

Chapter 5 Continuity and Differentiability

EXERCISE 5.1

Question 1:

Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, $x = -3$ and at $x = 5$.

Solution:

The given function is $f(x) = 5x - 3$

$$\text{At } x = 0, f(0) = 5(0) - 3 = -3$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (5x - 3) = 5(0) - 3 = -3$$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$.

$$\text{At } x = -3, f(-3) = 5(-3) - 3 = -18$$

$$\lim_{x \rightarrow -3} f(x) = \lim_{x \rightarrow -3} (5x - 3) = 5(-3) - 3 = -18$$

$$\therefore \lim_{x \rightarrow -3} f(x) = f(-3)$$

Therefore, f is continuous at $x = -3$.

$$\text{At } x = 5, f(5) = 5(5) - 3 = 22$$

$$\lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow 5} (5x - 3) = 5(5) - 3 = 22$$

$$\therefore \lim_{x \rightarrow 5} f(x) = f(5)$$

Therefore, f is continuous at $x = 5$.

Question 2:

Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

Solution:

The given function is $f(x) = 2x^2 - 1$

$$\text{At } x = 3, f(3) = 2(3)^2 - 1 = 17$$

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} (2x^2 - 1) = 2(3^2) - 1 = 17$$

$$\therefore \lim_{x \rightarrow 3} f(x) = f(3)$$

Therefore, f is continuous at $x = 3$.

Question 3:

Examine the following functions for continuity.

(i) $f(x) = x - 5$

- (ii) $f(x) = \frac{1}{x-5}, x \neq 5$
 (iii) $f(x) = \frac{x^2-25}{x+5}, x \neq -5$
 (iv) $f(x) = |x-5|, x \neq 5$

Solution:

- (i) The given function is $f(x) = x - 5$

It is evident that f is defined at every real number k and its value at k is $k - 5$.

It is also observed that

$$\lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (x - 5) = k - 5 = f(k)$$

$$\therefore \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, f is continuous at every real number and therefore, it is a continuous function.

- (ii) The given function is $f(x) = \frac{1}{x-5}, x \neq 5$

For any real number $k \neq 5$, we obtain

$$\lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} \frac{1}{x-5} = \frac{1}{k-5}$$

Also,

$$f(k) = \frac{1}{k-5} \quad (\text{As } k \neq 5)$$

$$\therefore \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

- (iii) The given function is $f(x) = \frac{x^2-25}{x+5}, x \neq -5$

For any real number $c \neq -5$, we obtain

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \frac{x^2-25}{x+5} = \lim_{x \rightarrow c} \frac{(x+5)(x-5)}{x+5} = \lim_{x \rightarrow c} (x-5) = (c-5)$$

Also,

$$f(c) = \frac{(c+5)(c-5)}{c+5} = (c-5)$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Hence, f is continuous at every point in the domain of f and therefore, it is a continuous function.

(iv) The given function is $f(x) = |x - 5| = \begin{cases} 5 - x, & \text{if } x < 5 \\ x - 5, & \text{if } x \geq 5 \end{cases}$

This function f is defined at all points of the real line. Let c be a point on a real line. Then, $c < 5$, $c = 5$ or $c > 5$

Case I: $c < 5$

Then, $f(c) = 5 - c$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (5 - x) = 5 - c$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all real numbers less than 5.

Case II: $c = 5$

Then, $f(c) = f(5) = (5 - 5) = 0$

$$\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5} (5 - x) = (5 - 5) = 0$$

$$\lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5} (x - 5) = 0$$

$$\therefore \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = f(c)$$

Therefore, f is continuous at $x = 5$

Case III: $c > 5$

Then, $f(c) = f(5) = c - 5$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x - 5) = c - 5$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all real numbers greater than 5.

Hence, f is continuous at every real number and therefore, it is a continuous function.

Question 4:

Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

Solution:

The given function is $f(x) = x^n$

It is observed that f is defined at all positive integers, n , and its value at n is n^n .

Then,

$$\lim_{x \rightarrow n} f(n) = \lim_{x \rightarrow n} (x^n) = x^n$$

$$\therefore \lim_{x \rightarrow n} f(x) = f(n)$$

Therefore, f is continuous at n , where n is a positive integer.

Question 5:

Is the function f defined by $f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 5, & \text{if } x > 1 \end{cases}$ continuous at $x = 0$? At $x = 1$? At $x = 2$?

Solution:

The given function is $f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 5, & \text{if } x > 1 \end{cases}$

At $x = 0$,

It is evident that f is defined at 0 and its value at 0 is 0.

Then,

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (x) = 0$$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$.

At $x = 1$,

It is evident that f is defined at 1 and its value at 1 is 1.

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x) = 1$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (5) = 5$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$$

Therefore, f is not continuous at $x = 1$.

At $x = 2$,

It is evident that f is defined at 2 and its value at 2 is 5.

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (5) = 5$$

$$\therefore \lim_{x \rightarrow 2} f(x) = f(2)$$

Therefore, f is continuous at $x = 2$.

Question 6:

Find all points of discontinuity of f , where f is defined by $f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$.

Solution:

The given function is $f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$

It is evident that the given function f is defined at all the points of the real line.

Let c be a point on the real line. Then, three cases arise.

$$c < 2$$

$$c > 2$$

$$c = 2$$

Case I: $c < 2$

$$f(c) = 2c + 3$$

Then,

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (2x + 3) = 2c + 3$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 2$.

Case II: $c > 2$

Then,

$$f(c) = 2c - 3$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (2x - 3) = 2c - 3$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 2$

Case III: $c = 2$

Then, the left hand limit of f at $x = 2$ is,

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2x + 3) = 2(2) + 3 = 7$$

The right hand limit of f at $x = 2$ is,

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (2x - 3) = 2(2) - 3 = 1$$

It is observed that the left and right hand limit of f at $x = 2$ do not coincide.

Therefore, f is not continuous at $x = 2$.

Hence, $x = 2$ is the only point of discontinuity of f .

Question 7:

$$f(x) = \begin{cases} |x| + 3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \geq 3 \end{cases}$$

Find all points of discontinuity of f , where f is defined by

Solution:

$$f(x) = \begin{cases} |x| + 3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \geq 3 \end{cases}$$

The given function is

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < -3$, then $f(c) = -c + 3$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-x + 3) = -c + 3$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < -3$.

Case II:

If $c = -3$, then $f(-3) = -(-3) + 3 = 6$

$$\lim_{x \rightarrow -3^-} f(x) = \lim_{x \rightarrow -3^-} (-x + 3) = -(-3) + 3 = 6$$

$$\lim_{x \rightarrow -3^+} f(x) = \lim_{x \rightarrow -3^+} (-2x) = -2(-3) = 6$$

$$\therefore \lim_{x \rightarrow -3} f(x) = f(-3)$$

Therefore, f is continuous at $x = -3$.

Case III:

If $-3 < c < 3$, then $f(c) = -2c$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-2x) = -2c$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous in $(-3, 3)$.

Case IV:

If $c = 3$, then the left hand limit of f at $x = 3$ is,

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (-2x) = -2(3) = -6$$

The right hand limit of f at $x = 3$ is,

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (6x + 2) = 6(3) + 2 = 20$$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$.

Case V:

If $c > 3$, then $f(c) = 6c + 2$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (6x + 2) = 6c + 2$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 3$.

Hence, $x = 3$ is the only point of discontinuity of f .

Question 8:

$$f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Find all points of discontinuity of f , where f is defined by

Solution:

$$f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

The given function is

It is known that, $x < 0 \Rightarrow |x| = -x$ and $x > 0 \Rightarrow |x| = x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \frac{|x|}{x} = \frac{-x}{x} = -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ \frac{|x|}{x} = \frac{x}{x} = 1, & \text{if } x > 0 \end{cases}$$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = -1$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-1) = -1$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points $x < 0$.

Case II:

If $c = 0$, then the left hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (-1) = -1$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (1) = 1$$

It is observed that the left and right hand limit of f at $x = 0$ do not coincide.

Therefore, f is not continuous at $x = 0$.

Case III:

If $c > 0$, then $f(c) = 1$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (1) = 1$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 0$.

Hence, $x = 0$ is the only point of discontinuity of f .

Question 9:

$$f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

Find all points of discontinuity of f , where f is defined by

Solution:

$$f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

The given function is

It is known that $x < 0 \Rightarrow |x| = -x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \frac{x}{|x|} = \frac{x}{-x} = -1, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

$$\Rightarrow f(x) = -1 \forall x \in \mathbb{R}$$

Let c be any real number.

$$\text{Then, } \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-1) = -1$$

$$\text{Also, } f(c) = -1 = \lim_{x \rightarrow c} f(x)$$

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

Question 10:

$$f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$$

Find all points of discontinuity of f , where f is defined by

Solution:

The given function is $f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c^2 + 1$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x^2 + 1) = c^2 + 1$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 1$.

Case II:

If $c = 1$, then $f(c) = f(1) = 1 + 1 = 2$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^2 + 1) = 1^2 + 1 = 2$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x + 1) = 1 + 1 = 2$$

$$\therefore \lim_{x \rightarrow 1} f(x) = f(1)$$

Therefore, f is continuous at $x = 1$.

Case III:

If $c > 1$, then $f(c) = c + 1$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x + 1) = c + 1$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 1$.

Hence, the given function f has no point of discontinuity.

Question 11:

Find all points of discontinuity of f , where f is defined by $f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$

Solution:

The given function is $f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 2$, then $f(c) = c^3 - 3$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x^3 - 3) = c^3 - 3$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 2$.

Case II:

If $c = 2$, then $f(c) = f(2) = 2^3 - 3 = 5$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x^3 - 3) = 2^3 - 3 = 5$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x^2 + 1) = 2^2 + 1 = 5$$

$$\therefore \lim_{x \rightarrow 2} f(x) = f(2)$$

Therefore, f is continuous at $x = 2$.

Case III:

If $c > 2$, then $f(c) = c^2 + 1$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x^2 + 1) = c^2 + 1$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 2$.

Thus, the given function f is continuous at every point on the real line.

Hence, f has no point of discontinuity.

Question 12:

Find all points of discontinuity of f , where f is defined by $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$.

Solution:

The given function is $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c^{10} - 1$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x^{10} - 1) = c^{10} - 1$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 1$.

Case II:

If $c = 1$, then the left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^{10} - 1) = 1^{10} - 1 = 1 - 1 = 0$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x^2) = 1^2 = 1$$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$.

Case III:

If $c > 1$, then $f(c) = c^2$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x^2) = c^2$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 1$.

Thus from the above observation, it can be concluded that $x = 1$ is the only point of discontinuity of f .

Question 13:

Is the function defined by $f(x) = \begin{cases} x+5, & \text{if } x \leq 1 \\ x-5, & \text{if } x > 1 \end{cases}$ a continuous function?

Solution:

The given function is $f(x) = \begin{cases} x+5, & \text{if } x \leq 1 \\ x-5, & \text{if } x > 1 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c + 5$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x + 5) = c + 5$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 1$.

Case II:

If $c = 1$, then $f(1) = 1 + 5 = 6$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x + 5) = 1 + 5 = 6$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x - 5) = 1 - 5 = -4$$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$.

Case III:

If $c > 1$, then $f(c) = c - 5$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x - 5) = c - 5$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 1$.

From the above observation it can be concluded that, $x = 1$ is the only point of discontinuity of f .

Question 14:

$$f(x) = \begin{cases} 3, & \text{if } 0 \leq x \leq 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \leq x \leq 10 \end{cases}$$

Discuss the continuity of the function f , where f is defined by

Solution:

$$f(x) = \begin{cases} 3, & \text{if } 0 \leq x \leq 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \leq x \leq 10 \end{cases}$$

The given function is

The given function f is defined at all the points of the interval $[0, 10]$.

Let c be a point in the interval $[0, 10]$.

Case I:

If $0 \leq c < 1$, then $f(c) = 3$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (3) = 3$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous in the interval $[0, 1)$.

Case II:

If $c = 1$, then $f(3) = 3$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3) = 3$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (4) = 4$$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$.

Case III:

If $1 < c < 3$, then $f(c) = 4$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (4) = 4$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at in the interval $(1, 3)$.

Case IV:

If $c = 3$, then $f(c) = 5$

The left hand limit of f at $x = 3$ is,

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (4) = 4$$

The right hand limit of f at $x = 3$ is,

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (5) = 5$$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.

Therefore, f is discontinuous at $x = 3$.

Case V:

If $3 < c \leq 10$, then $f(c) = 5$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (5) = 5$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval $(3, 10]$.

Hence, f is discontinuous at $x = 1$ and $x = 3$.

Question 15:

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \leq x \leq 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

Discuss the continuity of the function f , where f is defined by

Solution:

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \leq x \leq 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

The given function is

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = 2c$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (2x) = 2c$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 0$.

Case II:

If $c = 0$, then $f(c) = f(0) = 0$

The left hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (2x) = 2(0) = 0$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (0) = 0$$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

Case III:

If $0 < c < 1$, then $f(x) = 0$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (0) = 0$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous in the interval $(0,1)$.

Case IV:

If $c = 1$, then $f(c) = f(1) = 0$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (0) = 0$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (4x) = 4(1) = 4$$

It is observed that the left and right hand limit of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$.

Case V:

If $c < 1$, then $f(c) = 4c$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (4x) = 4c$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 1$.

Hence, f is not continuous only at $x = 1$.

Question 16:

$$f(x) = \begin{cases} -2, & \text{if } x \leq -1 \\ 2x, & \text{if } -1 < x \leq 1 \\ 2, & \text{if } x > 1 \end{cases}$$

Discuss the continuity of the function f , where f is defined by

Solution:

$$f(x) = \begin{cases} -2, & \text{if } x \leq -1 \\ 2x, & \text{if } -1 < x \leq 1 \\ 2, & \text{if } x > 1 \end{cases}$$

The given function is

The given function f is defined at all the points.

Let c be a point on the real line.

Case I:

If $c < -1$, then $f(c) = -2$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-2) = -2$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < -1$.

Case II:

$$\text{If } c = -1, \text{ then } f(c) = f(-1) = -2$$

The left hand limit of f at $x = -1$ is,

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (-2) = -2$$

The right hand limit of f at $x = -1$ is,

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (2x) = 2(-1) = -2$$

$$\therefore \lim_{x \rightarrow -1} f(x) = f(-1)$$

Therefore, f is continuous at $x = -1$

Case III:

$$\text{If } -1 < c < 1, \text{ then } f(c) = 2c$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (2x) = 2c$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous in the interval $(-1, 1)$.

Case IV:

$$\text{If } c = 1, \text{ then } f(c) = f(1) = 2(1) = 2$$

The left hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2x) = 2(1) = 2$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2) = 2$$

$$\therefore \lim_{x \rightarrow 1} f(x) = f(c)$$

Therefore, f is continuous at $x = 2$.

Case V:

$$\text{If } c > 1, \text{ then } f(c) = 2$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (2) = 2$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 1$.

Thus, from the above observations, it can be concluded that f is continuous at all points of the real line.

Question 17:

Find the relationship between a and b so that the function f defined by $f(x) = \begin{cases} ax+1, & \text{if } x \leq 3 \\ bx+3, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$.

Solution:

The given function is $f(x) = \begin{cases} ax+1, & \text{if } x \leq 3 \\ bx+3, & \text{if } x > 3 \end{cases}$

For f to be continuous at $x = 3$, then

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3) \quad \dots(1)$$

Also,

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (ax+1) = 3a+1$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (bx+3) = 3b+3$$

$$f(3) = 3a+1$$

Therefore, from (1), we obtain

$$3a+1 = 3b+3 = 3a+1$$

$$\Rightarrow 3a+1 = 3b+3$$

$$\Rightarrow 3a = 3b+2$$

$$\Rightarrow a = b + \frac{2}{3}$$

Therefore, the required relationship is given by, $a = b + \frac{2}{3}$.

Question 18:

For what value of λ is the function defined by $f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \leq 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$ is continuous at $x = 0$? What about continuity at $x = 1$?

Solution:

The given function is
$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \leq 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

If f is continuous at $x = 0$, then

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} \lambda(x^2 - 2x) = \lim_{x \rightarrow 0^+} (4x + 1) = \lambda(0^2 - 2 \times 0)$$

$$\Rightarrow \lambda(0^2 - 2 \times 0) = 4(0) + 1 = 0$$

$$\Rightarrow 0 = 1 = 0 \quad [\text{which is not possible}]$$

Therefore, there is no value of λ for which f is continuous at $x = 0$.

At $x = 1$

$$f(1) = 4x + 1 = 4(1) + 1 = 5$$

$$\lim_{x \rightarrow 1} (4x + 1) = 4(1) + 1 = 5$$

$$\therefore \lim_{x \rightarrow 1} f(x) = f(1)$$

Therefore, for any values of λ , f is continuous at $x = 1$.

Question 19:

Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral point. Here $[x]$ denotes the greatest integer less than or equal to x .

Solution:

The given function is $g(x) = x - [x]$

It is evident that g is defined at all integral points.

Let n be an integer.

Then,

$$g(n) = n - [n] = n - n = 0$$

The left hand limit of g at $x = n$ is,

$$\lim_{x \rightarrow n^-} g(x) = \lim_{x \rightarrow n^-} (x - [x]) = \lim_{x \rightarrow n^-} (x) - \lim_{x \rightarrow n^-} [x] = n - (n-1) = 1$$

The right hand limit of g at $x = n$ is,

$$\lim_{x \rightarrow n^+} g(x) = \lim_{x \rightarrow n^+} (x - [x]) = \lim_{x \rightarrow n^+} (x) - \lim_{x \rightarrow n^+} [x] = n - n = 0$$

It is observed that the left and right hand limit of g at $x = n$ do not coincide.

Therefore, g is not continuous at $x = n$.

Hence, g is discontinuous at all integral points.

Question 20:

Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

Solution:

The given function is $f(x) = x^2 - \sin x + 5$

It is evident that f is defined at $x = \pi$.

At $x = \pi$, $f(x) = f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

Consider $\lim_{x \rightarrow \pi} f(x) = \lim_{x \rightarrow \pi} (x^2 - \sin x + 5)$

Put $x = \pi + h$, it is evident that if $x \rightarrow \pi$, then $h \rightarrow 0$

$$\begin{aligned} \therefore \lim_{x \rightarrow \pi} f(x) &= \lim_{x \rightarrow \pi} (x^2 - \sin x) + 5 \\ &= \lim_{h \rightarrow 0} [(\pi + h)^2 - \sin(\pi + h) + 5] \\ &= \lim_{h \rightarrow 0} (\pi + h)^2 - \lim_{h \rightarrow 0} \sin(\pi + h) + \lim_{h \rightarrow 0} 5 \\ &= (\pi + 0)^2 - \lim_{h \rightarrow 0} [\sin \pi \cos h + \cos \pi \sin h] + 5 \\ &= \pi^2 - \lim_{h \rightarrow 0} \sin \pi \cos h - \lim_{h \rightarrow 0} \cos \pi \sin h + 5 \\ &= \pi^2 - \sin \pi \cos 0 - \cos \pi \sin 0 + 5 \\ &= \pi^2 - 0(1) - (-1)0 + 5 \\ &= \pi^2 + 5 \\ &= f(\pi) \end{aligned}$$

Therefore, the given function f is continuous at $x = \pi$.

Question 21:

Discuss the continuity of the following functions.

(i) $f(x) = \sin x + \cos x$

(ii) $f(x) = \sin x - \cos x$

(iii) $f(x) = \sin x \times \cos x$

Solution:

It is known that if g and h are two continuous functions, then $g + h$, $g - h$ and $g \cdot h$ are also continuous.

Let $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$g(c) = \sin c$$

$$\begin{aligned}\lim_{x \rightarrow c} g(x) &= \lim_{x \rightarrow c} \sin x \\ &= \lim_{h \rightarrow 0} \sin(c + h) \\ &= \lim_{h \rightarrow 0} [\sin c \cos h + \cos c \sin h] \\ &= \lim_{h \rightarrow 0} (\sin c \cos h) + \lim_{h \rightarrow 0} (\cos c \sin h) \\ &= \sin c \cos 0 + \cos c \sin 0 \\ &= \sin c(1) + \cos c(0) \\ &= \sin c\end{aligned}$$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, $g(x) = \sin x$ is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$h(c) = \cos c$$

$$\begin{aligned}
\lim_{x \rightarrow c} h(x) &= \lim_{x \rightarrow c} \cos x \\
&= \lim_{h \rightarrow 0} \cos(c+h) \\
&= \lim_{h \rightarrow 0} [\cos c \cos h - \sin c \sin h] \\
&= \lim_{h \rightarrow 0} (\cos c \cos h) - \lim_{h \rightarrow 0} (\sin c \sin h) \\
&= \cos c \cos 0 - \sin c \sin 0 \\
&= \cos c(1) - \sin c(0) \\
&= \cos c \\
\therefore \lim_{x \rightarrow c} h(x) &= h(c)
\end{aligned}$$

Therefore, $h(x) = \cos x$ is a continuous function.

Therefore, it can be concluded that,

- (i) $f(x) = g(x) + h(x) = \sin x + \cos x$ is a continuous function.
- (ii) $f(x) = g(x) - h(x) = \sin x - \cos x$ is a continuous function.
- (iii) $f(x) = g(x) \times h(x) = \sin x \times \cos x$ is a continuous function.

Question 22:

Discuss the continuity of the cosine, cosecant, secant, and cotangent functions.

Solution:

It is known that if g and h are two continuous functions, then

- (i) $\frac{h(x)}{g(x)}, g(x) \neq 0$ is continuous.
- (ii) $\frac{1}{g(x)}, g(x) \neq 0$ is continuous.
- (iii) $\frac{1}{h(x)}, h(x) \neq 0$ is continuous.

Let $g(x) = \sin x$ and $h(x) = \cos x$ are continuous functions.

