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Continuity and Differentiability

Short Answer Type Questions

Q. 1 Examine the continuity of the function $f(x) = x^3 + 2x^2 - 1$ at $x = 1$.

Thinking Process

We know that, function f will be continuous at $x = a$, if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$.

Sol. We have, $f(x) = x^3 + 2x^2 - 1$ at $x = 1$.

$\therefore$ $\lim_{x \to 1^+} f(x) = \lim_{h \to 0} (1 + h)^3 + 2(1 + h)^2 - 1 = 2$

and $\lim_{x \to 1^-} f(x) = \lim_{h \to 0} (1 - h)^3 + 2(1 - h)^2 - 1 = 2$

$\therefore$ $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$

and $f(1) = 1 + 2 - 1 = 2$

So, $f(x)$ is continuous at $x = 1$.

Note *Every polynomial function is continuous at any real point.*

Q. 2 $f(x) = \begin{cases} 3x + 5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$ at $x = 2$.

Sol. We have, $f(x) = \begin{cases} 3x + 5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$ at $x = 2$.

At $x = 2$, $\text{LHL} = \lim_{x \to 2^-} (x)^2$

$= \lim_{h \to 0} (2 - h)^2 = \lim_{h \to 0} (4 + h^2 - 4h) = 4$

and $\text{RHL} = \lim_{x \to 2^+} (3x + 5)$

$= \lim_{h \to 0} [3(2 + h) + 5] = 11$

Since, $\text{LHL} \neq \text{RHL}$ at $x = 2$

So, $f(x)$ is discontinuous at $x = 2$.

Q. 3 $f(x)=\begin{cases}\dfrac{1-\cos 2x}{x^2}, & \text{if } x\neq 0\\ 5, & \text{if } x=0\end{cases}$ at $x=0$.

Sol. We have, $f(x)=\begin{cases}\dfrac{1-\cos 2x}{x^2}, & \text{if } x\neq 0\\ 5, & \text{if } x=0\end{cases}$ at $x=0$.

At $x=0$,

$$\text{LHL}=\lim_{x\to 0^-}\frac{1-\cos 2x}{x^2}$$

$$=\lim_{h\to 0}\frac{1-\cos 2(0-h)}{(0-h)^2}$$

$$=\lim_{h\to 0}\frac{1-\cos 2h}{h^2} \qquad [\because \cos(-\theta)=\cos\theta]$$

$$=\lim_{h\to 0}\frac{1-1+2\sin^2 h}{h^2} \qquad [\because \cos 2\theta=1-2\sin^2\theta]$$

$$=\lim_{h\to 0}\frac{2(\sin h)^2}{(h)^2} \qquad \left[\because \lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

$$=2$$

$$\text{RHL}=\lim_{x\to 0^+}\frac{1-\cos 2x}{x^2}$$

$$=\lim_{h\to 0}\frac{1-\cos 2(0+h)}{(0+h)^2}$$

$$=\lim_{h\to 0}\frac{2\sin^2 h}{h^2}=2 \qquad \left[\because \lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

and $f(0)=5$

Since, LHL $=$ RHL $\neq f(0)$

Hence, $f(x)$ is not continuous at $x=0$.

Q. 4 $f(x)=\begin{cases}\dfrac{2x^2-3x-2}{x-2}, & \text{if } x\neq 2\\ 5, & \text{if } x=2\end{cases}$ at $x=2$.

Sol. We have, $f(x)=\begin{cases}\dfrac{2x^2-3x-2}{x-2}, & \text{if } x\neq 2\\ 5, & \text{if } x=2\end{cases}$ at $x=2$.

At $x=2$,

$$\text{LHL}=\lim_{x\to 2^-}\frac{2x^2-3x-2}{x-2}$$

$$=\lim_{h\to 0}\frac{2(2-h)^2-3(2-h)-2}{(2-h)-2}$$

$$=\lim_{h\to 0}\frac{8+2h^2-8h-6+3h-2}{-h}$$

$$=\lim_{h\to 0}\frac{2h^2-5h}{-h}=\lim_{h\to 0}\frac{h(2h-5)}{-h}=5$$

$$\text{RHL}=\lim_{x\to 2^+}\frac{2x^2-3x-2}{x-2}$$

$$= \lim_{h \to 0} \frac{2(2+h)^2 - 3(2+h) - 2}{(2+h) - 2}$$

$$= \lim_{h \to 0} \frac{8 + 2h^2 + 8h - 6 - 3h - 2}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 + 5h}{h} = \lim_{h \to 0} \frac{h(2h+5)}{h} = 5$$

and $f(2) = 5$

$\therefore$ LHL = RHL = $f(2)$

So, $f(x)$ is continuous at $x = 2$.

Q. 5 $f(x) = \begin{cases} \dfrac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases}$ **at** $x = 4$.

Sol. We have,

$$f(x) = \begin{cases} \dfrac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases} \text{ at } x = 4.$$

At $x = 4$,

$$\text{LHL} = \lim_{x \to 4^-} \frac{|x-4|}{2(x-4)}$$

$$= \lim_{h \to 0} \frac{|4-h-4|}{2[(4-h)-4]} = \lim_{h \to 0} \frac{|0-h|}{(8-2h-8)}$$

$$= \lim_{h \to 0} \frac{h}{-2h} = \frac{-1}{2} \quad \text{and} \quad f(4) = 0 \neq \text{LHL}$$

So, $f(x)$ is discontinuous at $x = 4$.

Q. 6 $f(x) = \begin{cases} |x| \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ **at** $x = 0$.

Sol. We have,

$$f(x) = \begin{cases} |x| \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \text{ at } x = 0$$

At $x = 0$,

$$\text{LHL} = \lim_{x \to 0^-} |x| \cos \frac{1}{x} = \lim_{h \to 0} |0-h| \cos \frac{1}{0-h}$$

$$= \lim_{h \to 0} h \cos \left(\frac{-1}{h}\right)$$

$$= 0 \times [\text{an oscillating number between } -1 \text{ and } 1] = 0$$

$$\text{RHL} = \lim_{x \to 0^+} |x| \cos \frac{1}{x}$$

$$= \lim_{h \to 0} |0+h| \cos \frac{1}{(0+h)}$$

$$= \lim_{h \to 0} h \cos \frac{1}{h}$$

$$= 0 \times [\text{an oscillating number between } -1 \text{ and } 1] = 0$$

and $f(0) = 0$

Since, LHL = RHL = $f(0)$

So, $f(x)$ is continuous at $x = 0$.

Q. 7 $f(x) = \begin{cases} |x-a| \sin \dfrac{1}{x-a}, & \text{if } x \neq 0 \\ 0, & \text{if } x = a \end{cases}$ at $x = a$.

Sol. We have, $f(x) = \begin{cases} |x-a| \sin \dfrac{1}{x-a}, & \text{if } x \neq 0 \\ 0, & \text{if } x = a \end{cases}$ at $x = a$

At $x = a$,

$$\text{LHL} = \lim_{x \to a^-} |x-a| \sin \frac{1}{x-a}$$

$$= \lim_{h \to 0} |a-h-a| \sin \left(\frac{1}{a-h-a}\right)$$

$$= \lim_{h \to 0} -h \sin \left(\frac{1}{h}\right) \qquad [\because \sin(-\theta) = -\sin\theta]$$

$= 0 \times$ [an oscillating number between -1 and 1] $= 0$

$$\text{RHL} = \lim_{x \to a^+} |x-a| \sin \left(\frac{1}{x-a}\right)$$

$$= \lim_{h \to 0} |a+h-a| \sin \left(\frac{1}{a+h-a}\right) = \lim_{h \to 0} h \sin \frac{1}{h}$$

$= 0 \times$ [an oscillating number between -1 and 1] $= 0$

and $f(a) = 0$

$\therefore$ LHL = RHL = $f(a)$

So, $f(x)$ is continuous at $x = a$.

