

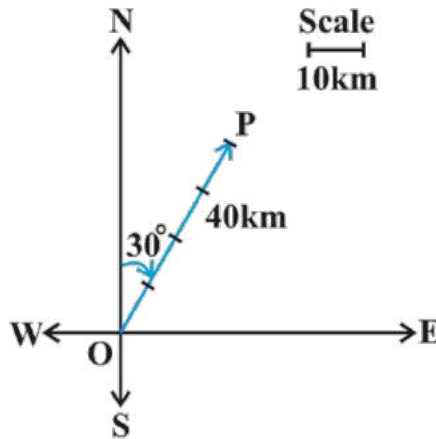
Chapter 10 Vector Algebra

EXERCISE 10.1

Question 1:

Represent graphically a displacement of 40km, 30° east of north.

Solution:



$\vec{OP}$ represents the displacement of 40km, 30° north-east.

Question 2:

Classify the following measures as scalars and vectors.

- | | | |
|--------------|--------------------------|------------------|
| (i) 10 kg | (ii) 2 meters north-east | (iii) 40° |
| (iv) 40 watt | (v) 10^{-19} coulomb | (vi) $20m / s^2$ |

Solution:

- (i) 10kg is a scalar.
- (ii) 2 meters north-west is a vector.
- (iii) 40° is a scalar.
- (iv) 40 watts is a scalar.
- (v) 10^{-19} Coulomb is a scalar.
- (vi) $20m / s^2$ is a vector

Question 3:

Classify the following as scalar and vector quantities.

- | | | |
|-----------------|----------------|-------------|
| (i) time period | (ii) distance | (iii) force |
| (iv) velocity | (v) work done. | |

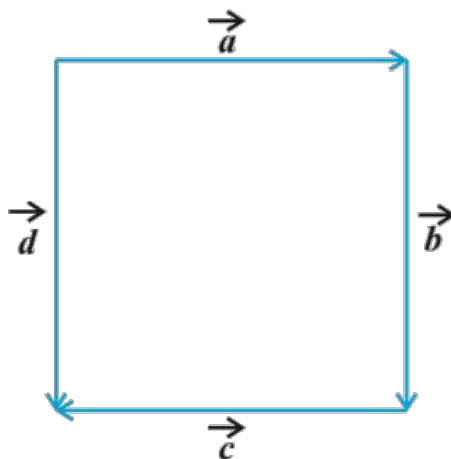
Solution:

- (i) Time period is a scalar.
- (ii) Distance is a scalar.

- (iii) Force is a vector.
- (iv) Velocity is a vector.
- (v) Work done is a scalar.

Question 4:

In figure, identify the following vectors.



- (i) Coinitial
- (ii) Equal
- (iii) Collinear but not equal.

Solution:

- (i) Vectors $\vec{a}$ and $\vec{d}$ are coinitial.
- (ii) Vectors $\vec{b}$ and $\vec{d}$ are equal.
- (iii) Vectors $\vec{a}$ and $\vec{c}$ are collinear but not equal.

Question 5:

Answer the following as true or false.

- (i) $\vec{a}$ and $-\vec{a}$ are collinear.
- (ii) Two collinear vectors are always equal in magnitude.
- (iii) Two vectors having same magnitude are collinear.
- (iv) Two collinear vectors having the same magnitude are equal.

Solution:

- (i) True.
- (ii) False.
- (iii) False.
- (iv) False

EXERCISE 10.2

Question 1:

Compute the magnitude of the following vectors:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}; \quad \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}; \quad \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

Solution:

$$|\vec{a}| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

$$|\vec{b}| = \sqrt{(2)^2 + (-7)^2 + (-3)^2} = \sqrt{4 + 49 + 9} = \sqrt{62}$$

$$|\vec{c}| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1$$

Question 2:

Write two different vectors having same magnitude.

Solution:

$$\text{Let } \vec{a} = \hat{i} - 2\hat{j} + 3\hat{k} \text{ and } \vec{b} = (2\hat{i} + \hat{j} - 3\hat{k})$$

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

But $\vec{a} \neq \vec{b}$

Question 3:

Write two different vectors having same direction.

Solution:

$$\text{Let } \vec{p} = (\vec{i} + \vec{j} + \vec{k}) \text{ and } \vec{q} = (2\vec{i} + 2\vec{j} + 2\vec{k})$$

The DCs of $\vec{p}$ are

$$l = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}$$

$$m = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}$$

$$n = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}$$

The DCs of $\vec{q}$ are

$$l = \frac{2}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{2}{2\sqrt{3}}$$

$$m = \frac{2}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{2}{2\sqrt{3}}$$

$$n = \frac{2}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{2}{2\sqrt{3}}$$

But $\vec{p} \neq \vec{q}$

Question 4:

Find the values of x and y so that the vectors $2\vec{i} + 3\vec{j}$ and $x\vec{i} + y\vec{j}$ are equal.

Solution:

It is given that the vectors $2\vec{i} + 3\vec{j}$ and $x\vec{i} + y\vec{j}$ are equal.

Therefore,

$$2\vec{i} + 3\vec{j} = x\vec{i} + y\vec{j}$$

On comparing the components of both sides

$$\Rightarrow x = 2$$

$$\Rightarrow y = 3$$

Question 5:

Find the scalar and vector components of the vector with initial point $(2,1)$ and terminal point $(-5,7)$.

Solution:

Let the points be $P(2,1)$ and $Q(-5,7)$

$$\begin{aligned}\overrightarrow{PQ} &= (-5-2)\hat{i} + (7-1)\hat{j} \\ &= -7\hat{i} + 6\hat{j}\end{aligned}$$

So, the scalar components are -7 and 6 , and the vector components are $-7\hat{i}$ and $6\hat{j}$.

Question 6:

Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} + 7\hat{k}$

Solution:

The given vectors are $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} + 7\hat{k}$.

Therefore,

$$\begin{aligned}\vec{a} + \vec{b} + \vec{c} &= (1-2+1)\hat{i} + (-2+4-6)\hat{j} + (1+5-7)\hat{k} \\ &= 0\hat{i} - 4\hat{j} - \hat{k} \\ &= -4\hat{j} - \hat{k}\end{aligned}$$

Question 7:

Find the unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$.

Solution:

We have $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$

Hence,

$$\begin{aligned}|\vec{a}| &= \sqrt{1^2 + 1^2 + 2^2} \\ &= \sqrt{1+1+4} \\ &= \sqrt{6}\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{a} &= \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{6}} \\ &= \frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}\end{aligned}$$

Question 8:

Find the unit vector in the direction of vector $\vec{PQ}$, where P and Q are the points $(1,2,3)$ and $(4,5,6)$ respectively.

Solution:

We have the given points $P(1,2,3)$ and $Q(4,5,6)$

Hence,

$$\begin{aligned}\vec{PQ} &= (4-1)\hat{i} + (5-2)\hat{j} + (6-3)\hat{k} \\ &= 3\hat{i} + 3\hat{j} + 3\hat{k} \\ |\vec{PQ}| &= \sqrt{3^2 + 3^2 + 3^2} \\ &= \sqrt{9+9+9} \\ &= \sqrt{27} \\ &= 3\sqrt{3}\end{aligned}$$

So, unit vector is

$$\begin{aligned}\frac{\vec{PQ}}{|\vec{PQ}|} &= \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{3\sqrt{3}} \\ &= \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\end{aligned}$$

Question 9:

For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$.

Solution:

The given vectors are $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$

Therefore,

$$\begin{aligned}\vec{a} + \vec{b} &= (2-1)\hat{i} + (-1+1)\hat{j} + (2-1)\hat{k} \\ &= 1\hat{i} + 0\hat{j} + 1\hat{k} \\ &= \hat{i} + \hat{k} \\ |\vec{a} + \vec{b}| &= \sqrt{1^2 + 1^2} = \sqrt{2}\end{aligned}$$

Thus, unit vector is

$$\begin{aligned}\frac{\vec{a+b}}{|\vec{a+b}|} &= \frac{\hat{i} + \hat{k}}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}\end{aligned}$$

Question 10:

Find a vector in the direction of vector $\hat{5i} - \hat{j} + \hat{2k}$ which has magnitude 8 units.

