REAL NUMBERS 1

1.1 Introduction

In Class IX, you began your exploration of the world of real numbers and encountered irrational numbers. We continue our discussion on real numbers in this chapter. We begin with two very important properties of positive integers in Sections 1.2 and 1.3, namely the Euclid's division algorithm and the Fundamental Theorem of Arithmetic.

Euclid's division algorithm, as the name suggests, has to do with divisibility of integers. Stated simply, it says any positive integer a can be divided by another positive integer b in such a way that it leaves a remainder r that is smaller than b. Many of you probably recognise this as the usual long division process. Although this result is quite easy to state and understand, it has many applications related to the divisibility properties of integers. We touch upon a few of them, and use it mainly to compute the HCF of two positive integers.

The Fundamental Theorem of Arithmetic, on the other hand, has to do something with multiplication of positive integers. You already know that every composite number can be expressed as a product of primes in a unique way—this important fact is the Fundamental Theorem of Arithmetic. Again, while it is a result that is easy to state and understand, it has some very deep and significant applications in the field of mathematics. We use the Fundamental Theorem of Arithmetic for two main applications. First, we use it to prove the irrationality of many of the numbers you studied in Class IX, such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$. Second, we apply this theorem to explore when exactly the decimal expansion of a rational number, say $\frac{p}{q}(q \neq 0)$, is terminating and when it is non-terminating repeating. We do so by looking at the prime factorisation of the denominator q of $\frac{p}{q}$. You will see that the prime factorisation of q will completely reveal the nature of the decimal expansion of $\frac{p}{q}$.

So let us begin our exploration.

1.2 Euclid's Division Lemma

Consider the following folk puzzle*.

A trader was moving along a road selling eggs. An idler who didn't have much work to do, started to get the trader into a wordy duel. This grew into a fight, he pulled the basket with eggs and dashed it on the floor. The eggs broke. The trader requested the Panchayat to ask the idler to pay for the broken eggs. The Panchayat asked the trader how many eggs were broken. He gave the following response:

If counted in pairs, one will remain;
If counted in threes, two will remain;
If counted in fours, three will remain;
If counted in fives, four will remain;
If counted in sixes, five will remain;
If counted in sevens, nothing will remain;
My basket cannot accomodate more than 150 eggs.

So, how many eggs were there? Let us try and solve the puzzle. Let the number of eggs be a. Then working backwards, we see that a is less than or equal to 150:

If counted in sevens, nothing will remain, which translates to $a = 7p + 0$, for some natural number p. If counted in sixes, $a = 6q + 5$, for some natural number q.

If counted in fives, four will remain. It translates to $a = 5w + 4$, for some natural number w.

If counted in fours, three will remain. It translates to $a = 4s + 3$, for some natural number s.

If counted in threes, two will remain. It translates to $a = 3t + 2$, for some natural number t.

If counted in pairs, one will remain. It translates to $a = 2u + 1$, for some natural number u.

That is, in each case, we have a and a positive integer b (in our example, b takes values 7, 6, 5, 4, 3 and 2, respectively) which divides a and leaves a remainder r (in our case, r is 0, 5, 4, 3, 2 and 1, respectively), that is smaller than b. The

* This is modified form of a puzzle given in 'Numeracy Counts!' by A. Rampal, and others.

moment we write down such equations we are using Euclid's division lemma, which is given in Theorem 1.1.

Getting back to our puzzle, do you have any idea how you will solve it? Yes! You must look for the multiples of 7 which satisfy all the conditions. By trial and error (using the concept of LCM), you will find he had 119 eggs.

In order to get a feel for what Euclid's division lemma is, consider the following pairs of integers:

17, 6; 5, 12; 20, 4

Like we did in the example, we can write the following relations for each such pair:

$17 = 6 \times 2 + 5$ (6 goes into 17 twice and leaves a remainder 5)

$5 = 12 \times 0 + 5$ (This relation holds since 12 is larger than 5)

$20 = 4 \times 5 + 0$ (Here 4 goes into 20 five-times and leaves no remainder)

That is, for each pair of positive integers a and b, we have found whole numbers q and r, satisfying the relation:

$$a = bq + r,\ 0 \le r < b$$

Note that q or r can also be zero.

Why don't you now try finding integers q and r for the following pairs of positive integers a and b?

(i) 10, 3; (ii) 4, 19; (iii) 81, 3

Did you notice that q and r are unique? These are the only integers satisfying the conditions $a = bq + r$, where $0 \le r < b$. You may have also realised that this is nothing but a restatement of the long division process you have been doing all these years, and that the integers q and r are called the *quotient* and *remainder,* respectively.

