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Complex Numbers and Quadratic Equations

Short Answer Type Questions

Q. 1 For a positive integer n, find the value of $(1-i)^n\left(1-\dfrac{1}{i}\right)^n$.

Sol. Given expression $= (1-i)^n\left(1-\dfrac{1}{i}\right)^n$

$$= (1-i)^n (i-1)^n \cdot i^{-n} = (1-i)^n (1-i)^n (-1)^n \cdot i^{-n}$$

$$= [(1-i)^2]^n (-1)^n \cdot i^{-n} = (1+i^2-2i)^n (-1)^n i^{-n} \qquad [\because i^2 = -1]$$

$$= (1-1-2i)^n (-1)^n i^{-n} = (-2)^n \cdot i^n (-1)^n i^{-n}$$

$$= (-1)^{2n} \cdot 2^n = 2^n$$

Q. 2 Evaluate $\displaystyle\sum_{n=1}^{13}(i^n + i^{n+1})$, where $n \in N$.

> 💡 **Thinking Process**
> Use $i^2 = -1, i^4 = (-1)^2 = 1, i^3 = -i$, and $i^5 = i$ to solve it

Sol. Given that, $\displaystyle\sum_{n=1}^{13}(i^n + i^{n+1}), n \in N$

$$= (i + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13})$$
$$+ (i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13} + i^{14})$$

$$= (i + 2i^2 + 2i^3 + 2i^4 + 2i^5 + 2i^6 + 2i^7 + 2i^8 + 2i^9 + 2i^{10} + 2i^{11} + 2i^{12} + 2i^{13} + i^{14})$$

$$= i - 2 - 2i + 2 + 2i + 2(i^4)i^2 + 2(i)^4 i^3 + 2(i^2)^4 + 2(i^2)^4 i + 2(i^2)^5$$
$$+ 2(i^2)^5 \cdot i + 2(i^2)^6 + 2(i^2)^6 \cdot i + (i^2)^7$$

$$= i - 2 - 2i + 2 + 2i - 2 - 2i + 2 + 2i - 2 - 2i + 2 + 2i - 1 - 1 + i$$

Alternate Method

$$\sum_{n=1}^{13} (i^n + i^{n+1}), n \in N = \sum_{h=1}^{13} i^n (1 + i)$$

$$= (1 + i)[i + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13}]$$

$$= (1 + i)[i^{13}] \quad [\because i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, \text{ where } n \in N \text{ i.e., } \sum_{n=1}^{12} i^n = 0$$

$$= (1 + i)i$$

$[\because (i^4)^3 \cdot i = i]$

$$= (i^2 + i) = i - 1$$

Q. 3 If $\left(\dfrac{1+i}{1-i}\right)^3 - \left(\dfrac{1-i}{1+i}\right)^3 = x + iy$, then find (x, y).

> 💡 **Thinking Process**
>
> *If two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal*
>
> *i.e., $z_1 = z_2 \implies x_1 + iy_1 = x_2 + iy_2$, then $x_1 = x_2$ and $y_1 = y_2$.*

Sol. Given that, $\left(\dfrac{1+i}{1-i}\right)^3 - \left(\dfrac{1-i}{1+i}\right)^3 = x + iy$ ...(i)

$\therefore \qquad \left(\dfrac{1+i}{1-i}\right)^3 = \dfrac{1 + i^3 + 3i(1+i)}{1 - i^3 - 3i(1-i)} = \dfrac{1 - i + 3i + 3i^2}{1 + i - 3i + 3i^2}$

$$= \dfrac{2i - 2}{-2i - 2} = \dfrac{i - 1}{-i - 1} = \dfrac{1 - i}{1 + i}$$

$$= \dfrac{(1-i)}{(1+i)} \dfrac{(1-i)}{(1-i)} = \dfrac{1 + i^2 - 2i}{1 + 1} = \dfrac{1 - 1 - 2i}{2}$$

$\implies \qquad \left(\dfrac{1+i}{1-i}\right)^3 = -i$ ...(ii)

Similarly, $\left(\dfrac{1-i}{1+i}\right)^3 = \dfrac{-1}{i} = \dfrac{i^2}{i} = i$ ...(iii)

Using Eqs. (ii) and (iii) in Eq. (i), we get

$$-i - i = x + iy$$

$\implies \qquad -2i = x + iy$

On comparing real and imaginary part of complex number, we get

$$x = 0 \text{ and } y = -2$$

So, $\qquad (x, y) = (0, -2)$

Q. 4 If $\dfrac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$.

Sol. Given that, $\qquad \dfrac{(1+i)^2}{2-i} = x + iy$

$\implies \qquad \dfrac{(1 + i^2 + 2i)}{2 - i} = x + iy \implies \dfrac{2i}{2 - i} = x + iy$

$\implies \qquad \dfrac{2i(2+i)}{(2-i)(2+i)} = x + iy \implies \dfrac{4i + 2i^2}{4 - i^2} = x + iy$

$\Rightarrow$ $\qquad\qquad \dfrac{4i-2}{4+1} = x + iy$ $\Rightarrow$ $\dfrac{-2}{5} + \dfrac{4i}{5} = x + iy$

On comparing both sides, we get

$$x = -2/5 \quad \Rightarrow \quad y = 4/5$$

$\Rightarrow$ $\qquad\qquad\qquad x + y = \dfrac{-2}{5} + \dfrac{4}{5} = 2/5$

Q. 5 If $\left(\dfrac{1-i}{1+i}\right)^{100} = a + ib$, then find (a, b).

Sol. Given that, $\left(\dfrac{1-i}{1+i}\right)^{100} = a + ib$

$\Rightarrow$ $\qquad \left[\dfrac{(1-i)}{(1+i)} \cdot \dfrac{(1-i)}{(1-i)}\right]^{100} = a + ib$ $\Rightarrow$ $\left(\dfrac{1+i^2-2i}{1-i^2}\right)^{100} = a + ib$

$\Rightarrow$ $\qquad\qquad\qquad \left(\dfrac{-2i}{2}\right)^{100} = a + ib$ $\qquad\qquad\qquad\qquad$ $[\because i^2 = -1]$

$\Rightarrow$ $\qquad\qquad\qquad (i^4)^{25} = a + ib \quad \Rightarrow \quad 1 = a + ib$

Then, $\qquad\qquad\qquad\qquad a = 1$ and $b = 0$ $\qquad\qquad\qquad\qquad$ $[\because i^4 = 1]$

$\therefore$ $\qquad\qquad\qquad\qquad\qquad (a, b) = (1, 0)$

Q. 6 If $a = \cos\theta + i\sin\theta$, then find the value of $\dfrac{1+a}{1-a}$.

$\bullet$ **Thinking Process**

To solve the above problem use the trigonometric formula $\cos\theta = 2\cos^2\theta/2$ $-1 = 1 - 2\sin^2\theta/2$ and $\sin\theta = 2\sin\theta/2 \cdot \cos\theta/2$.

Sol. Given that, $\quad a = \cos\theta + i\sin\theta$

$\therefore$ $\qquad \dfrac{1+a}{1-a} = \dfrac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}$

$\qquad\qquad = \dfrac{1 + 2\cos^2\theta/2 - 1 + 2i\sin\theta/2 \cdot \cos\theta/2}{1 - 1 + 2\sin^2\theta/2 - 2i\sin\theta/2 \cdot \cos\theta/2} = \dfrac{2\cos\theta/2(\cos\theta/2 + i\sin\theta/2)}{2\sin\theta/2(\sin\theta/2 - i\cos\theta/2)}$

$\qquad\qquad = -\dfrac{2\cos\theta/2(\cos\theta/2 + i\sin\theta/2)}{2i\sin\theta/2(\cos\theta/2 + i\sin\theta/2)} = -\dfrac{1}{i}\cot\theta/2$

$\qquad\qquad = \dfrac{+i^2}{i}\cot\theta/2 = i\cot\theta/2$ $\qquad\qquad\qquad\qquad$ $\left[\because \dfrac{-1}{i} = \dfrac{i^2}{i}\right]$

Q. 7 If $(1+i)z = (1-i)\bar{z}$, then show that $z = -i\bar{z}$.

Sol. We have, $\qquad (1+i)z = (1-i)\bar{z}$ $\Rightarrow$ $\dfrac{z}{\bar{z}} = \dfrac{(1-i)}{(1+i)}$

$\Rightarrow$ $\qquad\qquad \dfrac{z}{\bar{z}} = \dfrac{(1-i)}{(1+i)}\dfrac{(1-i)}{(1-i)}$ $\Rightarrow$ $\dfrac{z}{\bar{z}} = \dfrac{1+i^2-2i}{1-i^2}$ $\qquad\qquad$ $[\because i^2 = -1]$

$\Rightarrow$ $\qquad\qquad \dfrac{z}{\bar{z}} = \dfrac{1-1-2i}{2}$ $\Rightarrow$ $\dfrac{z}{\bar{z}} = -i$

$$\therefore \qquad\qquad z = -i\,\bar{z} \qquad\qquad\qquad \textbf{Hence proved.}$$

Q. 8 If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + b = 0$, where $b \in R$, represents a circle.

Sol. Given that,
$$z = x + iy$$
Then,
$$\bar{z} = x - iy$$
Now,
$$z\bar{z} + 2(z + \bar{z}) + b = 0$$
$$\Rightarrow \quad (x + iy)(x - iy) + 2(x + iy + x - iy) + b = 0$$
$$\Rightarrow \qquad\qquad x^2 + y^2 + 4x + b = 0, \text{ which is the equation of a circle.}$$

Q. 9 If the real part of $\dfrac{\bar{z} + 2}{\bar{z} - 1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.

Sol. Let
$$z = x + iy$$
Now,
$$\frac{\bar{z} + 2}{\bar{z} - 1} = \frac{x - iy + 2}{x - iy - 1}$$
$$= \frac{[(x+2) - i\,y][(x-1) + i\,y]}{[(x-1) - i\,y][(x-1) + i\,y]}$$
$$= \frac{(x-1)(x+2) - iy(x-1) + i\,y(x+2) + y^2}{(x-1)^2 + y^2}$$
$$= \frac{(x-1)(x+2) + y^2 + i[(x+2)y - (x-1)y]}{(x-1)^2 + y^2} \qquad [\because -i^2 = 1]$$

Taking real part, $\dfrac{(x-1)(x+2) + y^2}{(x-1)^2 + y^2} = 4$
$$\Rightarrow \qquad x^2 - x + 2x - 2 + y^2 = 4(x^2 - 2x + 1 + y^2)$$
$$\Rightarrow \qquad 3x^2 + 3y^2 - 9x + 6 = 0, \text{ which represents a circle.}$$
Hence, z lies on the circle.

