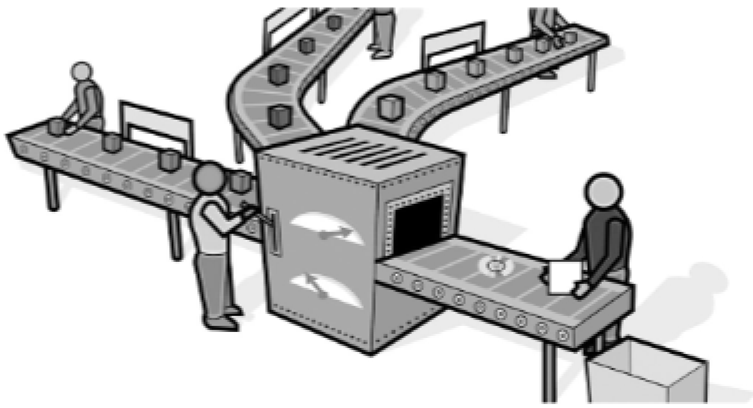
CHAPTER

4 Determinants

CASE STUDY / PASSAGE BASED QUESTIONS

1

A company produces three products every day. Their production on certain day is 45 tons. It is found that the production of third product exceeds the production of first product by 8 tons while the total production of first and third product is twice the production of second product.



Syllabus

Determinant of a square matrix (up to 3 x 3 matrices), minors, co-factors and applications of determinants in finding the area of a triangle. Adjoint and inverse of a square matrix. Solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix.

Using the concepts of matrices and determinants, answer the following questions.

(i) If x, y and z respectively denotes the quantity (in tons) of first, second and third product produced, then which of the following is true?

(a) $x + y + z = 45$ (b) $x + 8 = z$ (c) $x - 2y + z = 0$ (d) all of these

(ii) If $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 2 & 2 \\ 3 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}$, then the inverse of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}$ is

(a) $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{-1}{2} \\ \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \end{pmatrix}$ (c) $\begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{2} & \frac{1}{6} \end{pmatrix}$ (d) none of these

(iii) $x : y : z$ is equal to

(a) 12 : 13 : 20 (b) 11 : 15 : 19 (c) 15 : 19 : 11 (d) 13 : 12 : 20

(iv) Which of the following is not true?

(a) $|A| = |A'|$

(b) $(A')^{-1} = (A^{-1})'$

(c) A is skew symmetric matrix of odd order, then $|A| = 0$

(d) $|AB| = |A| + |B|$

(v) Which of the following is not true in the given determinant of A, where $A = [a_{ij}]_{3 \times 3}$?

(a) Order of minor is less than order of the det (A).

(b) Minor of an element can never be equal to cofactor of the same element.

(c) Value of a determinant is obtained by multiplying elements of a row or column by corresponding cofactors.

(d) Order of minors and cofactors of same elements of A is same.

2

If there is a statement involving the natural number n such that

(i) The statement is true for $n = 1$

(ii) When the statement is true for $n = k$ (where k is some positive integer), then the statement is also true for $n = k + 1$.

Then, the statement is true for all natural numbers n.

Also, if A is a square matrix of order n, then A^2 is defined as AA. In general, $A^m = AA \dots A$ (m times), where m is any positive integer.

Based on the above information, answer the following questions.

(i) If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then for any positive integer n,

(a) $A^n = \begin{bmatrix} 3n & -4n \\ n & -n \end{bmatrix}$ (b) $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ (c) $A^n = \begin{bmatrix} 3n & -8n \\ 1 & -n \end{bmatrix}$ (d) $A^n = \begin{bmatrix} 1+3n & -4n \\ n & 1-3n \end{bmatrix}$

(ii) If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, then $|A^n|$, where $n \in N$, is equal to

(a) 2^n (b) 3^n (c) n (d) 1

(iii) If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then which of the following holds for all natural numbers $n \geq 1$?