Solution:

Let $\vec{a} = \hat{5i} - \hat{j} + \hat{2k}$

Hence,

$$\begin{aligned}|\vec{a}| &= \sqrt{5^2 + (-1)^2 + (2)^2} \\ &= \sqrt{25 + 1 + 4} \\ &= \sqrt{30}\end{aligned}$$

Therefore,

$$\vec{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{5i} - \hat{j} + \hat{2k}}{\sqrt{30}}$$

Thus, a vector parallel to $\hat{5i} - \hat{j} + \hat{2k}$ with magnitude 8 units is

$$\begin{aligned}\hat{8a} &= 8 \left(\frac{\hat{5i} - \hat{j} + \hat{2k}}{\sqrt{30}} \right) \\ &= \frac{40}{\sqrt{30}}\hat{i} - \frac{8}{\sqrt{30}}\hat{j} + \frac{16}{\sqrt{30}}\hat{k}\end{aligned}$$

Question 11:

Show that the vectors $\hat{2i} - \hat{3j} + \hat{4k}$ and $-\hat{4i} + \hat{6j} - \hat{8k}$ are collinear.

Solution:

We have $\vec{a} = \hat{2i} - \hat{3j} + \hat{4k}$ and $\vec{b} = -\hat{4i} + \hat{6j} - \hat{8k}$

Now,

$$\begin{aligned}\vec{b} &= -\hat{4i} + \hat{6j} - \hat{8k} \\ &= -2(\hat{2i} - \hat{3j} + \hat{4k}) \\ &= -2\vec{a}\end{aligned}$$

Since, $\vec{b} = \lambda \vec{a}$

Therefore, $\lambda = -2$

So, the vectors are collinear.

Question 12:

Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$

Solution:

Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

Therefore,

$$\begin{aligned} |\vec{a}| &= \sqrt{1^2 + 2^2 + 3^2} \\ &= \sqrt{1 + 4 + 9} \\ &= \sqrt{14} \end{aligned}$$

Thus, the DCs of $\vec{a}$ are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)$

Question 13:

Find the direction of the cosines of the vectors joining the points $A(1, 2, -3)$ and $B(-1, -2, 1)$ directions from A to B.

Solution:

The given points are $A(1, 2, -3)$ and $B(-1, -2, 1)$.

Therefore,

$$\begin{aligned} \vec{AB} &= (-1-1)\hat{i} + (-2-2)\hat{j} + \{1-(-3)\}\hat{k} \\ &= -2\hat{i} - 4\hat{j} + 4\hat{k} \\ |\vec{AB}| &= \sqrt{(-2)^2 + (-4)^2 + 4^2} \\ &= \sqrt{4 + 16 + 16} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$

Thus, the DCs of $\vec{AB}$ are $\left(-\frac{2}{6}, -\frac{4}{6}, \frac{4}{6}\right) = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$

Question 14:

Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axis OX, OY and OZ.

Solution:

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

Therefore,

$$\begin{aligned} |\vec{a}| &= \sqrt{1^2 + 1^2 + 1^2} \\ &= \sqrt{3} \end{aligned}$$

Thus, the DCs of $\vec{a}$ are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

Now, let α, β and γ be the angles formed by $\vec{a}$ with the positive directions of x, y and z axes respectively

Then,

$$\cos \alpha = \frac{1}{\sqrt{3}}, \cos \beta = \frac{1}{\sqrt{3}}, \cos \gamma = \frac{1}{\sqrt{3}}$$

Hence, the vector is equally inclined to OX, OY and OZ.

Question 15:

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2:1.

- (i) Internally
- (ii) Externally

Solution:

Position vectors of P and Q are given as:

$$\vec{OP} = \hat{i} + 2\hat{j} - \hat{k} \text{ and } \vec{OQ} = -\hat{i} + \hat{j} + \hat{k}$$

- (i) The position vector of R which divides the line joining two points P and Q internally in the ratio 2:1 is

$$\begin{aligned}
 \vec{OR} &= \frac{2(-\hat{i} + \hat{j} + \hat{k}) + 1(\hat{i} + 2\hat{j} - \hat{k})}{2+1} \\
 &= \frac{(-2\hat{i} + 2\hat{j} + 2\hat{k}) + (\hat{i} + 2\hat{j} - \hat{k})}{3} \\
 &= \frac{-\hat{i} + 4\hat{j} + \hat{k}}{3} \\
 &= -\frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}
 \end{aligned}$$

- (ii) The position vector of R which divides the line joining two points P and Q externally in the ratio 2:1 is

$$\begin{aligned}
 \vec{OR} &= \frac{2(-\hat{i} + \hat{j} + \hat{k}) - 1(\hat{i} + 2\hat{j} - \hat{k})}{2-1} \\
 &= \frac{(-2\hat{i} + 2\hat{j} + 2\hat{k}) - (\hat{i} + 2\hat{j} - \hat{k})}{1} \\
 &= -3\hat{i} + 3\hat{k}
 \end{aligned}$$

Question 16:

Find the position vector of the mid-point of the vector joining the points $P(2,3,4)$ and $Q(4,1,-2)$.

Solution:

The position vector of the mid-point R is

$$\begin{aligned}
 \vec{OR} &= \frac{(\hat{2i} + \hat{3j} + \hat{4k}) + (\hat{4i} + \hat{1j} - \hat{2k})}{2} \\
 &= \frac{(2+4)\hat{i} + (3+1)\hat{j} + (4-2)\hat{k}}{2} \\
 &= \frac{6\hat{i} + 4\hat{j} + 2\hat{k}}{2} \\
 &= 3\hat{i} + 2\hat{j} + \hat{k}
 \end{aligned}$$

Question 17:

Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$, respectively form the vertices of a right angled triangle.

Solution:

Position vectors of points A, B, and C are respectively given as:

$$\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}, \quad \vec{b} = 2\hat{i} - \hat{j} + \hat{k} \quad \text{and} \quad \vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$$

Therefore,

$$\begin{aligned} \vec{AB} &= \vec{b} - \vec{a} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} \\ &= -\hat{i} + 3\hat{j} + 5\hat{k} \\ \vec{BC} &= \vec{c} - \vec{b} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} \\ &= -\hat{i} + 2\hat{j} - 6\hat{k} \\ \vec{CA} &= \vec{a} - \vec{c} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} \\ &= 2\hat{i} - \hat{j} + \hat{k} \end{aligned}$$

Now,

$$\begin{aligned} |\vec{AB}|^2 &= (-1)^2 + 3^2 + 5^2 = 1 + 9 + 25 = 35 \\ |\vec{BC}|^2 &= (-1)^2 + (-2)^2 + (-6)^2 = 1 + 4 + 36 = 41 \\ |\vec{CA}|^2 &= 2^2 + (-1)^2 + 1^2 = 4 + 1 + 1 = 6 \end{aligned}$$

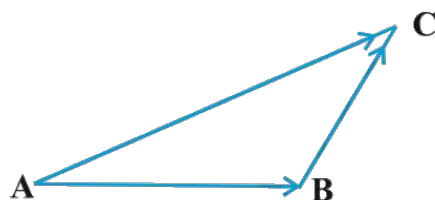
Also,

$$\begin{aligned} |\vec{AB}|^2 + |\vec{CA}|^2 &= 35 + 6 \\ &= 41 \\ &= |\vec{BC}|^2 \end{aligned}$$

Thus, ABC is a right-angled triangle.

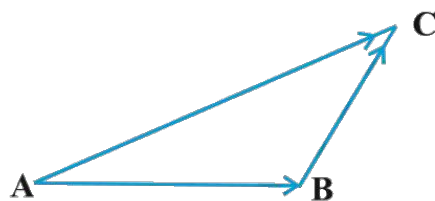
Question 18:

In triangle ABC which of the following is not true.



- (A) $\vec{AB} + \vec{BC} + \vec{CA} = 0$
 (B) $\vec{AB} + \vec{BC} - \vec{AC} = 0$
 (C) $\vec{AB} + \vec{BC} - \vec{CA} = 0$
 (D) $\vec{AB} - \vec{CB} + \vec{CA} = 0$

Solution:



On applying the triangle law of addition in the given triangle, we have:

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \quad \dots(1)$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = -\overrightarrow{CA}$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0 \quad \dots(2)$$

Hence, the equation given in option A is true.

Now, from equation (2)

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = 0$$

Hence, the equation given in option B is true.

Also,

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$$

$$\Rightarrow \overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = 0$$

Hence, the equation given in option D is true

Now, consider the equation given in option C,

$$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = 0$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{CA} \quad \dots(3)$$

From equations (1) and (2)

$$\Rightarrow \overrightarrow{AC} = \overrightarrow{CA}$$

$$\Rightarrow \overrightarrow{AC} = -\overrightarrow{AC}$$

$$\Rightarrow \overrightarrow{AC} + \overrightarrow{AC} = 0$$

$$\Rightarrow 2\overrightarrow{AC} = 0$$

$$\Rightarrow \overrightarrow{AC} = 0$$

Which is not true. So, the equation given in option C is incorrect.

Thus, the correct option is C.

Question 19:

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect?