It is evident that $g(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$g(c) = \sin c$$

$$\begin{aligned}\lim_{x \rightarrow c} g(x) &= \lim_{x \rightarrow c} \sin x \\ &= \lim_{h \rightarrow 0} \sin(c + h) \\ &= \lim_{h \rightarrow 0} [\sin c \cos h + \cos c \sin h] \\ &= \lim_{h \rightarrow 0} (\sin c \cos h) + \lim_{h \rightarrow 0} (\cos c \sin h) \\ &= \sin c \cos 0 + \cos c \sin 0 \\ &= \sin c(1) + \cos c(0) \\ &= \sin c\end{aligned}$$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, $g(x) = \sin x$ is a continuous function.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$h(c) = \cos c$$

$$\begin{aligned}\lim_{x \rightarrow c} h(x) &= \lim_{x \rightarrow c} \cos x \\ &= \lim_{h \rightarrow 0} \cos(c + h) \\ &= \lim_{h \rightarrow 0} [\cos c \cos h - \sin c \sin h] \\ &= \lim_{h \rightarrow 0} (\cos c \cos h) - \lim_{h \rightarrow 0} (\sin c \sin h) \\ &= \cos c \cos 0 - \sin c \sin 0 \\ &= \cos c(1) - \sin c(0) \\ &= \cos c\end{aligned}$$

$$\therefore \lim_{x \rightarrow c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function.

Therefore, it can be concluded that,

$$\operatorname{cosec} x = \frac{1}{\sin x}, \sin x \neq 0 \text{ is continuous.}$$

$$\Rightarrow \operatorname{cosec} x, x \neq n\pi (n \in \mathbb{Z}) \text{ is continuous.}$$

Therefore, cosecant is continuous except at $x = n\pi (n \in \mathbb{Z})$

$$\sec x = \frac{1}{\cos x}, \cos x \neq 0 \text{ is continuous.}$$

$$\Rightarrow \sec x, x \neq (2n+1)\frac{\pi}{2} (n \in \mathbb{Z}) \text{ is continuous.}$$

Therefore, secant is continuous except at $x = (2n+1)\frac{\pi}{2} (n \in \mathbb{Z})$

$\cot x = \frac{\cos x}{\sin x}, \sin x \neq 0$ is continuous.

$\Rightarrow \cot x, x \neq n\pi (n \in \mathbb{Z})$ is continuous.

Therefore, cotangent is continuous except at $x = n\pi (n \in \mathbb{Z})$.

Question 23:

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$$

Find the points of discontinuity of f , where

Solution:

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$$

The given function is

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c < 0$, then $f(c) = \frac{\sin c}{c}$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \left(\frac{\sin x}{x} \right) = \frac{\sin c}{c}$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x < 0$.

Case II:

If $c > 0$, then $f(c) = c+1$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x+1) = c+1$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x > 0$.

Case III:

If $c = 0$, then $f(c) = f(0) = 0 + 1 = 1$

The left hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left(\frac{\sin x}{x} \right) = 1$$

The right hand limit of f at $x = 0$ is,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x + 1) = 1$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at all points of the real line.

Thus, f has no point of discontinuity.

Question 24:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Determine if f defined by is a continuous function?

Solution:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

The given function is

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c \neq 0$, then $f(c) = c^2 \sin \frac{1}{c}$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \left(x^2 \sin \frac{1}{x} \right) = \left(\lim_{x \rightarrow c} x^2 \right) \left(\lim_{x \rightarrow c} \sin \frac{1}{x} \right) = c^2 \sin \frac{1}{c}$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x \neq 0$.

Case II:

If $c = 0$, then $f(0) = 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left(x^2 \sin \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right)$$

It is known that, $-1 \leq \sin \frac{1}{x} \leq 1, x \neq 0$

$$\Rightarrow -x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$

$$\Rightarrow \lim_{x \rightarrow 0} (-x^2) \leq \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) \leq \lim_{x \rightarrow 0} x^2$$

$$\Rightarrow 0 \leq \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) \leq 0$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) = 0$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = 0$$

Similarly,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(x^2 \sin \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left(x^2 \sin \frac{1}{x} \right) = 0$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = f(0) = \lim_{x \rightarrow 0^+} f(x)$$

Therefore, f is continuous at $x = 0$.

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 25:

Examine the continuity of f , where f is defined by $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$

Solution:

The given function is $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$

The given function f is defined at all the points of the real line.

Let c be a point on the real line.

Case I:

If $c \neq 0$, then $f(c) = \sin c - \cos c$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (\sin x - \cos x) = \sin c - \cos c$$

$$\therefore \lim_{x \rightarrow c} f(x) = f(c)$$

Therefore, f is continuous at all points x , such that $x \neq 0$.

Case II:

If $c = 0$, then $f(0) = -1$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} (\sin x - \cos x) = \sin 0 - \cos 0 = 0 - 1 = -1$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

Therefore, f is continuous at $x = 0$.

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

Question 26:

Find the values of k so that the function f is continuous at the indicated point

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

Solution:

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$$

The given function is

The given function f is continuous at $x = \frac{\pi}{2}$, if f is defined at $x = \frac{\pi}{2}$ and if the value of the f at $x = \frac{\pi}{2}$ equals the limit of f at $x = \frac{\pi}{2}$.

It is evident that f is defined at $x = \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = 3$

$$\lim_{x \rightarrow \frac{\pi}{2}} f(x) = \lim_{x \rightarrow \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x}$$

Put $x = \frac{\pi}{2} + h$

Then $x \rightarrow \frac{\pi}{2} \Rightarrow h \rightarrow 0$

$$\begin{aligned}
\therefore \lim_{x \rightarrow \frac{\pi}{2}} f(x) &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x} = \lim_{h \rightarrow 0} \frac{k \cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} \\
&= k \lim_{h \rightarrow 0} \frac{-\sin h}{-2h} = \frac{k}{2} \lim_{h \rightarrow 0} \frac{\sin h}{h} = \frac{k}{2} \cdot 1 = \frac{k}{2} \\
\therefore \lim_{x \rightarrow \frac{\pi}{2}} f(x) &= f\left(\frac{\pi}{2}\right) \\
\Rightarrow \frac{k}{2} &= 3 \\
\Rightarrow k &= 6
\end{aligned}$$

Therefore, the value of $k = 6$.

Question 27:

Find the values of k so that the function f is continuous at the indicated point.

$$f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases} \quad \text{at } x = 2$$

Solution:

The given function is
$$f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases}$$

The given function f is continuous at $x = 2$, if f is defined at $x = 2$ and if the value of the f at $x = 2$ equals the limit of f at $x = 2$.

It is evident that f is defined at $x = 2$ and $f(2) = k(2)^2 = 4k$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x \rightarrow 2^-} (kx^2) = \lim_{x \rightarrow 2^+} (3) = 4k$$

$$\Rightarrow k \times 2^2 = 3 = 4k$$

$$\Rightarrow 4k = 3$$

$$\Rightarrow k = \frac{3}{4}$$

Therefore, the value of $k = \frac{3}{4}$.

Question 28:

Find the values of k so that the function f is continuous at the indicated point

$$f(x) = \begin{cases} kx+1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases} \text{ at } x = \pi$$

Solution:

The given function is $f(x) = \begin{cases} kx+1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

The given function f is continuous at $x = \pi$, if f is defined at $x = \pi$ and if the value of the f at $x = \pi$ equals the limit of f at $x = \pi$.

It is evident that f is defined at $x = \pi$ and $f(\pi) = k\pi + 1$

$$\lim_{x \rightarrow \pi^-} f(x) = \lim_{x \rightarrow \pi^+} f(x) = f(\pi)$$

$$\Rightarrow \lim_{x \rightarrow \pi^-} (kx+1) = \lim_{x \rightarrow \pi^+} (\cos x) = k\pi + 1$$

$$\Rightarrow k\pi + 1 = \cos \pi = k\pi + 1$$

$$\Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow k = -\frac{2}{\pi}$$

Therefore, the value of $k = -\frac{2}{\pi}$.

Question 29:

Find the values of k so that the function f is continuous at the indicated point

$$f(x) = \begin{cases} kx+1, & \text{if } x \leq 5 \\ 3x-5, & \text{if } x > 5 \end{cases} \text{ at } x = 5.$$

Solution:

The given function is $f(x) = \begin{cases} kx+1, & \text{if } x \leq 5 \\ 3x-5, & \text{if } x > 5 \end{cases}$

The given function f is continuous at $x = 5$, if f is defined at $x = 5$ and if the value of the f at $x = 5$ equals the limit of f at $x = 5$.

It is evident that f is defined at $x = 5$ and $f(5) = kx+1 = 5k+1$

$$\begin{aligned}
\lim_{x \rightarrow 5^-} f(x) &= \lim_{x \rightarrow 5^+} f(x) = f(5) \\
\Rightarrow \lim_{x \rightarrow 5^-} (kx + 1) &= \lim_{x \rightarrow 5^+} (3x - 5) = 5k + 1 \\
\Rightarrow 5k + 1 &= 3(5) - 5 = 5k + 1 \\
\Rightarrow 5k + 1 &= 15 - 5 = 5k + 1 \\
\Rightarrow 5k + 1 &= 10 = 5k + 1 \\
\Rightarrow 5k + 1 &= 10 \\
\Rightarrow 5k &= 9 \\
\Rightarrow k &= \frac{9}{5}
\end{aligned}$$

Therefore, the value of $k = \frac{9}{5}$.

Question 30:

$$f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \geq 10 \end{cases}$$

Find the values of a & b such that the function defined by $f(x)$, is a continuous function.

Solution:

$$f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \geq 10 \end{cases}$$

The given function is

It is evident that f is defined at all points of the real line.

If f is a continuous function, then f is continuous at all real numbers.

In particular, f is continuous at $x = 2$ and $x = 10$

Since f is continuous at $x = 2$, we obtain

$$\begin{aligned}
\lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^+} f(x) = f(2) \\
\Rightarrow \lim_{x \rightarrow 2^-} (5) &= \lim_{x \rightarrow 2^+} (ax + b) = 5 \\
\Rightarrow 5 &= 2a + b = 5 \\
\Rightarrow 2a + b &= 5 \quad \dots(1)
\end{aligned}$$

Since f is continuous at $x = 10$, we obtain

$$\begin{aligned}
\lim_{x \rightarrow 10^-} f(x) &= \lim_{x \rightarrow 10^+} f(x) = f(10) \\
\Rightarrow \lim_{x \rightarrow 10^-} (ax + b) &= \lim_{x \rightarrow 10^+} (21) = 21 \\
\Rightarrow 10a + b &= 21 = 21 \\
\Rightarrow 10a + b &= 21 \quad \dots(2)
\end{aligned}$$

On subtracting equation (1) from equation (2), we obtain

$$8a = 16$$

$$\Rightarrow a = 2$$

By putting $a = 2$ in equation (1), we obtain

$$2(2) + b = 5$$

$$\Rightarrow 4 + b = 5$$

$$\Rightarrow b = 1$$

Therefore, the values of a and b for which f is a continuous function are 2 and 1 respectively.

Question 31:

Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

Solution:

The given function is $f(x) = \cos(x^2)$.

This function f is defined for every real number and f can be written as the composition of two functions as,

$$f = goh, \text{ where } g(x) = \cos x \text{ and } h(x) = x^2$$

$$[\because (goh)(x) = g(h(x)) = g(x^2) = \cos(x^2) = f(x)]$$

It has to be proved first that $g(x) = \cos x$ and $h(x) = x^2$ are continuous functions.

It is evident that g is defined for every real number.

Let c be a real number.

Let $g(c) = \cos c$. Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} \cos x$$

$$= \lim_{h \rightarrow 0} \cos(c + h)$$

$$= \lim_{h \rightarrow 0} [\cos c \cos h - \sin c \sin h]$$

$$= \lim_{h \rightarrow 0} (\cos c \cos h) - \lim_{h \rightarrow 0} (\sin c \sin h)$$

$$= \cos c \cos 0 - \sin c \sin 0$$

$$= \cos c(1) - \sin c(0)$$

$$= \cos c$$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, $g(x) = \cos x$ is a continuous function.

Let $h(x) = x^2$

It is evident that h is defined for every real number.

Let k be a real number, then $h(k) = k^2$

$$\lim_{x \rightarrow k} h(x) = \lim_{x \rightarrow k} x^2 = k^2$$

$$\therefore \lim_{x \rightarrow k} h(x) = h(k)$$

Therefore, h is a continuous function.

It is known that for real valued functions g and h , such that (goh) is defined at c , if g is continuous at c and if f is continuous at $g(c)$, then (fog) is continuous at c .

Therefore, $f(x) = (goh)(x) = \cos(x^2)$ is a continuous function.

Question 32:

Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Solution:

The given function is $f(x) = |\cos x|$.

This function f is defined for every real number and f can be written as the composition of two functions as,

$$f = goh, \text{ where } g(x) = |x| \text{ and } h(x) = \cos x$$

$$[\because (goh)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)]$$

It has to be proved first that $g(x) = |x|$ and $h(x) = \cos x$ are continuous functions.

$$g(x) = |x| \text{ can be written as } g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

It is evident that g is defined for every real number.

Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} (-x) = -c$$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, g is continuous at all points x , such that $x < 0$.

Case II:

If $c > 0$, then $g(c) = c$

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} (x) = c$$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, g is continuous at all points x , such that $x > 0$.

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (x) = 0$$

$$\therefore \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (x) = g(0)$$

Therefore, g is continuous at all $x = 0$.

From the above three observations, it can be concluded that g is continuous at all points.

Let $h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let c be a real number. Put $x = c + h$

If $x \rightarrow c$, then $h \rightarrow 0$

$$h(c) = \cos c$$

$$\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} \cos x$$

$$= \lim_{h \rightarrow 0} \cos(c + h)$$

$$= \lim_{h \rightarrow 0} [\cos c \cos h - \sin c \sin h]$$

$$= \lim_{h \rightarrow 0} (\cos c \cos h) - \lim_{h \rightarrow 0} (\sin c \sin h)$$

$$= \cos c \cos 0 - \sin c \sin 0$$

$$= \cos c(1) - \sin c(0)$$

$$= \cos c$$

$$\therefore \lim_{x \rightarrow c} h(x) = h(c)$$

Therefore, $h(x) = \cos x$ is a continuous function.

It is known that for real valued functions g and h , such that (goh) is defined at c , if g is continuous at c and if f is continuous at $g(c)$, then (fog) is continuous at c .

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\cos x) = |\cos x|$ is a continuous function.

Question 33:

Show that the function defined by $f(x) = |\sin x|$ is a continuous function.

Solution:

The given function is $f(x) = |\sin x|$.

This function f is defined for every real number and f can be written as the composition of two functions as,

$f = goh$, where $g(x) = |x|$ and $h(x) = \sin x$

$$[\because (goh)(x) = g(h(x)) = g(\sin x) = |\sin x| = f(x)]$$

It has to be proved first that $g(x) = |x|$ and $h(x) = \sin x$ are continuous functions.

$g(x) = |x|$ can be written as $g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$

It is evident that g is defined for every real number.

Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} (-x) = -c$$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, g is continuous at all points x , such that $x < 0$.

Case II:

If $c > 0$, then $g(c) = c$

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} (x) = c$$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, g is continuous at all points x , such that $x > 0$.

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (x) = 0$$

$$\therefore \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (x) = g(0)$$

Therefore, g is continuous at all $x = 0$.

From the above three observations, it can be concluded that g is continuous at all points.

Let $h(x) = \sin x$

It is evident that $h(x) = \sin x$ is defined for every real number.

Let c be a real number. Put $x = c + k$

If $x \rightarrow c$, then $k \rightarrow 0$

$$h(c) = \sin c$$

$$\begin{aligned}\lim_{x \rightarrow c} h(x) &= \lim_{x \rightarrow c} \sin x \\ &= \lim_{k \rightarrow 0} \sin(c+k) \\ &= \lim_{k \rightarrow 0} [\sin c \cos k + \cos c \sin k] \\ &= \lim_{k \rightarrow 0} (\sin c \cos k) + \lim_{k \rightarrow 0} (\cos c \sin k) \\ &= \sin c \cos 0 + \cos c \sin 0 \\ &= \sin c(1) + \cos c(0) \\ &= \sin c\end{aligned}$$

$$\therefore \lim_{x \rightarrow c} h(x) = h(c)$$

Therefore, $h(x) = \sin x$ is a continuous function.

It is known that for real valued functions g and h , such that (goh) is defined at c , if g is continuous at c and if f is continuous at $g(c)$, then (fog) is continuous at c .

Therefore, $f(x) = (goh)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Question 34:

Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$.

Solution:

The given function is $f(x) = |x| - |x+1|$.

The two functions, g and h are defined as $g(x) = |x|$ and $h(x) = |x+1|$.

Then, $f = g - h$

The continuity of g and h are examined first.

$g(x) = |x|$ can be written as $g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$

It is evident that g is defined for every real number.

Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} (-x) = -c$$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, g is continuous at all points x , such that $x < 0$.

Case II:

If $c > 0$, then $g(c) = c$

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} (x) = c$$

$$\therefore \lim_{x \rightarrow c} g(x) = g(c)$$

Therefore, g is continuous at all points x , such that $x > 0$.

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

$$\lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (x) = 0$$

$$\therefore \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} (x) = g(0)$$

Therefore, g is continuous at all $x = 0$.

From the above three observations, it can be concluded that g is continuous at all points.

$$h(x) = |x+1| \text{ can be written as } h(x) = \begin{cases} -(x+1), & \text{if } x < -1 \\ x+1, & \text{if } x \geq -1 \end{cases}$$

It is evident that h is defined for every real number.

Let c be a real number.

Case I:

If $c < -1$, then $h(c) = -(c+1)$

$$\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} [-(x+1)] = -(c+1)$$

$$\therefore \lim_{x \rightarrow c} h(x) = h(c)$$

Therefore, h is continuous at all points x , such that $x < -1$.

Case II:

If $c > -1$, then $h(c) = c+1$

$$\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} (x+1) = c+1$$

$$\therefore \lim_{x \rightarrow c} h(x) = h(c)$$

Therefore, h is continuous at all points x , such that $x > -1$.

Case III:

If $c = -1$, then $h(c) = h(-1) = -1+1 = 0$

$$\lim_{x \rightarrow -1^-} h(x) = \lim_{x \rightarrow -1^-} [-(x+1)] = -(-1+1) = 0$$

$$\lim_{x \rightarrow -1^+} h(x) = \lim_{x \rightarrow -1^+} (x+1) = (-1+1) = 0$$

$$\therefore \lim_{x \rightarrow -1} h(x) = \lim_{x \rightarrow -1} h(x) = h(-1)$$

Therefore, h is continuous at $x = -1$.

From the above three observations, it can be concluded that h is continuous at all points.

It concludes that g and h are continuous functions. Therefore, $f = g - h$ is also a continuous function.

Therefore, f has no point of discontinuity.

EXERCISE 5.2

Question 1:

Differentiate the function with respect to x .

$$\sin(x^2 + 5)$$

Solution:

Let $f(x) = \sin(x^2 + 5)$, $u(x) = x^2 + 5$ and $v(t) = \sin t$

Then, $(v \circ u)(x) = v(u(x)) = v(x^2 + 5) = \sin(x^2 + 5) = f(x)$

Thus, f is a composite of two functions.

Put $t = u(x) = x^2 + 5$

Then, we get

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(x^2 + 5)$$

$$\frac{dt}{dx} = \frac{d}{dx}(x^2 + 5) = \frac{d}{dx}(x^2) + \frac{d}{dx}(5) = 2x + 0 = 2x$$

By chain rule of derivative,

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(x^2 + 5) \times 2x = 2x \cos(x^2 + 5)$$

Alternate method:

$$\begin{aligned} \frac{d}{dx}[\sin(x^2 + 5)] &= \cos(x^2 + 5) \cdot \frac{d}{dx}(x^2 + 5) \\ &= \cos(x^2 + 5) \cdot \left[\frac{d}{dx}(x^2) + \frac{d}{dx}(5) \right] \\ &= \cos(x^2 + 5) \cdot [2x + 0] \\ &= 2x \cos(x^2 + 5) \end{aligned}$$

Question 2:

Differentiate the function with respect to x

$$\cos(\sin x)$$

Solution:

Let $f(x) = \cos(\sin x)$, $u(x) = \sin x$ and $v(t) = \cos t$

Then, $(v \circ u)(x) = v(u(x)) = v(\sin x) = \cos(\sin x) = f(x)$

Here, f is a composite function of two functions.

Put $t = u(x) = \sin x$

$$\therefore \frac{dv}{dt} = \frac{d}{dt}[\cos t] = -\sin t = -\sin(\sin x)$$

$$\frac{dt}{dx} = \frac{d}{dx}(\sin x) = \cos x$$

By chain rule,

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$$

Alternate method:

$$\begin{aligned} \frac{d}{dx}[\cos(\sin x)] &= -\sin(\sin x) \cdot \frac{d}{dx}(\sin x) \\ &= -\sin(\sin x) \times \cos x \\ &= -\cos x \sin(\sin x) \end{aligned}$$

Question 3:

Differentiate the function with respect to x

$$\sin(ax + b)$$

Solution:

Let $f(x) = \sin(ax + b)$, $u(x) = ax + b$ and $v(t) = \sin t$

Then, $(v \circ u)(x) = v(u(x)) = v(ax + b) = \sin(ax + b) = f(x)$

Here, f is a composite function of two functions u and v .

Put, $t = u(x) = ax + b$

Thus,

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax + b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax + b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Hence, by chain rule, we get

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax + b) \cdot a = a \cos(ax + b)$$

Alternate method:

$$\begin{aligned}
\frac{d}{dx}[\sin(ax+b)] &= \cos(ax+b) \cdot \frac{d}{dx}(ax+b) \\
&= \cos(ax+b) \cdot \left[\frac{d}{dx}(ax) + \frac{d}{dx}(b) \right] \\
&= \cos(ax+b) \cdot (a+0) \\
&= a \cos(ax+b)
\end{aligned}$$

Question 4:

Differentiate the function with respect to x

$$\sec(\tan(\sqrt{x}))$$

Solution:

Let $f(x) = \sec(\tan(\sqrt{x}))$, $u(x) = \sqrt{x}$, $v(t) = \tan t$ and $w(s) = \sec s$

Then, $(w \circ v \circ u)(x) = w[v(u(x))] = w[v(\sqrt{x})] = w(\tan \sqrt{x}) = \sec(\tan \sqrt{x}) = f(x)$

Here, f is a composite function of three functions u , v and w .

