Chapter 7 Integrals

EXERCISE 7.1

Question 1:
Find an anti-derivative (or integral) of the following functions by the method of inspection, $\sin 2x$.

Solution:

$$\Rightarrow \frac{d}{dx}(\cos 2x) = -2\sin 2x$$

$$\Rightarrow \sin 2x = -\frac{1}{2}\frac{d}{dx}(\cos 2x)$$

$$\Rightarrow \sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

Thus, the anti-derivative of sin $2x$ is $-\frac{1}{2}\cos 2x$

Question 2:
Find an anti-derivative (or integral) of the following functions by the method of inspection, $\cos 3x$.

Solution:

$$\Rightarrow \frac{d}{dx}(\sin 3x) = 3\cos 3x$$

$$\Rightarrow \cos 3x = \frac{1}{3}\frac{d}{dx}(\sin 3x)$$

$$\Rightarrow \cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

Thus, the anti-derivative of $\cos 3x$ is $\frac{1}{3}\sin 3x$

Question 3:
Find an anti-derivative (or integral) of the following functions by the method of inspection, e^{2x}.

Solution:

$$\Rightarrow \frac{d}{dx}\left(e^{2x}\right)=2e^{2x}$$

$$\Rightarrow e^{2x}=\frac{1}{2}\frac{d}{dx}\left(e^{2x}\right)$$

$$\Rightarrow e^{2x}=\frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Thus, the anti-derivative of e^{2x} is $\frac{1}{2}e^{2x}$.

Question 4:

Find an anti-derivative (or integral) of the following functions by the method of inspection, $(ax+b)^2$.

Solution:

$$\frac{d}{dx}(ax+b)^3=3a(ax+b)^2$$

$$\Rightarrow (ax+b)^2=\frac{1}{3a}\frac{d}{dx}(ax+b)^3$$

$$\Rightarrow (ax+b)^2=\frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$

Thus, the anti-derivative $(ax+b)^2$ of is $\frac{1}{3a}(ax+b)^3$

Question 5:

Find an anti-derivative (or integral) of the following functions by the method of inspection, $\sin 2x-4e^{3x}$

Solution:

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x-\frac{4}{3}e^{3x}\right)=\sin 2x-4e^{3x}$$

Thus, the anti-derivative of $\sin 2x-4e^{3x}$ is $\left(-\frac{1}{2}\cos 2x-\frac{4}{3}e^{3x}\right)$

Find the following integrals in Exercises 6 to 20:

Question 6:

$$\int\left(4e^{3x}+1\right)dx$$

Solution:

$$\int\left(4e^{3x}+1\right)dx=4\int e^{3x}dx+\int 1\,dx$$
$$=4\left(\frac{e^{3x}}{3}\right)+x+C$$
$$=\frac{4}{3}e^{3x}+x+C$$

Question 7:

$$\int x^2\left(1-\frac{1}{x^2}\right)dx$$

Solution:

$$\int x^2\left(1-\frac{1}{x^2}\right)dx=\int\left(x^2-1\right)dx$$
$$=\int x^2dx-\int 1\,dx$$
$$=\frac{x^3}{3}-x+C$$

Question 8:

$$\int\left(ax^2+bx+c\right)dx$$

Solution:

$$\int\left(ax^2+bx+c\right)dx=a\int x^2dx+b\int xdx+c\int 1dx$$
$$=a\left(\frac{x^3}{3}\right)+b\left(\frac{x^2}{2}\right)+cx+C$$
$$=\frac{ax^3}{3}+\frac{bx^2}{2}+cx+C$$

Question 9:

$$\int\left(2x^2+e^x\right)dx$$

Solution:

$$\int(2x^2+e^x)\,dx=2\int x^2dx+\int e^xdx$$
$$=2\left(\frac{x^3}{3}\right)+e^x+C$$
$$=\frac{2}{3}x^3+e^x+C$$

Question 10:

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx$$

Solution:

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx=\int\left(x+\frac{1}{x}-2\right)$$
$$=\int xdx+\int\frac{1}{x}dx-2\int 1dx$$
$$=\frac{x^2}{2}+\log|x|-2x+C$$

Question 11:

$$\int\frac{x^3+5x^2-4}{x^2}dx$$

Solution:

$$\int\frac{x^3+5x^2-4}{x^2}dx=\int\left(x+5-4x^{-2}\right)dx$$
$$=\int xdx+5\int 1dx-4\int x^{-2}dx$$
$$=\frac{x^2}{2}+5x-4\left(\frac{x^{-1}}{-1}\right)+C$$
$$=\frac{x^2}{2}+5x+\frac{4}{x}+C$$

Question 12:

$$\int\frac{x^3+3x+4}{\sqrt{x}}dx$$

Solution:

$$\int \frac{x^3+3x+4}{\sqrt{x}}dx = \int \left(x^{\frac{5}{2}} + 3x^{\frac{1}{2}} + 4x^{-\frac{1}{2}} \right) dx$$

$$= \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{3\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}} + \frac{4\left(x^{\frac{1}{2}}\right)}{\frac{1}{2}} + C$$

$$= \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + C$$

$$= \frac{2}{7}x^{\frac{7}{2}} + 2x^{\frac{3}{2}} + 8\sqrt{x} + C$$

Question 13:

$$\int \frac{x^3 - x^2 + x - 1}{x-1}dx$$

Solution:

$$\int \frac{x^3 - x^2 + x - 1}{x-1}dx = \int \left[\frac{\left(x^2+1\right)\left(x-1\right)}{x-1} \right] dx$$

$$= \int \left(x^2+1\right)dx$$

$$= \int x^2 dx + \int 1 dx$$

$$= \frac{x^3}{3} + x + C$$

Question 14:

$$\int (1-x)\sqrt{x}dx$$

Solution:

$$\int (1-x)\sqrt{x}dx = \int \left(\sqrt{x} - x^{\frac{3}{2}} \right) dx$$

$$= \int x^{\frac{1}{2}} dx - \int x^{\frac{3}{2}} dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$$

$$= \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + C$$

Question 15:

$$\int \sqrt{x}\left(3x^2+2x+3\right)dx$$

Solution:

$$\int \sqrt{x}\left(3x^2+2x+3\right)dx = \int\left(3x^{\frac{5}{2}}+2x^{\frac{3}{2}}+3x^{\frac{1}{2}}\right)dx$$

$$= 3\int x^{\frac{5}{2}}dx + 2\int x^{\frac{3}{2}}dx + 3\int x^{\frac{1}{2}}dx$$

$$= 3\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right) + 2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right) + 3\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{6}{7}x^{\frac{7}{2}} + \frac{4}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C$$

Question 16:

$$\int\left(2x-3\cos x+e^x\right)dx$$

Solution:

$$\int\left(2x-3\cos x+e^x\right)dx = 2\int x dx - 3\int \cos x dx + \int e^x dx$$

$$= \frac{2x^2}{2} - 3(\sin x) + e^x + C$$

$$= x^2 - 3\sin x + e^x + C$$

Question 17:

$$\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$$

Solution:

$$\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx = 2\int x^2 dx - 3\int \sin x dx + 5\int x^{\frac{1}{2}} dx$$

$$= \frac{2x^3}{3} - 3(-\cos x) + 5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{\frac{3}{2}} + C$$

Question 18:

$$\int \sec x(\sec x+\tan x)dx$$

Solution:

$$\begin{aligned}\int \sec x(\sec x+\tan x)dx &= \int(\sec^2 x+\sec x\tan x)dx\\ &= \int \sec^2 xdx+\int \sec x\tan xdx\\ &= \tan x+\sec x+C\end{aligned}$$

Question 19:

$$\int \frac{\sec^2 x}{\cos ec^2 x}dx$$

Solution:

$$\begin{aligned}\int \frac{\sec^2 x}{\cos ec^2 x}dx &= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}}dx\\ &= \int \frac{\sin^2 x}{\cos^2 x}dx\\ &= \int \tan^2 xdx\\ &= \int(\sec^2 x-1)dx\\ &= \int \sec^2 xdx-\int 1dx\\ &= \tan x-x+C\end{aligned}$$

Question 20:

$$\int \frac{2-3\sin x}{\cos^2 x}dx$$

Solution:

$$\int \frac{2-3\sin x}{\cos^2 x}dx = \int\left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x}\right)dx$$
$$= \int 2\sec^2 x dx - 3\int \tan x \sec x dx$$
$$= 2\tan x - 3\sec x + C$$

Choose the correct answer in Exercises 21 and 22

Question 21:

The anti-derivative of $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$ equals

(A) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$

(B) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$

(C) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

(D) $\frac{3}{3}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$

Solution:

$$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right) = \int x^{\frac{1}{2}}dx + \int x^{-\frac{1}{2}}dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$
$$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

Thus, the correct option is C.

Question 22:

If $\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$ such that $f(2)=0$, then $f(x)$ is

(A) $x^4+\frac{1}{x^3}-\frac{129}{8}$

(B) $x^3+\frac{1}{x^4}+\frac{129}{8}$

(C) $x^4+\frac{1}{x^3}+\frac{129}{8}$

(D) $x^3+\frac{1}{x^4}-\frac{129}{8}$

Solution:

Given, $\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$

Anti-derivative of $4x^3-\frac{3}{x^4}=f(x)$
Therefore,

$$f(x)=\int 4x^3-\frac{3}{x^4}dx$$

$$f(x)=4\int x^3dx-3\int\left(x^{-4}\right)dx$$

$$f(x)=4\left(\frac{x^4}{4}\right)-3\left(\frac{x^{-3}}{-3}\right)+C$$

$$f(x)=x^4+\frac{1}{x^3}+C$$

Also,

$$\Rightarrow f(2)=0$$

$$\Rightarrow f(2)=(2)^4+\frac{1}{(2)^3}+C=0$$

$$\Rightarrow 16+\frac{1}{8}+C=0$$

$$\Rightarrow C=-\left(16+\frac{1}{8}\right)$$

$$\Rightarrow C=-\frac{129}{8}$$

$$\Rightarrow f(x)=x^4+\frac{1}{x^3}-\frac{129}{8}$$

Thus, the correct option is A.

EXERCISE 7.2

Integrate the functions in Exercises 1 to 37:

Question 1:

$$\frac{2x}{1+x^2}$$

Solution:

Put $1+x^2=t$

Therefore, $2xdx=dt$

$$\int \frac{2x}{1+x^2}dx=\int \frac{1}{t}dt=\log|t|+C$$
$$=\log\left|1+x^2\right|+C$$
$$=\log\left(1+x^2\right)+C$$

Question 2:

$$\frac{(\log x)^2}{x}$$

Solution:

Put $\log|x|=t$

Therefore, $\frac{1}{x}dx=dt$

$$\int \frac{(\log|x|)^2}{x}dx=\int t^2dt$$
$$=\frac{t^3}{3}+C$$
$$=\frac{(\log|x|)^3}{3}+C$$

Question 3:

$$\frac{1}{x+x\log x}$$

Solution:

$$\frac{1}{x+x\log x}=\frac{1}{x(1+\log x)}$$

Put $1+\log x=t$

Therefore, $\frac{1}{x}dx = dt$

$$\int \frac{1}{x(1+\log x)}dx = \int \frac{1}{t}dt = \log|t| + C$$

$$= \log|1+\log x| + C$$

Question 4:

$\sin x \sin(\cos x)$

Solution:

Put $\cos x = t$

Therefore, $-\sin x dx = dt$

$$\int \sin x \sin(\cos x) dx = -\int \sin t dt = -[-\cos t] + C$$

$$= \cos t + C$$

$$= \cos(\cos x) + C$$

Question 5:

$\text{Sin}(ax+b)\cos(ax+b)$

Solution:

$$\sin(ax+b)\cos(ax+b) = \frac{2\sin(ax+b)\cos(ax+b)}{2}$$

$$= \frac{\sin 2(ax+b)}{2}$$

Put $2(ax+b) = t$

Therefore, $2adx = dt$

$$\int \frac{\sin 2(ax+b)}{2}dx = \frac{1}{2}\int \frac{\sin t dt}{2a}$$

$$= \frac{1}{4a}[-\cos t] + C$$

$$= \frac{-1}{4a}\cos 2(ax+b) + C$$

Question 6:

$\sqrt{ax+b}$

Solution:

Put $ax+b = t$

Therefore,

$$\Rightarrow adx = dt$$

$$\Rightarrow dx = \frac{1}{a}dt$$

$$\int (ax+b)^{\frac{1}{2}} dx = \frac{1}{a}\int t^{\frac{1}{2}} dt = \frac{1}{a}\left(\frac{t^{\frac{1}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{3a}(ax+b)^{\frac{3}{2}} + C$$

Question 7:

$x\sqrt{x+2}$

Solution:

Put, $x+2=t$

$\therefore dx = dt$

$$\Rightarrow \int x\sqrt{x+2} = \int (t-2)\sqrt{t}dt$$

$$= \int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right)dt$$

$$= \int t^{\frac{3}{2}}dt - 2\int t^{\frac{1}{2}}dt$$

$$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C$$

Question 8:

$x\sqrt{1+2x^2}$

Solution:

Put, $1+2x^2=t$

$\therefore 4xdx=dt$

$$\Rightarrow \int x\sqrt{1+2x^2}dx=\int \frac{\sqrt{t}}{4}dt$$

$$=\frac{1}{4}\int t^{\frac{1}{2}}dt$$

$$=\frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)+C$$

$$=\frac{1}{6}\left(1+2x^2\right)^{\frac{3}{2}}+C$$

Question 9:

$(4x+2)\sqrt{x^2+x+1}$

Solution:

Put, $x^2+x+1=t$

$\therefore (2x+1)dx=dt$

$$\int (4x+2)\sqrt{x^2+x+1}dx$$

$$=\int 2\sqrt{t}dt$$

$$=2\int \sqrt{t}dt$$

$$=2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)+C=\frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}}+C$$

Question 10:

$$\frac{1}{x-\sqrt{x}}$$

Solution:

$$\frac{1}{x-\sqrt{x}}=\frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}$$

Put, $\left(\sqrt{x}-1\right)=t$

$$\therefore \frac{1}{2\sqrt{x}}dx=dt$$

$$\Rightarrow \int \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}dx=\int \frac{2}{t}dt$$

$$=2\log|t|+C$$

$$=2\log\left|\sqrt{x}-1\right|+C$$

Question 11:

$$\frac{x}{\sqrt{x+4}}, x>0$$

Solution:

Put, $x+4=t$

$\therefore dx=dt$

$$\int \frac{x}{\sqrt{x+4}}dx=\int \frac{(t-4)}{\sqrt{t}}dt=\int\left(\sqrt{t}-\frac{4}{\sqrt{t}}\right)dt$$

$$=\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)-4\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right)+C=\frac{2}{3}(t)^{\frac{3}{2}}-8(t)^{\frac{1}{2}}+C$$

$$=\frac{2}{3}t.t^{\frac{1}{2}}-8t^{\frac{1}{2}}+C$$

$$=\frac{2}{3}t^{\frac{1}{2}}(t-12)+C$$

$$=\frac{2}{3}(x+4)^{\frac{1}{2}}(x+4-12)+C$$

$$=\frac{2}{3}\sqrt{x+4}(x-8)+C$$

Question 12:

$\left(x^3-1\right)^{\frac{1}{3}}x^5$

Solution:

Put, $x^3-1=t$

$\therefore 3x^2dx=dt$

$\Rightarrow \int\left(x^3-1\right)^{\frac{1}{3}}x^5dx=\int\left(x^3-1\right)^{\frac{1}{3}}x^3x^2dx$

$\Rightarrow \int t^{\frac{1}{3}}(t+1)\frac{dt}{3}=\frac{1}{3}\int\left(t^{\frac{4}{3}}+t^{\frac{1}{3}}\right)dt$

$=\frac{1}{3}\left[\frac{t^{\frac{7}{3}}}{\frac{7}{3}}+\frac{t^{\frac{4}{3}}}{\frac{4}{3}}\right]+C$

$=\frac{1}{3}\left[\frac{3}{7}t^{\frac{7}{3}}+\frac{3}{4}t^{\frac{4}{3}}\right]+C$

$=\frac{1}{7}\left(x^3-1\right)^{\frac{7}{3}}+\frac{1}{4}\left(x^3-1\right)^{\frac{4}{3}}+C$

Question 13:

$\frac{x^2}{\left(2+3x^3\right)^3}$

Solution:

Put, $2+3x^3=t$

$\therefore 9x^2dx=dt$

$\Rightarrow \int\frac{x^2}{\left(2+3x^3\right)^3}dx=\frac{1}{9}\int\frac{dt}{(t)^3}$

$=\frac{1}{9}\left[\frac{t^{-2}}{-2}\right]+C$

$=-\frac{1}{18}\left(\frac{1}{t^2}\right)+C$

$=\frac{-1}{18\left(2+3x^3\right)^2}+C$

Question 14:

$$\frac{1}{x(\log x)^m}, x > 0$$

Solution:

Put, $\log x = t$

$$\therefore \frac{1}{x} dx = dt$$

$$\Rightarrow \int \frac{1}{x(\log x)^m} dx = \int \frac{dt}{(t)^m} = \left(\frac{t^{-m-1}}{1-m}\right) + C$$

$$= \frac{(\log x)^{1-m}}{(1-m)} + C$$

Question 15:

$$\frac{x}{9-4x^2}$$

Solution:

Put, $9 - 4x^2 = t$

$$\therefore -8xdx = dt$$

$$\Rightarrow \int \frac{x}{9-4x^2} dx = \frac{-1}{8} \int \frac{1}{t} dt$$

$$= \frac{-1}{8} \log|t| + C$$

$$= \frac{-1}{8} \log\left|9 - 4x^2\right| + C$$

Question 16:

$$e^{2x+3}$$

Solution:

Put, $2x + 3 = t$

$$\therefore 2dx = dt$$

$$\Rightarrow \int e^{2x+3} dx = \frac{1}{2} \int e^t dt$$

$$= \frac{1}{2}\left(e^t\right) + C$$

$$= \frac{1}{2} e^{(2x+3)} + C$$

Question 17:

$\frac{x}{e^{x^2}}$

Solution:

Put, $x^2 = t$

$\therefore 2xdx = dt$

$\Rightarrow \int \frac{x}{e^{x^2}} dx = \frac{1}{2}\int \frac{1}{e^t} dt = \frac{1}{2}\int e^{-t} dt$

$= \frac{1}{2}\left(\frac{e^{-t}}{-1}\right) + C$

$= -\frac{1}{2}e^{-x^2} + C$

$= \frac{-1}{2e^{x^2}} + C$

Question 18:

$\frac{e^{\tan^{-1}x}}{1+x^2}$

Solution:

Put, $\tan^{-1} x = t$

$\therefore \frac{1}{1+x^2} dx = dt$

$\Rightarrow \int \frac{e^{\tan^{-1}x}}{1+x^2} dx = \int e^t dt$

$= e^t + C$

$= e^{\tan^{-1}x} + C$

Question 19:

$\frac{e^{2x}-1}{e^{2x}+1}$

Solution:

$\frac{e^{2x}-1}{e^{2x}+1}$

Dividing Nr and Dr by e^x, we get

$$\frac{\frac{e^{2x}-1}{e^x}}{\frac{e^{2x}+1}{e^x}}=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

Let $e^x+e^{-x}=t$

$\left(e^x-e^{-x}\right)dx=dt$

$\Rightarrow \int \frac{e^{2x}-1}{e^{2x}+1}dx=\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx$

$=\int \frac{dt}{t}$

$=\log|t|+C$

$=\log\left|e^x+e^{-x}\right|+C$

Question 20:

$\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

Solution:

Put, $e^{2x}+e^{-2x}=t$

$\left(2e^{2x}-2e^{-2x}\right)dx=dt$

$\Rightarrow 2\left(e^{2x}-e^{-2x}\right)dx=dt$

$\Rightarrow \int\left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right)dx=\int \frac{dt}{2t}$

$=\frac{1}{2}\int \frac{1}{t}dt$

$=\frac{1}{2}\log|t|+C$

$=\frac{1}{2}\log\left|e^{2x}+e^{-2x}\right|+C$

Question 21:

$\tan^2(2x-3)$

Solution:

$\tan^2(2x-3)=\sec^2(2x-3)-1$

Put, $2x-3=t$

$\therefore 2dx = dt$

$\Rightarrow \int \tan^2(2x-3)\,dx = \int \left[\sec^2(2x-3)-1\right]dx$

$= \frac{1}{2}\int (\sec^2 t)\,dt - \int 1dx = \frac{1}{2}\int \sec^2 t\,dt - \int 1dx$

$= \frac{1}{2}\tan t - x + C$

$= \frac{1}{2}\tan(2x-3) - x + C$

Question 22:

$\sec^2(7-4x)$

Solution:

Put, $7-4x = t$

$\therefore -4dx = dt$

$\therefore \int \sec^2(7-4x)\,dx = \frac{-1}{4}\int \sec^2 t\,dt$

$= \frac{-1}{4}(\tan t) + C$

$= \frac{-1}{4}\tan(7-4x) + C$

Question 23:

$\frac{\sin^{-1} x}{\sqrt{1-x^2}}$

Solution:

Put, $\sin^{-1} x = t$

$\frac{1}{\sqrt{1-x^2}}dx = dt$

$\Rightarrow \int \frac{\sin^{-1} x}{\sqrt{1-x^2}}dx = \int t\,dt$

$= \frac{t^2}{2} + C = \frac{(\sin^{-1} x)^2}{2} + C$

Question 24:

$$\frac{2\cos x-3\sin x}{6\cos x+4\sin x}$$

Solution:

$$\frac{2\cos x-3\sin x}{6\cos x+4\sin x}=\frac{2\cos x-3\sin x}{2(3\cos x+2\sin x)}$$

Let $3\cos x+2\sin x=t$

$$(-3\sin x+2\cos x)dx=dt$$

$$\int\frac{2\cos x-3\sin x}{6\cos x+4\sin x}dx=\int\frac{dt}{2t}$$

$$=\frac{1}{2}\int\frac{1}{t}dt$$

$$=\frac{1}{2}\log|t|+C$$

$$=\frac{1}{2}\log|2\sin x+3\cos x|+C$$

Question 25:

$$\frac{1}{\cos^2 x(1-\tan x)^2}$$

Solution:

$$\frac{1}{\cos^2 x(1-\tan x)^2}=\frac{\sec^2 x}{(1-\tan x)^2}$$

Let $(1-\tan x)=t$

$$-\sec^2 x dx=dt$$

$$\Rightarrow\int\frac{\sec^2 x}{(1-\tan x)^2}dx=\int\frac{-dt}{t^2}$$

$$=-\int t^{-2}dt$$

$$=\frac{1}{t}+C$$

$$=\frac{1}{(1-\tan x)}+C$$

Question 26:

$$\frac{\cos\sqrt{x}}{\sqrt{x}}$$

Solution:

Let $\sqrt{x}=t$

$$\frac{1}{2\sqrt{x}}dx=dt$$

$$\Rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx=2\int \cos t dt$$

$$=2\sin t+C$$

$$=2\sin\sqrt{x}+C$$

Question 27:

$$\sqrt{\sin 2x}\cos 2x$$

Solution:

Put, $\sin 2x=t$

So, $2\cos 2x dx=dt$

$$\Rightarrow \int \sqrt{\sin 2x}\cos 2x dx=\frac{1}{2}\int \sqrt{t}dt$$

$$=\frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)+C$$

$$=\frac{1}{3}t^{\frac{3}{2}}+C$$

$$=\frac{1}{3}(\sin 2x)^{\frac{3}{2}}+C$$

Question 28:

$$\frac{\cos x}{\sqrt{1+\sin x}}$$

Solution:

Put, $1+\sin x=t$

$\therefore \cos x dx = dt$

$\Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}} dx = \int \frac{dt}{\sqrt{t}}$

$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= 2\sqrt{t} + C$

$= 2\sqrt{1+\sin x} + C$

Question 29:

$\cot x \log \sin x$

Solution:

Let $\log \sin x = t$

$\Rightarrow \frac{1}{\sin x} \cos x dx = dt$

$\therefore \cot x dx = dt$

$\Rightarrow \int \cot x \log \sin x dx = \int t dt$

$= \frac{t^2}{2} + C$

$= \frac{1}{2}(\log \sin x)^2 + C$

Question 30:

$\frac{\sin x}{1+\cos x}$

Solution:

Put, $1+\cos x = t$

$\therefore -\sin x dx = dt$

$\Rightarrow \int \frac{\sin x}{1+\cos x} dx = \int -\frac{dt}{t}$

$= -\log|t| + C$

$= -\log|1+\cos x| + C$

Question 31:

$$\frac{\sin x}{(1+\cos x)^2}$$

Solution:

Put, $1+\cos x = t$

$\therefore -\sin x dx = dt$

$$\Rightarrow \int \frac{\sin x}{(1+\cos x)^2} dx = \int -\frac{dt}{t^2}$$

$$= -\int t^{-2} dt$$

$$= \frac{1}{t} + C$$

$$= \frac{1}{(1+\cos x)} + C$$

Question 32:

$$\frac{1}{1+\cos x}$$

Solution:

Let I $= \int \frac{1}{1+\cos x} dx$

$$= \int \frac{1}{1+\frac{\cos x}{\sin x}} dx$$

$$= \int \frac{\sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \frac{2\sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x + \cos x)} dx$$

$$= \frac{1}{2}\int 1 dx + \frac{1}{2}\int \frac{\sin x - \cos x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}(x) + \frac{1}{2}\int \frac{\sin x - \cos x}{\sin x + \cos x} dx$$

Let $\sin x + \cos x = t \Rightarrow (\cos x - \sin x) dx = dt$

$$\therefore \text{I} = \frac{x}{2} + \frac{1}{2}\int \frac{-(dt)}{t}$$

$$= \frac{x}{2} - \frac{1}{2}\log|t| + C = \frac{x}{2} - \frac{1}{2}\log|\sin x + \cos x| + C$$

Question 33:

$$\frac{1}{1-\tan x}$$

Solution:

Put, $I=\int\frac{1}{1-\tan x}dx$

$$=\int\frac{1}{1-\frac{\sin x}{\cos x}}dx=\int\frac{\cos x}{\cos x-\sin x}dx$$

$$=\frac{1}{2}\int\frac{2\cos x}{\cos x-\sin x}dx=\frac{1}{2}\int\frac{(\cos x-\sin x)+(\cos x+\sin x)}{(\cos x-\sin x)}dx$$

$$=\frac{1}{2}\int 1dx+\frac{1}{2}\int\frac{\cos x+\sin x}{\cos x-\sin x}dx=\frac{x}{2}+\frac{1}{2}\int\frac{\cos x+\sin x}{\cos x-\sin x}dx$$

Put, $\cos x-\sin x=t\Rightarrow(-\sin x-\cos x)dx=dt$

$$\therefore I=\frac{x}{2}+\frac{1}{2}\int\frac{-(dt)}{t}=\frac{x}{2}-\frac{1}{2}\log|t|+C$$

$$=\frac{x}{2}-\frac{1}{2}\log|\cos x-\sin x|+C$$

Question 34:

$$\frac{\sqrt{\tan x}}{\sin x\cos x}$$

Solution:

Let $I=\int\frac{\sqrt{\tan x}}{\sin x\cos x}dx=\int\frac{\sqrt{\tan x}\times\cos x}{\sin x\cos x\times\cos x}dx$

$$=\int\frac{\sqrt{\tan x}}{\tan x\cos^2 x}dx=\int\frac{\sec^2 xdx}{\sqrt{\tan x}}$$

Let $\tan x=t\Rightarrow\sec^2 xdx=dt$

$$\therefore I=\int\frac{dt}{\sqrt{t}}$$

$$=2\sqrt{t}+C$$

$$=2\sqrt{\tan x}+C$$

Question 35:

$$\frac{(1+\log x)^2}{x}$$

Solution:

Put, $1+\log x=t$

$\therefore \frac{1}{x}dx=dt$

$\Rightarrow \int \frac{(1+\log x)^2}{x}dx=\int t^2dt$

$=\frac{t^3}{3}+C$

$=\frac{(1+\log x)^3}{3}+C$

Question 36:

$\frac{(x+1)(x+\log x)^2}{x}$

Solution:

$\frac{(x+1)(x+\log x)^2}{x}=\left(\frac{x+1}{x}\right)(x+\log)^2=\left(1+\frac{1}{x}\right)(x+\log x)^2$

Put, $(x+\log x)=t$

$\therefore \left(1+\frac{1}{x}\right)dx=dt$

$\Rightarrow \int\left(1+\frac{1}{x}\right)(x+\log x)^2\,dx=\int t^2dt$

$=\frac{t^3}{3}+C$

$=\frac{1}{3}(x+\log x)^3+C$

Question 37:

$\frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}$

Solution:

Put, $x^4=t$

$\therefore 4x^3dx=dt$

$\Rightarrow \int \frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}dx=\frac{1}{4}\int\frac{\sin\left(\tan^{-1}t\right)}{1+t^2}dt \qquad \ldots(1)$

Let $\tan^{-1}t=u$

$$\therefore \frac{1}{1+t^2}dt = du$$

From (1), we get

$$\int \frac{x^3 \sin\left(\tan^{-1} x^4\right)dx}{1+x^8} = \frac{1}{4}\int \sin u du$$

$$= \frac{1}{4}(-\cos u) + C$$

$$= -\frac{1}{4}\cos\left(\tan^{-1} t\right) + C$$

$$= \frac{-1}{4}\cos\left(\tan^{-1} x^4\right) + C$$

Choose the correct answer in Exercises 38 and 39.