Q. 8 $f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$.

Sol. We have, $f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

At $x = 0$,

$$\text{LHL} = \lim_{x \to 0^-} \frac{e^{1/x}}{1+e^{1/x}} = \lim_{h \to 0} \frac{e^{1/0-h}}{1+e^{1/0-h}}$$

$$= \lim_{h \to 0} \frac{e^{-1/h}}{1+e^{-1/h}} = \lim_{h \to 0} \frac{1}{e^{1/h}(1+e^{-1/h})}$$

$$= \lim_{h \to 0} \frac{1}{e^{1/h}+1} = \frac{1}{e^{\infty}+1} = \frac{1}{\infty+1} \qquad [\because e^{\infty} = \infty]$$

$$= \frac{1}{\frac{1}{0}} = 0$$

$$\text{RHL} = \lim_{x \to 0^+} \frac{e^{1/x}}{1+e^{1/x}}$$

$$= \lim_{h \to 0} \frac{e^{1/0+h}}{1+e^{1/0+h}} = \lim_{h \to 0} \frac{e^{1/h}}{1+e^{1/h}}$$

$$= \lim_{h \to 0} \frac{1}{e^{-1/h} + 1} = \frac{1}{e^{-\infty} + 1}$$

$$= \frac{1}{0 + 1} = 1 \qquad [\because e^{-\infty} = 0]$$

Hence, LHL $\neq$ RHL at $x = 0$.

So, $f(x)$ is discontinuous at $x = 0$.

Q. 9 $f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$ **at $x = 1$.**

Sol. We have, $f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$ at $x = 1$

At $x = 1$,

$$\text{HL} = \lim_{x \to 1^-} \frac{x^2}{2} = \lim_{h \to 0} \frac{(1 - h)^2}{2}$$

$$= \lim_{h \to 0} \frac{1 + h^2 - 2h}{2} = \frac{1}{2}$$

$$\text{RHL} = \lim_{x \to 1^+} \left(2x^2 - 3x + \frac{3}{2}\right)$$

$$= \lim_{h \to 0} \left[2(1 + h)^2 - 3(1 + h) + \frac{3}{2}\right]$$

$$= \lim_{h \to 0} \left(2 + 2h^2 + 4h - 3 - 3h + \frac{3}{2}\right) = -1 + \frac{3}{2} = \frac{1}{2}$$

and $f(1) = \dfrac{1^2}{2} = \dfrac{1}{2}$

$\therefore$ LHL = RHL = $f(1)$

Hence, $f(x)$ is continuous at $x = 1$.

Q. 10 $f(x) = |x| + |x - 1|$ **at $x = 1$.**

Sol. We have, $f(x) = |x| + |x - 1|$ at $x = 1$

At $x = 1$, $\text{LHL} = \lim_{x \to 1^-} [|x| + |x - 1|]$

$$= \lim_{h \to 0} [|1 - h| + |1 - h - 1|] = 1 + 0 = 1$$

and $\text{RHL} = \lim_{x \to 1^-} [|x| + |x - 1|]$

$$= \lim_{h \to 0} [|1 + h| + |1 + h - 1|] = 1 + 0 = 1$$

and $f(1) = |1| + |0| = 1$

$\therefore$ LHL = RHL = $f(1)$

Hence, $f(x)$ is continuous at $x = 1$.

Note *Every modulus function is a continuous function at any real point.*

Q. 11 $f(x)=\begin{cases}3x-8, \text{ if } x\le 5\\ 2k, \quad \text{if } x>5\end{cases}$ at $x=5$.

Sol. We have, $f(x)=\begin{cases}3x-8, \text{ if } x\le 5\\ 2k, \quad \text{if } x>5\end{cases}$ at $x=5$

Since, $f(x)$ is continuous at $x=5$.

$\therefore$ LHL = RHL = $f(5)$

Now, LHL = $\lim_{x\to 5^-}(3x-8)=\lim_{h\to 0}[3(5-h)-8]$

$=\lim_{h\to 0}[15-3h-8]=7$

RHL = $\lim_{x\to 5^+} 2k = \lim_{h\to 0} 2k = 2k = 7$ [$\because$ LHL = RHL]

and $f(5)=3\times 5-8=7$

$\therefore$ $2k=7 \Rightarrow k=\frac{7}{2}$

Q. 12 $f(x)=\begin{cases}\dfrac{2^{x+2}-16}{4^x-16}, & \text{if } x\ne 2\\ k, & \text{if } x=2\end{cases}$ at $x=2$.

Sol. We have, $f(x)=\begin{cases}\dfrac{2^{x+2}-16}{4^x-16}, & \text{if } x\ne 2\\ k, & \text{if } x=2\end{cases}$ at $x=2$

Since, $f(x)$ is continuous at $x=2$.

$\therefore$ LHL = RHL = $f(2)$

At $x=2$, $\lim_{x\to 2}\dfrac{2^x\cdot 2^2-2^4}{4^x-4^2}=\lim_{x\to 2}\dfrac{4\cdot(2^x-4)}{(2^x)^2-(4)^2}$

$=\lim_{x\to 2}\dfrac{4\cdot(2^x-4)}{(2^x-4)(2^x+4)}$ [$\because a^2-b^2=(a+b)(a-b)$]

$=\lim_{x\to 2}\dfrac{4}{2^x+4}=\dfrac{4}{8}=\dfrac{1}{2}$

But $f(2)=k$

$\therefore$ $k=\frac{1}{2}$

Q.13 $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0\\ \dfrac{2x+1}{x-1}, & \text{if } 0\le x\le 1\end{cases}$ at $x=0$.

Sol. We have, $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1\le x<0\\ \dfrac{2x+1}{x-1}, & \text{if } 0\le x\le 1\end{cases}$ at $x=0$.

$\therefore$ $$\text{LHL} = \lim_{x\to 0^-} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}$$

$$= \lim_{x\to 0^-} \left(\frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}\right) \cdot \left(\frac{\sqrt{1+kx} + \sqrt{1-kx}}{\sqrt{1+kx} + \sqrt{1-kx}}\right)$$

$$= \lim_{x\to 0^-} \frac{1+kx-1+kx}{x[\sqrt{1+kx} + \sqrt{1-kx}]}$$

$$= \lim_{x\to 0^-} \frac{2kx}{x\sqrt{1+kx} + \sqrt{1-kx}}$$

$$= \lim_{h\to 0} \frac{2k}{\sqrt{1+k(0-h)} + \sqrt{1-k(0-h)}}$$

$$= \lim_{h\to 0} \frac{2k}{\sqrt{1-kh} + \sqrt{1+kh}} \quad = \frac{2k}{2} = k$$

and $$f(0) = \frac{2\times 0+1}{0-1} = -1$$

$\Rightarrow$ $k = -1$ $\quad [\because \text{LHL} = \text{RHL} = f(0)]$

Q. 14 $f(x) = \begin{cases} \dfrac{1-\cos kx}{x\sin x}, & \text{if } x \neq 0 \\ \dfrac{1}{2}, & \text{if } x = 0 \end{cases}$ at $x = 0$.

Sol. We have, $f(x) = \begin{cases} \dfrac{1-\cos kx}{x\sin x}, & \text{if } x \neq 0 \\ \dfrac{1}{2}, & \text{if } x = 0 \end{cases}$ at $x = 0$

At $x = 0$, $$\text{LHL} = \lim_{x\to 0^-} \frac{1-\cos kx}{x\sin x} = \lim_{h\to 0} \frac{1-\cos k(0-h)}{(0-h)\sin(0-h)}$$

$$= \lim_{h\to 0} \frac{1-\cos\ (-kh)}{-h\sin(-h)}$$

$$= \lim_{h\to 0} \frac{1-\cos kh}{h\sin h} \quad [\because \cos(-\theta) = \cos\theta,\ \sin(-\theta) = -\sin\theta]$$

$$= \lim_{h\to 0} \frac{1-1+2\sin^2 \frac{kh}{2}}{h\sin h} \quad \left[\because \cos\theta = 1-2\sin^2\frac{\theta}{2}\right]$$

$$= \lim_{h\to 0} \frac{2\sin^2 \frac{kh}{2}}{h\sin h}$$

$$= \lim_{h\to 0} \frac{2\sin\frac{kh}{2}}{\frac{kh}{2}} \cdot \frac{\sin\frac{kh}{2}}{\frac{kh}{2}} \cdot \frac{1}{\frac{\sin h}{h}} \cdot \frac{k^2h/4}{h}$$

$$= \frac{2k^2}{4} = \frac{k^2}{2} \quad \left[\because \lim_{h\to 0} \frac{\sin h}{h} = 1\right]$$

Also, $$f(0) = \frac{1}{2} \Rightarrow \frac{k^2}{2} = \frac{1}{2} \Rightarrow k = \pm 1 \qquad \text{p}$$

Q. 15 Prove that the function f defined by $f(x)=\begin{cases}\dfrac{x}{|x|+2x^2}, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$

remains discontinuous at $x=0$, regardless the choice of k.