A formal statement of this result is as follows :

Theorem 1.1 (Euclid's Division Lemma) : *Given positive integers a and b, there exist unique integers q and r satisfying $a = bq + r$, $0 \le r < b$.*

This result was perhaps known for a long time, but was first recorded in Book VII of Euclid's Elements. Euclid's division algorithm is based on this lemma.

An **algorithm** is a series of well defined steps which gives a procedure for solving a type of problem.

The word *algorithm* comes from the name of the 9th century Persian mathematician al-Khwarizmi. In fact, even the word 'algebra' is derived from a book, he wrote, called *Hisab al-jabr w'al-muqabala*.

A **lemma** is a proven statement used for proving another statement.

Muhammad ibn Musa al-Khwarizmi (C.E. 780 – 850)

Euclid's division algorithm is a technique to compute the Highest Common Factor (HCF) of two given positive integers. Recall that the HCF of two positive integers a and b is the largest positive integer d that divides both a and b.

Let us see how the algorithm works, through an example first. Suppose we need to find the HCF of the integers 455 and 42. We start with the larger integer, that is, 455. Then we use Euclid's lemma to get

$$455 = 42 \times 10 + 35$$

Now consider the divisor 42 and the remainder 35, and apply the division lemma to get

$$42 = 35 \times 1 + 7$$

Now consider the divisor 35 and the remainder 7, and apply the division lemma to get

$$35 = 7 \times 5 + 0$$

Notice that the remainder has become zero, and we cannot proceed any further. **We claim** that the HCF of 455 and 42 is the divisor at this stage, i.e., 7. You can easily verify this by listing all the factors of 455 and 42. Why does this method work? It works because of the following result.

So, let us state **Euclid's division algorithm** clearly.

To obtain the HCF of two positive integers, say c and d, with $c > d$, follow the steps below:

Step 1 : Apply Euclid's division lemma, to c and d. So, we find whole numbers, q and r such that $c = dq + r$, $0 \leq r < d$.

Step 2 : If $r = 0$, d is the HCF of c and d. If $r \neq 0$, apply the division lemma to d and r.

Step 3 : Continue the process till the remainder is zero. The divisor at this stage will be the required HCF.

This algorithm works because HCF (c, d) = HCF (d, r) where the symbol HCF (c, d) denotes the HCF of c and d, etc.

Example 1 : Use Euclid's algorithm to find the HCF of 4052 and 12576.

Solution :

Step 1 : Since 12576 > 4052, we apply the division lemma to 12576 and 4052, to get

$$12576 = 4052 \times 3 + 420$$

Step 2 : Since the remainder $420 \neq 0$, we apply the division lemma to 4052 and 420, to get

$$4052 = 420 \times 9 + 272$$

Step 3 : We consider the new divisor 420 and the new remainder 272, and apply the division lemma to get

$$420 = 272 \times 1 + 148$$

We consider the new divisor 272 and the new remainder 148, and apply the division lemma to get

$$272 = 148 \times 1 + 124$$

We consider the new divisor 148 and the new remainder 124, and apply the division lemma to get

$$148 = 124 \times 1 + 24$$

We consider the new divisor 124 and the new remainder 24, and apply the division lemma to get

$$124 = 24 \times 5 + 4$$

We consider the new divisor 24 and the new remainder 4, and apply the division lemma to get

$$24 = 4 \times 6 + 0$$

The remainder has now become zero, so our procedure stops. Since the divisor at this stage is 4, the HCF of 12576 and 4052 is 4.

Notice that 4 = HCF (24, 4) = HCF (124, 24) = HCF (148, 124) = HCF (272, 148) = HCF (420, 272) = HCF (4052, 420) = HCF (12576, 4052).

Euclid's division algorithm is not only useful for calculating the HCF of very large numbers, but also because it is one of the earliest examples of an algorithm that a computer had been programmed to carry out.

Remarks :

1. Euclid's division lemma and algorithm are so closely interlinked that people often call former as the division algorithm also.

2. Although Euclid's Division Algorithm is stated for only positive integers, it can be extended for all integers except zero, i.e., $b \neq 0$. However, we shall not discuss this aspect here.

Euclid's division lemma/algorithm has several applications related to finding properties of numbers. We give some examples of these applications below:

Example 2 : Show that every positive even integer is of the form $2q$, and that every positive odd integer is of the form $2q + 1$, where q is some integer.

Solution : Let a be any positive integer and $b = 2$. Then, by Euclid's algorithm, $a = 2q + r$, for some integer $q \geq 0$, and $r = 0$ or $r = 1$, because $0 \leq r < 2$. So, $a = 2q$ or $2q + 1$.