Put, $s = v(t) = \tan t$ and $t = u(x) = \sqrt{x}$

Then,

$$\begin{aligned}
\frac{dw}{ds} &= \frac{d}{ds}(\sec s) \\
&= \sec s \tan s \\
&= \sec(\tan t) \cdot \tan(\tan t) \quad [s = \tan t] \\
&= \sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \quad [t = \sqrt{x}]
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{ds}{dt} &= \frac{d}{dt}(\tan t) = \sec^2 t = \sec^2 \sqrt{x} \\
\frac{dt}{dx} &= \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}
\end{aligned}$$

Hence, by chain rule, we get

$$\begin{aligned}
\frac{d}{dx} \left[\sec(\tan \sqrt{x}) \right] &= \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx} \\
&= \sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \cdot \sec^2 \sqrt{x} \cdot \frac{1}{2\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}} \sec^2 \sqrt{x} \sec(\tan \sqrt{x}) \tan(\tan \sqrt{x}) \\
&= \frac{\sec^2 \sqrt{x} \sec(\tan \sqrt{x}) \tan(\tan \sqrt{x})}{2\sqrt{x}}
\end{aligned}$$

Alternate method:

$$\begin{aligned}
\frac{d}{dx} \left[\sec(\tan \sqrt{x}) \right] &= \sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \cdot \frac{d}{dx} (\tan \sqrt{x}) \\
&= \sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{d}{dx} (\sqrt{x}) \\
&= \sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \cdot \sec^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \\
&= \frac{\sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \cdot \sec^2(\sqrt{x})}{2\sqrt{x}}
\end{aligned}$$

Question 5:

Differentiate the function with respect to x

$$\frac{\sin(ax+b)}{\cos(cx+d)}$$

Solution:

Given, $f(x) = \frac{\sin(ax+b)}{\cos(cx+d)}$, where $g(x) = \sin(ax+b)$ and $h(x) = \cos(cx+d)$

$$\therefore f = \frac{g'h - gh'}{h^2}$$

Consider $g(x) = \sin(ax+b)$

Let $u(x) = ax+b$, $v(t) = \sin t$

Then $(v \circ u)(x) = v(u(x)) = v(ax+b) = \sin(ax+b) = g(x)$

$\therefore g$ is a composite function of two functions, u and v .

Put, $t = u(x) = ax+b$

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax + b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax + b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

Thus, by chain rule, we get

$$g' = \frac{dg}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax + b) \cdot a = a \cos(ax + b)$$

Consider $h(x) = \cos(cx + d)$

Let $p(x) = cx + d$, $q(y) = \cos y$

Then, $(q \circ p)(x) = q(p(x)) = q(cx + d) = \cos(cx + d) = h(x)$

$\therefore h$ is a composite function of two functions, p and q .

Put, $y = p(x) = cx + d$

$$\frac{dq}{dy} = \frac{d}{dy}(\cos y) = -\sin y = -\sin(cx + d)$$

$$\frac{dy}{dx} = \frac{d}{dx}(cx + d) = \frac{d}{dx}(cx) + \frac{d}{dx}(d) = c$$

Using chain rule, we get

$$\begin{aligned} h' &= \frac{dh}{dx} = \frac{dq}{dy} \cdot \frac{dy}{dx} \\ &= -\sin(cx + d) \times c \\ &= -c \sin(cx + d) \end{aligned}$$

Therefore,

$$\begin{aligned} f' &= \frac{a \cos(ax + b) \cdot \cos(cx + d) - \sin(ax + b) \{-c \sin(cx + d)\}}{[\cos(cx + d)]^2} \\ &= \frac{a \cos(ax + b)}{\cos(cx + d)} + c \sin(ax + b) \cdot \frac{\sin(cx + d)}{\cos(cx + d)} \times \frac{1}{\cos(cx + d)} \\ &= a \cos(ax + b) \sec(cx + d) + c \sin(ax + b) \tan(cx + d) \sec(cx + d) \end{aligned}$$

Question 6:

Differentiate the function with respect to x

$$\cos x^3 \cdot \sin^2(x^5)$$

Solution:

Given, $\cos x^3 \cdot \sin^2(x^5)$

$$\begin{aligned}
\frac{d}{dx} [\cos x^3 \cdot \sin^2(x^5)] &= \sin^2(x^5) \times \frac{d}{dx}(\cos x^3) + \cos x^3 \times \frac{d}{dx}[\sin^2(x^5)] \\
&= \sin^2(x^5) \times (-\sin x^3) \times \frac{d}{dx}(x^3) + \cos x^3 \times 2 \sin(x^5) \cdot \frac{d}{dx}[\sin x^5] \\
&= -\sin x^3 \sin^2(x^5) \times 3x^2 + 2 \sin x^5 \cos x^3 \cdot \cos x^5 \times \frac{d}{dx}(x^5) \\
&= -3x^2 \sin x^3 \cdot \sin^2(x^5) + 2 \sin x^5 \cos x^5 \cos x^3 \times 5x^4 \\
&= 10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2(x^5)
\end{aligned}$$

Question 7:

Differentiate the function with respect to x

$$2\sqrt{\cot(x^2)}$$

Solution:

$$\begin{aligned}
\frac{d}{dx} [2\sqrt{\cot(x^2)}] &= 2 \cdot \frac{1}{2\sqrt{\cot(x^2)}} \times \frac{d}{dx} [\cot(x^2)] \\
&= \frac{\frac{\sin(x^2)}{\cos(x^2)}}{\sqrt{\frac{\sin(x^2)}{\cos(x^2)}}} \times -\operatorname{cosec}^2(x^2) \times \frac{d}{dx}(x^2) \\
&= \frac{\frac{\sin(x^2)}{\cos(x^2)}}{\sqrt{\frac{\sin(x^2)}{\cos(x^2)}}} \times \frac{-1}{\sin^2(x^2)} \times (2x) \\
&= \frac{-2x}{\sin x^2 \sqrt{\cos x^2 \sin x^2}} \\
&= \frac{-2\sqrt{2}x}{\sin x^2 \sqrt{2 \sin x^2 \cos x^2}} \\
&= \frac{-2\sqrt{2}x}{\sin x^2 \sqrt{\sin 2x^2}}
\end{aligned}$$

Question 8:

Differentiate the function with respect to x

$$\cos(\sqrt{x})$$

Solution:

$$\text{Let } f(x) = \cos(\sqrt{x})$$

$$\text{Also, let } u(x) = \sqrt{x} \text{ and, } v(t) = \cos t$$

Then,

$$\begin{aligned}(v \circ u)(x) &= v(u(x)) \\ &= v(\sqrt{x}) \\ &= \cos \sqrt{x} \\ &= f(x)\end{aligned}$$

Since, f is a composite function of u and v .

$$t = u(x) = \sqrt{x}$$

Then,

$$\begin{aligned}\frac{dt}{dx} &= \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{-1}{2}} \\ &= \frac{1}{2\sqrt{x}} \\ \frac{dv}{dt} &= \frac{d}{dt}(\cos t) = -\sin t\end{aligned}$$

And, $= -\sin(\sqrt{x})$

Using chain rule, we get

$$\begin{aligned}\frac{df}{dx} &= \frac{dv}{dt} \cdot \frac{dt}{dx} \\ &= -\sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \\ &= -\frac{1}{2\sqrt{x}} \sin(\sqrt{x}) \\ &= -\frac{\sin(\sqrt{x})}{2\sqrt{x}}\end{aligned}$$

Alternate method:

$$\begin{aligned}\frac{d}{dx}[\cos(\sqrt{x})] &= -\sin(\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x}) \\ &= -\sin(\sqrt{x}) \times \frac{d}{dx}\left(x^{\frac{1}{2}}\right) \\ &= -\sin \sqrt{x} \times \frac{1}{2}x^{\frac{-1}{2}} \\ &= \frac{-\sin \sqrt{x}}{2\sqrt{x}}\end{aligned}$$

Question 9:

Prove that the function f given by

$f(x) = |x-1|, x \in \mathbf{R}$ is not differentiable at $x=1$.

Solution:

Given, $f(x) = |x-1|, x \in \mathbf{R}$

It is known that a function f is differentiable at a point $x=c$ in its domain if both

$\lim_{h \rightarrow 0^-} \frac{f(c) - f(c-h)}{h}$ and $\lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h}$ are finite and equal.

To check the differentiability of the given function at $x=1$,

Consider LHD at $x=1$

$$\begin{aligned}\lim_{h \rightarrow 0^-} \frac{f(1) - f(1-h)}{h} &= \lim_{h \rightarrow 0^-} \frac{f|1-1| - |1-h-1|}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{0 - |h|}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{-h}{h} \quad (h < 0 \Rightarrow |h| = -h) \\ &= -1\end{aligned}$$

Consider RHD at $x=1$

$$\begin{aligned}\lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} &= \lim_{h \rightarrow 0^+} \frac{f|1+h-1| - |1-1|}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{|h| - 0}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{h}{h} \quad (h > 0 \Rightarrow |h| = h) \\ &= 1\end{aligned}$$

Since LHD and RHD at $x=1$ are not equal,

Therefore, f is not differentiable at $x=1$.

Question 10:

Prove that the greatest integer function defined by $f(x) = [x], 0 < x < 3$ is not differentiable at $x=1$ and $x=2$.

Solution:

Given, $f(x) = [x], 0 < x < 3$

It is known that a function f is differentiable at a point $x = c$ in its domain if both

$\lim_{h \rightarrow 0^-} \frac{f(c) - f(c-h)}{h}$ and $\lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h}$ are finite and equal.

At $x = 1$,

Consider the LHD at $x = 1$

$$\begin{aligned}\lim_{h \rightarrow 0^-} \frac{f(1) - f(1-h)}{h} &= \lim_{h \rightarrow 0^-} \frac{[1] - [1-h]}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{1-0}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{1}{h} \\ &= \infty\end{aligned}$$

Consider RHD at $x = 1$

$$\begin{aligned}\lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} &= \lim_{h \rightarrow 0^+} \frac{[1+h] - [1]}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{1-1}{h} \\ &= \lim_{h \rightarrow 0^+} 0 \\ &= 0\end{aligned}$$

Since LHD and RHD at $x = 1$ are not equal,

Hence, f is not differentiable at $x = 1$.

To check the differentiability of the given function at $x = 2$,

Consider LHD at $x = 2$

$$\begin{aligned}\lim_{h \rightarrow 0^-} \frac{f(2) - f(2-h)}{h} &= \lim_{h \rightarrow 0^-} \frac{[2] - [2-h]}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{2-1}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{1}{h} \\ &= \infty\end{aligned}$$

Now, consider RHD at $x = 2$

$$\begin{aligned}
 \lim_{h \rightarrow 0^+} \frac{f(2+h) - f(2)}{h} &= \lim_{h \rightarrow 0^+} \frac{[2+h] - [2]}{h} \\
 &= \lim_{h \rightarrow 0^+} \frac{2-2}{h} \\
 &= \lim_{h \rightarrow 0^+} 0 \\
 &= 0
 \end{aligned}$$

Since, LHD and RHD at $x = 2$ are not equal.

Hence, f is not differentiable at $x = 2$.

EXERCISE 5.3

Question 1:

Find $\frac{dy}{dx}$: $2x + 3y = \sin x$

Solution:

Given, $2x + 3y = \sin x$

Differentiating with respect to x , we get

$$\frac{d}{dx}(2x + 3y) = \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \cos x$$

$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos x$$

$$\Rightarrow 3\frac{dy}{dx} = \cos x - 2$$

$$\therefore \frac{dy}{dx} = \frac{\cos x - 2}{3}$$

Question 2:

Find $\frac{dy}{dx}$: $2x + 3y = \sin y$

Solution:

Given, $2x + 3y = \sin y$

Differentiating with respect to x , we get

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(\sin y)$$

$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos y \frac{dy}{dx} \quad [\text{By using chain rule}]$$

$$\Rightarrow 2 = (\cos y - 3)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

Question 3:

Find $\frac{dy}{dx}$: $ax + by^2 = \cos y$

Solution:

Given, $ax + by^2 = \cos y$

Differentiating with respect to x , we get

$$\begin{aligned}\frac{d}{dx}(ax) + \frac{d}{dx}(by^2) &= \frac{d}{dx}(\cos y) \\ \Rightarrow a + b \frac{d}{dx}(y^2) &= \frac{d}{dx}(\cos y) \quad \dots(1)\end{aligned}$$

$$\frac{d}{dx}(y^2) = 2y \frac{dy}{dx} \text{ and } \frac{d}{dx}(\cos y) = -\sin y \frac{dy}{dx} \quad \dots(2)$$

From (1) and (2), we obtain

$$\begin{aligned}a + b \times 2y \frac{dy}{dx} &= -\sin y \frac{dy}{dx} \\ \Rightarrow (2by + \sin y) \frac{dy}{dx} &= -a \\ \therefore \frac{dy}{dx} &= \frac{-a}{2by + \sin y}\end{aligned}$$

Question 4:

Find $\frac{dy}{dx}$: $xy + y^2 = \tan x + y$

Solution:

Given, $xy + y^2 = \tan x + y$

Differentiating with respect to x , we get

$$\begin{aligned}\frac{d}{dx}(xy + y^2) &= \frac{d}{dx}(\tan x + y) \\ \Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(\tan x) + \frac{dy}{dx} \\ \Rightarrow \left[y \cdot \frac{d}{dx}(x) + x \cdot \frac{dy}{dx} \right] + 2y \frac{dy}{dx} &= \sec^2 x + \frac{dy}{dx} \quad \text{[using product rule and chain rule]} \\ \Rightarrow y \cdot 1 + x \frac{dy}{dx} + 2y \frac{dy}{dx} &= \sec^2 x + \frac{dy}{dx} \Rightarrow (x + 2y - 1) \frac{dy}{dx} = \sec^2 x - y \\ \therefore \frac{dy}{dx} &= \frac{\sec^2 x - y}{(x + 2y - 1)}\end{aligned}$$

Question 5:

Find $\frac{dy}{dx}$: $x^2 + xy + y^2 = 100$

Solution:

Given, $x^2 + xy + y^2 = 100$

Differentiating with respect to x , we get

$$\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(100)$$

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$

$$\Rightarrow 2x + \left[y \cdot \frac{d}{dx}(x) + x \cdot \frac{dy}{dx} \right] + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow 2x + y \cdot 1 + x \cdot \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow 2x + y + (x + 2y) \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

Question 6:

Find $\frac{dy}{dx}$: $x^3 + x^2y + xy^2 + y^3 = 81$

Solution:

Given, $x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating with respect to x , we get

$$\frac{d}{dx}(x^3 + x^2y + xy^2 + y^3) = \frac{d}{dx}(81)$$

$$\Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) + \frac{d}{dx}(y^3) = 0$$

$$\Rightarrow 3x^2 + \left[y \frac{d}{dx}(x^2) + x^2 \frac{dy}{dx} \right] + \left[y^2 \frac{d}{dx}(x) + x \frac{d}{dx}(y^2) \right] + 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + \left[y \cdot 2x + x^2 \frac{dy}{dx} \right] + \left[y^2 \cdot 1 + x \cdot 2y \cdot \frac{dy}{dx} \right] + 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow (x^2 + 2xy + 3y^2) \frac{dy}{dx} + (3x^2 + 2xy + y^2) = 0$$

$$\therefore \frac{dy}{dx} = \frac{-(3x^2 + 2xy + y^2)}{(x^2 + 2xy + 3y^2)}$$

Question 7:

Find $\frac{dx}{dy}$: $\sin^2 y + \cos xy = \pi$

Solution:

Given, $\sin^2 y + \cos xy = \pi$

Differentiating with respect to x , we get

$$\begin{aligned}\frac{d}{dx}(\sin^2 y + \cos xy) &= \frac{d}{dx}(\pi) \\ \Rightarrow \frac{d}{dx}(\sin^2 y) + \frac{d}{dx}(\cos xy) &= 0\end{aligned}\quad \dots(1)$$

Using chain rule, we obtain

$$\frac{d}{dx}(\sin^2 y) = 2 \sin y \frac{d}{dx}(\sin y) = 2 \sin y \cos y \frac{dy}{dx} \quad \dots(2)$$

$$\begin{aligned}\frac{d}{dx}(\cos xy) &= -\sin xy \frac{d}{dx}(xy) = -\sin xy \left[y \frac{d}{dx}(x) + x \frac{dy}{dx} \right] \\ &= -\sin xy \left[y \cdot 1 + x \frac{dy}{dx} \right] = -y \sin xy - x \sin xy \frac{dy}{dx}\end{aligned}\quad \dots(3)$$

From (1), (2) and (3), we obtain

$$2 \sin y \cos y \frac{dy}{dx} + \left(-y \sin xy - x \sin xy \frac{dy}{dx} \right) = 0$$

$$\Rightarrow (2 \sin y \cos y - x \sin xy) \frac{dy}{dx} = y \sin xy$$

$$\Rightarrow (\sin 2y - x \sin xy) \frac{dx}{dy} = y \sin xy$$

$$\therefore \frac{dx}{dy} = \frac{y \sin xy}{\sin 2y - x \sin xy}$$

Question 8:

Find $\frac{dy}{dx}$: $\sin^2 x + \cos^2 y = 1$

Solution:

Given, $\sin^2 x + \cos^2 y = 1$

Differentiating with respect to x , we get

$$\begin{aligned}
\frac{d}{dx}(\sin^2 x + \cos^2 y) &= \frac{d}{dx}(1) \\
\Rightarrow \frac{d}{dx}(\sin^2 x) + \frac{d}{dx}(\cos^2 y) &= 0 \\
\Rightarrow 2 \sin x \cdot \frac{d}{dx}(\sin x) + 2 \cos y \cdot \frac{d}{dx}(\cos y) &= 0 \\
\Rightarrow 2 \sin x \cos x + 2 \cos y (-\sin y) \cdot \frac{dy}{dx} &= 0 \\
\Rightarrow \sin 2x - \sin 2y \frac{dy}{dx} &= 0 \\
\therefore \frac{dy}{dx} &= \frac{\sin 2x}{\sin 2y}
\end{aligned}$$

Question 9:

Find $\frac{dy}{dx}$: $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution:

Given,

$$\begin{aligned}
y &= \sin^{-1}\left(\frac{2x}{1+x^2}\right) \\
\Rightarrow \sin y &= \frac{2x}{1+x^2}
\end{aligned}$$

Differentiating with respect to x , we get

$$\begin{aligned}
\frac{d}{dx}(\sin y) &= \frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \quad \dots(1) \\
\Rightarrow \cos y \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)
\end{aligned}$$

The function $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$

By quotient rule, we get

$$\begin{aligned}
 \frac{d}{dx} \left(\frac{2x}{1+x^2} \right) &= \frac{(1+x^2) \frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}(1+x^2)}{(1+x^2)^2} \\
 &= \frac{(1+x^2) \cdot 2 - 2x \cdot [0+2x]}{(1+x^2)^2} \\
 &= \frac{2+2x^2-4x^2}{(1+x^2)^2} \\
 &= \frac{2(1-x^2)}{(1+x^2)^2}
 \end{aligned}$$

Also, $\sin y = \frac{2x}{1+x^2}$

$$\begin{aligned}
 \cos y &= \sqrt{1 - \sin^2 y} = \sqrt{1 - \left(\frac{2x}{1+x^2} \right)^2} \\
 &= \sqrt{\frac{(1+x^2)^2 - 4x^2}{(1+x^2)^2}} \\
 &= \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}} \\
 &= \frac{1-x^2}{1+x^2}
 \end{aligned}$$

From (1), (2) and (3), we get

$$\begin{aligned}
 \frac{1-x^2}{1+x^2} \times \frac{dy}{dx} &= \frac{2(1-x^2)}{(1+x^2)^2} \\
 \Rightarrow \frac{dy}{dx} &= \frac{2}{1+x^2}
 \end{aligned}$$

Question 10:

Find $\frac{dy}{dx}$: $y = \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right)$, $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Solution:

Given, $y = \tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2} \right)$

$$\Rightarrow \tan y = \left(\frac{3x - x^3}{1 - 3x^2} \right) \quad \dots(1)$$

Since, we know that

$$\Rightarrow \tan y = \left(\frac{3 \tan \frac{y}{3} - \tan^3 \frac{y}{3}}{1 - 3 \tan^2 \frac{y}{3}} \right) \quad (2)$$

Comparing (1) and (2) we get,

$$x = \tan \frac{y}{3}$$

Differentiating with respect to x , we get

$$\frac{d}{dx}(x) = \frac{d}{dx} \left(\tan \frac{y}{3} \right)$$

$$\Rightarrow 1 = \sec^2 \frac{y}{3} \cdot \frac{d}{dx} \left(\frac{y}{3} \right)$$

$$\Rightarrow 1 = \sec^2 \frac{y}{3} \cdot \frac{1}{3} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{\sec^2 \frac{y}{3}} = \frac{3}{1 + \tan^2 \frac{y}{3}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{1 + x^2}$$

Question 11:

Find $\frac{dy}{dx}$: $y = \cos^{-1} \left(\frac{1 - x^2}{1 + x^2} \right)$, $0 < x < 1$

Solution:

Given, $y = \cos^{-1} \left(\frac{1 - x^2}{1 + x^2} \right)$

$$\Rightarrow \cos y = \left(\frac{1-x^2}{1+x^2} \right)$$

$$\Rightarrow \frac{1 - \tan^2 \frac{y}{2}}{1 + \tan^2 \frac{y}{2}} = \frac{1-x^2}{1+x^2}$$

Comparing LHS and RHS, we get

$$\tan \frac{y}{2} = x$$

Differentiating with respect to x , we get

$$\sec^2 \frac{y}{2} \cdot \frac{d}{dx} \left(\frac{y}{2} \right) = \frac{d}{dx} (x)$$

$$\Rightarrow \sec^2 \frac{y}{2} \times \frac{1}{2} \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sec^2 \frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1 + \tan^2 \frac{y}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{2}{1+x^2}$$

Question 12:

Find $\frac{dy}{dx} : y = \sin^{-1} \left(\frac{1-x^2}{1+x^2} \right), 0 < x < 1$

Solution:

Given, $y = \sin^{-1} \left(\frac{1-x^2}{1+x^2} \right)$

$$y = \sin^{-1} \left(\frac{1-x^2}{1+x^2} \right)$$

$$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$$

Differentiating with respect to x , we get

$$\frac{d}{dx} (\sin y) = \frac{d}{dx} \left(\frac{1-x^2}{1+x^2} \right) \quad \dots(1)$$

Using chain rule, we get

$$\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$$

$$\begin{aligned}\cos y &= \sqrt{1 - \sin^2 y} = \sqrt{1 - \left(\frac{1-x^2}{1+x^2}\right)^2} \\ &= \sqrt{\frac{(1+x^2)^2 - (1-x^2)^2}{(1+x^2)^2}} = \sqrt{\frac{4x^2}{(1+x^2)^2}} = \frac{2x}{1+x^2}\end{aligned}$$