Sol. We have, $$f(x)=\begin{cases}\dfrac{x}{|x|+2x^2}, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$$

At $x=0$, $$\text{LHL}=\lim_{x\to 0^-}\frac{x}{|x|+2x^2}=\lim_{h\to 0}\frac{(0-h)}{|0-h|+2(0-h)^2}$$

$$=\lim_{h\to 0}\frac{-h}{h+2h^2}=\lim_{h\to 0}\frac{-h}{h(1+2h)}=-1$$

$$\text{RHL}=\lim_{x\to 0^+}\frac{x}{|x|+2x^2}=\lim_{h\to 0}\frac{0+h}{|0+h|+2(0+h)^2}$$

$$=\lim_{h\to 0}\frac{h}{h+2h^2}=\lim_{h\to 0}\frac{h}{h(1+2h)}=1$$

and $f(0)=k$

Since, LHL $\neq$ RHL for any value of k.

Hence, $f(x)$ is discontinuous at $x=0$ regardless the choice of k.

Q. 16 Find the values of a and b such that the function f defined by

$$f(x)=\begin{cases}\dfrac{x-4}{|x-4|}+a, & \text{if } x<4\\ a+b, & \text{if } x=4\\ \dfrac{x-4}{|x-4|}+b, & \text{if } x>4\end{cases}$$

is a continuous function at $x=4$.

Sol. We have, $$f(x)=\begin{cases}\dfrac{x-4}{|x-4|}+a, & \text{if } x<4\\ a+b, & \text{if } x=4\\ \dfrac{x-4}{|x-4|}+b, & \text{if } x>4\end{cases}$$

At $x=4$, $$\text{LHL}=\lim_{x\to 4^-}\frac{x-4}{|x-4|}+a$$

$$=\lim_{h\to 0}\frac{4-h-4}{|4-h-4|}+a=\lim_{h\to 0}\frac{-h}{h}+a$$

$$=-1+a$$

$$\text{RHL}=\lim_{x\to 4^+}\frac{x-4}{|x-4|}+b$$

$$=\lim_{h\to 0}\frac{4+h-4}{|4+h-4|}+b=\lim_{h\to 0}\frac{h}{h}+b=1+b$$

$$f(4)=a+b\Rightarrow -1+a=1+b=a+b$$

$\Rightarrow$ $-1+a=a+b$ and $1+b=a+b$

$\therefore$ $b=-1$ and $a=1$

Q. 17 If the function $f(x) = \frac{1}{x+2}$, then find the points of discontinuity of the composite function $y = f\{f(x)\}$.

Sol. We have, $f(x) = \frac{1}{x+2}$

$\therefore$
$$y = f\{f(x)\}$$
$$= f\left(\frac{1}{x+2}\right) = \frac{1}{\frac{1}{x+2} + 2}$$
$$= \frac{1}{1 + 2x + 4} \cdot (x+2) = \frac{(x+2)}{(2x+5)}$$

So, the function y will not be continuous at those points, where it is not defined as it is a rational function.

Therefore, $y = \frac{x+2}{(2x+5)}$ is not defined, when $2x + 5 = 0$

$\therefore$
$$x = \frac{-5}{2}$$

Hence, y is discontinuous at $x = \frac{-5}{2}$.

Q. 18 Find all points of discontinuity of the function $f(t) = \frac{1}{t^2 + t - 2}$, where $t = \frac{1}{x-1}$.

Sol. We have, $f(t) = \frac{1}{t^2 + t - 2}$ and $t = \frac{1}{x-1}$

$\therefore$
$$f(t) = \frac{1}{\left(\frac{1}{x^2 + 1 - 2x}\right) + \left(\frac{1}{x-1}\right) - \frac{2}{1}}$$
$$= \frac{1}{\left(\frac{1 + x - 1 + [-2(x-1)^2]}{(x^2 + 1 - 2x)}\right)}$$
$$= \frac{x^2 + 1 - 2x}{x - 2x^2 - 2 + 4x}$$
$$= \frac{x^2 + 1 - 2x}{-2x^2 + 5x - 2}$$
$$= \frac{(x-1)^2}{-(2x^2 - 5x + 2)}$$
$$= \frac{(x-1)^2}{(2x-1)(2-x)}$$

So, $f(t)$ is discontinuous at $2x - 1 = 0 \Rightarrow x = 1/2$

and $2 - x = 0 \Rightarrow x = 2$.

Q. 19 Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x = \pi$.

Sol. We have, $f(x) = |\sin x + \cos x|$ at $x = \pi$

Let $g(x) = \sin x + \cos x$

and $h(x) = |x|$

$\therefore$ $hog(x) = h[g(x)]$

$= h(\sin x + \cos x)$

$= |\sin x + \cos x|$

Since, $g(x) = \sin x + \cos x$ is a continuous function as it is forming with addition of two continuous functions $\sin x$ and $\cos x$.

Also, $h(x) = |x|$ is also a continuous function. Since, we know that composite functions of two continuous functions is also a continuous function.

Hence, $f(x) = |\sin x + \cos x|$ is a continuous function everywhere.

So, $f(x)$ is continuous at $x = \pi$.

Q. 20 Examine the differentiability of f, where f is defined by

$$f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x, & \text{if } 2 \le x < 3 \end{cases} \text{ at } x = 2.$$

Thinking Process

We know that, a function f is differentiable at a point a in its domain, if both $Lf'(a)$ and $Rf'(a)$ are finite and equal, where $Lf'(c) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$ and $Rf'(c) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

Sol. We have, $f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x & \text{if } 2 \le x < 3 \end{cases}$ at $x = 2$.

At $x = 2$, $Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$

$= \lim_{h \to 0} \frac{(2-h)[2-h] - (2-1)2}{-h}$

$\{\because [a-h] = [a-1], \text{where } a \text{ is any positive number}\}$

$= \lim_{h \to 0} \frac{(2-h)(1) - 2}{-h}$

$= \lim_{h \to 0} \frac{2-h-2}{-h} = \lim_{h \to 0} \frac{-h}{-h} = 1$

$Rf'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

$= \lim_{h \to 0} \frac{(2+h-1)(2+h) - (2-1)\cdot 2}{h}$

$= \lim_{h \to 0} \frac{(1+h)(2+h) - 2}{h}$

$= \lim_{h \to 0} \frac{2+h+2h+h^2-2}{h}$

$= \lim_{h \to 0} \frac{h^2+3h}{h} = \lim_{h \to 0} \frac{h(h+3)}{h} = 3$

$\therefore$ $Lf'(2) \ne Rf'(2)$

So, $f(x)$ is not differentiable at $x = 2$.

Q. 21 $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$.

Sol. We have, $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

For differentiability at $x = 0$,

$$Lf'(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{x^2 \sin \frac{1}{x} - 0}{x - 0}$$

$$= \lim_{h \to 0} \frac{(0-h)^2 \sin\left(\frac{1}{0-h}\right)}{0-h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{-1}{h}\right)}{-h}$$

$$= \lim_{h \to 0} + h \sin\left(\frac{1}{h}\right) \qquad [\because \sin(-\theta) = -\sin\theta]$$

$= 0 \times$[an oscillating number between -1and1] $= 0$

$$Rf'(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^2 \sin \frac{1}{x} - 0}{x - 0}$$

$$= \lim_{h \to 0} \frac{(0+h)^2 \sin\left(\frac{1}{0+h}\right)}{0+h} = \lim_{h \to 0} \frac{h^2 \sin(1/h)}{h}$$

$$= \lim_{h \to 0} h \sin(1/h)$$

$= 0 \times$ [an oscillating number between -1 and 1] $= 0$

$\because \quad Lf'(0) = Rf'(0)$

So, $f(x)$ is differentiable at $x = 0$.