Question 38:

$\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$ equals

$(A)\ 10^x - x^{10} + C$ $(B)\ 10^x + x^{10} + C$

$(C)\ \left(10^x - x^{10}\right)^{-1} + C$ $(D)\ \log\left(10^x + x^{10}\right) + C$

Solution:

Put, $x^{10} + 10^x = t$

$$\therefore \left(10x^9 + 10^x \log_e 10\right)dx = \int \frac{dt}{t}$$

$$\Rightarrow \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10x} dx = \int \frac{dt}{t}$$

$$= \log t + C$$

$$= \log\left(10^x + x^{10}\right) + C$$

Thus, the correct option is D.

Question 39:

$\int \frac{dx}{\sin^2 x \cos^2 x}$ equals

$(A)\ \tan x + \cot x + C$ $(B)\ \tan x - \cot x + C$

$(C)\ \tan x \cot x + C$ $(D)\ \tan x - \cot 2x + C$

Solution:

Put, $\mathrm{I} = \int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{1}{\sin^2 x \cos^2 x} dx$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \sec^2 x dx + \int \cos ec^2 dx$$

$$= \tan x - \cot x + C$$

Thus, the correct option is B.

EXERCISE 7.3

Find the integrals of the functions in Exercises 1 to 22:

Question 1:

$\sin^2(2x+5)$

Solution:

$$\sin^2(2x+5)=\frac{1-\cos 2(2x+5)}{2}=\frac{1-\cos(4x+10)}{2}$$

$$\Rightarrow \int \sin^2(2x+5)\,dx=\int \frac{1-\cos(4x+10)}{2}\,dx$$

$$=\frac{1}{2}\int 1dx-\frac{1}{2}\int \cos(4x+10)\,dx$$

$$=\frac{1}{2}x-\frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right)+C$$

$$=\frac{1}{2}x-\frac{1}{8}\sin(4x+10)+C$$

Question 2:

$\sin 3x\cos 4x$

Solution:

Using, $\sin A\cos B=\frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$

$$\therefore \int \sin 3x\cos 4x dx=\frac{1}{2}\int\{\sin(3x+4x)+\sin(3x-4x)\}\,dx$$

$$=\frac{1}{2}\int\{\sin 7x+\sin(-x)\}dx$$

$$=\frac{1}{2}\int\{\sin 7x-\sin x\}\,dx$$

$$=\frac{1}{2}\int \sin 7x dx-\frac{1}{2}\int \sin x dx$$

$$=\frac{1}{2}\left(\frac{-\cos 7x}{7}\right)-\frac{1}{2}(-\cos x)+C$$

$$=\frac{-\cos 7x}{14}+\frac{\cos x}{2}+C$$

Question 3:

$\cos 2x\cos 4x\cos 6x$

Solution:

Using, $\cos A \cos B = \frac{1}{2}\{\cos(A+B)+\cos(A-B)\}$

$\therefore \int \cos 2x(\cos 4x \cos 6x)dx = \int \cos 2x\left[\frac{1}{2}\{\cos(4x+6x)+\cos(4x-6x)\}\right]dx$

$=\frac{1}{2}\int\{\cos 2x \cos 10x + \cos 2x \cos(-2x)\}dx$

$=\frac{1}{2}\int\{\cos 2x \cos 10x + \cos^2 2x\}\,dx$

$=\frac{1}{2}\int\left[\left\{\frac{1}{2}\cos(2x+10x)+\frac{1}{2}\cos(2x-10x)\right\}+\left(\frac{1+\cos 4x}{2}\right)\right]dx$

$=\frac{1}{4}\int(\cos 12x + \cos 8x + 1 + \cos 4x)dx$

$=\frac{1}{4}\left[\frac{\sin 12x}{12}+\frac{\sin 8x}{8}+x+\frac{\sin 4x}{4}+C\right]$

Question 4:

$\sin^3(2x+1)$

Solution:

Put, $I = \int \sin^3(2x+1)$

$\Rightarrow \int \sin^3(2x+1)dx = \int \sin^2(2x+1)\sin(2x+1)dx$

$=\int(1-\cos^2(2x+1))\sin(2x+1)dx$

Let $\cos(2x+1)=t$

$\Rightarrow -2\sin(2x+1)dx = dt$

$\Rightarrow \sin(2x+1)dx = \frac{-dt}{2}$

$\Rightarrow I = \frac{-1}{2}\int(1-t^2)dt$

$=\frac{-1}{2}\left\{t-\frac{t^3}{3}\right\}$

$=\frac{-1}{2}\left\{\cos(2x+1)-\frac{\cos^3(2x+1)}{3}\right\}$

$=\frac{-\cos(2x+1)}{2}+\frac{\cos^3(2x+1)}{6}+C$

Question 5:

$\sin^3 x \cos^3 x$

Solution:

Let $\mathrm{I} = \int \sin^3 x \cos^3 x dx$

$$= \int \cos^3 x \sin^2 x \sin x dx$$

$$= \int \cos^3 x \left(1 - \cos^2 x\right) \sin x dx$$

Let $\cos x = t$

$\Rightarrow -\sin x dx = dt$

$$\Rightarrow \mathrm{I} = -\int t^3 \left(1 - t^2\right) dt$$

$$= -\int \left(t^3 - t^5\right) dt = -\left\{\frac{t^4}{4} - \frac{t^6}{6}\right\} + C$$

$$= -\left\{\frac{\cos^4 x}{4} - \frac{\cos^6 x}{6}\right\} + C = \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$$

Question 6:

$\sin x \sin 2x \sin 3x$

Solution:

Using, $\sin A \sin B = \frac{1}{2}\left\{\cos(A - B) - \cos(A + B)\right\}$

$$\therefore \int \sin x \sin 2x \sin 3x dx = \int \left[\sin x \frac{1}{2}\left\{\cos(2x - 3x) - \cos(2x + 3x)\right\}\right] dx$$

$$= \frac{1}{2}\int \left(\sin x \cos(-x) - \sin x \cos 5x\right) dx$$

$$= \frac{1}{2}\int \left(\sin x \cos x - \sin x \cos 5x\right) dx$$

$$= \frac{1}{2}\int \frac{\sin 2x}{2} dx - \frac{1}{2}\int \sin x \cos 5x dx$$

$$= \frac{1}{4}\left[\frac{-\cos 2x}{2}\right] - \frac{1}{2}\int \left\{\frac{1}{2}\sin(x + 5x) + \frac{1}{2}\sin(x - 5x)\right\} dx$$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\int \left(\sin 6x + \sin(-4x)\right) dx$$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\left[\frac{-\cos 6x}{6} + \frac{\cos 4x}{4}\right] + C$$

$$= \frac{-\cos 2x}{8} - \frac{1}{8}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{2}\right] + C$$

$$= \frac{1}{8}\left[\frac{\cos 6x}{3} - \frac{\cos 4x}{2} - \cos 2x\right] + C$$

Question 7:

$\sin 4x \sin 8x$

Solution:

Using, $\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$

$$\therefore \int \sin 4x \sin 8x dx = \int \left\{\frac{1}{2}\cos(4x-8x) - \frac{1}{2}\cos(4x+8x)\right\} dx$$

$$= \frac{1}{2}\int (\cos(-4x) - \cos 12x) dx$$

$$= \frac{1}{2}\int (\cos 4x - \cos 12x) dx$$

$$= \frac{1}{2}\left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12}\right]$$

Question 8:

$$\frac{1-\cos x}{1+\cos x}$$

Solution:

$$\frac{1-\cos x}{1+\cos x} = \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} \qquad \left[2\sin^2 \frac{x}{2} = 1-\cos x \text{ and } 2\cos^2 \frac{x}{2} = 1+\cos x\right]$$

$$= \tan^2 \frac{x}{2}$$

$$= \left(\sec^2 \frac{x}{2} - 1\right)$$

$$\therefore \frac{1-\cos x}{1+\cos x} dx = \int \left(\sec^2 \frac{x}{2} - 1\right) dx$$

$$= \left[\frac{\tan \frac{x}{2}}{\frac{1}{2}} - x\right] + C$$

$$= 2\tan \frac{x}{2} - x + C$$

Question 9:

$$\frac{\cos x}{1+\cos x}$$

Solution:

$$\frac{\cos x}{1+\cos x}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} \qquad \left[\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2} \text{ and } \cos x=2\cos^2\frac{x}{2}-1\right]$$

$$=\frac{1}{2}\left[1-\tan^2\frac{x}{2}\right]$$

$$\therefore \int\frac{\cos x}{1+\cos x}dx=\frac{1}{2}\int\left(1-\tan^2\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\int\left(1-\sec^2\frac{x}{2}+1\right)dx$$

$$=\frac{1}{2}\int\left(2-\sec^2\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\left[2x-\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]+C$$

$$=x-\tan\frac{x}{2}+C$$

Question 10:

$$\sin^4 x$$

Solution:

$\sin^4 x = \sin^2 x \sin^2 x$

$= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$

$= \frac{1}{4}(1-\cos 2x)^2$

$= \frac{1}{4}\left[1+\cos^2 2x - 2\cos 2x\right]$

$= \frac{1}{4}\left[1+\left(\frac{1+\cos 4x}{2}\right) - 2\cos 2x\right]$

$= \frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$

$= \frac{1}{4}\left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]$

$\therefore \int \sin^4 x dx = \frac{1}{4}\int\left[\frac{3}{2}+\frac{1}{2}\cos 4x - 2\cos 2x\right]dx$

$= \frac{1}{4}\left[\frac{3}{2}x+\frac{1}{2}\left(\frac{\sin 4x}{4}\right) - 2\times\frac{\sin 2x}{2}\right]+C$

$= \frac{1}{8}\left[3x+\frac{\sin 4x}{4} - 2\sin 2x\right]+C$

$= \frac{3x}{8}-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x + C$

Question 11:

$\cos^4 2x$

Solution:

$\cos^4 2x = \left(\cos^2 2x\right)^2$

$= \left(\dfrac{1+\cos 4x}{2}\right)^2$

$= \dfrac{1}{4}\left[1+\cos^2 4x + 2\cos 4x\right]$

$= \dfrac{1}{4}\left[1+\left(\dfrac{1+\cos 8x}{2}\right)+2\cos 4x\right]$

$= \dfrac{1}{4}\left[1+\dfrac{1}{2}+\dfrac{\cos 8x}{2}+2\cos 4x\right]$

$= \dfrac{1}{4}\left[\dfrac{3}{2}+\dfrac{\cos 8x}{2}+2\cos 4x\right]$

$\therefore \int \cos^4 2x dx = \int \left(\dfrac{3}{8}+\dfrac{\cos 8x}{8}+\dfrac{\cos 4x}{2}\right)dx$

$= \dfrac{3}{8}x + \dfrac{1}{64}\sin 8x + \dfrac{1}{8}\sin 4x + C$

Question 12:

$\dfrac{\sin^2 x}{1+\cos x}$

Solution:

$\dfrac{\sin^2 x}{1+\cos x} = \dfrac{\left(2\sin\dfrac{x}{2}\cos\dfrac{x}{2}\right)^2}{2\cos^2\dfrac{x}{2}}$ $\left[\sin x = 2\sin\dfrac{x}{2}\cos\dfrac{x}{2}; \cos x = 2\cos^2\dfrac{x}{2}-1\right]$

$= \dfrac{4\sin^2\dfrac{x}{2}\cos^2\dfrac{x}{2}}{2\cos^2\dfrac{x}{2}}$

$= 2\sin^2\dfrac{x}{2}$

$= 1-\cos x$

$\therefore \int \dfrac{\sin^2 x}{1+\cos x}dx = \int (1-\cos x)dx$

$= x - \sin x + C$

Question 13:

$$\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}$$

Solution:

$$\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}=\frac{-2\sin\frac{2x+2\alpha}{2}\sin\frac{2x-2\alpha}{2}}{-2\sin\frac{x+\alpha}{2}\sin\frac{x-\alpha}{2}} \qquad \left[\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$=\frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$=\frac{\left[2\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right]\left[2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x+\alpha}{2}\right)\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$=4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

$$=2\left[\cos\left(\frac{x+\alpha}{2}+\frac{x-\alpha}{2}\right)+\cos\left(\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right)\right]$$

$$=2\left[\cos(x)+\cos\alpha\right]$$

$$=2\cos x+2\cos\alpha$$

$$\therefore\int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}dx=\int 2\cos x+2\cos\alpha\; dx$$

$$=2\left[\sin x+x\cos\alpha\right]+C$$

Question 14:

$$\frac{\cos x-\sin x}{1+\sin 2x}$$

Solution:

$$\frac{\cos x-\sin x}{1+\sin 2x}=\frac{\cos x-\sin x}{(\sin^2 x+\cos^2 x)+2\sin x\cos x} \qquad \left[\sin^2 x+\cos^2 x=1;\sin 2x=2\sin x\cos x\right]$$

$$=\frac{\cos x-\sin x}{(\sin x+\cos x)^2}$$

Let $\sin x+\cos x=t$

$\therefore (\cos x - \sin x)\, dx = dt$

$$\Rightarrow \int \frac{\cos x - \sin x}{1+\sin 2x} dx = \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2} dx$$

$$= \int \frac{dt}{t^2}$$

$$= \int t^{-2} dt$$

$$= -t^{-1} + C$$

$$= -\frac{1}{t} + C$$

$$= \frac{-1}{\sin x + \cos x} + C$$

Question 15:

$\tan^3 2x \sec 2x$

Solution:

$$\tan^3 2x \sec 2x = \tan^2 2x \tan 2x \sec 2x$$

$$= (\sec^2 2x - 1) \tan 2x \sec 2x$$

$$= \sec^2 2x \tan 2x \sec 2x - \tan 2x \sec 2x$$

$$\therefore \int \tan^3 2x \sec 2x dx = \int \sec^2 2x \tan 2x \sec 2x - \int \tan 2x \sec 2x$$

$$= \int \sec^2 2x \tan 2x \sec 2x - \frac{\sec 2x}{2} + C$$

Let $\sec 2x = t$

$\therefore 2 \sec 2x \tan 2x dx = dt$

$$\therefore \int \tan^3 2x \sec 2x dx = \frac{1}{2} \int t^2 dt - \frac{\sec 2x}{2} + C$$

$$= \frac{t^3}{6} - \frac{\sec 2x}{2} + C$$

$$= \frac{(\sec 2x)^3}{6} - \frac{\sec 2x}{2} + C$$

Question 16:

$\tan^4 x$

Solution:

$$\tan^4 x$$
$$= \tan^2 x \tan^2 x$$
$$= \left(\sec^2 x - 1\right) \tan^2 x$$
$$= \sec^2 x \tan^2 x - \tan^2 x$$
$$= \sec^2 x \tan^2 x - \left(\sec^2 x - 1\right)$$
$$= \sec^2 x \tan^2 x - \sec^2 x + 1$$
$$\therefore \int \tan^4 x dx = \int \sec^2 x \tan^2 x dx - \int \sec^2 x dx + \int 1 dx$$
$$= \int \sec^2 x \tan^2 x dx - \tan x + x + C \qquad \ldots(1)$$

Consider $\sec^2 x \tan^2 x dx$

Let $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\Rightarrow \int \sec^2 x \tan^2 x dx = \int t^2 dt = \frac{t^3}{3} = \frac{\tan^3 x}{3}$$

From equation (1), we get

$$\int \tan^4 x dx = \frac{1}{3} \tan^3 x - \tan x + x + C$$

Question 17:

$\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$

Solution:

$$\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x}$$
$$= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x}$$
$$= \tan x \sec x + \cot x \operatorname{cosec} x$$
$$\therefore \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx = \int \left(\tan x \sec x + \cot x \operatorname{cosec} x\right) dx$$
$$= \sec x - \operatorname{cosec} x + C$$

Question 18:

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$$

Solution:

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos 2x + (1 - \cos 2x)}{\cos^2 x} \quad \left[\cos 2x = 1 - 2\sin^2 x\right]$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$$\therefore \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx = \int \sec^2 x dx = \tan x + C$$

Question 19:

$$\frac{1}{\sin x \cos^3 x}$$

Solution:

$$\frac{1}{\sin x \cos^3 x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x}$$

$$= \frac{\sin x}{\cos^3 x} + \frac{1}{\sin x \cos x}$$

$$= \tan x \sec^2 x + \frac{\frac{1}{\cos^2 x}}{\frac{\sin x \cos x}{\cos^2 x}}$$

$$= \tan x \sec^2 x + \frac{\sec^2 x}{\tan x}$$

$$\therefore \int \frac{1}{\sin x \cos^3 x} dx = \int \tan x \sec^2 x dx + \int \frac{\sec^2 x}{\tan x} dx$$

Let $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\Rightarrow \int \frac{1}{\sin x \cos^3 x} dx = \int t dt + \int \frac{1}{t} dt$$

$$= \frac{t^2}{2} + \log|t| + C$$

$$= \frac{1}{2}\tan^2 x + \log|\tan x| + C$$

Question 20:

$$\frac{\cos 2x}{(\cos x+\sin x)^2}$$

Solution:

$$\frac{\cos 2x}{(\cos x+\sin x)^2}=\frac{\cos 2x}{\cos^2 x+\sin x^2+2\cos x\sin x}=\frac{\cos 2x}{1+\sin 2x}$$

$$\therefore \int \frac{\cos 2x}{(\cos x+\sin x)^2}dx=\int \frac{\cos 2x}{(1+\sin 2x)}dx$$

Let $1+\sin 2x=t$

$$\Rightarrow 2\cos 2x dx=dt$$

$$\therefore \int \frac{\cos 2x}{(\cos x+\sin x)^2}dx=\frac{1}{2}\int \frac{1}{t}dt$$

$$=\frac{1}{2}\log|t|+C$$

$$=\frac{1}{2}\log|1+\sin 2x|+C$$

$$=\frac{1}{2}\log\left|(\sin x+\cos x)^2\right|+C$$

$$=\log|\sin x+\cos x|+C$$

Question 21:

$$\sin^{-1}(\cos x)$$

Solution:

$$\sin^{-1}(\cos x)$$

Let $\cos x=t$

Then, $\sin x=\sqrt{1-t^2}$

$$\Rightarrow (-\sin x)dx=dt$$

$$dx=\frac{-dt}{\sin x}$$

$$dx=\frac{-dt}{\sqrt{1-t^2}}$$

$$\therefore \int \sin^{-1}(\cos x)dx=\int \sin^{-1}t\left(\frac{-dt}{\sqrt{1-t^2}}\right)$$

$$=-\int \frac{\sin^{-1}t}{\sqrt{1-t^2}}$$

Let $\sin^{-1} t = u$

$$\Rightarrow \frac{1}{\sqrt{1-t^2}} dt = du$$

$$\therefore \int \sin^{-1}(\cos x)\,dx = -\int u du$$

$$= -\frac{u^2}{2} + C$$

$$= \frac{-(\sin^{-1} t)^2}{2} + C$$

$$= \frac{-[\sin^{-1}(\cos x)]^2}{2} + C \quad \ldots(1)$$

We know that,

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\therefore \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x) = \left(\frac{\pi}{2} - x\right)$$

Substituting in equation (1), we get

$$\int \sin^{-1}(\cos x)\,dx = \frac{-\left[\frac{\pi}{2} - x\right]^2}{2} + C$$

$$= -\frac{1}{2}\left(\frac{\pi^2}{4} + x^2 - \pi x\right) + C$$

$$= -\frac{\pi^2}{8} - \frac{x^2}{2} + \frac{\pi x}{2} + C$$

$$= \frac{\pi x}{2} - \frac{x^2}{2} + \left(C - \frac{\pi^2}{8}\right)$$

$$= \frac{\pi x}{2} - \frac{x^2}{2} + C_1$$

Question 22:

$$\frac{1}{\cos(x-a)\cos(x-b)}$$

Solution:

$$\frac{1}{\cos(x-a)\cos(x-b)}=\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}\right]$$

$$=\frac{1}{\sin(a-b)}\frac{[\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)]}{\cos(x-a)\cos(x-b)}$$

$$=\frac{1}{\sin(a-b)}[\tan(x-b)-\tan(x-a)]$$

$$\Rightarrow \int\frac{1}{\cos(x-a)\cos(x-b)}dx=\frac{1}{\sin(a-b)}\int[\tan(x-b)-\tan(x-a)]dx$$

$$=\frac{1}{\sin(a-b)}[-\log|\cos(x-b)|+\log|\cos(x-a)|]$$

$$=\frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right]+C$$

Choose the correct answer in Exercises 23 and 24.

Question 23:

$\int\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx$ is equal to

(A) $\tan x+\cot x+C$ (B) $\tan x+\operatorname{cosec} x+C$

(C) $-\tan x+\cot x+C$ (D) $\tan x+\sec x+C$

Solution:

$$\int\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}dx=\int\left(\frac{\sin^2 x}{\sin^2 x\cos^2 x}-\frac{\cos^2 x}{\sin^2 x\cos^2 x}\right)dx$$

$$=\int(\sec^2 x-\operatorname{cosec}^2 x)dx$$

$$=\tan x+\cot x+C$$

Thus, the correct option is A.

Question 24:

$$\int \frac{e^x(1+x)}{\cos^2(e^x x)}dx$$ equals

$(A)\ -\cot(ex^x)+C$ $\qquad$ $(B)\ \tan(xe^x)+C$

$(C)\ \tan(e^x)+C$ $\qquad$ $(D)\ \cot(e^x)+C$

Solution:

$$\int \frac{e^x(1+x)}{\cos^2(e^x x)}dx$$

Put, $e^x x = t$

$\Rightarrow (e^x x + e^x.1)dx = dt$

$e^x(x+1)dx = dt$

$\therefore \int \frac{e^x(1+x)}{\cos^2(e^x x)}dx = \int \frac{dt}{\cos^2 t}$

$= \int \sec^2 t dt$

$= \tan t + C$

$= \tan(e^x x) + C$

Thus, the correct answer is B.

EXERCISE 7.4

Integrate the functions in Exercises 1 to 23

Question 1:

$$\frac{3x^2}{x^6+1}$$

Solution:

Put, $x^3 = t$

$\therefore 3x^2 dx = dt$

$$\Rightarrow \frac{3x^2}{x^6+1}dx = \int \frac{dt}{t^2+1}$$

$$= \tan^{-1} t + C$$

$$= \tan^{-1}\left(x^3\right) + C$$

Question 2:

$$\frac{1}{\sqrt{1+4x^2}}$$

Solution:

Put, $2x = t$

$\therefore 2dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$

$$= \frac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right] + C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right| + C$$

Question 3:

$$\frac{1}{\sqrt{(2-x)^2+1}}$$

Solution:

Put, $2 - x = t$

$\Rightarrow -dx = dt$

$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{1}{\sqrt{t^2+1}}dt$

$= -\log\left|t+\sqrt{t^2+1}\right|+C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$

$= -\log\left|2-x+\sqrt{(2-x)^2+1}\right|+C$

$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right|+C$

Question 4:

$\frac{1}{\sqrt{9-25x^2}}$

Solution:

Put, $5x = t$

$\therefore 5dx = dt$

$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}}dx = \frac{1}{5}\int \frac{1}{\sqrt{9-t^2}}dt$

$= \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}}dt$

$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right)+C$

$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right)+C$

Question 5:

$\frac{3x}{1+2x^4}$

Solution:

Let $\sqrt{2}x^2 = t$

$\therefore 2\sqrt{2}xdx = dt$

$\Rightarrow \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$

$= \frac{3}{2\sqrt{2}}\left[\tan^{-1} t\right] + C$

$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$

Question 6:

$\frac{x^2}{1-x^6}$

Solution:

Put, $x^3 = t$

$\therefore 3x^2 dx = dt$

$\Rightarrow \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2}$

$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$

$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$

Question 7:

$\frac{x-1}{\sqrt{x^2-1}}$

Solution:

$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \quad \ldots(1)$

For $\int \frac{x}{\sqrt{x^2-1}}dx,$ let $x^2 - 1 = t \Rightarrow 2xdx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\int t^{-\frac{1}{2}}dt$$

$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$

$$= \sqrt{t}$$

$$= \sqrt{x^2-1}$$

From (1), we get

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \qquad \left[\int \frac{x}{\sqrt{x^2-a^2}}dt = \log\left|x+\sqrt{x^2-a^2}\right|\right]$$

$$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C$$

Question 8:

$$\frac{x^2}{\sqrt{x^6+a^6}}$$

Solution:

Put, $x^3 = t \Rightarrow 3x^2 dx = dt$

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$

$$= \frac{1}{3}\log\left|t+\sqrt{t^2+a^6}\right| + C$$

$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right| + C$$

Question 9:

$$\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$$

Solution:

Put, $\tan x = t$

$\therefore \sec^2 x dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx = \int \frac{dt}{\sqrt{t^2 + 2^2}}$$

$$= \log\left|t + \sqrt{t^2 + 4}\right| + C$$

$$= \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$$

Question 10:

$$\frac{1}{\sqrt{x^2 + 2x + 2}}$$

Solution:

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{(x+1)^2 + (1)^2}} dx$$

Let $x + 1 = t$

$\therefore dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{t^2 + 1}} dt$$

$$= \log\left|t + \sqrt{t^2 + 1}\right| C$$

$$= \log\left|(x+1) + \sqrt{(x+1)^2 + 1}\right| + C$$

$$= \log\left|(x+1) + \sqrt{x^2 + 2x + 2}\right| + C$$

Question 11:

$$\frac{1}{\sqrt{9x^2 + 6x + 5}}$$

Solution:

$$\int \frac{1}{\sqrt{9x^2+6x+5}}dx = \int \frac{1}{(3x+1)^2+(2)^2}dx$$

Let $(3x+1)=t$

$\Rightarrow 3dx = dt$

$$\Rightarrow \int \frac{1}{(3x+1)^2+(2)^2}dx = \frac{1}{3}\int \frac{1}{t^2+2^2}dt$$

$$=\frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]+C$$

$$=\frac{1}{6}\left[\tan^{-1}\left(\frac{3x+1}{2}\right)\right]+C$$

Question 12:

$$\frac{1}{\sqrt{7-6x-x^2}}$$

Solution:

$7-6x-x^2$ can be written as $7-\left(x^2+6x+9-9\right)$

Thus,

$7-\left(x^2+6x+9-9\right)$

$=16-\left(x^2+6x+9\right)$

$=16-(x+3)^2$

$=(4)^2-(x+3)^2$

$$\therefore \int \frac{1}{\sqrt{7-6x-x^2}}dx = \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx$$

Let $x+3=t$

$\Rightarrow dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx = \int \frac{1}{\sqrt{(4)^2-(t)^2}}dt$$

$$=\sin^{-1}\left(\frac{t}{4}\right)+C$$

$$=\sin^{-1}\left(\frac{x+3}{4}\right)+C$$

Question 13:

$$\frac{1}{\sqrt{(x-1)(x-2)}}$$

Solution:

$(x-1)(x-2)$ can be written as x^2-3x+2

Thus,

$$x^2-3x+2$$

$$=x^2-3x+\frac{9}{4}-\frac{9}{4}+2$$

$$=\left(x-\frac{3}{2}\right)^2-\frac{1}{4}$$

$$=\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\therefore \int\frac{1}{\sqrt{(x-1)(x-2)}}dx=\int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

Let $\left(x-\frac{3}{2}\right)=t$

$$\therefore dx=dt$$

$$\Rightarrow \int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx=\int\frac{1}{\sqrt{t^2-\left(\frac{1}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{1}{2}\right)^2}\right|+C$$

$$=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$$

Question 14:

$$\frac{1}{\sqrt{8+3x-x^2}}$$

Solution:

$$8+3x-x^2=8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

Thus,

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

$$=\int\frac{1}{\sqrt{8+3x-x^2}}dx=\int\frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx$$

Let $\left(x-\frac{3}{2}\right)=t$

$\therefore dx=dt$

$$\Rightarrow\int\frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx=\int\frac{1}{\sqrt{\left(\frac{41}{4}\right)-t^2}}dt$$

$$=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

Question 15:

$$\frac{1}{\sqrt{(x-a)(x-b)}}$$

Solution:

$(x-a)(x-b)=x^2-(a+b)x+ab$

Thus,

$$x^2-(a+b)x+ab$$

$$=x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a+b)^2}{4}+ab$$

$$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}$$

$$\Rightarrow\int\frac{1}{\sqrt{(x-a)(x-b)}}dx=\int\frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a+b}{2}\right)^2}}dx$$

Let $x-\left(\frac{a+b}{2}\right)=t$

$\therefore dx=dt$

$$\Rightarrow\int\frac{1}{\sqrt{\left\{x-\left(\frac{a+b}{2}\right)\right\}^2-\left(\frac{a+b}{2}\right)^2}}dx=\int\frac{1}{\sqrt{t^2-\left(\frac{a+b}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{a+b}{2}\right)^2}\right|+C$$

$$=\log\left|\left\{x-\left(\frac{a+b}{2}\right)\right\}+\sqrt{(x-a)(x-b)}\right|+C$$

Question 16:

$$\frac{4x+1}{\sqrt{2x^2+x-3}}$$

Solution:

$$\text{Let, } 4x+1=\text{A}\frac{d}{dx}\left(2x^2+x-3\right)+\text{B}$$

$$\Rightarrow 4x+1=\text{A}(4x+1)+\text{B}$$

$$\Rightarrow 4x+1=4\text{A}x+\text{A}+\text{B}$$

Equating the coefficients of x and constant term on both sides, we get

$$4\text{A}=4\Rightarrow\text{A}=1$$

$$\text{A}+\text{B}=1\Rightarrow\text{B}=0$$

Let $2x^2+x-3=t$

$\therefore(4x+1)dx=dt$

$$\Rightarrow\int\frac{4x+1}{\sqrt{2x^2+x-3}}dx=\int\frac{1}{\sqrt{t}}dt$$

$$=2\sqrt{t}+C$$

$$=2\sqrt{2x^2+x-3}+C$$

Question 17:

$$\frac{x+2}{\sqrt{x^2-1}}$$

Solution:

Put, $x+2 = \text{A}\frac{d}{dx}(x^2-1)+\text{B} \quad \ldots(1)$

$\Rightarrow x+2 = \text{A}(2x)+\text{B}$

Equating the coefficients of x and constant term on both sides, we get

$2\text{A}=1 \Rightarrow \text{A}=\frac{1}{2}$

$\text{B}=2$

From (1) we get

$$\Rightarrow \int \frac{x+2}{\sqrt{x^2-1}}dx = \int \frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx + \int \frac{2}{\sqrt{x^2-1}}dx \quad \ldots(2)$$

In $\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx$, Let $x^2-1=t \Rightarrow 2xdx = dt$

$$\frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\left[2\sqrt{t}\right]$$

$$= \frac{1}{2}\left[2\sqrt{x^2-1}\right]$$

$$= \sqrt{x^2-1}$$

Then, $\int \frac{2}{\sqrt{x^2-1}}dx = 2\int \frac{1}{\sqrt{x^2-1}}dx = 2\log\left|x+\sqrt{x^2-1}\right|$

From equation (2) we get

$$\int \frac{x+2}{\sqrt{x^2-1}}dx = \sqrt{x^2-1} + 2\log\left|x+\sqrt{x^2-1}\right| + C$$

Question 18:

$$\frac{5x-2}{1+2x+3x^2}$$

Solution:

Let $5x-2=A\frac{d}{dx}\left(1+2x+3x^2\right)+B$

$\Rightarrow 5x-2=A\left(2+6x\right)+B$

Equating the coefficients of x and constant term on both sides, we get

$5=6A\Rightarrow A=\frac{5}{6}$

$2A+B=-2\Rightarrow B=-\frac{11}{3}$

$\therefore 5x-2=\frac{5}{6}(2+6x)+\left(-\frac{11}{3}\right)$

$\Rightarrow \int\frac{5x-2}{1+2x+3x^2}dx=\int\frac{\frac{5}{6}(2+6x)-\frac{11}{3}}{1+2x+3x^2}dx$

$\Rightarrow \frac{5}{6}\int\frac{2+6x}{1+2x+3x^2}dx-\frac{11}{3}\int\frac{1}{1+2x+3x^2}dx$

Let

$I_1=\int\frac{2+6x}{1+2x+3x^2}dx$ and $I_2=\int\frac{1}{1+2x+3x^2}dx$

$\therefore \int\frac{5x-2}{1+2x+3x^2}dx=\frac{5}{6}I_1-\frac{11}{3}I_2 \quad \ldots(1)$

$I_1=\int\frac{2+6x}{1+2x+3x^2}dx$

Put $1+2x+3x^2=t$

$\Rightarrow \left(2+6x\right)dx=dt$

$\therefore I_1=\int\frac{dt}{t}$

$I_1=\log\left|t\right|$

$I_1=\log\left|1+2x+3x^2\right| \quad \ldots(2)$

$I_2=\int\frac{1}{1+2x+3x^2}dx$

$1+2x+3x^2$ can be written as $1+3\left(x^2+\frac{2}{3}x\right)$

Thus,

$$1+3\left(x^2+\frac{2}{3}x\right)$$

$$=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$$

$$=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}$$

$$=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$I_2=\frac{1}{3}\int\frac{1}{\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]}dx$$

$$=\frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

$$=\frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \quad \ldots(3)$$

Substituting equations (2) and (3) in equation (1), we get

$$\int\frac{5x-2}{1+2x+3x^2}dx=\frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right]-\frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]+C$$

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$

Question 19:

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}$$

Solution:

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}=\frac{6x+7}{\sqrt{x^2-9x+20}}$$

Put, $6x+7=A\frac{d}{dx}\left(x^2-9x+20\right)+B$

$\Rightarrow 6x+7=A\left(2x-9\right)+B$

Equating the coefficients of *x* and constant term, we get

$2A=6\Rightarrow A=3$

$-9A+B=7\Rightarrow B=34$

$\therefore 6x+7=3\left(2x-9\right)+34$

$$\int\frac{6x+7}{\sqrt{x^2-9x+20}}=\int\frac{3\left(2x-9\right)+34}{\sqrt{x^2-9x+20}}dx$$

$$=3\int\frac{\left(2x-9\right)}{\sqrt{x^2-9x+20}}dx+34\int\frac{1}{\sqrt{x^2-9x+20}}dx$$

Let $I_1=\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx$ and $I_2=\int\frac{1}{\sqrt{x^2-9x+20}}dx$

$$\therefore\int\frac{6x+7}{\sqrt{x^2-9x+20}}=3I_1+34I_2 \qquad \ldots(1)$$

Then,

$$I_1=\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

Let $x^2-9x+20=t$

$\Rightarrow\left(2x-9\right)dx=dt$

$\Rightarrow I_1=\frac{dt}{\sqrt{t}}$

$I_1=2\sqrt{t}$

$I_1=2\sqrt{x^2-9x+20} \qquad \ldots(2)$

and

$$I_2=\int\frac{1}{\sqrt{x^2-9x+20}}dx$$

$$x^2-9x+20=x^2-9x+20+\frac{81}{4}-\frac{81}{4}$$

Thus,

$$x^2-9x+20+\frac{81}{4}-\frac{81}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\frac{1}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\Rightarrow I_2=\int\frac{1}{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}dx$$

$$I_2=\log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right| \qquad \ldots(3)$$

Substituting equations (2) and (3) in (1), we get

$$\int\frac{6x+7}{\sqrt{x^2-9x+20}}dx=3\left[2\sqrt{x^2-9x+20}\right]+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

$$=6\sqrt{x^2-9x+20}+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

Question 20:

$$\frac{x+2}{\sqrt{4x-x^2}}$$

Solution:

Consider, $x+2=A\frac{d}{dx}\left(4x-x^2\right)+B$

$$\Rightarrow x+2=A(4-2x)+B$$

Equating the coefficients of x and constant term on both sides, we get

$$-2A=1\Rightarrow A=-\frac{1}{2}$$

$$4A+B=2\Rightarrow B=4$$

$$\Rightarrow (x+2)=-\frac{1}{2}(4-2x)+4$$

$$\therefore\int\frac{x+2}{\sqrt{4x-x^2}}dx=\int\frac{-\frac{1}{2}(4-2x)+4}{\sqrt{\left(4x-x^2\right)}}dx$$

$$=-\frac{1}{2}\int\frac{(4-2x)}{\sqrt{\left(4x-x^2\right)}}dx+4\int\frac{1}{\sqrt{\left(4x-x^2\right)}}dx$$

Let $I_1=\int\frac{4-2x}{\sqrt{4x-x^2}}dx$ and $I_2=\int\frac{1}{\sqrt{4x-x^2}}dx$

$$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}I_1 + 4I_2 \qquad \ldots(1)$$

Then,

$$I_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}dx$$

Let $4x - x^2 = t$

$$\Rightarrow (4-2x)dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4x-x^2} \qquad \ldots(2)$$

$$I_2 = \int \frac{1}{\sqrt{4x-x^2}}dx$$

$$\Rightarrow 4x - x^2 = -(-4x + x^2)$$

$$= (-4x + x^2 + 4 - 4)$$

$$= 4 - (x-2)^2$$

$$= (2)^2 - (x-2)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(2)^2 - (x-2)^2}}dx = \sin^{-1}\left(\frac{x-2}{2}\right) \quad \ldots(3)$$

Using equations (2) and (3) in (1), we get

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right) + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

$$= -\sqrt{4x-x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$$

Question 21:

$$\frac{x+2}{\sqrt{x^2+2x+3}}$$

Solution:

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

Let $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+2x+3}} dx$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \frac{1}{2} I_1 + I_2 \quad \ldots(1)$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$

Put, $x^2+2x+3 = t$

$$\Rightarrow (2x+2)dx = dt$$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \quad \ldots(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}} dx$$

$$\Rightarrow x^2+2x+3 = x^2+2x+1+2 = (x+1)^2 + \left(\sqrt{2}\right)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2 + \left(\sqrt{2}\right)^2}} dx = \log\left|(x+1) + \sqrt{x^2+2x+3}\right| \quad \ldots(3)$$

Using equations (2) and (3) in (1), we get

$$\int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \frac{1}{2}\left[2\sqrt{x^2+2x+3}\right] + \log\left|(x+1) + \sqrt{x^2+2x+3}\right| + C$$

$$= \sqrt{x^2+2x+3} + \log\left|(x+1) + \sqrt{x^2+2x+3}\right| + C$$

Question 22:

$$\frac{x+3}{x^2-2x-5}$$

Solution:

Let $(x+3) = A\frac{d}{dx}(x^2-2x-5) + B$

$$(x+3) = A(2x-2) + B$$

Equating the coefficients of x and constant term on both sides, we get

$2A = 1 \Rightarrow A = \frac{1}{2}$

$-2A + B = 3 \Rightarrow B = 4$

$\therefore (x+3) = \frac{1}{2}(2x-2) + 4$

$$\Rightarrow \int \frac{x+3}{x^2-2x-5}dx = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$$

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx + 4\int \frac{1}{x^2-2x-5}dx$$

Let $I_1 = \int \frac{2x-2}{x^2-2x-5}dx$ and $I_2 = \int \frac{1}{x^2-2x-5}dx$

$$\therefore \int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}I_1 + 4I_2 \quad \ldots(1)$$

Then, $I_1 = \int \frac{2x-2}{x^2-2x-5}dx$

Put, $x^2 - 2x - 5 = t$

$\Rightarrow (2x-2)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t} = \log|t| = \log\left|x^2-2x-5\right| \quad \ldots(2)$$

$$I_2 = \int \frac{1}{x^2-2x-5}dx$$

$$= \int \frac{1}{(x^2-2x+1)-6}dx$$

$$= \int \frac{1}{(x-1)^2-\left(\sqrt{6}\right)^2}dx$$

$$= \frac{1}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) \quad \ldots(3)$$

Substituting (2) and (3) in (1), we get

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

$$= \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

Question 23:

$$\frac{5x+3}{\sqrt{x^2+4x+10}}$$

Solution:

Let $5x+3=A\frac{d}{dx}\left(x^2+4x+10\right)+B$

$\Rightarrow 5x+3=A(2x+4)+B$

Equating the coefficients of x and constant term, we get

$2A=5\Rightarrow A=\frac{5}{2}$

$4A+B=3\Rightarrow B=-7$

$\therefore 5x+3=\frac{5}{2}(2x+4)-7$

$$\Rightarrow \int\frac{5x+3}{\sqrt{x^2+4x+10}}dx=\int\frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$$

$$=\frac{5}{2}\int\frac{2x+4}{\sqrt{x^2+4x+10}}dx-7\int\frac{1}{\sqrt{x^2+4x+10}}dx$$

Let $I_1=\int\frac{2x+4}{\sqrt{x^2+4x+10}}dx$ and $I_2=\int\frac{1}{\sqrt{x^2+4x+10}}dx$

$$\therefore \int\frac{5x+3}{\sqrt{x^2+4x+10}}dx=\frac{5}{2}I_1-7I_2 \quad \ldots(1)$$

Then,

$$I_1=\int\frac{2x+4}{\sqrt{x^2+4x+10}}dx$$

Put, $x^2+4x+10=t$

$\therefore (2x+4)dx=dt$

$$\Rightarrow I_1=\int\frac{dt}{t}=2\sqrt{t}=2\sqrt{x^2+4x+10} \quad \ldots(2)$$

$$I_2=\int\frac{1}{\sqrt{x^2+4x+10}}dx$$

$$=\int\frac{1}{\sqrt{\left(x^2+4x+4\right)+6}}dx$$

$$=\int\frac{1}{\sqrt{(x+2)^2+\left(\sqrt{6}\right)^2}}dx$$

$$=\log\left|(x+2)\sqrt{x^2+4x+10}\right| \quad \ldots(3)$$

Using equations (2) and (3) in (1), we get

$$\int\frac{5x+3}{\sqrt{x^2+4x+10}}dx=\frac{5}{2}\left[2\sqrt{x^2+4x+10}\right]-7\log\left|(x+2)\sqrt{x^2+4x+10}\right|+C$$

$$=5\sqrt{x^2+4x+10}-7\log\left|(x+2)\sqrt{x^2+4x+10}\right|+C$$

Choose the correct answer in Exercises 24 and 25.

Question 24:

$\int \frac{dx}{x^2+2x+2}$ equals

(A) $x\tan^{-1}(x+1)+C$ (B) $\tan^{-1}(x+1)+C$

(C) $(x+1)\tan^{-1}x+C$ (D) $\tan^{-1}x+C$

Solution:

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x^2+2x+1)+1}$$

$$= \int \frac{1}{(x+1)^2+(1)^2}dx$$

$$= \left[\tan^{-1}(x+1)\right]+C$$

Hence, the correct option is B.

Question 25:

$\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$ (B) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{8}\right)+C$

(C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$ (D) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Solution:

$$\int \frac{dx}{\sqrt{9x-4x^2}}$$

$$=\int \frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}dx$$

$$=\int \frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x+\frac{81}{64}-\frac{81}{64}\right)}}dx$$

$$=\int \frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}dx$$

$$=\frac{1}{2}\int \frac{1}{\left(\frac{9}{8}\right)^2-\left(x-\frac{9}{8}\right)^2}dx$$

$$=\frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right]+C \qquad \left(\int \frac{dy}{\sqrt{a^2-y^2}}=\sin^{-1}\frac{y}{a}+C\right)$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

Hence, the correct option is B.

EXERCISE 7.5

Integrate the rational functions in Exercises 1 to 21.

Question 1:

$$\frac{x}{(x+1)(x+2)}$$

Solution:

Let $\frac{x}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

$\Rightarrow x=A(x+2)+B(x+1)$

Equating the coefficients of x and constant term, we get

$A+B=1$

$2A+B=0$

On solving, we get

$A=-1$ and $B=2$

$$\therefore \frac{x}{(x+1)(x+2)}=\frac{-1}{(x+1)}+\frac{2}{(x+2)}$$

$$\Rightarrow \int\frac{x}{(x+1)(x+2)}dx=\int\frac{-1}{(x+1)}+\frac{2}{(x+2)}dx$$

$$=-\log|x+1|+2\log|x+2|+C$$

$$=\log(x+2)^2-\log(x+1)+C$$

$$=\log\frac{(x+2)^2}{(x+1)}+C$$

Question 2:

$$\frac{1}{x^2-9}$$

Solution:

Let $\frac{1}{(x+3)(x-3)}=\frac{A}{(x+3)}+\frac{B}{(x-3)}$

$1=A(x-3)+B(x+3)$

Equating the coefficients of x and constants term, we get

$A+B=0$

$-3A+3B=1$

On solving, we get

$A=-\frac{1}{6}$ and $B=\frac{1}{6}$

$$\therefore \frac{1}{(x+3)(x-3)}=\frac{-1}{6(x+3)}+\frac{1}{6(x-3)}$$

$$\Rightarrow \int \frac{1}{(x^2-9)}dx=\int\left(\frac{-1}{6(x+3)}+\frac{1}{6(x-3)}\right)dx$$

$$=-\frac{1}{6}\log|x+3|+\frac{1}{6}\log|x-3|+C$$

$$=\frac{1}{6}\log\frac{|(x-3)|}{|(x+3)|}+C$$

Question 3:

$$\frac{3x-1}{(x-1)(x-2)(x-3)}$$

Solution:

Let $\frac{3x-1}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$

$$3x-1=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2) \quad \ldots(1)$$

Equating the coefficients of x^2, x and constant terms, we get

$A+B+C=0$

$-5A-4B-3C=3$

$6A+3B+2C=-1$

Solving these equations, we get

$A=1, B=-5$ and $C=4$

$$\therefore \frac{3x-1}{(x-1)(x-2)(x-3)}=\frac{1}{(x-1)}-\frac{5}{(x-2)}+\frac{4}{(x-3)}$$

$$\Rightarrow \int\frac{3x-1}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{(x-1)}-\frac{5}{(x-2)}+\frac{4}{(x-3)}\right\}dx$$

$$=\log|x-1|-5\log|x-2|+4\log|x-3|+C$$

Question 4:

$$\frac{x}{(x-1)(x-2)(x-3)}$$

Solution:

Let $\frac{x}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$

$$x=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2) \quad \ldots(1)$$

Equating the coefficients of x^2, x and constant terms, we get

$$A+B+C=0$$
$$-5A-4B-3C=1$$
$$6A+4B+2C=0$$

Solving these equations, we get

$$A=\frac{1}{2}, B=-2 \text{ and } C=\frac{3}{2}$$

$$\therefore \frac{x}{(x-1)(x-2)(x-3)}=\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}\right\}dx$$

$$=\frac{1}{2}\log|x-1|-2\log|x-2|+\frac{3}{2}\log|x-3|+C$$

Question 5:

$$\frac{2x}{x^2+3x+2}$$

Solution:

$$\text{Let } \frac{2x}{x^2+3x+2}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$$

$$2x=A(x+2)+B(x+1) \qquad \ldots(1)$$

Equating the coefficients of x and constant terms, we get

$$A+B=2$$
$$2A+B=0$$

Solving these equations, we get

$$A=-2 \text{ and } B=4$$

$$\therefore \frac{2x}{(x+1)(x+2)}=\frac{-2}{(x+1)}+\frac{4}{(x+2)}$$

$$\Rightarrow \int \frac{2x}{(x+1)(x+2)}dx=\int\left\{\frac{4}{(x+2)}-\frac{2}{(x+1)}\right\}dx$$

$$=4\log|x+2|-2\log|x+1|+C$$

Question 6:

$$\frac{1-x^2}{x(1-2x)}$$

Solution:

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(1-x^2)$ by $x(1-2x)$, we get

$$\frac{1-x^2}{x(1-2x)}=\frac{1}{2}+\frac{1}{2}\left(\frac{2-x}{x(1-2x)}\right)\ldots(1)$$

Let $\frac{2-x}{x(1-2x)}=\frac{A}{x}+\frac{B}{(1-2x)}$

$\Rightarrow(2-x)=A(1-2x)+Bx$

Equating the coefficients of x and constant term, we get

$-2A+B=-1$

And, $A=2$

Solving these equations, we get

$A=2$ and $B=3$

$\therefore\frac{2-x}{x(1-2x)}=\frac{2}{x}+\frac{3}{(1-2x)}$

Substituting in equation (1), we get

$$\frac{1-x^2}{x(1-2x)}=\frac{1}{2}+\frac{1}{2}\left\{\frac{2}{x}+\frac{3}{(1-2x)}\right\}$$

$$\Rightarrow\int\frac{1-x^2}{x(1-2x)}dx=\int\left\{\frac{1}{2}+\frac{1}{2}\left(\frac{2}{x}+\frac{3}{(1-2x)}\right)\right\}dx$$

$$=\frac{x}{2}+\log|x|+\frac{3}{2(-2)}\log|1-2x|+C$$

$$=\frac{x}{2}+\log|x|-\frac{3}{4}\log|1-2x|+C$$

Question 7:

$$\frac{x}{(x^2+1)(x-1)}$$

Solution:

Let $\frac{x}{(x^2+1)(x-1)}=\frac{Ax+B}{(x^2+1)}+\frac{C}{(x-1)}$ ......(1)

$x=(Ax+B)(x-1)+C(x^2+1)$

$x=Ax^2-Ax+Bx-B+Cx^2+C$

Equating the coefficients of x^2, x and constant term, we get

$A+C=0$

$-A+B=1$

$-B+C=0$

On solving these equations, we get

$A=-\frac{1}{2}, B=\frac{1}{2}$ and $C=\frac{1}{2}$

From equation (1), we get

$$\therefore \frac{x}{(x^2+1)(x-1)} = \frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1} + \frac{\frac{1}{2}}{(x-1)}$$

$$\Rightarrow \int \frac{x}{(x^2+1)(x-1)} = -\frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$$

$$= -\frac{1}{4}\int \frac{2x}{x^2+1}dx + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

Consider $\int \frac{2x}{x^2+1}dx$, let $(x^2+1) = t \Rightarrow 2xdx = dt$

$$\Rightarrow \int \frac{2x}{x^2+1}dx = \int \frac{dt}{t} = \log|t| = \log|x^2+1|$$

$$\therefore \int \frac{x}{(x^2+1)(x-1)} = -\frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + \frac{1}{2}\log|x-1| + C$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$$

Question 8:

$$\frac{x}{(x-1)^2(x+2)}$$

Solution:

Let $\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$

$$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Equating the coefficients of x^2, x and constant term, we get

$$A + C = 0$$

$$A + B - 2C = 1$$

$$-2A + 2B + C = 0$$

On solving these equations, we get

$$A = \frac{2}{9}, B = \frac{1}{3} \text{ and } C = -\frac{2}{9}$$

$$\therefore \frac{x}{(x-1)^2(x+2)} = \frac{2}{9(x+1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}$$

$$\Rightarrow \int \frac{x}{(x-1)^2(x+2)}dx = \frac{2}{9}\int \frac{1}{(x-1)}dx + \frac{1}{3}\int \frac{1}{(x-1)^2}dx - \frac{2}{9}\int \frac{1}{(x-2)}dx$$

$$= \frac{2}{9}\log|x-1| + \frac{1}{3}\left(\frac{-1}{x-1}\right) - \frac{2}{9}\log|x+2| + C$$

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C$$

Question 9:

$$\frac{3x+5}{x^3-x^2-x+1}$$

Solution:

$$\frac{3x+5}{x^3-x^2-x+1}=\frac{3x+5}{(x-1)^2(x+1)}$$

Let $\frac{3x+5}{(x-1)^2(x+1)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}$

$$3x+5=A(x-1)(x+1)+B(x+1)+C(x-1)^2$$

$$3x+5=A(x-1)(x+1)+B(x+1)+C(x^2-2x+1) \qquad \ldots(1)$$

Equating the coefficients of x^2, x and constant term, we get

$$A+C=0$$
$$B-2C=3$$
$$-A+B+C=5$$

On solving these equations, we get

$$A=-\frac{1}{2}, B=4 \text{ and } C=\frac{1}{2}$$

$$\therefore \frac{3x+5}{(x-1)^2(x+1)}=\frac{-1}{2(x-1)}+\frac{4}{(x-1)^2}+\frac{1}{2(x+1)}$$

$$\Rightarrow \int\frac{3x+5}{(x-1)^2(x+1)}dx=-\frac{1}{2}\int\frac{1}{(x-1)}dx+4\int\frac{1}{(x-1)^2}dx+\frac{1}{2}\int\frac{1}{(x+1)}dx$$

$$=-\frac{1}{2}\log|x-1|+4\left(\frac{-1}{x-1}\right)+\frac{1}{2}\log|x+1|+C$$

$$=\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{(x-1)}+C$$

Question 10:

$$\frac{2x-3}{(x^2-1)(2x+3)}$$

Solution:

$$\frac{2x-3}{(x^2-1)(2x+3)}=\frac{2x-3}{(x-1)(x+1)(2x+3)}$$

Let $\frac{2x-3}{(x-1)(x+1)(2x+3)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{C}{(2x+3)}$

$\Rightarrow (2x-3) = A(x-1)(2x-3) + B(x+1)(2x+3) + C(x+1)(x-1)$

$\Rightarrow (2x-3) = A(2x^2+x-3) + B(2x^2+5x+3) + C(x^2-1)$

$\Rightarrow (2x-3) = (2A+2B+C)x^2 + (A+5B)x + (-3A+3B-C)$

Equating the coefficients of x^2, x and constant term, we get

$2A+2B+C=0$

$A+5B=2$

$-3A+3B-C=-3$

On solving, we get

$A=\frac{5}{2}, B=-\frac{1}{10}$ and $C=-\frac{24}{5}$

$\therefore \frac{2x-3}{(x+1)(x-1)(2x+3)} = \frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$

$\Rightarrow \int \frac{2x-3}{(x+1)(x-1)(x+1)}dx = \frac{5}{2}\int \frac{1}{(x+1)}dx - \frac{1}{10}\int \frac{1}{(x-1)}dx - \frac{24}{5}\int \frac{1}{(2x+3)}dx$

$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{24}{5\times 2}\log|2x+3| + C$

$= \frac{5}{2}\log|x+1| - \frac{1}{10}\log|x-1| - \frac{12}{5}\log|2x+3| + C$

Question 11:

$\frac{5x}{(x+1)(x^2-4)}$

Solution:

$\frac{5x}{(x+1)(x^2-4)} = \frac{5x}{(x+1)(x+2)(x-2)}$

Let $\frac{5x}{(x+1)(x^2-4)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-2)}$