Therefore,

$$\frac{d}{dx}(\sin y) = \frac{2x}{1+x^2} \frac{dy}{dx} \quad \dots(2)$$

$$\begin{aligned}\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) &= \frac{(1+x^2) \cdot \frac{d}{dx}(1-x^2) - (1-x^2) \cdot \frac{d}{dx}(1+x^2)}{(1+x^2)^2} && \text{[using quotient rule]} \\ &= \frac{(1+x^2)(-2x) - (1-x^2)(2x)}{(1+x^2)^2} \\ &= \frac{-2x - 2x^3 - 2x + 2x^3}{(1+x^2)^2} \\ &= \frac{-4x}{(1+x^2)^2} \quad \dots(3)\end{aligned}$$

From equation (1), (2) and (3), we get

$$\begin{aligned}\frac{2x}{1+x^2} \frac{dy}{dx} &= \frac{-4x}{(1+x^2)^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{-2}{1+x^2}\end{aligned}$$

Question 13:

Find $\frac{dy}{dx}$: $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$, $-1 < x < 1$

Solution:

$$\text{Given, } y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\cos y = \left(\frac{2x}{1+x^2}\right)$$

Differentiating with respect to x , we get

$$\begin{aligned}\frac{d}{dx}(\cos y) &= \frac{d}{dx}\left(\frac{2x}{1+x^2}\right) \\ \Rightarrow -\sin y \cdot \frac{dy}{dx} &= \frac{(1+x^2) \cdot \frac{d}{dx}(2x) - 2x \cdot \frac{d}{dx}(1+x^2)}{(1+x^2)^2} \\ \Rightarrow -\sqrt{1-\cos^2 y} \frac{dy}{dx} &= \frac{(1+x^2) \times 2 - 2x \times 2x}{(1+x^2)^2} \\ \Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2} \right] \frac{dy}{dx} &= -\left[\frac{2(1-x^2)}{(1+x^2)^2} \right] \\ \Rightarrow \sqrt{\frac{(1+x^2)^2 - 4x^2}{(1+x^2)^2}} \cdot \frac{dy}{dx} &= \frac{-2(1-x^2)}{(1+x^2)^2} \\ \Rightarrow \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}} \frac{dy}{dx} &= \frac{-2(1-x^2)}{(1+x^2)^2} \\ \Rightarrow \frac{1-x^2}{1+x^2} \cdot \frac{dy}{dx} &= \frac{-2(1-x^2)}{(1+x^2)^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{-2}{1+x^2}\end{aligned}$$

Question 14:

Find $\frac{dy}{dx}$: $y = \sin^{-1}(2x\sqrt{1-x^2})$, $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Solution:

$$\begin{aligned}\text{Given, } y &= \sin^{-1}(2x\sqrt{1-x^2}) \\ y &= \sin^{-1}(2x\sqrt{1-x^2}) \\ \Rightarrow \sin y &= (2x\sqrt{1-x^2})\end{aligned}$$

Differentiating with respect to x , we get

$$\begin{aligned}\cos y \cdot \frac{dy}{dx} &= 2 \left[x \frac{d}{dx} (\sqrt{1-x^2}) + \sqrt{1-x^2} \frac{dx}{dx} \right] \\ \Rightarrow \sqrt{1-\sin^2 y} \frac{dy}{dx} &= 2 \left[\frac{x}{2} \cdot \frac{-2x}{\sqrt{1-x^2}} + \sqrt{1-x^2} \right] \\ \Rightarrow \sqrt{1-\left(2x\sqrt{1-x^2}\right)^2} \cdot \frac{dy}{dx} &= 2 \left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}} \right] \\ \Rightarrow \sqrt{1-4x^2(1-x^2)} \frac{dy}{dx} &= 2 \left[\frac{1-2x^2}{\sqrt{1-x^2}} \right] \\ \Rightarrow \sqrt{(1-2x^2)^2} \frac{dy}{dx} &= 2 \left[\frac{1-2x^2}{\sqrt{1-x^2}} \right] \\ \Rightarrow (1-2x^2) \frac{dy}{dx} &= 2 \left[\frac{1-2x^2}{\sqrt{1-x^2}} \right] \\ \Rightarrow \frac{dy}{dx} &= \frac{2}{\sqrt{1-x^2}}\end{aligned}$$

Question 15:

Find $\frac{dy}{dx}$: $y = \sec^{-1} \left(\frac{1}{2x^2-1} \right), 0 < x < \frac{1}{\sqrt{2}}$

Solution:

Given, $y = \sec^{-1} \left(\frac{1}{2x^2-1} \right)$

$$\Rightarrow y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

$$\Rightarrow \sec y = \left(\frac{1}{2x^2-1}\right)$$

$$\Rightarrow \cos y = 2x^2 - 1$$

$$\Rightarrow 2x^2 = 1 + \cos y$$

$$\Rightarrow 2x^2 = 2 \cos^2 \frac{y}{2}$$

$$\Rightarrow x = \cos \frac{y}{2}$$

Differentiating with respect to x , we get

$$\frac{d}{dx}(x) = \frac{d}{dx}\left(\cos \frac{y}{2}\right)$$

$$\Rightarrow 1 = \sin \frac{y}{2} \cdot \frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow \frac{-1}{\sin \frac{y}{2}} = \frac{1}{2} \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin \frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - \cos^2 \frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - x^2}}$$

EXERCISE 5.4

Question 1:

Differentiating the following wrt x : $\frac{e^x}{\sin x}$

Solution:

Let $y = \frac{e^x}{\sin x}$

By using the quotient rule, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sin x \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(\sin x)}{\sin^2 x} \\ &= \frac{\sin x \cdot (e^x) - e^x \cdot (\cos x)}{\sin^2 x} \\ &= \frac{e^x (\sin x - \cos x)}{\sin^2 x}\end{aligned}$$

Question 2:

Differentiating the following $e^{\sin^{-1} x}$

Solution:

Let $y = e^{\sin^{-1} x}$

By using the quotient rule, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(e^{\sin^{-1} x}) \\ &= e^{\sin^{-1} x} \cdot \frac{d}{dx}(\sin^{-1} x) \\ &= e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} \\ &= \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} \\ &= \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}, x \in (-1, 1)\end{aligned}$$

Question 3:

Differentiating the following wrt x : e^{x^3}

Solution:

Let $y = e^{x^3}$

By using the quotient rule, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(e^{x^3}) \\ &= e^{x^3} \cdot \frac{d}{dx}(x^3) \\ &= e^{x^3} \cdot 3x^2 \\ &= 3x^2 e^{x^3}\end{aligned}$$

Question 4:

Differentiate the following wrt $x : \sin(\tan^{-1} e^{-x})$

Solution:

Let $y = \sin(\tan^{-1} e^{-x})$

By using the chain rule, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[\sin(\tan^{-1} e^{-x})] \\ &= \cos(\tan^{-1} e^{-x}) \cdot \frac{d}{dx}(\tan^{-1} e^{-x}) \\ &= \cos(\tan^{-1} e^{-x}) \cdot \frac{1}{1+(e^{-x})^2} \cdot \frac{d}{dx}(e^{-x}) \\ &= \frac{\cos(\tan^{-1} e^{-x})}{1+e^{-2x}} \cdot e^{-x} \cdot \frac{d}{dx}(-x) \\ &= \frac{e^{-x} \cos(\tan^{-1} e^{-x})}{1+e^{-2x}} \times (-1) \\ &= \frac{-e^{-x} \cos(\tan^{-1} e^{-x})}{1+e^{-2x}}\end{aligned}$$

Question 5:

Differentiate the following wrt $x : \log(\cos e^x)$

Solution:

Let $y = \log(\cos e^x)$

By using the chain rule, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} [\log(\cos e^x)] \\
&= \frac{1}{\cos e^x} \cdot \frac{d}{dx} (\cos e^x) \\
&= \frac{1}{\cos e^x} \cdot (-\sin e^x) \cdot \frac{d}{dx} (e^x) \\
&= \frac{-\sin e^x}{\cos e^x} \cdot e^x \\
&= -e^x \tan e^x, e^x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{N}
\end{aligned}$$

Question 6:

Differentiate the following wrt x : $e^x + e^{x^2} + \dots + e^{x^5}$

Solution:

$$\frac{d}{dx} (e^x + e^{x^2} + \dots + e^{x^5})$$

Differentiating wrt x , we get

$$\begin{aligned}
\frac{d}{dx} (e^x + e^{x^2} + \dots + e^{x^5}) &= \frac{d}{dx} (e^x) + \frac{d}{dx} (e^{x^2}) + \frac{d}{dx} (e^{x^3}) + \frac{d}{dx} (e^{x^4}) + \frac{d}{dx} (e^{x^5}) \\
&= e^x + \left[e^{x^2} \times \frac{d}{dx} (x^2) \right] + \left[e^{x^3} \times \frac{d}{dx} (x^3) \right] + \left[e^{x^4} \times \frac{d}{dx} (x^4) \right] + \left[e^{x^5} \times \frac{d}{dx} (x^5) \right] \\
&= e^x + (e^{x^2} \times 2x) + (e^{x^3} \times 3x^2) + (e^{x^4} \times 4x^3) + (e^{x^5} \times 5x^4) \\
&= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}
\end{aligned}$$

Question 7:

Differentiating the following wrt x : $\sqrt{e^{\sqrt{x}}}, x > 0$

Solution:

$$\text{Let } y = \sqrt{e^{\sqrt{x}}}$$

$$\text{Then, } y^2 = e^{\sqrt{x}}$$

Differentiating wrt x , we get

$$y^2 = e^{\sqrt{x}}$$

$$\begin{aligned}
\frac{d}{dx}(y^2) &= \frac{d}{dx}(e^{\sqrt{x}}) \\
\Rightarrow 2y \frac{dy}{dx} &= e^{\sqrt{x}} \frac{d}{dx}(\sqrt{x}) \\
\Rightarrow 2y \frac{dy}{dx} &= e^{\sqrt{x}} \frac{1}{2} \cdot \frac{1}{\sqrt{x}} \\
\Rightarrow \frac{dy}{dx} &= \frac{e^{\sqrt{x}}}{4y\sqrt{x}} \\
\Rightarrow \frac{dy}{dx} &= \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}} \\
\Rightarrow \frac{dy}{dx} &= \frac{e^{\sqrt{x}}}{4\sqrt{x}e^{\sqrt{x}}}, x > 0
\end{aligned}$$

Question 8:

Differentiating the following wrt $x: \log(\log x), x > 1$

Solution:

Let $y = \log(\log x)$

By using the chain rule, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}[\log(\log x)] \\
&= \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) \\
&= \frac{1}{\log x} \cdot \frac{1}{x} \\
&= \frac{1}{x \log x}, x > 1
\end{aligned}$$

Question 9:

Differentiating the following wrt $x: \frac{\cos x}{\log x}, x > 0$

Solution:

Let $y = \frac{\cos x}{\log x}$

By using the quotient rule, we get

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{\frac{d}{dx}(\cos x) \cdot \log x - \cos x \cdot \frac{d}{dx}(\log x)}{(\log x)^2} \\
 &= \frac{-\sin x \log x - \cos x \cdot \frac{1}{x}}{(\log x)^2} \\
 &= -\left[\frac{x \log x \cdot \sin x + \cos x}{x(\log x)^2} \right], x > 0
 \end{aligned}$$

Question 10:

Differentiate the following wrt x : $\cos(\log x + e^x), x > 0$

Solution:

Let $y = \cos(\log x + e^x)$

By using the chain rule, we get

$$\begin{aligned}
 \frac{dy}{dx} &= -\sin[\log x + e^x] \cdot \frac{d}{dx}(\log x + e^x) \\
 &= -\sin(\log x + e^x) \cdot \left[\frac{d}{dx}(\log x) + \frac{d}{dx}(e^x) \right] \\
 &= -\sin(\log x + e^x) \cdot \left(\frac{1}{x} + e^x \right) \\
 &= -\left(\frac{1}{x} + e^x \right) \sin(\log x + e^x), x > 0
 \end{aligned}$$

EXERCISE 5.5

Question 1:

Differentiate the function with respect to x : $\cos x \cdot \cos 2x \cdot \cos 3x$

Solution:

Let $y = \cos x \cdot \cos 2x \cdot \cos 3x$

Taking logarithm on both the sides, we obtain

$$\log y = \log(\cos x \cdot \cos 2x \cdot \cos 3x)$$

$$\Rightarrow \log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) + \frac{1}{\cos 2x} \cdot \frac{d}{dx}(\cos 2x) + \frac{1}{\cos 3x} \cdot \frac{d}{dx}(\cos 3x)$$

$$\Rightarrow \frac{dy}{dx} = y \left[-\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x} \cdot \frac{d}{dx}(2x) - \frac{\sin 3x}{\cos 3x} \cdot \frac{d}{dx}(3x) \right]$$

$$\therefore \frac{dy}{dx} = -\cos x \cdot \cos 2x \cdot \cos 3x [\tan x + 2 \tan 2x + 3 \tan 3x]$$

Question 2:

Differentiate the function with respect to x : $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Solution:

$$\text{Let } y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Taking logarithm on both the sides, we obtain

$$\log y = \log \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$\Rightarrow \log y = \frac{1}{2} \log \left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)} \right]$$

$$\Rightarrow \log y = \frac{1}{2} [\log \{(x-1)(x-2)\} - \log \{(x-3)(x-4)(x-5)\}]$$

$$\Rightarrow \log y = \frac{1}{2} [\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)]$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{1}{2} \left[\frac{1}{x-1} \cdot \frac{d}{dx}(x-1) + \frac{1}{x-2} \cdot \frac{d}{dx}(x-2) - \frac{1}{x-3} \cdot \frac{d}{dx}(x-3) \right. \\ &\quad \left. - \frac{1}{x-4} \cdot \frac{d}{dx}(x-4) - \frac{1}{x-5} \cdot \frac{d}{dx}(x-5) \right] \\ \Rightarrow \frac{dy}{dx} &= \frac{y}{2} \left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right] \\ \therefore \frac{dy}{dx} &= \frac{1}{2} \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5} \right]\end{aligned}$$

Question 3:

Differentiate the function with respect to x : $(\log x)^{\cos x}$

Solution:

Let $y = (\log x)^{\cos x}$

Taking logarithm on both the sides, we obtain

$$\log y = \cos x \cdot \log(\log x)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\frac{1}{y} \cdot \frac{dy}{dx} &= \frac{d}{dx}(\cos x) \cdot \log(\log x) + \cos x \cdot \frac{d}{dx}[\log(\log x)] \\ \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= -\sin x \log(\log x) + \cos x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) \\ \Rightarrow \frac{dy}{dx} &= y \left[-\sin x \log(\log x) + \frac{\cos x}{\log x} \cdot \frac{1}{x} \right] \\ \therefore \frac{dy}{dx} &= (\log x)^{\cos x} \left[\frac{\cos x}{x \log x} - \sin x \log(\log x) \right]\end{aligned}$$

Question 4:

Differentiate the function with respect to x : $x^x - 2^{\sin x}$

Solution:

$$\text{Let } y = x^x - 2^{\sin x}$$

$$\text{Also, let } x^x = u \text{ and } 2^{\sin x} = v$$

$$\therefore y = u - v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

$$u = x^x$$

Taking logarithm on both the sides, we obtain

$$\log u = x \log x$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{u} \cdot \frac{du}{dx} = \left[\frac{d}{dx}(x) \times \log x + x \times \frac{d}{dx}(\log x) \right]$$

$$\Rightarrow \frac{du}{dx} = u \left[1 \times \log x + x \times \frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^x (\log x + 1)$$

$$\Rightarrow \frac{du}{dx} = x^x (1 + \log x)$$

$$v = 2^{\sin x}$$

Taking logarithm on both the sides, we obtain

$$\log v = \sin x \cdot \log 2$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log 2 \cdot \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$

$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2$$

$$\therefore \frac{dy}{dx} = x^x (1 + \log x) - 2^{\sin x} \cos x \log 2$$

Question 5:

Differentiate the function with respect to x : $(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$

Solution:

$$\text{Let } y = (x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$$

Taking logarithm on both the sides, we obtain

$$\begin{aligned}\log y &= \log(x+3)^2 + \log(x+4)^3 + \log(x+5)^4 \\ \Rightarrow \log y &= 2 \log(x+3) + 3 \log(x+4) + 4 \log(x+5)\end{aligned}$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\frac{1}{y} \cdot \frac{dy}{dx} &= 2 \cdot \frac{1}{x+3} \cdot \frac{d}{dx}(x+3) + 3 \cdot \frac{1}{x+4} \cdot \frac{d}{dx}(x+4) + 4 \cdot \frac{1}{x+5} \cdot \frac{d}{dx}(x+5) \\ \Rightarrow \frac{dy}{dx} &= y \left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5} \right] \\ \Rightarrow \frac{dy}{dx} &= (x+3)^2 (x+4)^3 (x+5)^4 \cdot \left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5} \right] \\ \Rightarrow \frac{dy}{dx} &= (x+3)^2 (x+4)^3 (x+5)^4 \cdot \left[\frac{2(x+4)(x+5) + 3(x+3)(x+5) + 4(x+3)(x+4)}{(x+3)(x+4)(x+5)} \right] \\ \Rightarrow \frac{dy}{dx} &= (x+3)(x+4)^2 (x+5)^3 \cdot \left[\frac{2(x^2 + 9x + 20) + 3(x^2 + 8x + 15) + 4(x^2 + 7x + 12)}{(x+3)(x+4)(x+5)} \right] \\ \therefore \frac{dy}{dx} &= (x+3)(x+4)^2 (x+5)^3 (9x^2 + 70x + 133)\end{aligned}$$

Question 6:

Differentiate the function with respect to x : $\left(x + \frac{1}{x}\right)^x + x^{\left(1 + \frac{1}{x}\right)}$

Solution:

$$\text{Let } y = \left(x + \frac{1}{x}\right)^x + x^{\left(1 + \frac{1}{x}\right)}$$

$$\text{Also, let } u = \left(x + \frac{1}{x}\right)^x \text{ and } v = x^{\left(1 + \frac{1}{x}\right)}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

Then, $u = \left(x + \frac{1}{x}\right)^x$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u = \log \left(x + \frac{1}{x}\right)^x$$

$$\Rightarrow \log u = x \log \left(x + \frac{1}{x}\right)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned} \frac{1}{u} \cdot \frac{du}{dx} &= \frac{d}{dx} (x) \times \log \left(x + \frac{1}{x}\right) + x \times \frac{d}{dx} \left[\log \left(x + \frac{1}{x}\right) \right] \\ \Rightarrow \frac{1}{u} \cdot \frac{du}{dx} &= 1 \times \log \left(x + \frac{1}{x}\right) + x \times \frac{1}{\left(x + \frac{1}{x}\right)} \cdot \frac{d}{dx} \left(x + \frac{1}{x}\right) \\ \Rightarrow \frac{du}{dx} &= u \left[\log \left(x + \frac{1}{x}\right) + \frac{x}{\left(x + \frac{1}{x}\right)} \times \left(1 - \frac{1}{x^2}\right) \right] \\ \Rightarrow \frac{du}{dx} &= \left(x + \frac{1}{x}\right)^x \left[\log \left(x + \frac{1}{x}\right) + \frac{\left(x - \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)} \right] \\ \Rightarrow \frac{du}{dx} &= \left(x + \frac{1}{x}\right)^x \left[\log \left(x + \frac{1}{x}\right) + \frac{x^2 - 1}{x^2 + 1} \right] \\ \Rightarrow \frac{du}{dx} &= \left(x + \frac{1}{x}\right)^x \left[\frac{x^2 - 1}{x^2 + 1} + \log \left(x + \frac{1}{x}\right) \right] \quad \dots(2) \end{aligned}$$

Now, $v = x^{\left(1 + \frac{1}{x}\right)}$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log v = \log \left[x^{\left(1 + \frac{1}{x}\right)} \right]$$

$$\Rightarrow \log v = \left(1 + \frac{1}{x}\right) \log x$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
 \frac{1}{v} \cdot \frac{dv}{dx} &= \left[\frac{d}{dx} \left(1 + \frac{1}{x} \right) \right] \times \log x + \left(1 + \frac{1}{x} \right) \cdot \frac{d}{dx} \log x \\
 \Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} &= \left(-\frac{1}{x^2} \right) \log x + \left(1 + \frac{1}{x} \right) \cdot \frac{1}{x} \\
 \Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} &= -\frac{\log x}{x^2} + \frac{1}{x} + \frac{1}{x^2} \\
 \Rightarrow \frac{dv}{dx} &= v \left[\frac{-\log x + x + 1}{x^2} \right] \\
 \Rightarrow \frac{dv}{dx} &= x^{\left(1 + \frac{1}{x}\right)} \left[\frac{-\log x + x + 1}{x^2} \right] \quad \dots(3)
 \end{aligned}$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx} = \left(x + \frac{1}{x} \right)^x \left[\frac{x^2 - 1}{x^2 + 1} + \log \left(x + \frac{1}{x} \right) \right] + x^{\left(1 + \frac{1}{x}\right)} \left[\frac{x + 1 - \log x}{x^2} \right]$$

Question 7:

Differentiate the function with respect to x : $(\log x)^x + x^{\log x}$

Solution:

Let $y = (\log x)^x + x^{\log x}$

Also, let $u = (\log x)^x$ and $v = x^{\log x}$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