Q. 10 Show that the complex number z, satisfying the condition arg $\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$ lies on a circle.

> 💡 **Thinking Process**
>
> *First use,* $arg\left(\dfrac{z_1}{z_2}\right) = arg(z_1) - arg(z_2).$ *Also apply* $arg(z) = \theta = \tan^{-1}\dfrac{y}{x},$ *where* $z = x + iy$
>
> *and then use the property* $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

Sol. Let
$$z = x + iy$$
Given that,
$$\arg\left(\frac{z-1}{z+1}\right) = \pi/4$$
$$\Rightarrow \qquad \arg(z-1) - \arg(z+1) = \pi/4$$
$$\Rightarrow \qquad \arg(x + iy - 1) - \arg(x + iy + 1) = \pi/4$$
$$\Rightarrow \qquad \arg(x - 1 + iy) - \arg(x + 1 + iy) = \pi/4$$

$$\Rightarrow \qquad \tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x+1} = \pi/4$$

$$\Rightarrow \qquad \tan^{-1}\left[\frac{\dfrac{y}{x-1} - \dfrac{y}{x+1}}{1+\left(\dfrac{y}{x-1}\right)\left(\dfrac{y}{x+1}\right)}\right] = \pi/4$$

$$\Rightarrow \qquad \frac{y\left[\dfrac{x+1-x+1}{x^2-1}\right]}{\dfrac{x^2-1+y^2}{x^2-1}} = \tan \pi/4$$

$$\Rightarrow \qquad \frac{2y}{x^2+y^2-1} = 1$$

$$\Rightarrow \qquad x^2+y^2-1 = 2y$$

$$\Rightarrow \qquad x^2+y^2-2y-1 = 0, \text{ which represents a circle.}$$

Q. 11 Solve the equation $|z| = z + 1 + 2i$.

Sol. The given equation is $|z| = z + 1 + 2i$ \hfill ...(i)

Let $\qquad\qquad z = x + iy$

From Eq. (i), $\qquad |x + iy| = x + iy + 1 + 2i$

$$\Rightarrow \qquad \sqrt{x^2+y^2} = x + iy + 1 + 2i \qquad\qquad \left[\because\; |z| + \sqrt{x^2 = y^2}\right]$$

$$\Rightarrow \qquad \sqrt{x^2+y^2} = (x+1) + i\,(y+2)$$

On squaring both sides, we get

$$x^2+y^2 = (x+1)^2 + i^2(y+2)^2 + 2i\,(x+1)(y+2)$$

$$\Rightarrow \qquad x^2+y^2 = x^2 + 2x + 1 - y^2 - 4y - 4 + 2i\,(x+1)(y+2)$$

On comparing real and imaginary parts,

$$x^2+y^2 = x^2 + 2x + 1 - y^2 - 4y - 4$$

i.e., $\qquad\qquad 2y^2 = 2x - 4y - 3$ \hfill ...(ii)

and $\qquad\qquad 2\,(x+1)(y+2) = 0$

$$(x+1) = 0 \text{ or } (y+2) = 0$$

$$\Rightarrow \qquad\qquad x = -1 \text{ or } y = -2$$

For $x = -1$, $\qquad\qquad 2y^2 = -2 - 4y - 3$

$$2y^2 + 4y + 5 = 0 \qquad\qquad\qquad \text{[using Eq. (ii)]}$$

$$\Rightarrow \qquad\qquad y = \frac{-4 \pm \sqrt{16 - 2 \times 4 \times 5}}{4}$$

$$\Rightarrow \qquad\qquad y = \frac{-4 \pm \sqrt{-24}}{4} \notin R$$

For $y = -2$, $\qquad\qquad 2(-2)^2 = 2x - 4(-2) - 3 \qquad\qquad \text{[usign Eq. (ii)]}$

$$\Rightarrow \qquad\qquad 8 = 2x + 8 - 3$$

$$\Rightarrow \qquad\qquad 2x = 3 \Rightarrow x = 3/2$$

$$\therefore \qquad\qquad z = x + iy = 3/2 - 2i$$

Long Answer Type Questions

Q. 12 If $|z + 1| = z + 2(1 + i)$, then find the value of z.

Sol. Given that, $|z + 1| = z + 2(1 + i)$ ...(i)

$$z = x + iy$$

Then, $|x + iy + 1| = x + iy + 2(1 + i)$

$\Rightarrow$ $|x + 1 + iy| = (x + 2) + i(y + 2)$

$\Rightarrow$ $\sqrt{(x + 1)^2 + y^2} = (x + 2) + i(y + 2)$

On squaring both sides, we get

$$(x + 1)^2 + y^2 = (x + 2)^2 + i^2(y + 2)^2 + 2i(x + 2)(y + 2)$$

$\Rightarrow$ $x^2 + 2x + 1 + y^2 = x^2 + 4x + 4 - y^2 - 4y - 4 + 2i(x + 2)(y + 2)$

$\Rightarrow$ $x^2 + y^2 + 2x + 1 = x^2 - y^2 + 4x - 4y + 2i(x + 2)(y + 2)$

On comparing real and imaginary parts, we get

$$x^2 + y^2 + 2x + 1 = x^2 - y^2 + 4x - 4y$$

$\Rightarrow$ $2y^2 - 2x + 4y + 1 = 0$ ...(ii)

and $2(x + 2)(y + 2) = 0$

$\Rightarrow$ $x + 2 = 0$ or $y + 2 = 0$

$x = -2$ or $y = -2$ ...(iii)

For $x = -2$, $2y^2 + 4 + 4y + 1 = 0$ [using Eq. (ii)]

$\Rightarrow$ $2y^2 + 4y + 5 = 0$

$\Rightarrow$ $16 - 4 \times 2 \times 5 < 0$

$\therefore$ Discriminant, $D = b^2 - 4ac < 0$

$\Rightarrow$ $2y^2 + 4y + 5$ has no real roots.

For $y = -2$, $2(-2)^2 - 2x + 4(-2) + 1 = 0$ [using Eq. (ii)]

$\Rightarrow$ $8 - 2x - 8 + 1 = 0$

$\Rightarrow$ $x = 1/2$

$\therefore$ $z = x + iy = \dfrac{1}{2} - 2i$

Q. 13 If $\arg(z - 1) = \arg(z + 3i)$, then find $x - 1 : y$, where $z = x + iy$.

Sol. Given that, $\arg(z - 1) = \arg(z + 3i)$

and   let $z = x + iy$

Now, $\arg(z - 1) = \arg(z + 3i)$

$\Rightarrow$ $\arg(x + iy - 1) = \arg(x + iy + 3i)$

$\Rightarrow$ $\arg(x - 1 + iy) = \arg[x + i(y + 3)]$

$\Rightarrow$ $\tan^{-1}\dfrac{y}{x - 1} = \tan^{-1}\dfrac{y + 3}{x}$

$\Rightarrow$ $\dfrac{y}{x - 1} = \dfrac{y + 3}{x}$ $\Rightarrow$ $xy = (x - 1)(y + 3)$

$\Rightarrow$ $xy = xy - y + 3x - 3$ $\Rightarrow$ $3x - 3 = y$

$$\Rightarrow \qquad \frac{3(x-1)}{y} = 1 \quad \Rightarrow \quad \frac{x-1}{y} = \frac{1}{3}$$

$$\therefore \qquad (x-1) : y = 1 : 3$$

Q. 14 Show that $\left|\dfrac{z-2}{z-3}\right| = 2$ represents a circle. Find its centre and radius.

    💡 **Thinking Process**

       If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, (z_2 \neq 0)$, use

       this concept to solve the above problem. Also, we know that general equation of a circle

       is $x^2 + y^2 + 2gx + 2fy + c = 0$, with centre $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

Sol. Let $z = x + iy$

Given, equation is $\left|\dfrac{z-2}{z-3}\right| = 2 \quad \Rightarrow \quad \left|\dfrac{z-2}{z-3}\right| = 2$

$$\Rightarrow \qquad \left|\frac{x+iy-2}{x+iy-3}\right| = 2$$

$$\Rightarrow \qquad |x-2+iy| = 2\,|x-3+iy|$$

$$\Rightarrow \qquad \sqrt{(x-2)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2} \qquad\qquad \left[\because |x+iy| = \sqrt{x^2+y^2}\right]$$

On squaring both sides, we get

$$x^2 - 4x + 4 + y^2 = 4\,(x^2 - 6x + 9 + y^2)$$

$$\Rightarrow \qquad 3x^2 + 3y^2 - 20x + 32 = 0$$

$$\Rightarrow \qquad x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0 \qquad\qquad\qquad\qquad \ldots(i)$$