- (A) $\vec{b} = \lambda \vec{a}$, for some scalar λ
- (B) $\vec{a} = \pm \vec{b}$
- (C) the respective components of $\vec{a}$ and $\vec{b}$ are proportional.
- (D) both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes

Solution:

If $\vec{a}$ and $\vec{b}$ are collinear vectors, they are parallel.

Therefore, for some scalar λ

$$\vec{b} = \lambda \vec{a}$$

If $\lambda = \pm 1$, then $\vec{a} = \pm \vec{b}$

If $\vec{a} = \hat{a}_1 i + \hat{a}_2 j + \hat{a}_3 k$ and $\vec{b} = \hat{b}_1 i + \hat{b}_2 j + \hat{b}_3 k$

Then,

$$\begin{aligned} \Rightarrow \vec{b} &= \lambda \vec{a} \\ \Rightarrow \hat{b}_1 i + \hat{b}_2 j + \hat{b}_3 k &= \lambda (\hat{a}_1 i + \hat{a}_2 j + \hat{a}_3 k) \\ \Rightarrow \hat{b}_1 i + \hat{b}_2 j + \hat{b}_3 k &= (\lambda \hat{a}_1) i + (\lambda \hat{a}_2) j + (\lambda \hat{a}_3) k \end{aligned}$$

Comparing the components of both the sides

$$\begin{aligned} \Rightarrow \vec{b}_1 &= \lambda \vec{a}_1 \\ \Rightarrow \vec{b}_2 &= \lambda \vec{a}_2 \\ \Rightarrow \vec{b}_3 &= \lambda \vec{a}_3 \end{aligned}$$

Therefore,

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

Thus, the respective components of $\vec{a}$ and $\vec{b}$ are proportional.

However, $\vec{a}$ and $\vec{b}$ may have different directions.

Hence, that statement given in D is incorrect.

Thus, the correct option is D.

EXERCISE 10.3

Question 1:

Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitude $\sqrt{3}$ and 2, respectively have $\vec{a} \cdot \vec{b} = \sqrt{6}$

Solution:

It is given that

$$\begin{aligned} |\vec{a}| &= \sqrt{3} \\ |\vec{b}| &= 2 \\ \vec{a} \cdot \vec{b} &= \sqrt{6} \end{aligned}$$

Therefore,

$$\begin{aligned} \Rightarrow \sqrt{6} &= \sqrt{3} \times 2 \cos \theta \\ \Rightarrow \cos \theta &= \frac{\sqrt{6}}{\sqrt{3} \times 2} \\ \Rightarrow \cos \theta &= \frac{1}{\sqrt{2}} \\ \Rightarrow \theta &= \frac{\pi}{4} \end{aligned}$$

Question 2:

Find the angle between the vectors $\hat{i} - \hat{2}j + \hat{3}k$ and $\hat{3}i - \hat{2}j + \hat{k}$.

Solution:

Let $\vec{a} = \hat{i} - \hat{2}j + \hat{3}k$ and $\vec{b} = \hat{3}i - \hat{2}j + \hat{k}$.

Hence,

$$\begin{aligned} |\vec{a}| &= \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14} \\ |\vec{b}| &= \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14} \\ \vec{a} \cdot \vec{b} &= (\hat{i} - \hat{2}j + \hat{3}k) \cdot (\hat{3}i - \hat{2}j + \hat{k}) \\ &= 1 \times 3 + (-2)(-2) + 3 \times 1 \\ &= 3 + 4 + 3 \\ &= 10 \end{aligned}$$

Therefore,

$$\Rightarrow 10 = \sqrt{14}\sqrt{14} \cos \theta$$

$$\Rightarrow \cos \theta = \frac{10}{14}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

Question 3:

Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$

Solution:

Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$

Projection of $\vec{a}$ on $\vec{b}$ is

$$\begin{aligned} \frac{1}{|\vec{b}|}(\vec{a} \cdot \vec{b}) &= \frac{1}{\sqrt{1+1}} \{(1)(1) + (-1)(1)\} \\ &= \frac{1}{\sqrt{2}}(1-1) \\ &= 0 \end{aligned}$$

Question 4:

Find the projection of vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$

Solution:

Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ is

$$\begin{aligned} \frac{1}{|\vec{b}|}(\vec{a} \cdot \vec{b}) &= \frac{1}{\sqrt{7^2 + (-1)^2 + 8^2}} \{(1)(7) + 3(-1) + 7(8)\} \\ &= \frac{1}{\sqrt{49 + 1 + 64}}(7 - 3 + 56) \\ &= \frac{60}{\sqrt{114}} \end{aligned}$$

Question 5:

Show that each of the given three vectors is a unit vector which are mutually perpendicular to each other.

$$\frac{1}{7}(\hat{2}i + \hat{3}j + \hat{6}k), \frac{1}{7}(\hat{3}i - \hat{6}j + \hat{2}k), \frac{1}{7}(\hat{6}i + \hat{2}j - \hat{3}k)$$

Solution:

Let

$$\vec{a} = \frac{1}{7}(\hat{2}i + \hat{3}j + \hat{6}k) = \frac{2}{7}i + \frac{3}{7}j + \frac{6}{7}k$$

$$\vec{b} = \frac{1}{7}(\hat{3}i - \hat{6}j + \hat{2}k) = \frac{3}{7}i - \frac{6}{7}j + \frac{2}{7}k$$

$$\vec{c} = \frac{1}{7}(\hat{6}i + \hat{2}j - \hat{3}k) = \frac{6}{7}i + \frac{2}{7}j - \frac{3}{7}k$$

Now,

$$|\vec{a}| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = 1$$

$$|\vec{b}| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(-\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2} = \sqrt{\frac{9}{49} + \frac{36}{49} + \frac{4}{49}} = 1$$

$$|\vec{c}| = \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(-\frac{3}{7}\right)^2} = \sqrt{\frac{36}{49} + \frac{4}{49} + \frac{9}{49}} = 1$$

So, each of the vector is a unit vector.

Hence,

$$\vec{a} \cdot \vec{b} = \frac{2}{7} \times \frac{3}{7} + \frac{3}{7} \times \left(-\frac{6}{7}\right) + \frac{6}{7} \times \frac{2}{7} = \frac{6}{49} - \frac{18}{49} + \frac{12}{49} = 0$$

$$\vec{b} \cdot \vec{c} = \frac{3}{7} \times \frac{6}{7} + \frac{2}{7} \times \left(-\frac{6}{7}\right) + \left(-\frac{3}{7}\right) \times \frac{2}{7} = \frac{18}{49} - \frac{12}{49} - \frac{6}{49} = 0$$

$$\vec{c} \cdot \vec{a} = \frac{2}{7} \times \frac{6}{7} + \frac{3}{7} \times \frac{2}{7} + \frac{6}{7} \times \left(-\frac{3}{7}\right) = \frac{12}{49} + \frac{6}{49} - \frac{18}{49} = 0$$

So, the vectors are mutually perpendicular to each other.

Question 6:

Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a} + \vec{b})(\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$

Solution:

It is given that $(\vec{a} + \vec{b})(\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$

Therefore,

$$\begin{aligned}
&\Rightarrow (\vec{a} + \vec{b})(\vec{a} - \vec{b}) = 8 \\
&\Rightarrow \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b} = 8 \\
&\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 8 \\
&\Rightarrow (8|\vec{b}|)^2 - |\vec{b}|^2 = 8 \\
&\Rightarrow 64|\vec{b}|^2 - |\vec{b}|^2 = 8 \\
&\Rightarrow 63|\vec{b}|^2 = 8 \\
&\Rightarrow |\vec{b}|^2 = \frac{8}{63} \\
&\Rightarrow |\vec{b}| = \sqrt{\frac{8}{63}} \\
&\Rightarrow |\vec{b}| = \frac{2\sqrt{2}}{3\sqrt{7}}
\end{aligned}$$

Now,

$$\begin{aligned}
|\vec{a}| &= 8|\vec{b}| \\
&= \frac{8 \times 2\sqrt{2}}{3\sqrt{7}} \\
&= \frac{16\sqrt{2}}{3\sqrt{7}}
\end{aligned}$$

Question 7:

Evaluate the product $(3\vec{a} - 5\vec{b})(2\vec{a} + 7\vec{b})$

Solution:

$$\begin{aligned}
(3\vec{a} - 5\vec{b})(2\vec{a} + 7\vec{b}) &= 3\vec{a} \cdot 2\vec{a} + 3\vec{a} \cdot 7\vec{b} - 5\vec{b} \cdot 2\vec{a} - 5\vec{b} \cdot 7\vec{b} \\
&= 6\vec{a} \cdot \vec{a} + 21\vec{a} \cdot \vec{b} - 10\vec{b} \cdot \vec{a} - 35\vec{b} \cdot \vec{b} \\
&= 6|\vec{a}|^2 + 11\vec{a} \cdot \vec{b} - 35|\vec{b}|^2
\end{aligned}$$

