligned}$$

simplifying $\sin^2 t \cdot \cos^2 t$

on multiplying and dividing by 4 we get $\sin^2 t \cdot \cos^2 t$ as $\sin^2 2t$

$$\begin{aligned}
&= \frac{1}{2} \int \frac{1 - \cos 2t}{2} dt - \frac{1}{2} \int \frac{\sin^2 2t}{4} \\
&= \frac{1}{2} \int \frac{1 - \cos 2t}{2} dt - \frac{1}{2} \int \frac{1 - \cos 4t}{4.2} \\
&= \frac{1}{4} \int 1 - \cos 2t dt - \frac{1}{16} \int 1 - \cos 4t dt \\
&= \frac{t}{4} - \frac{\sin 2t}{8} - \frac{t}{8} + \frac{\sin 4t}{64} + c
\end{aligned}$$

Hence our final answer is

$$= \frac{t}{8} - \frac{\sin 2t}{8} + \frac{\sin 4t}{64} + c$$

12. Question

Evaluate $\int \cos^3 3x dx$

Answer

We can write $\int \cos^3 3x dx$ as:

$$\int \cos 3x (\cos 3x)^2 dx \quad \int \cos 3x (\cos^2 3x) dx \text{ and}$$

further as:

$$= \cos 3x(1 - \sin^2 3x) dx$$

$$= \int \cos 3x dx - \int \cos 3x (\sin^2 3x) dx$$

$$\text{Taking } A = \int \cos 3x dx$$

Solving for A

$$A = \frac{\sin 3x}{3}$$

$$\text{Taking } B = \int \cos 3x (\sin^2 3x) dx$$

In this taking $\sin 3x = t$

Differentiating on both sides we get

$$3 \cos 3x dx = dt$$

Solving by putting these values we get

$$B = \int \frac{t^2}{3} dt$$

$$= \frac{t^3}{9} + c$$

Substituting values we get

$$B = \frac{\sin^3 3x}{9} + c$$

Our final answer is A+B i.e

$$= \frac{\sin 3x}{3} + \frac{\sin^3 3x}{9} + c$$

12. Question

Evaluate $\int \cos^3 3x dx$

Answer

We can write $\int \cos^3 3x dx$ as:

$$\int \cos 3x (\cos 3x)^2 dx \quad \int \cos 3x (\cos^2 3x) dx \text{ and}$$

further as:

$$= \cos 3x(1 - \sin^2 3x) dx$$

$$= \int \cos 3x dx - \int \cos 3x (\sin^2 3x) dx$$

$$\text{Taking } A = \int \cos 3x dx$$

Solving for A

$$A = \frac{\sin 3x}{3}$$

$$\text{Taking } B = \int \cos 3x (\sin^2 3x) dx$$

In this taking $\sin 3x = t$

Differentiating on both sides we get

$$3 \cos 3x dx = dt$$

Solving by putting these values we get

$$B = \int \frac{t^2}{3} dt$$

$$= \frac{t^3}{9} + c$$

Substituting values we get

$$B = \frac{\sin^3 3x}{9} + c$$

Our final answer is A+B i.e

$$= \frac{\sin 3x}{3} + \frac{\sin 3x}{3} + c$$

13. Question

Evaluate $\int \frac{\sin 2x}{a^2 + b^2 \sin^2 x}$

Answer

Taking b^2 common, we get,

$$\int \frac{\sin 2x}{b^2(\frac{a^2}{b^2} + \sin^2 x)} dx$$

taking $\frac{a^2}{b^2} + \sin^2 x = t$

on differentiating both sides we get

$$2 \sin x \cos x dx = dt$$

Means $\sin 2x dx = dt$

putting $\frac{a^2}{b^2} + \sin^2 x = t$ and $\sin 2x dx = dt$ in equation we get our equation as

$$\int \frac{dt}{b^2(t)}$$

On solving this we get

$$= \frac{\ln(t)}{b^2} + c$$

Substituting value of t we get our answer as

$$= \frac{\ln(\frac{a^2}{b^2} + \sin^2 x)}{b^2} + c$$

13. Question

Evaluate $\int \frac{\sin 2x}{a^2 + b^2 \sin^2 x}$

Answer

Taking b^2 common, we get,

$$\int \frac{\sin 2x}{b^2(\frac{a^2}{b^2} + \sin^2 x)} dx$$

taking $\frac{a^2}{b^2} + \sin^2 x = t$

on differentiating both sides we get

$$2\sin x \cos x dx = dt$$

$$\text{Means } \sin^2 x dx = dt$$

putting $\frac{a^2}{b^2} + \sin^2 x = t$ and $\sin^2 x dx = dt$ in equation we get our equation as

$$\int \frac{dt}{b^2(t)}$$

On solving this we get

$$= \frac{\ln(t)}{b^2} + c$$

Substituting value of t we get our answer as

$$= \frac{\ln\left(\frac{a^2}{b^2} + \sin^2 x\right)}{b^2} + c$$

14. Question

$$\text{Evaluate } \int \frac{1}{(\sin^{-1} x) \sqrt{1-x^2}} dx$$

Answer

$$\text{Taking } \sin^{-1} x = t$$

Differentiating both sides,

$$\text{We get } \frac{1}{\sqrt{1-x^2}} dx = dt$$

Our new equation has become

$$\int \frac{dt}{t}$$

On solving this we get

$$\int \frac{dt}{t} = \ln(t) + c$$

$$\text{Substituting value of } t = \sin^{-1} x$$

We get our answer as

$$= \ln(\sin^{-1} x) + c$$

14. Question

$$\text{Evaluate } \int \frac{1}{(\sin^{-1} x) \sqrt{1-x^2}} dx$$

Answer

$$\text{Taking } \sin^{-1} x = t$$

Differentiating both sides,

$$\text{We get } \frac{1}{\sqrt{1-x^2}} dx = dt$$

Our new equation has become

$$\int \frac{dt}{t}$$

On solving this we get

$$\int \frac{dt}{t} = \ln(t) + c$$

Substituting value of $t = \sin^{-1}x$

We get our answer as

$$= \ln(\sin^{-1}x) + c$$

15. Question

Evaluate $\int \frac{(\sin^{-1}x)^3}{\sqrt{1-x^2}} dx$

Answer

Taking $\sin^{-1}x = t$

Differentiating both sides,

We get $\frac{1}{\sqrt{1-x^2}} dx = dt$

Our new equation has become

$$\int t^3 dt$$

On solving this we get

$$\int t^3 dt = \frac{t^4}{4} + c$$

Substituting value of $t = \sin^{-1}x$

We get our answer as

$$= \frac{(\sin^{-1}x)^4}{4} + c$$

15. Question

Evaluate $\int \frac{(\sin^{-1}x)^3}{\sqrt{1-x^2}} dx$

Answer

Taking $\sin^{-1}x = t$

Differentiating both sides,

We get $\frac{1}{\sqrt{1-x^2}} dx = dt$

Our new equation has become

$$\int t^3 dt$$

On solving this we get

$$\int t^3 dt = \frac{t^4}{4} + c$$

Substituting value of $t = \sin^{-1}x$

We get our answer as

$$= \frac{(\sin^{-1}x)^4}{4} + c$$