$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2) \quad \ldots(1)$

Equating the coefficients of x^2, x and constant term, we get

$A+B+C=0$

$-B+3C=5$

$-4A-2B+2C=0$

On solving, we get

$A=\frac{5}{3}, B=-\frac{5}{2}$ and $C=\frac{5}{6}$

$$\therefore \frac{5x}{(x+1)(x+2)(x-2)}=\frac{5}{3(x+1)}-\frac{5}{2(x+2)}+\frac{5}{6(x-2)}$$

$$\Rightarrow \int \frac{5x}{(x+1)(x+2)(x-2)}dx=\frac{5}{3}\int \frac{1}{(x+1)}dx-\frac{5}{2}\int \frac{1}{(x+2)}dx+\frac{5}{6}\int \frac{1}{(x-2)}dx$$

$$=\frac{5}{3}\log|x+1|-\frac{5}{2}\log|x+2|+\frac{5}{6}\log|x-2|+C$$

Question 12:

$$\frac{x^3+x+1}{x^2-1}$$

Solution:

On dividing (x^3+x+1) by x^2-1, we get

$$\frac{x^3+x+1}{x^2-1}=x+\frac{2x+1}{x^2-1}$$

Let $\frac{2x+1}{x^2-1}=\frac{A}{(x+1)}+\frac{B}{(x+1)}$

$$2x+1=A(x-1)+B(x+1) \quad \ldots(1)$$

Equating the coefficients of x and constant term, we get

$$A+B=2$$

$$-A+B=1$$

On solving, we get

$A=\frac{1}{2}$ and $B=\frac{3}{2}$

$$\therefore \frac{x^3+x+1}{x^2-1}=x+\frac{1}{2(x+1)}+\frac{3}{2(x-1)}$$

$$\Rightarrow \int \frac{x^3+x+1}{x^2-1}dx=\int x dx+\frac{1}{2}\int \frac{1}{(x+1)}dx+\frac{3}{2}\int \frac{1}{(x-1)}dx$$

$$=\frac{x^2}{2}+\frac{1}{2}\log|x+1|+\frac{3}{2}\log|x-1|+C$$

Question 13:

$$\frac{2}{(1-x)(1+x^2)}$$

Solution:

Let $\frac{2}{(1-x)(1+x^2)}=\frac{A}{(1-x)}+\frac{Bx+C}{(1+x^2)}$

$2=A(1+x^2)+(Bx+C)(1-x)$

$2=A+Ax^2+Bx-Bx^2+C-Cx$

Equating the coefficients of x^2, x and constant term, we get

$A-B=0$

$B-C=0$

$A+C=2$

On solving these equations, we get

$A=1, B=1$ and $C=1$

$\therefore \frac{2}{(1-x)(1+x^2)}=\frac{1}{1-x}+\frac{x+1}{1+x^2}$

$\Rightarrow \int\frac{2}{(1-x)(1+x^2)}dx=\int\frac{1}{1-x}dx+\int\frac{x}{1+x^2}dx+\int\frac{1}{1+x^2}dx$

$=-\int\frac{1}{1-x}dx+\frac{1}{2}\int\frac{2x}{1+x^2}dx+\int\frac{1}{1+x^2}dx$

$=-\log|x-1|+\frac{1}{2}\log\left|1+x^2\right|+\tan^{-1}x+C$

Question 14:

$\frac{3x-1}{(x+2)^2}$

Solution:

Let $\frac{3x-1}{(x+2)^2}=\frac{A}{(x+2)}+\frac{B}{(x+2)^2}$

$\Rightarrow 3x-1=A(x+2)+B$

Equating the coefficient of x and constant term, we get

$A = 3$

$2A + B = -1 \Rightarrow B = -7$

$\therefore \frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} - \frac{7}{(x+2)^2}$

$\Rightarrow \int \frac{3x-1}{(x+2)^2} dx = 3\int \frac{1}{(x+2)} dx - 7\int \frac{x}{(x+2)^2} dx$

$= 3\log|x+2| - 7\left(\frac{-1}{(x+2)}\right) + C$

$= 3\log|x+2| + \frac{7}{(x+2)} + C$

Question 15:

$\frac{1}{x^4 - 1}$

Solution:

$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(x^2+1)}$

Let $\frac{1}{(x+1)(x-1)(x^2+1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+1)}$

$1 = A(x-1)(1+x^2) + B(x+1)(1+x^2) + (Cx+D)(x^2-1)$

$1 = A(x^3+x-x^2-1) + B(x^3+x+x^2+1) + Cx^3 + Dx^2 - Cx - D$

$1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$

Equating the coefficients of x^3, x^2, x and constant term, we get

$A + B + C = 0$

$-A + B + D = 0$

$A + B - C = 0$

$-A + B - D = 1$

On solving, we get

$A=\frac{-1}{4}, B=\frac{1}{4}, C=0$ and $D=-\frac{1}{2}$

$$\therefore \frac{1}{(x^4-1)}=\frac{-1}{4(x+1)}+\frac{1}{4(x-1)}+\frac{1}{2(1+x^2)}$$

$$\Rightarrow \int \frac{1}{(x^4-1)}dx=\int \frac{-1}{4(x+1)}dx+\int \frac{1}{4(x-1)}dx-\int \frac{1}{2(1+x^2)}dx$$

$$\Rightarrow \int \frac{1}{(x^4-1)}dx=-\frac{1}{4}\log|x+1|+\frac{1}{4}\log|x-1|-\frac{1}{2}\tan^{-1}x+C$$

$$=\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$$

Question 16:

$$\frac{1}{x(x^n+1)}$$

[Hint: multiply numerator and denominator by x^{n-1} and put $x^n=t$]

Solution:

$$\frac{1}{x(x^n+1)}$$

Multiplying numerator and denominator by x^{n-1}, we get

$$\frac{1}{x(x^n+1)}=\frac{x^{n-1}}{x^{n-1}x(x^n+1)}=\frac{x^{n-1}}{x^n(x^n+1)}$$

Let $x^n=t \Rightarrow nx^{n-1}dx=dt$

$$\therefore \int \frac{1}{x(x^n+1)}dx=\int \frac{x^{n-1}}{x^n(x^n+1)}dx=\frac{1}{n}\int \frac{1}{t(t+1)}dt$$

Let $\frac{1}{t(t+1)}=\frac{A}{t}+\frac{B}{(t+1)}$

$1=A(1+t)+Bt \quad \ldots(1)$

Equating the coefficients of t and constant term, we get

$A=1$ and $B=-1$

$$\therefore \frac{1}{t(t+1)}=\frac{1}{t}-\frac{1}{(1+t)}$$

$$\Rightarrow \int \frac{1}{x(x^n+1)}dx=\frac{1}{n}\int\left\{\frac{1}{t}-\frac{1}{(1+t)}\right\}dx$$

$$=\frac{1}{n}\left[\log|t|-\log|t+1|\right]+C$$

$$=\frac{1}{n}\left[\log|x^n|-\log|x^n+1|\right]+C$$

$$=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C$$

Question 17:

$\frac{\cos x}{(1-\sin x)(2-\sin x)}$ [Hint: Put $\sin x=t$]

Solution:

$\frac{\cos x}{(1-\sin x)(2-\sin x)}$ Put, $\sin x=t\Rightarrow \cos x dx=dt$

$$\therefore \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx=\int\frac{dt}{(1-t)(2-t)}$$

Let $\frac{1}{(1-t)(2-t)}=\frac{A}{(1-t)}+\frac{B}{(2-t)}$

$1=A(2-t)+B(1-t) \quad \ldots(1)$

Equating the coefficients of t and constant, we get

$-2A-B=0$, and $2A+B=1$

On solving, we get

$A=1$ and $B=-1$

$$\therefore \frac{1}{(1-t)(2-t)}=\frac{1}{(1-t)}-\frac{1}{(2-t)}$$

$$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx=\int\left\{\frac{1}{1-t}-\frac{1}{(2-t)}\right\}dt$$

$$=-\log|1-t|+\log|2-t|+C$$

$$=\log\left|\frac{2-t}{1-t}\right|+C$$

$$=\log\left|\frac{2-\sin x}{1-\sin x}\right|+C$$

Question 18:

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$$

Solution:

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}=\frac{(4x^2+10)}{(x^2+3)(x^2+4)}$$

Let $\frac{(4x^2+10)}{(x^2+3)(x^2+4)}=\frac{Ax+B}{(x^2+3)}+\frac{Cx+D}{(x^2+4)}$

$$4x^2+10=(Ax+B)(x^2+4)+(Cx+D)(x^2+3)$$

$$4x^2+10=Ax^3+4Ax+Bx^2+4B+Cx^3+3Cx+Dx^2+3D$$

$$4x^2+10=(A+C)x^3+(B+D)x^2+(4A+3C)x+(4B+3D)$$

Equating the coefficients of x^3, x^2, x and constant term, we get

$A+C=0$

$B+D=4$

$4A+3C=0$

$4B+3D=10$

On solving these equations, we get

$A=0, B=-2, C=0$ and $D=6$

$$\therefore \frac{(4x^2+10)}{(x^2+3)(x^2+4)}=\frac{-2}{(x^2+3)}+\frac{6}{(x^2+4)}$$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}=\left(\frac{-2}{(x^2+3)}+\frac{6}{(x^2+4)}\right)$$

$$\Rightarrow \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx=\int 1-\left\{\frac{-2}{(x^2+3)}+\frac{6}{(x^2+4)}\right\}dx$$

$$=\int\left\{1+\frac{2}{x^2+(\sqrt{3})^2}-\frac{6}{x^2+2^2}\right\}dx$$

$$=x+2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right)-6\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right)+C$$

$$=x+\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}-3\tan^{-1}\frac{x}{2}+C$$

Question 19:

$$\frac{2x}{(x^2+1)(x^2+3)}$$

Solution:

$$\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}$$

Put, $x^2=t \Rightarrow 2xdx=dt$

$$\therefore \int \frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}dx=\int \frac{dt}{(t+1)(t+3)} \quad \ldots(1)$$

Let $\frac{1}{(t+1)(t+3)}=\frac{A}{(t+1)}+\frac{B}{(t+3)}$

$1=A(t+3)+B(t+1) \quad \ldots(2)$

Equating the coefficients of t and constant, we get

$A+B=0$ and $3A+B=1$

On solving, we get

$A=\frac{1}{2}$ and $B=-\frac{1}{2}$

$$\therefore \frac{1}{(t+1)(t+3)}=\frac{1}{2(t+1)}+\frac{1}{2(t+3)}$$

$$\Rightarrow \int \frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}dx=\int \left\{\frac{1}{2(t+1)}-\frac{1}{2(t+3)}\right\}dt$$

$$=\frac{1}{2}\log|(t+1)|-\frac{1}{2}\log|t+3|+C$$

$$=\frac{1}{2}\log\left|\frac{t+1}{t+3}\right|+C=\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right|+C$$

Question 20:

$$\frac{1}{x\left(x^4-1\right)}$$

Solution:

$$\frac{1}{x\left(x^4-1\right)}$$

Multiplying Nr and Dr by x^3, we get

$$\frac{1}{x\left(x^4-1\right)}=\frac{x^3}{x^4\left(x^4-1\right)}$$

$$\therefore \int \frac{1}{x\left(x^4-1\right)}dx=\int \frac{x^3}{x^4\left(x^4-1\right)}dx$$

Put, $x^4=t \Rightarrow 4x^3=dt$

$$\therefore \int \frac{1}{x\left(x^4-1\right)}dx=\frac{1}{4}\int \frac{dt}{t(t-1)}$$

Let $\frac{1}{t(t-1)}=\frac{A}{t}+\frac{B}{(t-1)}$

$1=A(t-1)+Bt \quad \ldots(1)$

Equating the coefficients of t and constant, we get

$A=-1$ and $B=1$

$$\Rightarrow \frac{1}{t(t-1)}=\frac{-1}{t}+\frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{x\left(x^4-1\right)}dx=\frac{1}{4}\int\left\{\frac{-1}{t}+\frac{1}{t-1}\right\}dt$$

$$=\frac{1}{4}\left[-\log|t|+\log|t-1|\right]+C$$

$$=\frac{1}{4}\log\left|\frac{t-1}{t}\right|+C=\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right|+C$$

Question 21:

$\frac{1}{\left(e^x-1\right)}$ [Hint: Put $e^x=t$]

Solution:

Put $e^x=t \Rightarrow e^x dx=dt$

$$\Rightarrow \int \frac{1}{\left(e^x-1\right)}dx=\int \frac{1}{t-1}\times\frac{dt}{t}=\int \frac{1}{t(t-1)}dt$$

Let $\frac{1}{t(t-1)}=\frac{A}{t}+\frac{B}{t-1}$

$1=A(t-1)+Bt \qquad \ldots(1)$

Equating the coefficients of t and constant, we get

$A=-1$ and $B=1$

$$\therefore \frac{1}{t(t-1)}=\frac{-1}{t}+\frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{t(t-1)}dt=\log\left|\frac{t-1}{t}\right|+C$$

$$=\log\left|\frac{e^x-1}{e^x}\right|+C$$

Question 22:

$\int \frac{x\,dx}{(x-1)(x-2)}$ equals

A. $\log\left|\frac{(x-1)^2}{(x-2)}\right|+C$

B. $\log\left|\frac{(x-2)^2}{(x-1)}\right|+C$

C. $\log\left|\left(\frac{x-1}{x-2}\right)^2\right|+C$

D. $\log\left|(x-1)(x-2)\right|+C$

Solution:

Let $\frac{x}{(x-1)(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}$

$x=A(x-2)+B(x-1) \quad \ldots(1)$

Equating the coefficients of x and constant, we get

$A=-1$ and $B=2$

$$\therefore \frac{x}{(x-1)(x-2)}=\frac{-1}{(x-1)}+\frac{2}{(x-2)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)}dx=\int\left\{\frac{-1}{(x-1)}+\frac{2}{(x-2)}\right\}dx$$

$$=-\log|x-1|+2\log|x-2|+C$$

$$=\log\left|\frac{(x-2)^2}{x-1}\right|+C$$

Thus, the correct option is *B*.

Question 23:

$\int \frac{dx}{x\left(x^2+1\right)}$ equals

$A.\ \log|x|-\frac{1}{2}\log\left(x^2+1\right)+C$

$B.\ \log|x|+\frac{1}{2}\log\left(x^2+1\right)+C$

$C.\ -\log|x|+\frac{1}{2}\log\left(x^2+1\right)+C$

$D.\ \frac{1}{2}\log|x|+\log\left(x^2+1\right)+C$

Solution:

Let $\frac{1}{x\left(x^2+1\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$

$1=A\left(x^2+1\right)+\left(Bx+C\right)x$

Equating the coefficients of x^2, x and constant terms, we get

$A+B=0$

$C=0$

$A=1$

On solving these equations, we get

$A=1\ B=-1$ and $C=0$

$\therefore \frac{1}{x\left(x^2+1\right)}=\frac{1}{x}+\frac{-x}{x^2+1}$

$\Rightarrow \int \frac{1}{x\left(x^2+1\right)}dx=\int\left\{\frac{1}{x}-\frac{x}{x^2-1}\right\}dx$

$=\log|x|-\frac{1}{2}\log\left|x^2+1\right|+C$

Thus, the correct option is A.

EXERCISE 7.6

Integrate the functions in Exercises 1 to 22.

Question 1:

$x\sin x$

Solution:

Let $I = \int x\sin x dx$

Taking $u = x$ and $v = \sin x$ and integrating by parts,

$$I = x\int \sin x dx - \int\left\{\left(\frac{d}{dx}(x)\right)\int \sin x dx\right\}dx$$

$$= x(-\cos x) - \int 1\cdot(-\cos x)dx$$

$$= -x\cos x + \sin x + C$$

Question 2:

$x\sin 3x$

Solution:

Let $I = \int x\sin 3x dx$

Taking $u = x$ and $v = \sin 3x$ and integrating by parts,

$$I = x\int \sin 3x dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sin 3x dx\right\}dx$$

$$= x\left(\frac{-\cos 3x}{3}\right) - \int 1\cdot\left(\frac{-\cos 3x}{3}\right)dx$$

$$= \frac{-x\cos 3x}{3} + \frac{1}{3}\int \cos 3x dx$$

$$= \frac{-x\cos 3x}{3} + \frac{1}{9}\sin 3x + C$$

Question 3:

$x^2 e^x$

Solution:

Let $I = \int x^2 e^x dx$

Taking $u = x^2$ and $v = e^x$ and integrating by parts, we get

$$I = x^2\int e^x dx - \int\left\{\left(\frac{d}{dx}x^2\right)\int e^x dx\right\}dx$$
$$= x^2e^x - \int 2x.e^x dx$$
$$= x^2e^x - 2\int x.e^x dx$$

Again using integration by parts, we get

$$= x^2e^x - 2\left[x\int e^x dx - \int\left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\}dx\right]$$
$$= x^2e^x - 2\left[xe^x - \int e^x dx\right]$$
$$= x^2e^x - 2\left[xe^x - e^x\right]$$
$$= x^2e^x - 2xe^x + 2e^x + C$$
$$= e^x\left(x^2 - 2x + 2\right) + C$$

Question 4:

$x\log x$

Solution:

Let $I = \int x\log x dx$

Taking $u = \log x$ and $v = x$ and integrating by parts, we get

$$I = \log x\int x dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int x dx\right\}dx$$
$$= \log x.\frac{x^2}{2} - \int\frac{1}{x}.\frac{x^2}{2}dx$$
$$= \frac{x^2\log x}{2} - \int\frac{x}{2}dx$$
$$= \frac{x^2\log x}{2} - \frac{x^2}{4} + C$$

Question 5:

$x\log 2x$

Solution:

Let $I = \int x\log 2x dx$

Taking $u = \log 2x$ and $v = x$ and integrating by parts, we get

$$I = \log 2x \int x dx - \int \left\{ \left(\frac{d}{dx} \log 2x \right) \int x dx \right\} dx$$

$$= \log 2x . \frac{x^2}{2} - \int \frac{2}{2x} . \frac{x^2}{2} dx$$

$$= \frac{x^2 \log 2x}{2} - \int \frac{x}{2} dx$$

$$= \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + C$$

Question 6:

$x^2 \log x$

Solution:

Let $I = \int x^2 \log x dx$

Taking $u = \log x$ and $v = x^2$ and integrating by parts, we get

$$I = \log x \int x^2 dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x^2 dx \right\} dx$$

$$= \log x . \left(\frac{x^3}{3} \right) - \int \frac{1}{x} . \frac{x^3}{3} dx$$

$$= \frac{x^3 \log x}{3} - \int \frac{x^2}{3} dx$$

$$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C$$

Question 7:

$x \sin^{-1} x$

Solution:

Let $I = \int x \sin^{-1} x dx$

Taking $u = \sin^{-1} x$ and $v = x$ and integrating by parts, we get

$$I = \sin^{-1} x \int x dx - \int \left\{ \left(\frac{d}{dx} \sin^{-1} x \right) \int x dx \right\} dx$$

$$= \sin^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right\} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right\} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \int \sqrt{1-x^2} dx - \int \frac{1}{\sqrt{1-x^2}} dx \right\}$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x - \sin^{-1} x \right\} + C$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{x}{4} \sqrt{1-x^2} + \frac{1}{4} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + C$$

$$= \frac{1}{4} \left(2x^2 - 1 \right) \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} + C$$

Question 8:

$x \tan^{-1} x$

Solution:

Let $I = \int x \tan^{-1} x dx$

Taking $u = \tan^{-1} x$ and $v = x$ and integrating by parts, we get

$$I = \tan^{-1} x \int x dx - \int \left\{ \left(\frac{d}{dx} \tan^{-1} x \right) \int x dx \right\} dx$$

$$= \tan^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{1}{1+x^2} . \frac{x^2}{2} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left\{ \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} \right\} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \left(x - \tan^{-1} x \right) + C$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C$$

Question 9:

$x \cos^{-1} x$

Solution:

Let $I = \int x \cos^{-1} x dx$

Taking $u = \cos^{-1} x$ and $v = x$ and integrating by parts, we get

$$I = \cos^{-1} x \int x dx - \int \left\{ \left(\frac{d}{dx} \cos^{-1} x \right) \int x dx \right\} dx$$

$$= \cos^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{-1}{\sqrt{1-x^2}} . \frac{x^2}{2} dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \left\{ \sqrt{1-x^2} + \left(\frac{-1}{\sqrt{1-x^2}} \right) \right\} dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} \int \sqrt{1-x^2} dx - \frac{1}{2} \int \left(\frac{-1}{\sqrt{1-x^2}} \right) dx$$

$$= \frac{x^2 \cos^{-1} x}{2} - \frac{1}{2} I_1 - \frac{1}{2} \cos^{-1} x .... (1)$$

Where, $I_1 = \int \sqrt{1-x^2}dx$

$$\Rightarrow I_1 = \sqrt{1-x^2}\int 1dx - \int \frac{d}{dx}\sqrt{1-x^2}\int 1dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{-2x}{2\sqrt{1-x^2}}xdx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \left\{\int \sqrt{1-x^2}dx + \int \frac{-dx}{\sqrt{1-x^2}}\right\}$$

$$\Rightarrow I_1 = x\sqrt{1-x^2} - \left\{I_1 + \cos^{-1}x\right\}$$

$$\Rightarrow 2I_1 = x\sqrt{1-x^2} - \cos^{-1}x$$

$$\therefore I_1 = \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1}x$$

Substituting in (1),

$$I = \frac{x^2\cos^{-1}x}{2} - \frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1}x\right) - \frac{1}{2}\cos^{-1}x$$

$$= \frac{(2x^2-1)}{4}\cos^{-1}x - \frac{x}{4}\sqrt{1-x^2} + C$$

Question 10:

$(\sin^{-1}x)^2$

Solution:

Let $I = \int (\sin^{-1}x)^2 .1\, dx$

Taking $u = (\sin^{-1}x)^2$ and $v = 1$ and integrating by parts, we get

$$I=\int\left(\sin^{-1}x\right)^2.\int 1dx-\int\left\{\frac{d}{dx}\left(\sin^{-1}x\right)^2.\int 1dx\right\}dx$$

$$=\left(\sin^{-1}x\right)^2.x-\int\frac{2\sin^{-1}x}{\sqrt{1-x^2}}.xdx$$

$$=x\left(\sin^{-1}x\right)^2+\int\sin^{-1}x\left(\frac{-2x}{\sqrt{1-x^2}}\right)dx$$

$$=x\left(\sin^{-1}x\right)^2+\left[\sin^{-1}x\int\frac{-2x}{\sqrt{1-x^2}}dx-\int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$$

$$=x\left(\sin^{-1}x\right)^2+\left[\sin^{-1}x.2\sqrt{1-x^2}-\int\frac{1}{\sqrt{1-x^2}}.2\sqrt{1-x^2}dx\right]$$

$$=x\left(\sin^{-1}x\right)^2+2\sqrt{1-x^2}\sin^{-1}x-\int 2dx$$

$$=x\left(\sin^{-1}x\right)^2+2\sqrt{1-x^2}\sin^{-1}x-2x+C$$

Question 11:

$$\frac{x\cos^{-1}x}{\sqrt{1-x^2}}$$

Solution:

Let $I=\int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

$$I=\frac{-1}{2}\int\frac{-2x}{\sqrt{1-x^2}}.\cos^{-1}xdx$$

Taking $u=\cos^{-1}x$ and $v=\left(\frac{-2x}{\sqrt{1-x^2}}\right)$ and integrating by parts, we get

$$I=\frac{-1}{2}\left[\cos^{-1}x\int\frac{-2x}{\sqrt{1-x^2}}dx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$$

$$=\frac{-1}{2}\left[\cos^{-1}x.2\sqrt{1-x^2}-\int\frac{-1}{\sqrt{1-x^2}}.2\sqrt{1-x^2}dx\right]$$

$$=\frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x+\int 2dx\right]$$

$$=\frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x+2x\right]+C$$

$$=-\left[\sqrt{1-x^2}\cos^{-1}x+x\right]+C$$

Question 12:

$$x\sec^2 x$$

Solution:

Let $I=\int x\sec^2 xdx$

Taking $u = x$ and $v = \sec^2 x$ and integrating by parts, we get

$$I = x\int \sec^2 x dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sec^2 x dx\right\}dx$$

$$= x\tan x - \int 1.\tan x dx$$

$$= x\tan x + \log\left|\cos x\right| + C$$

Question 13:

$\tan^{-1} x$

Solution:

Let $I = \int 1.\tan^{-1} x dx$

Taking $u = \tan^{-1} x$ and $v = 1$ and integrating by parts, we get

$$I = \tan^{-1} x\int 1 dx - \int\left\{\left(\frac{d}{dx}\tan^{-1} x\right)\int 1.dx\right\}dx$$

$$= \tan^{-1} x.x - \int\frac{1}{1+x^2}x dx$$

$$= x\tan^{-1} x - \frac{1}{2}\int\frac{2}{1+x^2}dx$$

$$= x\tan^{-1} x - \frac{1}{2}\log\left|1+x^2\right| + C$$

$$= x\tan^{-1} x - \frac{1}{2}\log\left(1+x^2\right) + C$$

Question 14:

$x\left(\log x\right)^2$

Solution:

Let $I = \int x\left(\log x\right)^2 dx$

Taking $u = \left(\log x\right)^2$ and $v = x$ and integrating by parts, we get

$$I = \left(\log x\right)^2\int x dx - \int\left[\left\{\frac{d}{dx}\left(\log x\right)^2\right\}\int x dx\right]dx$$

$$= \frac{x^2}{2}\left(\log x\right)^2 - \left[\int 2\log x.\frac{1}{x}.\frac{x^2}{2}dx\right]$$

$$= \frac{x^2}{2}\left(\log x\right)^2 - \int x\log x dx$$

Again, using integration by parts, we get

$$I=\frac{x^2}{2}(\log x)^2-\left[\log x\int xdx-\int\left\{\left(\frac{d}{dx}\log x\right)\int xdx\right\}dx\right]$$
$$=\frac{x^2}{2}(\log x)^2-\left[\frac{x^2}{2}\log x-\int\frac{1}{x}.\frac{x^2}{2}dx\right]$$
$$=\frac{x^2}{2}(\log x)^2-\frac{x^2}{2}\log x+\frac{1}{2}\int xdx$$
$$=\frac{x^2}{2}(\log x)^2-\frac{x^2}{2}\log x+\frac{x^2}{4}+C$$

Question 15:

$(x^2+1)\log x$

Solution:

Let $I=\int(x^2+1)\log xdx=\int x^2\log xdx+\int\log xdx$

Let $I=I_1+I_2$...........(1)

Where, $I_1=\int x^2\log xdx$ and $I_2=\int\log xdx$

$I_1=\int x^2\log xdx$

Taking $u=\log x$ and $v=x^2$ and integrating by parts, we get

$$I_1=\log x\int x^2dx-\int\left[\left(\frac{d}{dx}\log x\right)\int x^2dx\right]dx$$
$$=\log x.\frac{x^3}{3}-\int\frac{1}{x}.\frac{x^3}{3}dx$$
$$=\frac{x^3}{3}\log x-\frac{1}{3}\left(\int x^2dx\right)$$
$$=\frac{x^3}{3}\log x-\frac{x^3}{9}+C_1\text{...........}(2)$$

$I_2=\int\log xdx$

Taking $u=\log x$ and $v=1$ and integrating by parts,

$$I_2=\log x\int 1.dx-\int\left[\left(\frac{d}{dx}\log x\right)\int 1.dx\right]$$
$$=\log x.x-\int\frac{1}{x}.xdx$$
$$=x\log x-\int 1.dx$$
$$=x\log x-x+C_2\text{...........}(3)$$

Using equations (2) and (3) in (1),

$$I=\frac{x^3}{3}\log x-\frac{x^3}{9}+C_1+x\log x-x+C_2$$
$$=\frac{x^3}{3}\log x-\frac{x^3}{9}+x\log x-x+(C_1+C_2)$$
$$=\left(\frac{x^3}{3}+x\right)\log x-\frac{x^3}{9}-x+C$$

Question 16:

$e^x(\sin x+\cos x)$

Solution:

Let $I=\int e^x(\sin x+\cos x)dx$

Let $f(x)=\sin x$

$f'(x)=\cos x$

$I=\int e^x\{f(x)+f'(x)\}dx$

Since, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

$\therefore I=e^x\sin x+C$

Question 17:

$$\frac{xe^x}{(1+x)^2}$$

Solution:

Let, $I=\int\frac{xe^x}{(1+x)^2}dx=\int e^x\left\{\frac{x}{(1+x)^2}\right\}dx$

$$=\int e^x\left\{\frac{1+x-1}{(1+x)^2}\right\}dx=\int e^x\left\{\frac{1}{1+x}-\frac{1}{(1+x)^2}\right\}dx$$

Here, $f(x)=\frac{1}{1+x}$ $\quad f'(x)=\frac{-1}{(1+x)^2}$

$\Rightarrow\int\frac{xe^x}{(1+x)^2}dx=\int e^x\{f(x)+f'(x)\}dx$

Since, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

$\therefore\int\frac{xe^x}{(1+x)^2}dx=e^x\frac{1}{1+x}+C$

Question 18:

$$e^x\left(\frac{1+\sin x}{1+\cos x}\right)$$

Solution:

$$e^x\left(\frac{1+\sin x}{1+\cos x}\right)=e^x\left(\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

$$=\frac{e^x\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}=\frac{1}{2}e^x\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^2$$

$$=\frac{1}{2}e^x\left[\tan\frac{x}{2}+1\right]^2$$

$$=\frac{1}{2}e^x\left[1+\tan\frac{x}{2}\right]^2$$

$$=\frac{1}{2}e^x\left[1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$

$$=\frac{1}{2}e^x\left[\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$

$$\frac{e^x(1+\sin x)dx}{(1+\cos x)}=e^x\left[\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right]\text{............}(1)$$

Let $\tan\frac{x}{2}=f(x)\quad$ so $f'(x)=\frac{1}{2}\sec^2\frac{x}{2}$

It is known that, $\int e^x\left\{f(x)+f'(x)\right\}dx=e^x f(x)+C$

From equation (1), we get

$$\int\frac{e^x(1+\sin x)}{(1+\cos x)}dx=e^x\tan\frac{x}{2}+C$$

Question 19:

$$e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$$

Solution:

Let $I=\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$

Here, $\frac{1}{x}=f(x)\qquad f'(x)=\frac{-1}{x^2}$

It is known that,

$$\int e^x \{f(x)+f'(x)\}dx$$
$$=e^x f(x)+C$$
$$\therefore I=\frac{e^x}{x}+C$$

Question 20:

$$\frac{(x-3)e^x}{(x-1)^3}$$

Solution:

$$\int e^x\left\{\frac{x-3}{(x-1)^3}\right\}dx=\int e^x\left\{\frac{x-1-2}{(x-1)^3}\right\}dx$$
$$=\int e^x\left\{\frac{1}{(x-1)^2}-\frac{2}{(x-1)^3}\right\}dx$$

Let $f(x)=\frac{1}{(x-1)^2}\quad f'(x)=\frac{-2}{(x-1)^3}$

It is known that,

$$\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$$
$$\therefore \int e^x\left\{\frac{(x-3)}{(x-1)^2}\right\}dx=\frac{e^x}{(x-1)^2}+C$$

Question 21:

$e^{2x}\sin x$

Solution:

Let $I=e^{2x}\sin xdx$................(1)

Taking $u=\sin x$ and $v=e^{2x}$ and integrating by parts, we get

$$I=\sin x\int e^{2x}dx-\int\left\{\left(\frac{d}{dx}\sin x\right)\int e^{2x}dx\right\}dx$$
$$\Rightarrow I=\sin x.\frac{e^{2x}}{2}-\int\cos x.\frac{e^{2x}}{2}dx$$
$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\int e^{2x}\cos xdx$$

Again, using integration by parts, we get

$$I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left[\cos x\int e^{2x}dx-\int\left\{\left(\frac{d}{dx}\cos x\right)\int e^{2x}dx\right\}dx\right]$$

$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left[\cos x.\frac{e^{2x}}{2}-\int(-\sin x)\frac{e^{2x}}{2}dx\right]$$

$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left[\frac{e^{2x}\cos x}{2}+\frac{1}{2}\int e^{2x}\sin xdx\right]$$

$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{e^{2x}\cos x}{4}-\frac{1}{4}I \qquad [\text{From }(1)]$$

$$\Rightarrow I+\frac{1}{4}I=\frac{e^{2x}\sin x}{2}-\frac{e^{2x}\cos x}{4}$$

$$\Rightarrow \frac{5}{4}I=\frac{e^{2x}\sin x}{2}-\frac{e^{2x}\cos x}{4}$$

$$\Rightarrow I=\frac{4}{5}\left[\frac{e^{2x}\sin x}{2}-\frac{e^{2x}\cos x}{4}\right]+C$$

$$\Rightarrow I=\frac{e^{2x}}{5}[2\sin x-\cos x]+C$$

Question 22:

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Solution:

Let $x=\tan\theta \qquad dx=\sec^2\theta d\theta$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)=2\theta$$

$$\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx=\int 2\theta.\sec^2\theta d\theta=2\int\theta.\sec^2\theta d\theta$$

Using integration by parts, we get

$$2\left[\theta.\int \sec^2\theta d\theta-\int\left\{\left(\frac{d}{d\theta}\theta\right)\int \sec^2\theta d\theta\right\}d\theta\right]$$

$$=2\left[\theta.\tan\theta-\int \tan\theta d\theta\right]$$

$$=2\left[\theta.\tan\theta+\log\left|\cos\theta\right|\right]+C$$

$$=2\left[x\tan^{-1}x+\log\left|\frac{1}{\sqrt{1+x^2}}\right|\right]+C$$

$$=2x\tan^{-1}x+2\log\left(1+x^2\right)^{\frac{-1}{2}}+C$$

$$=2x\tan^{-1}x+2\left[\frac{-1}{2}\log\left(1+x^2\right)\right]+C$$

$$=2x\tan^{-1}x-\log\left(1+x^2\right)+C$$

Question 23:

$\int x^2e^{x^3}dx$ equals

A. $\frac{1}{3}e^{x^3}+C$

B. $\frac{1}{3}e^{x^2}+C$

C. $\frac{1}{2}e^{x^3}+C$

D. $\frac{1}{2}e^{x^2}+C$

Solution:

Let $I=\int x^2e^{x^3}dx$

Also, let $x^3=t$ so, $3x^2dx=dt$

$$\Rightarrow I=\frac{1}{3}\int e^t dt$$

$$=\frac{1}{3}\left(e^t\right)+C$$

$$=\frac{1}{3}e^{x^3}+C$$

Thus, the correct option is A.

Question 24:

$\int e^x \sec x(1+\tan x)\,dx$ equals

A. $e^x \cos x + C$
B. $e^x \sec x + C$
C. $e^x \sin x + C$
D. $e^x \tan x + C$

Solution:

$\int e^x \sec x(1+\tan x)\,dx$

Consider, $I = \int e^x \sec x(1+\tan x)\,dx = \int e^x (\sec x + \sec x \tan x)\,dx$

Let $\sec x = f(x) \quad \sec x \tan x = f'(x)$

It is known that, $\int e^x \{f(x) + f'(x)\}dx = e^x f(x) + C$

$\therefore I = e^x \sec x + C$

Thus, the correct option is B.

EXERCISE 7.7

Integrate the functions in Exercises 1 to 9.

Question 1:

$\sqrt{4-x^2}$

Solution:

Let $I=\int\sqrt{4-x^2}dx=\int\sqrt{(2)^2-(x)^2}dx$

Since, $\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$$\therefore I=\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}+C$$

$$=\frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+C$$

Question 2:

$\sqrt{1-4x^2}$

Solution:

Let, $I=\int\sqrt{1-4x^2}dx=\int\sqrt{(1)^2-(2x)^2}dx$

Put, $2x=t\Rightarrow 2dx=dt$

$$\therefore I=\frac{1}{2}\int\sqrt{(1)^2-(t)^2}$$

Since, $\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$$\Rightarrow I=\frac{1}{2}\left[\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right]+C$$

$$=\frac{t}{4}\sqrt{1-t^2}+\frac{1}{4}\sin^{-1}t+C$$

$$=\frac{2x}{4}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+C$$

$$=\frac{x}{2}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+C$$

Question 3:

$\sqrt{x^2+4x+6}$

Solution:

$$\text{Let } I=\int\sqrt{x^2+4x+6}dx$$
$$=\int\sqrt{x^2+4x+4+2}dx$$
$$=\int\sqrt{\left(x^2+4x+4\right)+2}dx$$
$$=\int\sqrt{(x+2)^2+\left(\sqrt{2}\right)^2}\,dx$$

Since, $\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$$I=\frac{(x+2)}{2}\sqrt{x^2+4x+6}+\frac{2}{2}\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$$
$$=\frac{(x+2)}{2}\sqrt{x^2+4x+6}+\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$$

Question 4:

$\sqrt{x^2+4x+1}$

Solution:

Consider,

$$I=\int\sqrt{x^2+4x+1}dx$$
$$=\int\sqrt{\left(x^2+4x+4\right)-3}dx$$
$$=\int\sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}\,dx$$

Since, $\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$$\therefore I=\frac{(x+2)}{2}\sqrt{x^2+4x+1}-\frac{3}{2}\log\left|(x-2)+\sqrt{x^2+4x+1}\right|+C$$

Question 5:

$\sqrt{1-4x-x^2}$

Solution:

Consider, $I=\int\sqrt{1-4x-x^2}dx$

$=\int\sqrt{1-\left(x^2+4x+4-4\right)}dx$

$=\int\sqrt{1+4-(x+2)^2}dx$

$=\int\sqrt{\left(\sqrt{5}\right)^2-(x+2)^2}dx$

Since, $\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$\therefore I=\frac{(x+2)}{2}\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$

Question 6:

$\sqrt{x^2+4x-5}$

Solution:

Let $I=\int\sqrt{x^2+4x-5dx}$

$=\int\sqrt{\left(x^2+4x+4\right)-9dx}=\int\sqrt{(x+2)^2-(3)^2}dx$

Since, $\int\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\therefore I=\frac{(x+2)}{2}\sqrt{x^2+4x-5}-\frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right|+C$

Question 7:

$\sqrt{1+3x-x^2}$

Solution:

Put, $I=\int\sqrt{1+3x-x^2}dx$

$=\int\sqrt{1-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)}dx$

$=\int\sqrt{\left(1+\frac{9}{4}\right)-\left(x-\frac{3}{2}\right)^2}dx=\int\sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}dx$

Since, $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

$$\therefore I = \frac{x-\frac{3}{2}}{2}\sqrt{1+3x-x^2} + \frac{13}{4\times 2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{13}}{2}}\right) + C$$

$$= \frac{2x-3}{4}\sqrt{1+3x-x^2} + \frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + C$$

Question 8:

$\sqrt{x^2+3x}$

Solution:

Let $I = \int \sqrt{x^2+3x}\,dx$

$$= \int \sqrt{x^2+3x+\frac{9}{4}-\frac{9}{4}}\,dx$$

$$= \int \sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\,dx$$

Since, $\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$

$$\therefore I = \frac{\left(x+\frac{3}{2}\right)}{2}\sqrt{x^2+3x} - \frac{\frac{9}{4}}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right| + C$$

$$= \frac{(2x+3)}{4}\sqrt{x^2+3x} - \frac{9}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x}\right| + C$$

Question 9:

$\sqrt{1+\frac{x^2}{9}}$

Solution:

Let $I = \int \sqrt{1+\frac{x^2}{9}}\,dx = \frac{1}{3}\int \sqrt{9+x^2}\,dx = \frac{1}{3}\int \sqrt{(3)^2+x^2}\,dx$

Since, $\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$

$$\therefore I = \frac{1}{3}\left[\frac{x}{2}\sqrt{x^2+9} + \frac{9}{2}\log\left|x+\sqrt{x^2+9}\right|\right] + C$$

$$= \frac{x}{6}\sqrt{x^2+9} + \frac{3}{2}\log\left|x+\sqrt{x^2+9}\right| + C$$

Question 10:

$\int \sqrt{1+x^2}$ is equal to

A. $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$

B. $\frac{2}{3}\left(1+x^2\right)^{\frac{2}{3}}+C$

C. $\frac{2}{3}x\left(1+x^2\right)^{\frac{2}{3}}+C$

D. $\frac{x^3}{2}\sqrt{1+x^2}+\frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right|+C$

Solution:

Since. $\sqrt{a^2+x^2}dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$\therefore \int\sqrt{1+x^2}dx=\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$

Thus, the correct option is A.

Question 11:

$\int\sqrt{x^2-8x+7}dx$ is equal to

A. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

B. $\frac{1}{2}(x+4)\sqrt{x^2-8x+7}+9\log\left|x+4+\sqrt{x^2-8x+7}\right|+C$

C. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

D. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

Solution:

Let $I=\int\sqrt{x^2-8x+7}dx$

$=\int\sqrt{\left(x^2-8x+16\right)-9}dx$

$=\int\sqrt{(x-4)^2-(3)^2}dx$

Since, $\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\therefore I=\frac{(x-4)}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|(x-4)+\int\sqrt{x^2-8x+7}\right|+C$

Thus, the correct option is D.

EXERCISE 7.8

Evaluate the following definite integrals as limit of sums.

Question 1:

$\int_a^b x\,dx$

Solution:

Since, $\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right]$ where $h=\frac{b-a}{n}$

Here, $a=a, b=b$ and $f(x)=x$

$$\therefore \int_a^b x\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[a+(a+h)\ldots(a+2h)\ldots a+(n-1)h\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{a+a+a+\ldots+a}_{n\,times}\right)+\left(h+2h+3h+\ldots(n-1)h\right)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left(1+2+3+\ldots+(n-1)\right)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+\frac{n(n-1)h}{2}\right]=(b-a)\lim_{n\to\infty}\frac{n}{n}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)h}{2}\right]=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)(b-a)}{2n}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{\left(1-\frac{1}{n}\right)(b-a)}{2}\right]=(b-a)\left[a+\frac{(b-a)}{2}\right]$$

$$=(b-a)\left[\frac{2a+b-a}{2}\right]$$

$$=\frac{(b-a)(b+a)}{2}$$

$$=\frac{1}{2}\left(b^2-a^2\right)$$

Question 2:

$\int_0^b (x+1)\,dx$

Solution:

Let $I=\int_0^b (x+1)\,dx$

Since, $\int_a^b f(x)\,dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right]$, where $h=\frac{b-a}{n}$

Here, $a=0, b=5$ and $f(x)=(x+1)$

$\Rightarrow h=\frac{5-0}{n}=\frac{5}{n}$

$\therefore \int_0^5 (x+1)dx=(5-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{5}{n}\right)+\ldots+f\left((n-1)\frac{5}{n}\right)\right]$

$=5\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{5}{n}+1\right)+\ldots\left\{1+\left(\frac{5(n-1)}{n}\right)\right\}\right]$

$=5\lim_{n\to\infty}\frac{1}{n}\left[\left(1+\underset{n\ times}{1}+1\ldots1\right)+\left[\frac{5}{n}+2.\frac{5}{n}+3.\frac{5}{n}+\ldots+(n-1)\frac{5}{n}\right]\right]$

$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\{1+2+3\ldots(n-1)\}\right]$

$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}.\frac{(n-1)n}{2}\right]=5\lim_{n\to\infty}\left[1+\frac{5(n-1)}{2n}\right]$

$=5\lim_{n\to\infty}\left[1+\frac{5}{2}\left(1-\frac{1}{n}\right)\right]=5\left[1+\frac{5}{2}\right]$

$=5\left[\frac{7}{2}\right]$

$=\frac{35}{2}$

Question 3:

$\int_2^3 x^2\,dx$

Solution:

Since,

$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right]$, where $h=\frac{b-a}{n}$

Here, $a=2, b=3$ and $f(x)=x^2$

$$\Rightarrow h=\frac{3-2}{n}=\frac{1}{n}$$

$$\therefore \int_2^3 x^2dx=(3-2)\lim_{n\to\infty}\frac{1}{n}\left[f(2)+f\left(2+\frac{1}{n}\right)+f\left(2+\frac{2}{n}\right)\dots f\left\{2+(n-1)\frac{1}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[(2)^2+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+\dots\left(2+\frac{(n-1)^2}{n}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[2^2+\left\{2^2+\left(\frac{1}{n}\right)^2+2.2\frac{1}{n}\right\}+\dots+\left\{(2)^2+\frac{(n-1)^2}{n^2}+2.2.\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\left(2^2+\underset{n\,times}{\dots}+2^2\right)+\left\{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\dots+\left(\frac{n-1}{n}\right)^2\right\}+2.2.\left\{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\dots+\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{1^2+2^2+3^2\dots+(n-1)^2\right\}+\frac{4}{n}\left\{1+2+\dots+(n-1)\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}+\frac{4}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{n\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}+\frac{4n-4}{2}\right]=\lim_{n\to\infty}\left[4+\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+2-\frac{2}{n}\right]$$

$$=4+\frac{2}{6}+2$$

$$=\frac{19}{3}$$

Question 4:

$$\int_1^4\left(x^2-x\right)dx$$

Solution:

Let $I=\int_1^4\left(x^2-x\right)dx$

$$=\int_1^4 x^2dx-\int_1^4 xdx$$

Let $I=I_1-I_2$, where $I_1=\int_1^4 x^2dx$ and $I_2=\int_1^4 xdx \quad \dots(1)$

Since, $\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+f(a+(n-1)h)\right]$, where $h=\frac{b-a}{n}$

For, $I_1=\int_1^4 x^2dx,$

$a=1, b=4$ and $f(x)=x^2$

$\therefore h=\frac{4-1}{n}=\frac{3}{n}$

$I_1=\int_1^4 x^2 dx=(4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+\ldots+f(1+(n-1)h)\right]$

$=3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left(1+\frac{3}{n}\right)^2+\left(1+2.\frac{3}{n}\right)^2+\ldots\left(1+\frac{(n-1)3}{n}\right)^2\right]$

$=3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left\{1^2+\left(\frac{3}{n}\right)^2+2.\frac{3}{n}\right\}+\ldots+\left\{1^2+\left(\frac{(n-1)3}{n}\right)^2+\frac{2.(n-1).3}{2}\right\}\right]$

$=3\lim_{n\to\infty}\frac{1}{n}\left[\left(1^2+\underset{n\,times}{\ldots}+1^2\right)+\left(\frac{3}{n}\right)^2\left\{1^2+2^2+\ldots+(n-1)^2\right\}+2.\frac{3}{n}\left\{1+2+\ldots+(n-1)\right\}\right]$

$=3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9}{n^2}\left\{\frac{(n-1)(n)(2n-1)}{6}\right\}+\frac{6}{n}\left\{\frac{(n-1)(n)}{2}\right\}\right]$

$=3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9n}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{6n-6}{2}\right]$

$=3\lim_{n\to\infty}\left[1+\frac{9}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+3-\frac{3}{n}\right]$

$=3[1+3+3]$

$=3[7]$

$I_1=21 \quad \ldots(2)$

For $I_2=\int_1^4 x dx$

$a=1, b=4$ and $f(x)=x$

$\Rightarrow h=\dfrac{4-1}{n}=\dfrac{3}{n}$

$\therefore I_2=(4-1)\lim\limits_{n\to\infty}\dfrac{1}{n}\left[f(1)+f(1+h)+\ldots+f(a+(n-1)h)\right]$

$=3\lim\limits_{n\to\infty}\dfrac{1}{n}\left[1+(1+h)+\ldots+(1+(n-1)h)\right]$

$=3\lim\limits_{n\to\infty}\dfrac{1}{n}\left[1+\left(1+\dfrac{3}{n}\right)+\ldots+\left\{1+(n-1)\dfrac{3}{n}\right\}\right]$

$=3\lim\limits_{n\to\infty}\dfrac{1}{n}\left[\left(1+1+\underset{n\,times}{\ldots}+1\right)+\dfrac{3}{n}(1+2+\ldots+(n-1))\right]$

$=3\lim\limits_{n\to\infty}\dfrac{1}{n}\left[n+\dfrac{3}{n}\left\{\dfrac{(n-1)n}{2}\right\}\right]$

$=3\lim\limits_{n\to\infty}\left[1+\dfrac{3}{2}\left(1-\dfrac{1}{n}\right)\right]$

$=3\left[1+\dfrac{3}{2}\right]=3\left[\dfrac{5}{2}\right]$

$I_2=\dfrac{15}{2}\quad\ldots(3)$

From equations (2) and (3), we get

$I=I_1-I_2=21-\dfrac{15}{2}=\dfrac{27}{2}$

Question 5:

$\int_{-1}^{1} e^x dx$

Solution:

Let $I=\int_{-1}^{1} e^x dx\quad\ldots(1)$

Since, $\int_a^b f(x)dx=(b-a)\lim\limits_{n\to\infty}\dfrac{1}{n}\left[f(a)+f(a+h)+f(a+(n-1)h)\right]$, where $h=\dfrac{b-a}{n}$

Here, $a=-1, b=1$ and $f(x)=e^x$

$$\therefore h=\frac{1+1}{n}=\frac{2}{n}$$

$$\therefore I=(1+1)\lim_{n\to\infty}\frac{1}{n}\left[f(-1)+f\left(-1+\frac{2}{n}\right)+f\left(-1+2.\frac{2}{n}\right)+\ldots+f\left(-1+\frac{(n-1)2}{n}\right)\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}+e^{\left(-1+\frac{2}{n}\right)}+e^{\left(-1+2\frac{2}{n}\right)}+e^{\left(-1+(n-1)\frac{2}{n}\right)}\right]$$

$$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}\left\{1+e^{\frac{2}{n}}+e^{\frac{4}{n}}+e^{\frac{6}{n}}+e^{(n-1)\frac{2}{n}}\right\}\right]$$

$$=2\lim_{n\to\infty}\frac{e^{-1}}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right]=e^{-1}\times 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^{2}-1}{e^{\frac{2}{n}}-1}\right]$$

$$=\frac{e^{-1}\times 2\left(e^{2}-1\right)}{\lim_{\frac{2}{n}\to 0}\left(\frac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right)\times 2}\qquad\left[\lim_{h\to 0}\left(\frac{e^{h}-1}{h}\right)=1\right]$$

$$=\frac{e^{2}-1}{e}$$

$$=\left(e-\frac{1}{e}\right)$$

Question 6:

$$\int_0^4\left(x+e^{2x}\right)dx$$

Solution:

Since,

$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+....+f(a+(n-1)h)\right]$, where $h=\frac{b-a}{n}$

Here, $a=0, b=4$ and $f(x)=x+e^{2x}$

$$\therefore h = \frac{4-0}{n} = \frac{4}{n}$$

$$\Rightarrow \int_0^4 \left(x + e^{2x}\right)dx = (4-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0) + f(h) + f(2h) + \ldots + f((n-1)h)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\left(0 + e^0\right) + \left(h + e^{2h}\right) + \left(2h + e^{2.2h}\right) + \ldots + \left\{(n-1)h + e^{2(n-1)h}\right\}\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[1 + \left(h + e^{2h}\right) + \left(2h + e^{4h}\right) + \ldots + \left\{(n-1)h + e^{2(n-1)h}\right\}\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[\left\{h + 2h + 3h + \ldots\ldots + (n-1)h\right\} + \left(1 + e^{2h} + e^{4h} + \ldots\ldots + e^{2(n-1)h}\right)\right]$$

$$= 4\lim_{n\to\infty}\frac{1}{n}\left[h\left\{1 + 2 + \ldots(n-1)\right\} + \left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right] = 4\lim_{n\to\infty}\frac{1}{n}\left[\frac{\left[h(n-1)n\right]}{2} + \left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$= 4\lim_{x\to\infty}\frac{1}{n}\left[\frac{4}{n}\frac{(n-1)n}{2} + \left(\frac{e^8-1}{e^{\frac{8}{n}}-1}\right)\right] = 4(2) + 4\lim_{n\to\infty}\frac{\left(e^8-1\right)}{\left(\frac{e^{\frac{8}{n}}-1}{\frac{8}{n}}\right)8}$$

$$= 8 + \frac{4\left(e^8-1\right)}{8} \qquad \left(\lim_{x\to 0}\frac{e^x-1}{x} = 1\right)$$

$$= 8 + \frac{e^8-1}{2} = \frac{15+e^8}{2}$$

EXERCISE 7.9

Evaluate the definite integrals in Exercises 1 to 20.