Question 8:

Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that angle between them is 60° and their scalar product is $\frac{1}{2}$

Solution:

Let θ be the angle between $\vec{a}$ and $\vec{b}$

It is given that $|\vec{a}| = |\vec{b}|$, $\vec{a} \cdot \vec{b} = \frac{1}{2}$ and $\theta = 60^\circ$

Therefore,

$$\Rightarrow \frac{1}{2} = |\vec{a}| |\vec{b}| \cos 60^\circ$$

$$\Rightarrow \frac{1}{2} = |\vec{a}|^2 \times \frac{1}{2}$$

$$\Rightarrow |\vec{a}|^2 = 1$$

$$\Rightarrow |\vec{a}| = |\vec{b}| = 1$$

Question 9:

Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$

Solution:

$$\Rightarrow (\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$$

$$\Rightarrow \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{a} - \vec{a} \cdot \vec{x} - \vec{a} \cdot \vec{a} = 12$$

$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12$$

$$\Rightarrow |\vec{x}|^2 - 1 = 12$$

$$\Rightarrow |\vec{x}|^2 = 13$$

$$\Rightarrow |\vec{x}| = \sqrt{13}$$

Question 10:

If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{j}$ are such that $\vec{a} + \lambda \vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ .

Solution:

We have $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{j}$ are such that $\vec{a} + \lambda \vec{b}$ is perpendicular to $\vec{c}$

Then,

$$\begin{aligned} \vec{a} + \lambda \vec{b} &= (\hat{i} + \hat{j} + \hat{k}) + \lambda(-\hat{i} + \hat{j} + \hat{k}) \\ &= (1 - \lambda)\hat{i} + (1 + \lambda)\hat{j} + (1 + \lambda)\hat{k} \end{aligned}$$

Now,

$$\begin{aligned}
 &\Rightarrow (\vec{a} + \lambda \vec{b}) \cdot \vec{c} = 0 \\
 &\Rightarrow [(2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}] \cdot (3\hat{i} + \hat{j}) = 0 \\
 &\Rightarrow 3(2 - \lambda) + (2 + 2\lambda) + 0(3 + \lambda) = 0 \\
 &\Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0 \\
 &\Rightarrow -\lambda + 8 = 0 \\
 &\Rightarrow \lambda = 8
 \end{aligned}$$

Question 11:

Show that $|\vec{a}\vec{b} + \vec{b}\vec{a}|$ is perpendicular to $|\vec{a}\vec{b} - \vec{b}\vec{a}|$, for any non-zero vectors $\vec{a}$ and $\vec{b}$.

Solution:

$$\begin{aligned}
 (|\vec{a}\vec{b} + \vec{b}\vec{a}|) \cdot (|\vec{a}\vec{b} - \vec{b}\vec{a}|) &= |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}| |\vec{b}| |\vec{b}\vec{a}| + |\vec{b}| |\vec{a}| |\vec{a}\vec{b}| - |\vec{b}|^2 |\vec{a}|^2 \\
 &= |\vec{a}|^2 |\vec{b}|^2 - |\vec{b}|^2 |\vec{a}|^2 \\
 &= 0
 \end{aligned}$$

Question 12:

If $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$, then what can be concluded about the vector $\vec{b}$?

Solution:

We have $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$

Hence,

$$\begin{aligned}
 &\Rightarrow |\vec{a}|^2 = 0 \\
 &\Rightarrow |\vec{a}| = 0
 \end{aligned}$$

Therefore, $\vec{a}$ is the zero vector

Thus, any vector $\vec{b}$ can satisfy $\vec{a} \cdot \vec{b} = 0$

Question 13:

If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$, find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$.

Solution:

We have $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$

Therefore,

$$\begin{aligned} |\vec{a} + \vec{b} + \vec{c}|^2 &= (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) \\ 0 &= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) \\ 0 &= 1 + 1 + 1 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) \\ (\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) &= \frac{-3}{2} \end{aligned}$$

Question 14:

If either vector $\vec{a} = 0$ or $\vec{b} = 0$, then $\vec{a} \cdot \vec{b} = 0$. But the converse need not be true. Justify the answer with an example.

Solution:

Let $\vec{a} = 2\hat{i} + 4\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 3\hat{j} - 6\hat{k}$

Therefore,

$$\begin{aligned} \vec{a} \cdot \vec{b} &= 2(3) + 4(3) + 3(-6) \\ &= 6 + 12 - 18 \\ &= 0 \end{aligned}$$

Now,

$$\begin{aligned} |\vec{a}| &= \sqrt{2^2 + 4^2 + 3^2} = \sqrt{29} \\ &\Rightarrow \vec{a} \neq 0 \\ |\vec{b}| &= \sqrt{3^2 + 3^2 + (-6)^2} = \sqrt{54} \\ &\Rightarrow \vec{b} \neq 0 \end{aligned}$$

So, the converse of the statement need not to be true.

Question 15:

If the vertices A, B, C of a triangle ABC are $(1, 2, 3), (-1, 0, 0), (0, 1, 2)$ respectively, then find $\angle ABC$. [$\angle ABC$ is the angle between the vectors $\vec{BA}$ and $\vec{BC}$]

Solution:

Vertices of the triangle are $A(1, 2, 3), B(-1, 0, 0)$ and $C(0, 1, 2)$.

Hence,

$$\begin{aligned}
 \overrightarrow{BA} &= \{1 - (1)\}\hat{i} + (2 - 0)\hat{j} + (3 - 0)\hat{k} \\
 &= 2\hat{i} + 2\hat{j} + 3\hat{k} \\
 \overrightarrow{BC} &= \{0 - (-1)\}\hat{i} + (1 - 0)\hat{j} + (2 - 0)\hat{k} \\
 &= \hat{i} + \hat{j} + 2\hat{k} \\
 \overrightarrow{BA} \cdot \overrightarrow{BC} &= (2\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (\hat{i} + \hat{j} + 2\hat{k}) \\
 &= 2 \times 1 + 2 \times 1 + 3 \times 2 \\
 &= 2 + 2 + 6 \\
 &= 10 \\
 |\overrightarrow{BA}| &= \sqrt{2^2 + 2^2 + 3^2} \\
 &= \sqrt{4 + 4 + 9} \\
 &= \sqrt{17} \\
 |\overrightarrow{BC}| &= \sqrt{1^2 + 1^2 + 2^2} \\
 &= \sqrt{6} \\
 \overrightarrow{BA} \cdot \overrightarrow{BC} &= |\overrightarrow{BA}| |\overrightarrow{BC}| \cos(\angle ABC)
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \Rightarrow 10 &= \sqrt{17} \times \sqrt{6} (\cos \angle ABC) \\
 \Rightarrow \cos(\angle ABC) &= \frac{10}{\sqrt{17} \times \sqrt{6}} \\
 \Rightarrow (\angle ABC) &= \cos^{-1} \left(\frac{10}{\sqrt{102}} \right)
 \end{aligned}$$

Question 16:

Show that the points $A(1, 2, 7)$, $B(2, 6, 3)$ and $C(3, 10, -1)$ are collinear.

Solution:

The given points are $A(1, 2, 7)$, $B(2, 6, 3)$ and $C(3, 10, -1)$.

Hence,

$$\begin{aligned}
\vec{AB} &= (2-1)\hat{i} + (6-2)\hat{j} + (3-7)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k} \\
\vec{BC} &= (3-2)\hat{i} + (10-6)\hat{j} + (-1-3)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k} \\
\vec{AC} &= (3-1)\hat{i} + (10-2)\hat{j} + (-1-7)\hat{k} = 2\hat{i} + 8\hat{j} - 8\hat{k} \\
|\vec{AB}| &= \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33} \\
|\vec{BC}| &= \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33} \\
|\vec{AC}| &= \sqrt{2^2 + 8^2 + 8^2} = \sqrt{4+64+64} = 2\sqrt{33}
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\vec{AB}| + |\vec{BC}| &= \sqrt{33} + \sqrt{33} \\
&= 2\sqrt{33} \\
&= |\vec{AC}|
\end{aligned}$$

Hence, the points are collinear.

Question 17:

Show that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ form the vertices of a right angled triangle.

Solution:

Let $\vec{OA} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{OB} = \hat{i} - 3\hat{j} - 5\hat{k}$ and $\vec{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$

Hence,

$$\begin{aligned}
\vec{AB} &= (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k} \\
\vec{BC} &= (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k} \\
\vec{AC} &= (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k} \\
|\vec{AB}| &= \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41} \\
|\vec{BC}| &= \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6} \\
|\vec{AC}| &= \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{1+9+25} = \sqrt{35}
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\vec{BC}|^2 + |\vec{AC}|^2 &= 6 + 35 \\
&= 41 \\
&= |\vec{AB}|^2
\end{aligned}$$

Thus, $\triangle ABC$ is a right-angled triangle.