(a) $A^n = nA - (n-1)I$ (b) $A^n = 2^{n-1}A - (n-1)I$

(c) $A^n = nA + (n-1)I$ (d) $A^n = 2^{n-1}A + (n-1)I$

(iv) Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ and $A^n = [a_{ij}]_{3 \times 3}$ for some positive integer n, then the cofactor of a_{13} is

(a) a^n (b) $-a^n$ (c) $2a^n$ (d) 0

(v) If A is a square matrix such that $|A| = 2$, then for any positive integer n, $|A^n|$ is equal to

(a) 0 (b) $2n$ (c) 2^n (d) n^2

3

Each triangular face of the Pyramid of Peace in Kazakhstan is made up of 25 smaller equilateral triangles as shown in the figure.

Using the above information and concept of determinants, answer the following questions.

(i) If the vertices of one of the smaller equilateral triangle are $(0, 0)$, $(3, \sqrt{3})$ and $(3, -\sqrt{3})$, then the area of such triangle is

(a) $\sqrt{3}$ sq. units (b) $2\sqrt{3}$ sq. units (c) $3\sqrt{3}$ sq. units (d) none of these

(ii) The area of a face of the Pyramid is

(a) $25\sqrt{3}$ sq. units (b) $50\sqrt{3}$ sq. units (c) $75\sqrt{3}$ sq. units (d) $35\sqrt{3}$ sq. units

(iii) The length of a altitude of a smaller equilateral triangle is

(a) 2 units (b) 3 units (c) $\sqrt{3}$ units (d) 4 units

(iv) If (2, 4), (2, 6) are two vertices of a smaller equilateral triangle, then the third vertex will lie on the line represented by

(a) $x + y = 5$ (b) $x = 1 + \sqrt{3}$ (c) $x = 2 + \sqrt{3}$ (d) $2x + y = 5$

(v) Let $A(a, 0)$, $B(0, b)$ and $C(1, 1)$ be three points. If $\frac{1}{a} + \frac{1}{b} = 1$, then the three points are

(a) vertices of an equilateral triangle (b) vertices of a right angled triangle
(c) collinear (d) vertices of an isosceles triangle

4

Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, and U_1, U_2 are first and second columns respectively of a 2×2 matrix U. Also, let the column matrices U_1 and U_2 satisfying $AU_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $AU_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Based on the above information, answer the following questions.

(i) The matrix $U_1 + U_2$ is equal to

(a) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$ (d) $\begin{bmatrix} 4 \\ -4 \end{bmatrix}$

(ii) The value of $|U|$ is

(a) 2 (b) –2 (c) 3 (d) –3

(iii) If $X = \begin{bmatrix} 3 & 2 \end{bmatrix} U \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then the value of $|X| =$

(a) 3 (b) –3 (c) –5 (d) 5

(iv) The minor of element at the position a_{22} in U is

(a) 1 (b) 2 (c) –2 (d) –1

(v) If $U = [a_{ij}]_{2 \times 2}$, then the value of $a_{11}A_{11} + a_{12}A_{12}$, where A_{ij} denotes the cofactor of a_{ij}, is

(a) 1 (b) 2 (c) –3 (d) 3

5

The upward speed $v(t)$ of a rocket at time t is approximated by $v(t) = at^2 + bt + c$, $0 \le t \le 100$, where a, b and c are constants. It has been found that the speed at times $t = 3$, $t = 6$ and $t = 9$ seconds are respectively 64, 133 and 208 miles per second.

If $\begin{pmatrix} 9 & 3 & 1 \\ 36 & 6 & 1 \\ 81 & 9 & 1 \end{pmatrix}^{-1} = \frac{1}{18}\begin{pmatrix} 1 & -2 & 1 \\ -15 & 24 & -9 \\ 54 & -54 & 18 \end{pmatrix}$, then answer the following questions.

(i) The value of $b + c$ is

(a) 20 (b) 21 (c) 3/4 (d) 4/3

(ii) The value of $a + c$ is

(a) 1 (b) 20 (c) 4/3 (d) none of these

(iii) $v(t)$ is given by

(a) $t^2 + 20t + 1$ (b) $\frac{1}{3}t^2 + 20t + 1$ (c) $t^2 + \frac{1}{3}t + 20$ (d) $t^2 + t + 1$

(iv) The speed at time $t = 15$ seconds is

(a) 346 miles/sec (b) 356 miles/sec (c) 366 miles/sec (d) 376 miles/sec

(v) The time at which the speed of rocket is 784 miles/sec is

(a) 20 seconds (b) 30 seconds (c) 25 seconds (d) 27 seconds

6

Two schools A and B want to award their selected students on the values of Honesty, Hard work and Punctuality. The school A wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to its 3, 2 and 1 students respectively with a total award money of ₹ 2200. School B wants to spend ₹ 3100 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as school A). The total amount of award for one prize on each value is ₹ 1200.