Q. 22 $f(x) = \begin{cases} 1 + x, & \text{if } x \leq 2 \\ 5 - x, & \text{if } x > 2 \end{cases}$ at $x = 2$.

Sol. We have, $f(x) = \begin{cases} 1 + x, \text{ if } x \leq 2 \\ 5 - x, \text{ if } x > 2 \end{cases}$ at $x = 2$.

For differentiability at $x = 2$,

$$Lf'(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{(1+x) - (1+2)}{x - 2}$$

$$= \lim_{h \to 0} \frac{(1 + 2 - h) - 3}{2 - h - 2} = \lim_{h \to 0} \frac{-h}{-h} = 1$$

$$Rf'(2) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{(5 - x) - 3}{x - 2}$$

$$= \lim_{h \to 0} \frac{5 - (2 + h) - 3}{2 + h - 2}$$

$$= \lim_{h \to 0} \frac{5 - 2 - h - 3}{h} = \lim_{h \to 0} \frac{-h}{+h}$$

$= -1$

$\because \quad Lf'(2) \neq Rf'(2)$

So, $f(x)$ is not differentiable at $x = 2$.

Q. 23 Show that $f(x)=|x-5|$ is continuous but not differentiable at $x=5$.

Sol. We have, $f(x)=|x-5|$

$\therefore$ $$f(x)=\begin{cases}-(x-5), & \text{if } x<5\\ x-5, & \text{if } x\geq 5\end{cases}$$

For continuity at $x=5$,

$$\text{LHL}=\lim_{x\to 5^-}(-x+5)$$
$$=\lim_{h\to 0}[-(5-h)+5]=\lim_{h\to 0}h=0$$
$$\text{RHL}=\lim_{x\to 5^+}(x-5)$$
$$=\lim_{h\to 0}(5+h-5)=\lim_{h\to 0}h=0$$

$\therefore$ $f(5)=5-5=0$

$\Rightarrow$ $\text{LHL}=\text{RHL}=f(5)$

Hence, $f(x)$ is continuous at $x=5$.

Now, $$Lf'(5)=\lim_{x\to 5^-}\frac{f(x)-f(5)}{x-5}$$
$$=\lim_{x\to 5^-}\frac{-x+5-0}{x-5}=-1$$
$$Rf'(5)=\lim_{x\to 5^+}\frac{f(x)-f(5)}{x-5}$$
$$=\lim_{x\to 5^+}\frac{x-5-0}{x-5}=1$$

$\therefore$ $Lf'(5)\neq Rf'(5)$

So, $f(x)=|x-5|$ is not differentiable at $x=5$.

Q. 24 A function $f:R\to R$ satisfies the equation $f(x+y)=f(x)\cdot f(y)$ for all $x,y\in R$, $f(x)\neq 0$. Suppose that the function is differentiable at $x=0$ and $f'(0)=2$, then prove that $f'(x)=2f(x)$.

Sol. Let $f:R\to R$ satisfies the equation $f(x+y)=f(x)\cdot f(y)$, $\forall\, x,y\in R$, $f(x)\neq 0$.

Let $f(x)$ is differentiable at $x=0$ and $f'(0)=2$.

$\Rightarrow$ $$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$$

$\Rightarrow$ $$2=\lim_{x\to 0}\frac{f(x)-f(0)}{x}$$

$\Rightarrow$ $$2=\lim_{h\to 0}\frac{f(0+h)-f(0)}{0+h}$$

$\Rightarrow$ $$2=\lim_{h\to 0}\frac{f(0)\cdot f(h)-f(0)}{h}$$

$\Rightarrow$ $$2=\lim_{h\to 0}\frac{f(0)\,[f(h)-1]}{h}$$ $[\because f(0)=f(h)]$ ...(i)

Also, $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h}$$ $[\because f(x+y)=f(x)\cdot f(y)]$
$$=\lim_{h\to 0}\frac{f(x)\,[f(h)-1]}{h}=2f(x)$$ [using Eq. (i)]

$\therefore$ $f'(x)=2f(x)$

Q. 25 $2^{\cos^2 x}$

Sol. Let $$y = 2^{\cos^2 x}$$

$\therefore$ $$\log y = \log 2^{\cos^2 x} = \cos^2 x \cdot \log 2$$

On differentiating w.r.t. x, we get

$$\frac{d}{dy}\log y.\frac{dy}{dx} = \frac{d}{dx}\log 2 \cdot \cos^2 x$$

$\Rightarrow$ $$\frac{1}{y}\cdot\frac{dy}{dx} = \log 2\frac{d}{dx}(\cos x)^2$$

$\Rightarrow$ $$\frac{1}{y}\cdot\frac{dy}{dx} = \log 2 \cdot [2\cos x]\cdot\frac{d}{dx}\cos x$$

$$= \log 2 \cdot 2\cos x \cdot (-\sin x)$$

$$= \log 2 \cdot [-(\sin 2x)]$$

$\therefore$ $$\frac{dy}{dx} = -y \cdot \log 2\,(\sin 2x)$$

$$= -2^{\cos^2 x}\cdot \log 2\,(\sin 2x)$$

Q. 26 $\frac{8^x}{x^8}$

Sol. Let $$y = \frac{8^x}{x^8} \Rightarrow \log y = \log\frac{8^x}{x^8}$$

$\Rightarrow$ $$\frac{d}{dy}\log y \cdot \frac{dy}{dx} = \frac{d}{dx}[\log 8^x - \log x^8]$$

$\Rightarrow$ $$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}[x \cdot \log 8 - 8\cdot\log x]$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \log 8 \cdot 1 - 8\cdot\frac{1}{x}$$

$\Rightarrow$ $$\frac{1}{y}\cdot\frac{dy}{dx} = \log 8 - \frac{8}{x}$$

$\therefore$ $$\frac{dy}{dx} = y\left(\log 8 - \frac{8}{x}\right) = \frac{8^x}{x^8}\left(\log 8 - \frac{8}{x}\right)$$

Q. 27 $\log(x + \sqrt{x^2 + a})$

Sol. Let $$y = \log(x + \sqrt{x^2 + a})$$

$\therefore$ $$\frac{dy}{dx} = \frac{d}{dx}\log(x + \sqrt{x^2 + a})$$

$$= \frac{1}{(x + \sqrt{x^2 + a})}.\frac{d}{dx}[x + \sqrt{x^2 + a}]$$

$$= \frac{1}{(x + \sqrt{x^2 + a})}\left[1 + \frac{1}{2}(x^2 + a)^{-1/2}\cdot 2x\right]$$

$$= \frac{1}{(x + \sqrt{x^2 + a})}\cdot\left(1 + \frac{x}{\sqrt{x^2 + a}}\right)$$

$$= \frac{(\sqrt{x^2 + a} + x)}{(x + \sqrt{x^2 + a})(\sqrt{x^2 + a})} = \frac{1}{(\sqrt{x^2 + a})}$$

Q. 28 $\log[\log(\log x^5)]$

Sol. Let $y = \log[\log(\log x^5)]$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[\log(\log \log x^5)]$$

$$= \frac{1}{\log\log x^5} \cdot \frac{d}{dx}(\log \cdot \log x^5)$$

$$= \frac{1}{\log\log x^5} \cdot \left(\frac{1}{\log x^5}\right) \cdot \frac{d}{dx}\log x^5$$

$$= \frac{1}{\log\log x^5} \cdot \frac{1}{\log x^5} \cdot \frac{d}{dx}(5\log x) = \frac{5}{x \cdot \log(\log x^5) \cdot \log(x^5)}$$

Q. 29 $\sin\sqrt{x} + \cos^2\sqrt{x}$

Sol. Let $y = \sin\sqrt{x} + (\cos\sqrt{x})^2$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\sin(x^{1/2}) + \frac{d}{dx}[\cos(x^{1/2})]^2$$

$$= \cos x^{1/2} \cdot \frac{d}{dx}x^{1/2} + 2\cos(x^{1/2})\ \frac{d}{dx}[\cos(x^{1/2})]$$

$$= \cos(x^{1/2})\ \frac{1}{2}x^{-1/2} + 2 \cdot \cos(x^{1/2}) \cdot \left[-\sin(x^{1/2}) \cdot \frac{d}{dx}x^{1/2}\right]$$

$$= \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}}[-2\cos(x^{1/2})] \cdot \sin x^{1/2} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}[\cos(\sqrt{x}) - \sin(2\sqrt{x})]$$