Question 1:

$\int_{-1}^{1}(x+1)dx$

Solution:

Let $I=\int_{-1}^{1}(x+1)dx$

$\int(x+1)dx=\frac{x^2}{2}+x=F(x)$

Using second fundamental theorem of calculus, we get

$I=F(1)-F(-1)$

$=\left(\frac{1}{2}+1\right)-\left(\frac{1}{2}-1\right)$

$=\frac{1}{2}+1-\frac{1}{2}+1$

$=2$

Question 2:

$\int_{2}^{3}\frac{1}{x}dx$

Solution:

Let $I=\int_{2}^{3}\frac{1}{x}dx$

$\int\frac{1}{x}dx=\log|x|=F(x)$

Using second fundamental theorem of calculus, we get

$I=F(3)-F(2)$

$=\log|3|-\log|2|=\log\frac{3}{2}$

Question 3:

$\int_{1}^{2}\left(4x^3-5x^2+6x+9\right)dx$

Solution:

Let $I=\int_{1}^{2}\left(4x^3-5x^2+6x+9\right)dx$

$$\int\left(4x^3-5x^2+6x+9\right)dx=4\left(\frac{x^4}{4}\right)-5\left(\frac{x^3}{3}\right)+6\left(\frac{x^2}{2}\right)+9\left(x\right)$$

$$=x^4-\frac{5x^3}{3}+3x^2+9x=F\left(x\right)$$

Using second fundamental theorem of calculus, we get

$$I=F\left(2\right)-F\left(1\right)$$

$$I=\left\{2^4-\frac{5(2)^3}{3}+3(2)^2+9(2)\right\}-\left\{(1)^4-\frac{5(1)^3}{3}+3(1)^2+9(1)\right\}$$

$$=\left(16-\frac{40}{3}+12+18\right)-\left(1-\frac{5}{3}+3+9\right)$$

$$=16-\frac{40}{3}+12+18-1+\frac{5}{3}-3-9$$

$$=33-\frac{35}{3}$$

$$=\frac{99-35}{3}$$

$$=\frac{64}{3}$$

Question 4:

$$\int_0^{\frac{\pi}{4}}\sin 2x\,dx$$

Solution:

Let $I=\int_0^{\frac{\pi}{4}}\sin 2x\,dx$

$$\int\sin 2x\,dx=\left(\frac{-\cos 2x}{2}\right)=F\left(x\right)$$

Using second fundamental theorem of calculus, we get

$$I=F\left(\frac{\pi}{4}\right)-F\left(0\right)$$

$$=-\frac{1}{2}\left[\cos 2\left(\frac{\pi}{4}\right)-\cos 0\right]=-\frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right)-\cos 0\right]$$

$$=-\frac{1}{2}\left[0-1\right]$$

$$=\frac{1}{2}$$

Question 5:

$\int_0^{\frac{\pi}{2}} \cos 2x dx$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \cos 2x dx$

$\int \cos 2x dx = \left(\frac{\sin 2x}{2}\right) = F(x)$

Using second fundamental theorem of calculus, we get

$I = F\left(\frac{\pi}{2}\right) - F(0)$

$= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right] = \frac{1}{2}[\sin \pi - \sin 0]$

$= \frac{1}{2}[0 - 0] = 0$

Question 6:

$\int_4^5 e^x dx$

Solution:

Let $I = \int_4^5 e^x dx$

$\int e^x dx = e^x = F(x)$

Using second fundamental theorem of calculus, we get

$I = F(5) - F(4)$

$= e^5 - e^4$

$= e^4(e - 1)$

Question 7:

$\int_0^{\frac{\pi}{4}} \tan x dx$

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \tan x dx$

$\int \tan x dx = -\log|\cos x| = F(x)$

Using second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= -\log\left|\cos\frac{\pi}{4}\right| + \log|\cos 0| = -\log\left|\frac{1}{\sqrt{2}}\right| + \log|1|$$

$$= -\log(2)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\log 2$$

Question 8:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ecx dx$$

Solution:

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos ecx dx$

$$\int \cos ecx dx = \log|\cos ecx - \cot x| = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{4}\right) - F\left(\frac{\pi}{6}\right)$$

$$= \log\left|\cos ec\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\cos ec\frac{\pi}{6} - \cot\frac{\pi}{6}\right|$$

$$\log\left|\sqrt{2}-1\right| - \log\left|2-\sqrt{3}\right| = \log\left(\frac{\sqrt{2}-1}{2-\sqrt{3}}\right)$$

Question 9:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

Solution:

Let $I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x = F(x)$$

Using second fundamental theorem of calculus, we get

$I = F(1) - F(0)$

$= \sin^{-1}(1) - \sin^{-1}(0)$

$= \frac{\pi}{2} - 0$

$= \frac{\pi}{2}$

Question 10:

$\int_0^1 \frac{dx}{1+x^2}$

Solution:

Let $I = \int_0^1 \frac{dx}{1+x^2}$

$\int \frac{dx}{1+x^2} = \tan^{-1} x = F(x)$

Using second fundamental theorem of calculus, we get

$I = F(1) - F(0)$

$= \tan^{-1}(1) - \tan^{-1}(0)$

$= \frac{\pi}{4}$

Question 11:

$\int_2^3 \frac{dx}{x^2 - 1}$

Solution:

Let $I = \int_2^3 \frac{dx}{x^2 - 1}$

$\int \frac{dx}{x^2 - 1} = \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| = F(x)$

Using second fundamental theorem of calculus, we get

$I = F(3) - F(2)$

$= \frac{1}{2}\left[\log \left| \frac{3-1}{3+1} \right| - \log \left| \frac{2-1}{2+1} \right| \right] = \frac{1}{2}\left[\log \left| \frac{2}{4} \right| - \log \left| \frac{1}{3} \right| \right]$

$= \frac{1}{2}\left[\log \frac{1}{2} - \log \frac{1}{3} \right]$

$= \frac{1}{2}\left[\log \frac{3}{2} \right]$

Question 12:

$\int_0^{\frac{\pi}{2}} \cos^2 x dx$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \cos^2 x dx$

$\int \cos^2 x dx = \int \left(\frac{1+\cos 2x}{2}\right) dx = \frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) = F(x)$

Using second fundamental theorem of calculus, we get

$I = \left[F\left(\frac{\pi}{2}\right) - F(0)\right] = \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2}\right) - \left(0 + \frac{\sin \pi}{2}\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0 - 0\right]$

$= \frac{\pi}{4}$

Question 13:

$\int_2^3 \frac{x}{x^2+1} dx$

Solution:

Let $I = \int_2^3 \frac{x}{x^2+1} dx$

$\int \frac{x}{x^2+1} dx = \frac{1}{2}\int \frac{2x}{x^2+1} dx = \frac{1}{2}\log\left(1+x^2\right) = F(x)$

Using second fundamental theorem of calculus, we get

$I = F(3) - F(2)$

$= \frac{1}{2}\left[\log\left(1+(3)^2\right) - \log\left(1+(2)^2\right)\right]$

$= \frac{1}{2}\left[\log(10) - \log(5)\right]$

$= \frac{1}{2}\log\left(\frac{10}{5}\right) = \frac{1}{2}\log 2$

Question 14:

$\int_0^1 \frac{2x+3}{5x^2+1} dx$

Solution:

Let $I = \int_0^1 \frac{2x+3}{5x^2+1} dx$

$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int \frac{5(2x+3)}{5x^2+1}dx = \frac{1}{5}\int \frac{10x+15}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5\left(x^2+\frac{1}{5}\right)}dx = \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{5}.\frac{1}{\frac{1}{\sqrt{5}}}\tan^{-1}\frac{x}{\frac{1}{\sqrt{5}}}$$

$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)x$$

$$= F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F(1) - F(0)$$

$$= \left\{\frac{1}{5}\log(5+1) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)\right\} - \left\{\frac{1}{5}\log(5\times 0+1) + \frac{3}{\sqrt{5}}\tan^{-1}(0)\right\}$$

$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$$

Question 15:

$$\int_0^1 xe^{x^2}dx$$

Solution:

Let $I = \int_0^1 xe^{x^2}dx$

Put, $x^2 = t \Rightarrow 2xdx = dt$

As $x \to 0, t \to 0$ and as $x \to 1, t \to 1$

$$\therefore I = \frac{1}{2}\int_0^1 e^t dt$$

$$\frac{1}{2}\int e^t dt = \frac{1}{2}e^t = F(t)$$

Using second fundamental theorem of calculus, we get

$$I = F(1) - F(0)$$

$$= \frac{1}{2}e - \frac{1}{2}e^0$$

$$= \frac{1}{2}(e-1)$$

Question 16:

$$\int_1^2 \frac{5x^2}{x^2+4x+3}dx$$

Solution:

Let $I = \int_1^2 \frac{5x^2}{x^2+4x+3}dx$

Dividing $5x^2$ by x^2+4x+3, we get

$$I = \int_1^2 \left\{5 - \frac{20x+15}{x^2+4x+3}\right\}dx$$

$$= \int_1^2 5dx - \int_1^2 \frac{20x+15}{x^2+4x+3}dx$$

$$= [5x]_1^2 - \int_1^2 \frac{20x+15}{x^2+4x+3}dx$$

$$I = 5 - I_1, \text{ where } I = \int_1^2 \frac{20x+15}{x^2+4x+3}dx \quad \ldots(1)$$

Let $20x+15 = A\frac{d}{dx}\left(x^2+4x+3\right)+B$

$= 2Ax + (4A+B)$

Equating the coefficients of x and constant term, we get

$A = 10$ and $B = -25$

Let $x^2+4x+3 = t$

$\Rightarrow (2x+4)dx = dt$

$$\Rightarrow I_1 = 10\int \frac{dt}{t} - 25\int \frac{dx}{(x+2)^2-1^2}$$

$$= 10\log t - 25\left[\frac{1}{2}\log\left(\frac{x+2-1}{x+2+1}\right)\right] = \left[10\log\left(x^2+4x+3\right)\right]_1^2 - 25\left[\frac{1}{2}\log\left(\frac{x+1}{x+3}\right)\right]_1^2$$

$$= [10\log 15 - 10\log 8] - 25\left[\frac{1}{2}\log\frac{3}{5} - \frac{1}{2}\log\frac{2}{4}\right]$$

$$= [10\log(5\times 3) - 10\log(4\times 2)] - \frac{25}{2}[\log 3 - \log 5 - \log 2 + \log 4]$$

$$= [10\log 5 + 10\log 3 - 10\log 4 - 10\log 2] - \frac{25}{2}[\log 3 - \log 5 - \log 2 + \log 4]$$

$$= \left[10 + \frac{25}{2}\right]\log 5 + \left[-10 - \frac{25}{2}\right]\log 4 + \left[10 - \frac{25}{2}\right]\log 3 + \left[-10 + \frac{25}{2}\right]\log 2$$

$$= \frac{45}{2}\log 5 - \frac{45}{2}\log 4 - \frac{5}{2}\log 3 + \frac{5}{2}\log 2$$

$$= \frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}$$

Substituting the value I_1 in (1), we get

$$I = 5 - \left[\frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}\right]$$

$$= 5 - \frac{5}{2}\left[9\log\frac{5}{4} - \log\frac{3}{2}\right]$$

Question 17:

$$\int_0^{\frac{\pi}{4}} \left(2\sec^2 x + x^3 + 2\right) dx$$

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \left(2\sec^2 x + x^3 + 2\right) dx$

$$\int \left(2\sec^2 x + x^3 + 2\right) dx = 2\tan x + \frac{x^4}{4} + 2x = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{4}\right) - F(0) = \left\{\left(2\tan\frac{\pi}{4} + \frac{1}{4}\left(\frac{\pi}{4}\right)^4 + 2\left(\frac{\pi}{4}\right)\right) - \left(2\tan 0 + 0 + 0\right)\right\}$$

$$= 2\tan\frac{\pi}{4} + \frac{\pi^4}{4^5} + \frac{\pi}{2}$$

$$= 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$$

Question 18:

$$\int_0^{\pi} \left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right) dx$$

Solution:

Let $I = \int_0^{\pi} \left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right) dx = -\int_0^{\pi} \left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) dx$

$$= -\int_0^{\pi} \cos x dx$$

$$\int \cos x dx = \sin x = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F(\pi) - F(0)$$

$$= \sin\pi - \sin 0$$

$$= 0$$

Question 19:

$$\int_0^2 \frac{6x+3}{x^2+4}dx$$

Solution:

Let $I = \int_0^2 \frac{6x+3}{x^2+4}dx$

$$\int \frac{6x+3}{x^2+4}dx = 3\int \frac{2x+1}{x^2+4}dx$$

$$= 3\int \frac{2x}{x^2+4}dx + 3\int \frac{1}{x^2+4}dx$$

$$= 3\log\left(x^2+4\right) + \frac{3}{2}\tan^{-1}\frac{x}{2} = F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F(2) - F(0)$$

$$= \left\{3\log\left(2^2+4\right) + \frac{3}{2}\tan^{-1}\left(\frac{2}{2}\right)\right\} - \left\{3\log(0+4) + \frac{3}{2}\tan^{-1}\left(\frac{0}{2}\right)\right\}$$

$$= 3\log 8 + \frac{3}{2}\tan^{-1}1 - 3\log 4 - \frac{3}{2}\tan^{-1}0$$

$$= 3\log 8 + \frac{3}{2}\left(\frac{\pi}{4}\right) - 3\log 4 - 0$$

$$= 3\log\left(\frac{8}{4}\right) + \frac{3\pi}{8}$$

$$= 3\log 2 + \frac{3\pi}{8}$$

Question 20:

$$\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx$$

Solution:

Let $I = \int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx$

$$\int_0^1\left(xe^x+\sin\frac{\pi x}{4}\right)dx = x\int e^x dx-\int\left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\}dx+\left\{\frac{-\cos\frac{\pi x}{4}}{\frac{\pi}{4}}\right\}$$

$$= xe^x-\int e^x dx-\frac{4}{\pi}\cos\frac{\pi x}{4}$$

$$= xe^x-e^x-\frac{4}{\pi}\cos\frac{\pi x}{4}$$

$$= F(x)$$

Using second fundamental theorem of calculus, we get

$$I = F(1)-F(0)$$

$$=\left(1.e^1-e^1-\frac{4}{\pi}\cos\frac{\pi}{4}\right)-\left(0.e^0-e^0-\frac{4}{\pi}\cos 0\right)$$

$$= e-e-\frac{4}{\pi}\left(\frac{1}{\sqrt{2}}\right)+1+\frac{4}{\pi}=1+\frac{4}{\pi}-\frac{2\sqrt{2}}{\pi}$$

Question 21:

$$\int_1^{\sqrt{3}}\frac{dx}{1+x^2}$$

A. $\frac{\pi}{3}$

B. $\frac{2\pi}{3}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{12}$ equals

Solution:

$$\int\frac{dx}{1+x^2}=\tan^{-1}x=F(x)$$

Using second fundamental theorem of calculus, we get

$$\int_1^{\sqrt{3}}\frac{dx}{1+x^2}=F\left(\sqrt{3}\right)-F(1)$$

$$=\tan^{-1}\sqrt{3}-\tan^{-1}1$$

$$=\frac{\pi}{3}-\frac{\pi}{4}$$

$$=\frac{\pi}{12}$$

Thus, the correct option is D.

Question 22:

$$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$$

$A.\ \frac{\pi}{6}$

$B.\ \frac{\pi}{12}$

$C.\ \frac{\pi}{24}$

$D.\ \frac{\pi}{4}$ equals

Solution:

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2+(3x)^2}$$

Put $3x = t \Rightarrow 3dx = dt$

$$\therefore \int \frac{dx}{(2)^2+(3x)^2} = \frac{1}{3}\int \frac{dt}{(2)^2+t^2}$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)$$

$$= F(x)$$

Using second fundamental theorem of calculus, we get

$$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = F\left(\frac{2}{3}\right) - F(0)$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3}{2}.\frac{2}{3}\right) - \frac{1}{6}\tan^{-1}0$$

$$= \frac{1}{6}\tan^{-1}1 - 0$$

$$= \frac{1}{6}\times\frac{\pi}{4}$$

$$= \frac{\pi}{24}$$

Thus, the correct option is C.

EXERCISE 7.10

Evaluate the integrals in Exercises 1 to 8 using substitution.

Question 1:

$$\int_0^1 \frac{x}{x^2+1}dx$$

Solution:

$$\int_0^1 \frac{x}{x^2+1}dx$$

Put, $x^2+1=t \Rightarrow 2xdx=dt$

When, $x=0, t=1$ and when $x=1, t=2$

$$\therefore \int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$=\frac{1}{2}\left[\log|t|\right]_1^2$$

$$=\frac{1}{2}\left[\log 2-\log 1\right]$$

$$=\frac{1}{2}\log 2$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi$$

Solution:

Consider, $I=\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi\cos\phi d\phi$

Let $\sin\phi = t \Rightarrow \cos\phi d\phi = dt$

When $\phi=0, t=0$ and when $\phi=\frac{\pi}{2}, t=1$

$\therefore I=\int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt$

$=\int_0^1 t^{\frac{1}{2}}\left(1+t^4-2t^2\right)dt$

$=\int_0^1\left[t^{\frac{1}{2}}+t^{\frac{9}{2}}-2t^{\frac{5}{2}}\right]dt$

$=\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+\frac{t^{\frac{11}{2}}}{\frac{11}{2}}-\frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$

$=\frac{2}{3}+\frac{2}{11}-\frac{4}{7}$

$=\frac{154+42-132}{231}=\frac{64}{231}$

Question 3:

$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Solution:

Consider, $I=\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Let $x=\tan\theta \Rightarrow dx=\sec^2\theta d\theta$

When $x=0,\theta=0$ and when $x=1,\theta=\frac{\pi}{4}$

$I=\int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta d\theta$

$=\int_0^{\frac{\pi}{4}} \sin^{-1}\left(\sin 2\theta\right)\sec^2\theta d\theta$

$=\int_0^{\frac{\pi}{4}} 2\theta\sec^2\theta d\theta$

$=2\int_0^{\frac{\pi}{4}} \theta\sec^2\theta d\theta$

Taking $u=\theta$ and $v=\sec^2\theta$ and integrating by parts, we get

$$I=2\left[\theta\int\sec^2\theta d\theta-\int\left\{\left(\frac{d}{d\theta}\theta\right)\int\sec^2\theta d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$=2\left[\theta\tan\theta-\int\tan\theta d\theta\right]_0^{\frac{\pi}{4}}$$

$$=2\left[\theta\tan\theta+\log|\cos\theta|\right]_0^{\frac{\pi}{4}}$$

$$=2\left[\frac{\pi}{4}\tan\frac{\pi}{4}+\log\left|\cos\frac{\pi}{4}\right|-\log|\cos 0|\right]=2\left[\frac{\pi}{4}+\log\left(\frac{1}{\sqrt{2}}\right)-\log 1\right]$$

$$=2\left[\frac{\pi}{4}-\frac{1}{2}\log 2\right]$$

$$=\frac{\pi}{2}-\log 2$$

Question 4:

$$\int_0^2 x\sqrt{x+2}\quad\left(\text{Put } x+2=t^2\right)$$

Solution:

$$\int_0^2 x\sqrt{x+2}dx$$

Put, $x+2=t^2\Rightarrow dx=2tdt$

When $x=0,t=\sqrt{2}$ and when $x=2,t=2$

$$\therefore\int_0^2 x\sqrt{x+2}dx=\int_{\sqrt{2}}^2\left(t^2-2\right)\sqrt{t^2}2tdt$$

$$=2\int_{\sqrt{2}}^2\left(t^2-2\right)t^2dt$$

$$=2\int_{\sqrt{2}}^2\left(t^4-2t^2\right)dt$$

$$=2\left[\frac{t^5}{5}-\frac{2t^3}{3}\right]_{\sqrt{2}}^2$$

$$=2\left[\frac{32}{5}-\frac{16}{3}-\frac{4\sqrt{2}}{5}+\frac{4\sqrt{2}}{3}\right]=2\left[\frac{96-80-12\sqrt{2}+20\sqrt{2}}{15}\right]=2\left[\frac{16+8\sqrt{2}}{15}\right]$$

$$=\frac{16\left(2+\sqrt{2}\right)}{15}$$

$$=\frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$$

Question 5:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$$

Put, $\cos x = t \Rightarrow -\sin x dx = dt$

When $x = 0, t = 1$ and when $x = \frac{\pi}{2}, t = 0$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx = -\int_1^0 \frac{dt}{1+t^2}$$

$$= -\left[\tan^{-1} t\right]_1^0$$

$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$= -\left[-\frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}$$

Question 6:

$$\int_0^2 \frac{dx}{x+4-x^2}$$

Solution:

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-(x^2-x-4)}$$

$$= \int \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)} = \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}$$

Let $x - \frac{1}{2} = t \Rightarrow dx = dt$

when $x = 0, t = -\frac{1}{2}$ and when $x = 2, t = \frac{3}{2}$

$$\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2} = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2 - t^2}$$

$$=\left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)}\log\frac{\frac{\sqrt{17}}{2}+t}{\frac{\sqrt{17}}{2}-t}\right]_{-\frac{1}{2}}^{\frac{3}{2}}=\frac{1}{\sqrt{17}}\left[\log\frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}}-\frac{\log\frac{\sqrt{17}}{2}-\frac{1}{2}}{\log\frac{\sqrt{17}}{2}+\frac{1}{2}}\right]$$

$$=\frac{1}{\sqrt{17}}\left[\log\frac{\sqrt{17}+3}{\sqrt{17}-3}-\log\frac{\sqrt{17}-1}{\sqrt{17}+1}\right]=\frac{1}{\sqrt{17}}\log\frac{\sqrt{17}+3}{\sqrt{17}-3}\times\frac{\sqrt{17}+1}{\sqrt{17}-1}$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right]=\frac{1}{\sqrt{17}}\log\left[\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)=\frac{1}{\sqrt{17}}\log\left[\frac{\left(5+\sqrt{17}\right)\left(5+\sqrt{17}\right)}{25-17}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{25+17+10\sqrt{17}}{8}\right]=\frac{1}{\sqrt{17}}\log\left(\frac{42+10\sqrt{17}}{8}\right)$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right)$$

Question 7:

$$\int_{-1}^{1}\frac{dx}{x^2+2x+5}$$

Solution:

$$\int_{-1}^{1}\frac{dx}{x^2+2x+5}=\int_{-1}^{1}\frac{dx}{\left(x^2+2x+1\right)+4}=\int_{-1}^{1}\frac{dx}{\left(x+1\right)^2+\left(2\right)^2}$$

Put, $x+1=t\Rightarrow dx=dt$

When $x=-1, t=0$ and when $x=1, t=2$

$$\int_{-1}^{1}\frac{dx}{\left(x-1\right)^2+\left(2\right)^2}=\int_{0}^{2}\frac{dt}{t^2+2^2}$$

$$=\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2=\frac{1}{2}\tan^{-1}1-\frac{1}{2}\tan^{-1}0$$

$$=\frac{1}{2}\left(\frac{\pi}{4}\right)=\frac{\pi}{8}$$

Question 8:

$$\int_1^2\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$$

Solution:

$$\int_1^2\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$$

Put, $2x=t \Rightarrow 2dx=dt$

When $x=1, t=2$ and when $x=2, t=4$

$$\therefore \int_1^2\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx=\frac{1}{2}\int_2^4\left(\frac{2}{t}-\frac{2}{t^2}\right)e^t dt$$

$$=\int_2^4\left(\frac{1}{t}-\frac{1}{t^2}\right)e^t dt$$

Let $\frac{1}{t}=f(t)$

Then, $f'(t)=-\frac{1}{t^2}$

$$\Rightarrow \int_2^4\left(\frac{1}{t}-\frac{1}{t^2}\right)e^t dt=\int_2^4 e^t\left[f(t)+f'(t)\right]dt$$

$$=\left[e^t f(t)\right]_2^4$$

$$=\left[e^t.\frac{1}{t}\right]_2^4$$

$$=\left[\frac{e^t}{t}\right]_2^4$$

$$=\frac{e^4}{4}-\frac{e^2}{2}$$

$$=\frac{e^2\left(e^2-2\right)}{4}$$

Question 9:

The value of the integral $\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$ is

A. 6

B. 0

C. 3

D. 4

Solution:

Consider, $I=\int_{\frac{1}{3}}^{1}\frac{\left(x-x^{3}\right)^{\frac{1}{3}}}{x^{4}}dx$

Let $x=\sin\theta \Rightarrow dx=\cos\theta d\theta$

When $x=\frac{1}{3},\theta=\sin^{-1}\left(\frac{1}{3}\right)$ and when $x=1,\theta=\frac{\pi}{2}$

$$\Rightarrow I=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta-\sin^{3}\theta\right)^{\frac{1}{3}}}{\sin^{4}\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(1-\sin^{2}\theta\right)^{\frac{1}{3}}}{\sin^{4}\theta}\cos\theta d\theta=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^{4}\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\sin\theta\right)^{\frac{1}{3}}\left(\cos\theta\right)^{\frac{2}{3}}}{\sin^{2}\theta\sin^{2}\theta}\cos\theta d\theta=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{\left(\cos\theta\right)^{\frac{5}{3}}}{\left(\sin\theta\right)^{\frac{5}{3}}}\cos ec^{2}\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\left(\cot\theta\right)^{\frac{5}{3}}\cos ec^{2}\theta d\theta$$

Put $\cot\theta=t\Rightarrow-\cos ec^{2}\theta d\theta=dt$

When $\theta=\sin^{-1}\left(\frac{1}{3}\right),t=2\sqrt{2}$ and when $\theta=\frac{\pi}{2},t=0$

$$\therefore I=-\int_{2\sqrt{2}}^{0}\left(t\right)^{\frac{5}{3}}dt$$

$$=-\left[\frac{3}{8}\left(t\right)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$=-\frac{3}{8}\left[-\left(2\sqrt{2}\right)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}=\frac{3}{8}\left[\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$$

$$=\frac{3}{8}\left[\left(8\right)^{\frac{4}{3}}\right]$$

$$=\frac{3}{8}[16]$$

$$=6$$

Thus, the correct option is A.

Question 10:

If $f(x)=\int_{0}^{x}t\sin t dt$, then $f'(x)$ is

$A.\ \cos x + x\sin x$

$B.\ x\sin x$

$C.\ x\cos x$

$D.\ \sin x + x\cos x$

Solution:

$f(x) = \int_0^x t\sin t\,dt$

Using integration by parts, we get

$f(x) = t\int_0^x \sin t\,dt - \int_0^x \left\{\left(\frac{d}{dt}t\right)\int \sin t\,dt\right\}dt$

$= \left[t(-\cos t)\right]_0^x - \int_0^x (-\cos t)dt$

$= \left[-t\cos t + \sin t\right]_0^x$

$= -x\cos x + \sin x$

$\Rightarrow f'(x) = -\left[\{x(-\sin x)\} + \cos x\right] + \cos x$

$= x\sin x - \cos x + \cos x$

$= x\sin x$

Thus, the correct option is B.

EXERCISE 7.11

By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

Question 1:

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx$$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x dx \quad \ldots(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right) dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^2 x\right) dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1.dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 2:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Consider, $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \ldots(1)$

$$\Rightarrow I = I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx \quad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1.dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Question 3:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} xdx}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}dx$$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} xdx}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}dx \quad \ldots(1)$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)dx}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)+\cos^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}dx \quad \left(\int_0^a f(x) = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} xdx}{\cos^{\frac{3}{2}} x+\sin^{\frac{3}{2}} x}dx \quad \ldots(3)$$

Adding (1) and (2), we get

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}1.dx\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$$

Question 4:

$$\int_0^{\frac{\pi}{2}}\frac{\cos^5 xdx}{\sin^5 x+\cos^5 x}dx$$

Solution:

Consider,

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos^5 xdx}{\sin^5 x+\cos^5 x}dx \quad \ldots(1)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^5\left(\frac{\pi}{2}-x\right)dx}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_0^a f(x)=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin^5 x}{\cos^5 x+\sin^5 x}dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^5 x+\cos^5 x}{\sin^5 x+\cos^5 x}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}1.dx\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$$

Question 5:

$$\int_{-5}^{5}|x+2|dx$$

Solution:

Let $I=\int_{-5}^{5}|x+2|dx$

As, $(x+2)\le 0$ on $[-5,-2]$ and $(x+2)\ge 0$ on $[-2,5]$

$$\therefore \int_{-5}^{5}|x+2|dx=\int_{-5}^{-2}-(x+2)dx+\int_{-2}^{5}(x+2)dx \qquad \left(\int_a^b f(x)=\int_a^c f(x)+\int_c^b f(x)\right)$$

$$I=-\left[\frac{x^2}{2}+2x\right]_{-5}^{-2}+\left[\frac{x^2}{2}+2x\right]_{-2}^{5}$$

$$=-\left[\frac{(-2)^2}{2}+2(-2)-\frac{(-5)^2}{2}-2(-5)\right]+\left[\frac{(5)^2}{2}+2(5)-\frac{(-2)^2}{2}-2(-2)\right]$$

$$=-\left[2-4-\frac{25}{2}+10\right]+\left[\frac{25}{2}+10-2+4\right]$$

$$=-2+4+\frac{25}{2}-10+\frac{25}{2}+10-2+4$$

$$=29$$

Question 6:

$$\int_2^8|x-5|dx$$

Solution:

Consider, $I=\int_2^8|x-5|dx$

As $(x-5)\le 0$ on $[2,5]$ and $(x-5)\ge 0$ on $[5,8]$

$$I=\int_2^5-(x-5)dx+\int_2^8(x-5)dx \quad \left(\int_a^b f(x)=\int_a^c f(x)+\int_c^b f(x)\right)$$

$$=-\left[\frac{x^2}{2}-5x\right]_2^5+\left[\frac{x^2}{2}-5x\right]_5^8$$

$$=-\left[\frac{25}{2}-25-2+10\right]+\left[32-40-\frac{25}{2}+25\right]=9$$

Question 7:

$$\int_0^1 x(1-x)^n\,dx$$

Solution:

Consider, $I=\int_0^1 x(1-x)^n\,dx$

$$\therefore I=\int_0^1(1-x)(1-(1-x))^n\,dx$$

$$=\int_0^1(1-x)(x)^n dx=\int_0^1\left(x^n-x^{n+1}\right)dx$$

$$=\left[\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right]_0^1 \qquad \left(\int_1^a f(x)dx=\int_0^a f(a-x)\,dx\right)$$

$$==\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$$

$$=\frac{(n+2)-(n+1)}{(n+1)(n+2)}$$

$$=\frac{1}{(n+1)(n+2)}$$

Question 8:

$$\int_0^{\frac{\pi}{4}}\log(1+\tan x)\,dx$$

Solution:

Let $I=\int_0^{\frac{\pi}{4}}\log(1+\tan x)\,dx \quad \ldots(1)$

$$\therefore I=I=\int_0^{\frac{\pi}{4}}\log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log\left\{1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right\}dx \qquad \left(\tan(a-b)=\frac{\tan a-\tan b}{1+\tan a\tan b}\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log\left\{1+\frac{1-\tan x}{1+\tan x}\right\}dx \Rightarrow I=\int_0^{\frac{\pi}{4}}\log\frac{2}{(1+\tan x)}dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log 2dx-\int_0^{\frac{\pi}{4}}\log(1+\tan x)dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log 2dx-I \quad [\text{from}(1)]$$

$$\Rightarrow 2I=\log 2[x]_0^{\frac{\pi}{4}}$$

$$\Rightarrow 2I=\log 2\left[\frac{\pi}{4}-0\right]$$

$$I=\frac{\pi}{8}\log 2$$

Question 9:

$$\int_0^2 x\sqrt{2-x}dx$$

Solution:

Consider, $I=\int_0^2 x\sqrt{2-x}dx$

$I=\int_0^2 (2-x)\sqrt{2-(2-x)}dx \quad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$

$=\int_0^2 (2-x)\sqrt{x}dx$

$=\int_0^2 \left\{2x^{\frac{1}{2}}-x^{\frac{3}{2}}\right\}dx=\left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)-\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^2$

$=\left[\frac{4}{3}x^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}\right]_0^2=\frac{4}{3}(2)^{\frac{3}{2}}-\frac{2}{5}(2)^{\frac{5}{2}}$

$=\frac{4\times 2\sqrt{2}}{3}-\frac{2}{5}\times 4\sqrt{2}=\frac{8\sqrt{2}}{3}-\frac{8\sqrt{2}}{5}$

$=\frac{40\sqrt{2}-24\sqrt{2}}{15}=\frac{16\sqrt{2}}{15}$

Question 10:

$$\int_0^{\frac{\pi}{2}} (2\log\sin x-\log\sin 2x)dx$$

Solution:

Consider, $I=\int_0^{\frac{\pi}{2}} (2\log\sin x-\log\sin 2x)dx$

$I=\int_0^{\frac{\pi}{2}} (2\log\sin x-\log(2\sin x\cos x))dx$

$I=\int_0^{\frac{\pi}{2}} (2\log\sin x-\log\sin x-\log\cos x-\log 2)dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}} \{\log\sin x-\log\cos x-\log 2\} \quad \ldots(1)$

Since, $\left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$

$\Rightarrow I=\int_0^{\frac{\pi}{2}} \{\log\cos x-\log\sin x-\log 2\}dx \quad \ldots(2)$

Adding (1) and (2), we get

$$2I=\int_0^{\frac{\pi}{2}}(-\log 2-\log 2)\,dx$$

$$\Rightarrow 2I=-2\log 2\int_0^{\frac{\pi}{2}}1.dx$$

$$\Rightarrow I=-\log 2\left[\frac{\pi}{2}\right]$$

$$\Rightarrow I=\frac{\pi}{2}(-\log 2)$$

$$\Rightarrow I=\frac{\pi}{2}\left[\log\frac{1}{2}\right]$$

$$\Rightarrow I=\frac{\pi}{2}\log\frac{1}{2}$$

Question 11:

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\,dx$$

Solution:

Let $I=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\,dx$

As $\sin^2(-x)=(\sin(-x))^2=(-\sin x)^2=\sin^2 x$, therefore $\sin^2 x$ is an even function.