Using the concept of matrices and determinants, answer the following questions.

(i) What is the award money for Honesty?

(a) ₹ 350 (b) ₹ 300 (c) ₹ 500 (d) ₹ 400

(ii) What is the award money for Punctuality?

(a) ₹ 300 (b) ₹ 280 (c) ₹ 450 (d) ₹ 500

(iii) What is the award money for Hard work?

(a) ₹ 500 (b) ₹ 400 (c) ₹ 300 (d) ₹ 550

(iv) If a matrix P is both symmetric and skew-symmetric, then $|P|$ is equal to

(a) 1 (b) –1 (c) 0 (d) none of these

(v) If P and Q are two matrices such that $PQ = Q$ and $QP = P$, then $|Q^2|$ is equal to

(a) $|Q|$ (b) $|P|$ (c) 1 (d) 0

7

Three shopkeepers Salim, Vijay and Venket are using polythene bags, handmade bags (prepared by prisoners) and newspaper's envelope as carry bags. It is found that the shopkeepers Salim, Vijay and Venket are using (20, 30, 40), (30, 40, 20) and (40, 20, 30) polythene bags, handmade bags and newspaper's envelopes respectively. The shopkeepers Salim, Vijay and Venket spent ₹ 250, ₹ 270 and ₹ 200 on these carry bags respectively.

Using the concept of matrices and determinants, answer the following questions.

(i) What is the cost of one polythene bag?

(a) ₹ 1 (b) ₹ 2 (c) ₹ 3 (d) ₹ 5

(ii) What is the cost of one handmade bag?

(a) ₹ 1 (b) ₹ 2 (c) ₹ 3 (d) ₹ 5

(iii) What is the cost of one newspaper envelope?

(a) ₹ 1 (b) ₹ 2 (c) ₹ 3 (d) ₹ 5

(iv) Keeping in mind the social conditions, which shopkeeper is better?

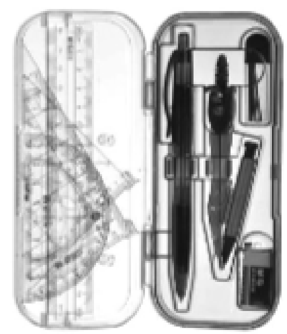
(a) Salim (b) Vijay (c) Venket (d) None of these

(v) Keeping in mind the environmental conditions, which shopkeeper is better?

(a) Salim (b) Vijay (c) Venket (d) None of these

8

Gaurav purchased 5 pens, 3 bags and 1 instrument box and pays ₹ 16. From the same shop, Dheeraj purchased 2 pens, 1 bag and 3 instrument boxes and pays ₹ 19, while Ankur purchased 1 pen, 2 bags and 4 instrument boxes and pays ₹ 25.



Using the concept of matrices and determinants, answer the following questions.

(i) The cost of one pen is

(a) ₹ 2 (b) ₹ 5 (c) ₹ 1 (d) ₹ 3

(ii) What is the cost of one pen and one bag?

(a) ₹ 3 (b) ₹ 5 (c) ₹ 7 (d) ₹ 8

(iii) What is the cost of one pen and one instrument box?

(a) ₹ 7 (b) ₹ 6 (c) ₹ 8 (d) ₹ 9

(iv) Which of the following is correct?

(a) Determinant is a square matrix.

(b) Determinant is a number associated to a matrix.

(c) Determinant is a number associated to a square matrix.

(d) All of the above

(v) From the matrix equation $AB = AC$, it can be concluded that $B = C$ provided

(a) A is singular (b) A is non-singular

(c) A is symmetric (d) A is square.

9

Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by the determinant

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Since, area is a positive quantity, so we always take the absolute value of the determinant Δ. Also, the area of the triangle formed by three collinear points is zero.
Based on the above information, answer the following questions.