On comparing the above equation with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$$\Rightarrow \qquad 2g = \frac{-20}{3} \quad \Rightarrow \quad g = \frac{-10}{3}$$

and $\qquad\qquad 2f = 0 \quad \Rightarrow \quad f = 0 \text{ and } c = \frac{32}{3}$

$$\therefore \qquad \text{Centre} = (-g, -f) = (10/3, 0)$$

Also, $\qquad\qquad$ radius $(r) = \sqrt{(10/3)^2 + 0 - 32/3} \qquad\qquad [\because r = \sqrt{g^2 + f^2 - c}\,]$

$$= \frac{1}{3}\sqrt{(100 - 96)} = 2/3$$

Q. 15 If $\dfrac{z-1}{z+1}$ is a purely imaginary number $(z \neq -1)$, then find the value of $\left|z\right|$.

    💡 **Thinking Process**

       If $z = x + iy$ is a purely imaginary number, then its real part must be equal to zero i.e.,

       $x = 0$,

Sol. Let $\qquad\qquad\qquad z = x + iy$

$$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1}, \ z \neq -1$$

$$= \frac{x-1+iy}{x+1+iy} = \frac{(x-1+iy)\,(x+1-iy)}{(x+1+iy)(x+1-iy)}$$

$$= \frac{(x^2 - 1) + iy\,(x + 1) - iy\,(x - 1) - i^2 y^2}{(x + 1)^2 - (iy)^2}$$

$\Rightarrow \qquad \dfrac{z - 1}{z + 1} = \dfrac{(x^2 - 1) + y^2 + i\,[y\,(x + 1) - y\,(x - 1)]}{(x + 1)^2 + y^2}$

Given that, $\dfrac{z - 1}{z + 1}$ is a purely imaginary numbers.

Then, $\qquad \dfrac{(x^2 - 1) + y^2}{(x + 1)^2 + y^2} = 0$

$\Rightarrow \qquad\qquad x^2 - 1 + y^2 = 0 \;\Rightarrow\; x^2 + y^2 = 1$

$\Rightarrow \qquad\qquad \sqrt{x^2 + y^2} = \sqrt{1} \;\Rightarrow\; |z| = 1 \qquad\qquad [\because |z| = \sqrt{x^2 + y^2}\,]$

Q. 16 z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then show that $z_1 = -\bar{z}_2$.

Sol. Let $z_1 = r_1\,(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2\,(\cos\theta_2 + i\sin\theta_2)$ are two complex numbers.

Given that, $\qquad\qquad |z_1| = |z_2|$

and $\qquad\qquad \arg(z_1) + \arg(z_2) = \pi$

If $\qquad\qquad |z_1| = |z_2|$

$\Rightarrow \qquad\qquad r_1 = r_2$ $\qquad\qquad\qquad\qquad$ ...(i)

and if $\qquad \arg(z_1) + \arg(z_2) = \pi$

$\Rightarrow \qquad\qquad \theta_1 + \theta_2 = \pi$

$\Rightarrow \qquad\qquad \theta_1 = \pi - \theta_2$

Now, $\qquad z_1 = r_1\,(\cos\theta_1 + i\sin\theta_1)$

$\Rightarrow \qquad z_1 = r_2\,[\cos(\pi - \theta_2) + i\sin(\pi - \theta_2)] \qquad [\because r_1 = r_2 \text{ and } \theta_1 = (\pi - \theta_2)]$

$\Rightarrow \qquad z_1 = r_2\,(-\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow \qquad z_1 = -r_2\,(\cos\theta_2 - i\sin\theta_2)$

$\Rightarrow \qquad z_1 = -[r_2\,(\cos\theta_2 - i\sin\theta_2)]$

$\Rightarrow \qquad z_1 = -\bar{z}_2 \qquad\qquad\qquad [\because \bar{z}_2 = r_2(\cos\theta_2 - i\sin\theta_2)]$

Q. 17 If $|z_1| = 1\,(z_1 \neq -1)$ and $z_2 = \dfrac{z_1 - 1}{z_1 + 1}$, then show that the real part of z_2 is zero.

Sol. Let $\qquad\qquad z_1 = x + iy$

$\Rightarrow \qquad |z_1| = \sqrt{x^2 + y^2} = 1 \qquad\qquad [\because |z_1| = 1, \text{ given}] \text{ ...(i)}$

Now, $\qquad z_2 = \dfrac{z_1 - 1}{z_1 + 1} = \dfrac{x + iy - 1}{x + iy + 1}$

$\qquad\qquad = \dfrac{x - 1 + iy}{x + 1 + iy} = \dfrac{(x - 1 + iy)\,(x + 1 - iy)}{(x + 1 + iy)(x + 1 - iy)}$

$\qquad\qquad = \dfrac{x^2 - 1 + iy\,(x + 1) - iy\,(x - 1) - i^2 y^2}{(x + 1)^2 - i^2 y^2}$

$\qquad\qquad = \dfrac{x^2 - 1 + ixy + iy - ixy + iy + y^2}{(x + 1)^2 + y^2}$

$$= \frac{x^2 + y^2 - 1 + 2iy}{(x+1)^2 + y^2} = \frac{1 - 1 + 2iy}{(x+1)^2 + y^2} \qquad [\because x^2 + y^2 = 1]$$

$$= 0 + \frac{2yi}{(x+1)^2 + y^2}$$

Hence, the real part of z_2 is zero.

Q. 18 If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find $\arg\left(\dfrac{z_1}{z_4}\right) + \arg\left(\dfrac{z_2}{z_3}\right)$.

> 💡 **Thinking Process**
> First let, $z = r(\cos\theta + i\sin\theta)$, then conjugate of z i.e., $\bar{z} = r(\cos\theta - i\sin\theta)$. Use the property $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

Sol. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$,

Then, $\qquad z_2 = \bar{z}_1 = r_1(\cos\theta_1 - i\sin\theta_1) = r_1[\cos(-\theta_1) + \sin(-\theta_1)]$

Also, $\qquad$ let $z_3 = r_2(\cos\theta_2 + i\sin\theta_2)$,

Then, $\qquad z_4 = \bar{z}_3 = r_2(\cos\theta_2 - i\sin\theta_2)$

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3)$$

$$= \theta_1 - (-\theta_2) + (-\theta_1) - \theta_2 \qquad [\because \arg(z) = \theta]$$

$$= \theta_1 + \theta_2 - \theta_1 - \theta_2 = 0$$

Q. 19 If $|z_1| = |z_2| = \cdots = |z_n| = 1$, then show that

$$|z_1 + z_2 + z_3 + \cdots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \cdots + \frac{1}{z_n}\right|.$$

Sol. Given that, $\qquad\qquad\qquad |z_1| = |z_2| = \cdots = |z_n| = 1$

$$\Rightarrow \qquad\qquad\qquad |z_1|^2 = |z_2|^2 = \cdots = |z_n|^2 = 1$$

$$\Rightarrow \qquad\qquad\qquad z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 = \cdots = z_n\bar{z}_n = 1$$

$$\Rightarrow \qquad\qquad\qquad z_1 = \frac{1}{\bar{z}_1}, z_2 = \frac{1}{\bar{z}_2} = \cdots = z_n = \frac{1}{\bar{z}_n}$$

Now, $\quad |z_1 + z_2 + z_3 + z_4 + \cdots + z_n|$

$$= \left|\frac{z_1\bar{z}_1}{\bar{z}_1} + \frac{z_2\bar{z}_2}{\bar{z}_2} + \frac{z_3\bar{z}_3}{\bar{z}_3} + \cdots + \frac{z_n\bar{z}_n}{\bar{z}_n}\right| \quad \left[\because z_1\bar{z} = 1, \text{ where } z_1 = \frac{1}{z}, z_1 = \frac{\bar{z}}{\bar{z} - \bar{z}}, z_1 = \bar{z}\right]$$

$$= \left|\frac{|z_1|^2}{\bar{z}_1} + \frac{|z_2|^2}{\bar{z}_2} + \frac{|z_3|^2}{\bar{z}_3} + \cdots + \frac{|z_n|^2}{\bar{z}_n}\right|$$

$$= \left|\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3} + \cdots + \frac{1}{\bar{z}_n}\right| = \sqrt{\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3}}$$

$$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n}\right| \qquad\qquad\qquad\qquad\qquad \textbf{Hence proved.}$$

Q. 20 If the complex numbers z_1 and z_2, arg $(z_1) -$ arg $(z_2) = 0$, then show that $|z_1 - z_2| = |z_1| - |z_2|$.

Sol. Let
$$z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$$
and
$$z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$$
$$\Rightarrow \quad \text{arg}(z_1) = \theta_1 \text{ and arg}(z_2) = \theta_2$$
Given that, arg $(z_1) -$ arg $(z_2) = 0$
$$\theta_1 - \theta_2 = 0 \Rightarrow \theta_1 = \theta_2$$
$$z_2 = r_2(\cos\theta_1 + i \sin \theta_1) \qquad\qquad [\because \theta_1 = \theta_2]$$
$$z_1 - z_2 = (r_1\cos\theta_1 - r_2\cos\theta_1) + i (r_1\sin\theta_1 - r_2\sin\theta_1)$$
$$|z_1 - z_2| = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_1)^2 + (r_1\sin\theta_1 - r_2\sin\theta_1)^2}$$
$$= \sqrt{r_1^2 + r_2^2 - 2\,r_1 r_2 \cos^2\theta_1 - 2r_1 r_2 \sin^2\theta_1}$$
$$= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2(\sin^2\theta_1 + \cos^2\theta_1)}$$
$$= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2} = \sqrt{(r_1 - r_2)^2}$$
$$\Rightarrow \qquad |z_1 - z_2| = r_1 - r_2 \qquad\qquad [\because r = |z|]$$
$$= |z_1| - |z_2| \qquad\qquad \textbf{Hence proved.}$$