Question 18:

If $\vec{a}$ is a nonzero vector of magnitude ' a ' and λ a nonzero scalar, then $\lambda \vec{a}$ is a unit vector if

- (A) $\lambda = 1$ (B) $\lambda = -1$ (C) $a = |\lambda|$ (D) $a = \frac{1}{|\lambda|}$

Solution:

$$\Rightarrow |\lambda \vec{a}| = 1$$

$$\Rightarrow |\lambda| |\vec{a}| = 1$$

$$\Rightarrow |\vec{a}| = \frac{1}{|\lambda|}$$

$$\Rightarrow a = \frac{1}{|\lambda|}$$

Hence the correct option is D.

EXERCISE 10.4

Question 1:

Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

Solution:

We have, $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

Hence,

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 3 & -2 & 2 \end{vmatrix} \\ &= \hat{i}(-1 \cdot 2 - 1 \cdot (-2)) - \hat{j}(1 \cdot 2 - 3 \cdot 1) + \hat{k}(1 \cdot (-2) - 3 \cdot (-1)) \\ &= 19\hat{j} + 19\hat{k}\end{aligned}$$

Therefore,

$$\begin{aligned}|\vec{a} \times \vec{b}| &= \sqrt{(19)^2 + (19)^2} \\ &= \sqrt{2 \times (19)^2} \\ &= 19\sqrt{2}\end{aligned}$$

Question 2:

Find a unit vector perpendicular to each of the vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

Solution:

We have $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

Hence,

$$\begin{aligned}\vec{a} + \vec{b} &= 4\hat{i} + 4\hat{j} \\ \vec{a} - \vec{b} &= 2\hat{i} + 4\hat{k}\end{aligned}$$

Therefore,

$$\begin{aligned}
 (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix} \\
 &= \hat{i}(16) - \hat{j}(16) + \hat{k}(-8) \\
 &= 16\hat{i} - 16\hat{j} - 8\hat{k}
 \end{aligned}$$

$$\begin{aligned}
 |(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})| &= \sqrt{16^2 + (-16)^2 + (-8)^2} = \sqrt{2^2 \times 8^2 + 2^2 \times 8^2 + 8^2} \\
 &= 8\sqrt{2^2 + 2^2 + 1} = 8\sqrt{9} \\
 &= 8 \times 3 = 24
 \end{aligned}$$

So, the unit vector is

$$\begin{aligned}
 \pm \frac{(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})}{|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})|} &= \pm \frac{16\hat{i} - 16\hat{j} - 8\hat{k}}{24} \\
 &= \pm \frac{2\hat{i} - 2\hat{j} - \hat{k}}{3} \\
 &= \pm \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}
 \end{aligned}$$

Question 3:

If a unit vector $\vec{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\vec{a}$.

Solution:

Let the unit vector $\vec{a} = \hat{a}_1\hat{i} + \hat{a}_2\hat{j} + \hat{a}_3\hat{k}$

Then, $|\vec{a}| = 1$

Now,

$$\cos \frac{\pi}{3} = \frac{a_1}{|a|} \Rightarrow a_1 = \frac{1}{2}$$

$$\cos \frac{\pi}{4} = \frac{a_2}{|a|} \Rightarrow a_2 = \frac{1}{\sqrt{2}}$$

$$\cos \theta = \frac{a_3}{|a|} \Rightarrow a_3 = \cos \theta$$

Therefore,

$$\Rightarrow \sqrt{a_1^2 + a_2^2 + a_3^2} = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2 \theta = 1$$

$$\Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2 \theta = 1$$

$$\Rightarrow \frac{3}{4} + \cos^2 \theta = 1$$

$$\Rightarrow \cos^2 \theta = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\Rightarrow \cos \theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

Hence,

$$a_3 = \cos \frac{\pi}{3} = \frac{1}{2}$$

So, $\theta = \frac{\pi}{3}$ and components of $\vec{a}$ are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

Question 4:

Show that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$

Solution:

$$\begin{aligned}
LHS &= (\vec{a}-\vec{b}) \times (\vec{a}+\vec{b}) \\
&= (\vec{a}-\vec{b}) \times \vec{a} + (\vec{a}-\vec{b}) \times \vec{b} \\
&= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b} \\
&= 0 + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} - 0 \\
&= 2\vec{a} \times \vec{b} \\
&= RHS
\end{aligned}$$

Question 5:

Find λ and μ if $(\hat{2}i + \hat{6}j + 27\hat{k}) \times (\hat{i} + \hat{\lambda}j + \hat{\mu}k) = 0$

Solution:

We have $(\hat{2}i + \hat{6}j + 27\hat{k}) \times (\hat{i} + \hat{\lambda}j + \hat{\mu}k) = 0$

Therefore,

$$\begin{aligned}
&\Rightarrow (\hat{2}i + \hat{6}j + 27\hat{k}) \times (\hat{i} + \hat{\lambda}j + \hat{\mu}k) = 0 \\
&\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = \hat{0}i + \hat{0}j + \hat{0}k \\
&\Rightarrow \hat{i}(6\mu - 27\lambda) - \hat{j}(2\mu - 27) + \hat{k}(2\lambda - 6) = \hat{0}i + \hat{0}j + \hat{0}k
\end{aligned}$$

On comparing the corresponding components, we have:

$$6\mu - 27\lambda = 0$$

$$2\mu - 27 = 0$$

$$2\lambda - 6 = 0$$

Now,

$$2\lambda - 6 = 0 \Rightarrow \lambda = 3$$

$$2\mu - 27 = 0 \Rightarrow \mu = \frac{27}{2}$$

Question 6:

Given that $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = 0$. What can you conclude about $\vec{a}$ and $\vec{b}$?

Solution:

When $\vec{a} \cdot \vec{b} = 0$

Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$

Or $\vec{a} \perp \vec{b}$ (if $|\vec{a}| \neq 0$ and $|\vec{b}| \neq 0$)

When $\vec{a} \times \vec{b} = 0$

Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$

Or $\vec{a} \parallel \vec{b}$ (if $|\vec{a}| \neq 0$ and $|\vec{b}| \neq 0$)

Since, $\vec{a}$ and $\vec{b}$ cannot be perpendicular and parallel simultaneously.

So, $\vec{a} = 0$ or $\vec{b} = 0$.

Question 7:

Let the vectors $\vec{a}, \vec{b}, \vec{c}$ given as $\hat{a}_1 i + \hat{a}_2 j + \hat{a}_3 k$, $\hat{b}_1 i + \hat{b}_2 j + \hat{b}_3 k$, $\hat{c}_1 i + \hat{c}_2 j + \hat{c}_3 k$. Then show that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

Solution:

We have

$$\begin{aligned}\vec{a} &= \hat{a}_1 i + \hat{a}_2 j + \hat{a}_3 k \\ \vec{b} &= \hat{b}_1 i + \hat{b}_2 j + \hat{b}_3 k \\ \vec{c} &= \hat{c}_1 i + \hat{c}_2 j + \hat{c}_3 k\end{aligned}$$

Then,

$$(\vec{b} + \vec{c}) = (\hat{b}_1 + \hat{c}_1)i + (\hat{b}_2 + \hat{c}_2)j + (\hat{b}_3 + \hat{c}_3)k$$

Now,

$$\begin{aligned}\vec{a} \times (\vec{b} + \vec{c}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \\ \hat{b}_1 + \hat{c}_1 & \hat{b}_2 + \hat{c}_2 & \hat{b}_3 + \hat{c}_3 \end{vmatrix} = \begin{pmatrix} \hat{i} [a_2(b_3 + c_3) - a_3(b_2 + c_2)] - \hat{j} [a_1(b_3 + c_3) - a_3(b_1 + c_1)] \\ \hat{k} [a_1(b_2 + c_2) - a_2(b_1 + c_1)] \end{pmatrix} \\ &= \begin{pmatrix} \hat{i} [a_2 b_3 + a_2 c_3 - a_3 b_2 - a_3 c_2] + \hat{j} [-a_1 b_3 - a_1 c_3 + a_3 b_1 + a_3 c_1] \\ \hat{k} [a_1 b_2 + a_1 c_2 - a_2 b_1 - a_2 c_1] \end{pmatrix}\end{aligned}$$

Also,

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \hat{i}[a_2b_3 - a_3b_2] + \hat{j}[a_3b_1 - a_1b_3] + \hat{k}[a_2b_2 - a_2b_1]\end{aligned}$$

$$\begin{aligned}\vec{a} \times \vec{c} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \hat{i}[a_2c_3 - a_3c_2] + \hat{j}[a_3c_1 - a_1c_3] + \hat{k}[a_2c_2 - a_2c_1]\end{aligned}$$

Therefore,

$$\begin{aligned}(\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) &= \begin{pmatrix} \hat{i}[a_2b_3 - a_3b_2] + \hat{j}[a_3b_1 - a_1b_3] + \hat{k}[a_2b_2 - a_2b_1] \\ \hat{i}[a_2c_3 - a_3c_2] + \hat{j}[a_3c_1 - a_1c_3] + \hat{k}[a_2c_2 - a_2c_1] \end{pmatrix} \\ &= \begin{pmatrix} \hat{i}[a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2] + \hat{j}[-a_1b_3 - a_1c_3 + a_3b_1 + a_3c_1] \\ \hat{k}[a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1] \end{pmatrix}\end{aligned}$$

Thus,

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

Hence proved.

