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Inverse Trigonometric Functions

Short Answer Type Questions

Q. 1 Find the value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Thinking Process

Use the property, $\tan^{-1}\tan x = x,\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ *and* $\cos^{-1}(\cos x)=x,\ x\in[0,\pi]$ *to get the answer.*

Sol. We know that, $\tan^{-1}\tan x = x;\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\cos^{-1}\cos x = x;\ x\in[0,\pi]$

$$\therefore \quad \tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$$

$$= \tan^{-1}\left[\tan\left(\pi-\frac{\pi}{6}\right)\right]+\cos^{-1}\left[\cos\left(\pi+\frac{7\pi}{6}\right)\right]$$

$$= \tan^{-1}\left(-\tan\frac{\pi}{6}\right)+\cos^{-1}\left(-\cos\frac{7\pi}{6}\right) \qquad [\because \cos(\pi+\theta)=-\cos\theta]$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\left[\cos^{-1}\cos\left(\frac{7\pi}{6}\right)\right]$$

$$\{\because \tan^{-1}(-x)=-\tan^{-1}x;\ x\in R \text{ and } \cos^{-1}(-x)=\pi-\cos^{-1}x;\ x\in[-1,1]\}$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\cos^{-1}\left[\cos\left(\pi+\frac{\pi}{6}\right)\right]$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\left[\cos^{-1}\left(-\cos\frac{\pi}{6}\right)\right] \qquad [\because \cos(\pi+\theta)=-\cos\theta]$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\pi-\pi+\cos^{-1}\left(\cos\frac{\pi}{6}\right) \qquad [\because \cos^{-1}(-x)=\pi-\cos^{-1}x]$$

$$= -\frac{\pi}{6}+0+\frac{\pi}{6}=0$$

Note *Remember that,* $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)\neq\frac{5\pi}{6}$ *and* $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)\neq\frac{13\pi}{6}$

Since, $\frac{5\pi}{6}\notin\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ *and* $\frac{13\pi}{6}\notin[0,\pi]$

Q. 2 Evaluate $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$.

Sol. We have, $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]=\cos\left[\cos^{-1}\left(\cos\frac{5\pi}{6}\right)+\frac{\pi}{6}\right]$ $\left[\because \cos\frac{5\pi}{6}=\frac{-\sqrt{3}}{2}\right]$

$$=\cos\left(\frac{5\pi}{6}+\frac{\pi}{6}\right) \qquad \{\because \cos^{-1}\cos x = x;\ x\in[0,\pi]\}$$

$$=\cos\left(\frac{6\pi}{6}\right)$$

$$=\cos(\pi)=-1$$

Q. 3 Prove that $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)=7.$

Sol. We have to prove, $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)=7$

$\Rightarrow$ $\left(\frac{\pi}{4}-2\cot^{-1}3\right)=\cot^{-1}7$

$\Rightarrow$ $(2\cot^{-1}3)=\frac{\pi}{4}-\cot^{-1}7$

$\Rightarrow$ $2\tan^{-1}\frac{1}{3}=\frac{\pi}{4}-\tan^{-1}\frac{1}{7}$

$\Rightarrow$ $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\frac{2/3}{1-(1/3)^2}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\frac{2/3}{8/9}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}=\frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\frac{(21+4)/28}{(28-3)/28}=\frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}\frac{25}{25}=\frac{\pi}{4}$

$\Rightarrow$ $1=\tan\frac{\pi}{4}$

$\Rightarrow$ $1=1$

$\Rightarrow$ LHS = RHS **Hence proved.**

Q. 4 Find the value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$.

Sol. We have, $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$

$$= \tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cot^{-1}\left(\cot\frac{\pi}{3}\right)+\tan^{-1}(-1)$$

$$= \tan^{-1}\left[\tan\left(\pi-\frac{\pi}{6}\right)\right]+\cot^{-1}\left[\cot\left(\frac{\pi}{3}\right)\right]+\tan^{-1}\left[\tan\left(\pi-\frac{\pi}{4}\right)\right]$$

$$= \tan^{-1}\left(-\tan\frac{\pi}{6}\right)+\cot^{-1}\left(\cot\frac{\pi}{3}\right)+\tan^{-1}\left(-\tan\frac{\pi}{4}\right)$$

$\left[\because \tan^{-1}(\tan x) = x,\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\ \cot^{-1}(\cot x) = x, x\in(0,\pi) \text{ and } \tan^{-1}(-x) = -\tan^{-1}x\right]$

$$= -\frac{\pi}{6}+\frac{\pi}{3}-\frac{\pi}{4} = \frac{-2\pi+4\pi-3\pi}{12}$$

$$= \frac{-5\pi+4\pi}{12} = -\frac{\pi}{12}$$

Q. 5 Find the value of $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$.

Sol. We have, $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}\tan\left(\pi-\frac{\pi}{3}\right)$

$$= \tan^{-1}\left(-\tan\frac{\pi}{3}\right) \qquad [\because \tan^{-1}(-x) = -\tan^{-1}x]$$

$$= -\tan^{-1}\tan\frac{\pi}{3} = -\frac{\pi}{3} \qquad \left[\because \tan^{-1}(\tan x) = x,\ x\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)\right]$$

Note *Remember that,* $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)\neq\frac{2\pi}{3}$

Since, $\tan^{-1}(\tan x) = x$, *if* $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ *and* $\frac{2\pi}{3}\notin\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

Q. 6 Show that $2\tan^{-1}(-3) = \frac{-\pi}{2}+\tan^{-1}\left(\frac{-4}{3}\right)$.

Sol. LHS $= 2\tan^{-1}(-3) = -2\tan^{-1}3$ $\qquad [\because \tan^{-1}(-x) = -\tan^{-1}x, x\in R]$

$$= -\left[\cos^{-1}\frac{1-3^2}{1+3^2}\right] \qquad \left[\because 2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}, x\geq 0\right]$$

$$= -\left[\cos^{-1}\left(\frac{-8}{10}\right)\right] = -\left[\cos^{-1}\left(\frac{-4}{5}\right)\right]$$

$$= -\left[\pi-\cos^{-1}\left(\frac{4}{5}\right)\right] \qquad \{\because \cos^{-1}(-x) = \pi-\cos^{-1}x, x\in[-1,1]\}$$

$$= -\pi+\cos^{-1}\left(\frac{4}{5}\right) \qquad \left[\text{let } \cos^{-1}\left(\frac{4}{5}\right) = \theta \Rightarrow \cos\theta = \frac{4}{5} \Rightarrow \tan\theta = \frac{3}{4} \Rightarrow \theta = \tan^{-1}\frac{3}{4}\right]$$

$$= -\pi + \tan^{-1}\left(\frac{3}{4}\right) = -\pi + \left[\frac{\pi}{2} - \cot^{-1}\left(\frac{3}{4}\right)\right]$$

$$= -\frac{\pi}{2} - \cot^{-1}\frac{3}{4} = -\frac{\pi}{2} - \tan^{-1}\frac{4}{3}$$

$$= -\frac{\pi}{2} + \tan^{-1}\left(\frac{-4}{3}\right) \qquad [\because \tan^{-1}(-x) = -\tan^{-1}x]$$

= RHS **Hence proved.**

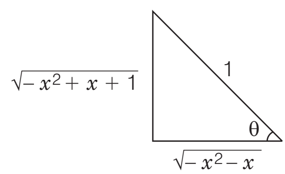
Q. 7 Find the real solution of

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2 + x + 1} = \frac{\pi}{2}.$$

Thinking Process

Convert the $\sin^{-1}\sqrt{x^2 + x + 1}$ *into inverse of tangent function and then use the property* $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$.