Then, $u = (\log x)^x$

Taking logarithm on both the sides, we obtain

$$\begin{aligned}
 \Rightarrow \log u &= \log \left[(\log x)^x \right] \\
 \Rightarrow \log u &= x \log (\log x)
 \end{aligned}$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
\frac{1}{u} \cdot \frac{du}{dx} &= \frac{d}{dx}(x) \times \log(\log x) + x \cdot \frac{d}{dx}[\log(\log x)] \\
\Rightarrow \frac{du}{dx} &= u \left[1 \times \log(\log x) + x \cdot \frac{1}{(\log x)} \cdot \frac{d}{dx}(\log x) \right] \\
\Rightarrow \frac{du}{dx} &= (\log x)^x \left[\log(\log x) + \frac{x}{(\log x)} \cdot \frac{1}{x} \right] \\
\Rightarrow \frac{du}{dx} &= (\log x)^x \left[\log(\log x) + \frac{1}{(\log x)} \right] \\
\Rightarrow \frac{du}{dx} &= (\log x)^x \left[\frac{\log(\log x) \cdot \log x + 1}{\log x} \right] \\
\Rightarrow \frac{du}{dx} &= (\log x)^{x-1} [1 + \log x \cdot \log(\log x)] \quad \dots(2)
\end{aligned}$$

$$v = x^{\log x}$$

Taking logarithm on both the sides, we obtain

$$\begin{aligned}
\Rightarrow \log v &= \log(x^{\log x}) \\
\Rightarrow \log v &= \log x \log x = (\log x)^2
\end{aligned}$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
\frac{1}{v} \cdot \frac{dv}{dx} &= \frac{d}{dx}[(\log x)^2] \\
\Rightarrow \frac{1}{v} \cdot \frac{dv}{dx} &= 2(\log x) \cdot \frac{d}{dx}(\log x) \\
\Rightarrow \frac{dv}{dx} &= 2v(\log x) \cdot \frac{1}{x} \\
\Rightarrow \frac{dv}{dx} &= 2x^{\log x} \frac{\log x}{x} \\
\Rightarrow \frac{dv}{dx} &= 2x^{\log x - 1} \cdot \log x \quad \dots(3)
\end{aligned}$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx} = (\log x)^{x-1} [1 + \log x \cdot \log(\log x)] + 2x^{\log x - 1} \cdot \log x$$

Question 8:

Differentiate the function with respect to x : $(\sin x)^x + \sin^{-1} \sqrt{x}$

Solution:

$$\text{Let } y = (\sin x)^x + \sin^{-1} \sqrt{x}$$

$$\text{Also, let } u = (\sin x)^x \text{ and } v = \sin^{-1} \sqrt{x}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

$$\text{Then, } u = (\sin x)^x$$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u = \log (\sin x)^x$$

$$\Rightarrow \log u = x \log (\sin x)$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x) \times \log (\sin x) + x \cdot \frac{d}{dx} [\log (\sin x)]$$

$$\Rightarrow \frac{du}{dx} = u \left[1 \times \log (\sin x) + x \cdot \frac{1}{(\sin x)} \cdot \frac{d}{dx} (\sin x) \right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x \left[\log (\sin x) + \frac{x}{(\sin x)} \cdot \cos x \right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x [x \cot x + \log \sin x] \quad \dots(2)$$

$$v = \sin^{-1} \sqrt{x}$$

Differentiating both sides with respect to x , we obtain

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx} (\sqrt{x})$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}} \quad \dots(3)$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx} = (\sin x)^x [x \cot x + \log \sin x] + \frac{1}{2\sqrt{x-x^2}}$$

Question 9:

Differentiate the function with respect to x : $x^{\sin x} + (\sin x)^{\cos x}$

Solution:

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Also, let $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

Then, $u = x^{\sin x}$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u = \log(x^{\sin x})$$

$$\Rightarrow \log u = \sin x \log x$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(\sin x) \cdot \log x + \sin x \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u \left[\cos x \log x + \sin x \cdot \frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x} \left[\cos x \log x + \frac{\sin x}{x} \right] \quad \dots(2)$$

$$v = (\sin x)^{\cos x}$$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log v = \log(\sin x)^{\cos x}$$

$$\Rightarrow \log v = \cos x \log(\sin x)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned} \frac{1}{v} \frac{dv}{dx} &= \frac{d}{dx}(\cos x) \times \log(\sin x) + \cos x \times \frac{d}{dx}[\log(\sin x)] \\ \Rightarrow \frac{dv}{dx} &= v \left[-\sin x \cdot \log(\sin x) + \cos x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) \right] \\ \Rightarrow \frac{dv}{dx} &= (\sin x)^{\cos x} \left[-\sin x \log(\sin x) + \frac{\cos x}{\sin x} \cos x \right] \\ \Rightarrow \frac{dv}{dx} &= (\sin x)^{\cos x} \left[-\sin x \log(\sin x) + \cot x \cos x \right] \\ \Rightarrow \frac{dv}{dx} &= (\sin x)^{\cos x} \left[\cot x \cos x - \sin x \log(\sin x) \right] \quad \dots(3) \end{aligned}$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx} = x^{\sin x} \left[\cos x \log x + \frac{\sin x}{x} \right] + (\sin x)^{\cos x} \left[\cot x \cos x - \sin x \log(\sin x) \right]$$

Question 10:

Differentiate the function with respect to x : $x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$

Solution:

$$\text{Let } y = x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$$

$$\text{Also, let } u = x^{x \cos x} \text{ and } v = \frac{x^2 + 1}{x^2 - 1}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

$$\text{Then, } u = x^{x \cos x}$$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u = \log(x^{x \cos x})$$

$$\Rightarrow \log u = x \cos x \log x$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\frac{1}{u} \cdot \frac{du}{dx} &= \frac{d}{dx}(x) \cdot \cos x \cdot \log x + x \cdot \frac{d}{dx}(\cos x) \cdot \log x + x \cos x \cdot \frac{d}{dx}(\log x) \\ \Rightarrow \frac{du}{dx} &= u \left[1 \cdot \cos x \cdot \log x + x \cdot (-\sin x) \log x + x \cos x \cdot \frac{1}{x} \right] \\ \Rightarrow \frac{du}{dx} &= x^{x \cos x} [\cos x \log x - x \cdot \sin x \log x + \cos x] \\ \Rightarrow \frac{du}{dx} &= x^{x \cos x} [\cos x (1 + \log x) - x \cdot \sin x \log x] \quad \dots (2)\end{aligned}$$

$$v = \frac{x^2 + 1}{x^2 - 1}$$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log v = \log(x^2 + 1) - \log(x^2 - 1)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\frac{1}{v} \frac{dv}{dx} &= \frac{2x}{x^2 + 1} - \frac{2x}{x^2 - 1} \\ \Rightarrow \frac{dv}{dx} &= v \left[\frac{2x(x^2 - 1) - 2x(x^2 + 1)}{(x^2 + 1)(x^2 - 1)} \right] \\ \Rightarrow \frac{dv}{dx} &= \frac{x^2 + 1}{x^2 - 1} \times \left[\frac{-4x}{(x^2 + 1)(x^2 - 1)} \right] \\ \Rightarrow \frac{dv}{dx} &= \frac{-4x}{(x^2 - 1)^2} \quad \dots (3)\end{aligned}$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx} = x^{x \cos x} [\cos x (1 + \log x) - x \cdot \sin x \log x] - \frac{4x}{(x^2 - 1)^2}$$

Question 11:

Differentiate the function with respect to x : $(x \cos x)^x + (x \sin x)^{\frac{1}{x}}$

Solution:

$$\text{Let } y = (x \cos x)^x + (x \sin x)^{\frac{1}{x}}$$

$$\text{Also, let } u = (x \cos x)^x \text{ and } v = (x \sin x)^{\frac{1}{x}}$$

$$\therefore y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

$$\text{Then, } u = (x \cos x)^x$$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u = \log (x \cos x)^x$$

$$\Rightarrow \log u = x \log (x \cos x)$$

$$\Rightarrow \log u = x [\log x + \log \cos x]$$

$$\Rightarrow \log u = x \log x + x \log \cos x$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x \log x) + \frac{d}{dx}(x \log \cos x)$$

$$\Rightarrow \frac{du}{dx} = u \left[\left\{ \log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x) \right\} + \left\{ \log \cos x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log \cos x) \right\} \right]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x \left[\left(\log x \cdot 1 + x \cdot \frac{1}{x} \right) + \left\{ \log \cos x \cdot 1 + x \cdot \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) \right\} \right]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x \left[(\log x + 1) + \left\{ \log \cos x + \frac{x}{\cos x} \cdot (-\sin x) \right\} \right]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x \left[(1 + \log x) + (\log \cos x - x \tan x) \right]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x \left[(1 - x \tan x) + (\log x + \log \cos x) \right]$$

$$\Rightarrow \frac{du}{dx} = (x \cos x)^x \left[1 - x \tan x + \log(x \cos x) \right] \quad \dots(2)$$

$$v = (x \sin x)^{\frac{1}{x}}$$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log v = \log (x \sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \frac{1}{x} \log (x \sin x)$$

$$\Rightarrow \log v = \frac{1}{x} (\log x + \log \sin x)$$

$$\Rightarrow \log v = \frac{1}{x} \log x + \frac{1}{x} \log \sin x$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned} \frac{1}{v} \frac{dv}{dx} &= \frac{d}{dx} \left(\frac{1}{x} \log x \right) + \frac{d}{dx} \left[\frac{1}{x} \log (\sin x) \right] \\ \Rightarrow \frac{1}{v} \frac{dv}{dx} &= \left[\log x \cdot \frac{d}{dx} \left(\frac{1}{x} \right) + \frac{1}{x} \cdot \frac{d}{dx} (\log x) \right] + \left[\log (\sin x) \cdot \frac{d}{dx} \left(\frac{1}{x} \right) + \frac{1}{x} \cdot \frac{d}{dx} \{ \log (\sin x) \} \right] \\ \Rightarrow \frac{1}{v} \frac{dv}{dx} &= \left[\log x \cdot \left(-\frac{1}{x^2} \right) + \frac{1}{x} \cdot \frac{1}{x} \right] + \left[\log (\sin x) \cdot \left(-\frac{1}{x^2} \right) + \frac{1}{x} \cdot \frac{1}{\sin x} \cdot \frac{d}{dx} (\sin x) \right] \\ \Rightarrow \frac{1}{v} \frac{dv}{dx} &= \frac{1}{x^2} (1 - \log x) + \left[-\frac{\log (\sin x)}{x^2} + \frac{1}{x \sin x} \cdot \cos x \right] \\ \Rightarrow \frac{dv}{dx} &= (x \sin x)^{\frac{1}{x}} \left[\frac{1 - \log x}{x^2} + \frac{-\log (\sin x) + x \cot x}{x^2} \right] \\ \Rightarrow \frac{dv}{dx} &= (x \sin x)^{\frac{1}{x}} \left[\frac{1 - \log x - \log (\sin x) + x \cot x}{x^2} \right] \\ \Rightarrow \frac{dv}{dx} &= (x \sin x)^{\frac{1}{x}} \left[\frac{1 - \log (x \sin x) + x \cot x}{x^2} \right] \quad \dots (3) \end{aligned}$$

Therefore, from (1), (2) and (3);

$$\frac{dy}{dx} = (x \cos x)^x [1 - x \tan x + \log (x \cos x)] + (x \sin x)^{\frac{1}{x}} \left[\frac{x \cot x + 1 - \log (x \sin x)}{x^2} \right]$$

Question 12:

Find $\frac{dy}{dx}$ of the function $x^y + y^x = 1$

Solution:

The given function is $x^y + y^x = 1$

Let, $x^y = u$ and $y^x = v$

$$\therefore u + v = 1$$

$$\Rightarrow \frac{du}{dx} + \frac{dv}{dx} = 0 \quad \dots(1)$$

Then, $u = x^y$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log u = \log(x^y)$$

$$\Rightarrow \log u = y \log x$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{u} \cdot \frac{du}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u \left[\log x \frac{dy}{dx} + y \cdot \frac{1}{x} \right]$$

$$\Rightarrow \frac{du}{dx} = x^y \left[\log x \frac{dy}{dx} + \frac{y}{x} \right] \quad \dots(2)$$

Now, $v = y^x$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log v = \log(y^x)$$

$$\Rightarrow \log v = x \log y$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{v} \frac{dv}{dx} = \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y)$$

$$\Rightarrow \frac{dv}{dx} = v \left[\log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} \right]$$

$$\Rightarrow \frac{dv}{dx} = y^x \left[\log y + \frac{x}{y} \cdot \frac{dy}{dx} \right] \quad \dots(3)$$

Therefore, from (1), (2) and (3);

$$\begin{aligned}
& x^y \left[\log x \frac{dy}{dx} + \frac{y}{x} \right] + y^x \left[\log y + \frac{x}{y} \cdot \frac{dy}{dx} \right] = 0 \\
& \Rightarrow (x^y \log x + xy^{x-1}) \frac{dy}{dx} = -(yx^{y-1} + y^x \log y) \\
& \therefore \frac{dy}{dx} = \frac{-(yx^{y-1} + y^x \log y)}{(x^y \log x + xy^{x-1})}
\end{aligned}$$

Question 13:

Find $\frac{dy}{dx}$ of the function $y^x = x^y$

Solution:

The given function is $y^x = x^y$

Taking logarithm on both the sides, we obtain

$$x \log y = y \log x$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
& \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y) = \log x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(\log x) \\
& \Rightarrow \log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} = \log x \cdot \frac{dy}{dx} + y \cdot \frac{1}{x} \\
& \Rightarrow \log y + \frac{x}{y} \cdot \frac{dy}{dx} = \log x \cdot \frac{dy}{dx} + \frac{y}{x} \\
& \Rightarrow \left(\frac{x}{y} - \log x \right) \frac{dy}{dx} = \frac{y}{x} - \log y \\
& \Rightarrow \left(\frac{x - y \log x}{y} \right) \frac{dy}{dx} = \frac{y - x \log y}{x} \\
& \therefore \frac{dy}{dx} = \frac{y}{x} \left(\frac{y - x \log y}{x - y \log x} \right)
\end{aligned}$$

Question 14:

Find $\frac{dy}{dx}$ of the function $(\cos x)^y = (\cos y)^x$

Solution:

The given function is $(\cos x)^y = (\cos y)^x$

Taking logarithm on both the sides, we obtain

$$y \log \cos x = x \log \cos y$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log \cos x) &= \log \cos y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log \cos y) \\ \Rightarrow \log \cos x \cdot \frac{dy}{dx} + y \cdot \frac{1}{\cos x} \cdot \frac{d}{dx}(\cos x) &= \log \cos y \cdot 1 + x \cdot \frac{1}{\cos y} \cdot \frac{d}{dx}(\cos y) \\ \Rightarrow \log \cos x \cdot \frac{dy}{dx} + \frac{y}{\cos x} \cdot (-\sin x) &= \log \cos y + \frac{x}{\cos y} \cdot (-\sin y) \cdot \frac{dy}{dx} \\ \Rightarrow \log \cos x \cdot \frac{dy}{dx} - y \tan x &= \log \cos y - x \tan y \frac{dy}{dx} \\ \Rightarrow (\log \cos x + x \tan y) \frac{dy}{dx} &= y \tan x + \log \cos y \\ \therefore \frac{dy}{dx} &= \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}\end{aligned}$$

Question 15:

Find $\frac{dy}{dx}$ of the function $xy = e^{(x-y)}$

Solution:

The given function is $xy = e^{(x-y)}$

Taking logarithm on both the sides, we obtain

$$\begin{aligned}\log(xy) &= \log(e^{x-y}) \\ \Rightarrow \log x + \log y &= (x-y) \log e \\ \Rightarrow \log x + \log y &= (x-y) \times 1 \\ \Rightarrow \log x + \log y &= (x-y)\end{aligned}$$

Differentiating both sides with respect to x , we obtain

$$\frac{d}{dx}(\log x) + \frac{d}{dx}(\log y) = \frac{d}{dx}(x) - \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \left(1 + \frac{1}{y}\right) \frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\Rightarrow \left(\frac{y+1}{y}\right) \frac{dy}{dx} = \frac{x-1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$$

Question 16:

Find the derivative of the function given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.

Solution:

The given function is $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$

Taking logarithm on both the sides, we obtain

$$\log f(x) = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
 \frac{1}{f(x)} \cdot \frac{d}{dx} [f(x)] &= \frac{d}{dx} \log(1+x) + \frac{d}{dx} \log(1+x^2) \\
 &\quad + \frac{d}{dx} \log(1+x^4) + \frac{d}{dx} \log(1+x^8) \\
 \Rightarrow \frac{1}{f(x)} \cdot f'(x) &= \frac{1}{1+x} \cdot \frac{d}{dx} (1+x) + \frac{1}{1+x^2} \cdot \frac{d}{dx} (1+x^2) \\
 &\quad + \frac{1}{1+x^4} \cdot \frac{d}{dx} (1+x^4) + \frac{1}{1+x^8} \cdot \frac{d}{dx} (1+x^8) \\
 \Rightarrow f'(x) &= f(x) \left[\frac{1}{1+x} + \frac{1}{1+x^2} \cdot 2x + \frac{1}{1+x^4} \cdot 4x^3 + \frac{1}{1+x^8} \cdot 8x^7 \right] \\
 \therefore f'(x) &= (1+x)(1+x^2)(1+x^4)(1+x^8) \left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} \right]
 \end{aligned}$$

Hence,

$$\begin{aligned}
 f'(1) &= (1+1)(1+1^2)(1+1^4)(1+1^8) \left[\frac{1}{1+1} + \frac{2(1)}{1+1^2} + \frac{4(1)^3}{1+1^4} + \frac{8(1)^7}{1+1^8} \right] \\
 &= 2 \times 2 \times 2 \times 2 \left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2} \right] \\
 &= 16 \left(\frac{15}{2} \right) \\
 &= 120
 \end{aligned}$$

Question 17:

Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below.

- (i) By using product rule
- (ii) By expanding the product to obtain a single polynomial.
- (iii) By logarithmic differentiation.

Do they all give the same answer?

Solution:

$$\text{Let } y = (x^2 - 5x + 8)(x^3 + 7x + 9)$$

- (i) By using product rule

Let $u = (x^2 - 5x + 8)$ and $v = x^3 + 7x + 9$

$$\therefore y = uv$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx} \quad (\text{product rule})$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2 - 5x + 8) \cdot (x^3 + 7x + 9) + (x^2 - 5x + 8) \cdot \frac{d}{dx}(x^3 + 7x + 9)$$

$$\Rightarrow \frac{dy}{dx} = (2x - 5)(x^3 + 7x + 9) + (x^2 - 5x + 8)(3x^2 + 7)$$

$$\Rightarrow \frac{dy}{dx} = 2x(x^3 + 7x + 9) - 5(x^3 + 7x + 9) + x^2(3x^2 + 7)$$

$$- 5x(3x^2 + 7) + 8(3x^2 + 7)$$

$$\Rightarrow \frac{dy}{dx} = (2x^4 + 14x^2 + 18x) - 5x^3 - 35x - 45 + (3x^4 + 7x^2)$$

$$\Rightarrow \frac{dy}{dx} = -15x^3 - 35x + 24x^2 + 56$$

$$\therefore \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(ii) By expanding the product to obtain a single polynomial.

$$y = (x^2 - 5x + 8)(x^3 + 7x + 9)$$

$$= x^2(x^3 + 7x + 9) - 5x(x^3 + 7x + 9) + 8(x^3 + 7x + 9)$$

$$= x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$$

$$= x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$$

Therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72)$$

$$= \frac{d}{dx}(x^5) - 5 \frac{d}{dx}(x^4) + 15 \frac{d}{dx}(x^3) - 26 \frac{d}{dx}(x^2) + 11 \frac{d}{dx}(x) + \frac{d}{dx}(72)$$

$$= 5x^4 - 5(4x^3) + 15(3x^2) - 26(2x) + 11(1) + 0$$

$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(iii) By logarithmic differentiation.

$$y = (x^2 - 5x + 8)(x^3 + 7x + 9)$$

Taking logarithm on both the sides, we obtain

$$\log y = \log(x^2 - 5x + 8) + \log(x^3 + 7x + 9)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
 \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{d}{dx} \log(x^2 - 5x + 8) + \frac{d}{dx} \log(x^3 + 7x + 9) \\
 \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{1}{x^2 - 5x + 8} \cdot \frac{d}{dx}(x^2 - 5x + 8) + \frac{1}{x^3 + 7x + 9} \cdot \frac{d}{dx}(x^3 + 7x + 9) \\
 \Rightarrow \frac{dy}{dx} &= y \left[\frac{1}{x^2 - 5x + 8} \cdot (2x - 5) + \frac{1}{x^3 + 7x + 9} \cdot (3x^2 + 7) \right] \\
 \Rightarrow \frac{dy}{dx} &= (x^2 - 5x + 8)(x^3 + 7x + 9) \left[\frac{2x - 5}{x^2 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9} \right] \\
 \Rightarrow \frac{dy}{dx} &= (x^2 - 5x + 8)(x^3 + 7x + 9) \left[\frac{(2x - 5)(x^3 + 7x + 9) + (3x^2 + 7)(x^2 - 5x + 8)}{(x^2 - 5x + 8)(x^3 + 7x + 9)} \right] \\
 \Rightarrow \frac{dy}{dx} &= 2x(x^3 + 7x + 9) - 5(x^3 + 7x + 9) + 3x^2(x^2 - 5x + 8) + 7(x^2 - 5x + 8) \\
 \Rightarrow \frac{dy}{dx} &= 2x^4 + 14x^2 + 18x - 5x^3 - 35x - 45 + 3x^5 - 15x^3 + 24x^2 + 7x^2 - 35x + 56 \\
 \Rightarrow \frac{dy}{dx} &= 5x^4 - 20x^3 + 45x^2 - 52x + 11
 \end{aligned}$$

From the above three observations, it can be concluded that all the results of $\frac{dy}{dx}$ are same.

Question 18:

If u , v and w are functions of x , then show that

$$\frac{d}{dx}(u.v.w) = \frac{du}{dx}v.w + u \cdot \frac{dv}{dx}.w + u.v \cdot \frac{dw}{dx}$$

in two ways - first by repeated application of product rule, second by logarithmic differentiation.