Q. 30 $\sin^n(ax^2 + bx + c)$

Sol. Let $y = \sin^n(ax^2 + bx + c)$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[\sin(ax^2 + bx + c)]^n$$

$$= n \cdot [\sin(ax^2 + bx + c)]^{n-1} \cdot \frac{d}{dx}\sin(ax^2 + bx + c)$$

$$= n \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c) \cdot \frac{d}{dx}(ax^2 + bx + c)$$

$$= n \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c) \cdot (2ax + b)$$

$$= n \cdot (2ax + b) \cdot \sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c)$$

Q. 31 $\cos(\tan\sqrt{x+1})$

Sol. Let $y = \cos(\tan\sqrt{x+1})$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\cos(\tan\sqrt{x+1}) = -\sin(\tan\sqrt{x+1}) \cdot \frac{d}{dx}(\tan\sqrt{x+1})$$

$$= -\sin(\tan\sqrt{x+1}) \cdot \sec^2\sqrt{x+1} \cdot \frac{d}{dx}(x+1)^{1/2} \qquad \left[\because \frac{d}{dx}(\tan x) = \sec^2 x\right]$$

$$= -\sin(\tan\sqrt{x+1}) \cdot (\sec\sqrt{x+1})^2 \cdot \frac{1}{2}(x+1)^{-1/2} \cdot \frac{d}{dx}(x+1)$$

$$= \frac{-1}{2\sqrt{x+1}} \cdot \sin(\tan\sqrt{x+1}) \cdot \sec^2(\sqrt{x+1})$$

Q. 32 $\sin x^2 + \sin^2 x + \sin^2 (x^2)$

Sol. Let $y = \sin x^2 + \sin^2 x + \sin^2 (x^2)$

$\therefore$
$$\frac{dy}{dx} = \frac{d}{dx}\sin (x^2) + \frac{d}{dx}(\sin x)^2 + \frac{d}{dx}(\sin x^2)^2$$
$$= \cos (x^2)\frac{d}{dx}(x^2) + 2\sin x \cdot \frac{d}{dx}\sin x + 2\sin x^2 \cdot \frac{d}{dx}\sin x^2$$
$$= \cos x^2 2x + 2\cdot \sin x \cdot \cos x + 2\sin x^2 \cos x^2 \cdot \frac{d}{dx}x^2$$
$$= 2x\cos (x)^2 + 2\cdot \sin x \cdot \cos x + 2\sin x^2 \cdot \cos x^2 \cdot 2x$$
$$= 2x\cos (x)^2 + \sin 2x + \sin 2(x)^2 \cdot 2x$$
$$= 2x\cos (x^2) + 2x \cdot \sin 2(x^2) + \sin 2x$$

Q. 33 $\sin^{-1}\frac{1}{\sqrt{x+1}}$

Sol. Let $y = \sin^{-1}\frac{1}{\sqrt{x+1}}$

$\therefore$
$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\frac{1}{\sqrt{x+1}}$$
$$= \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{x+1}}\right)^2}} \cdot \frac{d}{dx}\frac{1}{(x+1)^{1/2}} \qquad \left[\because \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}\right]$$
$$= \frac{1}{\sqrt{\frac{x+1-1}{x+1}}} \cdot \frac{d}{dx} \cdot (x+1)^{-1/2}$$
$$= \sqrt{\frac{x+1}{x}} \cdot \frac{-1}{2}(x+1)^{-\frac{1}{2}-1} \cdot \frac{d}{dx}(x+1)$$
$$= \frac{(x+1)^{1/2}}{x^{1/2}} \cdot \left(-\frac{1}{2}\right)(x+1)^{-3/2} = \frac{-1}{2\sqrt{x}} \cdot \left(\frac{1}{x+1}\right)$$

Q. 34 $(\sin x)^{\cos x}$

Sol. Let $y = (\sin x)^{\cos x}$

$\Rightarrow$ $\log y = \log(\sin x)^{\cos x} = \cos x \log \sin x$

$\therefore$ $\frac{d}{dy}\log y \cdot \frac{dy}{dx} = \frac{d}{dx}(\cos x \cdot \log \sin x)$

$\Rightarrow$
$$\frac{1}{y}\cdot\frac{dy}{dx} = \cos x \cdot \frac{d}{dx}\log \sin x + \log \sin x \cdot \frac{d}{dx}\cos x$$
$$= \cos x \cdot \frac{1}{\sin x}\cdot\frac{d}{dx}\sin x + \log \sin x \cdot (-\sin x)$$
$$= \cot x \cdot \cos x - \log (\sin x)\cdot \sin x \qquad \left[\because \cot x = \frac{\cos x}{\sin x}\right]$$

$\therefore$
$$\frac{dy}{dx} = y\left[\frac{\cos^2 x}{\sin x} - \sin x \cdot \log (\sin x)\right]$$
$$= \sin x^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \cdot \log (\sin x)\right]$$

Q. 35 $\sin^m x \cdot \cos^n x$

Sol. Let $y = \sin^m x \cdot \cos^n x$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left[(\sin x)^m \cdot (\cos x)^n\right]$$

$$= (\sin x)^m \cdot \frac{d}{dx}(\cos x)^n + (\cos x)^n \cdot \frac{d}{dx}(\sin x)^m$$

$$= (\sin x)^m \cdot n\,(\cos x)^{n-1} \cdot \frac{d}{dx}\cos x + (\cos x)^n\; m\,(\sin x)^{m-1} \cdot \frac{d}{dx}\sin x$$

$$= (\sin x)^m \cdot n(\cos x)^{n-1}\;(-\sin x) + (\cos x)^n \cdot m\,(\sin x)^{m-1}\cos x$$

$$= -n\sin^m x \cdot \cos^{n-1} x \cdot (\sin x) + m\cos^n x \cdot \sin^{m-1} x \cdot \cos x$$

$$= -\,n \cdot \sin^m x \cdot \sin x \cdot \cos^n x \cdot \frac{1}{\cos x} + m \cdot \sin^m x . \frac{1}{\sin x} \cdot \cos^n x \cdot \cos x$$

$$= -\,n \cdot \sin^m x \cdot \cos^n x \cdot \tan x + m\sin^m x \cdot \cos^n x \cdot \cot x$$

$$= \sin^m x \cdot \cos^n x\,[-\,n\tan x + m\cot x]$$

Q. 36 $(x+1)^2(x+2)^3(x+3)^4$

Sol. Let $y = (x+1)^2\,(x+2)^3\,(x+3)^4$

$$\therefore \quad \log y = \log\{(x+1)^2 \cdot (x+2)^3 (x+3)^4\}$$

$$= \log(x+1)^2 + \log(x+2)^3 + \log(x+3)^4$$

and $$\frac{d}{dy}\log y \cdot \frac{dy}{dx} = \frac{d}{dx}[2\log(x+1)] + \frac{d}{dx}[3\log(x+2)] + \frac{d}{dx}[4\log(x+3)]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2}{(x+1)} \cdot \frac{d}{dx}(x+1) + 3 \cdot \frac{1}{(x+2)} \cdot \frac{d}{dx}(x+2) + 4 \cdot \frac{1}{(x+3)} \cdot \frac{d}{dx}(x+3) \quad \left[\because \frac{d}{dx}(\log x) = \frac{1}{x}\right]$$

$$= \left[\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right]$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{2}{(x+1)} + \frac{3}{(x+2)} + \frac{4}{(x+3)}\right]$$

$$= (x+1)^2 \cdot (x+2)^3 \cdot (x+3)^4\left[\frac{2}{(x+1)} + \frac{3}{(x+2)} + \frac{4}{(x+3)}\right]$$

$$= (x+1)^2 \cdot (x+2)^3 \cdot (x+3)^4 \left[\frac{2\,(x+2)(x+3) + 3\,(x+1)(x+3) + 4(x+1)(x+2)}{(x+1)(x+2)(x+3)}\right]$$

$$= \frac{(x+1)^2\,(x+2)^3\,(x+3)^4}{(x+1)(x+2)(x+3)}\,[2\,(x^2+5x+6) + 3\,(x^2+4x+3) + 4\,(x^2+3x+2)]$$