Sol. We have, $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2 + x + 1} = \frac{\pi}{2}$ ...(i)



Let $\sin^{-1}\sqrt{x^2 + x + 1} = \theta$

$$\Rightarrow \quad \sin\theta = \sqrt{\frac{x^2 + x + 1}{1}}$$

$$\Rightarrow \quad \tan\theta = \frac{\sqrt{x^2 + x + 1}}{\sqrt{-x^2 - x}} \qquad \left[\because \tan\theta = \frac{\sin\theta}{\cos\theta}\right]$$

$$\because \quad \theta = \tan^{-1}\frac{\sqrt{x^2 + x + 1}}{\sqrt{-x^2 - x}}$$

$$= \sin^{-1}\sqrt{x^2 + x + 1}$$

On putting the value of θ in Eq. (i), we get

$$\tan^{-1}\sqrt{x(x+1)} + \tan^{-1}\frac{\sqrt{x^2 + x + 1}}{\sqrt{-x^2 - x}} = \frac{\pi}{2}$$

We know that, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1$

$$\therefore \quad \tan^{-1}\left[\frac{\sqrt{x(x+1)} + \sqrt{\frac{x^2 + x + 1}{-x^2 - x}}}{1 - \sqrt{x(x+1)} \cdot \sqrt{\frac{x^2 + x + 1}{-x^2 - x}}}\right] = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\sqrt{x^2 + x} + \sqrt{\frac{x^2 + x + 1}{-1(x^2 + x)}}}{1 - \sqrt{(x^2 + x) \cdot \frac{(x^2 + x + 1)}{-1(x^2 + x)}}}\right] = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{x^2 + x + \sqrt{-(x^2 + x + 1)}}{[1 - \sqrt{-(x^2 + x + 1}]\sqrt{(x^2 + x)}} = \tan\frac{\pi}{2} = \frac{1}{0}$$

$$\Rightarrow \quad [1-\sqrt{-(x^2+x+1)}]\sqrt{(x^2+x)}=0$$
$$\Rightarrow \quad -(x^2+x+1)=1 \quad \text{or} \quad x^2+x=0$$
$$\Rightarrow \quad -x^2-x-1=1 \quad \text{or} \quad x(x+1)=0$$
$$\Rightarrow \quad x^2+x+2=0 \quad \text{or} \quad x(x+1)=0$$
$$\therefore \quad x=\frac{-1\pm\sqrt{1-4\times 2}}{2}$$
$$\Rightarrow \quad x=0 \quad \text{or} \quad x=-1$$

For real solution, we have $x = 0, -1$.

Q. 8 Find the value of $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos(\tan^{-1}2\sqrt{2})$.

Sol. We have, $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos(\tan^{-1}2\sqrt{2})$

$$=\sin\left[\sin^{-1}\left\{\frac{2\times\frac{1}{3}}{1+\left(\frac{1}{3}\right)^2}\right\}\right]+\cos\left(\cos^{-1}\frac{1}{3}\right) \quad \left[\because \tan^{-1}x=\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$

$$\left[\because 2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}, -1\le x\le 1 \text{ and } \tan^{-1}(2\sqrt{2})=\cos^{-1}\frac{1}{3}\right]$$

$$=\sin\left[\sin^{-1}\left(\frac{\frac{2}{3}}{1+\frac{1}{9}}\right)\right]+\frac{1}{3} \quad \{\because \cos(\cos^{-1}x)=x;\ x\in[-1,1]\}$$

$$=\sin\left[\sin^{-1}\left(\frac{2\times 9}{3\times 10}\right)\right]+\frac{1}{3}=\sin\left[\sin^{-1}\left(\frac{3}{5}\right)\right]+\frac{1}{3} \quad [\because \sin(\sin^{-1}x)=x]$$

$$=\frac{3}{5}+\frac{1}{3}=\frac{9+5}{15}=\frac{14}{15}$$

Q. 9 If $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$, then show that $\theta=\frac{\pi}{4}$, where n is any integer.

Thinking Process

Use the property, $2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ *to prove the desired result.*

Sol. We have, $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos\theta}{1-\cos^2\theta}\right)=\tan^{-1}(2\operatorname{cosec}\theta)$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \quad \left(\frac{2\cos\theta}{\sin^2\theta}\right)=(2\operatorname{cosec}\theta)$$

$$\Rightarrow \quad (\cot\theta\cdot 2\operatorname{cosec}\theta)=(2\operatorname{cosec}\theta) \Rightarrow \cot\theta=1$$

$$\Rightarrow \quad \cot\theta=\cot\frac{\pi}{4} \Rightarrow \theta=\frac{\pi}{4}$$

Q. 10 Show that $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$.

Thinking Process

Use the property $2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}$ *and* $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$, *to prove LHS = RHS.*

Sol. We have, $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{1-\left(\frac{1}{7}\right)^2}{1+\left(\frac{1}{7}\right)^2}\right)\right]=\sin\left[2\cdot 2\tan^{-1}\frac{1}{3}\right] \qquad \left[\because 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{\frac{48}{49}}{\frac{50}{49}}\right)\right]=\sin\left[2\cdot\left(\tan^{-1}\frac{\frac{2}{3}}{1-\left(\frac{1}{3}\right)^2}\right)\right] \qquad \left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{48\times 49}{50\times 49}\right)\right]=\sin\left[2\tan^{-1}\left(\frac{18}{24}\right)\right]$$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{24}{25}\right)\right]=\sin\left(2\tan^{-1}\frac{3}{4}\right)$$

$$\Rightarrow \cos\left[\cos^{-1}\left(\frac{24}{25}\right)\right]=\sin\left(\sin^{-1}\frac{2\times\frac{3}{4}}{1+\frac{9}{16}}\right) \qquad \left[\because 2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}\right]$$

$$\Rightarrow \frac{24}{25}=\sin\left(\sin^{-1}\frac{3/2}{25/16}\right)$$

$$\Rightarrow \frac{24}{25}=\frac{48}{50} \Rightarrow \frac{24}{25}=\frac{24}{25}$$

$\therefore$ LHS = RHS **Hence proved.**

Q. 11 Solve the equation $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$.

Sol. We have, $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$

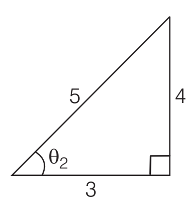
$$\Rightarrow \cos\left(\cos^{-1}\frac{1}{\sqrt{x^2+1}}\right)=\sin\left(\sin^{-1}\frac{4}{5}\right)$$