If a is of the form $2q$, then a is an even integer. Also, a positive integer can be either even or odd. Therefore, any positive odd integer is of the form $2q + 1$.

Example 3 : Show that any positive odd integer is of the form $4q + 1$ or $4q + 3$, where q is some integer.

Solution : Let us start with taking a, where a is a positive odd integer. We apply the division algorithm with a and $b = 4$.

Since $0 \leq r < 4$, the possible remainders are 0, 1, 2 and 3.

That is, a can be $4q$, or $4q + 1$, or $4q + 2$, or $4q + 3$, where q is the quotient. However, since a is odd, a cannot be $4q$ or $4q + 2$ (since they are both divisible by 2).

Therefore, any odd integer is of the form $4q + 1$ or $4q + 3$.

Example 4 : A sweetseller has 420 *kaju barfis* and 130 *badam barfis*. She wants to stack them in such a way that each stack has the same number, and they take up the least area of the tray. What is the number of that can be placed in each stack for this purpose?

Solution : This can be done by trial and error. But to do it systematically, we find HCF (420, 130). Then this number will give the maximum number of *barfis* in each stack and the number of stacks will then be the least. The area of the tray that is used up will be the least.

Now, let us use Euclid's algorithm to find their HCF. We have :

$$420 = 130 \times 3 + 30$$

$$130 = 30 \times 4 + 10$$

$$30 = 10 \times 3 + 0$$

So, the HCF of 420 and 130 is 10.

Therefore, the sweetseller can make stacks of 10 for both kinds of *barfi.*

EXERCISE 1.1

1. Use Euclid's division algorithm to find the HCF of :

(i) 135 and 225 (ii) 196 and 38220 (iii) 867 and 255

2. Show that any positive odd integer is of the form $6q + 1$, or $6q + 3$, or $6q + 5$, where q is some integer.

3. An army contingent of 616 members is to march behind an army band of 32 members in a parade. The two groups are to march in the same number of columns. What is the maximum number of columns in which they can march?

4. Use Euclid's division lemma to show that the square of any positive integer is either of the form $3m$ or $3m + 1$ for some integer m.

[**Hint :** Let x be any positive integer then it is of the form $3q$, $3q + 1$ or $3q + 2$. Now square each of these and show that they can be rewritten in the form $3m$ or $3m + 1$.]

5. Use Euclid's division lemma to show that the cube of any positive integer is of the form $9m$, $9m + 1$ or $9m + 8$.

1.3 The Fundamental Theorem of Arithmetic

In your earlier classes, you have seen that any natural number can be written as a product of its prime factors. For instance, $2 = 2$, $4 = 2 \times 2$, $253 = 11 \times 23$, and so on. Now, let us try and look at natural numbers from the other direction. That is, can any natural number be obtained by multiplying prime numbers? Let us see.

Take any collection of prime numbers, say 2, 3, 7, 11 and 23. If we multiply some or all of these numbers, allowing them to repeat as many times as we wish, we can produce a large collection of positive integers (In fact, infinitely many). Let us list a few :

$7 \times 11 \times 23 = 1771$ $3 \times 7 \times 11 \times 23 = 5313$

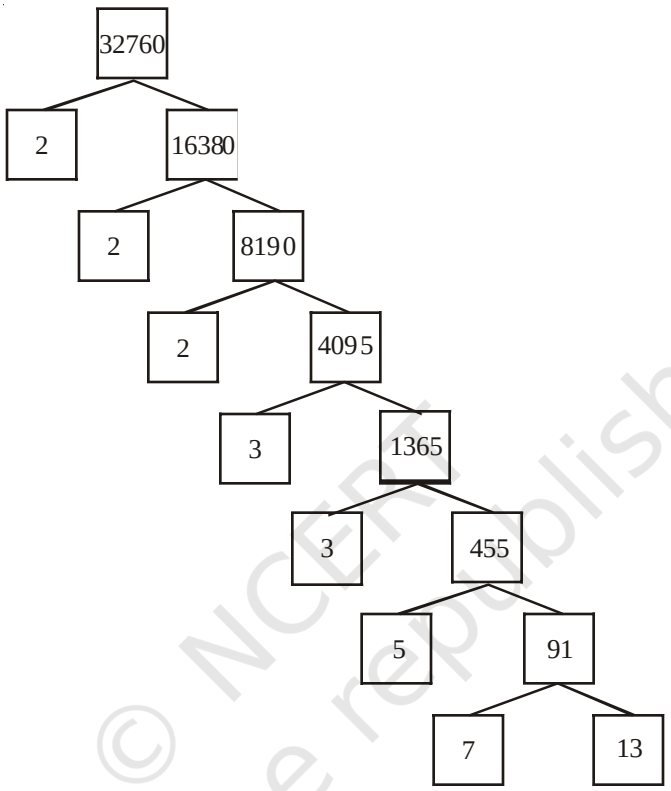
$2 \times 3 \times 7 \times 11 \times 23 = 10626$ $2^3 \times 3 \times 7^3 = 8232$

$2^2 \times 3 \times 7 \times 11 \times 23 = 21252$

and so on.