$$= (x+1)(x+2)^2\,(x+3)^3\,[2x^2+10x+12+3x^2+12x+9+4x^2+12x+8]$$

$$= (x+1)(x+2)^2\,(x+3)^3\,[9x^2+34x+29]$$

Q. 37 $\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right), -\frac{\pi}{4}<x<\frac{\pi}{4}$

Sol. Let $$y=\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$\therefore$ $$\frac{dy}{dx}=\frac{d}{dx}\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$$=\frac{-1}{\sqrt{1-\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)^2}}\cdot\frac{d}{dx}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$$

$$\left[\because \frac{d}{dx}(\cos x)=-\frac{1}{\sqrt{1-x^2}}\right]$$

$$=\frac{-1}{\sqrt{4-\frac{(\sin^2 x+\cos^2 x+2\sin x\cdot\cos x)}{2}}}\cdot\frac{1}{\sqrt{2}}(\cos x-\sin x)$$

$$=\frac{-1\cdot\sqrt{2}}{\sqrt{1-\sin 2x}}\cdot\frac{1}{\sqrt{2}}(\cos x-\sin x)$$

$$[\because 1-\sin 2x=(\cos x-\sin x)^2=\cos^2 x+\sin^2 x-2\sin x\cos x]$$

$$=\frac{-1(\cos x-\sin x)}{(\cos x-\sin x)}=-1$$

Q. 38 $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}, -\frac{\pi}{4}<x<\frac{\pi}{4}$

Sol. Let $$y=\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$\therefore$ $$\frac{dy}{dx}=\frac{d}{dx}\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$=\frac{1}{1+\sqrt{\left(\frac{1-\cos x}{1+\cos x}\right)^2}}\cdot\frac{d}{dx}\left[\frac{1-\cos x}{1+\cos x}\right]^{1/2} \qquad \left[\because \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}\right]$$

$$=\frac{1}{1+\frac{1-\cos x}{1+\cos x}}\cdot\frac{1}{2}\left[\frac{1-\cos x}{1+\cos x}\right]^{-1/2}\cdot\frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$=\frac{1}{\frac{1+\cos x+1-\cos x}{1+\cos x}}\cdot\frac{1}{2}\left[\frac{(1-\cos x)}{(1+\cos x)}\cdot\frac{(1-\cos x)}{(1-\cos x)}\right]^{-1/2}$$

$$\cdot\frac{(1+\cos x)\cdot\sin x+(1-\cos x)\cdot\sin x}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)}{2}\cdot\frac{1}{2}\left[\frac{(1-\cos x)^2}{(1-\cos^2 x)}\right]^{-1/2}\left[\frac{\sin x\,(1+\cos x+1-\cos x)}{(1+\cos x)^2}\right]$$

$$=\frac{(1+\cos x)}{2}\cdot\frac{1}{2}\left[\frac{(1-\cos x)^2}{(1-\cos^2 x)}\right]^{-1/2}\left[\frac{\sin x\,(1+\cos x+1-\cos x)}{(1+\cos x)^2}\right]$$

$$= \frac{(1+\cos x)}{2} \cdot \frac{1}{2}\left[\frac{(1-\cos x)^2}{\sin x}\right]^{-1/2} \cdot \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)}{2} \cdot \frac{1}{2} \cdot \frac{\sin x}{(1-\cos x)} \cdot \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{2\sin^2 x}{4(1+\cos x)(1-\cos x)} = \frac{1}{2} \cdot \frac{\sin^2 x}{(1-\cos^2 x)}$$

$$= \frac{1}{2} \cdot \frac{\sin^2 x}{\sin^2 x} = \frac{1}{2}$$

Alternate Method

Let $\quad y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

$$= \tan^{-1}\left(\sqrt{\frac{1-1+2\sin^2\frac{x}{2}}{1+2\cos^2\frac{x}{2}-1}}\right) \qquad \left[\because \cos x = 1-2\sin^2\frac{x}{2} = 2\cos^2\frac{x}{2}-1\right]$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}$$

Q. 39 $\tan^{-1}(\sec x + \tan x), \frac{-\pi}{2} < x < \frac{\pi}{2}$

Sol. Let $\quad y = \tan^{-1}(\sec x + \tan x)$

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\tan^{-1}(\sec x + \tan x)$

$$= \frac{1}{1+(\sec x + \tan x)^2} \cdot \frac{d}{dx}(\sec x + \tan x) \qquad \left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

$$= \frac{1}{1+\sec^2 x + \tan^2 x + 2\sec x \cdot \tan x} \cdot [\sec x \cdot \tan x + \sec^2 x]$$

$$= \frac{1}{(\sec^2 x + \sec^2 x + 2\sec x \cdot \tan x)} \cdot \sec x \cdot (\sec x + \tan x)$$

$$= \frac{1}{2\sec x(\tan x + \sec x)} \cdot \sec x(\sec x + \tan x) \quad = \frac{1}{2}$$

Q. 40 $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right), \frac{-\pi}{2} < x < \frac{\pi}{2}$ and $\frac{a}{b}\tan x > -1.$

Sol. Let $\quad y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$

$$= \tan^{-1}\left[\frac{\frac{a\cos x}{b\cos x} - \frac{b\sin x}{b\cos x}}{\frac{b\cos x}{b\cos x} + \frac{a\sin x}{b\cos x}}\right] = \tan^{-1}\left[\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right]$$

$$= \tan^{-1}\frac{a}{b} - \tan^{-1}\tan x \qquad \left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\frac{a}{b} - x$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{a}{b}\right) - \frac{d}{dx}(x)$$

$$= 0 - 1 \qquad \left[\because \frac{d}{dx}\left(\frac{a}{b}\right) = 0\right]$$

$$= -1$$

Q. 41 $\sec^{-1}\left(\dfrac{1}{4x^3 - 3x}\right),\ 0 < x < \dfrac{1}{\sqrt{2}}$

Sol. Let
$$y = \sec^{-1}\left(\frac{1}{4x^3 - 3x}\right) \qquad \text{...(i)}$$

On putting $x = \cos\theta$ in Eq. (i), we get

$$y = \sec^{-1}\frac{1}{4\cos^3\theta - 3\cos\theta}$$

$$= \sec^{-1}\frac{1}{\cos 3\theta}$$

$$= \sec^{-1}(\sec 3\theta) = 3\theta$$

$$= 3\cos^{-1}x \qquad [\because \theta = \cos^{-1}x]$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(3\cos^{-1}x)$$

$$= 3\cdot\frac{-1}{\sqrt{1-x^2}}$$

Q. 42 $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right),\ \dfrac{-1}{\sqrt{3}} < \dfrac{x}{a} < \dfrac{1}{\sqrt{3}}$

Sol. Let
$$y = \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$$

Put
$$x = a\tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a}$$

$$\therefore \quad y = \tan^{-1}\left[\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right] \qquad \left[\because \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right]$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta$$

$$= 3\tan^{-1}\frac{x}{a} \qquad \left[\because \theta = \tan^{-1}\frac{x}{a}\right]$$

$$\therefore \quad \frac{dy}{dx} = 3\cdot\frac{d}{dx}\tan^{-1}\frac{x}{a} = 3\cdot\left[\frac{1}{1+\frac{x^2}{a^2}}\right]\cdot\frac{d}{dx}\cdot\left(\frac{x}{a}\right)$$

$$= 3\cdot\frac{a^2}{a^2+x^2}\cdot\frac{1}{a} = \frac{3a}{a^2+x^2}$$

Q. 43 $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right], -1 < x < 1, x \neq 0$

Sol. Let $y = \tan^{-1}\left[\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$

Put $x^2 = \cos 2\theta$

$$\therefore \quad y = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1+2\cos^2\theta-1}+\sqrt{1-1+2\sin^2\theta}}{\sqrt{1+2\cos^2\theta-1}-\sqrt{1-1+2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right) = \tan^{-1}\left[\frac{\sqrt{2}(\cos\theta+\sin\theta)}{\sqrt{2}(\cos\theta-\sin\theta)}\right]$$

$$= \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) = \tan^{-1}\left(\frac{\frac{\cos\theta+\sin\theta}{\cos\theta}}{\frac{\cos\theta-\sin\theta}{\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}+\theta\right) \qquad \left[\because \tan(a+b) = \frac{\tan a+\tan b}{1-\tan a\cdot\tan b}\right]$$

$$= \frac{\pi}{4}+\theta = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2 \qquad \left[\because 2\theta = \cos^{-1}x^2 \Rightarrow \theta = \frac{1}{2}\cos^{-1}x^2\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}\right)+\frac{d}{dx}\left(\frac{1}{2}\cos^{-1}x^2\right)$$

$$= 0+\frac{1}{2}\cdot\frac{-1}{\sqrt{1-x^4}}\cdot\frac{d}{dx}x^2 = \frac{1}{2}\cdot\frac{-2x}{\sqrt{1-x^4}} = \frac{-x}{\sqrt{1-x^4}}$$

Find $\dfrac{dy}{dx}$ of each of the functions expressed in parametric form.