Let $\tan^{-1}x=\theta_1 \Rightarrow \tan\theta_1=\frac{x}{1}$

$\Rightarrow \cos\theta_1=\frac{1}{\sqrt{x^2+1}} \Rightarrow \theta_1=\cos^{-1}\frac{1}{\sqrt{x^2+1}}$

and $\cot^{-1}\frac{3}{4}=\theta_2 \Rightarrow \cot\theta_2=\frac{3}{4}$

$\Rightarrow \sin\theta_2=\frac{4}{5} \Rightarrow \theta_2=\sin^{-1}\frac{4}{5}$



$$\Rightarrow \quad \frac{1}{\sqrt{x^2+1}} = \frac{4}{5}$$

$\{\because \cos(\cos^{-1} x) = x, x \in [-1, 1]$ and $\sin(\sin^{-1} x) = x, x \in [-1, 1]\}$

On squaring both sides, we get

$$16(x^2 + 1) = 25$$

$$\Rightarrow \quad 16x^2 = 9$$

$$\Rightarrow \quad x^2 = \left(\frac{3}{4}\right)^2$$

$$\therefore \quad x = \pm\frac{3}{4} = \frac{-3}{4}, \frac{3}{4}$$

Long Answer Type Questions

Q. 12 Prove that $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$.

Sol. We have,

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$$

$$\therefore \quad \text{LHS} = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) \quad \text{...(i)}$$

[let $x^2 = \cos 2\theta = (\cos^2\theta - \sin^2\theta) = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$]

$$\Rightarrow \quad \cos^{-1} x^2 = 2\theta \Rightarrow \theta = \frac{1}{2}\cos^{-1} x^2$$

$$\therefore \quad \sqrt{1+x^2} = \sqrt{1+\cos 2\theta}$$

$$= \sqrt{1 + 2\cos^2\theta - 1} = \sqrt{2}\cos\theta$$

and

$$\sqrt{1-x^2} = \sqrt{1-\cos 2\theta}$$

$$= \sqrt{1 - 1 + 2\sin^2\theta} = \sqrt{2}\sin\theta$$

$$\therefore \quad \text{LHS} = \tan^{-1}\left(\frac{\sqrt{2}\cos\theta + \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) = \tan^{-1}\left(\frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\cdot\tan\theta}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right] \quad \left[\because \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}\right]$$

$$= \frac{\pi}{4} + \theta = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$$

$$= \text{RHS}$$

Hence proved.

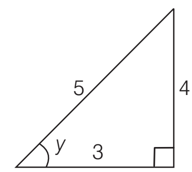
Q. 13 Find the simplified form of

$$\cos^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right), \text{ where } x \in \left[\frac{-3\pi}{4}, \frac{\pi}{4}\right].$$

Sol. We have, $\cos^{-1}\left[\frac{3}{5}\cos x + \frac{4}{5}\sin x\right], x \in \left[\frac{-3\pi}{4}, \frac{\pi}{4}\right]$

Let $\cos y = \frac{3}{5}$

$\Rightarrow$ $\sin y = \frac{4}{5}$



$\Rightarrow$ $y = \cos^{-1}\frac{3}{5} = \sin^{-1}\frac{4}{5} = \tan^{-1}\left(\frac{4}{3}\right)$

$\therefore$ $\cos^{-1}[\cos y \cdot \cos x + \sin y \cdot \sin x]$

$= \cos^{-1}[\cos(y - x)]$ $\quad [\because \cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B]$

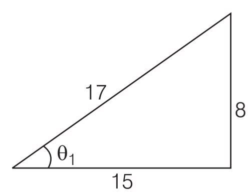
$= y - x = \tan^{-1}\frac{4}{3} - x$ $\quad \left[\because y = \tan^{-1}\frac{4}{3}\right]$

Q. 14 Prove that $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}$.

Sol. We have, $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}$

$\therefore$ $\text{LHS} = \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$

$= \tan^{-1}\frac{8}{15} + \tan^{-1}\frac{3}{4}$



Let $\sin^{-1}\frac{8}{17} = \theta_1 \Rightarrow \sin\theta_1 = \frac{8}{17}$

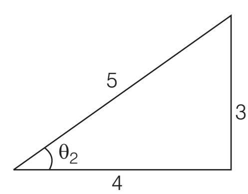
$\Rightarrow$ $\tan\theta_1 = \frac{8}{15} \Rightarrow \theta_1 = \tan^{-1}\frac{8}{15}$

and $\sin^{-1}\frac{3}{5} = \theta_2 \Rightarrow \sin\theta_2 = \frac{3}{5}$

$\Rightarrow$ $\tan\theta_2 = \frac{3}{4} \Rightarrow \theta_2 = \tan^{-1}\frac{3}{4}$

$= \tan^{-1}\left[\dfrac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{8}{15} \times \frac{3}{4}}\right]$ $\quad \left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)\right]$

$= \tan^{-1}\left[\dfrac{\frac{32 + 45}{60}}{\frac{60 - 24}{60}}\right] = \tan^{-1}\left(\frac{77}{36}\right)$



Let $\theta_3 = \tan^{-1}\frac{77}{36} \Rightarrow \tan\theta_3 = \frac{77}{36}$

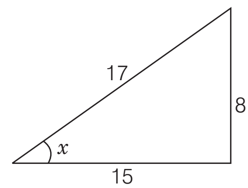
$\Rightarrow$ $\sin\theta_3 = \frac{77}{\sqrt{5929 + 1296}} = \frac{77}{85}$

$\therefore$ $\theta_3 = \sin^{-1}\frac{77}{85}$

$= \sin^{-1}\frac{77}{85} = \text{RHS}$ **Hence proved.**

Alternate Method

To prove, $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{77}{85}$

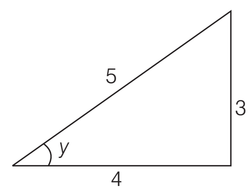


Let $\sin^{-1}\frac{8}{17}=x$

$\Rightarrow$ $\sin x=\frac{8}{17}$

$\Rightarrow$ $\cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\left(\frac{8}{17}\right)^2}$

$=\sqrt{\frac{289-64}{289}}=\sqrt{\frac{225}{289}}=\frac{15}{17}$



Let $\sin^{-1}\frac{3}{5}=y$

$\Rightarrow$ $\sin y=\frac{3}{5}$ $\Rightarrow$ $\sin^2 y=\frac{9}{25}$

$\therefore$ $\cos^2 y=1-\frac{9}{25}$

$\Rightarrow$ $\cos^2 y=\left(\frac{4}{5}\right)^2$ $\Rightarrow$ $\cos y=\frac{4}{5}$

Now, $\sin(x+y)=\sin x\cdot\cos y+\cos x\cdot\sin y$

$=\frac{8}{17}\cdot\frac{4}{5}+\frac{15}{17}\cdot\frac{3}{5}$

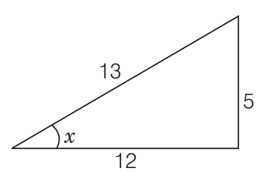
$=\frac{32}{85}+\frac{45}{85}=\frac{77}{85}$

$\Rightarrow$ $(x+y)=\sin^{-1}\left(\frac{77}{85}\right)$

$\Rightarrow$ $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{77}{85}$

Q. 15 Show that $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$.

Sol. We have, $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$ ...(i)