Question 8:

If either $\vec{a} = 0$ or $\vec{b} = 0$, then $\vec{a} \times \vec{b} = 0$. Is the converse true? Justify your answer with an example.

Solution:

Let $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = 4\hat{i} + 6\hat{j} + 8\hat{k}$

Therefore,

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{vmatrix} \\ &= \hat{i}(24 - 24) - \hat{j}(16 - 16) + \hat{k}(12 - 12) \\ &= 0\end{aligned}$$

Now,

$$|\vec{a}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

Thus,

$$\vec{a} \neq 0$$

Also,

$$|\vec{b}| = \sqrt{4^2 + 6^2 + 8^2} = \sqrt{116}$$

Thus,

$$\vec{b} \neq 0$$

Hence, converse of the statement need not to be true.

Question 9:

Find the area of triangle with vertices $A(1,1,2)$ $B(2,3,5)$ and $C(1,5,5)$.

Solution:

Vertices of the triangle are $A(1,1,2)$ $B(2,3,5)$ and $C(1,5,5)$

Hence,

$$\begin{aligned}\vec{AB} &= (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k} \\ &= \hat{i} + 2\hat{j} + 3\hat{k} \\ \vec{BC} &= (1-2)\hat{i} + (5-3)\hat{j} + (5-5)\hat{k} \\ &= -\hat{i} + 2\hat{j}\end{aligned}$$

Therefore,

$$\text{Area of the triangle } ar(\triangle ABC) = \frac{1}{2} |\vec{AB} \times \vec{BC}|$$

Now,

$$\begin{aligned}\vec{AB} \times \vec{BC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} \\ &= \hat{i}(-6) - \hat{j}(3) + \hat{k}(2+2) \\ &= -6\hat{i} - 3\hat{j} + 4\hat{k} \\ |\vec{AB} \times \vec{BC}| &= \sqrt{(-6)^2 + (-3)^2 + 4^2} \\ &= \sqrt{36+9+16} \\ &= \sqrt{61}\end{aligned}$$

Therefore,

$$\begin{aligned}ar(\Delta ABC) &= \frac{1}{2}\sqrt{61} \\ &= \frac{\sqrt{61}}{2}\end{aligned}$$

Question 10:

Find the area of the parallelogram whose adjacent sides are determined by the vector $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Solution:

We have $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$

Hence,

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} \\ &= \hat{i}(-1+21) - \hat{j}(1-6) + \hat{k}(-7+2) \\ &= 20\hat{i} + 5\hat{j} - 5\hat{k} \\ |\vec{a} \times \vec{b}| &= \sqrt{20^2 + 5^2 + 5^2} \\ &= \sqrt{400 + 25 + 25} \\ &= 15\sqrt{2}\end{aligned}$$

Thus, the area of parallelogram is $15\sqrt{2}$ square units.

Question 11:

Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

- (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Solution:

We have $|\vec{a}| = 3$, $|\vec{b}| = \frac{\sqrt{2}}{3}$ and $|\vec{a} \times \vec{b}| = 1$

Therefore,

$$\begin{aligned}
&\Rightarrow \|\vec{a}\|\|\vec{b}\|\sin\theta = 1 \\
&\Rightarrow 3 \times \frac{\sqrt{2}}{3} \times \sin\theta = 1 \\
&\Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \\
&\Rightarrow \theta = \frac{\pi}{4}
\end{aligned}$$

Question 12:

Area of the rectangle having vertices A, B, C and D with position vectors $\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, respectively is

- (A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

Solution:

We have vertices $A\left(\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$, $B\left(\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$, $C\left(\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}\right)$ and $D\left(-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$.

Therefore,

$$\begin{aligned}
\vec{AB} &= (1+1)\hat{i} + \left(\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = 2\hat{i} \\
\vec{BC} &= (1-1)\hat{i} + \left(-\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = -\hat{j}
\end{aligned}$$

Now,

$$\begin{aligned}
\vec{AB} \times \vec{BC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} \\
&= \hat{k}(-2) \\
&= -2\hat{k} \\
|\vec{AB} \times \vec{BC}| &= \sqrt{(-2)^2} \\
&= 2
\end{aligned}$$

So, area of the rectangle is 2 square units.

MISCELLANEOUS EXERCISE

Question 1:

Write down a unit vector in XY-plane, making an angle of 30° with the positive direction x -axis.

Solution:

Unit vector is $\vec{r} = \cos \hat{\theta} i + \sin \hat{\theta} j$, where θ is angle with positive x -axis.
Therefore,

$$\begin{aligned}\vec{r} &= \cos 30^\circ i + \sin 30^\circ j \\ &= \frac{\sqrt{3}}{2} i + \frac{1}{2} j\end{aligned}$$

Question 2:

Find the scalar components and magnitude of the vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Solution:

We have $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

Therefore,

$$\begin{aligned}\vec{PQ} &= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \\ |\vec{PQ}| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\end{aligned}$$

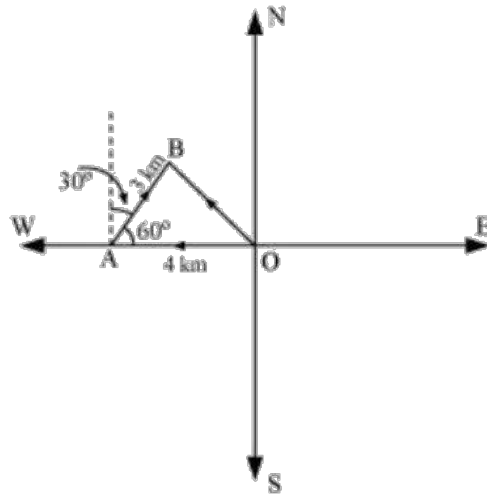
Hence, the scalar components of the vector is $\{(x_2 - x_1) + (y_2 - y_1) + (z_2 - z_1)\}$ and magnitude of the vector is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Question 3:

A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Solution:

Let O and B be the initial and final positions of the girl, respectively.
Then, the girl's position can be shown by the below diagram:



We have:

$$\begin{aligned}\vec{OA} &= -4\hat{i} \\ \vec{AB} &= |\vec{AB}| \cos 60^\circ \hat{i} + |\vec{AB}| \sin 60^\circ \hat{j} \\ &= 3 \times \frac{1}{2} \hat{i} + 3 \times \frac{\sqrt{3}}{2} \hat{j} \\ &= \frac{3}{2} \hat{i} + \frac{3\sqrt{3}}{2} \hat{j}\end{aligned}$$

By the triangle law of addition for vector,

$$\begin{aligned}\vec{OB} &= \vec{OA} + \vec{AB} \\ &= (-4\hat{i}) + \left(\frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j} \right) \\ &= \left(-4 + \frac{3}{2} \right) \hat{i} + \frac{3\sqrt{3}}{2} \hat{j} \\ &= \left(\frac{-8+3}{2} \right) \hat{i} + \frac{3\sqrt{3}}{2} \hat{j} \\ &= \frac{-5}{2} \hat{i} + \frac{3\sqrt{3}}{2} \hat{j}\end{aligned}$$

Hence, the girl's displacement from her initial point of departure is $\frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$.

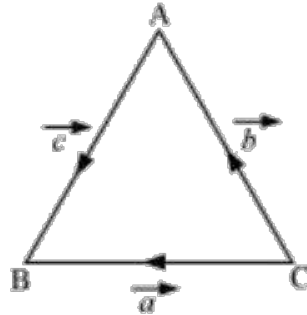
Question 4:

If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.