Solution:

Let $y = u.v.w = u.(v.w)$

By applying product rule, we get

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{du}{dx} \cdot (v.w) + u \cdot \frac{d}{dx}(v.w) \\
 \Rightarrow \frac{dy}{dx} &= \frac{du}{dx} \cdot (v.w) + u \left[\frac{dv}{dx} \cdot w + v \cdot \frac{dw}{dx} \right] \quad (\text{Again applying product rule}) \\
 \Rightarrow \frac{dy}{dx} &= \frac{du}{dx} \cdot v.w + u \cdot \frac{dv}{dx} \cdot w + u.v \cdot \frac{dw}{dx}
 \end{aligned}$$

Taking logarithm on both the sides of the equation $y = u.v.w$, we obtain
 $\log y = \log u + \log v + \log w$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(\log u) + \frac{d}{dx}(\log v) + \frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx} \right)$$

$$\Rightarrow \frac{dy}{dx} = u.v.w \left(\frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx} \right)$$

$$\therefore \frac{d}{dx}(u.v.w) = \frac{du}{dx} v.w + u \cdot \frac{dv}{dx} .w + u.v \cdot \frac{dw}{dx}$$

EXERCISE 5.6

Question 1:

If x and y are connected parametrically by the equations $x = 2at^2, y = at^4$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = 2at^2, y = at^4$

Then,

$$\frac{dx}{dt} = \frac{d}{dt}(2at^2) = 2a \cdot \frac{d}{dt}(t^2) = 2a \cdot 2t = 4at$$

$$\frac{dy}{dt} = \frac{d}{dt}(at^4) = a \cdot \frac{d}{dt}(t^4) = a \cdot 4t^3 = 4at^3$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4at^3}{4at} = t^2$$

Question 2:

If x and y are connected parametrically by the equations $x = a \cos \theta, y = b \cos \theta$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = a \cos \theta, y = b \cos \theta$

Then,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a \cos \theta) = a(-\sin \theta) = -a \sin \theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b \cos \theta) = b(-\sin \theta) = -b \sin \theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-b \sin \theta}{-a \sin \theta} = \frac{b}{a}$$

Question 3:

If x and y are connected parametrically by the equations $x = \sin t, y = \cos 2t$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = \sin t, y = \cos 2t$

$$\text{Then, } \frac{dx}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2t) = -\sin 2t, \frac{d}{dt}(2t) = 2 \Rightarrow -2 \sin 2t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2 \sin 2t}{\cos t} = \frac{-2 \cdot 2 \sin t \cos t}{\cos t} = -4 \sin t$$

Question 4:

If x and y are connected parametrically by the equations $x = 4t, y = \frac{4}{t}$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = 4t, y = \frac{4}{t}$

$$\frac{dx}{dt} = \frac{d}{dt}(4t) = 4$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right) = 4 \cdot \frac{d}{dt}\left(\frac{1}{t}\right) = 4 \cdot \left(\frac{-1}{t^2}\right) = \frac{-4}{t^2}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-4}{t^2}\right)}{4} = \frac{-1}{t^2}$$

Question 5:

If x and y are connected parametrically by the equations $x = \cos \theta - \cos 2\theta, y = \sin \theta - \sin 2\theta$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = \cos \theta - \cos 2\theta, y = \sin \theta - \sin 2\theta$

Then,

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\cos \theta - \cos 2\theta) = \frac{d}{d\theta}(\cos \theta) - \frac{d}{d\theta}(\cos 2\theta)$$

$$= -\sin \theta - (-2 \sin 2\theta) = 2 \sin 2\theta - \sin \theta$$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin \theta - \sin 2\theta) = \frac{d}{d\theta}(\sin \theta) - \frac{d}{d\theta}(\sin 2\theta)$$

$$= \cos \theta - 2 \cos 2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\cos \theta - 2 \cos 2\theta}{2 \sin 2\theta - \sin \theta}$$

Question 6:

If x and y are connected parametrically by the equations $x = a(\theta - \sin \theta), y = a(1 + \cos \theta)$,

without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = a(\theta - \sin \theta), y = a(1 + \cos \theta)$

$$\text{Then, } \frac{dx}{d\theta} = a \left[\frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin \theta) \right] = a(1 - \cos \theta)$$

$$\frac{dy}{d\theta} = a \left[\frac{d}{d\theta}(1) + \frac{d}{d\theta}(\cos \theta) \right] = a[0 + (-\sin \theta)] = -a \sin \theta$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-a \sin \theta}{a(1 - \cos \theta)} = \frac{-2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} = \frac{-\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = -\cot \frac{\theta}{2}$$

Question 7:

If x and y are connected parametrically by the equations $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$, without

eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

Then,

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt} \left[\frac{\sin^3 t}{\sqrt{\cos 2t}} \right] \\&= \frac{\sqrt{\cos 2t} \cdot \frac{d}{dt}(\sin^3 t) - \sin^3 t \cdot \frac{d}{dt} \sqrt{\cos 2t}}{\cos 2t} \\&= \frac{\sqrt{\cos 2t} \cdot 3 \sin^2 t \cdot \frac{d}{dt}(\sin t) - \sin^3 t \times \frac{1}{2\sqrt{\cos 2t}} \cdot \frac{d}{dt}(\cos 2t)}{\cos 2t} \\&= \frac{3\sqrt{\cos 2t} \cdot \sin^2 t \cdot \cos t - \frac{\sin^3 t}{2\sqrt{\cos 2t}} \cdot (-2 \sin 2t)}{\cos 2t} \\&= \frac{3 \cos 2t \cdot \sin^2 t \cos t + \sin^3 t \cdot \sin 2t}{\cos 2t \sqrt{\cos 2t}} \\ \frac{dy}{dt} &= \frac{d}{dt} \left[\frac{\cos^3 t}{\sqrt{\cos 2t}} \right] \\&= \frac{\sqrt{\cos 2t} \cdot \frac{d}{dt}(\cos^3 t) - \cos^3 t \cdot \frac{d}{dt}(\sqrt{\cos 2t})}{\cos 2t} \\&= \frac{\sqrt{\cos 2t} \cdot 3 \cos^2 t \cdot \frac{d}{dt}(\cos t) - \cos^3 t \cdot \frac{1}{2\sqrt{\cos 2t}} \cdot \frac{d}{dt}(\cos 2t)}{\cos 2t} \\&= \frac{3\sqrt{\cos 2t} \cdot \cos^2 t (-\sin t) - \cos^3 t \cdot \frac{1}{\sqrt{\cos 2t}} \cdot (-2 \sin 2t)}{\cos 2t} \\&= \frac{-3 \cos 2t \cdot \cos^2 t \cdot \sin t + \cos^3 t \cdot \sin 2t}{\cos 2t \sqrt{\cos 2t}}\end{aligned}$$

$$\begin{aligned}
\therefore \frac{dy}{dx} &= \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{-3 \cos 2t \cdot \cos^2 t \cdot \sin t + \cos^3 t \sin 2t}{\cos 2t \cdot \sqrt{\cos 2t}}}{\frac{3 \cos 2t \cdot \sin^2 t \cdot \cos t + \sin^3 t \sin 2t}{\cos 2t \cdot \sqrt{\cos 2t}}} \\
&= \frac{-3 \cos 2t \cdot \cos^2 t \cdot \sin t + \cos^3 t \sin 2t}{3 \cos 2t \cdot \sin^2 t \cdot \cos t + \sin^3 t \sin 2t} \\
&= \frac{-3 \cos 2t \cdot \cos^2 t \cdot \sin t + \cos^3 t (2 \sin t \cos t)}{3 \cos 2t \cdot \sin^2 t \cdot \cos t + \sin^3 t (2 \sin t \cos t)} \\
&= \frac{\sin t \cos t [-3 \cos 2t \cdot \cos t + 2 \cos^3 t]}{\sin t \cos t [3 \cos 2t \sin t + 2 \sin^3 t]} \\
&= \frac{[-3(2 \cos^2 t - 1) \cos t + 2 \cos^3 t]}{[3(1 - 2 \sin^2 t) \sin t + 2 \sin^3 t]} \quad \begin{bmatrix} \cos 2t = (2 \cos^2 t - 1) \\ \cos 2t = (1 - 2 \sin^2 t) \end{bmatrix} \\
&= \frac{-4 \cos^3 t + 3 \cos t}{3 \sin t - 4 \sin^3 t} \quad \begin{bmatrix} \cos 3t = 4 \cos^3 t - 3 \cos t \\ \sin 3t = 3 \sin t - 4 \sin^3 t \end{bmatrix} \\
&= \frac{-\cos 3t}{\sin 3t} = -\cot 3t
\end{aligned}$$

Question 8:

If x and y are connected parametrically by the equations $x = a \left(\cos t + \log \tan \frac{t}{2} \right)$, $y = a \sin t$,
without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = a \left(\cos t + \log \tan \frac{t}{2} \right)$, $y = a \sin t$

Then,

$$\begin{aligned}
 \frac{dx}{dt} &= a \left[\frac{d}{dt}(\cos t) + \frac{d}{dt} \left(\log \tan \frac{t}{2} \right) \right] \\
 &= a \left[-\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \frac{d}{dt} \left(\tan \frac{t}{2} \right) \right] \\
 &= a \left[-\sin t + \cot \frac{t}{2} \cdot \sec^2 \frac{t}{2} \cdot \frac{d}{dt} \left(\frac{t}{2} \right) \right] \\
 &= a \left[-\sin t + \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \times \frac{1}{\cos^2 \frac{t}{2}} \times \frac{1}{2} \right] \\
 &= a \left[-\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}} \right] \\
 &= a \left(-\sin t + \frac{1}{\sin t} \right) \\
 &= a \left(\frac{-\sin^2 t + 1}{\sin t} \right) \\
 &= a \left(\frac{\cos^2 t}{\sin t} \right)
 \end{aligned}$$

$$\frac{dy}{dt} = a \frac{d}{dt}(\sin t) = a \cos t$$

Therefore,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)} = \frac{a \cos t}{\left(a \frac{\cos^2 t}{\sin t} \right)} = \frac{\sin t}{\cos t} = \tan t$$

Question 9:

If x and y are connected parametrically by the equations $x = a \sec \theta, y = b \tan \theta$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = a \sec \theta, y = b \tan \theta$

Then,

$$\frac{dx}{d\theta} = a \cdot \frac{d}{d\theta}(\sec \theta) = a \sec \theta \tan \theta$$

$$\frac{dy}{d\theta} = b \cdot \frac{d}{d\theta}(\tan \theta) = b \sec^2 \theta$$

Therefore,

$$\begin{aligned} \frac{dy}{dx} &= \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} \\ &= \frac{b \sec^2 \theta}{a \sec \theta \tan \theta} \\ &= \frac{b}{a} \sec \theta \cot \theta \\ &= \frac{b \cos \theta}{a \cos \theta \sin \theta} \\ &= \frac{b}{a} \times \frac{1}{\sin \theta} \\ &= \frac{b}{a} \operatorname{cosec} \theta \end{aligned}$$

Question 10:

If x and y are connected parametrically by the equations

$x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$, without eliminating the parameter, find $\frac{dy}{dx}$

Solution:

Given, $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$

Then,

$$\begin{aligned} \frac{dx}{d\theta} &= a \left[\frac{d}{d\theta} \cos \theta + \frac{d}{d\theta} (\theta \sin \theta) \right] \\ &= a \left[-\sin \theta + \theta \frac{d}{d\theta} (\sin \theta) + \sin \theta \frac{d}{d\theta} (\theta) \right] \\ &= a [-\sin \theta + \theta \cos \theta + \sin \theta] \\ &= a\theta \cos \theta \\ \frac{dy}{d\theta} &= a \left[\frac{d}{d\theta} (\sin \theta) - \frac{d}{d\theta} (\theta \cos \theta) \right] = a \left[\cos \theta - \left\{ \theta \frac{d}{d\theta} (\cos \theta) + \cos \theta \cdot \frac{d}{d\theta} (\theta) \right\} \right] \\ &= a [\cos \theta + \theta \sin \theta - \cos \theta] \\ &= a\theta \sin \theta \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} \\ &= \frac{a\theta \sin \theta}{a\theta \cos \theta} \\ &= \tan \theta\end{aligned}$$

Question 11:

If $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

Solution:

Given, $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$

Hence,

$$x = \sqrt{a^{\sin^{-1} t}} = \left(a^{\sin^{-1} t}\right)^{\frac{1}{2}} = a^{\frac{1}{2}\sin^{-1} t} \quad \text{and} \quad y = \sqrt{a^{\cos^{-1} t}} = \left(a^{\cos^{-1} t}\right)^{\frac{1}{2}} = a^{\frac{1}{2}\cos^{-1} t}$$

Consider $x = a^{\frac{1}{2}\sin^{-1} t}$

Taking log on both sides, we get

$$\log x = \frac{1}{2} \sin^{-1} t \log a$$

Therefore,

$$\begin{aligned}\Rightarrow \frac{1}{x} \cdot \frac{dx}{dt} &= \frac{1}{2} \log a \cdot \frac{d}{dt} (\sin^{-1} t) \\ \Rightarrow \frac{dx}{dt} &= \frac{x}{2} \log a \cdot \frac{1}{\sqrt{1-t^2}} \\ \Rightarrow \frac{dx}{dt} &= \frac{x \log a}{2\sqrt{1-t^2}}\end{aligned}$$

Now, $y = a^{\frac{1}{2}\cos^{-1} t}$

Taking log on both sides, we get

$$\log y = \frac{1}{2} \cos^{-1} t \log a$$

Therefore,

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dt} = \frac{1}{2} \log a \cdot \frac{d}{dt} (\cos^{-1} t)$$

$$\Rightarrow \frac{dy}{dt} = \frac{y}{2} \log a \cdot \frac{-1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dy}{dt} = \frac{-y \log a}{2\sqrt{1-t^2}}$$

Hence,

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\left(\frac{-y \log a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x \log a}{2\sqrt{1-t^2}}\right)} = -\frac{y}{x}$$

EXERCISE 5.7

Question 1:

Find the second order derivative of the function $x^2 + 3x + 2$

Solution:

Consider, $y = x^2 + 3x + 2$

Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(2) \\ &= 2x + 3 + 0 \\ &= 2x + 3\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}(2x + 3) \\ &= \frac{d}{dx}(2x) + \frac{d}{dx}(3) \\ &= 2 + 0 \\ &= 2\end{aligned}$$

Question 2:

Find the second order derivative of the function x^{20}

Solution:

Consider, $y = x^{20}$

Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^{20}) \\ &= 20x^{19}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}(20x^{19}) \\ &= 20 \frac{d}{dx}(x^{19}) \\ &= 20 \cdot 19 \cdot x^{18} \\ &= 380x^{18}\end{aligned}$$

Question 3:

Find the second order derivative of the function $x \cos x$

Solution:

Consider, $y = x \cos x$

Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x \cdot \cos x) \\ &= \cos x \cdot \frac{d}{dx}(x) + x \frac{d}{dx}(\cos x) \\ &= \cos x \cdot 1 + x(-\sin x) \\ &= \cos x - x \sin x\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}[\cos x - x \sin x] \\ &= \frac{d}{dx}(\cos x) - \frac{d}{dx}(x \sin x) \\ &= -\sin x - \left[\sin x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\sin x) \right] \\ &= -\sin x - (\sin x + x \cos x) \\ &= -(x \cos x + 2 \sin x)\end{aligned}$$

Question 4:

Find the second order derivative of the function $\log x$

Solution:

Let $y = \log x$

Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\log x) \\ &= \frac{1}{x}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{1}{x}\right) \\ &= \frac{-1}{x^2}\end{aligned}$$

Question 5:

Find the second order derivative of the function $x^3 \log x$

Solution:

Let $y = x^3 \log x$

Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [x^3 \log x] \\ &= \log x \cdot \frac{d}{dx} (x^3) + x^3 \cdot \frac{d}{dx} (\log x) \\ &= \log x \cdot 3x^2 + x^3 \cdot \frac{1}{x} = \log x \cdot 3x^2 + x^2 \\ &= x^2 (1 + 3 \log x)\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d^2 y}{dx^2} &= \frac{d}{dx} [x^2 (1 + 3 \log x)] \\ &= (1 + 3 \log x) \cdot \frac{d}{dx} (x^2) + x^2 \cdot \frac{d}{dx} (1 + 3 \log x) \\ &= (1 + 3 \log x) \cdot 2x + x^2 \cdot \frac{3}{x} \\ &= 2x + 6 \log x + 3x \\ &= 5x + 6x \log x \\ &= x(5 + 6 \log x)\end{aligned}$$

Question 6:

Find the second order derivative of the function $e^x \sin 5x$

Solution:

Let $y = e^x \sin 5x$

Then,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(e^x \sin 5x) \\
&= \sin 5x \times \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(\sin 5x) \\
&= \sin 5x \cdot e^x + e^x \cdot \cos 5x \cdot \frac{d}{dx}(5x) \\
&= e^x \sin 5x + e^x \cos 5x \cdot 5 \\
&= e^x (\sin 5x + 5 \cos 5x)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d^2 y}{dx^2} &= \frac{d}{dx} [e^x (\sin 5x + 5 \cos 5x)] \\
&= (\sin 5x + 5 \cos 5x) \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(\sin 5x + 5 \cos 5x) \\
&= (\sin 5x + 5 \cos 5x) e^x + e^x \left[\cos 5x \cdot \frac{d}{dx}(5x) + 5(-\sin 5x) \cdot \frac{d}{dx}(5x) \right] \\
&= e^x (\sin 5x + 5 \cos 5x) + e^x (5 \cos 5x - 25 \sin 5x) \\
&= e^x (10 \cos 5x - 24 \sin 5x) \\
&= 2e^x (5 \cos 5x - 12 \sin 5x)
\end{aligned}$$

Question 7:

Find the second order derivative of the function $e^{6x} \cos 3x$

Solution:

Let $y = e^{6x} \cos 3x$

Then,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(e^{6x} \cos 3x) = \cos 3x \cdot \frac{d}{dx}(e^{6x}) + e^{6x} \cdot \frac{d}{dx}(\cos 3x) \\
&= \cos 3x \cdot e^{6x} \cdot \frac{d}{dx}(6x) + e^{6x} \cdot (-\sin 3x) \cdot \frac{d}{dx}(3x) \\
&= 6e^{6x} \cos 3x - 3e^{6x} \sin 3x \quad \dots(1)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d^2 y}{dx^2} &= \frac{d}{dx} (6e^{6x} \cos 3x - 3e^{6x} \sin 3x) = 6 \cdot \frac{d}{dx} (e^{6x} \cos 3x) - 3 \cdot \frac{d}{dx} (e^{6x} \sin 3x) \\
&= 6 \cdot [6e^{6x} \cos 3x - 3e^{6x} \sin 3x] - 3 \cdot \left[\sin 3x \cdot \frac{d}{dx} (e^{6x}) + e^{6x} \cdot \frac{d}{dx} (\sin 3x) \right] \quad [\text{using (1)}] \\
&= 36e^{6x} \cos 3x - 18e^{6x} \sin 3x - 3 \left[\sin 3x \cdot e^{6x} \cdot 6 + e^{6x} \cdot \cos 3x \cdot 3 \right] \\
&= 36e^{6x} \cos 3x - 18e^{6x} \sin 3x - 18e^{6x} \sin 3x - 9e^{6x} \cos 3x \\
&= 27e^{6x} \cos 3x - 36e^{6x} \sin 3x \\
&= 9e^{6x} (3 \cos 3x - 4 \sin 3x)
\end{aligned}$$

Question 8:

Find the second order derivative of the function $\tan^{-1} x$

Solution:

Let $y = \tan^{-1} x$

Then,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} (\tan^{-1} x) \\
&= \frac{1}{1+x^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d^2 y}{dx^2} &= \frac{d}{dx} \left(\frac{1}{1+x^2} \right) = \frac{d}{dx} (1+x^2)^{-1} \\
&= (-1) \cdot (1+x^2)^{-2} \cdot \frac{d}{dx} (1+x^2) = \frac{-1}{(1+x^2)^2} \times 2x \\
&= \frac{-2x}{(1+x^2)^2}
\end{aligned}$$

Question 9:

Find the second order derivative of the function $\log(\log x)$

Solution:

Consider, $y = \log(\log x)$

Then,

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} [\log(\log x)] \\
 &= \frac{1}{\log x} \cdot \frac{d}{dx} (\log x) \\
 &= \frac{1}{\log x} \cdot \frac{1}{x} = (x \log x)^{-1}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \frac{d^2 y}{dx^2} &= \frac{d}{dx} [(x \log x)^{-1}] \\
 &= (-1) \cdot (x \log x)^{-2} \frac{d}{dx} (x \log x) \\
 &= \frac{-1}{(x \log x)^2} \cdot \left[\log x \cdot \frac{d}{dx} (x) + x \cdot \frac{d}{dx} (\log x) \right] \\
 &= \frac{-1}{(x \log x)^2} \cdot \left[\log x \cdot 1 + x \cdot \frac{1}{x} \right] \\
 &= \frac{-(1 + \log x)}{(x \log x)^2}
 \end{aligned}$$

Question 10:

Find the second order derivative of the function $\sin(\log x)$

Solution:

Let $y = \sin(\log x)$

Then,

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} [\sin x(\log x)] \\
 &= \cos(\log x) \cdot \frac{d}{dx} (\log x) \\
 &= \frac{\cos(\log x)}{x}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left[\frac{\cos(\log x)}{x} \right] \\
 &= \frac{x \cdot \frac{d}{dx} [\cos(\log x)] - \cos(\log x) \cdot \frac{d}{dx} (x)}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \frac{d^2 y}{dx^2} &= \frac{x \left[-\sin(\log x) \cdot \frac{d}{dx}(\log x) \right] - \cos(\log x) \cdot 1}{x^2} \\
 &= \frac{-x \sin(\log x) \cdot \frac{1}{x} - \cos(\log x)}{x^2} \\
 &= \frac{-[\sin(\log x) + \cos(\log x)]}{x^2}
 \end{aligned}$$

Question 11:

If $y = 5 \cos x - 3 \sin x$, prove that $\frac{d^2 y}{dx^2} + y = 0$

Solution:

Given, $y = 5 \cos x - 3 \sin x$

Then,

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx}(5 \cos x) - \frac{d}{dx}(3 \sin x) \\
 &= 5 \frac{d}{dx}(\cos x) - 3 \frac{d}{dx}(\sin x) \\
 &= 5(-\sin x) - 3 \cos x \\
 &= -(5 \sin x + 3 \cos x)
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \frac{d^2 y}{dx^2} &= \frac{d}{dx}[-(5 \sin x + 3 \cos x)] \\
 &= -\left[5 \cdot \frac{d}{dx}(\sin x) + 3 \cdot \frac{d}{dx}(\cos x) \right] \\
 &= -[5 \cos x + 3(-\sin x)] \\
 &= -[5 \cos x - 3 \sin x] \\
 &= -y
 \end{aligned}$$

Thus, $\frac{d^2 y}{dx^2} + y = 0$

Hence proved.

Question 12:

If $y = \cos^{-1} x$, find $\frac{d^2 y}{dx^2}$ in terms of y alone.