Let $\sin^{-1}\frac{5}{13}=x$

$\Rightarrow$ $\sin x=\frac{5}{13}$

and $\cos^2 x=1-\sin^2 x$

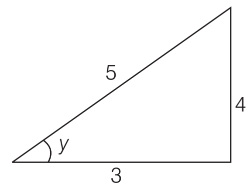
$=1-\frac{25}{169}=\frac{144}{169}$

$\Rightarrow$ $\cos x=\sqrt{\frac{144}{169}}=\frac{12}{13}$

$\therefore$ $\tan x=\frac{\sin x}{\cos x}=\frac{5/13}{12/13}=\frac{5}{12}$ ...(ii)

$\Rightarrow$ $\tan x=5/12$ ...(iii)

Again, let $\cos^{-1}\frac{3}{5} = y \Rightarrow \cos y = \frac{3}{5}$



$\therefore$ $\sin y = \sqrt{1-\cos^2 y}$

$= \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}}$

$\sin y = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\Rightarrow \tan y = \frac{\sin y}{\cos y} = \frac{4/5}{3/5} = \frac{4}{3}$ ...(iii)

We know that,

$$\tan(x+y) = \frac{\tan x + \tan y}{1-\tan x \cdot \tan y}$$

$\Rightarrow \tan(x+y) = \dfrac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\cdot\frac{4}{3}} \Rightarrow \tan(x+y) = \dfrac{\frac{15+48}{36}}{\frac{36-20}{36}}$

$\Rightarrow \tan(x+y) = \frac{63/36}{16/36}$

$\Rightarrow \tan(x+y) = \frac{63}{16}$

$\Rightarrow x + y = \tan^{-1}\frac{63}{16}$

$\Rightarrow \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{4}{3} = \tan^{-1}\frac{63}{16}$ **Hence proved.**

Q. 16 Prove that $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \sin^{-1}\frac{1}{\sqrt{5}}$.

Sol. We have, $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \sin^{-1}\frac{1}{\sqrt{5}}$ ...(i)

Let $\tan^{-1}\frac{1}{4} = x$

$\Rightarrow \tan x = \frac{1}{4}$

$\Rightarrow \tan^2 x = \frac{1}{16}$

$\Rightarrow \sec^2 x - 1 = \frac{1}{16}$

$\Rightarrow \sec^2 x = 1 + \frac{1}{16} = \frac{17}{16}$

$\Rightarrow \frac{1}{\cos^2 x} = \frac{17}{16}$

$\Rightarrow \cos^2 x = \frac{16}{17}$

$\Rightarrow \cos x = \frac{4}{\sqrt{17}}$

$\Rightarrow \sin^2 x = 1 - \cos^2 x = 1 - \frac{16}{17} = \frac{1}{17}$

$\Rightarrow \sin x = \frac{1}{\sqrt{17}}$ ...(ii)

Again, let $\tan^{-1}\frac{2}{9} = y$

$\Rightarrow \quad \tan y = \frac{2}{9} \Rightarrow \tan^2 y = \frac{4}{81}$

$\Rightarrow \quad \sec^2 y - 1 = \frac{4}{81}$

$\Rightarrow \quad \sec^2 y = \frac{4}{81} + 1 = \frac{85}{81}$

$\Rightarrow \quad \cos^2 y = \frac{81}{85} \Rightarrow \cos y = \frac{9}{\sqrt{85}}$

$\Rightarrow \quad \sin^2 y = 1 - \cos^2 y = 1 - \frac{81}{85} = \frac{4}{85}$

$\Rightarrow \quad \sin y = \frac{2}{\sqrt{85}}$ ...(iii)

We know that, $\sin(x + y) = \sin x \cdot \cos y + \cos x \cdot \sin y$

$$= \frac{1}{\sqrt{17}} \cdot \frac{9}{\sqrt{85}} + \frac{4}{\sqrt{17}} \cdot \frac{2}{\sqrt{85}}$$

$$= \frac{17}{\sqrt{17} \cdot \sqrt{85}} = \frac{\sqrt{17}}{\sqrt{17} \cdot \sqrt{5}} = \frac{1}{\sqrt{5}}$$

$\Rightarrow \quad (x + y) = \sin^{-1}\frac{1}{\sqrt{5}}$

$\Rightarrow \quad \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \sin^{-1}\frac{1}{\sqrt{5}}$ **Hence proved.**

Q. 17 Find the value of $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$.

Thinking Process

Use the properties $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ *and* $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ *to get the desired value.*

Sol. We have, $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

$$= 2 \cdot 2\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

$$= 2 \cdot \left[\tan^{-1}\frac{\frac{2}{5}}{1-\left(\frac{1}{5}\right)^2}\right] - \tan^{-1}\frac{1}{239} \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= 2 \cdot \left[\tan^{-1}\left(\frac{\frac{2}{5}}{1-\frac{1}{25}}\right)\right] - \tan^{-1}\frac{1}{239}$$

$$= 2 \cdot \left[\tan^{-1}\left(\frac{2/5}{24/25}\right)\right] - \tan^{-1}\frac{1}{239}$$

$$= 2\tan^{-1}\frac{5}{12} - \tan^{-1}\frac{1}{239}$$

$$= \tan^{-1} \frac{2 \cdot \frac{5}{12}}{1 - \left(\frac{5}{12}\right)^2} - \tan^{-1} \frac{1}{239} \qquad \left[\because\ 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1} \left(\frac{\frac{5}{6}}{1 - \frac{25}{144}}\right) - \tan^{-1} \frac{1}{239}$$

$$= \tan^{-1}\left(\frac{144 \times 5}{119 \times 6}\right) - \tan^{-1} \frac{1}{239}$$

$$= \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1} \frac{1}{239}$$

$$= \tan^{-1}\left(\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}}\right) \qquad \left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{120 \times 239 - 119}{119 \times 239 + 120}\right)$$

$$= \tan^{-1}\left[\frac{28680 - 119}{28441 + 120}\right] = \tan^{-1} \frac{28561}{28561}$$

$$= \tan^{-1}(1) = \tan^{-1}\left(\tan \frac{\pi}{4}\right) = \frac{\pi}{4}$$

Q. 18 Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$ and justify why the other value $\frac{4+\sqrt{7}}{3}$ is ignored?