(i) Find the area of the triangle whose vertices are $(-2, 6)$, $(3, -6)$ and $(1, 5)$.

(a) 30 sq. units (b) 35 sq. units (c) 40 sq. units (d) 15.5 sq. units

(ii) If the points $(2, -3)$, $(k, -1)$ and $(0, 4)$ are collinear, then find the value of $4k$.

(a) 4 (b) $\frac{7}{140}$ (c) 47 (d) $\frac{40}{7}$

(iii) If the area of a triangle ABC, with vertices $A(1, 3)$, $B(0, 0)$ and $C(k, 0)$ is 3 sq. units, then a value of k is

(a) 2 (b) 3 (c) 4 (d) 5

(iv) Using determinants, find the equation of the line joining the points $A(1, 2)$ and $B(3, 6)$.

(a) $y = 2x$ (b) $x = 3y$ (c) $y = x$ (d) $4x - y = 5$

(v) If $A \equiv (11, 7)$, $B \equiv (5, 5)$ and $C \equiv (-1, 3)$, then

(a) ΔABC is scalene triangle (b) ΔABC is equilateral triangle
(c) A, B and C are collinear (d) None of these

10

Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its i^{th} row and j^{th} column in which element a_{ij} lies and is denoted by M_{ij}.
Cofactor of an element a_{ij}, denoted by A_{ij}, is defined by $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is minor of a_{ij}.
Also, the determinant of a square matrix A is the sum of the products of the elements of any row (or column) with their corresponding cofactors. For example, if $A = [a_{ij}]_{3 \times 3}$, then $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.
Based on the above information, answer the following questions.

(i) Find the sum of the cofactors of all the elements of $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$.

(a) 2 (b) -2 (c) 4 (d) 1

(ii) Find the minor of a_{21} of $\begin{vmatrix} 5 & 6 & -3 \\ -4 & 3 & 2 \\ -4 & -7 & 3 \end{vmatrix}$.

(a) 3 (b) -3 (c) 39 (d) -39

(iii) In the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, find the value of $a_{32} \cdot A_{32}$.

(a) 27 (b) –110 (c) 110 (d) –27

(iv) If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, then write the minor of a_{23}.

(a) –10 (b) –7 (c) 10 (d) 7

(v) If $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then find the value of $|\Delta|$.

(a) 26 (b) 28 (c) 72 (d) 46

HINTS & EXPLANATIONS

1. (i) (d) : According to given condition, we have the following system of linear equations.

$x + y + z = 45$

$x + 8 = z$ or $x + 0 \cdot y - z = -8$

and $x + z = 2y$ or $x - 2y + z = 0$

(ii) (c) : Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$ then we have,

$$A^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 2 & 2 \\ 3 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

Now, $(A')^{-1} = (A^{-1})'$

$$= \frac{1}{6}\begin{pmatrix} 2 & 3 & 1 \\ 2 & 0 & -2 \\ 2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & 0 & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{2} & \frac{1}{6} \end{pmatrix}$$

(iii) (b) : The above system of equations can be written in matrix form as

$A'X = B$, where,

$$A' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 45 \\ -8 \\ 0 \end{bmatrix}$$

$$\Rightarrow X = (A')^{-1}B = \frac{1}{6}\begin{bmatrix} 2 & 3 & 1 \\ 2 & 0 & -2 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 45 \\ -8 \\ 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 90-24 \\ 90 \\ 90+24 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 66 \\ 90 \\ 114 \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \\ 19 \end{bmatrix}$$

Thus, $x : y : z = 11 : 15 : 19$

(iv) (d) : Clearly, $|AB| = |A| \cdot |B|$

(v) (b)

2. (i) (b) : We have, $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$\therefore A^2 = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$, which can be obtained from $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ for $n = 2$.

(ii) (d) : We have, $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

Also, $|A^n| = |A \cdot A \ldots\ldots A(n \text{ times})| = |A|^n = 1^n = 1$

(iii) (a) : For $n = 1$, all options are true.

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

and $A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

Putting $n = 3$, in (a), we get $A^3 = 3A - 2I$

$$= 3\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \text{ which is true.}$$

All other options are different from $A^3 = 3A - 2I$ for $n = 3$.