Q. 21 Solve the system of equations Re $(z^2) = 0$, $|z| = 2$.

Sol. Given that,
$$\text{Re}(z^2) = 0, |z| = 2$$
Let
$$z = x + iy$$
$$|z| = \sqrt{x^2 + y^2}$$
$$\because \qquad \sqrt{x^2 + y^2} = 2$$
$$\Rightarrow \qquad x^2 + y^2 = 4 \qquad\qquad \text{...(i)}$$
and
$$\text{Re}(z) = x$$
Also,
$$z = x + iy$$
$$\Rightarrow \qquad z^2 = x^2 + 2ixy - y^2$$
$$\Rightarrow \qquad z^2 = (x^2 - y^2) + 2ixy$$
$$\Rightarrow \qquad \text{Re}(z^2) = x^2 - y^2 \qquad\qquad [\because \text{Re}(z^2) = 0]$$
$$\Rightarrow \qquad x^2 - y^2 = 0 \qquad\qquad \text{...(ii)}$$
From Eqs. (i) and (ii),
$$x^2 + x^2 = 4$$
$$\Rightarrow \qquad 2x^2 = 4 \Rightarrow x^2 = 2$$
$$\Rightarrow \qquad x = \pm \sqrt{2}$$
$$\therefore \qquad y = \pm\sqrt{2}$$
$$\because \qquad z = x + iy$$
$$\Rightarrow \qquad z = \sqrt{2} \pm i\sqrt{2}, -\sqrt{2} \pm i\sqrt{2}$$

Q. 22 Find the complex number satisfying the equation $z + \sqrt{2}\,\big|(z+1)\big| + i = 0$.

Sol. Given equation is $\qquad z + \sqrt{2}\,\big|(z+1)\big| + i = 0$ $\qquad\qquad\qquad\qquad$ ...(i)

Let $\qquad\qquad\qquad\qquad\qquad\qquad z = x + iy$

$\Rightarrow \qquad\qquad x + iy + \sqrt{2}\,\big|x + iy + 1\big| + i = 0$

$\Rightarrow \qquad x + i(1+y) + \sqrt{2}\left[\sqrt{(x+1)^2 + y^2}\right] = 0$

$\Rightarrow \qquad x + i\,(1+y) + \sqrt{2}\,\sqrt{(x^2 + 2x + 1 + y^2)} = 0$

$\Rightarrow \qquad\qquad x + \sqrt{2}\,\sqrt{x^2 + 2x + 1 + y^2} = 0$

$\Rightarrow \qquad\qquad\qquad\qquad x^2 = 2(x^2 + 2x + 1 + y^2$

$\Rightarrow \qquad\qquad\qquad x^2 + 4x + 2y^2 + 2 = 0$ $\qquad\qquad\qquad\qquad$ ...(ii)

$\qquad\qquad\qquad\qquad\qquad 1 + y = 0$

$\Rightarrow \qquad\qquad\qquad\qquad\qquad y = -1$

For $y = -1$, $\qquad x^2 + 4x + 2 + 2 = 0$ $\qquad\qquad\qquad$ [using Eq. (ii)]

$\Rightarrow \qquad\qquad x^2 + 4x + 4 = 0 \ \Rightarrow \ (x+2)^2 = 0$

$\Rightarrow \qquad\qquad x + 2 = 0 \ \Rightarrow \ x = -2$

$\therefore \qquad\qquad\qquad z = x + iy = -2 - i$

Q. 23 Write the complex number $z = \dfrac{1-i}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}}$ in polar form.

Sol. Given that, $\quad z = \dfrac{1-i}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}} = \dfrac{-\sqrt{2}\left[\dfrac{-1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}\right]}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}}$

$\qquad\qquad\qquad = \dfrac{-\sqrt{2}[\cos(\pi - \pi/4) + i\sin(\pi - \pi/4)]}{\cos\pi/3 + i\sin\pi/3}$

$\qquad\qquad\qquad = \dfrac{-\sqrt{2}[\cos 3\pi/4 + i\sin 3\pi/4]}{\cos\pi/3 + i\sin\,\pi/3}$

$\qquad\qquad\qquad = -\sqrt{2}\left[\cos\left(\dfrac{3\pi}{4} - \dfrac{\pi}{3}\right) + i\sin\left(\dfrac{3\pi}{4} - \dfrac{\pi}{3}\right)\right]$

$\qquad\qquad\qquad = -\sqrt{2}\left[\cos\dfrac{5\pi}{12} + i\sin\dfrac{5\pi}{12}\right]$

Q. 24 If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \dfrac{\pi}{2}$, then show that $\bar{z}\,w = -i$.

Sol. Let $\qquad\qquad z = r_1\,(\cos\theta_1 + i\sin\theta_1)$ and $w = r_2\,(\cos\theta_2 + i\sin\theta_2)$

Also, $\qquad\qquad |zw| = |z|\,|w| = r_1\,r_2 = 1$

$\therefore \qquad\qquad\qquad r_1 r_2 = 1$

Further, $\qquad \arg(z) = \theta_1 \ $ and $\arg(w) = \theta_2$

But $\quad\quad\quad\quad \arg(z) - \arg(w) = \dfrac{\pi}{2}$

$\Rightarrow \quad\quad\quad\quad\quad \theta_1 - \theta_2 = \dfrac{\pi}{2}$

$\Rightarrow \quad\quad\quad\quad\quad \arg\left(\dfrac{z}{w}\right) = \dfrac{\pi}{2}$

Now, $\quad\quad\quad$ to prove $\bar{z}w = -i$

$\quad\quad\quad\quad\quad\quad \text{LHS} = \bar{z}w$

$\quad\quad\quad\quad\quad\quad = r_1(\cos\theta_1 - i\sin\theta_1)\, r_2\, (\cos\theta_2 + i\sin\theta_2)$

$\quad\quad\quad\quad\quad\quad = r_1 r_2 [\cos(\theta_2 - \theta_1) + i\sin(\theta_2 - \theta_1)]$

$\quad\quad\quad\quad\quad\quad = r_1 r_2 [\cos(-\pi/2) + i\sin(-\pi/2)]$

$\quad\quad\quad\quad\quad\quad = 1\,[0 - i]$

$\quad\quad\quad\quad\quad\quad = -i = \text{RHS}$ **Hence proved.**

Q. 25 Fill in the blanks of the following.

(i) For any two complex numbers z_1, z_2 and any real numbers a, b,
$\left|az_1 - bz_2\right|^2 + \left|bz_1 + az_2\right|^2 = \cdots$

(ii) The value of $\sqrt{-25} \times \sqrt{-9}$ is ...

(iii) The number $\dfrac{(1-i)^3}{1-i^3}$ is equal to ...

(iv) The sum of the series $i + i^2 + i^3 + \cdots$ upto 1000 terms is ...

(v) Multiplicative inverse of $1 + i$ is ...

(vi) If z_1 and z_2 are complex numbers such that $z_1 + z_2$ is a real number, then $z_1 = \cdots$

(vii) $\arg(z) + \arg \bar{z}$ where, $(\bar{z} \neq 0)$ is ...

(viii) If $\left|z + 4\right| \leq 3$, then the greatest and least values of $\left|z + 1\right|$ are ... and ...

(ix) If $\left|\dfrac{z-2}{z+2}\right| = \dfrac{\pi}{6}$, then the locus of z is ...

(x) If $|z| = 4$ and $\arg(z) = \dfrac{5\pi}{6}$, then $z = \cdots$

Sol. (i) $\left|az_1 - bz_2\right|^2 + \left|bz_1 + az_2\right|^2$

$\quad\quad = \left|az_1\right|^2 + \left|bz_2\right|^2 - 2\,\mathrm{Re}\,(az_1 \cdot b\bar{z}_2) + \left|bz_1\right|^2 + \left|az_2\right|^2 + 2\,\mathrm{Re}\,(az_1 \cdot b\bar{z}_2)$

$\quad\quad = (a^2 + b^2)\left|z_1\right|^2 + (a^2 + b^2)\left|z_2\right|^2$

$\quad\quad = (a^2 + b^2)\,(\left|z_1\right|^2 + \left|z_2\right|^2)$

(ii) $\sqrt{-25} \times \sqrt{-9} = i\sqrt{25} \times i\sqrt{9} = i^2\,(5 \times 3) = -15$

(iii) $\dfrac{(1-i)^3}{1-i^3} = \dfrac{(1-i)^3}{(1-i)(1+i+i^2)}$

$\quad\quad = \dfrac{(1-i)^2}{i} = \dfrac{1+i^2-2i}{i} = \dfrac{-2i}{i} = -2$

(iv) $i + i^2 + i^3 + \ldots$ upto 1000 terms $= i + i^2 + i^3 + i^4 + \ldots i^{1000} = 0$

$$\left[\because i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, \text{ where } n \in Ni.e., \sum_{n=1}^{1000} i^n = 0 \right]$$

(v) Multiplicative inverse of $1 + i = \dfrac{1}{1+i} = \dfrac{1-i}{1-i^2} = \dfrac{1}{2}(1-i)$

(vi) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, which is real.

If $z_1 + z_2$ is real, then $y_1 + y_2 = 0$

$\Rightarrow$ $y_1 = -y_2$

$\because$ $z_2 = x_2 - iy_1$

$\Rightarrow$ $z_2 = \overline{z_1}$ [when $x_1 = x_2$]

(vii) $\arg(z) + \arg(\overline{z}), (\overline{z} \neq 0)$

$\Rightarrow$ $\theta + (-\theta) = 0$

(viii) Given that, $|z + 4| \leq 3$

For the greatest value of $|z + 1|$.

$\Rightarrow$ $|z + 1| = |z + 4 - 3| \leq |z + 4| + |-3|$

$= |z + 4 - 3| \leq 3 + 3$

$= |z + 4 - 3| \leq 6$

So, greatest value of $|z + 1| = 6$

For, now, least value of $|z + 1|$, we know that the least value of the modulus of a complex number is zero. So, the least value of $|z + 1|$ is zero.