Solution:

Given, $y = \cos^{-1} x$

Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\cos^{-1} x) \\ &= \frac{-1}{\sqrt{1-x^2}} \\ &= -(1-x^2)^{-\frac{1}{2}}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d^2 y}{dx^2} &= \frac{d}{dx} \left[-(1-x^2)^{-\frac{1}{2}} \right] \\ &= - \left(-\frac{1}{2} \right) \cdot (1-x^2)^{-\frac{3}{2}} \cdot \frac{d}{dx}(1-x^2) \\ &= \frac{1}{2\sqrt{(1-x^2)^3}} \times (-2x) \\ \frac{d^2 y}{dx^2} &= \frac{-x}{\sqrt{(1-x^2)^3}} \quad \dots(1)\end{aligned}$$

But we need to calculate $\frac{d^2 y}{dx^2}$ in terms of y

$$\Rightarrow y = \cos^{-1} x$$

$$\Rightarrow x = \cos y$$

Putting $x = \cos y$ in equation (1), we get

$$\begin{aligned}
\frac{d^2 y}{dx^2} &= \frac{-\cos y}{\sqrt{(1 - \cos^2 y)^3}} \\
&= \frac{-\cos y}{\sqrt{(\sin^2 y)^3}} \\
&= \frac{-\cos y}{\sin^3 y} \\
&= \frac{-\cos y}{\sin y} \times \frac{1}{\sin^2 y} \\
&= -\cot y \cdot \operatorname{cosec}^2 y
\end{aligned}$$

Question 13:

If $y = 3 \cos(\log x) + 4 \sin(\log x)$, show that $x^2 y_2 + x y_1 + y = 0$

Solution:

Given, $y = 3 \cos(\log x) + 4 \sin(\log x)$

Then,

$$\begin{aligned}
y_1 &= 3 \cdot \frac{d}{dx} [\cos(\log x)] + 4 \cdot \frac{d}{dx} [\sin(\log x)] \\
&= 3 \cdot \left[-\sin(\log x) \cdot \frac{d}{dx} (\log x) \right] + 4 \cdot \left[\cos(\log x) \cdot \frac{d}{dx} (\log x) \right] \\
&= \frac{-3 \sin(\log x)}{x} + \frac{4 \cos(\log x)}{x} \\
&= \frac{4 \cos(\log x) - 3 \sin(\log x)}{x}
\end{aligned}$$

Therefore,

$$\begin{aligned}
y_2 &= \frac{d}{dx} \left(\frac{4 \cos(\log x) - 3 \sin(\log x)}{x} \right) \\
&= \frac{x \{4 \cos(\log x) - 3 \sin(\log x)\}' - \{4 \cos(\log x) - 3 \sin(\log x)\} \{x\}'}{x^2} \\
&= \frac{x [4 \{ \cos(\log x) \}' - 3 \{ \sin(\log x) \}'] - \{4 \cos(\log x) - 3 \sin(\log x)\} \cdot 1}{x^2} \\
&= \frac{x [-4 \sin(\log x) \cdot (\log x)' - 3 \cos(\log x) \cdot (\log x)'] - 4 \cos(\log x) + 3 \sin(\log x)}{x^2} \\
&= \frac{x \left[-4 \sin(\log x) \frac{1}{x} - 3 \cos(\log x) \frac{1}{x} \right] - 4 \cos(\log x) + 3 \sin(\log x)}{x^2} \\
&= \frac{-4 \sin(\log x) - 3 \cos(\log x) - 4 \cos(\log x) + 3 \sin(\log x)}{x^2} \\
&= \frac{-\sin(\log x) - 7 \cos(\log x)}{x^2}
\end{aligned}$$

Thus,

$$\begin{aligned}
x^2 y_2 + x y_1 + y &= \left[x^2 \left(\frac{-\sin(\log x) - 7 \cos(\log x)}{x^2} \right) + x \left(\frac{4 \cos(\log x) - 3 \sin(\log x)}{x} \right) \right] \\
&\quad + 3 \cos(\log x) + 4 \sin(\log x) \\
&= \left[-\sin(\log x) - 7 \cos(\log x) + 4 \cos(\log x) - 3 \sin(\log x) \right] \\
&\quad + 3 \cos(\log x) + 4 \sin(\log x) \\
&= 0
\end{aligned}$$

Hence proved.

Question 14:

If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2 y}{dx^2} - (m+n) \frac{dy}{dx} + mny = 0$.

Solution:

Given, $y = Ae^{mx} + Be^{nx}$

Then,

$$\begin{aligned}
\frac{dy}{dx} &= A \cdot \frac{d}{dx} (e^{mx}) + B \cdot \frac{d}{dx} (e^{nx}) \\
&= A \cdot e^{mx} \cdot \frac{d}{dx} (mx) + B \cdot e^{nx} \cdot \frac{d}{dx} (nx) \\
&= A m e^{mx} + B n e^{nx}
\end{aligned}$$

Therefore,

$$\begin{aligned}
 \frac{d^2 y}{dx^2} &= \frac{d}{dx} (Ame^{mx} + Bne^{nx}) \\
 &= Am \cdot \frac{d}{dx} (e^{mx}) + Bn \cdot \frac{d}{dx} (e^{nx}) \\
 &= Am \cdot e^{mx} \cdot \frac{d}{dx} (mx) + Bn \cdot e^{nx} \cdot \frac{d}{dx} (nx) \\
 &= Am^2 e^{mx} + Bn^2 e^{nx}
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \frac{d^2 y}{dx^2} - (m+n) \frac{dy}{dx} + mny &= Am^2 e^{mx} + Bn^2 e^{nx} - (m+n) \cdot (Ame^{mx} + Bne^{nx}) + mn(Ae^{mx} + Be^{nx}) \\
 &= Am^2 e^{mx} + Bn^2 e^{nx} - Am^2 e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2 e^{nx} + Amne^{mx} + Bmne^{nx} \\
 &= 0
 \end{aligned}$$

Hence proved.

Question 15:

If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2 y}{dx^2} = 49y$

Solution:

Given, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\begin{aligned}
 \frac{dy}{dx} &= 500 \cdot \frac{d}{dx} (e^{7x}) + 600 \cdot \frac{d}{dx} (e^{-7x}) \\
 &= 500 \cdot e^{7x} \cdot \frac{d}{dx} (7x) + 600 \cdot e^{-7x} \cdot \frac{d}{dx} (-7x) \\
 &= 3500e^{7x} - 4200e^{-7x}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \frac{d^2 y}{dx^2} &= 3500e^{7x} \cdot \frac{d}{dx} (e^{7x}) - 4200 \cdot \frac{d}{dx} (e^{-7x}) \\
 &= 3500 \cdot e^{7x} \cdot \frac{d}{dx} (7x) - 4200 \cdot e^{-7x} \cdot \frac{d}{dx} (-7x) \\
 &= 7 \times 3500 \cdot e^{7x} + 7 \times 4200 \cdot e^{-7x} \\
 &= 49 \times 500 \cdot e^{7x} + 49 \times 600 \cdot e^{-7x} \\
 &= 49(500e^{7x} + 600e^{-7x}) \\
 &= 49y
 \end{aligned}$$

Hence proved.

Question 16:

If $e^y(x+1)=1$, show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$

Solution:

Given, $e^y(x+1)=1$

$$\Rightarrow e^y(x+1)=1$$

$$\Rightarrow e^y = \frac{1}{x+1}$$

Taking log on both sides, we get

$$y = \log \frac{1}{(x+1)}$$

Differentiating with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= (x+1) \frac{d}{dx} \left(\frac{1}{x+1} \right) \\ &= (x+1) \cdot \frac{-1}{(x+1)^2} \\ &= \frac{-1}{x+1} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{-1}{x+1} \right) = - \left(\frac{-1}{(x+1)^2} \right) \\ &= \frac{1}{(x+1)^2} = \left(\frac{-1}{x+1} \right)^2 \\ &= \left(\frac{dy}{dx} \right)^2 \end{aligned}$$

Hence proved.

Question 17:

If $y = (\tan^{-1} x)^2$, show that $(x^2+1)^2 y_2 + 2x(x^2+1)y_1 = 2$

Solution:

Given, $y = (\tan^{-1} x)^2$

Then,

$$\Rightarrow y_1 = 2 \tan^{-1} x \frac{d}{dx} (\tan^{-1} x)$$

$$\Rightarrow y_1 = 2 \tan^{-1} x \cdot \left(\frac{1}{1+x^2} \right)$$

$$\Rightarrow (1+x^2) y_1 = 2 \tan^{-1} x$$

Again, differentiating with respect to x , we get

$$\Rightarrow (1+x^2) y_2 + 2xy_1 = 2 \left(\frac{1}{1+x^2} \right)$$

$$\Rightarrow (1+x^2)^2 y_2 + 2x(1+x^2) y_1 = 2$$

Hence proved.

EXERCISE 5.8

Question 1:

Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8, x \in [-4, 2]$

Solution:

Given, $f(x) = x^2 + 2x - 8$, being polynomial function is continuous in $[-4, 2]$ and also differentiable in $(-4, 2)$.

$$\begin{aligned} f(-4) &= (-4)^2 + 2(-4) - 8 \\ &= 16 - 8 - 8 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(2) &= (2)^2 + 2 \times 2 - 8 \\ &= 4 + 4 - 8 \\ &= 0 \end{aligned}$$

Therefore, $f(-4) = f(2) = 0$

The value of $f(x)$ at -4 and 2 coincides.

Rolle's Theorem states that there is a point $c \in (-4, 2)$ such that $f'(c) = 0$

$$f(x) = x^2 + 2x - 8$$

Therefore, $f'(x) = 2x + 2$

Hence,

$$f'(c) = 0$$

$$2c + 2 = 0$$

$$c = -1$$

Thus, $c = -1 \in (-4, 2)$

Hence, Rolle's Theorem is verified.

Question 2:

Examine if Rolle's Theorem is applicable to any of the following functions. Can you say something about the converse of Rolle's Theorem from these examples?

(i) $f(x) = [x]$ for $x \in [5, 9]$

- (ii) $f(x) = [x]$ for $x \in [-2, 2]$
 (iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

Solution:

By Rolle's Theorem, $f: [a, b] \rightarrow \mathbf{R}$,

If

- (a) f is continuous on $[a, b]$
 (b) f is continuous on (a, b)
 (c) $f(a) = f(b)$

Then, there exists some $c \in (a, b)$ such that $f'(c) = 0$

Thus, Rolle's Theorem is not applicable to those functions that do not satisfy any of three conditions of the hypothesis.

- (i) $f(x) = [x]$ for $x \in [5, 9]$

Since, the given function $f(x)$ is not continuous at every integral point.

In general, $f(x)$ is not continuous at $x = 5$ and $x = 9$

Therefore, $f(x)$ is not continuous in $[5, 9]$

Also, $f(5) = [5] = 5$ and $f(9) = [9] = 9$

Thus, $f(5) \neq f(9)$

The differentiability of f in $(5, 9)$ is checked as follows.

Let n be an integer such that $n \in (5, 9)$

The LHD of f at $x = n$ is

$$\lim_{h \rightarrow 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^-} \frac{n-1-n}{h} = \lim_{h \rightarrow 0^-} \frac{-1}{h} = \infty$$

The RHD of f at $x = n$ is

$$\lim_{h \rightarrow 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^+} \frac{n-n}{h} = \lim_{h \rightarrow 0^+} 0 = 0$$

Since LHD and RHD of f at $x = n$ are not equal, f is not differentiable at $x = n$

Therefore, f is not differentiable in $(5, 9)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Thus, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$.

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

Since, the given function $f(x)$ is not continuous at every integral point.

In general, $f(x)$ is not continuous at $x = -2$ and $x = 2$

Therefore, $f(x)$ is not continuous in $[-2, 2]$

Also, $f(-2) = [-2] = -2$ and $f(2) = [2] = 2$

Thus, $f(-2) \neq f(2)$

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$

The LHD of f at $x = n$ is

$$\lim_{h \rightarrow 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^-} \frac{n-1-n}{h} = \lim_{h \rightarrow 0^-} \frac{-1}{h} = \infty$$

The RHD of f at $x = n$ is

$$\lim_{h \rightarrow 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^+} \frac{n-1-n}{h} = \lim_{h \rightarrow 0^+} 0 = 0$$

Since LHD and RHD of f at $x = n$ are not equal, f is not differentiable at $x = n$

Therefore, f is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Thus, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

Since, f being a polynomial function is continuous in $[1, 2]$ and is differentiable in $(1, 2)$

Thus,

$$f(1) = (1)^2 - 1 = 0$$

$$f(2) = (2)^2 - 1 = 3$$

Therefore, $f(1) \neq f(2)$

Since, f does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Question 3:

If $f: [-5, 5] \rightarrow \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.

Solution:

Given, $f: [-5, 5] \rightarrow \mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(i) f is continuous on $[-5, 5]$

(ii) f is continuous on $(-5, 5)$

Thus, by the Mean Value Theorem, there exists $c \in (-5, 5)$ such that

$$\Rightarrow f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

$$\Rightarrow 10f'(c) = f(5) - f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

Therefore, $f'(c) \neq 0$

Thus,

$$\Rightarrow 10f'(c) \neq 0$$

$$\Rightarrow f(5) - f(-5) \neq 0$$

$$\Rightarrow f(5) \neq f(-5)$$

Hence proved.

Question 4:

Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.

Solution:

Given, $f(x) = x^2 - 4x - 3$

f , being a polynomial function, is continuous in $[1, 4]$ and is differentiable in $(1, 4)$, whose derivative is $2x - 4$.

Thus,

$$f(1) = 1^2 - 4 \times 1 - 3 = -6$$

$$f(4) = 4^2 - 4 \times 4 - 3 = -3$$

Therefore,

$$\begin{aligned}\frac{f(b) - f(a)}{b - a} &= \frac{f(4) - f(1)}{4 - 1} \\ &= \frac{-3 - (-6)}{3} \\ &= \frac{3}{3} \\ &= 1\end{aligned}$$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c) = 1$

Hence,

$$\Rightarrow f'(c) = 1$$

$$\Rightarrow 2c - 4 = 1$$

$$\Rightarrow c = \frac{5}{2} \quad \left[\text{where } c = \frac{5}{2} \in (1, 4) \right]$$

Thus, mean value theorem is verified for the given function.

Question 5:

Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$ where $a = 1$ and $b = 3$.

Find all $c \in (1, 3)$ for which $f'(c) = 0$.

Solution:

Given, f is $f(x) = x^3 - 5x^2 - 3x$

f , being a polynomial function, is continuous in $[1, 3]$ and is differentiable in $(1, 3)$, whose derivative is $3x^2 - 10x - 3$

Thus,

$$f(1) = 1^3 - 5 \times 1^2 - 3 \times 1 = -7$$

$$f(3) = 3^3 - 5 \times 3^2 - 3 \times 3 = -27$$

Therefore,

$$\begin{aligned}\frac{f(b)-f(a)}{b-a} &= \frac{f(3)-f(1)}{3-1} \\ &= \frac{-27-(-7)}{3-1} \\ &= -10\end{aligned}$$

Mean Value Theorem states that there exists a point $c \in (1, 3)$ such that $f'(c) = -10$
Hence,

$$\begin{aligned}\Rightarrow f'(c) &= -10 \\ \Rightarrow 3c^2 - 10c - 3 &= -10 \\ \Rightarrow 3c^2 - 10c + 7 &= 0 \\ \Rightarrow 3c^2 - 3c - 7c + 7 &= 0 \\ \Rightarrow 3c(c-1) - 7(c-1) &= 0 \\ \Rightarrow (c-1)(3c-7) &= 0 \\ \Rightarrow c = 1, \frac{7}{3} &\quad \left[\text{where } c = \frac{7}{3} \in (1, 3) \right]\end{aligned}$$

Thus, Mean Value Theorem is verified for the given function and $c = \frac{7}{3} \in (1, 3)$ is the only point for which $f'(c) = 0$.

Question 6:

Examine the applicability of Mean Value Theorem for all three functions given

- (i) $f(x) = [x]$ for $x \in [5, 9]$
- (ii) $f(x) = [x]$ for $x \in [-2, 2]$
- (iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

Solution:

Mean Value Theorem states that for a function $f: [a, b] \rightarrow \mathbf{R}$, if

- (a) f is continuous on $[a, b]$
- (b) f is continuous on (a, b)

Then there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Thus, Mean Value Theorem is not applicable to those functions that do not satisfy any of three conditions of the hypothesis.

- (i) $f(x) = [x]$ for $x \in [5, 9]$

Since, the given function $f(x)$ is not continuous at every integral point.

In general, $f(x)$ is not continuous at $x = 5$ and $x = 9$

Therefore, $f(x)$ is not continuous in $[5, 9]$

The differentiability of f in $(5, 9)$ is checked as follows.

Let n be an integer such that $n \in (5, 9)$

The LHD of f at $x = n$ is

$$\lim_{h \rightarrow 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^-} \frac{n-1-n}{h} = \lim_{h \rightarrow 0^-} \frac{-1}{h} = \infty$$

The RHD of f at $x = n$ is

$$\lim_{h \rightarrow 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^+} \frac{n-n}{h} = \lim_{h \rightarrow 0^+} 0 = 0$$

Since LHD and RHD of f at $x = n$ are not equal, f is not differentiable at $x = n$

Therefore, f is not differentiable in $(5, 9)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Thus, Mean Value Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

Since, the given function $f(x)$ is not continuous at every integral point.

In general, $f(x)$ is not continuous at $x = -2$ and $x = 2$

Therefore, $f(x)$ is not continuous in $[-2, 2]$

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$

The LHD of f at $x = n$ is

$$\lim_{h \rightarrow 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \rightarrow 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^-} \frac{n-1-n}{h} = \lim_{h \rightarrow 0^-} \frac{-1}{h} = \infty$$

The RHD of f at $x = n$ is

$$\lim_{h \rightarrow 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{x \rightarrow 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \rightarrow 0^+} \frac{n-n}{h} = \lim_{h \rightarrow 0^+} 0 = 0$$

Since LHD and RHD of f at $x = n$ are not equal, f is not differentiable at $x = n$

Therefore, f is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Thus, Mean Value Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

Since, f being a polynomial function is continuous in $[1, 2]$ and is differentiable in $(1, 2)$

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

It can be proved as follows.

We have, $f(x) = x^2 - 1$

Then,

$$f(1) = (1)^2 - 1 = 0,$$

$$f(2) = (2)^2 - 1 = 3$$

Therefore,

$$\frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(1)}{2 - 1} = \frac{3 - 0}{1} = 3$$

Hence, $f'(x) = 2x$

Thus,

$$\Rightarrow f'(c) = 3$$

$$\Rightarrow 2c = 3$$

$$\Rightarrow c = \frac{3}{2}$$

$$\Rightarrow c = 1.5 \quad [\text{where } 1.5 \in [1, 2]]$$

MISCELLANEOUS EXERCISE

Question 1:

Differentiate with respect to x the function $(3x^2 - 9x + 5)^9$.

Solution:

$$\text{Let } y = (3x^2 - 9x + 5)^9$$

Using chain rule, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} (3x^2 - 9x + 5)^9 \\ &= 9(3x^2 - 9x + 5)^8 \cdot \frac{d}{dx} (3x^2 - 9x + 5) \\ &= 9(3x^2 - 9x + 5)^8 \cdot (6x - 9) \\ &= 9(3x^2 - 9x + 5)^8 \cdot 3(2x - 3) \\ &= 27(3x^2 - 9x + 5)^8 (2x - 3)\end{aligned}$$

Question 2:

Differentiate with respect to x the function $\sin^3 x + \cos^6 x$.

Solution:

$$\text{Let } y = \sin^3 x + \cos^6 x$$

Using chain rule, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} (\sin^3 x) + \frac{d}{dx} (\cos^6 x) \\ &= 3\sin^2 x \cdot \frac{d}{dx} (\sin x) + 6\cos^5 x \cdot \frac{d}{dx} (\cos x) \\ &= 3\sin^2 x \cdot \cos x + 6\cos^5 x \cdot (-\sin x) \\ &= 3\sin x \cos x (\sin x - 2\cos^4 x)\end{aligned}$$

Question 3:

Differentiate with respect to x the function $(5x)^{3\cos 2x}$.

Solution:

Let $y = (5x)^{3\cos 2x}$

Taking logarithm on both the sides, we obtain

$$\log y = 3 \cos 2x \log 5x$$

Differentiating both sides with respect to x , we get

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= 3 \left[\log 5x \cdot \frac{d}{dx}(\cos 2x) + \cos 2x \cdot \frac{d}{dx}(\log 5x) \right] \\ \frac{dy}{dx} &= 3y \left[\log 5x \cdot (-\sin 2x) \cdot \frac{d}{dx}(2x) + \cos 2x \cdot \frac{1}{5x} \cdot \frac{d}{dx}(5x) \right] \\ &= 3y \left[-2 \sin 2x \cdot \log 5x + \frac{\cos 2x}{x} \right] \\ &= y \left[\frac{3 \cos 2x}{x} - 6 \sin 2x \log 5x \right] \\ &= (5x)^{3 \cos 2x} \left[\frac{3 \cos 2x}{x} - 6 \sin 2x \log 5x \right] \end{aligned}$$

Question 4:

Differentiate with respect to x the function $\sin^{-1}(x\sqrt{x})$, $0 \leq x \leq 1$.

Solution:

Let $y = \sin^{-1}(x\sqrt{x})$

Using chain rule, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \sin^{-1}(x\sqrt{x}) \\
&= \frac{1}{\sqrt{1-(x\sqrt{x})^2}} \times \frac{d}{dx}(x\sqrt{x}) \\
&= \frac{1}{\sqrt{1-x^3}} \cdot \frac{d}{dx}\left(x^{\frac{3}{2}}\right) \\
&= \frac{1}{\sqrt{1-x^3}} \cdot \frac{3}{2} x^{\frac{1}{2}} \\
&= \frac{3\sqrt{x}}{2\sqrt{1-x^3}} \\
&= \frac{3}{2} \sqrt{\frac{x}{1-x^3}}
\end{aligned}$$

Question 5:

Differentiate with respect to x the function $\frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2$.

Solution:

Let $y = \frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}}$

Using quotient rule, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\sqrt{2x+7} \cdot \frac{d}{dx} \left(\cos^{-1} \frac{x}{2} \right) - \left(\cos^{-1} \frac{x}{2} \right) \cdot \frac{d}{dx} (\sqrt{2x+7})}{(\sqrt{2x+7})^2} \\
&= \frac{\sqrt{2x+7} \left[\frac{-1}{\sqrt{1 - \left(\frac{x}{2} \right)^2}} \cdot \frac{d}{dx} \left(\frac{x}{2} \right) \right] - \left(\cos^{-1} \frac{x}{2} \right) \cdot \frac{1}{2\sqrt{2x+7}} \cdot \frac{d}{dx} (2x+7)}{2x+7} \\
&= \frac{\sqrt{2x+7} \cdot \frac{-1}{\sqrt{4-x^2}} - \left(\cos^{-1} \frac{x}{2} \right) \cdot \frac{2}{2\sqrt{2x+7}}}{2x+7} \\
&= \frac{-\sqrt{2x+7}}{(\sqrt{4-x^2}) \cdot (2x+7)} - \frac{\cos^{-1} \frac{x}{2}}{(\sqrt{2x+7})(2x+7)} \\
&= - \left[\frac{1}{\sqrt{4-x^2} \sqrt{2x+7}} + \frac{\cos^{-1} \frac{x}{2}}{(2x+7)^{\frac{3}{2}}} \right]
\end{aligned}$$

Question 6:

Differentiate with respect to x the function $\cot^{-1} \left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right], 0 < x < \frac{\pi}{2}$.