Now, let us suppose your collection of primes includes all the possible primes. What is your guess about the size of this collection? Does it contain only a finite number of integers, or infinitely many? Infact, there are infinitely many primes. So, if we combine all these primes in all possible ways, we will get an infinite collection of numbers, all the primes and all possible products of primes. The question is – can we produce all the composite numbers this way? What do you think? Do you think that there may be a composite number which is not the product of powers of primes? Before we answer this, let us factorise positive integers, that is, do the opposite of what we have done so far.

We are going to use the factor tree with which you are all familiar. Let us take some large number, say, 32760, and factorise it as shown :



So we have factorised 32760 as $2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13$ as a product of primes, i.e., $32760 = 2^3 \times 3^2 \times 5 \times 7 \times 13$ as a product of powers of primes. Let us try another number, say, 123456789. This can be written as $3^2 \times 3803 \times 3607$. Of course, you have to check that 3803 and 3607 are primes! (Try it out for several other natural numbers yourself.) This leads us to a conjecture that every composite number can be written as the product of powers of primes. In fact, this statement is true, and is called the **Fundamental Theorem of Arithmetic** because of its basic crucial importance to the study of integers. Let us now formally state this theorem.

Theorem 1.2 (Fundamental Theorem of Arithmetic) : *Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique, apart from the order in which the prime factors occur.*

An equivalent version of Theorem 1.2 was probably first recorded as Proposition 14 of Book IX in Euclid's Elements, before it came to be known as the Fundamental Theorem of Arithmetic. However, the first correct proof was given by Carl Friedrich Gauss in his *Disquisitiones Arithmeticae*.

Carl Friedrich Gauss is often referred to as the 'Prince of Mathematicians' and is considered one of the three greatest mathematicians of all time, along with Archimedes and Newton. He has made fundamental contributions to both mathematics and science.

Carl Friedrich Gauss (1777 – 1855)

The Fundamental Theorem of Arithmetic says that every composite number can be factorised as a product of primes. Actually it says more. It says that given any composite number it can be factorised as a product of prime numbers in a **'unique'** way, except for the order in which the primes occur. That is, given any composite number there is one and only one way to write it as a product of primes, as long as we are not particular about the order in which the primes occur. So, for example, we regard $2 \times 3 \times 5 \times 7$ as the same as $3 \times 5 \times 7 \times 2$, or any other possible order in which these primes are written. This fact is also stated in the following form:

The prime factorisation of a natural number is unique, except for the order of its factors.

In general, given a composite number x, we factorise it as $x = p_1 p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes and written in ascending order, i.e., $p_1 \leq p_2 \leq \ldots \leq p_n$. If we combine the same primes, we will get powers of primes. For example,

$$32760 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13 = 2^3 \times 3^2 \times 5 \times 7 \times 13$$

Once we have decided that the order will be ascending, then the way the number is factorised, is unique.

The Fundamental Theorem of Arithmetic has many applications, both within mathematics and in other fields. Let us look at some examples.

Example 5 : Consider the numbers 4^n, where n is a natural number. Check whether there is any value of n for which 4^n ends with the digit zero.

Solution : If the number 4^n, for any n, were to end with the digit zero, then it would be divisible by 5. That is, the prime factorisation of 4^n would contain the prime 5. This is

not possible because $4^n = (2)^{2n}$; so the only prime in the factorisation of 4^n is 2. So, the uniqueness of the Fundamental Theorem of Arithmetic guarantees that there are no other primes in the factorisation of 4^n. So, there is no natural number n for which 4^n ends with the digit zero.

You have already learnt how to find the HCF and LCM of two positive integers using the Fundamental Theorem of Arithmetic in earlier classes, without realising it! This method is also called the *prime factorisation method*. Let us recall this method through an example.

Example 6 : Find the LCM and HCF of 6 and 20 by the prime factorisation method.

Solution : We have : $6 = 2^1 \times 3^1$ and $20 = 2 \times 2 \times 5 = 2^2 \times 5^1$.