(ix) Given that, $\left| \dfrac{z-2}{z+2} \right| = \dfrac{\pi}{6}$

$\Rightarrow$ $\dfrac{|x + iy - 2|}{|x + iy + 2|} = \dfrac{\pi}{6}$ $\Rightarrow$ $\dfrac{|x - 2 + iy|}{|x + 2 + iy|} = \dfrac{\pi}{6}$

$\Rightarrow$ $6|x - 2 + iy| = \pi |x + 2 + iy|$

$\Rightarrow$ $6\sqrt{(x-2)^2 + y^2} = \pi \sqrt{(x+2)^2 + y^2}$

$\Rightarrow$ $36[x^2 + 4 - 4x + y^2] = \pi^2 [x^2 + 4x + 4 + y^2]$

$\Rightarrow (36 - \pi^2) x^2 + (36 - \pi^2) y^2 - (144 + 4\pi^2) x + 144 + 4\pi^2 = 0$, which is a circle.

(x) Given that, $|z| = 4$ and $\arg (z) = \dfrac{5\pi}{6}$

Let $z = x + iy = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $|z| = r = 4$ and $\arg (z) = \theta$

$\because$ $\tan\theta = \dfrac{5\pi}{6}$

$\Rightarrow$ $z = 4\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right) = 4[\cos(\pi - \pi/6) + i\sin(\pi - \pi/6)]$

$= 4\left[-\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right] = 4\left[\dfrac{-\sqrt{3}}{2} + i\dfrac{1}{2}\right] = -2\sqrt{3} + 2i$

True/False

Q. 26 State true or false for the following.

(i) The order relation is defined on the set of complex numbers.

(ii) Multiplication of a non-zero complex number by $-i$ rotates the point about origin through a right angle in the anti-clockwise direction.

(iii) For any complex number z, the minimum value of $|z| + |z - 1|$ is 1.

(iv) The locus represented by $|z - 1| = |z - i|$ is a line perpendicular to the join of the points (1, 0) and (0, 1).

(v) If z is a complex number such that $z \neq 0$ and Re $(z) = 0$, then, Im $(z^2) = 0$.

(vi) The inequality $|z - 4| < |z - 2|$ represents the region given by $x > 3$.

(vii) Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then arg $(z_1 - z_2) = 0$.

(viii) 2 is not a complex number.

Sol. (i) *False*

We can compare two complex numbers when they are purely real. Otherwise comparison of complex number is not possible.

(ii) *False*

$$(x, y)\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix}, \text{ which is false.}$$

(iii) *True*

Let
$$z = x + iy$$
$$|z| + |z - 1| = \sqrt{x^2 + y^2} + \sqrt{(x-1)^2 + y^2}$$

If $x = 0, y = 0$, then the value of $|z| + |z - 1| = 1$.

(iv) *True*

Let
$$z = x + iy$$
$$|z - 1| = |z - i|$$

Then,
$$|x - 1 + iy| = |x - i(1 - y)|$$
$$(x - 1)^2 + y^2 = x^2 + (1 - y)^2$$
$$x^2 - 2x + 1 + y^2 = x^2 + 1 + y^2 - 2y$$
$$-2x + 1 = 1 - 2y$$
$$-2x + 2y = 0$$
$$x - y = 0 \qquad\qquad ...(i)$$

Equation of a line through the points (1, 0) and (0, 1),
$$y - 0 = \frac{1 - 0}{0 - 1}(x - 1)$$
$$\Rightarrow \qquad\qquad y = -(x - 1) \quad \Rightarrow \quad x + y = 1 \qquad\qquad ...(ii)$$

which is perpendicular to the line $x - y = 0$.

(v) *False*

Let $z = x + iy$, $z \neq 0$ and Re $(z) = 0$

i.e., $\qquad\qquad\qquad\qquad\qquad x = 0$

$\therefore \qquad\qquad\qquad\qquad\qquad z = iy$

$\text{Im}(z^2) = i^2 y^2 = -y^2$ which is real.

(vi) *True*

Given inequality, $\qquad\qquad\qquad |z - 4| < |z - 2|$

Let $\qquad\qquad\qquad\qquad\qquad z = x + iy$

$\therefore \qquad\qquad\qquad\qquad |x - 4 + iy| < |x - 2 + iy|$

$\Rightarrow \qquad\qquad \sqrt{(x-4)^2 + y^2} < \sqrt{(x-2)^2 + y^2}$

$\Rightarrow \qquad\qquad\quad (x-4)^2 + y^2 < (x-2)^2 + y^2$

$\Rightarrow \qquad\qquad x^2 - 8x + 16 + y^2 < x^2 - 4x + 4 + y^2$

$\Rightarrow \qquad\qquad\qquad -8x + 16 < -4x + 4$

$\Rightarrow \qquad\qquad\qquad\qquad -8x < -4x - 12$

$\Rightarrow \qquad\qquad\qquad\qquad\quad -4x < -12$

$\Rightarrow \qquad\qquad\qquad\qquad\qquad 4x > 12$

$\Rightarrow \qquad\qquad\qquad\qquad\qquad x > 3$

(vii) *False*

Let $\qquad z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Given that, $\qquad\qquad\qquad |z_1 + z_2| = |z_1| + |z_2|$

$$|x_1 + iy_1 + x_2 + iy_2| = |x_1 + iy_1| + |x_2 + iy_2|$$

$\Rightarrow \qquad \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} = \sqrt{(x_1^2 + y_1^2)} + \sqrt{(x_2^2 + y_2^2)}$

On squaring both sides, we get

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

$\Rightarrow\ x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$

$\Rightarrow \qquad\qquad 2x_1x_2 + 2y_1y_2 = 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$

$\Rightarrow \qquad\qquad\quad x_1x_2 + y_1y_2 = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$

On squaring both sides, we get

$$x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2 = x_1^2x_2^2 + y_1^2x_2^2 + x_1^2y_2^2 + y_1^2y_2^2$$

$\Rightarrow \qquad\qquad\qquad (x_1y_2 - x_2y_1)^2 = 0$

$\Rightarrow \qquad\qquad\qquad\qquad x_1y_2 = x_2y_1$

$\Rightarrow \qquad\qquad\qquad\qquad \dfrac{y_1}{x_1} = \dfrac{y_2}{x_2}$

$\Rightarrow \qquad\qquad\qquad \left(\dfrac{y_1}{x_1}\right) - \left(\dfrac{y_2}{x_2}\right) = 0$

$\Rightarrow \qquad\qquad \arg(z_1) - \arg(z_2) = 0$

(viii) *True*

We know that, 2 is a real number.

Since, 2 is not a complex number.

Q. 27 Match the statements of Column A and Column B.

	Column A		Column B
(i)	The polar form of $i + \sqrt{3}$ is	(a)	Perpendicular bisector of segment joining $(-2, 0)$ and $(2, 0)$.
(ii)	The amplitude of $-1 + \sqrt{-3}$ is	(b)	On or outside the circle having centre at $(0, -4)$ and radius 3.
(iii)	It $|z + 2| = |z - 2|$, then locus of z is	(c)	$\dfrac{2\pi}{3}$
(iv)	It $|z + 2i| = |z - 2i|$, then locus of z is	(d)	Perpendiculor bisectar of segment joining $(0, -2)$ and $(0, 2)$.
(v)	Region represented by	(e)	$2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$
(vi)	Region represented by $|z + 4| \le 3$ is	(f)	On or inside the circle having centre $(-4, 0)$ and radius 3 units.
(vii)	Conjugate of $\dfrac{1 + 2i}{1 - i}$ lies in	(g)	First quadrant
(viii)	Reciprocal of $1 - i$ lies in	(h)	Third quadrant

Sol. **(i)** Given,

$$z = i + \sqrt{3} = r(\cos\theta + \sin\theta)$$

$\because$ $\qquad\qquad r\cos\theta = \sqrt{3}, r\sin\theta = 1$

$\Rightarrow$ $\qquad\qquad r^2 = 1 + 3 = 4 \Rightarrow r = 2$ $\qquad\qquad [\because r > 0]$

$\Rightarrow$ $\qquad\qquad \tan\alpha = \left|\dfrac{r\sin\theta}{r\cos\theta}\right| = \dfrac{1}{\sqrt{3}}$

$\Rightarrow$ $\qquad\qquad \tan\alpha = \dfrac{1}{\sqrt{3}} \Rightarrow \alpha = \dfrac{\pi}{6}$

$\because$ $\qquad\qquad x > 0, \quad y > 0$

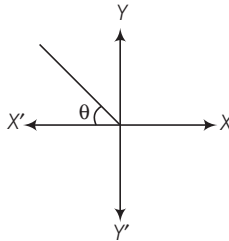and $\qquad\qquad \arg(z) = \theta = \dfrac{\pi}{6}$

So the polar form of z is $2\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$.

(ii) Given that, $\qquad\qquad z = -1 + \sqrt{-3} = -1 + i\sqrt{3}$

$\therefore$ $\qquad\qquad \tan\alpha = \left|\dfrac{\sqrt{3}}{-1}\right| = \sqrt{3}$

$\Rightarrow$ $\qquad\qquad \tan\alpha = \tan\dfrac{\pi}{3} \Rightarrow \alpha = \dfrac{\pi}{3}$

$\because$ $\qquad\qquad x < 0, \quad y > 0$

$\Rightarrow$ $\qquad\qquad \theta = \pi - \alpha = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

(iii) Given that,
$$|z + 2| = |z - 2|$$
$$\Rightarrow \quad |x + 2 + iy| = |x - 2 + iy|$$
$$\Rightarrow \quad (x + 2)^2 + y^2 = (x - 2)^2 + y^2$$
$$\Rightarrow \quad x^2 + 4x + 4 = x^2 - 4x + 4 \quad \Rightarrow \quad 8x = 0$$
$$\therefore \quad x = 0$$
It is a straight line which is a perpendicular bisector of segment joining the points $(-2, 0)$ and $(2, 0)$.