If $f(x)$ is an even function, then $\int_{-a}^{a}f(x)\,dx=2\int_0^a f(x)\,dx$

$$I=2\int_0^{\frac{\pi}{2}}\sin^2 x\,dx=2\int_0^{\frac{\pi}{2}}\frac{1-\cos 2x}{2}dx$$

$$=\int_0^{\frac{\pi}{2}}(1-\cos 2x)\,dx=\left[x-\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$=\left[\frac{\pi}{2}-\frac{\sin 2\left(\frac{\pi}{2}\right)}{2}\right]-\left[0-\frac{\sin 2(0)}{2}\right]$$

$$=\frac{\pi}{2}-\frac{\sin\pi}{2}-0$$

$$=\frac{\pi}{2}$$

Question 12:

$$\int_0^{\pi} \frac{xdx}{1+\sin x}$$

Solution:

Let $I = \int_0^{\pi} \frac{xdx}{1+\sin x} \quad \ldots(1)$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin(\pi - x)} dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin x} dx \ \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x}{1+\sin x} dx + \int_0^{\pi} \frac{\pi - x}{1+\sin x} dx$$

$$\Rightarrow 2I = \int_0^{\pi} \frac{\pi}{1+\sin x} dx$$

Multiplying and Dividig by $(1-\sin x)$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \{\sec^2 x - \tan x \sec x\} dx$$

$$\Rightarrow 2I = \pi \left[[\tan x]_0^{\pi} - [\sec x]_0^{\pi}\right]$$

$$\Rightarrow 2I = \pi \left[(\tan(\pi) - \tan(0)) - (\sec(\pi) - \sec(0))\right]$$

$$\Rightarrow 2I = \pi [2]$$

$$\Rightarrow I = \pi$$

Question 13:

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$$

Solution:

Let $I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx \ \ldots(1)$

As $\sin^7(-x) = (\sin(-x))^7 = (-\sin x)^7 = -\sin^7 x$, thus $\sin^2 x$ is an odd function.

$f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

$\therefore I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx=0$

Question 14:

$\int_0^{2\pi} \cos^5 x dx$

Solution:

Let $I=\int_0^{2\pi} \cos^5 x dx \quad \ldots(1)$

$\cos^5(2\pi-x)=\cos^5 x$

We know that,

$\int_0^{2a} f(x)dx=2\int_0^{a} f(x)dx$, if $f(2a-x)=f(x)$

$=0$ if $f(2a-x)=-f(x)$

$\therefore I=2\int_0^{2\pi} \cos^5 x dx$

$\Rightarrow I=2(0)=0 \qquad \left[\cos^5(\pi-x)=-\cos^5 x\right]$

Question 15:

$\int_0^{\frac{\pi}{2}} \frac{\sin x-\cos x}{1+\sin x\cos x}$

Solution:

Consider, $I=\int_0^{\frac{\pi}{2}} \frac{\sin x-\cos x}{1+\sin x\cos x}dx \quad \ldots(1)$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx \quad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{\cos x-\sin x}{1+\sin x\cos x}dx \quad \ldots(2)$

Adding (1) and (2), we get

$\Rightarrow 2I=\int_0^{\frac{\pi}{2}} \frac{0}{1+\sin x\cos x}dx \Rightarrow I=0$

Question 16:

$\int_0^{\pi} \log(1+\cos x)dx$

Solution:

Consider, $I = \int_0^{\pi} \log(1+\cos x)dx \quad \ldots(1)$

$\Rightarrow I = \int_0^{\pi} \log(1+\cos(\pi - x))dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$\Rightarrow I = \int_0^{\pi} \log(1-\cos x)dx \quad \ldots(2)$

Adding (1) and (2), we get

$2I = \int_0^{\pi} \{\log(1+\cos x) + \log(1-\cos x)\} dx$

$\Rightarrow 2I = \int_0^{\pi} \log(1-\cos^2 x)dx$

$\Rightarrow 2I = \int_0^{\pi} \log(\sin^2 x)dx$

$\Rightarrow 2I = 2\int_0^{\pi} \log(\sin x)dx$

$\Rightarrow I = \int_0^{\pi} \log(\sin x)dx \qquad \ldots(3)$

$\therefore \ \sin(\pi - x) = \sin x$

We know that,

$\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \text{ if } f(2a-x) = f(x)$

$\therefore I = \int_0^{\frac{\pi}{2}} \log\sin x dx \quad \ldots(4)$

$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2} - x\right)dx = 2\int_0^{\frac{\pi}{2}} \log\cos x dx \qquad \ldots(5)$

Adding (4) and (5), we get

$$2I=\int_0^{\frac{\pi}{2}}(\log\sin x+\log\cos xdx)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}(\log\sin x+\log\cos x+\log 2-\log 2)\,dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}(\log 2\sin x\cos x-\log 2)dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\log\sin 2xdx-\int_0^{\frac{\pi}{2}}\log 2dx$$

put, $2x=t\Rightarrow 2dx=dt$

When $x=0,t=0$

$$\therefore I=\frac{1\pi}{2}\int_0^{a}\log\sin tdt-\frac{\pi}{2}\log 2$$

$$\Rightarrow I=\frac{1}{2}-\frac{\pi}{2}\log 2$$

$$\Rightarrow I=-\pi\log 2$$

Question 17:

$$\int_0^a\frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$$

Solution:

Let $I=\int_0^a\frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$ …(1)

We know that, $\left(\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$

$$I=\int\frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx\quad\cdots(2)$$

Adding (1) and (2), we get

$$2I=\int_0^a\frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx$$

$$\Rightarrow 2I=\int_0^a 1.dx$$

$$\Rightarrow 2I=[x]_0^a$$

$$\Rightarrow 2I=a$$

$$\Rightarrow I=\frac{a}{2}$$

Question 18:

$$\int_0^4 |x-1|\,dx$$

Solution:

$$\int_0^4 |x-1|\,dx$$

Since,

$(x-1) \le 0$ when $0 \le x \le 1$ and $(x-1) \ge 0$ when $1 \le x \le 4$

$$I = \int_0^1 |x-1|\,dx + \int_1^4 |x-1|\,dx \qquad \left(\int_b^b f(x)\,dx = \int_b^c f(x)\,dx + \int_c^b f(x)\,dx\right)$$

$$I = \int_0^1 -(x-1)\,dx + \int_0^4 (x-1)\,dx$$

$$= \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4 = 1 - \frac{1}{2} + \frac{(4)^2}{2} - 4 - \frac{1}{2} + 1$$

$$= 1 - \frac{1}{2} + 8 - 4 - \frac{1}{2} + 1$$

$$= 5$$

Question 19:

Show that $\int_0^a f(x)g(x)\,dx = 2\int_0^a f(x)dx$ if f and g are defined as $f(x) = f(a-x)$ and $g(x) = (a-x) = 4$

Solution:

Let

$$I = \int_0^a f(x)g(x)\,dx \quad \ldots(1)$$

$$\Rightarrow \int_0^a f(a-x)g(a-x)\,dx \qquad \left(\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow \int_0^a f(x)g(a-x)\,dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^a \{f(x)g(x) + f(x)g(a-x)\}\,dx$$

$$\Rightarrow 2I = \int_0^a f(x)\{g(x) + g(a-x)\}\,dx$$

$$\Rightarrow 2I = \int_0^a f(x) \times 4\,dx \qquad [g(x) + g(a-x) = 4]$$

$$\Rightarrow I = 2\int_0^a f(x)\,dx$$

Question 20:

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3 + x\cos x + \tan^5 x + 1\right)dx$ is

A. 0

B. 2

C. π

D. 1

Solution:

Consider, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3 + x\cos x + \tan^5 x + 1\right)dx$

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^5 x dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.dx$$

For $f(x)$ an even function, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx$

And

$$I = 0 + 0 + 0 + 2\int_{0}^{\frac{\pi}{2}} 1.dx$$

$$= 2[x]_0^{\frac{\pi}{2}}$$

$$= \frac{2\pi}{2}$$

$$= \pi$$

Thus, the correct is option C.

Question 21:

The value of $\int_0^{\frac{\pi}{2}}\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ is

A. 2

B. $\frac{3}{4}$

C. 0

D. -2

Solution:

Let $I=\int_0^{\frac{\pi}{2}}\left(\frac{4+3\sin x}{4+3\cos x}\right)dx \quad \dots(1)$

$$\Rightarrow I=I=\int_0^{\frac{\pi}{2}}\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right)dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\log\left(\frac{4+3\cos x}{4+3\sin x}\right) \quad \dots(2)$$

Adding (1) and (2), we get

$$2I=\int_0^{\frac{\pi}{2}}\left\{\log\left(\frac{4+3\sin x}{4+3\cos x}\right)+\log\left(\frac{4+3\cos x}{4+3\sin x}\right)\right\}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}\left(\frac{4+3\sin x}{4+3\cos x}\times\frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}\log 1dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}0dx$$

$$\Rightarrow I=0$$

Thus, the correct option is C.

MISCELLANEOUS EXERCISE

Integrate the functions in Exercises 1 to 24.

Question 1:

$$\frac{1}{x-x^3}$$

Solution:

$$\frac{1}{x-x^3}=\frac{1}{x\left(1-x^2\right)}=\frac{1}{x(1-x)(1+x)}$$

Let $\frac{1}{x(1-x)(1+x)}=\frac{A}{x}+\frac{B}{(1-x)}+\frac{C}{(1+x)} \quad \ldots(1)$

$$\Rightarrow 1=A\left(1-x^2\right)+Bx(1+x)+Cx(1-x)$$

$$\Rightarrow 1=A-Ax^2+Bx+Bx^2+Cx-Cx^2$$

Equating the coefficients of x^2, x and constant terms, we get

$$-A+B-C=0$$

$$B+C=0$$

$$A=1$$

On solving these equations, we get

$$A=1$$

$$B=\frac{1}{2}$$

$$C=-\frac{1}{2}$$

From equation (1), we get

$$\frac{1}{x(1-x)(1+x)}=\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}$$

$$\Rightarrow \int \frac{1}{x(1-x)(1+x)}dx=\int \frac{1}{x}dx+\frac{1}{2}\int \frac{1}{(1-x)}dx-\frac{1}{2}\int \frac{1}{(1+x)}dx$$

$$=\log|x|-\frac{1}{2}\log|(1-x)|-\frac{1}{2}\log|(1+x)|$$

$$=\log|x|-\log\left|(1-x)^{\frac{1}{2}}\right|-\log\left|(1+x)^{\frac{1}{2}}\right|=\log\left|\frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}}\right|+C$$

$$=\log\left|\left(\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}\right|+C=\frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right|+C$$

Question 2:

$$\frac{1}{\sqrt{x+a}+\sqrt{x+b}}$$

Solution:

$$\frac{1}{\sqrt{x+a}+\sqrt{x+b}}=\frac{1}{\sqrt{x+a}+\sqrt{x+b}}\times\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$=\frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}=\frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{a-b}$$

$$\Rightarrow\int\frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx=\frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$

$$=\frac{1}{(a-b)}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right]=\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]+C$$

Question 3:

$$\frac{1}{x\sqrt{ax-x^2}}\qquad\left[\text{Hint: } x=\frac{a}{t}\right]$$

Solution:

$$\frac{1}{x\sqrt{ax-x^2}}$$

Let $x=\frac{a}{t}\Rightarrow dx=-\frac{a}{t^2}dt$

$$\Rightarrow\int\frac{1}{x\sqrt{ax-x^2}}dx=\int\frac{1}{\frac{a}{t}\sqrt{a.\frac{a}{t}-\left(\frac{a}{t}\right)^2}}\left(-\frac{a}{t^2}dt\right)$$

$$=-\int\frac{1}{at}\frac{1}{\sqrt{\frac{1}{t}-\frac{1}{t^2}}}dt=-\frac{1}{a}\int\frac{1}{\sqrt{\frac{t^2}{t}-\frac{t^2}{t^2}}}dt$$

$$=-\frac{1}{a}\int\frac{1}{\sqrt{t-1}}dt$$

$$=-\frac{1}{a}\left[2\sqrt{t-1}\right]+C$$

$$=-\frac{1}{a}\left[2\sqrt{\frac{a}{x}-1}\right]+C$$

$$=-\frac{2}{a}\left(\sqrt{\frac{a-x}{x}}\right)+C$$

Question 4:

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Solution:

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Multiplying and dividing by x^{-3}, we get

$$\frac{x^{-3}}{x^2x^{-3}\left(x^4+1\right)^{\frac{3}{4}}}=\frac{x^{-3}\left(x^4+1\right)^{\frac{-3}{4}}}{x^2x^{-3}}$$

$$=\frac{\left(x^4+1\right)^{\frac{-3}{4}}}{x^5\left(x^4\right)^{-\frac{3}{4}}}=\frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{-\frac{3}{4}}$$

$$=\frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}$$

Let

$$\frac{1}{x^4}=t\Rightarrow-\frac{4}{x^5}dx=dt\Rightarrow\frac{1}{x^5}dx=-\frac{dt}{4}$$

$$\therefore\int\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx=\int\frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}dx=-\frac{1}{4}\int\left(1+t\right)^{-\frac{3}{4}}dt$$

$$=-\frac{1}{4}\left[\frac{\left(1+t\right)^{\frac{1}{4}}}{\frac{1}{4}}\right]+C=-\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}}+C$$

$$=-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}+C$$

Question 5:

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}} \quad \left[\text{Hint: } \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)} \text{ put } x=t^6\right]$$

Solution:

$$\frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}=\frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$$

Let $x=t^6 \Rightarrow dx=6t^5dt$

$$\therefore \int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx=\int \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}dx=\int \frac{6t^5}{t^2(1+t)}dt$$

$$=6\int \frac{t^3}{(1+t)}dt$$

Adding and Substracting 1 in Numerator

$$=6\int \frac{t^3+1-1}{1+t}dt$$

$$=6\int \left(\frac{t^3+1}{1+t}-\frac{1}{1+t}\right)dt$$

Using $a^3+b^3=(a+b)(a^2+b^2-ab)$

$$=6\int \left\{\frac{(1+t)(t^2+1^2-1\times t)}{1+t}-\frac{1}{1+t}\right\}dt$$

$$=6\int \left\{(t^2-t+1)-\frac{1}{1+t}\right\}dt$$

$$=6\left[\left(\frac{t^3}{3}\right)-\left(\frac{t^2}{2}\right)+t-\log|1+t|\right]$$

$$=2x^{\frac{1}{2}}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

$$=2\sqrt{x}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\log\left(1+x^{\frac{1}{6}}\right)+C$$

Question 6:

$$\frac{5x}{(x+1)(x^2+9)}$$

Solution:

Consider, $\frac{5x}{(x+1)(x^2+9)}=\frac{A}{(x+1)}+\frac{Bx+C}{(x^2+9)} \qquad \ldots(1)$

$$\Rightarrow 5x=A(x^2+9)+(Bx+C)(x+1)$$

$$\Rightarrow 5x=Ax^2+9A+Bx^2+Bx+Cx+C$$

Equating the coefficients of x^2, x and constant term, we get

$A+B=0$

$B+C=5$

$9A+C=0$

On solving these equations, we get

$A=-\frac{1}{2}$

$B=\frac{1}{2}$

$C=\frac{9}{2}$

From equation (1), we get

$$\frac{5x}{(x+1)(x^2+9)}=\frac{-1}{2(x+1)}+\frac{\frac{x}{2}+\frac{9}{2}}{(x^2+9)}$$

$$\int\frac{5x}{(x+1)(x^2+9)}dx=\int\left\{\frac{-1}{2(x+1)}+\frac{(x+9)}{2(x^2+9)}\right\}dx$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{2}\int\frac{x}{x^2+9}dx+\frac{9}{2}\int\frac{1}{x^2+9}dx=-\frac{1}{2}\log|x+1|+\frac{1}{4}\int\frac{2x}{x^2+9}dx+\frac{9}{2}\int\frac{1}{x^2+9}dx$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log|x^2+9|+\frac{9}{2}.\frac{1}{3}\tan^{-1}\frac{x}{3}+C$$

$$==-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+9)+\frac{3}{2}\tan^{-1}\frac{x}{3}+C$$

Question 7:

$$\frac{\sin x}{\sin(x-a)}$$

Solution:

$$\frac{\sin x}{\sin(x-a)}$$

Put, $x-a=t \Rightarrow dx=dt$

$$\int \frac{\sin x}{\sin(x-a)}dx = \int \frac{\sin(t+a)}{\sin t}dt$$

$$= \int \frac{\sin t\cos a+\cos t\sin a}{\sin t}dt = \int(\cos a+\cot t\sin a)dt$$

$$= t\cos a+\sin a\log|\sin t|+C_1$$

$$= (x-a)\cos a+\sin a\log|\sin(x-a)|+C_1$$

$$= x\cos a+\sin a\log|\sin(x-a)|-a\cos a+C_1$$

$$= \sin a\log|\sin(x-a)|+x\cos a+C$$

Question 8:

$$\frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}$$

Solution:

$$\frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}=\frac{e^{4\log x}\left(e^{\log x}-1\right)}{e^{2\log x}\left(e^{\log x}-1\right)}$$

$$=e^{2\log x}$$

$$=e^{\log x^2}$$

$$=x^2$$

$$\therefore\int\frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}dx=\int x^2dx=\frac{x^3}{3}+C$$

Question 9:

$$\frac{\cos x}{\sqrt{4-\sin^2 x}}$$

Solution:

$$\frac{\cos x}{\sqrt{4-\sin^2 x}}$$

Put, $\sin x=t\Rightarrow\cos x dx=dt$

$$\Rightarrow\int\frac{\cos x}{\sqrt{4-\sin^2 x}}dx=\int\frac{dt}{\sqrt{(2)^2-(t)^2}}$$

$$=\sin^{-1}\left(\frac{t}{2}\right)+C$$

$$=\sin^{-1}\left(\frac{\sin x}{2}\right)+C$$

$$=\frac{x}{2}+C$$

Question 10:

$$\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$$

Solution:

$$\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}=\frac{\left(\sin^4 x-\cos^4 x\right)\left(\sin^4 x+\cos^4 x\right)}{\sin^2 x+\cos^2 x-\sin^2 x\cos^2 x-\sin^2 x\cos^2 x}$$

$$=\frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x-\cos^2 x\right)\left(\sin^2 x+\cos^2 x\right)}{\left(\sin^2 x-\sin^2 x\cos^2 x\right)+\left(\cos^2 x-\sin^2 x\cos^2 x\right)}$$

$$=\frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x-\cos^2 x\right)}{\sin^2 x\left(1-\cos^2 x\right)+\cos^2 x\left(1-\sin^2 x\right)}$$

$$=\frac{-\left(\sin^4 x+\cos^4 x\right)\left(\cos^2 x-\sin^2 x\right)}{\left(\sin^4 x+\cos^4 x\right)}$$

$$=-\cos 2x$$

$$\therefore \int\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx=\int-\cos 2x dx=-\frac{\sin 2x}{2}+C$$

Question 11:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Solution:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by $\sin(a-b)$, we get

$$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)}-\frac{\sin(x+b)}{\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\tan(x+a)-\tan(x+b)\right]$$

$$\int\frac{1}{\cos(x+a)\cos(x+b)}dx=\frac{1}{\sin(a-b)}\int\left[\tan(x+a)-\tan(x+b)\right]dx$$

$$=\frac{1}{\sin(a-b)}\left[-\log\left|\cos(x+a)\right|+\log\left|\cos(x+b)\right|\right]+C$$

$$=\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$$

Question 12:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Solution:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Put, $x^4=t \Rightarrow 4x^3dx=dt$

$$\Rightarrow\int\frac{x^3}{\sqrt{1-x^8}}dx=\frac{1}{4}\int\frac{dt}{\sqrt{1-t^2}}$$

$$=\frac{1}{4}\sin^{-1}t+C$$

$$=\frac{1}{4}\sin^{-1}\left(x^4\right)+C$$

Question 13:

$$\frac{e^x}{(1+e^x)(2+e^x)}$$

Solution:

$$\frac{e^x}{(1+e^x)(2+e^x)}$$

Put $e^x = t \Rightarrow e^x dx = dt$

$$\Rightarrow \int \frac{e^x}{(1+e^x)(2+e^x)} dx = \int \frac{dt}{(t+1)(t+2)}$$

$$= \int \left[\frac{1}{(t+1)} - \frac{1}{(t+2)} \right] dt$$

$$= \log|t+1| - \log|t+2| + C$$

$$= \log\left|\frac{t+1}{t+2}\right| + C$$

$$= \log\left|\frac{1+e^x}{2+e^x}\right| + C$$

Question 14:

$$\frac{1}{(x^2+1)(x^2+4)}$$

Solution:

$$\therefore \frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)}$$

$$\Rightarrow 1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

$$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

Equating the coefficients of x^3, x^2, x and constant term, we get

$A + C = 0$

$B + D = 0$

$4A + C = 0$

$4B + D = 1$

On solving these equations, we get

$A=0$

$B=\frac{1}{3}$

$C=0$

$D=-\frac{1}{3}$

From equation (1), we get

$$\frac{1}{\left(x^2+1\right)\left(x^2+4\right)}=\frac{1}{3\left(x^2+1\right)}-\frac{1}{3\left(x^2+4\right)}$$

$$\int\frac{1}{\left(x^2+1\right)\left(x^2+4\right)}dx=\frac{1}{3}\int\frac{1}{x^2+1}dx-\frac{1}{3}\int\frac{1}{x^2+4}dx$$

$$=\frac{1}{3}\tan^{-1}x-\frac{1}{3}.\frac{1}{2}\tan^{-1}\frac{x}{2}+C$$

$$=\frac{1}{3}\tan^{-1}x-\frac{1}{6}\tan^{-1}\frac{x}{2}+C$$

Question 15:

$\cos^3 xe^{\log\sin x}$

Solution:

$\cos^3 xe^{\log\sin x}=\cos^3 x\times\sin x$

Let $\cos x=t\Rightarrow-\sin xdx=dt$

$$\Rightarrow\int\cos^3 xe^{\log\sin x}dx=\int\cos^3 x\sin xdx$$

$$=-\int t^3dt$$

$$=-\frac{t^4}{4}+C$$

$$=-\frac{\cos^4 x}{4}+C$$

Question 16:

$e^{3\log x}\left(x^4+1\right)^{-1}$

Solution:

$$e^{3\log x}\left(x^4+1\right)^{-1}=e^{\log x^3}\left(x^4+1\right)^{-1}=\frac{x^3}{\left(x^4+1\right)}$$

Let $x^4+1=t\Rightarrow 4x^3dx=dt$

$$\Rightarrow \int e^{3\log x}\left(x^4+1\right)^{-1}dx = \int \frac{x^3}{\left(x^4+1\right)}dx$$

$$= \frac{1}{4}\int \frac{dt}{t}$$

$$= \frac{1}{4}\log|t| + C$$

$$= \frac{1}{4}\log\left|x^4+1\right| + C$$

$$= \frac{1}{4}\log\left(x^4+1\right) + C$$

Question 17:

$$f'(ax+b)\left[f(ax+b)\right]^n$$

Solution:

$$f'(ax+b)\left[f(ax+b)\right]^n$$

Put, $f(ax+b) = t \Rightarrow af'(ax+b)dx = dt$

$$\Rightarrow \int f'(ax+b)\left[f(ax+b)\right]^n dx = \frac{1}{a}\int t^n dt$$

$$= \frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right] = \frac{1}{a(n+1)}\left(f(ax+b)\right)^{n+1} + C$$

Question 18:

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$$

Solution:

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} = \frac{1}{\sqrt{\sin^3 x\left(\sin x\cos\alpha + \cos x\sin\alpha\right)}}$$

$$= \frac{1}{\sqrt{\sin^4 x\cos\alpha + \sin^3 x\cos x\sin\alpha}}$$

$$= \frac{1}{\sin^2 x\sqrt{\cos\alpha + \cot x\sin\alpha}} = \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x\sin\alpha}}$$

Put, $\cos\alpha + \cot x\sin\alpha = t \Rightarrow -\operatorname{cosec}^2 x\sin\alpha\, dx = dt$

$$\therefore \int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}dx = \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}}dx$$

$$= \frac{-1}{\sin\alpha}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{-1}{\sin\alpha}\left[2\sqrt{t}\right] + C$$

$$= \frac{-1}{\sin\alpha}\left[2\sqrt{\cos\alpha + \cot x \sin\alpha}\right] + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\cos\alpha + \frac{\cos x \sin\alpha}{\sin x}} + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\frac{\sin x \cos\alpha + \cos x \sin\alpha}{\sin x}} + C = \frac{-2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$$

Question 19:

$$\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}, x \in [0,1]$$

Solution:

Let $I = \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}dx$

As we know that, $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}$

$$\Rightarrow I = \int \frac{\left(\frac{\pi}{2} - \cos^{-1}\sqrt{x}\right) - \cos^{-1}\sqrt{x}}{\frac{\pi}{2}}dx$$

$$= \frac{2}{\pi}\int\left(\frac{\pi}{2} - 2\cos^{-1}\sqrt{x}\right)dx$$

$$= \frac{2}{\pi}.\frac{\pi}{2}\int 1.dx - \frac{4}{\pi}\int \cos^{-1}\sqrt{x}dx$$

$$= x - \frac{4}{\pi}\int \cos^{-1}\sqrt{x}dx \qquad \ldots(1)$$

Let $I_1 = \int \cos^{-1}\sqrt{x}dx$

Also, let $\sqrt{x} = t \Rightarrow dx = 2tdt$

$$\Rightarrow I_1 = 2\int \cos^{-1} t.tdt$$

$$= 2\left[\cos^{-1} t.\frac{t^2}{2} - \int \frac{-1}{\sqrt{1-t^2}}.\frac{t^2}{2}dt\right]$$

$$= t^2\cos^{-1} t + \int \frac{t^2}{\sqrt{1-t^2}}dt$$

$$= t^2\cos^{-1} t - \int \frac{1-t^2-1}{\sqrt{1-t^2}}dt$$

$$= t^2\cos^{-1} t - \int \sqrt{1-t^2}dt + \int \frac{1}{\sqrt{1-t^2}}dt$$

$$= t^2\cos^{-1} t - \frac{1}{2}\sqrt{1-t^2} - \frac{1}{2}\sin^{-1} t + \sin^{-1} t$$

$$= t^2\cos^{-1} t - \frac{1}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t$$