You can find HCF(6, 20) = 2 and LCM(6, 20) = $2 \times 2 \times 3 \times 5 = 60$, as done in your earlier classes.

Note that HCF(6, 20) = 2^1 = **Product of the smallest power of each common prime factor in the numbers.**

LCM (6, 20) = $2^2 \times 3^1 \times 5^1$ = **Product of the greatest power of each prime factor, involved in the numbers.**

From the example above, you might have noticed that HCF(6, 20) × LCM(6, 20) = 6 × 20. In fact, we can verify that **for any two positive integers *a* and *b*, HCF (*a*, *b*) × LCM (*a*, *b*) = *a* × *b*.** We can use this result to find the LCM of two positive integers, if we have already found the HCF of the two positive integers.

Example 7 : Find the HCF of 96 and 404 by the prime factorisation method. Hence, find their LCM.

Solution : The prime factorisation of 96 and 404 gives :

$$96 = 2^5 \times 3,\ 404 = 2^2 \times 101$$

Therefore, the HCF of these two integers is $2^2 = 4$.

Also, $$\text{LCM}(96, 404) = \frac{96 \times 404}{\text{HCF}(96, 404)} = \frac{96 \times 404}{4} = 9696$$

Example 8 : Find the HCF and LCM of 6, 72 and 120, using the prime factorisation method.

Solution : We have :

$$6 = 2 \times 3,\ 72 = 2^3 \times 3^2,\ 120 = 2^3 \times 3 \times 5$$

Here, 2^1 and 3^1 are the smallest powers of the common factors 2 and 3, respectively.

So, HCF (6, 72, 120) = $2^1 \times 3^1 = 2 \times 3 = 6$

2^3, 3^2 and 5^1 are the greatest powers of the prime factors 2, 3 and 5 respectively involved in the three numbers.

So, LCM (6, 72, 120) = $2^3 \times 3^2 \times 5^1 = 360$

Remark : Notice, $6 \times 72 \times 120 \neq$ HCF (6, 72, 120) $\times$ LCM (6, 72, 120). So, the product of three numbers is not equal to the product of their HCF and LCM.

EXERCISE 1.2

1. Express each number as a product of its prime factors:
(i) 140 (ii) 156 (iii) 3825 (iv) 5005 (v) 7429

2. Find the LCM and HCF of the following pairs of integers and verify that LCM × HCF = product of the two numbers.
(i) 26 and 91 (ii) 510 and 92 (iii) 336 and 54

3. Find the LCM and HCF of the following integers by applying the prime factorisation method.
(i) 12, 15 and 21 (ii) 17, 23 and 29 (iii) 8, 9 and 25

4. Given that HCF (306, 657) = 9, find LCM (306, 657).

5. Check whether 6^n can end with the digit 0 for any natural number n.

6. Explain why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ are composite numbers.

7. There is a circular path around a sports field. Sonia takes 18 minutes to drive one round of the field, while Ravi takes 12 minutes for the same. Suppose they both start at the same point and at the same time, and go in the same direction. After how many minutes will they meet again at the starting point?

1.4 Revisiting Irrational Numbers

In Class IX, you were introduced to irrational numbers and many of their properties. You studied about their existence and how the rationals and the irrationals together made up the real numbers. You even studied how to locate irrationals on the number line. However, we did not prove that they were irrationals. In this section, we will prove that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and, in general, $\sqrt{p}$ is irrational, where p is a prime. One of the theorems, we use in our proof, is the Fundamental Theorem of Arithmetic.

Recall, a number 's' is called *irrational* if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Some examples of irrational numbers, with

which you are already familiar, are :

$$\sqrt{2}, \sqrt{3}, \sqrt{15}, \pi, -\frac{\sqrt{2}}{\sqrt{3}}, 0.101101110111110\ldots, \text{ etc.}$$

Before we prove that $\sqrt{2}$ is irrational, we need the following theorem, whose proof is based on the Fundamental Theorem of Arithmetic.

Theorem 1.3 : *Let p be a prime number. If p divides a^2, then p divides a, where a is a positive integer.*

***Proof :** Let the prime factorisation of a be as follows :

$a = p_1 p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes, not necessarily distinct.

Therefore, $a^2 = (p_1 p_2 \ldots p_n)(p_1 p_2 \ldots p_n) = p_1^2 p_2^2 \ldots p_n^2$.

Now, we are given that p divides a^2. Therefore, from the Fundamental Theorem of Arithmetic, it follows that p is one of the prime factors of a^2. However, using the uniqueness part of the Fundamental Theorem of Arithmetic, we realise that the only prime factors of a^2 are $p_1, p_2, \ldots, p_n$. So p is one of $p_1, p_2, \ldots, p_n$.