(iv) Given that,
$$|z + 2i| = |z - 2i|$$
$$\Rightarrow \quad |x + i(y + 2)| = |x + i(y - 2)|$$
$$\Rightarrow \quad x^2 + (y + 2)^2 = x^2 + (y - 2)^2$$
$$\Rightarrow \quad 4y = 0 \quad \Rightarrow \quad y = 0$$
It is a straight line, which is a perpendicular bisector of segment joining $(0, -2)$ and $(0, 2)$.

(v) Given that,
$$|z + 4i| \geq 3 = |x + iy + 4i| \geq 3$$
$$\Rightarrow \qquad = |x + i(y + 4)| \geq 3$$
$$\Rightarrow \qquad = \sqrt{x^2 + (y + 4)^2} \geq 3$$
$$\Rightarrow \qquad = x^2 + (y + 4)^2 \geq 9$$
$$\Rightarrow \qquad = x^2 + y^2 + 8y + 16 \geq 9$$
$$\qquad = x^2 + y^2 + 8y + 7 \geq 0$$
Which represent a circle. On or outside having centre $(0, -4)$ and radius 3.

(vi) Given that,
$$|z + 4| \leq 3$$
$$\Rightarrow \qquad |x + iy + 4| \leq 3$$
$$\Rightarrow \qquad |x + 4 + iy| \leq 3$$
$$\Rightarrow \qquad \sqrt{(x + 4)^2 + y^2} \leq 3$$
$$\Rightarrow \qquad (x + 4)^2 + y^2 \leq 9$$
$$\Rightarrow \qquad x^2 + 8x + 16 + y^2 \leq 9$$
$$\Rightarrow \qquad x^2 + 8x + y^2 + 7 \leq 0$$
It represent the region which is on or inside the circle having the centre $(-4, 0)$ and radius 3.

(vii) Given that,
$$z = \frac{1 + 2i}{1 - i} = \frac{(1 + 2i)(1 + i)}{(1 - i)(1 + i)}$$
$$= \frac{1 + 2i + i + 2i^2}{1 - i^2} = \frac{1 - 2 + 3i}{1 + 1} = \frac{-1 + 3i}{2}$$
$$\therefore \qquad \bar{z} = \frac{-1}{2} - \frac{3i}{2}$$
Hence, $\left(\dfrac{-1}{2}, \dfrac{-3}{2}\right)$ lies in third quadrant.

(viii) Given that, $z = 1 - i$
$$\therefore \qquad \frac{1}{z} = \frac{1}{1 - i} = \frac{1 + i}{(1 - i)(1 + i)} = \frac{1 + i}{1 - i^2} = \frac{1}{2}(1 + i)$$
Hence, $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ lies in first quadrant.

Hence, the correct matches are **(a)** $\rightarrow$ (v), **(b)** $\rightarrow$ (iii), **(c)** $\rightarrow$ (i), **(d)** $\rightarrow$ (iv), **(e)** $\rightarrow$ (ii), **(f)** $\rightarrow$ (vi), **(g)** $\rightarrow$ (viii), **(h)** $\rightarrow$ (vii)

Q. 28 What is the conjugate of $\dfrac{2-i}{(1-2i)^2}$?

Sol.   Given that, $z = \dfrac{2-i}{(1-2i)^2} = \dfrac{2-i}{1+4i^2-4i}$

$$= \dfrac{2-i}{1-4-4i} = \dfrac{2-i}{-3-4i}$$

$$= \dfrac{(2-i)}{-(3+4i)} = -\left[\dfrac{(2-i)(3-4i)}{(3+4i)(3-4i)}\right]$$

$$= -\left(\dfrac{6-8i-3i+4i^2}{9+16}\right) = -\dfrac{(-11i+2)}{25}$$

$$= \dfrac{-1}{25}(2-11i) \Rightarrow z = \dfrac{1}{25}(-2+11i)$$

∴ $\bar{z} = \dfrac{1}{25}(-2-11i) = \dfrac{-2}{25} - \dfrac{11}{25}i$

Q. 29 If $|z_1| = |z_2|$, is it necessary that $z_1 = z_2$.

Sol. Given that, $|z_1| = |z_2|$
Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$
$\Rightarrow$ $|x_1 + iy_1| = |x_2 + iy_2|$
$\Rightarrow$ $x_1^2 + y_1^2 = x_2^2 + y_2^2$
$\Rightarrow$ $x_1^2 = x_2^2,\ y_1^2 = y_2^2$
$\Rightarrow$ $x_1 = \pm x_2,\ y_1 = \pm y_2$
$\Rightarrow$ $z_1 = x_1 + iy_1$ or $z_1 = \pm x_2 \pm iy_2$
Hence, it is not neccessary that $z_1 = z_2$.

Q. 30 If $\dfrac{(a^2+1)^2}{2a-i} = x+iy$, then what is the value of $x^2 + y^2$?

Sol. Given that, $\dfrac{(a^2+1)^2}{2a-i} = x+iy \Rightarrow \dfrac{(a^2+1)^2}{(2a-i)} = x+iy$

$\Rightarrow$ $\dfrac{(a^2+1)^2(2a+i)}{(2a-i)(2a+i)} = x+iy$

$\Rightarrow$ $\dfrac{(a^2+1)^2(2a+i)}{4a^2+1} = x+iy$

$\Rightarrow$ $x = \dfrac{2a(a^2+1)^2}{4a^2+1}$ and $y = \dfrac{(a^2+1)^2}{4a^2+1}$

∴ $x^2 + y^2 = 4a^2\left[\dfrac{(a^2+1)^2}{4a^2+1}\right]^2 + \left[\dfrac{(a^2+1)^2}{4a^2+1}\right]^2$

$$= \dfrac{(4a^2+1)(a^2+1)^4}{(4a^2+1)^2} = \dfrac{(a^2+1)^4}{(4a^2+1)}$$

Q. 31 Find the value of z, if $|z| = 4$ and arg $(z) = \dfrac{5\pi}{6}$.

Sol. Let $\qquad\qquad\qquad\qquad z = r\,(\cos\theta + i\sin\theta)$

Also, $\qquad\qquad\qquad |z| = r = 4$ and $\theta = \dfrac{5\pi}{6}$ $\qquad\qquad\qquad\qquad$ [∵ arg $(z) = \theta$]

∴ $\qquad\qquad\qquad\qquad z = 4\left[\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right]$ $\qquad\qquad$ [∵ $z = r\,(\cos\theta + i\sin\theta)$]

$\qquad\qquad\qquad\qquad = 4\left[\cos\left(\pi - \dfrac{\pi}{6}\right) + i\sin\left(\pi - \dfrac{\pi}{6}\right)\right]$

$\qquad\qquad\qquad\qquad = 4\left[-\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right]$

$\qquad\qquad\qquad\qquad = 4\left[-\dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}\right] = -2\sqrt{3} + 2i$

Q. 32 Find the value of $\left|(1+i)\dfrac{(2+i)}{(3+i)}\right|$.

> **💡 Thinking Process**
>
> *First, convert the given expression in the formed $a + ib$, then use $|a + ib| = \sqrt{a^2 + b^2}$.*

Sol. Given that, $\quad \left|(1+i)\dfrac{(2+i)}{(3+i)}\right| = \left|\dfrac{(2 + i + 2i + i^2)}{(3+i)}\right| = \left|\dfrac{2 + 3i - 1}{3 + i}\right|$

$\qquad\qquad\qquad\qquad = \left|\dfrac{1 + 3i}{3 + i}\right| = \left|\dfrac{(1 + 3i)(3 - i)}{(3 + i)(3 - i)}\right|$

$\qquad\qquad\qquad\qquad = \left|\dfrac{3 + 9i - i - 3i^2}{9 - i^2}\right| = \left|\dfrac{3 + 8i + 3}{9 + 1}\right| = \left|\dfrac{6 + 8i}{10}\right|$

$\qquad\qquad\qquad\qquad = \sqrt{\dfrac{6^2}{100} + \dfrac{8^2}{100}} = \sqrt{\dfrac{36 + 64}{100}} = \sqrt{\dfrac{100}{100}} = 1$

Q. 33 Find the principal argument of $(1 + i\sqrt{3})^2$.

> **💡 Thinking Process**
>
> *Let $z = a + ib$, then the polar form of z is $r\,(\cos\theta + i\sin\theta)$, where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = \dfrac{b}{a}$. Here, θ is argument or amplitude of z i.e., arg $(z) = \theta$. The principal argument is a unique value of θ such that $-\pi \le \theta \le \pi$.*

Sol. Given that, $\qquad z = (1 + i\sqrt{3})^2$

$\Rightarrow \qquad\qquad z = 1 - 3 + 2i\sqrt{3} \quad \Rightarrow \quad z = -2 + i2\sqrt{3}$

$\Rightarrow \qquad\qquad \tan\alpha = \left|\dfrac{2\sqrt{3}}{-2}\right| = \left|-\sqrt{3}\right| = \sqrt{3}$ $\qquad\qquad \left[∵ \tan\alpha = \left|\dfrac{\text{Im}\,(z)}{\text{Re}\,(z)}\right|\right]$

$\Rightarrow \qquad\qquad \tan\alpha = \tan\dfrac{\pi}{3} \quad \Rightarrow \quad \alpha = \dfrac{\pi}{3}$