Solution:

In $\triangle ABC$

$$\overrightarrow{CB} = \vec{a}, \overrightarrow{CA} = \vec{b} \text{ and } \overrightarrow{AB} = \vec{c}$$



By triangle law of addition for vectors

$$\vec{a} = \vec{b} + \vec{c}$$

By triangle inequality law of lengths

$$|\vec{a}| < |\vec{b}| + |\vec{c}|$$

Hence, it is not true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$

Question 5:

Find the value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

Solution:

We have a unit vector $x(\hat{i} + \hat{j} + \hat{k})$

Therefore,

$$\Rightarrow |x(\hat{i} + \hat{j} + \hat{k})| = 1$$

$$\Rightarrow \sqrt{x^2 + x^2 + x^2} = 1$$

$$\Rightarrow \sqrt{3x^2} = 1$$

$$\Rightarrow \sqrt{3}x = 1$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{3}}$$

Question 6:

Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Solution:

We have $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$

Hence,

$$\begin{aligned}\vec{c} &= \vec{a} + \vec{b} \\ &= (2+1)\hat{i} + (3-2)\hat{j} + (-1+1)\hat{k} \\ &= 3\hat{i} + \hat{j} \\ |\vec{c}| &= \sqrt{3^2 + 1^2} \\ &= \sqrt{9+1} \\ &= \sqrt{10}\end{aligned}$$

Therefore,

$$\vec{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{(3\hat{i} + \hat{j})}{\sqrt{10}}$$

So, a vector of magnitude 5 and parallel to the resultant of $\vec{a}$ and $\vec{b}$ is

$$\begin{aligned}\pm(\vec{c}) &= \pm 5 \left(\frac{1}{\sqrt{10}} (3\hat{i} + \hat{j}) \right) \\ &= \pm \frac{3\sqrt{10}}{2} \hat{i} \pm \frac{\sqrt{10}}{2} \hat{j}\end{aligned}$$

Question 7:

If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$ find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.

Solution:

We have $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$

Therefore,

$$\begin{aligned}
\vec{2a} - \vec{b} + 3\vec{c} &= 2(\hat{i} + \hat{j} + \hat{k}) - (\hat{2i} - \hat{j} + \hat{3k}) + 3(\hat{i} - 2\hat{j} + \hat{k}) \\
&= \hat{2i} + \hat{2j} + \hat{2k} - \hat{2i} + \hat{j} - \hat{3k} + \hat{3i} - \hat{6j} + \hat{3k} \\
&= \hat{3i} - \hat{3j} + \hat{2k} \\
|\vec{2a} - \vec{b} + 3\vec{c}| &= \sqrt{3^2 + (-3)^2 + 2^2} \\
&= \sqrt{9 + 9 + 4} \\
&= \sqrt{22}
\end{aligned}$$

So, the required unit vector is

$$\begin{aligned}
\frac{\vec{2a} - \vec{b} + 3\vec{c}}{|\vec{2a} - \vec{b} + 3\vec{c}|} &= \frac{\hat{3i} - \hat{3j} + \hat{2k}}{\sqrt{22}} \\
&= \frac{3}{\sqrt{22}}\hat{i} - \frac{3}{\sqrt{22}}\hat{j} + \frac{2}{\sqrt{22}}\hat{k}
\end{aligned}$$

Question 8:

Show that the points $A(1, -2, -8)$, $B(5, 0, -2)$ and $C(11, 3, 7)$ are collinear and find the ratio in which B divides AC.

Solution:

We have points $A(1, -2, -8)$, $B(5, 0, -2)$ and $C(11, 3, 7)$

Therefore,

$$\begin{aligned}
\vec{AB} &= (5-1)\hat{i} + (0+2)\hat{j} + (-2+8)\hat{k} = \hat{4i} + \hat{2j} + \hat{6k} \\
\vec{BC} &= (11-5)\hat{i} + (3-0)\hat{j} + (7+2)\hat{k} = \hat{6i} + \hat{3j} + \hat{9k} \\
\vec{AC} &= (11-1)\hat{i} + (3+2)\hat{j} + (7+8)\hat{k} = \hat{10i} + \hat{5j} + \hat{15k} \\
|\vec{AB}| &= \sqrt{4^2 + 2^2 + 6^2} = \sqrt{16 + 4 + 36} = \sqrt{56} = 2\sqrt{14} \\
|\vec{BC}| &= \sqrt{6^2 + 3^2 + 9^2} = \sqrt{36 + 9 + 81} = \sqrt{126} = 3\sqrt{14} \\
|\vec{AC}| &= \sqrt{10^2 + 5^2 + 15^2} = \sqrt{100 + 25 + 225} = \sqrt{350} = 5\sqrt{14}
\end{aligned}$$

Now,

$$\begin{aligned}
|\vec{AB}| + |\vec{BC}| &= 2\sqrt{14} + 3\sqrt{14} \\
&= 5\sqrt{14} \\
&= |\vec{AC}|
\end{aligned}$$

Thus, the points are collinear.

Let B divides AC in the ratio $\lambda : 1$

Therefore,

$$\begin{aligned}\vec{OB} &= \frac{\lambda \vec{OC} + \vec{OA}}{(\lambda + 1)} \\ \Rightarrow \hat{5}i - \hat{2}k &= \frac{\lambda(11\hat{i} + 3\hat{j} + 7\hat{k}) + (\hat{i} - 2\hat{j} - 8\hat{k})}{\lambda + 1} \\ \Rightarrow (\lambda + 1)(\hat{5}i - \hat{2}k) &= 11\lambda\hat{i} + 3\lambda\hat{j} + 7\lambda\hat{k} + \hat{i} - 2\hat{j} - 8\hat{k} \\ \Rightarrow 5(\lambda + 1)\hat{i} - 2(\lambda + 1)\hat{k} &= (11\lambda + 1)\hat{i} + (3\lambda - 2)\hat{j} + (7\lambda - 8)\hat{k}\end{aligned}$$

On equating the corresponding components, we get

$$\begin{aligned}\Rightarrow 5(\lambda + 1) &= (11\lambda + 1) \\ \Rightarrow 5\lambda + 5 &= 11\lambda + 1 \\ \Rightarrow 6\lambda &= 4 \\ \Rightarrow \lambda &= \frac{2}{3}\end{aligned}$$

Thus, the ratio is $2 : 3$.

Question 9:

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ externally in the ratio $1 : 2$. Also show that P is midpoint of the line segment RQ.

Solution:

We have $\vec{OP} = 2\vec{a} + \vec{b}$, $\vec{OQ} = \vec{a} - 3\vec{b}$

It is given that point R divides a line segment joining two points P and Q externally in the ratio $1 : 2$.

Then, on using the section formula, we get:

$$\begin{aligned}\vec{OR} &= \frac{2(2\vec{a} + \vec{b}) - (\vec{a} - 3\vec{b})}{2 - 1} \\ &= \frac{4\vec{a} + 2\vec{b} - \vec{a} + 3\vec{b}}{1} \\ &= 3\vec{a} + 5\vec{b}\end{aligned}$$

Hence, the position vector of R is $3\vec{a} + 5\vec{b}$

Thus, the position vector of midpoint of $RQ = \frac{\vec{OQ} + \vec{OR}}{2}$

$$\begin{aligned} \frac{\vec{OQ} + \vec{OR}}{2} &= \frac{(\vec{a} - 3\vec{b}) + (3\vec{a} + 5\vec{b})}{2} \\ &= \frac{2\vec{a} + 2\vec{b}}{2} \\ &= \vec{a} + \vec{b} \\ &= \vec{OP} \end{aligned}$$

Thus, P is the midpoint of line segment RQ.

Question 10:

The two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

Solution:

Diagonal of a parallelogram is $\vec{a} + \vec{b}$

$$\begin{aligned} \vec{a} + \vec{b} &= (2 + 1)\hat{i} + (-4 - 2)\hat{j} + (5 - 3)\hat{k} \\ &= 3\hat{i} - 6\hat{j} + 2\hat{k} \end{aligned}$$

So, the unit vector parallel to the diagonal is

$$\begin{aligned} \frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} &= \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{3^2 + (-6)^2 + 2^2}} \\ &= \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{9 + 36 + 4}} \\ &= \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{7} \\ &= \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) \end{aligned}$$

Area of the parallelogram is $|\vec{a} + \vec{b}|$

Now,

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix} \\
&= \hat{i}(12+10) - \hat{j}(-6-5) + \hat{k}(-4+4) \\
&= 22\hat{i} + 11\hat{j} \\
&= 11(\hat{i} + \hat{j}) \\
|\vec{a} + \vec{b}| &= 11\sqrt{2^2 + 1^2} = 11\sqrt{5}
\end{aligned}$$

So, area of parallelogram is $11\sqrt{5}$ square units.

Question 11:

Show that the direction cosines of a vector equally inclined to the axis OX, OY and OZ are

$$\pm \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

Solution:

Let a vector be equally inclined to OX, OY and OZ at an angle α

So, the DCs of the vectors are $\cos \alpha$, $\cos \alpha$ and $\cos \alpha$.