Solution:

Let $y = \cot^{-1} \left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right] \quad \dots(1)$

Then,

$$\begin{aligned}
\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} &= \frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})^2}{(\sqrt{1+\sin x} - \sqrt{1-\sin x})(\sqrt{1+\sin x} + \sqrt{1-\sin x})} \\
&= \frac{(1+\sin x) + (1-\sin x) + 2\sqrt{(1+\sin x)(1-\sin x)}}{(1+\sin x) - (1-\sin x)} \\
&= \frac{2 + 2\sqrt{1-\sin^2 x}}{2\sin x} = \frac{1 + \cos x}{\sin x} \\
&= \frac{1 + 2\cos^2 \frac{x}{2} - 1}{2\sin \frac{x}{2} \cos \frac{x}{2}} = \frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}} \\
&= \cot \frac{x}{2}
\end{aligned}$$

Therefore, equation (1) becomes,

$$y = \cot^{-1} \left(\cot \frac{x}{2} \right)$$

$$\Rightarrow y = \frac{x}{2}$$

Thus,

$$\begin{aligned}
\Rightarrow \frac{dy}{dx} &= \frac{1}{2} \frac{d}{dx} (x) \\
&= \frac{1}{2}
\end{aligned}$$

Question 7:

Differentiate with respect to x the function $(\log x)^{\log x}$, $x > 1$.

Solution:

Let $y = (\log x)^{\log x}$

Taking logarithm on both the sides, we obtain

$$\log y = \log x \cdot \log(\log x)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
 \Rightarrow \frac{1}{y} \frac{dy}{dx} &= \frac{d}{dx} [\log x \cdot \log(\log x)] \\
 \Rightarrow \frac{1}{y} \frac{dy}{dx} &= \log(\log x) \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx} [\log(\log x)] \\
 \Rightarrow \frac{dy}{dx} &= y \left[\log(\log x) \cdot \frac{1}{x} + \log x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) \right] \\
 \Rightarrow \frac{dy}{dx} &= y \left[\frac{1}{x} \cdot \log(\log x) + \frac{1}{x} \right] \\
 \Rightarrow \frac{dy}{dx} &= (\log x)^{\log x} \left[\frac{1}{x} + \frac{\log(\log x)}{x} \right]
 \end{aligned}$$

Question 8:

Differentiate with respect to x the function $\cos(a \cos x + b \sin x)$, for some constant a and b .

Solution:

Let $y = \cos(a \cos x + b \sin x)$

Using chain rule, we get

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \cos(a \cos x + b \sin x) \\
 &= -\sin(a \cos x + b \sin x) \cdot \frac{d}{dx}(a \cos x + b \sin x) \\
 &= -\sin(a \cos x + b \sin x) \cdot [a(-\sin x) + b \cos x] \\
 &= (a \sin x - b \cos x) \cdot \sin(a \cos x + b \sin x)
 \end{aligned}$$

Question 9:

Differentiate with respect to x the function $(\sin x - \cos x)^{(\sin x - \cos x)}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$

Solution:

Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

Taking log on both the sides, we obtain

$$\begin{aligned}\log y &= \log \left[(\sin x - \cos x)^{(\sin x - \cos x)} \right] \\ &= (\sin x - \cos x) \log (\sin x - \cos x)\end{aligned}$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{d}{dx} [(\sin x - \cos x) \log (\sin x - \cos x)] \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= \log (\sin x - \cos x) \cdot \frac{d}{dx} (\sin x - \cos x) + (\sin x - \cos x) \cdot \frac{d}{dx} \log (\sin x - \cos x) \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= \log (\sin x - \cos x) \cdot (\cos x + \sin x) + (\sin x - \cos x) \cdot \frac{1}{(\sin x - \cos x)} \cdot \frac{d}{dx} (\sin x - \cos x) \\ \Rightarrow \frac{dy}{dx} &= (\sin x - \cos x)^{(\sin x - \cos x)} [(\cos x + \sin x) \cdot \log (\sin x - \cos x) + (\cos x + \sin x)] \\ \Rightarrow \frac{dy}{dx} &= (\sin x - \cos x)^{(\sin x - \cos x)} (\cos x + \sin x) [1 + \log (\sin x - \cos x)]\end{aligned}$$

Question 10:

Differentiate with respect to x the function $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$.

Solution:

Let $y = x^x + x^a + a^x + a^a$

Also, let $x^x = u$, $x^a = v$, $a^x = w$ and $a^a = s$

Therefore,

$$\begin{aligned}\Rightarrow y &= u + v + w + s \\ \Rightarrow \frac{dy}{dx} &= \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx} \quad \dots(1)\end{aligned}$$

Now, $u = x^x$

Taking logarithm on both the sides, we obtain

$$\begin{aligned}\Rightarrow \log u &= \log x^x \\ \Rightarrow \log u &= x \log x\end{aligned}$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\frac{1}{u} \frac{du}{dx} &= \log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x) \\ \frac{du}{dx} &= u \left[\log x \cdot 1 + x \cdot \frac{1}{x} \right] \\ &= x^x [\log x + 1] = x^x (1 + \log x) \quad \dots(2)\end{aligned}$$

Now, $v = x^a$

Hence,

$$\begin{aligned}\frac{dv}{dx} &= \frac{d}{dx}(x^a) \\ &= ax^{a-1} \quad \dots(3)\end{aligned}$$

Now, $w = a^x$

Taking logarithm on both the sides, we obtain

$$\begin{aligned}\Rightarrow \log w &= \log a^x \\ \Rightarrow \log w &= x \log a\end{aligned}$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\frac{1}{w} \frac{dw}{dx} &= \log a \cdot \frac{d}{dx}(x) \\ \frac{dw}{dx} &= w \log a \\ &= a^x \log a \quad \dots(4)\end{aligned}$$

Now, $s = a^a$

Since a is constant, a^a is also a constant.

Hence,

$$\frac{ds}{dx} = 0 \quad \dots(5)$$

From (1), (2), (3), (4) and (5), we obtain

$$\begin{aligned}\frac{dy}{dx} &= x^x (1 + \log x) + ax^{a-1} + a^x \log a + 0 \\ &= x^x (1 + \log x) + ax^{a-1} + a^x \log a\end{aligned}$$

Question 11:

Differentiate with respect to x the function $x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$.

Solution:

Let $y = x^{x^2-3} + (x-3)^{x^2}$

Also, let $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$

Therefore,

$$y = u + v$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

Now, $u = x^{x^2-3}$

Taking logarithm on both the sides, we obtain

$$\begin{aligned} \log u &= \log(x^{x^2-3}) \\ &= (x^2-3) \log x \end{aligned}$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned} \frac{1}{u} \frac{du}{dx} &= \log x \cdot \frac{d}{dx}(x^2-3) + (x^2-3) \cdot \frac{d}{dx}(\log x) \\ \Rightarrow \frac{1}{u} \frac{du}{dx} &= \log x \cdot 2x + (x^2-3) \cdot \frac{1}{x} \\ \Rightarrow \frac{du}{dx} &= x^{x^2-3} \left[\frac{x^2-3}{x} + 2x \log x \right] \quad \dots(2) \end{aligned}$$

Now, $v = (x-3)^{x^2}$

Taking logarithm on both the sides, we obtain

$$\begin{aligned} \log v &= \log(x-3)^{x^2} \\ &= x^2 \log(x-3) \end{aligned}$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned} \frac{1}{v} \frac{dv}{dx} &= \log(x-3) \cdot \frac{d}{dx}(x^2) + (x^2) \cdot \frac{d}{dx}[\log(x-3)] \\ \Rightarrow \frac{1}{v} \frac{dv}{dx} &= \log(x-3) \cdot 2x + x^2 \cdot \frac{1}{x-3} \cdot \frac{d}{dx}(x-3) \\ \Rightarrow \frac{dv}{dx} &= v \left[2x \log(x-3) + \frac{x^2}{x-3} \cdot 1 \right] \\ \Rightarrow \frac{dv}{dx} &= (x-3)^{x^2} \left[\frac{x^2}{x-3} + 2x \log(x-3) \right] \quad \dots(3) \end{aligned}$$

From (1), (2), and (3), we obtain

$$\frac{dy}{dx} = x^{x^2-3} \left[\frac{x^2-3}{x} + 2x \log x \right] + (x-3)^{x^2} \left[\frac{x^2}{x-3} + 2x \log(x-3) \right]$$

Question 12:

Find $\frac{dy}{dx}$, if $y = 12(1 - \cos t)$, $x = 10(t - \sin t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

Solution:

The given function is $y = 12(1 - \cos t)$, $x = 10(t - \sin t)$

Hence,

$$\frac{dx}{dt} = \frac{d}{dt} [10(t - \sin t)]$$

$$= 10 \cdot \frac{d}{dt} (t - \sin t)$$

$$= 10(1 - \cos t)$$

$$\frac{dy}{dt} = \frac{d}{dt} [12(1 - \cos t)]$$

$$= 12 \cdot \frac{d}{dt} (1 - \cos t)$$

$$= 12 \cdot [0 - (-\sin t)]$$

$$= 12 \sin t$$

Therefore,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12 \sin t}{10(1 - \cos t)}$$

$$= \frac{12.2 \sin \frac{t}{2} \cdot \cos \frac{t}{2}}{10.2 \sin^2 \frac{t}{2}}$$

$$= \frac{6}{5} \cot \frac{t}{2}$$

Question 13:

Find $\frac{dy}{dx}$, if $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}$, $-1 \leq x \leq 1$.

Solution:

The given function is $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}$

Hence,

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx} \left[\sin^{-1} x + \sin^{-1} \sqrt{1-x^2} \right] \\
 &= \frac{d}{dx} (\sin^{-1} x) + \frac{d}{dx} (\sin^{-1} \sqrt{1-x^2}) \\
 \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{d}{dx} (\sqrt{1-x^2}) \\
 &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{x} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx} (1-x^2) \\
 &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{2x\sqrt{1-x^2}} (-2x) \\
 &= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \\
 &= 0
 \end{aligned}$$

Question 14:

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ for $-1 < x < 1$, prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

Solution:

The given function is $x\sqrt{1+y} + y\sqrt{1+x} = 0$
 $\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides, we obtain

$$\begin{aligned}
 x^2(1+y) &= y^2(1+x) \\
 \Rightarrow x^2 + x^2y &= y^2 + xy^2 \\
 \Rightarrow x^2 - y^2 &= xy^2 - x^2y \\
 \Rightarrow x^2 - y^2 &= xy(y-x) \\
 \Rightarrow (x+y)(x-y) &= xy(y-x) \\
 \Rightarrow x+y &= -xy \\
 \Rightarrow (1+x)y &= -x \\
 \Rightarrow y &= \frac{-x}{(1+x)}
 \end{aligned}$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
 \frac{dy}{dx} &= - \left[\frac{(1+x) \frac{d}{dx}(x) - (x) \cdot \frac{d}{dx}(1+x)}{(1+x)^2} \right] \\
 &= - \frac{(1+x) - x}{(1+x)^2} \\
 &= - \frac{1}{(1+x)^2}
 \end{aligned}$$

Hence proved.

Question 15:

If $(x-a)^2 + (y-b)^2 = c^2$ for $c > 0$, prove that $\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b .

Solution:

The given function is $(x-a)^2 + (y-b)^2 = c^2$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}
 \frac{d}{dx}[(x-a)^2] + \frac{d}{dx}[(y-b)^2] &= \frac{d}{dx}(c^2) \\
 \Rightarrow 2(x-a) \cdot \frac{d}{dx}(x-a) + 2(y-b) \cdot \frac{d}{dx}(y-b) &= 0 \\
 \Rightarrow 2(x-a) \cdot 1 + 2(y-b) \cdot \frac{dy}{dx} &= 0 \\
 \Rightarrow \frac{dy}{dx} &= \frac{-(x-a)}{y-b} \quad \dots(1)
 \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx} \left[\frac{-(x-a)}{y-b} \right] \\
&= - \left[\frac{(y-b) \cdot \frac{d}{dx}(x-a) - (x-a) \cdot \frac{d}{dx}(y-b)}{(y-b)^2} \right] \\
&= - \left[\frac{(y-b) - (x-a) \cdot \frac{dy}{dx}}{(y-b)^2} \right] \\
&= - \left[\frac{(y-b) - (x-a) \cdot \left\{ \frac{-(x-a)}{y-b} \right\}}{(y-b)^2} \right] \quad [\text{Using (1)}] \\
&= - \left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^3} \right]
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} &= \frac{\left[1 + \frac{(x-a)^2}{(y-b)^2} \right]^{\frac{3}{2}}}{- \left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^3} \right]} \\
&= \frac{\left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^2} \right]^{\frac{3}{2}}}{- \left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^3} \right]} \\
&= \frac{\left[\frac{c^2}{(y-b)^2} \right]^{\frac{3}{2}}}{- \frac{c^2}{(y-b)^3}} \\
&= \frac{\frac{c^3}{(y-b)^3}}{- \frac{c^2}{(y-b)^3}} \\
&= -c
\end{aligned}$$

$-c$ is a constant and is independent of a and b .

Hence proved.

Question 16:

If $\cos y = x \cos(a + y)$ with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a + y)}{\sin a}$.

Solution:

The given function is $\cos y = x \cos(a + y)$

Therefore,

$$\begin{aligned}\Rightarrow \frac{d}{dx}[\cos y] &= \frac{d}{dx}[x \cos(a + y)] \\ \Rightarrow -\sin y \frac{dy}{dx} &= \cos(a + y) \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}[\cos(a + y)] \\ \Rightarrow -\sin y \frac{dy}{dx} &= \cos(a + y) + x \cdot [-\sin(a + y)] \frac{dy}{dx} \\ \Rightarrow [x \sin(a + y) - \sin y] \frac{dy}{dx} &= \cos(a + y) \quad \dots(1)\end{aligned}$$

$$\text{Since, } \cos y = x \cos(a + y) \Rightarrow x = \frac{\cos y}{\cos(a + y)}$$

Then, equation (1) becomes,

$$\begin{aligned}\left[\frac{\cos y}{\cos(a + y)} \cdot \sin(a + y) - \sin y \right] \frac{dy}{dx} &= \cos(a + y) \\ \Rightarrow [\cos y \cdot \sin(a + y) - \sin y \cdot \cos(a + y)] \cdot \frac{dy}{dx} &= \cos^2(a + y) \\ \Rightarrow \sin(a + y - y) \frac{dy}{dx} &= \cos^2(a + y) \\ \Rightarrow \frac{dy}{dx} &= \frac{\cos^2(a + y)}{\sin a}\end{aligned}$$

Hence proved.

Question 17:

If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$, find $\frac{d^2 y}{dx^2}$.

Solution:

The given function is $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$

Therefore,

$$\begin{aligned}\frac{dx}{dt} &= a \cdot \frac{d}{dt}(\cos t + t \sin t) \\ &= a \left[-\sin t + \sin t + t \cdot \frac{d}{dt}(t) + t \cdot \frac{d}{dt}(\sin t) \right] \\ &= a[-\sin t + \sin t + t \cos t] \\ &= at \cos t\end{aligned}$$

$$\begin{aligned}\frac{dy}{dt} &= a \cdot \frac{d}{dt}(\sin t - t \cos t) \\ &= a \left[\cos t - \left\{ \cos t \cdot \frac{d}{dt}(t) + t \cdot \frac{d}{dt}(\cos t) \right\} \right] \\ &= a[\cos t - \{\cos t - t \sin t\}] \\ &= at \sin t\end{aligned}$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at \sin t}{at \cos t} = \tan t$$

$$\begin{aligned}\frac{d^2 y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx}(\tan t) = \sec^2 t \cdot \frac{dt}{dx} \\ &= \sec^2 t \cdot \frac{1}{at \cos t} \quad \left[\frac{dx}{dt} = at \cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at \cos t} \right] \\ &= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}\end{aligned}$$

Question 18:

If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x , and find it.

Solution:

It is known that $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Therefore, when $x \geq 0$, $f(x) = |x|^3 = x^3$

In this case, $f'(x) = 3x^2$ and hence, $f''(x) = 6x$

When $x < 0$, $f(x) = |x|^3 = (-x)^3 = -x^3$

In this case, $f'(x) = -3x^2$ and hence, $f''(x) = -6x$

Thus, for $f(x) = |x|^3$, $f''(x)$ exists for all real x and is given by,

$$f''(x) = \begin{cases} 6x, & \text{if } x \geq 0 \\ -6x, & \text{if } x < 0 \end{cases}$$

Question 19:

Using mathematical induction prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n .

Solution:

To prove: $P(n): \frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n .

For $n = 1$,

$$P(1): \frac{d}{dx}(x) = 1 = 1 \cdot x^{1-1}$$

Therefore, $P(n)$ is true for $n = 1$.

Let $P(k)$ is true for some positive integer k .

$$\text{That is, } P(k): \frac{d}{dx}(x^k) = kx^{k-1}$$

It has to be proved that $P(k+1)$ is also true.

Consider

$$\begin{aligned} \frac{d}{dx}(x^{k+1}) &= \frac{d}{dx}(x \cdot x^k) \\ &= x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x^k) && [\text{By applying product rule}] \\ &= x^k \cdot 1 + x \cdot k \cdot x^{k-1} \\ \frac{d}{dx}(x^{k+1}) &= x^k + kx^k \\ &= (k+1) \cdot x^k \\ &= (k+1) \cdot x^{(k+1)-1} \end{aligned}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for every positive integer n .

Hence, proved.

Question 20:

Using the fact that $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and the differentiation, obtain the sum formula for cosines.

Solution:

Given, $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Differentiating both sides with respect to x , we obtain

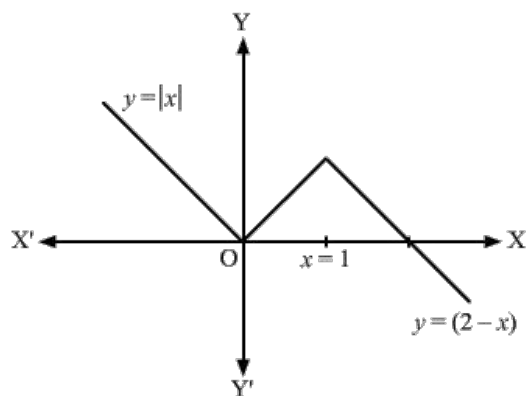
$$\begin{aligned} \frac{d}{dx}[\sin(A+B)] &= \frac{d}{dx}(\sin A \cos B) + \frac{d}{dx}(\cos A \sin B) \\ \Rightarrow \cos(A+B) \cdot \frac{d}{dx}(A+B) &= \cos B \cdot \frac{d}{dx}(\sin A) + \sin A \cdot \frac{d}{dx}(\cos B) + \sin B \cdot \frac{d}{dx}(\cos A) + \cos A \cdot \frac{d}{dx}(\sin B) \\ \Rightarrow \cos(A+B) \cdot \frac{d}{dx}(A+B) &= \cos B \cdot \cos A \frac{dA}{dx} + \sin A (-\sin B) \frac{dB}{dx} + \sin B (-\sin A) \frac{dA}{dx} + \cos A \cos B \frac{dB}{dx} \\ \Rightarrow \cos(A+B) \cdot \left[\frac{dA}{dx} + \frac{dB}{dx} \right] &= (\cos A \cos B - \sin A \sin B) \cdot \left[\frac{dA}{dx} + \frac{dB}{dx} \right] \\ \Rightarrow \cos(A+B) &= \cos A \cos B - \sin A \sin B \end{aligned}$$

Question 21:

Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer?

Solution:

Consider, $y = \begin{cases} |x| & -\infty < x \leq 1 \\ 2-x & 1 \leq x \leq \infty \end{cases}$



It can be seen from the above graph that the given function is continuous everywhere but not differentiable at exactly two points which are 0 and 1.

Question 22:

If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Solution:

Given, $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

$$\Rightarrow y = (mc - nb)f(x) - (lc - na)g(x) + (lb - ma)h(x)$$

Then,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}[(mc - nb)f(x)] - \frac{d}{dx}[(lc - na)g(x)] + \frac{d}{dx}[(lb - ma)h(x)] \\ &= (mc - nb)f'(x) - (lc - na)g'(x) + (lb - ma)h'(x) \\ &= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} \end{aligned}$$

Thus, $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$ proved.

Question 23:

If $y = e^{a \cos^{-1} x}$, $-1 \leq x \leq 1$, show that $(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$

Solution:

The given function is $y = e^{a \cos^{-1} x}$

Taking logarithm on both the sides, we obtain

$$\Rightarrow \log y = a \cos^{-1} x \log e$$

$$\Rightarrow \log y = a \cos^{-1} x$$

Differentiating both sides with respect to x , we obtain

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = a \cdot \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}}$$

By squaring both the sides, we obtain

$$\Rightarrow \left(\frac{dy}{dx} \right)^2 = \frac{a^2 y^2}{1-x^2}$$

$$\Rightarrow (1-x^2) \left(\frac{dy}{dx} \right)^2 = a^2 y^2$$

Again, differentiating both sides with respect to x , we obtain

$$\Rightarrow \left(\frac{dy}{dx} \right)^2 \frac{d}{dx} (1-x^2) + (1-x^2) \times \frac{d}{dx} \left[\left(\frac{dy}{dx} \right)^2 \right] = a^2 \frac{d}{dx} (y^2)$$

$$\Rightarrow \left(\frac{dy}{dx} \right)^2 (-2x) + (1-x^2) \times 2 \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2} = a^2 \cdot 2y \cdot \frac{dy}{dx}$$

$$\Rightarrow -x \frac{dy}{dx} + (1-x^2) \frac{d^2 y}{dx^2} = a^2 \cdot y \quad \left[\frac{dy}{dx} \neq 0 \right]$$

$$\Rightarrow (1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$$

Hence proved.