∵ $\qquad\qquad$ Re $(z) < 0$ and Im $(z) > 0$

$\Rightarrow \qquad\qquad$ arg $(z) = \pi - \dfrac{\pi}{3} \quad \Rightarrow \quad = \dfrac{2\pi}{3}$

Q. 34 Where does z lie, if $\left|\dfrac{z-5i}{z+5i}\right| = 1$?

> **Thinking Process**
>
> If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $|z_1| = \sqrt{x_1^2 + y_1^2}$ and $|z_2| = \sqrt{x_2^2 + y_2^2}$.
>
> Also, use the modulus property i.e., $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$,

Sol. Let $\qquad\qquad z = x + iy$

Given that, $\qquad \left|\dfrac{z-5i}{z+5i}\right| = \left|\dfrac{x+iy-5i}{x+iy+5i}\right|$

$\Rightarrow \qquad \left|\dfrac{z-5i}{z+5i}\right| = \dfrac{|x+i(y-5)|}{|x+i(y+5)|} \qquad\qquad \left[\because \left|\dfrac{z-5i}{z+5i}\right| = 1\right]$

$\Rightarrow \qquad \left|\dfrac{z-5i}{z+5i}\right| = \dfrac{\sqrt{x^2+(y-5)^2}}{\sqrt{x^2+(y+5)^2}}$

On squaring both sides, we get

$$x^2 + (y-5)^2 = x^2 + (y+5)^2$$

$\Rightarrow \qquad\qquad -10y = +10y$

$\Rightarrow \qquad\qquad 20y = 0$

$\therefore \qquad\qquad y = 0$

So, z lies on real axis.

Objective Type Questions

Q. 35 $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other for

(a) $x = n\pi$

(b) $x = \left(n + \dfrac{1}{2}\right)\dfrac{\pi}{2}$

(c) $x = 0$

(d) No value of x

Sol. *(d)* Let $\qquad\qquad z = \sin x + i\cos 2x$

and $\qquad\qquad \bar{z} = \sin x - i\cos 2x \qquad\qquad\qquad …(i)$

Given that, $\qquad\qquad \bar{z} = \cos x - i\sin 2x \qquad\qquad\qquad …(ii)$

$\therefore \qquad \sin x - i\cos 2x = \cos x - i\sin 2x$

$\Rightarrow \qquad \sin x = \cos x$ and $\cos 2x = \sin 2x$

$\Rightarrow \qquad \tan x = 1$ and $\tan 2x = 1$

$\Rightarrow \qquad \tan x = \tan \dfrac{\pi}{4}$ and $\tan 2x = \tan \dfrac{\pi}{4}$

$\Rightarrow \qquad x = n\pi + \dfrac{\pi}{4}$ and $2x = n\pi + \dfrac{\pi}{4}$

$\Rightarrow \qquad 2x - x = 0 \Rightarrow x = 0$

Q. 36 The real value of α for which the expression $\dfrac{1 - i \sin\alpha}{1 + 2i \sin\alpha}$ is purely real is

(a) $(n + 1)\dfrac{\pi}{2}$ (b) $(2n + 1)\dfrac{\pi}{2}$ (c) $n\pi$ (d) None of these

where, $n \in N$

💡 **Thinking Process**

First, convert the given expansion into $a + ib$ form and then check whether the complex number $a + ib$ is purely real.

Sol. *(c)*   Given expression, $z = \dfrac{1 - i \sin\alpha}{1 + 2i \sin\alpha}$ [let]

$$= \frac{(1 - i \sin\alpha)(1 - 2i \sin\alpha)}{(1 + 2i \sin\alpha)(1 - 2i \sin\alpha)}$$

$$= \frac{1 - i \sin\alpha - 2i \sin\alpha + 2i^2 \sin^2\alpha}{1 - 4i^2 \sin^2\alpha}$$

$$= \frac{1 - 3i \sin\alpha - 2\sin^2\alpha}{1 + 4\sin^2\alpha}$$

$$= \frac{1 - 2\sin^2\alpha}{1 + 4\sin^2\alpha} - \frac{3i \sin\alpha}{1 + 4\sin^2\alpha}$$

It is given that z is a purely real.

$\therefore$ $\dfrac{-3\sin\alpha}{1 + 4\sin^2\alpha} = 0$

$\Rightarrow$ $-3\sin\alpha = 0 \Rightarrow \sin\alpha = 0$

                 $\alpha = n\pi$

Q. 37 If $z = x + iy$ lies in the third quadrant, then $\dfrac{\bar{z}}{z}$ also lies in the third quadrant, if

(a) $x > y > 0$ (b) $x < y < 0$ (c) $y < x < 0$ (d) $y > x > 0$

Sol. *(b)*   Given that, $z = x + iy$ lies in third quadrant.

                     $x < 0$ and $y < 0$.

Now, $\dfrac{\bar{z}}{z} = \dfrac{x - iy}{x + iy} = \dfrac{(x - iy)(x - iy)}{(x + iy)(x - iy)} = \dfrac{x^2 - y^2 - 2ixy}{x^2 + y^2}$

               $\dfrac{\bar{z}}{z} = \dfrac{x^2 - y^2}{x^2 + y^2} - \dfrac{2ixy}{x^2 + y^2}$

Since, $\dfrac{\bar{z}}{z}$ also lies in third quadrant.

$\therefore$ $\dfrac{x^2 - y^2}{x^2 + y^2} < 0$ and $\dfrac{-2xy}{x^2 + y^2} < 0$

                  $x^2 - y^2 < 0$ and $-2xy < 0$

$\Rightarrow$ $x^2 < y^2$ and $xy > 0$

So, $x < y < 0$

Q. 38 The value of $(z + 3)(\bar{z} + 3)$ is equivalent to

    (a) $\left| z + 3 \right|^2$ (b) $\left| z - 3 \right|$ (c) $z^2 + 3$ (d) None of these

Sol. *(a)* Given that, $(z + 3)(\bar{z} + 3)$

Let $\qquad\qquad z = x + iy$

$\Rightarrow \qquad (z + 3)(\bar{z} + 3) = (x + iy + 3)(x + 3 - iy)$

$\qquad\qquad\qquad\qquad = (x + 3)^2 - (iy)^2 = (x + 3)^2 + y^2$

$\qquad\qquad\qquad\qquad = \left| x + 3 + iy \right|^2 = \left| z + 3 \right|^2$

Q. 39 If $\left(\dfrac{1 + i}{1 - i} \right)^x = 1$, then

    (a) $x = 2n + 1$ (b) $x = 4n$ (c) $x = 2n$ (d) $x = 4n + 1$

    where, $n \in N$

Sol. *(b)* Given that, $\qquad \left(\dfrac{1 + i}{1 - i} \right)^x = 1$

$\Rightarrow \qquad \left[\dfrac{(1 + i)(1 + i)}{(1 - i)(1 + i)} \right]^x = 1 \Rightarrow \left[\dfrac{1 + 2i + i^2}{1 - i^2} \right]^x = 1$

$\Rightarrow \qquad \left[\dfrac{2i}{1 + 1} \right]^x = 1 \Rightarrow \left[\dfrac{2i}{2} \right]^x = 1$

$\Rightarrow \qquad i^x = 1 \Rightarrow i^x = (i^{4n}) \qquad\qquad\qquad [\because i^{4n} = 1, n \in N]$

$\Rightarrow \qquad x = 4n$

Q. 40 A real value of x satisfies the equation $\left(\dfrac{3 - 4ix}{3 + 4ix} \right) = \alpha - i\beta$ $(\alpha, \beta \in R)$, if

$\alpha^2 + \beta^2$ is equal to

    (a) 1   (b) -1 (c) 2   (d) -2

Sol. *(a)* Given equation, $\left(\dfrac{3 - 4ix}{3 + 4ix} \right) = \alpha - i\beta$ $(\alpha, \beta \in R)$

$\Rightarrow \qquad \left[\dfrac{3 - 4ix}{3 + 4ix} \right] = \alpha - i\beta$

Now, $\qquad (\alpha - i\beta) = \dfrac{(3 - 4ix)(3 - 4ix)}{(3 + 4ix)(3 - 4ix)} = \dfrac{9 + 16i^2x^2 - 24ix}{9 - 16i^2x^2}$

$\Rightarrow \qquad \alpha - i\beta = \dfrac{9 - 16x^2 - 24ix}{9 + 16x^2}$

$\Rightarrow \qquad \alpha - i\beta = \dfrac{9 - 16x^2}{9 + 16x^2} - \dfrac{i24x}{9 + 16x^2} \qquad\qquad\qquad \dots\text{(i)}$

$\therefore \qquad \alpha + i\beta = \dfrac{9 - 16x^2}{9 + 16x^2} + \dfrac{i24x}{9 + 16x^2} \qquad\qquad\qquad \dots\text{(ii)}$

So, $(\alpha - i\beta)(\alpha + i\beta) = \left(\dfrac{9 - 16x^2}{9 + 16x^2}\right)^2 - \left(\dfrac{i24x}{9 + 16x^2}\right)^2$

∴ $\alpha^2 + \beta^2 = \dfrac{81 + 256x^4 - 288x^2 + 576x^2}{(9 + 16x^2)^2}$

$= \dfrac{81 + 256x^4 + 288x^2}{(9 + 16x^2)^2}$

$= \dfrac{(9 + 16x^2)^2}{(9 + 16x^2)^2} = 1$

Q. 41 Which of the following is correct for any two complex numbers z_1 and z_2?

(a) $\left| z_1 z_2 \right| = \left| z_1 \right| \left| z_2 \right|$ (b) $\arg (z_1 z_2) = \arg (z_1) \cdot \arg (z_2)$

(c) $\left| z_1 + z_2 \right| = \left| z_1 \right| + \left| z_2 \right|$ (d) $\left| z_1 + z_2 \right| \geq \left| z_1 \right| - \left| z_2 \right|$

Sol. *(a)* Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$

⇒ $\left| z_1 \right| = r_1$ ...(i)

and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

⇒ $\left| z_2 \right| = r_2$ ...(ii)

Now, $z_1 z_2 = r_1 r_2 [\cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i^2 \sin \theta_1 \sin \theta_2]$

$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$

⇒ $\left| z_1 z_2 \right| = r_1 r_2$

∴ $\left| z_1 z_2 \right| = \left| z_1 \right| \left| z_2 \right|$ [using Eqs. (i) and (ii)]

Q. 42 The point represented by the complex number $(2 - i)$ is rotated about origin through an angle $\dfrac{\pi}{2}$ in the clockwise direction, the new position of point is

(a) $1 + 2i$ (b) $-1 - 2i$ (c) $2 + i$ (d) $-1 + 2i$

💡 **Thinking Process**

Here, $z < i\alpha$ is a complex number, where modulus is r and argument $(\theta + \alpha)$. If P (z) rotates in clockwise sense through an angle α, then its new position will be $z(\theta - i\alpha)$.