Sol. We have, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$

$\therefore$ $\text{LHS} = \tan\left[\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)\right]$

Let $\frac{1}{2}\sin^{-1}\frac{3}{4} = \theta \Rightarrow \sin^{-1}\frac{3}{4} = 2\theta$

$\Rightarrow$ $\sin 2\theta = \frac{3}{4} \Rightarrow \frac{2\tan\theta}{1+\tan^2\theta} = \frac{3}{4}$

$\Rightarrow$ $3 + 3\tan^2\theta = 8\tan\theta$

$\Rightarrow$ $3\tan^2\theta - 8\tan\theta + 3 = 0$

Let $\tan\theta = y$

$\therefore$ $3y^2 - 8y + 3 = 0$

$\Rightarrow$ $y = \frac{+8 \pm \sqrt{64 - 4 \times 3 \times 3}}{2 \times 3} = \frac{8 \pm \sqrt{28}}{6}$

$= \frac{2[4 \pm \sqrt{7}]}{2 \cdot 3}$

$\Rightarrow$ $\tan\theta = \frac{4 \pm \sqrt{7}}{3}$

$\Rightarrow \qquad \theta = \tan^{-1}\left[\frac{4 \pm \sqrt{7}}{3}\right]$

$$\left\{\text{but } \frac{4+\sqrt{7}}{3} > \frac{1}{2}.\frac{\pi}{2}, \text{ since max}\left[\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)\right] = 1\right\}$$

$\therefore \qquad \text{LHS} = \tan\tan^{-1}\left(\frac{4-\sqrt{7}}{3}\right) = \frac{4-\sqrt{7}}{3} = \text{RHS}$

Note *Since,* $\quad -\frac{\pi}{2} \le \sin^{-1}\frac{3}{4} \le \pi/2$

$\Rightarrow \qquad \frac{-\pi}{4} \le \frac{1}{2}\sin^{-1}\frac{3}{4} \le \pi/4$

$\therefore \qquad \tan\left(\frac{-\pi}{4}\right) \le \tan\frac{1}{2}\left(\sin^{-1}\frac{3}{4}\right) \le \tan\frac{\pi}{4}$

$\Rightarrow \qquad -1 \le \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) \le 1$

Q. 19 If $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic progression with common difference d, then evaluate the following expression.

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \tan^{-1}\left(\frac{d}{1+a_3a_4}\right) + \ldots + \tan^{-1}\left(\frac{d}{1+a_{n-1}\,a_n}\right)\right]$$

Sol. We have, $\quad a_1 = a, a_2 = a+d, a_3 = a+2d$

and $\quad d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots = a_n - a_{n-1}$

Given that, $\quad \tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \tan^{-1}\left(\frac{d}{1+a_3a_4}\right) + \ldots + \tan^{-1}\left(\frac{d}{1+a_{n-1}\cdot a_n}\right)\right]$

$$= \tan\left[\tan^{-1}\frac{a_2-a_1}{1+a_2\cdot a_1} + \tan^{-1}\frac{a_3-a_2}{1+a_3\cdot a_2} + \ldots + \tan^{-1}\frac{a_n-a_{n-1}}{1+a_n\cdot a_{n-1}}\right]$$

$$= \tan\left[(\tan^{-1}a_2 - \tan^{-1}a_1) + (\tan^{-1}a_3 - \tan^{-1}a_2) + \ldots + (\tan^{-1}a_n - \tan^{-1}a_{n-1})\right]$$

$$= \tan[\tan^{-1}a_n - \tan^{-1}a_1]$$

$$= \tan\left[\tan^{-1}\frac{a_n - a_1}{1+a_n\cdot a_1}\right] \qquad \left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

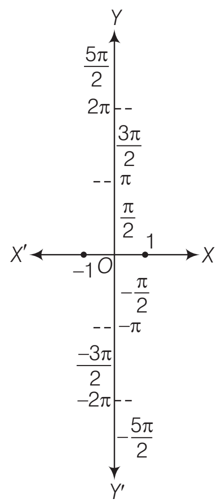
$$= \frac{a_n - a_1}{1+a_n\cdot a_1} \qquad [\because \tan(\tan^{-1}x) = x]$$

Objective Type Questions

Q. 20 Which of the following is the principal value branch of $\cos^{-1} x$?

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) $(0, \pi)$ (c) $[0, \pi]$ (d) $(0, \pi) - \left\{\frac{\pi}{2}\right\}$

Sol. *(c)* We know that, the principal value branch of $\cos^{-1} x$ is $[0, \pi]$.

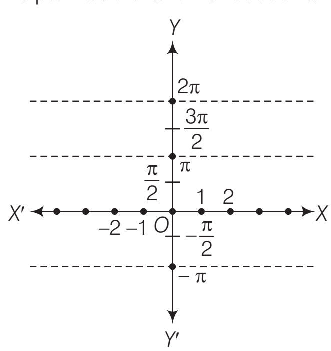


$\therefore$ $y = \cos^{-1} x$

Q. 21 Which of the following is the principal value branch of $\text{cosec}^{-1} x$?

(a) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ (b) $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ (c) $\left[\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - [0]$

Sol. *(d)* We know that, the principal value branch of $\text{cosec}^{-1} x$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - 0.$



$\therefore$ $y = \text{cosec}^{-1} x$

Q. 22 If $3\tan^{-1}x + \cot^{-1}x = \pi$, then x equals to

(a) 0 (b) 1 (c) −1 (d) $\frac{1}{2}$

Sol. *(b)* Given that, $3\tan^{-1}x + \cot^{-1}x = \pi$ ...(i)

$$\Rightarrow \quad 2\tan^{-1}x + \tan^{-1}x + \cot^{-1}x = \pi$$

$$\Rightarrow \quad 2\tan^{-1}x = \pi - \frac{\pi}{2} \qquad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

$$\Rightarrow \quad 2\tan^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{2} \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}, \forall x \in (-1, 1)\right]$$

$$\Rightarrow \quad \frac{2x}{1-x^2} = \tan\frac{\pi}{2}$$

$$\Rightarrow \quad \frac{2x}{1-x^2} = \frac{1}{0} \Rightarrow 1 - x^2 = 0$$

$$\Rightarrow \quad x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow x = 1$$

Hence, only $x = 1$ satisfies the given equation.

Note *Here, putting $x = -1$ in the given equation, we get*

$$3\tan^{-1}(-1) + \cot^{-1}(-1) = \pi$$

$$\Rightarrow \quad 3\tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right] + \cot^{-1}\left[\cot\left(\frac{-\pi}{4}\right)\right] = \pi$$

$$\Rightarrow \quad 3\tan^{-1}\left(-\tan\frac{\pi}{4}\right) + \cot^{-1}\left(-\cot\frac{\pi}{4}\right) = \pi$$

$$\Rightarrow \quad -3\tan^{-1}\left(\tan\frac{\pi}{4}\right) + \pi - \cot^{-1}\left(\cot\frac{\pi}{4}\right) = \pi$$

$$\Rightarrow \quad -3\cdot\frac{\pi}{4} + \pi - \frac{\pi}{4} = \pi$$

$$\Rightarrow \quad -\pi + \pi = \pi \Rightarrow 0 \neq \pi$$

Hence, $x = -1$ does not satisfy the given equation.