Q. 44 $x = t+\dfrac{1}{t},\ y = t-\dfrac{1}{t}$

Sol. $\because \quad x = t+\dfrac{1}{t}$ and $y = t-\dfrac{1}{t}$

$$\therefore \quad \frac{dx}{dt} = \frac{d}{dt}\left(t+\frac{1}{t}\right) \text{ and } \frac{dy}{dt} = \frac{d}{dt}\left(t-\frac{1}{t}\right)$$

$$\Rightarrow \quad \frac{dx}{dt} = 1+(-1)t^{-2} \text{ and } \frac{dy}{dt} = 1-(-1)t^{-2}$$

$$\Rightarrow \quad \frac{dx}{dt} = 1-\frac{1}{t^2} \text{ and } \frac{dy}{dt} = 1+\frac{1}{t^2}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{t^2-1}{t^2} \text{ and } \frac{dy}{dt} = \frac{t^2+1}{t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t^2+1/t^2}{t^2-1/t^2} = \frac{t^2+1}{t^2-1}$$

Q. 45 $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right), y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

Sol. $\because$ $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$ and $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

$\therefore$
$$\frac{dx}{d\theta} = \frac{d}{d\theta}\left[e^{\theta}\cdot\left(\theta + \frac{1}{\theta}\right)\right]$$
$$= e^{\theta}\cdot\frac{d}{d\theta}\left(\theta + \frac{1}{\theta}\right) + \left(\theta + \frac{1}{\theta}\right)\cdot\frac{d}{d\theta}e^{\theta}$$
$$= e^{\theta}\left(1 - \frac{1}{\theta^2}\right) + \left(\theta + \frac{1}{\theta}\right)e^{\theta}$$
$$= e^{\theta}\left(1 - \frac{1}{\theta^2} + \theta + \frac{1}{\theta}\right)$$
$$= e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right) \quad \text{...(i)}$$

and
$$\frac{dy}{d\theta} = \frac{d}{d\theta}\left[e^{-\theta}\cdot\left(\theta - \frac{1}{\theta}\right)\right]$$
$$= e^{-\theta}\cdot\frac{d}{d\theta}\left(\theta - \frac{1}{\theta}\right) + \frac{d}{d\theta}e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$$
$$= e^{-\theta}\left(1 + \frac{1}{\theta^2}\right) + \left(\theta - \frac{1}{\theta}\right)e^{-\theta}\cdot\frac{d}{d\theta}(-\theta)$$
$$= e^{-\theta}\left[\frac{\theta^2 + 1}{\theta^2} - \frac{\theta^2 - 1}{\theta}\right] = e^{-\theta}\left[\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right] \quad \text{...(ii)}$$

$\therefore$
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{e^{-\theta}\left(\frac{\theta^2 + 1 - \theta^3 + \theta}{\theta^2}\right)}{e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right)}$$
$$= e^{-2\theta}\left(\frac{-\theta^3 + \theta^2 + \theta + 1}{\theta^3 + \theta^2 + \theta - 1}\right)$$

Q. 46 $x = 3\cos\theta - 2\cos^3\theta, \; y = 3\sin\theta - 2\sin^3\theta$

Sol. $\because$ $x = 3\cos\theta - 2\cos^3\theta$ and $y = 3\sin\theta - 2\sin^3\theta$

$\therefore$
$$\frac{dx}{d\theta} = \frac{d}{d\theta}(3\cos\theta) - \frac{d}{d\theta}(2\cos^3\theta)$$
$$= 3\cdot(-\sin\theta) - 2\cdot 3\cos^2\theta\cdot\frac{d}{d\theta}\cdot\cos\theta$$
$$= -3\sin\theta + 6\cos^2\theta\sin\theta$$

and
$$\frac{dy}{d\theta} = 3\cos A - 2\cdot 3\sin^2\theta\cdot\frac{d}{d\theta}\cdot\sin\theta$$
$$= 3\cos\theta - 6\sin^2\theta\cdot\cos\theta$$

Now,
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta - 6\sin^2\theta\cos\theta}{-3\sin\theta + 6\cos^2\theta\sin\theta}$$
$$= \frac{3\cos\theta(1 - 2\sin^2\theta)}{3\sin\theta(-1 + 2\cos^2\theta)} = \cot\theta\cdot\frac{\cos 2\theta}{\cos 2\theta} = \cot\theta$$

Q. 47 $\sin x = \dfrac{2t}{1+t^2}, \tan y = \dfrac{2t}{1-t^2}$

Sol. $\because$ $$\sin x = \frac{2t}{1+t^2} \quad \text{...(i)}$$

and $$\tan y = \frac{2t}{1-t^2} \quad \text{...(ii)}$$

$$\therefore \quad \frac{d}{dx}\sin x \cdot \frac{dx}{dt} = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$

$$\Rightarrow \quad \cos x \frac{dx}{dt} = \frac{(1+t^2)\cdot \frac{d}{dt}(2t) - (2t)\cdot \frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{2(1+t^2) - 2t \cdot 2t}{(1+t^2)^2} = \frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \cdot \frac{1}{\cos x}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \cdot \frac{1}{\sqrt{1-\sin^2 x}} = \frac{2(1-t^2)}{(1+t^2)^2} \cdot \frac{1}{\sqrt{1-\left(\frac{2t}{1+t^2}\right)^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \cdot \frac{(1+t^2)}{(1-t^2)} = \frac{2}{1+t^2} \quad \text{...(iii)}$$

Also, $$\frac{d}{dy}\tan y \cdot \frac{dy}{dt} = \frac{d}{dt}\left(\frac{2t}{1-t^2}\right)$$

$$\sec^2 y \frac{dy}{dt} = \frac{(1-t^2)\frac{d}{dt}\cdot(2t) - 2t \cdot \frac{d}{dt}(1-t^2)}{(1-t^2)^2}$$

$$\frac{dy}{dt} = \frac{2-2t^2+4t^2}{(1-t^2)^2} \cdot \frac{1}{\sec^2 y}$$

$$= \frac{2(1+t^2)}{(1-t^2)^2} \cdot \frac{1}{(1+\tan^2 y)} = \frac{2(1+t^2)}{(1-t^2)^2} \cdot \frac{1}{1+\frac{4t^2}{(1-t^2)^2}}$$

$$= \frac{2(1+t^2)}{(1-t^2)^2} \cdot \frac{(1-t^2)^2}{(1+t^2)^2} = \frac{2}{1+t^2} \quad \text{...(iv)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2/1+t^2}{2/1+t^2} = 1 \quad \text{[from Eqs. (iii) and (iv)]}$$

Q. 48 $x = \dfrac{1+\log t}{t^2}, y = \dfrac{3+2\log t}{t}$

Sol. $\because$ $$x = \frac{1+\log t}{t^2} \text{ and } y = \frac{3+2\log t}{t}$$

$$\therefore \quad \frac{dx}{dt} = \frac{t^2 . \frac{d}{dt}(1+\log t) - (1+\log t). \frac{d}{dt}t^2}{(t^2)^2}$$

$$= \frac{t^2 \cdot \frac{1}{t} - (1 + \log t) \cdot 2t}{t^4} = \frac{t - (1 + \log t) \cdot 2t}{t^4}$$

$$= \frac{t}{t^4} [1 - 2 (1 + \log t) = \frac{-1 - 2 \log t}{t^3} \quad \text{... (i)}$$

and $$\frac{dy}{dt} = \frac{t \cdot \frac{d}{dt} (3 + 2 \log t) - (3 + 2 \log t) \cdot \frac{d}{dt} t}{t^2}$$

$$= \frac{t \cdot 2 \cdot \frac{1}{t} - (3 + 2 \log t) \cdot 1}{t^2}$$

$$= \frac{2 - 3 - 2 \log t}{t^2} = \frac{-1 - 2 \log t}{t^2} \quad \text{...(ii)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1 - 2 \log t / t^2}{-1 - 2 \log t / t^3} = t$$

Q. 49 If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, then prove that $\dfrac{dy}{dx} = -\dfrac{y \log x}{x \log y}$.