(iv) (d) : We have, $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

$$\therefore \quad A^2 = A \cdot A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

$$= \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix}$$

Similarly, $A^n = \begin{bmatrix} a^n & 0 & 0 \\ 0 & a^n & 0 \\ 0 & 0 & a^n \end{bmatrix}$

Now, cofactor of $a_{13} = (-1)^{1+3}\begin{vmatrix} 0 & a^n \\ 0 & 0 \end{vmatrix} = 0$

(v) (c) : We have, $|A| = 2$

and $|A^n| = |A \cdot A \;.....\; A(n\text{-times})|$

$= |A|\;|A| \;.....\; |A|(n\text{-times}) = |A|^n = 2^n$

3. (i) (c) : Area of triangle is given by $\left|\dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$.

$$\therefore \quad \text{Required area} = \left|\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 3 & \sqrt{3} & 1 \\ 3 & -\sqrt{3} & 1 \end{vmatrix}\right|$$

$$= \left|\frac{1}{2}[1(-3\sqrt{3} - 3\sqrt{3})]\right| \quad [\text{Expanding along } R_1]$$

$= 3\sqrt{3}$ sq. units

(ii) (c) : Since a face of the Pyramid consist of 25 smaller equilateral triangles.

$\therefore$ Required area $= 25 \times 3\sqrt{3} = 75\sqrt{3}$ sq. units

(iii) (b) : Area of equilateral triangle $= \dfrac{\sqrt{3}}{4}a^2$

$$\therefore \; 3\sqrt{3} = \frac{\sqrt{3}}{4}a^2 \Rightarrow a^2 = 12 \Rightarrow a = 2\sqrt{3}$$

Let h be the length of altitude of a smaller equilateral triangle. Then,

$$\frac{1}{2} \times \text{base} \times \text{height} = 3\sqrt{3}$$

$$\Rightarrow \frac{1}{2} \times 2\sqrt{3} \times h = 3\sqrt{3} \Rightarrow h = 3 \text{ units}$$

(iv) (c) : Let the third vertex be (x, y), then we get

$$\frac{1}{2}\begin{vmatrix} 2 & 4 & 1 \\ 2 & 6 & 1 \\ x & y & 1 \end{vmatrix} = \pm 3\sqrt{3}$$

$$\Rightarrow \frac{1}{2}[2(6-y) - 4(2-x) + 1(2y - 6x)] = \pm 3\sqrt{3}$$

$$\Rightarrow 12 - 2y - 8 + 4x + 2y - 6x = \pm 6\sqrt{3}$$

$$\Rightarrow 4 - 2x = \pm 6\sqrt{3}$$

$$\Rightarrow 2 - x = \pm 3\sqrt{3} \Rightarrow x = 2 \pm 3\sqrt{3}$$

(v) (c) : Area of $\Delta ABC = \dfrac{1}{2}\begin{vmatrix} a & 0 & 1 \\ 0 & b & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$$= \frac{1}{2}[a(b-1) - 0 + 1(0 - b)] = \frac{1}{2}(ab - a - b) = 0$$

$$\left[\because \frac{1}{a} + \frac{1}{b} = 1 \Rightarrow b + a = ab\right]$$

$\therefore$ Points A, B and C are collinear.

4. (i) (c) : We have, $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Let $U_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ then $AU_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ 2a + b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\Rightarrow a = 1$ and $2a + b = 0 \Rightarrow a = 1$ and $b = -2$

Let $U_2 = \begin{bmatrix} c \\ d \end{bmatrix}$ then $AU_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} c \\ 2c+d \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\Rightarrow$ $c = 2$ and $2c + d = 3$

$\Rightarrow$ $c = 2$ and $d = 3 - 4 = -1$

Thus, $U_1 + U_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

(ii) **(c)** : Clearly, $U = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

$\therefore \quad |U| = \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = -1 + 4 = 3$

(iii) **(d)** : We have, $X = \begin{bmatrix} 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$= [3 \quad 2]\begin{bmatrix} 7 \\ -8 \end{bmatrix} = [21 - 16] = [5]$

$\therefore \quad |X| = 5.$

(iv) **(a)** : a_{22} in U is -1 and its minor is 1.