Q. 23 The value of $\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right]$ is

(a) $\frac{3\pi}{5}$ (b) $\frac{-7\pi}{5}$ (c) $\frac{\pi}{10}$ (d) $\frac{-\pi}{10}$

Sol. *(d)* We have,

$$\sin^{-1}\left(\cos\frac{33\pi}{5}\right) = \sin^{-1}\left[\cos\left(6\pi + \frac{3\pi}{5}\right)\right] = \sin^{-1}\left[\cos\left(\frac{3\pi}{5}\right)\right] \qquad [\because \cos(2n\pi + \theta) = \cos\theta]$$

$$= \sin^{-1}\left[\cos\left(\frac{\pi}{2} + \frac{\pi}{10}\right)\right] = \sin^{-1}\left(-\sin\frac{\pi}{10}\right)$$

$$= -\sin^{-1}\left(\sin\frac{\pi}{10}\right) \qquad [\because \sin^{-1}(-x) = -\sin^{-1}x]$$

$$= -\frac{\pi}{10} \qquad \left[\because \sin^{-1}(\sin x) = x, x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right]$$

Q. 24 The domain of the function $\cos^{-1}(2x-1)$ is

(a) [0, 1] (b) [−1, 1] (c) (−1, 1) (d) [0, π]

Sol. *(a)* We have, $f(x) = \cos^{-1}(2x-1)$

$\therefore \quad -1 \le 2x - 1 \le 1$

$\Rightarrow \quad 0 \le 2x \le 2$

$\Rightarrow \quad 0 \le x \le 1$

$\therefore \quad x \in [0, 1]$

Q. 25 The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is

(a) [1, 2] (b) [−1, 1] (c) [0, 1] (d) None of these

Sol. *(a)* $\because \quad f(x) = \sin^{-1}\sqrt{x-1}$

$\Rightarrow \quad 0 \le x - 1 \le 1 \qquad [\because \sqrt{x-1} \ge 0 \text{ and } -1 \le \sqrt{x-1} \le 1]$

$\Rightarrow \quad 1 \le x \le 2$

$\therefore \quad x \in [1, 2]$

Q. 26 If $\cos\left(\sin^{-1}\frac{2}{5} + \cos^{-1}x\right) = 0$, then x is equal to

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) 0 (d) 1

Sol. *(b)* We have, $\cos\left(\sin^{-1}\frac{2}{5} + \cos^{-1}x\right) = 0$

$\Rightarrow \quad \sin^{-1}\frac{2}{5} + \cos^{-1}x = \cos^{-1}0$

$\Rightarrow \quad \sin^{-1}\frac{2}{5} + \cos^{-1}x = \cos^{-1}\cos\frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}\frac{2}{5} + \cos^{-1}x = \frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}\frac{2}{5}$

$\Rightarrow \quad \cos^{-1}x = \cos^{-1}\frac{2}{5} \qquad \left[\because \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}\right]$

$\therefore \quad x = \frac{2}{5}$

Q. 27 The value of $\sin[2\tan^{-1}(0.75)]$ is

(a) 0.75 (b) 1.5 (c) 0.96 (d) sin 1.5

Sol. *(c)* We have, $\sin[2\tan^{-1}(0.75)] = \sin\left(2\tan^{-1}\frac{3}{4}\right) \qquad \left[\because 0.75 = \frac{75}{100} = \frac{3}{4}\right]$

$$= \sin\left(\sin^{-1}\frac{2\cdot\frac{3}{4}}{1+\frac{9}{16}}\right) = \sin\left[\sin^{-1}\frac{3/2}{25/16}\right]$$

$$= \sin\left[\sin^{-1}\left(\frac{48}{50}\right)\right] = \sin\left[\sin^{-1}\left(\frac{24}{25}\right)\right] = \frac{24}{25} = 0.96$$

Q. 28 The value of $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $\frac{5\pi}{2}$ (d) $\frac{7\pi}{2}$

Sol. *(a)* We have, $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$

$$= \cos^{-1}\cos\left(2\pi - \frac{\pi}{2}\right) \qquad \left[\because \cos\left(2\pi - \frac{\pi}{2}\right) = \cos\frac{\pi}{2}\right]$$

$$= \cos^{-1}\cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \qquad \{\because \cos^{-1}(\cos x) = x,\ x \in [0, \pi]\}$$

Note *Remember that,* $\cos^{-1}\left(\cos\frac{3\pi}{2}\right) \neq \frac{3\pi}{2}$

$\because$ $\frac{3\pi}{2} \notin (0, \pi)$

Q. 29 The value of $2\sec^{-1}2 + \sin^{-1}\left(\frac{1}{2}\right)$ is

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{7\pi}{6}$ (d) 1

Sol. *(b)* We have, $2\sec^{-1}2 + \sin^{-1}\frac{1}{2} = 2\sec^{-1}\sec\frac{\pi}{3} + \sin^{-1}\sin\frac{\pi}{6}$

$$= 2\cdot\frac{\pi}{3} + \frac{\pi}{6} \qquad [\because \sec^{-1}(\sec x) = x \text{ and } \sin^{-1}(\sin x) = x]$$

$$= \frac{4\pi + \pi}{6} = \frac{5\pi}{6}$$

Q. 30 If $\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$, then $\cot^{-1}x + \cot^{-1}y$ equals to

(a) $\frac{\pi}{5}$ (b) $\frac{2\pi}{5}$ (c) $\frac{3\pi}{5}$ (d) π

Sol. *(a)* We have, $\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$

$$\Rightarrow \frac{\pi}{2} - \cot^{-1}x + \frac{\pi}{2} - \cot^{-1}y = \frac{4\pi}{5}$$

$$\Rightarrow -(\cot^{-1}x + \cot^{-1}y) = \frac{4\pi}{5} - \pi \qquad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

$$\Rightarrow \cot^{-1}x + \cot^{-1}y = -\left(-\frac{\pi}{5}\right)$$

$$\Rightarrow \cot^{-1}x + \cot^{-1}y = \frac{\pi}{5}$$

Q. 31 If $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $a, x \in]0, 1[$, then the value of x is

(a) 0 (b) $\frac{a}{2}$ (c) a (d) $\frac{2a}{1-a^2}$

Sol. *(d)* We have, $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Let $a=\tan\theta \Rightarrow \theta=\tan^{-1}a$

$$\therefore \quad \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)+\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\tan^{-1}\frac{2x}{1-x^2}$$

$$\Rightarrow \quad \sin^{-1}\sin2\theta+\cos^{-1}\cos2\theta=\tan^{-1}\frac{2x}{1-x^2}$$

$$\Rightarrow \quad 2\theta+2\theta=\tan^{-1}\frac{2x}{1-x^2}$$

$$\Rightarrow \quad 4\tan^{-1}a=\tan^{-1}\frac{2x}{1-x^2}$$

$$\Rightarrow \quad 2\cdot2\tan^{-1}a=\tan^{-1}\frac{2x}{1-x^2}$$

$$\Rightarrow \quad 2\cdot\tan^{-1}\frac{2a}{1-a^2}=\tan^{-1}\frac{2x}{1-x^2} \qquad \left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\Rightarrow \quad \tan^{-1}\frac{2\cdot\left(\frac{2a}{1-a^2}\right)}{1-\left(\frac{2a}{1-a^2}\right)^2}=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\therefore \quad x=\frac{2a}{1-a^2}$$

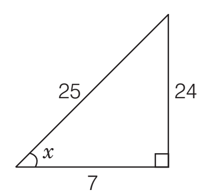
Q. 32 The value of $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right]$ is

(a) $\frac{25}{24}$ (b) $\frac{25}{7}$ (c) $\frac{24}{25}$ (d) $\frac{7}{24}$

Sol. *(d)* We have, $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right]$