Sol. *(b)* Given that, $z = 2 - i$

It is rotated about origin through an angle $\dfrac{\pi}{2}$ in the clockwise direction

∴   New position $= ze^{-i\pi/2} = (2 - i)e^{-i\pi/2}$

$= (2 - i)\left[\cos\left(\dfrac{-\pi}{2}\right) + i \sin\left(\dfrac{-\pi}{2}\right)\right] = (2 - i)[0 - i]$

$= -2i - 1 = -1 - 2i$

Q. 43 If $x, y \in R$, then $x + iy$ is a non-real complex number, if

(a) $x = 0$ (b) $y = 0$ (c) $x \neq 0$ (d) $y \neq 0$

Sol. *(d)* Given that, $x, y \in R$

Then, $x + iy$ is non-real complex number if and only if $y \neq 0$.

Q. 44 If $a + ib = c + id$, then

(a) $a^2 + c^2 = 0$ (b) $b^2 + c^2 = 0$

(c) $b^2 + d^2 = 0$ (d) $a^2 + b^2 = c^2 + d^2$

💡 **Thinking Process**

If two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, then

$$|z_1| = |z_2| \implies \sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$$

Sol. *(d)* Given that, $a + ib = c + id$

$\implies$ $|a + ib| = |c + id|$

$\implies$ $\sqrt{a^2 + b^2} = \sqrt{c^2 + d^2}$

On squaring both sides, we get

$$a^2 + b^2 = c^2 + d^2$$

Q. 45 The complex number z which satisfies the condition $\left|\dfrac{i + z}{i - z}\right| = 1$ lies on

(a) circle $x^2 + y^2 = 1$ (b) the X-axis

(c) the Y-axis   (d) the line $x + y = 1$

Sol. *(b)* Given that, $\left|\dfrac{i + z}{i - z}\right| = 1$

Let $z = x + iy$

$\therefore$ $\left|\dfrac{x + i(y + 1)}{-x - i(y - 1)}\right| = 1 \implies \dfrac{x^2 + (y + 1)^2}{x^2 + (y - 1)^2} = 1$

$\implies$ $x^2 + (y + 1)^2 = x^2 + (y - 1)^2$

$\implies$ $4y = 0 \implies y = 0$

So, z lies on X-axis (real axis).

Q. 46 If z is a complex number, then

(a) $|z^2| > |z|$ (b) $|z^2| = |z|^2$ (c) $|z^2| < |z|^2$ (d) $|z^2| \geq |z|^2$

Sol. *(b)* If z is a complex number, then $z = x + iy$

$$|z| = |x + iy| \quad \text{and} \quad |z|^2 = |x + iy|^2$$

$\implies$ $|z|^2 = x^2 + y^2$ ...(i)

and $z^2 = (x + iy)^2 = x^2 + i^2 y^2 + i2xy$

                 $z^2 = x^2 - y^2 + i2xy$

$\implies$ $|z^2| = \sqrt{(x^2 - y^2)^2 + (2xy)^2}$

$\implies$ $|z^2| = \sqrt{x^4 + y^4 - 2x^2 y^2 + 4x^2 y^2}$

$\implies$ $|z^2| = \sqrt{x^4 + y^4 + 2x^2 y^2} = \sqrt{(x^2 + y^2)^2}$

$\implies$ $|z^2| = x^2 + y^2$ ...(ii)

From Eqs. (i) and (ii),

$$|z|^2 = |z^2|$$

Q. 47 $|z_1 + z_2| = |z_1| + |z_2|$ is possible, if

(a) $z_2 = \bar{z}_1$

(b) $z_2 = \dfrac{1}{z_1}$

(c) $\arg(z_1) = \arg(z_2)$

(d) $|z_1| = |z_2|$

Sol. *(c)*   Given that, $|z_1 + z_2| = |z_1| + |z_2|$

$\Rightarrow$ $|r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)| = |r_1(\cos\theta_1 + i\sin\theta_1)|$
$$+ |r_2(\cos\theta_2 + i\sin\theta_2)|$$

$\Rightarrow$ $|(r_1\cos\theta_1 + r_2\cos\theta) + i(r_1\sin\theta_1 + r_2\sin\theta_2)| = r_1 + r_2$

$\Rightarrow$ $\sqrt{r_1^2\cos^2\theta_1 + r_2^2\cos^2\theta_2 + 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\sin^2\theta_1 + r_2^2\sin^2\theta_2}$
$$\sqrt{+ 2r_1r_2\sin\theta_1\sin\theta_2} = r_1 + r_2$$

$\Rightarrow$ $\sqrt{r_1^2 + r_2^2 + 2r_1r_2[\cos(\theta_1 - \theta_2)]} = r_1 + r_2$

On squaring both sides, we get
$$r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 + 2r_1r_2$$

$\Rightarrow$ $2r_1r_2[1 - \cos(\theta_1 - \theta_2)] = 0$

$\Rightarrow$ $1 - \cos(\theta_1 - \theta_2) = 0$

$\Rightarrow$ $\cos(\theta_1 - \theta_2) = 1$

$\Rightarrow$ $\cos(\theta_1 - \theta_2) = \cos 0°$

$\Rightarrow$ $\theta_1 - \theta_2 = 0°$

$\Rightarrow$ $\theta_1 = \theta_2$

$\therefore$ $\arg(z_1) = \arg(z_2)$

Q. 48 The real value of θ for which the expression $\dfrac{1 + i\cos\theta}{1 - 2i\cos\theta}$ is a real number is

(a) $n\pi + \dfrac{\pi}{4}$

(b) $n\pi + (-1)^n\,\dfrac{\pi}{4}$

(c) $2n\pi \pm \dfrac{\pi}{2}$

(d) None of these

Sol. *(c)*   Given expression $= \dfrac{1 + i\cos\theta}{1 - 2i\cos\theta} = \dfrac{(1 + i\cos\theta)(1 + 2i\cos\theta)}{(1 - 2i\cos\theta)(1 + 2i\cos\theta)}$

$$= \dfrac{1 + i\cos\theta + 2i\cos\theta + 2i^2\cos^2\theta}{1 - 4i^2\cos^2\theta}$$

$$= \dfrac{1 + 3i\cos\theta - 2\cos^2\theta}{1 + 4\cos^2\theta}$$

For real value of θ, $\dfrac{3\cos\theta}{1 + 4\cos^2\theta} = 0$

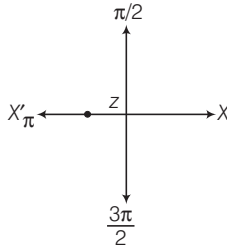$\Rightarrow$ $3\cos\theta = 0$

$\Rightarrow$ $\cos\theta = \cos\dfrac{\pi}{2}$

$\Rightarrow$ $\theta = 2n\pi \pm \dfrac{\pi}{2}$

Q. 49 The value of arg (x), when $x < 0$ is

 (a) 0 (b) $\dfrac{\pi}{2}$ (c) π (d) None of these

Sol. (c) Let $\qquad\qquad z = x + 0i$ and $x < 0$

$$|z| = \sqrt{(-1)^2 + (0^2)} = 1$$

Since, the point $(x, 0)$ represent $z = x + 0i$ lies on the negative side of real axis.

∴ Principal arg $(z) = \pi$

Q. 50 If $f(z) = \dfrac{7 - z}{1 - z^2}$, where $z = 1 + 2i$, then $\big|f(z)\big|$ is equal to

 (a) $\dfrac{|z|}{2}$ (b) $\big|z\big|$

 (c) $2\big|z\big|$ (d) None of these

Sol. (a) Let $\qquad\qquad z = 1 + 2i$

$\Rightarrow \qquad\qquad |z| = \sqrt{1 + 4} = \sqrt{5}$

Now, $\qquad\qquad f(z) = \dfrac{7 - z}{1 - z^2} = \dfrac{7 - 1 - 2i}{1 - (1 + 2i)^2}$

$$= \dfrac{6 - 2i}{1 - 1 - 4i^2 - 4i} = \dfrac{6 - 2i}{4 - 4i}$$

$$= \dfrac{(3 - i)(2 + 2i)}{(2 - 2i)(2 + 2i)}$$

$$= \dfrac{6 - 2i + 6i - 2i^2}{4 - 4i^2} = \dfrac{6 + 4i + 2}{4 + 4}$$

$$= \dfrac{8 + 4i}{8} = 1 + \dfrac{1}{2}i$$

$$f(z) = 1 + \dfrac{1}{2}i$$

∴ $\qquad\qquad \big|f(z)\big| = \sqrt{1 + \dfrac{1}{4}} = \sqrt{\dfrac{4 + 1}{4}} = \dfrac{\sqrt{5}}{2} = \dfrac{|z|}{2}$