(v) **(d)** : Since, the sum of products of elements of any row (or column) with their corresponding cofactors is equal to the value of determinant.

$\therefore \quad a_{11}A_{11} + a_{12}A_{12} = |U| = 3$

5. Since $v(3) = 64$, $v(6) = 133$ and $v(9) = 208$, we get the following system of linear equations

$9a + 3b + c = 64$

$36a + 6b + c = 133$

$81a + 9b + c = 208$

This can be written in matrix form as

$$\begin{bmatrix} 9 & 3 & 1 \\ 36 & 6 & 1 \\ 81 & 9 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 64 \\ 133 \\ 208 \end{bmatrix}$$

or $AX = B$

Since, $A^{-1} = \frac{1}{18}\begin{pmatrix} 1 & -2 & 1 \\ -15 & 24 & -9 \\ 54 & -54 & 18 \end{pmatrix}$

$\therefore \quad X = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^{-1}B = \frac{1}{18}\begin{pmatrix} 1 & -2 & 1 \\ -15 & 24 & -9 \\ 54 & -54 & 18 \end{pmatrix}\begin{pmatrix} 64 \\ 133 \\ 208 \end{pmatrix}$

$= \frac{1}{18}\begin{pmatrix} 64 - 266 + 208 \\ -960 + 3192 - 1872 \\ 3456 - 7182 + 3744 \end{pmatrix} = \frac{1}{18}\begin{pmatrix} 6 \\ 360 \\ 18 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 20 \\ 1 \end{pmatrix}$

Thus, $a = \frac{1}{3}$, $b = 20$ and $c = 1$

(i) **(b)** **(ii)** **(c)**

(iii) **(b)** : $v(t) = \frac{1}{3}t^2 + 20t + 1$

(iv) **(d)** : Clearly, required speed = $v(15)$

$= \frac{1}{3} \times 225 + 20 \times 15 + 1$

$= 75 + 300 + 1 = 376$ miles per second

(v) **(d)** : Consider, $v(t) = 784$

$\Rightarrow \frac{1}{3}t^2 + 20t + 1 = 784$

$\Rightarrow t^2 + 60t = 2349$

$\Rightarrow t^2 + (87 - 27)t - 2349 = 0$

$\Rightarrow t(t + 87) - 27(t + 87) = 0$

$\Rightarrow (t - 27)(t + 87) = 0$

$\Rightarrow t = 27$ seconds [$\because$ Time can't be negative]

6. Three equations are formed from the given statements :

$3x + 2y + z = 2200$

$4x + y + 3z = 3100$

and $x + y + z = 1200$

Converting the system of equations in matrix form, we get

$$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2200 \\ 3100 \\ 1200 \end{bmatrix}$$

i.e., $PX = Q$,

where $P = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $Q = \begin{bmatrix} 2200 \\ 3100 \\ 1200 \end{bmatrix}$

$|P| = 3(1 - 3) - 2(4 - 3) + 1(4 - 1) = -6 - 2 + 3 = -5 \neq 0$

$\Rightarrow X = P^{-1}Q$, provided P^{-1} exists.

$\therefore \quad \text{adj } P = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$

$\therefore \quad P^{-1} = \frac{1}{|P|}(\text{adj } P)$

$= \frac{1}{-5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 1 & -5 \\ 1 & -2 & 5 \\ -3 & 1 & 5 \end{bmatrix}$

$$\therefore \quad X = \frac{1}{5}\begin{bmatrix} 2 & 1 & -5 \\ 1 & -2 & 5 \\ -3 & 1 & 5 \end{bmatrix}\begin{bmatrix} 2200 \\ 3100 \\ 1200 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 4400+3100-6000 \\ 2200-6200+6000 \\ -6600+3100+6000 \end{bmatrix} = \begin{bmatrix} 300 \\ 400 \\ 500 \end{bmatrix}$$

$\Rightarrow \quad x = 300, y = 400$ and $z = 500$

Hence the money awarded for Honesty, Hardwork and Punctuality are ₹ 300, ₹ 400 and ₹ 500 respectively.