From equation (1), we get

$$I = x - \frac{4}{\pi}\left[t^2\cos^{-1} t - \frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1} t\right]$$

$$= x - \frac{4}{\pi}\left[x\cos^{-1}\sqrt{x} - \frac{\sqrt{x}}{2}\sqrt{1-x} + \frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$x - \frac{4}{\pi}\left[x\left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right) - \frac{\sqrt{x-x^2}}{2} + \frac{\pi}{2}\sin^{-1}\sqrt{x}\right]$$

$$= x - 2x + \frac{4x}{\pi}\sin^{-1}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - \frac{2}{\pi}\sin^{-1}\sqrt{x}$$

$$-x + \frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}\right] + \frac{2}{\pi}\sqrt{x-x^2} + C$$

$$= \frac{2(2x-1)}{\pi}\sin^{-}\sqrt{x} + \frac{2}{\pi}\sqrt{x-x^2} - x + C$$

Question 20:

$$\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Solution:

$$I = \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

Put, $x = \cos^2\theta \Rightarrow dx = -2\sin\theta\cos\theta d\theta$

$$I=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\left(-2\sin\theta\cos\theta\right)d\theta=-\int\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin 2\theta d\theta$$

$$=-2\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\theta d\theta$$

$$=-4\int\sin^2\frac{\theta}{2}\cos\theta d\theta$$

$$=-4\int\sin^2\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)d\theta$$

$$=-4\int\left(2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)d\theta$$

$$=-8\int\sin^2\frac{\theta}{2}.\cos^2\frac{\theta}{2}d\theta+4\int\sin^2\frac{\theta}{2}d\theta$$

$$=-2\int\sin^2\frac{\theta}{2}d\theta+4\int\sin^2\frac{\theta}{2}d\theta$$

$$=-2\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta+4\int\frac{1-\cos\theta}{2}d\theta$$

$$=-2\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]+4\left[\frac{\theta}{2}-\frac{\sin\theta}{2}\right]+C$$

$$=-\theta+\frac{\sin 2\theta}{2}+2\theta-2\sin\theta+C$$

$$=\theta+\frac{\sin 2\theta}{2}+2\sin\theta+C$$

$$=\theta+\frac{2\sin\theta\cos\theta}{2}-2\sin\theta+C$$

$$=\theta+\sqrt{1-\cos^2\theta}.\cos\theta-2\sqrt{1-\cos^2\theta}+C$$

$$=\cos^{-1}\sqrt{x}+\sqrt{1-x}.\sqrt{x}-2\sqrt{1-x}+C$$

$$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x(1-x)}+C$$

$$=-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x-x^2}+C$$

Question 21:

$$\frac{2+\sin 2x}{1+\cos 2x}e^x$$

Solution:

$$I=\int\left(\frac{2+\sin 2x}{1+\cos 2x}\right)e^x$$

$$=\int\left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right)e^x$$

$$=\int\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)e^x$$

$$=\int\left(\sec^2 x+\tan x\right)e^x$$

Let $f(x)=\tan x \Rightarrow f'(x)=\sec^2 x$

$$\therefore I=\int\left(f(x)+f'(x)\right)e^x dx$$

$$=e^x f(x)+C$$

$$=e^x \tan x+C$$

Question 22:

$$\frac{x^2+x+1}{(x+1)^2(x+2)}$$

Solution:

Let $\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+1)^2}+\frac{C}{(x+2)} \qquad \cdots(1)$

$$\Rightarrow x^2+x+1=A(x+1)(x+2)+B(x+2)+C\left(x^2+2x+1\right)$$

$$\Rightarrow x^2+x+1=A\left(x^2+3x+2\right)+B(x+2)+C\left(x^2+2x+1\right)$$

$$\Rightarrow x^2+x+1=(A+C)x^2+(3A+B+2C)x+(2A+2B+C)$$

Equating the coefficients of x^2, x and constant term, we get

$A+C=1$

$3A+B+2C=1$

$2A+2B+C=1$

On solving these equations, we get

$A=-2$

$B=1$

$C=3$

From equation (1), we get

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{(x+1)}+\frac{3}{(x+2)}+\frac{1}{(x+1)^2}$$

$$\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx=-2\int\frac{1}{x+1}dx+3\int\frac{1}{(x+2)}dx+\int\frac{1}{(x+1)^2}dx$$

$$=-2\log|x+1|+3\log|x+2|-\frac{1}{(x+1)}+C$$

Question 23:

$$\tan^{-1}\sqrt{\frac{1-x}{1+x}}$$

Solution:

$$I=\tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$$

Let $x=\cos\theta \Rightarrow dx=-\sin\theta d\theta$

$$I=\int\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-\sin\theta)d\theta$$

$$=-\int\tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin\theta d\theta=-\int\tan^{-1}\tan\frac{\theta}{2}\sin\theta d\theta$$

$$=-\frac{1}{2}\int\theta.\sin\theta d\theta=-\frac{1}{2}\left[\theta.(-\cos\theta)-\int 1.(-\cos\theta)d\theta\right]$$

$$=-\frac{1}{2}[-\theta\cos\theta+\sin\theta]$$

$$=\frac{1}{2}\theta\cos\theta-\frac{1}{2}\sin\theta$$

$$=\frac{1}{2}\cos^{-1}x.x-\frac{1}{2}\sqrt{1-x^2}+C=\frac{x}{2}\cos^{-1}x-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{1}{2}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)+C$$

Question 24:

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}$$

Solution:

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(x^2+1\right)-\log x^2\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

Let $1+\frac{1}{x^2}=t \Rightarrow \frac{-2}{x^3}dx=dt$

$$\therefore I=\int\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)dx$$

$$=-\frac{1}{2}\int\sqrt{t}\log t\,dt=-\frac{1}{2}\int t^{\frac{1}{2}}\log t\,dt$$

Using integration by parts, we get

$$I=-\frac{1}{2}\left[\log t.\int t^{\frac{1}{2}}dt-\left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}}dt\right\}dt\right]$$

$$=-\frac{1}{2}\left[\log t.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}-\int\frac{1}{t}.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{2}{3}\int t^{\frac{1}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{4}{9}t^{\frac{3}{2}}\right]$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\log t+\frac{2}{9}t^{\frac{3}{2}}$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\left[\log t-\frac{2}{3}\right]$$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+C$$

Question 25:

$$\int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$$

Solution:

$$I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right) dx = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{\operatorname{cosec}^2\frac{x}{2}}{2} - \cot\frac{x}{2}\right) dx$$

Let $f(x) = -\cot\frac{x}{2}$

$$\Rightarrow f'(x) = -\left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right) = \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(f(x) + f'(x)\right) dx$$

$$= \left[e^x f(x)\, dx\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^x \cot\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^x \times \cot\frac{\pi}{2} - e^{\frac{\pi}{2}} \times \cot\frac{\pi}{4}\right]$$

$$= -\left[e^{\pi} \times 0 - e^{\frac{\pi}{2}} \times 1\right]$$

$$= e^{\frac{\pi}{2}}$$

Question 26:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x\cos x}{\cos^4 x + \sin^4 x} dx$$

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x\cos x}{\cos^4 x + \sin^4 x} dx$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}}\frac{\frac{(\sin x\cos x)}{\cos^4 x}}{\frac{(\cos^4 x+\sin^4 x)}{\cos^4 x}}dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}}\frac{\tan x\sec^2 x}{1+\tan^4 x}dx$$

Put, $\tan^2 x=t\Rightarrow 2\tan x\sec^2 xdx=dt$

When $x=0,t=0$ and when $x=\frac{\pi}{4},t=1$

$$\therefore I=\frac{1}{2}\int_0^1\frac{dt}{1+t^2}=\frac{1}{2}\left[\tan^{-1}t\right]_0^1$$

$$=\frac{1}{2}\left[\tan^{-1}1-\tan^{-1}0\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{4}\right]$$

$$=\frac{\pi}{8}$$

Question 27:

$$\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4\sin^2 x}dx$$

Solution:

Consider, $I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4\sin^2 x}dx$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4\left(1-\cos^2 x\right)}dx$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4-4\cos^2 x}dx$$

$$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}\frac{-3\cos^2 x}{4-3\cos^2 x}dx$$

$$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}\frac{4-3\cos^2 x-4}{4-3\cos^2 x}dx$$

$$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}\frac{4-3\cos^2 x}{4-3\cos^2 x}dx+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4}{4-3\cos^2 x}dx$$

$$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}1dx+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4\sec^2 x}{4\sec^2 x-3}dx$$

$$\Rightarrow I=-\frac{1}{3}\left[x\right]_0^{\frac{\pi}{2}}+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4\sec^2 x}{4\left(1+\tan^2 x\right)-3}dx$$

$$\Rightarrow I=-\frac{\pi}{6}+\frac{2}{3}\int_0^{\frac{\pi}{2}}\frac{2\sec^2 x}{1+4\tan^2 x}dx \qquad \ldots(1)$$

Consider, $\int_0^{\frac{\pi}{2}}\frac{2\sec^2 x}{1+4\tan^2 x}dx$

Put, $2\tan x=t\Rightarrow 2\sec^2 xdx=dt$

When $x=0,t=0$ and when $x=\frac{\pi}{2},t=\infty$

$$\Rightarrow\int_0^{\frac{\pi}{2}}\frac{2\sec^2 x}{1+4\tan^2 x}dx=\int_0^{\infty}\frac{dt}{1+t^2}$$

$$=\left[\tan^{-1}t\right]_0^{\infty}$$

$$=\left[\tan^{-1}(\infty)-\tan^{-1}(0)\right]$$

$$=\frac{\pi}{2}$$

Therefore, from (1), we get

$$I=-\frac{\pi}{6}+\frac{2}{3}\left[\frac{\pi}{2}\right]=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$$

Question 28:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx$$

Solution:

Consider, $I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx$

$$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{-(-\sin 2x)}}dx \Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{-(-1+1-2\sin\cos x)}}dx$$

$$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}dx$$

Let $(\sin x-\cos x)=t \Rightarrow (\sin x+\cos x)dx=dt$

When $x=\frac{\pi}{6}, t=\left(\frac{1-\sqrt{3}}{2}\right)$ and when $x=\frac{\pi}{3}, t=\left(\frac{\sqrt{3}-1}{2}\right)$

$$I=\int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\Rightarrow I=\int_{-\left(\frac{1+\sqrt{3}}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

As $\frac{1}{\sqrt{1-(-t)^2}}=\frac{1}{\sqrt{1-t^2}}$, therefore, $\frac{1}{\sqrt{1-t^2}}$ is an even function

$$\int_{-a}^{a} f(x)dx=2\int_0^a f(x)dx$$

We know that if $f(x)$ is an even function, then

$$\Rightarrow I=2\int_0^{\frac{\sqrt{3}-1}{0}} \frac{dt}{\sqrt{1-t^2}}$$

$$=\left[2\sin^{-1}t\right]_0^{\frac{\sqrt{3}-1}{2}}$$

$$=2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

Question 29:

$$\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

Solution:

Consider, $I=\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

$$I=\int_0^1 \frac{1}{\left(\sqrt{1+x}-\sqrt{x}\right)}\times\frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{\left(\sqrt{1+x}+\sqrt{x}\right)}dx$$

$$=\int_0^1 \frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{1+x-x}dx$$

$$=\int_0^1 \sqrt{1+x}dx+\int_0^1 \sqrt{x}dx$$

$$=\left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1+\left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$$

$$=\frac{2}{3}\left[(2)^{\frac{3}{2}}-1\right]+\frac{2}{3}[1]$$

$$=\frac{2}{3}(2)^{\frac{3}{2}}=\frac{2.2\sqrt{2}}{3}$$

$$=\frac{4\sqrt{2}}{3}$$

Question 30:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$$

Solution:

Consider, $I=\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}dx$

Put, $\sin x-\cos x=t \Rightarrow (\cos x+\sin x)dx=dt$

When $x=0, t=-1$ and when $x=\frac{\pi}{4}, t=0$

$$\Rightarrow (\sin x - \cos x)^2 = t^2$$

$$\Rightarrow \sin^2 x + \cos^2 x - 2\sin x \cos x = t^2$$

$$\Rightarrow 1 - \sin 2x = t^2$$

$$\Rightarrow \sin 2x = 1 - t^2$$

$$\therefore I = \int_{-1}^{0} \frac{dt}{9 + 16(1 - t^2)}$$

$$= \int_{-1}^{0} \frac{dt}{9 + 16 - 16t^2}$$

$$= \int_{-1}^{0} \frac{dt}{25 - 16t^2} = \int_{-1}^{0} \frac{dt}{(5)^2 - (4t)^2}$$

$$= \frac{1}{4}\left[\frac{1}{2(5)} \log\left|\frac{5 + 4t}{5 - 4t}\right|\right]_{-1}^{0}$$

$$= \frac{1}{40}\left[\log(1) - \log\left|\frac{1}{9}\right|\right]$$

$$= \frac{1}{40} \log 9$$

Question 31:

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$$

Solution:

Consider, $I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx = \int_0^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x) dx$

Put, $\sin x = t \Rightarrow \cos x dx = dt$

When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = 1$

$$\Rightarrow I = 2\int_0^1 t \tan^{-1}(t) dt \qquad \ldots(1)$$

Consider $\int t . \tan^{-1} t dt = \tan^{-1} t \int t dt - \int \left\{\frac{d}{dt}(\tan^{-1} t) \int t dt\right\} dt$

$$= \tan^{-1} t . \frac{t^2}{2} - \int \frac{1}{1 + t^2} . \frac{t^2}{2} dt$$

$$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2}\int \frac{t^2 + 1 - 1}{1 + t^2} dt$$

$$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2}\int 1 . dt + \frac{1}{2}\int \frac{1}{1 + t^2} dt$$

$$= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2} t + \frac{1}{2} \tan^{-1} t$$

From equation (1), we get

$$\Rightarrow 2\int_0^1 t.\tan^{-1} t dt = 2\left[\frac{t^2 \tan^{-1} t}{2} - \frac{t}{2} + \frac{1}{2}\tan^{-1} t\right]_0^1$$

$$= \left[\frac{\pi}{4} - 1 + \frac{\pi}{4}\right] = \frac{\pi}{2} - 1$$

Question 32:

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$$

Solution:

Let $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx \qquad \ldots(1)$

$$I = \int_0^{\pi} \left\{\frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}\right\} dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\pi} \left\{\frac{-(\pi - x)\tan x}{-(\sec x + \tan x)}\right\} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\tan x}{(\sec x + \tan x)} dx \qquad \ldots(2)$$

Adding (1) and (2), we get

$$2I=\int_0^{\pi}\frac{\pi\tan x}{\sec x+\tan x}dx \Rightarrow 2I=\pi\int_0^{\pi}\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{\sin x+1-1}{1+\sin x}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}1.dx-\pi\int_0^{\pi}\frac{1}{1+\sin x}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}1.dx-\pi\int_0^{\pi}\frac{(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$$

$$\Rightarrow 2I=\pi[x]_0^{\pi}-\pi\int_0^{\pi}\frac{1-\sin x}{\cos^2 x}dx$$

$$\Rightarrow 2I=\pi^2-\pi\int_0^{\pi}(\sec^2 x-\tan x\sec x)dx$$

$$\Rightarrow 2I=\pi^2-\pi[\tan x-\sec x]_0^{\pi}$$

$$\Rightarrow 2I=\pi^2-\pi[\tan\pi-\sec\pi-\tan 0+\sec 0]$$

$$\Rightarrow 2I=\pi^2-\pi[0-(-1)-0+1]$$

$$\Rightarrow 2I=\pi^2-2\pi$$

$$\Rightarrow 2I=\pi(\pi-2)$$

$$I=\frac{\pi}{2}(\pi-2)$$

Question 33:

$$\int_1^4[|x-1|+|x-2|+|x-3|]dx$$

Solution:

Consider, $I=\int_1^4[|x-1|+|x-2|+|x-3|]dx$

$$\Rightarrow I=\int_1^4|x-1|dx+\int_1^4|x+2|dx+\int_1^4|x+3|dx$$

$$I=I_1+I_2+I_3 \qquad \ldots(1)$$

Where, $I_1=\int_1^4|x-1|dx, I_2=\int_1^4|x+2|dx$ and $I_3=\int_1^4|x+3|dx$

$$I_1=\int_1^4|x-1|dx$$

$(x-1) \geq 0 \;\; for \;\; 1 \leq x \leq 4$

$\therefore I_1 = \int_1^4 (x-1)\, dx$

$\Rightarrow I_1 = \left[\frac{x^2}{2} - x\right]_1^4$

$\Rightarrow I_1 = \left[8 - 4 - \frac{1}{2} + 1\right] = \frac{9}{2} \qquad \ldots(2)$

$I_2 = \int_1^4 |x-2|\, dx$

$x - 2 \geq 0 \;\; for \;\; 2 \leq x \leq 4$ and $x - 2 \leq 0 \;\; for \;\; 1 \leq x \leq 2$

$\therefore I_2 = \int_1^2 (2-x)\, dx + \int_2^4 (x-2)\, dx$

$\Rightarrow I_2 = \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4 \Rightarrow I_2 = \left[4 - 2 - 2 + \frac{1}{2}\right] + \left[8 - 8 - 2 + 4\right]$

$\Rightarrow I_2 = \frac{1}{2} + 2 = \frac{5}{2}$

$\Rightarrow I_3 = \int_1^4 |x-3|\, dx$

$x - 3 \geq 0 \;\; for \;\; 3 \leq x \leq 4$ and $x - 3 \leq 0 \;\; for \;\; 1 \leq x \leq 2$

$\therefore I_3 = \int_1^3 (3-x) dx + \int_3^4 (x-3)\, dx$

$\Rightarrow I_3 = \left[3 - \frac{x^2}{2}\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$

$\Rightarrow I_3 = \left[9 - \frac{9}{2} - 3 + \frac{1}{2}\right] + \left[8 - 12 - \frac{9}{2} + 9\right]$

$\Rightarrow I_3 = \left[6 - 4\right] + \left[\frac{1}{2}\right] = \frac{5}{2} \qquad \ldots(4)$

From equations (1), (2), (3) and (4), we get

$I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$

Question 34:

$\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$

Solution:

Consider, $\int_1^3 \frac{dx}{x^2(x+1)}$

Let, $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$$

$$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$$

Equating the coefficients of x^2, x and constant terms, we get

$A + C = 0$

$A + B = 0$

$B = 1$

On solving these equations, we get

$A = -1$

$C = 1$

$B = 1$

$$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$$

$$\Rightarrow I = \int_1^3 \left\{ -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx = \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^3$$

$$= \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x} \right]_1^3 = \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log\left(\frac{2}{1}\right) + 1$$

$$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$$

$$= \log 2 - \log 3 + \frac{2}{3}$$

$$= \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

Hence proved.

Question 35:

$$\int_0^1 xe^x dx = 1$$

Solution:

Let $I = \int_0^1 xe^x dx$

Using integration by parts, we get

$$I = x\int_0^1 e^x dx - \int_0^1 \left\{ \left(\frac{d}{dx}(x) \right) \int e^x dx \right\} dx$$

$$= \left[xe^x \right]_0^1 - \int_0^1 e^x dx$$

$$= \left[xe^x \right]_0^1 - \left[e^x \right]_0^1$$

$$= e - e + 1$$

$$= 1$$

Hence proved.

Question 36:

$$\int_{-1}^{1} x^{17}\cos^4 x dx = 0$$

Solution:

Consider, $I = \int_{-1}^{1} x^{17}\cos^4 x dx$

Let $f(x) = x^{17}\cos^4 x$

$\Rightarrow f(-x) = (-x)^{17}\cos^4(-x) = -x^{17}\cos^4 x = -f(x)$

$f(x)$ is an odd function.

We know that if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

$\therefore I = \int_{-1}^{1} x^{17}\cos^4 x dx = 0$

Hence proved.

Question 37:

$$\int_0^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}$$

Solution:

Consider, $I = \int_0^{\frac{\pi}{2}} \sin^3 x dx$

$I = \int_0^{\frac{\pi}{2}} \sin^2 x.\sin x dx$

$= \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\sin x dx$

$= \int_0^{\frac{\pi}{2}} \sin x dx - \int_0^{\frac{\pi}{2}} \cos^2 x.\sin x dx$

$= [-\cos x]_0^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}}$

$= 1 + \frac{1}{3}[-1] = 1 - \frac{1}{3} = \frac{2}{3}$

Hence proved.

Question 38:

$\int_0^{\frac{\pi}{4}} 2\tan^3 x dx = 1 - \log 2$

Solution:

Consider, $I = \int_0^{\frac{\pi}{4}} 2\tan^3 x dx$

$I = \int_0^{\frac{\pi}{4}} 2\tan^2 x.\tan x dx = 2\int_0^{\frac{\pi}{4}} \left(\sec^2 - 1\right)\tan x dx$

$= 2\int_0^{\frac{\pi}{4}} \sec^2 x \tan x dx - 2\int_0^{\frac{\pi}{4}} \tan x dx$

$= 2\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + 2\left[\log\cos x\right]_0^{\frac{\pi}{4}} = 1 + 2\left[\log\cos\frac{\pi}{4} - \log\cos 0\right]$

$= 1 + 2\left[\log\frac{1}{\sqrt{2}} - \log 1\right] = 1 - \log 2 - \log 1 = 1 - \log 2$

Hence proved.

Question 39:

$\int_0^1 \sin^{-1} x dx = \frac{\pi}{2} - 1$

Solution:

Let $\int_0^1 \sin^{-1} x dx$

$\Rightarrow I = \int_0^1 \sin^{-1} x.1.dx$

Using integration by parts, we get

$I = \left[\sin^{-1} x.x\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} x dx$

$= \left[x\sin^{-1} x\right]_0^1 + \frac{1}{2}\int_0^1 \frac{(-2x)}{\sqrt{1-x^2}} dx$

Put, $1 - x^2 = t \Rightarrow -2x dx = dt$

When $x = 0, t = 1$ and when $x = 1, t = 0$

$I = \left[x\sin^{-1} x\right]_0^1 + \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{t}}$

$= \left[x\sin^{-1} x\right]_0^1 + \frac{1}{2}\left[2\sqrt{t}\right]_1^0$

$= \sin^{-1}(1) + \left[-\sqrt{1}\right]$

$= \frac{\pi}{2} - 1$

Hence proved.

Question 40:

Evaluate $\int_0^1 e^{2-3x}dx$ as a limit of a sum.

Solution:

Let $I=\int_0^1 e^{2-3x}dx$

We know that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right]$$

Where, $h=\frac{b-a}{n}$

Here, $a=0, b=1$ and $f(x)=e^{2-3x}$

$$\Rightarrow h=\frac{1-0}{n}=\frac{1}{n}$$

$$\therefore \int_0^1 e^{2-3x}dx=(1-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(0+h)+\ldots+f(0+(n-1)h)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2+e^{2-3x}+\ldots+e^{2-3(n-1)h}\right]=\lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{1+e^{-3h}+e^{-6h}+e^{-9h}+\ldots+e^{-3(n-1)h}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-\left(e^{-3h}\right)^n}{1-\left(e^{-3h}\right)}\right\}\right]=\lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-e^{-\frac{3}{n}n}}{1-e^{-\frac{3}{n}}}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2\left(1-e^{-3}\right)}{1-e^{-\frac{3}{n}}}\right]=e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{e^{-\frac{3}{n}}-1}\right]$$

$$=e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\left(-\frac{1}{3}\right)\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]=\frac{e^2\left(e^{-3}-1\right)}{3}\lim_{n\to\infty}\left[\frac{\frac{-3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$=\frac{-e^2\left(e^{-3}-1\right)}{3}(1)$$

$$=\frac{-e^{-1}+e^2}{3}$$

$$=\frac{1}{3}\left(e^2-\frac{1}{e}\right)$$

Question 41:

$\int\frac{dx}{e^x+e^{-x}}$ is equal to

$A.\ \tan^{-1}(e^x)+C$

$B.\ \tan^{-1}(e^{-x})+C$

$C.\ \log(e^x-e^{-x})+C$

$D.\ \log(e^x+e^{-x})+C$

Solution:

Consider, $I=\int\frac{dx}{e^x+e^{-x}}dx=\int\frac{e^x}{e^{2x}+1}dx$

Put, $e^x=t\Rightarrow e^x dx=dt$

$\therefore I=\int\frac{dt}{1+t^2}$

$=\tan^{-1}t+C$

$=\tan^{-1}(e^x)+C$

Thus, the correct option is A.

Question 42:

$\int\frac{\cos 2x}{(\sin x+\cos x)^2}dx$ is

$A.\ \frac{-1}{\sin x+\cos x}+C$

$B.\ \log|\sin x+\cos x|+C$

$C.\ \log|\sin x-\cos x|+C$

$D.\ \frac{1}{(\sin x+\cos x)^2}+C$

Equals to

Solution:

Consider, $I=\int\frac{\cos 2x}{(\sin x+\cos x)^2}dx\Rightarrow I=\int\frac{\cos^2 x-\sin^2 x}{(\sin x+\cos x)}dx$

$=\int\frac{(\cos x+\sin x)(\cos x-\sin x)}{(\sin x+\cos x)^2}dx=\int\frac{\cos x-\sin x}{\cos x+\sin x}dx$

Let $\cos x+\sin x=t\Rightarrow(\cos x-\sin x)dx=dt$

$$\therefore I=\int\frac{dt}{t}$$
$$=\log|t|+C$$
$$=\log|\cos x+\sin x|+C$$

Thus, the correct option is B.

Question 43:

If $f(a+b-x)=f(x)$, then $\int_a^b xf(x)\,dx$ is equal to

$A.\ \frac{a+b}{2}\int_a^b f(b-x)dx$

$B.\ \frac{a+b}{2}\int_a^b f(b+x)dx$

$C.\ \frac{b-a}{2}\int_a^b f(x)dx$

$D.\ \frac{a+b}{2}\int_a^b f(x)dx$

Solution:

Consider, $I=\int_a^b xf(x)\,dx \qquad \ldots(1)$

$$I=\int_a^b(a+b-x)f(a+b-x)dx \qquad \left(\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right)$$
$$\Rightarrow I=\int_a^b(a+b-x)f(x)\,dx$$
$$\Rightarrow I=(a+b)\int_a^b f(x)\,dx-I\ldots\ldots(\text{Using equation ( )})$$
$$\Rightarrow I+I=(a+b)\int_a^b f(x)\,dx$$
$$\Rightarrow 2I=(a+b)\int_a^b f(x)\,dx$$
$$\Rightarrow I=\left(\frac{a+b}{2}\right)\int_a^b f(x)\,dx$$

Thus, the correct option is D.

Question 44:

The value of $\int_0^1\tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is

A. 1

B. 0
C. -1
D. $\frac{\pi}{4}$

Solution:

Consider, $I=\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

$\Rightarrow I=\int_0^1 \tan^{-1}\left(\frac{x-(1-x)}{1+x(1-x)}\right)dx$

$\Rightarrow I=\int_0^1\left[\tan^{-1}x-\tan^{-1}(1-x)\right]dx \quad \ldots(1)$

$\Rightarrow I=\int_0^1\left[\tan^{-1}(1-x)-\tan^{-1}(1-1+x)\right]dx$

$\Rightarrow I=\int_0^1\left[\tan^{-1}(1-x)-\tan^{-1}x\right]dx$

$\Rightarrow I=\int_0^1\left[\tan^{-1}(1-x)-\tan^{-1}(x)\right]dx \quad \ldots(2)$

Adding (1) and (2), we get

$\Rightarrow 2I=\int_0^1\left(\tan^{-1}x-\tan^{-1}(1-x)-\tan^{-1}(1-x)-\tan^{-1}x\right)dx$

$\Rightarrow 2I=0$

$\Rightarrow I=0$

Thus, the correct option is B.