Now, since $a = p_1 p_2 \ldots p_n$, p divides a. ■

We are now ready to give a proof that $\sqrt{2}$ is irrational.

The proof is based on a technique called 'proof by contradiction'. (This technique is discussed in some detail in Appendix 1).

Theorem 1.4 : $\sqrt{2}$ *is irrational.*

Proof : Let us assume, to the contrary, that $\sqrt{2}$ is rational.

So, we can find integers r and s ($\neq 0$) such that $\sqrt{2} = \frac{r}{s}$.

Suppose r and s have a common factor other than 1. Then, we divide by the common factor to get $\sqrt{2} = \frac{a}{b}$, where a and b are coprime.

So, $b\sqrt{2} = a$.

Squaring on both sides and rearranging, we get $2b^2 = a^2$. Therefore, 2 divides a^2.

Now, by Theorem 1.3, it follows that 2 divides a.

So, we can write $a = 2c$ for some integer c.

* Not from the examination point of view.

Substituting for a, we get $2b^2 = 4c^2$, that is, $b^2 = 2c^2$.

This means that 2 divides b^2, and so 2 divides b (again using Theorem 1.3 with $p = 2$).

Therefore, a and b have at least 2 as a common factor.

But this contradicts the fact that a and b have no common factors other than 1.

This contradiction has arisen because of our incorrect assumption that $\sqrt{2}$ is rational.

So, we conclude that $\sqrt{2}$ is irrational. ■

Example 9 : Prove that $\sqrt{3}$ is irrational.

Solution : Let us assume, to the contrary, that $\sqrt{3}$ is rational.

That is, we can find integers a and b ($\neq 0$) such that $\sqrt{3} = \frac{a}{b}$.

Suppose a and b have a common factor other than 1, then we can divide by the common factor, and assume that a and b are coprime.

So, $b\sqrt{3} = a$.

Squaring on both sides, and rearranging, we get $3b^2 = a^2$.

Therefore, a^2 is divisible by 3, and by Theorem 1.3, it follows that a is also divisible by 3.

So, we can write $a = 3c$ for some integer c.

Substituting for a, we get $3b^2 = 9c^2$, that is, $b^2 = 3c^2$.

This means that b^2 is divisible by 3, and so b is also divisible by 3 (using Theorem 1.3 with $p = 3$).

Therefore, a and b have at least 3 as a common factor.

But this contradicts the fact that a and b are coprime.

This contradiction has arisen because of our incorrect assumption that $\sqrt{3}$ is rational.

So, we conclude that $\sqrt{3}$ is irrational.

In Class IX, we mentioned that :

- the sum or difference of a rational and an irrational number is irrational and
- the product and quotient of a non-zero rational and irrational number is irrational.

We prove some particular cases here.

Example 10 : Show that $5 - \sqrt{3}$ is irrational.

Solution : Let us assume, to the contrary, that $5 - \sqrt{3}$ is rational.

That is, we can find coprime a and b ($b \neq 0$) such that $5 - \sqrt{3} = \frac{a}{b}$.

Therefore, $5 - \frac{a}{b} = \sqrt{3}$.

Rearranging this equation, we get $\sqrt{3} = 5 - \frac{a}{b} = \frac{5b - a}{b}$.

Since a and b are integers, we get $5 - \frac{a}{b}$ is rational, and so $\sqrt{3}$ is rational.

But this contradicts the fact that $\sqrt{3}$ is irrational.

This contradiction has arisen because of our incorrect assumption that $5 - \sqrt{3}$ is rational.

So, we conclude that $5 - \sqrt{3}$ is irrational.

Example 11 : Show that $3\sqrt{2}$ is irrational.

Solution : Let us assume, to the contrary, that $3\sqrt{2}$ is rational.

That is, we can find coprime a and b ($b \neq 0$) such that $3\sqrt{2} = \frac{a}{b}$.

Rearranging, we get $\sqrt{2} = \frac{a}{3b}$.

Since 3, a and b are integers, $\frac{a}{3b}$ is rational, and so $\sqrt{2}$ is rational.

But this contradicts the fact that $\sqrt{2}$ is irrational.

So, we conclude that $3\sqrt{2}$ is irrational.