(i) **(b)** **(ii)** **(d)** **(iii)** **(b)**

(iv) **(c)** : If a matrix P is both symmetric and skew-symmetric then it will be a zero matrix. So, $|P| = 0$.

(v) **(a)** : We have, $Q^2 = QQ = Q(PQ)$

$= (QP)\, Q = PQ = Q$

$\therefore \quad |Q^2| = |Q|$

7. Let the cost of a polythene bag = ₹ x,

the cost of a handmade bag = ₹ y

and the cost of a newspaper bag = ₹ z

According to question,

$20x + 30y + 40z = 250$, $30x + 40y + 20z = 270$

$40x + 20y + 30z = 200$

This system can be written as $AX = B$, where

$$A = \begin{bmatrix} 20 & 30 & 40 \\ 30 & 40 & 20 \\ 40 & 20 & 30 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 250 \\ 270 \\ 200 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 20 & 30 & 40 \\ 30 & 40 & 20 \\ 40 & 20 & 30 \end{vmatrix}$$

$= 20(1200 - 400) - 30(900 - 800) + 40(600 - 1600)$

$= 20(800) - 30(100) + 40(-1000)$

$= 16000 - 3000 - 40000 = -27000 \neq 0$

So, A^{-1} exists and system has a solution given by $X = A^{-1}B$.

$$\text{Now, adj}\, A = \begin{bmatrix} 800 & -100 & -1000 \\ -100 & -1000 & 800 \\ -1000 & 800 & -100 \end{bmatrix}'$$

$$= \begin{bmatrix} 800 & -100 & -1000 \\ -100 & -1000 & 800 \\ -1000 & 800 & -100 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{|A|}(\text{adj}\, A) = \frac{1}{-27000}\begin{bmatrix} 800 & -100 & -1000 \\ -100 & -1000 & 800 \\ -1000 & 800 & -100 \end{bmatrix}$$

$$\text{Now } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{27000}\begin{bmatrix} -800 & 100 & 1000 \\ 100 & 1000 & -800 \\ 1000 & -800 & 100 \end{bmatrix}\begin{bmatrix} 250 \\ 270 \\ 200 \end{bmatrix}$$

$$= \frac{1}{27000}\begin{bmatrix} 27000 \\ 135000 \\ 54000 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \Rightarrow x = 1, y = 5, z = 2$$

Hence, cost of a polythene bag, a handmade bag and a newspaper envelope is ₹ 1, ₹ 5 and ₹ 2 respectively.

(i) **(a)** **(ii)** **(d)** **(iii)** **(b)**

(iv) **(b)** : Vijay investing most of the money on hand -made bags.

(v) **(a)** : Salim investing less amount of money on polythene bags.

8. Let the cost of 1 pen = ₹ x,

the cost of 1 bag = ₹ y,

and the cost of 1 instrument box = ₹ z

According to the question, we have

$5x + 3y + z = 16$, $2x + y + 3z = 19$, $x + 2y + 4z = 25$

This system of equation can be written as $AX = B$,

$$\text{where } A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix},\ B = \begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$|A| = 5(4 - 6) - 3(8 - 3) + 1(4 - 1)$

$= -10 - 3(5) + 3 = -22 \neq 0$

$\therefore \quad A^{-1}$ exists.

Now, $X = A^{-1}B$, where $A^{-1} = \frac{1}{|A|}\text{adj}\, A$.

$$\text{Here, adj}\, A = \begin{bmatrix} -2 & -5 & 3 \\ -10 & 19 & -7 \\ 8 & -13 & -1 \end{bmatrix}' = \begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{-22}\begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$$

$$\therefore \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-22}\begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}\begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix}$$

$$= \frac{1}{-22}\begin{bmatrix} -32-190+200 \\ -80+361-325 \\ 48-133-25 \end{bmatrix} = \frac{-1}{22}\begin{bmatrix} -22 \\ -44 \\ -110 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$\therefore \quad x = 1, y = 2, z = 5$

Hence, cost of one pen, one bag and an instrument box is ₹ 1, ₹ 2 and ₹ 5 respectively.

(i) (c) : Cost of one pen is ₹ 1.