Let $\cos^{-1}\frac{7}{25}=x$

$$\Rightarrow \quad \cos x=\frac{7}{25}$$



$$\therefore \quad \sin x=\sqrt{1-\cos^2x}=\sqrt{1-\left(\frac{7}{25}\right)^2}$$

$$=\sqrt{\frac{625-49}{625}}=\frac{24}{25}$$

$$\therefore \quad \cot x = \frac{\cos x}{\sin x} = \frac{\frac{7}{25}}{\frac{24}{25}} = \frac{7}{24} \quad \text{...(i)}$$

$$\Rightarrow \quad x = \cot^{-1}\left(\frac{7}{24}\right) = \cos^{-1}\left(\frac{7}{25}\right)$$

$$\therefore \quad \cot\left(\cos^{-1}\frac{7}{25}\right) = \cot\left(\cot^{-1}\frac{7}{24}\right) = \frac{7}{24} \quad \left[\because \cot^{-1}\frac{7}{24} = \cos^{-1}\frac{7}{25}\right]$$

Q. 33 The value of $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)$ is

(a) $2+\sqrt{5}$ (b) $\sqrt{5}-2$ (c) $\frac{\sqrt{5}+2}{2}$ (d) $5+\sqrt{2}$

Sol. *(b)* We have, $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)$

Let $\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}} = \theta$

$$\Rightarrow \quad \cos^{-1}\frac{2}{\sqrt{5}} = 2\theta \Rightarrow \cos 2\theta = \frac{2}{\sqrt{5}}$$

$$\therefore \quad (1 - 2\sin^2\theta) = \frac{2}{\sqrt{5}}$$

$$\Rightarrow \quad 2\sin^2\theta = 1 - \frac{2}{\sqrt{5}}$$

$$\Rightarrow \quad \sin^2\theta = \frac{1}{2} - \frac{1}{\sqrt{5}}$$

$$\Rightarrow \quad \sin\theta = \sqrt{\frac{1}{2} - \frac{1}{\sqrt{5}}}$$

$$\therefore \quad \cos^2\theta = 1 - \sin^2\theta$$

$$= 1 - \frac{1}{2} + \frac{1}{\sqrt{5}} = \frac{1}{2} + \frac{1}{\sqrt{5}}$$

$$\Rightarrow \quad \cos\theta = \sqrt{\frac{1}{2} + \frac{1}{\sqrt{5}}}$$

$$\therefore \quad \tan\theta = \sqrt{\frac{\frac{1}{2} - \frac{1}{\sqrt{5}}}{\frac{1}{2} + \frac{1}{\sqrt{5}}}} = \sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}} \quad \left[\because \tan\theta = \frac{\sin\theta}{\cos\theta}\right]$$

$$\Rightarrow \quad \theta = \tan^{-1}\sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}} = \frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}$$

$$\therefore \quad \tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right) = \tan\tan^{-1}\sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}}$$

$$= \sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}\cdot\frac{\sqrt{5}-2}{\sqrt{5}-2}}$$

$$= \sqrt{\frac{(\sqrt{5}-2)^2}{5-4}} = \sqrt{5}-2$$

Q. 34 If $|x| \le 1$, then $2\tan^{-1} x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is equal to

(a) $4\tan^{-1} x$ (b) 0 (c) $\frac{\pi}{2}$ (d) π

Sol. *(a)* We have, $2\tan^{-1} x + \sin^{-1}\frac{2x}{1+x^2}$

Let $x = \tan\theta$

$\therefore$ $2\tan^{-1}\tan\theta + \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}$ $[\because \tan^{-1}(\tan x) = x]$

$= 2\theta + \sin^{-1}\sin 2\theta$ $\left[\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$

$= 2\theta + 2\theta$ $[\because \sin^{-1}(\sin x) = x]$

$= 4\theta$ $[\because \theta = \tan^{-1} x]$

$= 4\tan^{-1} x$

Q. 35 If $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$, then $\alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$ equals to

(a) 0 (b) 1 (c) 6 (d) 12

Sol. *(c)* We have, $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$

We know that, $0 \le \cos^{-1} x \le \pi$

$\Rightarrow$ $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$

If and only if, $\cos^{-1}\alpha = \cos^{-1}\beta = \cos^{-1}\gamma = \pi$

$\Rightarrow$ $\cos\pi = \alpha = \beta = \gamma$

$\Rightarrow$ $-1 = \alpha = \beta = \gamma$

$\Rightarrow$ $\alpha = \beta = \gamma = -1$

$\therefore$ $\alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$

$= -1(-1-1) - 1(-1-1) - 1(-1-1)$

$= 2 + 2 + 2 = 6$

Q. 36 The number of real solutions of the equation

$$\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x) \text{ in } \left[\frac{\pi}{2}, \pi\right] \text{ is}$$

(a) 0 (b) 1 (c) 2 (d) ∞

Sol. *(a)* We have, $\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x), \left[\frac{\pi}{2}, \pi\right]$

$\Rightarrow$ $\sqrt{1+2\cos^2 x - 1} = \sqrt{2}\cos^{-1}(\cos x)$

$\Rightarrow$ $\sqrt{2}\cos x = \sqrt{2}\cos^{-1}(\cos x)$

$\Rightarrow$ $\cos x = \cos^{-1}(\cos x)$

$\Rightarrow$ $\cos x = x$ $[\because \cos^{-1}(\cos x) = x]$

which is not true for any real value of x.

Hence, there is no solution possible for the given equation.

Q. 37 If $\cos^{-1} x > \sin^{-1} x$, then

(a) $\frac{1}{\sqrt{2}} < x \le 1$ (b) $0 \le x < \frac{1}{\sqrt{2}}$ (c) $-1 \le x < \frac{1}{\sqrt{2}}$ (d) $x > 0$

Sol. *(c)* We have, $\cos^{-1} x > \sin^{-1} x$, where $x \in [-1, 1]$

$\Rightarrow \quad x < \cos(\sin^{-1} x)$

$\Rightarrow \quad x < \cos [\cos^{-1} \sqrt{1 - x^2}] \qquad \left[\text{let } \sin^{-1} x = \theta \Rightarrow \sin \theta = \frac{x}{1}\right]$

$\left[\because \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - x^2} \Rightarrow \theta = \cos^{-1} \sqrt{1 - x^2}\right]$

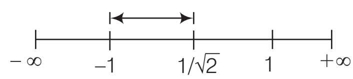
$\Rightarrow \quad x < \sqrt{1 - x^2}$

$\Rightarrow \quad x^2 < 1 - x^2 \Rightarrow 2x^2 < 1$

$\Rightarrow \quad x^2 < \frac{1}{2} \Rightarrow x < \pm\left(\frac{1}{\sqrt{2}}\right) \qquad \text{...(i)}$

Also, $\quad -1 \le x \le 1 \qquad \text{...(ii)}$

$\therefore \quad -1 \le x \le \frac{1}{\sqrt{2}}$



Alternate Method

$$\frac{\pi}{2} - \sin^{-1} x > \sin^{-1} x$$

$$\frac{\pi}{2} > 2\sin^{-1} x \Rightarrow \frac{\pi}{4} > \sin^{-1} x$$

$$\frac{1}{\sqrt{2}} > x \Rightarrow \frac{1}{\sqrt{2}} < x \le 1$$

We know that, $\sin^{-1} x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Fillers