EXERCISE 1.3

1. Prove that $\sqrt{5}$ is irrational.

2. Prove that $3 + 2\sqrt{5}$ is irrational.

3. Prove that the following are irrationals :

(i) $\frac{1}{\sqrt{2}}$ (ii) $7\sqrt{5}$ (iii) $6 + \sqrt{2}$

1.5 Revisiting Rational Numbers and Their Decimal Expansions

In Class IX, you studied that rational numbers have either a terminating decimal expansion or a non-terminating repeating decimal expansion. In this section, we are going to consider a rational number, say $\frac{p}{q}(q \neq 0)$, and explore exactly when the decimal expansion of $\frac{p}{q}$ is terminating and when it is non-terminating repeating (or recurring). We do so by considering several examples.

Let us consider the following rational numbers :

(i) 0.375 (ii) 0.104 (iii) 0.0875 (iv) 23.3408.

Now (i) $0.375 = \frac{375}{1000} = \frac{375}{10^3}$ (ii) $0.104 = \frac{104}{1000} = \frac{104}{10^3}$

(iii) $0.0875 = \frac{875}{10000} = \frac{875}{10^4}$ (iv) $23.3408 = \frac{233408}{10000} = \frac{233408}{10^4}$

As one would expect, they can all be expressed as rational numbers whose denominators are powers of 10. Let us try and cancel the common factors between the numerator and denominator and see what we get :

(i) $0.375 = \frac{375}{10^3} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{3}{2^3}$ (ii) $0.104 = \frac{104}{10^3} = \frac{13 \times 2^3}{2^3 \times 5^3} = \frac{13}{5^3}$

(iii) $0.0875 = \frac{875}{10^4} = \frac{7}{2^4 \times 5}$ (iv) $23.3408 = \frac{233408}{10^4} = \frac{2^2 \times 7 \times 521}{5^4}$

Do you see any pattern? It appears that, we have converted a real number whose decimal expansion terminates into a rational number of the form $\frac{p}{q}$, where p and q are coprime, and the prime factorisation of the denominator (that is, q) has only powers of 2, or powers of 5, or both. We should expect the denominator to look like this, since powers of 10 can only have powers of 2 and 5 as factors.

Even though, we have worked only with a few examples, you can see that any real number which has a decimal expansion that terminates can be expressed as a rational number whose denominator is a power of 10. Also the only prime factors of 10 are 2 and 5. So, cancelling out the common factors between the numerator and the denominator, we find that this real number is a rational number of the form $\frac{p}{q}$, where the prime factorisation of q is of the form 2^n5^m, and n, m are some non-negative integers.

Let us write our result formally:

Theorem 1.5 : *Let x be a rational number whose decimal expansion terminates. Then x can be expressed in the form $\frac{p}{q}$, where p and q are coprime, and the prime factorisation of q is of the form 2^n5^m, where n, m are non-negative integers.*

You are probably wondering what happens the other way round in Theorem 1.5. That is, if we have a rational number of the form $\frac{p}{q}$, and the prime factorisation of q is of the form 2^n5^m, where n, m are non negative integers, then does $\frac{p}{q}$ have a terminating decimal expansion?

Let us see if there is some obvious reason why this is true. You will surely agree that any rational number of the form $\frac{a}{b}$, where b is a power of 10, will have a terminating decimal expansion. So it seems to make sense to convert a rational number of the form $\frac{p}{q}$, where q is of the form 2^n5^m, to an equivalent rational number of the form $\frac{a}{b}$, where b is a power of 10. Let us go back to our examples above and work backwards.

(i) $\frac{3}{8} = \frac{3}{2^3} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{375}{10^3} = 0.375$

(ii) $\frac{13}{125} = \frac{13}{5^3} = \frac{13 \times 2^3}{2^3 \times 5^3} = \frac{104}{10^3} = 0.104$

(iii) $\frac{7}{80} = \frac{7}{2^4 \times 5} = \frac{7 \times 5^3}{2^4 \times 5^4} = \frac{875}{10^4} = 0.0875$

(iv) $\frac{14588}{625} = \frac{2^2 \times 7 \times 521}{5^4} = \frac{2^6 \times 7 \times 521}{2^4 \times 5^4} = \frac{233408}{10^4} = 23.3408$

So, these examples show us how we can convert a rational number of the form $\frac{p}{q}$, where q is of the form 2^n5^m, to an equivalent rational number of the form $\frac{a}{b}$, where b is a power of 10. Therefore, the decimal expansion of such a rational number terminates. Let us write down our result formally.