(ii) (a) : Cost of one pen and one bag = ₹ (1 + 2) = ₹ 3

(iii) (b) : Cost of one pen and one instrument box = ₹ (1 + 5) = ₹ 6

(iv) (c) : According to the definition of determinant, determinant is a number associated to a square matrix.

(v) (b) : Given matrix equation is $AB = AC$

Pre-multiplying by A^{-1} on both sides, we get

$A^{-1}AB = A^{-1}AC \Rightarrow (A^{-1}A)B = (A^{-1}A)C$

$\Rightarrow IB = IC$ $\quad (\because AA^{-1} = A^{-1}A = I)$

$\Rightarrow B = C$

Since A^{-1} exists only if A is non-singular.

$\therefore$ For $B = C$, A should be non-singular.

9. (i) (d) : Let Δ be the area of the triangle then,

$$\Delta = \frac{1}{2}\left|\begin{vmatrix} -2 & 6 & 1 \\ 3 & -6 & 1 \\ 1 & 5 & 1 \end{vmatrix}\right|$$

$$= \frac{1}{2}\left|-2(-6-5)-6(3-1)+1(15+6)\right|$$ [Expanding along R_1]

$\Rightarrow \Delta = \frac{1}{2}|43-12| = 15.5$ sq. units

(ii) (d) : The given points are collinear.

$$\therefore \frac{1}{2}\begin{vmatrix} 2 & -3 & 1 \\ k & -1 & 1 \\ 0 & 4 & 1 \end{vmatrix} = 0$$

Expanding along R_1, we get

$2(-1-4) + 3(k) + 1(4k) = 0$

$\Rightarrow 7k - 10 = 0 \Rightarrow k = \frac{10}{7} \Rightarrow 4k = \frac{40}{7}$

(iii) (a) : Area of ΔABC = 3 sq. units [Given]

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3 \Rightarrow \begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 6$$

$\Rightarrow 1(0-0) - 3(0-k) + 1(0-0) = \pm 6$

$\Rightarrow 3k = \pm 6 \Rightarrow k = \pm 2.$

(iv) (a) : Let $Q(x, y)$ be any point on the line joining $A(1, 2)$ and $B(3, 6)$. Then, area of $\Delta ABQ = 0$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$\Rightarrow 1(6-y) - 2(3-x) + 1(3y-6x) = 0$

$\Rightarrow 6 - y - 6 + 2x + 3y - 6x = 0$

$\Rightarrow -4x = -2y \Rightarrow 2x = y.$

(v) (c) : Area of ΔABC is given by

$$\frac{1}{2}\begin{vmatrix} 11 & 7 & 1 \\ 5 & 5 & 1 \\ -1 & 3 & 1 \end{vmatrix} = \frac{1}{2}\left[11(5-3) - 7(5+1) + 1(15+5)\right]$$

$$= \frac{1}{2}\left[22 - 42 + 20\right] = 0$$

$\therefore$ Points are collinear.

10. (i) (a) : Let $\Delta = \begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

Cofactor of 1 = 3, cofactor of –2 = –4

Cofactor of 4 = 2, cofactor of 3 = 1

$\therefore$ Required sum = 3 – 4 + 2 + 1 = 2

(ii) (b) : Let $\Delta = \begin{vmatrix} 5 & 6 & -3 \\ -4 & 3 & 2 \\ -4 & -7 & 3 \end{vmatrix}$

Minor of $a_{21} = \begin{vmatrix} 6 & -3 \\ -7 & 3 \end{vmatrix} = 18 - 21 = -3$

(iii) (c) : Let $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

Clearly, $a_{32} = 5$

and A_{32} = cofactor of a_{32} in $\Delta = (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix}$

$= (-1)(8-30) = 22$

$\therefore a_{32} \cdot A_{32} = 5 \times 22 = 110$

(iv) (d) : Here, $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

$\therefore$ Minor of $a_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7$

(v) (b) : Here, $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

$A_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} = 1(0-20) = -20,$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -1(-42-4) = 46,$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = 1(30-0) = 30$

$\therefore \Delta = a_{11}A_{11} + a_{12}a_{12} + a_{13}A_{13}$

$= 2(-20) - 3(46) + 5(30) = -28$

$\Rightarrow |\Delta| = 28$