Q. 38 The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is ....

Sol. $\because \quad 0 \le \cos^{-1} x \le \pi$

$$\cos^{-1}\left(-\frac{1}{2}\right) = \cos^{-1}\left(\cos \frac{2\pi}{3}\right) = \frac{2\pi}{3}$$

Q. 39 The value of $\sin^{-1}\left(\sin \frac{3\pi}{5}\right)$ is ....

Sol. $\because \quad -\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$\therefore \quad \sin^{-1}\left(\sin \frac{3\pi}{5}\right) = \sin^{-1} \sin\left(\pi - \frac{2\pi}{5}\right) = \sin^{-1}\left(\sin \frac{2\pi}{5}\right) = \frac{2\pi}{5}$

Q. 40 If $\cos(\tan^{-1} x + \cot^{-1} \sqrt{3}) = 0$, then the value of x is ....

Sol. We have, $\cos(\tan^{-1} x + \cot^{-1} \sqrt{3}) = 0$

$\Rightarrow \quad \tan^{-1} x + \cot^{-1} \sqrt{3} = \cos^{-1} 0$

$\Rightarrow \quad \tan^{-1} x + \cot^{-1} \sqrt{3} = \cos^{-1} \cos \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1} x + \cot^{-1} \sqrt{3} = \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1} x = \frac{\pi}{2} - \cot^{-1} \sqrt{3}$

$\Rightarrow \quad \tan^{-1} x = \tan^{-1} \sqrt{3} \qquad \left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$

$\therefore \quad x = \sqrt{3}$

Q. 41 The set of values of $\sec^{-1} \frac{1}{2}$ is ....

Sol. Since, domain of $\sec^{-1} x$ is $R - (-1, 1)$.

$\Rightarrow \quad (-\infty, -1] \cup [1, \infty)$

So, there is no set of values exist for $\sec^{-1} \frac{1}{2}$.

So, ϕ is the answer.

Q. 42 The principal value of $\tan^{-1} \sqrt{3}$ is ....

Sol. $\because \quad \tan^{-1} \sqrt{3} = \tan^{-1} \tan\left(\frac{\pi}{3}\right)$

$= \left(\frac{\pi}{3}\right) \qquad \left[\because \tan^{-1}(\tan x) = x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$

Q. 43 The value of $\cos^{-1}\left(\cos \frac{14\pi}{3}\right)$ is ....

Sol. We have, $\cos^{-1}\left(\cos \frac{14\pi}{3}\right) = \cos^{-1} \cos\left(4\pi + \frac{2\pi}{3}\right)$

$= \cos^{-1} \cos \frac{2\pi}{3} \qquad [\because \cos(2n\pi + \theta) = \cos \theta]$

$= \frac{2\pi}{3} \qquad \{\because \cos^{-1}(\cos x) = x, x \in [0, \pi]\}$

Note *Remember that,* $\cos^{-1}\left(\cos \frac{14\pi}{3}\right) \neq \frac{14\pi}{3}$

Since, $\frac{14\pi}{3} \notin [0, \pi]$

Q. 44 The value of $\cos(\sin^{-1} x + \cos^{-1} x)$, where $|x| \leq 1$, is ....

Sol. $\cos(\sin^{-1} x + \cos^{-1} x)$

$= \cos \frac{\pi}{2} = 0 \qquad \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$

Q. 45 The value of $\tan\left(\frac{\sin^{-1}x+\cos^{-1}x}{2}\right)$, when $x=\frac{\sqrt{3}}{2}$, is ....

Sol. $\because$ $\tan\left(\frac{\sin^{-1}x+\cos^{-1}x}{2}\right) = \tan\left(\frac{\pi/2}{2}\right)$ $\left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$

$$=\tan\frac{\pi}{4}=1$$

Q. 46 If $y=2\tan^{-1}x+\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then .... $< y <$ ....

Sol. We have, $y=2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}$

$\therefore$ $y=2\tan^{-1}\tan\theta+\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}$ [let $x=\tan\theta$]

$\Rightarrow$ $y=2\theta+\sin^{-1}\sin 2\theta$ $\left[\because \sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}\right]$

$\Rightarrow$ $y=2\theta+2\theta=4\theta$ [$\because$ $\theta=\tan^{-1}x$]

$\Rightarrow$ $y=4\tan^{-1}x$

$\because$ $-\pi/2<\tan^{-1}x<\pi/2$

$\therefore$ $-\frac{4\pi}{2}<4\tan^{-1}x<4\pi/2$

$\Rightarrow$ $-2\pi<4\tan^{-1}x<2\pi$

$\Rightarrow$ $-2\pi<y<2\pi$ [$\because$ $y=4\tan^{-1}x$]

Q. 47 The result $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$ is true when the value of xy is ....

Sol. We know that, $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$

where, $xy>-1$

Q. 48 The value of $\cot^{-1}(-x)$ $x\in R$ in terms of $\cot^{-1}x$ is ....

Sol. We know that,

$$\cot^{-1}(-x)=\pi-\cot^{-1}x,\ x\in R$$

True/False

Q. 49 All trigonometric functions have inverse over their respective domains.

Sol. ***False***

We know that, all trigonometric functions have inverse over their restricted domains.

Q. 50 The value of the expression $(\cos^{-1} x)^2$ is equal to $\sec^2 x$.

Sol. ***False***

$$\because \quad [\cos^{-1} x]^2 = \left[\sec^{-1} \frac{1}{x}\right]^2 \neq \sec^2 x$$

Q. 51 The domain of trigonometric functions can be restricted to any one of their branch (not necessarily principal value) in order to obtain their inverse functions.

Sol. ***True***

We know that, the domain of trigonometric functions are restricted in their domain to obtain their inverse functions.

Q. 52 The least numerical value, either positive or negative of angle θ is called principal value of the inverse trigonometric function.

Sol. ***True***

We know that, the smallest numerical value, either positive or negative of θ is called the principal value of the function.

Q. 53 The graph of inverse trigonometric function can be obtained from the graph of their corresponding function by interchanging X and Y-axes.

Sol. ***True***

We know that, the graph of an inverse function can be obtained from the corresponding graph of original function as a mirror image (*i.e.,* reflection) along the line $y = x$.

Q. 54 The minimum value of n for which $\tan^{-1} \frac{n}{\pi} > \frac{\pi}{4}$, $n \in N$, is valid is 5.

Sol. ***False***

$$\because \quad \tan^{-1} \frac{n}{\pi} > \frac{\pi}{4} \Rightarrow \frac{n}{\pi} > \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{n}{\pi} > 1 \qquad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$\Rightarrow \quad n > \pi$$

So, the minimum value of n is 4. $[\because n \in N \text{ and } \pi = 3.14...]$

Q. 55 The principal value of $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right]$ is $\frac{\pi}{3}$.

Sol. ***True***

Given that, $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right] = \sin^{-1}\left[\cos \sin^{-1}\left(\sin\frac{\pi}{6}\right)\right]$

$$= \sin^{-1}\left[\cos\frac{\pi}{6}\right] \qquad [\because \sin^{-1}(\sin x) = x]$$

$$= \sin^{-1}\frac{\sqrt{3}}{2}$$

$$= \sin^{-1}\sin\frac{\pi}{3} = \frac{\pi}{3}$$