Theorem 1.6 : *Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of q is of the form 2^n5^m, where n, m are non-negative integers. Then x has a decimal expansion which terminates.*

We are now ready to move on to the rational numbers whose decimal expansions are non-terminating and recurring. Once again, let us look at an example to see what is going on. We refer to Example 5, Chapter 1, from your Class IX textbook, namely, $\frac{1}{7}$. Here, remainders are 3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5, 1, . . . and divisor is 7.

$$
\begin{array}{r|l}
 & 0.1428571 \\
\hline
7 & 10 \\
 & 7 \\
\hline
 & 30 \\
 & 28 \\
\hline
 & 20 \\
 & 14 \\
\hline
 & 60 \\
 & 56 \\
\hline
 & 40 \\
 & 35 \\
\hline
 & 50 \\
 & 49 \\
\hline
 & 10 \\
 & 7 \\
\hline
 & 30
\end{array}
$$

Notice that the denominator here, i.e., 7 is clearly not of the form 2^n5^m. Therefore, from Theorems 1.5 and 1.6, we know that $\frac{1}{7}$ will not have a terminating decimal expansion. Hence, 0 will not show up as a remainder (Why?), and the remainders will start repeating after a certain stage. So, we will have a block of digits, namely, 142857, repeating in the quotient of $\frac{1}{7}$.

What we have seen, in the case of $\frac{1}{7}$, is true for any rational number not covered by Theorems 1.5 and 1.6. For such numbers we have :

Theorem 1.7 : *Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of q is not of the form 2^n5^m, where n, m are non-negative integers. Then, x has a decimal expansion which is non-terminating repeating (recurring).*

From the discussion above, we can conclude that *the decimal expansion of every rational number is either terminating or non-terminating repeating.*

EXERCISE 1.4

1. Without actually performing the long division, state whether the following rational numbers will have a terminating decimal expansion or a non-terminating repeating decimal expansion:

(i) $\frac{13}{3125}$ (ii) $\frac{17}{8}$ (iii) $\frac{64}{455}$ (iv) $\frac{15}{1600}$

(v) $\frac{29}{343}$ (vi) $\frac{23}{2^3 5^2}$ (vii) $\frac{129}{2^2 5^7 7^5}$ (viii) $\frac{6}{15}$

(ix) $\frac{35}{50}$ (x) $\frac{77}{210}$

2. Write down the decimal expansions of those rational numbers in Question 1 above which have terminating decimal expansions.

3. The following real numbers have decimal expansions as given below. In each case, decide whether they are rational or not. If they are rational, and of the form $\frac{p}{q}$, what can you say about the prime factors of q?

(i) 43.123456789 (ii) 0.120120012000120000... (iii) $43.\overline{123456789}$

1.6 Summary

In this chapter, you have studied the following points:

1. Euclid's division lemma :

Given positive integers a and b, there exist whole numbers q and r satisfying $a = bq + r$, $0 \le r < b$.

2. Euclid's division algorithm : This is based on Euclid's division lemma. According to this, the HCF of any two positive integers a and b, with $a > b$, is obtained as follows:

Step 1 : Apply the division lemma to find q and r where $a = bq + r$, $0 \le r < b$.

Step 2 : If $r = 0$, the HCF is b. If $r \neq 0$, apply Euclid's lemma to b and r.

Step 3 : Continue the process till the remainder is zero. The divisor at this stage will be HCF (a, b). Also, HCF(a, b) = HCF(b, r).

3. The Fundamental Theorem of Arithmetic :

Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique, apart from the order in which the prime factors occur.

4. If p is a prime and p divides a^2, then p divides a, where a is a positive integer.

5. To prove that $\sqrt{2}, \sqrt{3}$ are irrationals.

6. Let x be a rational number whose decimal expansion terminates. Then we can express x in the form $\frac{p}{q}$, where p and q are coprime, and the prime factorisation of q is of the form 2^n5^m, where n, m are non-negative integers.

7. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of q is of the form 2^n5^m, where n, m are non-negative integers. Then x has a decimal expansion which terminates.

8. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of q is not of the form $2^n 5^m$, where n, m are non-negative integers. Then x has a decimal expansion which is non-terminating repeating (recurring).

A NOTE TO THE READER

You have seen that :

HCF (p, q, r) × LCM (p, q, r) ≠ $p \times q \times r$, where p, q, r are positive integers (see Example 8). However, the following results hold good for three numbers p, q and r :

$$\text{LCM } (p, q, r) = \frac{p \cdot q \cdot r \cdot \text{HCF}(p, q, r)}{\text{HCF}(p, q) \cdot \text{HCF}(q, r) \cdot \text{HCF}(p, r)}$$

$$\text{HCF } (p, q, r) = \frac{p \cdot q \cdot r \cdot \text{LCM}(p, q, r)}{\text{LCM}(p, q) \cdot \text{LCM}(q, r) \cdot \text{LCM}(p, r)}$$