Therefore,

$$\begin{aligned}
\cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha &= 1 \\
\Rightarrow 3 \cos^2 \alpha &= 1 \\
\Rightarrow \cos^2 \alpha &= \frac{1}{3} \\
\Rightarrow \cos \alpha &= \pm \frac{1}{\sqrt{3}}
\end{aligned}$$

Thus, the DCs of the vector are $\pm \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$.

Question 12:

Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $c \cdot \vec{d} = 15$.

Solution:

Let $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$

Since, $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, we have

$$\begin{aligned}\vec{d} \cdot \vec{a} &= 0 \\ \Rightarrow d_1 + 4d_2 + 2d_3 &= 0 \quad \dots(1)\end{aligned}$$

And

$$\begin{aligned}\vec{d} \cdot \vec{b} &= 0 \\ \Rightarrow 3d_1 - 2d_2 + 7d_3 &= 0 \quad \dots(2)\end{aligned}$$

Also, it is given that

$$\begin{aligned}\vec{c} \cdot \vec{d} &= 15 \\ \Rightarrow 2d_1 - d_2 + 4d_3 &= 15 \quad \dots(3)\end{aligned}$$

On solving equations (1), (2) and (3), we get

$$d_1 = \frac{160}{3}, d_2 = -\frac{5}{3}, d_3 = -\frac{70}{3}$$

Therefore,

$$\begin{aligned}\vec{d} &= \frac{160}{3}\hat{i} - \frac{5}{3}\hat{j} - \frac{70}{3}\hat{k} \\ &= \frac{1}{3}(160\hat{i} - 5\hat{j} - 70\hat{k})\end{aligned}$$

Question 13:

The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ .

Solution:

$$(2\hat{i} + 4\hat{j} - 5\hat{k}) + (\lambda\hat{i} + 2\hat{j} + 3\hat{k}) = (2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}$$

Therefore, unit vector along $(2\hat{i} + 4\hat{j} - 5\hat{k}) + (\lambda\hat{i} + 2\hat{j} + 3\hat{k})$ is

$$\frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2 + \lambda)^2 + 6^2 + (-2)^2}} = \frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{4 + 4\lambda + \lambda^2 + 36 + 4}} = \frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}$$

Scalar product of $\hat{i} + \hat{j} + \hat{k}$ with this unit vector is 1.

$$\begin{aligned}
&\Rightarrow (\hat{i} + \hat{j} + \hat{k}) \cdot \left(\frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}} \right) = 1 \\
&\Rightarrow \frac{(2+\lambda) + 6 - 2}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1 \\
&\Rightarrow \sqrt{\lambda^2 + 4\lambda + 44} = \lambda + 6 \\
&\Rightarrow \lambda^2 + 4\lambda + 44 = (\lambda + 6)^2 \\
&\Rightarrow \lambda^2 + 4\lambda + 44 = \lambda^2 + 12\lambda + 36 \\
&\Rightarrow 8\lambda = 8 \\
&\Rightarrow \lambda = 1
\end{aligned}$$

Question 14:

If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a} + \vec{b} + \vec{c}$ is inclined to $\vec{a}, \vec{b}, \vec{c}$.

Solution:

Since, $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes
Therefore,

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$$

And

$$|\vec{a}| = |\vec{b}| = |\vec{c}|$$

Let $\vec{a} + \vec{b} + \vec{c}$ be inclined to $\vec{a}, \vec{b}, \vec{c}$ at angles $\theta_1, \theta_2, \theta_3$ respectively.

$$\begin{aligned}
\cos \theta_1 &= \frac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{a}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{a}|} = \frac{\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{a}|} = \frac{|\vec{a}|^2}{|\vec{a} + \vec{b} + \vec{c}| |\vec{a}|} = \frac{|\vec{a}|}{|\vec{a} + \vec{b} + \vec{c}|} \\
\cos \theta_2 &= \frac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{b}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{b}|} = \frac{\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{b}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{b}|} = \frac{|\vec{b}|^2}{|\vec{a} + \vec{b} + \vec{c}| |\vec{b}|} = \frac{|\vec{b}|}{|\vec{a} + \vec{b} + \vec{c}|} \\
\cos \theta_3 &= \frac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{c}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{c}|} = \frac{\vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{c}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{c}|} = \frac{|\vec{c}|^2}{|\vec{a} + \vec{b} + \vec{c}| |\vec{c}|} = \frac{|\vec{c}|}{|\vec{a} + \vec{b} + \vec{c}|}
\end{aligned}$$

Since $|\vec{a}| = |\vec{b}| = |\vec{c}|$, $\cos \theta_1 = \cos \theta_2 = \cos \theta_3$

Thus, $\theta_1 = \theta_2 = \theta_3$

Question 15:

Prove that $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular, given $\vec{a} \neq 0, \vec{b} \neq 0$.

Solution:

$$\begin{aligned}
 (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) &= |\vec{a}|^2 + |\vec{b}|^2 \\
 \Rightarrow \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} &= |\vec{a}|^2 + |\vec{b}|^2 \\
 \Rightarrow |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 \\
 \Rightarrow 2\vec{a} \cdot \vec{b} &= 0 \\
 \Rightarrow \vec{a} \cdot \vec{b} &= 0
 \end{aligned}$$

Thus, $\vec{a}$ and $\vec{b}$ are perpendicular.

Question 16:

If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a} \cdot \vec{b} \geq 0$ only when

- (A) $0 < \theta < \frac{\pi}{2}$ (B) $0 \leq \theta \leq \frac{\pi}{2}$
 (C) $0 < \theta < \pi$ (D) $0 \leq \theta \leq \pi$

Solution:

$$\begin{aligned}
 \vec{a} \cdot \vec{b} &\geq 0 \\
 \Rightarrow |\vec{a}| |\vec{b}| \cos \theta &\geq 0 \\
 \Rightarrow \cos \theta &\geq 0 \quad \left[\because |\vec{a}| \geq 0 \text{ and } |\vec{b}| \geq 0 \right] \\
 \Rightarrow 0 \leq \theta &\leq \frac{\pi}{2}
 \end{aligned}$$

Hence $\vec{a} \cdot \vec{b} \geq 0$ if $0 \leq \theta \leq \frac{\pi}{2}$

Thus, the correct option is B.

Question 17:

Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the right angle between them. Then $\vec{a} + \vec{b}$ is a unit vector

- (A) $\theta = \frac{\pi}{4}$ (B) $\theta = \frac{\pi}{3}$ (C) $\theta = \frac{\pi}{2}$ (D) $\theta = \frac{2\pi}{3}$

Solution:

We have $\vec{a}$ and $\vec{b}$, two unit vectors and θ is the angle between them.

Then,

$$|\vec{a}| = |\vec{b}| = 1$$

Now, $\vec{a} + \vec{b}$ is a unit vector if $|\vec{a} + \vec{b}| = 1$

Therefore,

$$\begin{aligned} &\Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = 1 \\ &\Rightarrow \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} = 1 \\ &\Rightarrow |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = 1 \\ &\Rightarrow 1^2 + 2|\vec{a}||\vec{b}|\cos\theta + 1^2 = 1 \\ &\Rightarrow 1 + 2(1)(1)\cos\theta + 1 = 1 \\ &\Rightarrow \cos\theta = -\frac{1}{2} \\ &\Rightarrow \theta = \frac{2\pi}{3} \end{aligned}$$

Hence, $\vec{a} + \vec{b}$ is a unit vector if $\theta = \frac{2\pi}{3}$

Thus, the correct option is D.

Question 18:

The value of $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$ is

- (A) 0 (B) -1 (C) 1 (D) 3

Solution:

$$\begin{aligned} \hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j}) &= \hat{i} \cdot \hat{i} + \hat{j} \cdot (-\hat{j}) + \hat{k} \cdot \hat{k} \\ &= 1 - 1 + 1 \\ &= 1 \end{aligned}$$

Thus, the correct option is C.

Question 19:

If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to

- (A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) n

Solution:

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors, so that $|\vec{a}|$ and $|\vec{b}|$ are positive.

Now,

$$\begin{aligned} \Rightarrow |\vec{a} \cdot \vec{b}| &= |\vec{a} \times \vec{b}| \\ \Rightarrow |\vec{a}| |\vec{b}| \cos \theta &= |\vec{a}| |\vec{b}| \sin \theta \\ \Rightarrow \cos \theta &= \sin \theta \\ \Rightarrow \tan \theta &= 1 \\ \Rightarrow \theta &= \frac{\pi}{4} \end{aligned}$$

So, $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$ when $\theta = \frac{\pi}{4}$

Thus, the correct option is B.