Sol. $\because$ $$x = e^{\cos 2t} \text{ and } y = e^{\sin 2t}$$

$\therefore$ $$\frac{dx}{dt} = \frac{d}{dt} e^{\cos 2t} = e^{\cos 2t} \cdot \frac{d}{dt} \cos 2t$$

$$= e^{\cos 2t} \cdot (-\sin 2t) \cdot \frac{d}{dt} (2t)$$

$$\frac{dx}{dt} = -2 e^{\cos 2t} \cdot \sin 2t \quad \text{...(i)}$$

and $$\frac{dy}{dt} = \frac{d}{dt} e^{\sin 2t} = e^{\sin 2t} \cdot \frac{d}{dt} \sin 2t$$

$$= e^{\sin 2t} \cos 2t \cdot \frac{d}{dt} 2t$$

$$= 2e^{\sin 2t} \cdot \cos 2t \quad \text{...(ii)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{\sin 2t} \cdot \cos 2t}{-2e^{\cos 2t} \cdot \sin 2t}$$

$$= \frac{e^{\sin 2t} \cdot \cos 2t}{e^{\cos 2t} \cdot \sin 2t} \quad \text{...(iii)}$$

We know that, $$\log x = \cos 2t \cdot \log e = \cos 2t \quad \text{...(iv)}$$

and $$\log y = \sin 2t \cdot \log e = \sin 2t \quad \text{...(v)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{-y \log x}{x \log y}$$

[using Eqs. (iv) and (v) in Eq. (iii) and $x = e^{\cos 2t}$, $y = e^{\sin 2t}$]

Hence proved.

Q. 50 If $x = a \sin 2t (1 + \cos 2t)$ and $y = b \cos 2t (1 - \cos 2t)$, then show that $\left(\dfrac{dy}{dx}\right)_{t = \pi/4} = \dfrac{b}{a}$.

Sol. $\because$ $$x = a \sin 2t (1 + \cos 2t) \text{ and } y = b \cos 2t (1 - \cos 2t)$$

$\therefore$ $$\frac{dx}{dt} = a \left[\sin 2t \cdot \frac{d}{dt} (1 + \cos 2t) + (1 + \cos 2t) \cdot \frac{d}{dt} \sin 2t \right]$$

$$= a\left[\sin 2t \cdot (-\sin 2t)\cdot \frac{d}{dt}2t + (1+\cos 2t)\cdot \cos 2t \cdot \frac{d}{dt}2t\right]$$

$$= -2a\sin^2 2t + 2a\cos 2t\,(1+\cos 2t)$$

$$\Rightarrow \quad \frac{dx}{dt} = -2a\,[\sin^2 2t - \cos 2t\,(1+\cos 2t)] \quad \text{...(i)}$$

and
$$\frac{dy}{dt} = b\left[\cos 2t \cdot \frac{d}{dt}(1-\cos 2t) + (1-\cos 2t)\cdot \frac{d}{dt}\cos 2t\right]$$

$$= b\left[\cos 2t \cdot (\sin 2t)\frac{d}{dt}2t + (1-\cos 2t)(-\sin 2t)\cdot \frac{d}{dt}2t\right]$$

$$= b\,[2\sin 2t \cdot \cos 2t + 2\,(1-\cos 2t)(-\sin 2t)]$$

$$= 2b\,[\sin 2t \cdot \cos 2t - (1-\cos 2t)\sin 2t] \quad \text{...(ii)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2b\,[-\sin 2t \cdot \cos 2t + (1-\cos 2t)\sin 2t]}{-2a\,[\sin^2 2t - \cos 2t\,(1+\cos 2t)]}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{t=\pi/4} = \frac{b}{a}\,\frac{\left[-\sin\frac{\pi}{2}\cos\frac{\pi}{2} + \left(1-\cos\frac{\pi}{2}\right)\sin\frac{\pi}{2}\right]}{\left[\sin^2\frac{\pi}{2} - \cos\frac{\pi}{2}\left(1+\cos\frac{\pi}{2}\right)\right]}$$

$$= \frac{b}{a}\cdot\frac{(0+1)}{(1-0)} \quad \left[\because \sin\frac{\pi}{2} = 1 \text{ and } \cos\frac{\pi}{2} = 0\right]$$

$$= \frac{b}{a}$$

Hence proved.

Q. 51 If $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, then find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$.

Sol. $\because$
$$x = 3\sin t - \sin 3t \text{ and } y = 3\cos t - \cos 3t$$

$$\therefore \quad \frac{dx}{dt} = 3\cdot\frac{d}{dt}\sin t - \frac{d}{dt}\sin 3t$$

$$= 3\cos t - \cos 3t \cdot \frac{d}{dt}3t = 3\cos t - 3\cos 3t \quad \text{...(i)}$$

and
$$\frac{dy}{dt} = 3\cdot\frac{d}{dt}\cos t - \frac{d}{dt}\cos 3t$$

$$= -3\sin t + \sin 3t \cdot \frac{d}{dt}3t$$

$$\frac{dy}{dt} = 3\sin 3t - 3t\sin t \quad \text{...(ii)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\,(\sin 3t - \sin t)}{3\,(\cos t - \cos 3t)}$$

Now,
$$\left(\frac{dy}{dx}\right)_{t=\pi/3} = \frac{\sin\frac{3\pi}{3} - \sin\frac{\pi}{3}}{\left(\cos\frac{\pi}{3} - \cos 3\frac{\pi}{3}\right)} = \frac{0 - \sqrt{3}/2}{\frac{1}{2} - (-1)}$$

$$= \frac{-\sqrt{3}/2}{3/2} = \frac{-\sqrt{3}}{3} = \frac{-1}{\sqrt{3}}$$

Q. 52 Differentiate $\dfrac{x}{\sin x}$ w.r.t. $\sin x$.

Sol. Let $u = \dfrac{x}{\sin x}$ and $v = \sin x$

$\therefore \quad \dfrac{du}{dx} = \dfrac{\sin x \cdot \dfrac{d}{dx} x - x \cdot \dfrac{d}{dx} \sin x}{(\sin x)^2}$

$= \dfrac{\sin x - x \cos x}{\sin^2 x}$ ...(i)

and $\quad \dfrac{dv}{dx} = \dfrac{d}{dx} \sin x = \cos x$ ...(ii)

$\therefore \quad \dfrac{du}{dv} = \dfrac{du/dx}{dv/dx} = \dfrac{\sin x - x \cos x / \sin^2 x}{\cos x}$

$= \dfrac{\sin x - x \cos x}{\sin^2 x \cos x} = \dfrac{\dfrac{\sin x - x \cos x}{\cos x}}{\dfrac{\sin^2 x \cos x}{\cos x}}$

[dividing by $\cos x$ in both numerator and denominator]

$= \dfrac{\tan x - x}{\sin^2 x}$

Q. 53 Differentiate $\tan^{-1} \dfrac{\sqrt{1 + x^2} - 1}{x}$ w.r.t. $\tan^{-1} x$, when $x \neq 0$.

Sol. Let $u = \tan^{-1} \dfrac{\sqrt{1 + x^2} - 1}{x}$ and $v = \tan^{-1} x$

$\therefore \quad x = \tan \theta$

$\Rightarrow \quad u = \tan^{-1} \dfrac{\sqrt{1 + \tan^2 \theta} - 1}{\tan \theta}$

$= \tan^{-1} \dfrac{(\sec \theta - 1) \cos \theta}{\sin \theta}$

$= \tan^{-1} \left(\dfrac{1 - \cos \theta}{\sin \theta} \right)$

$= \tan^{-1} \left[\dfrac{1 - 1 + 2 \sin^2 \theta/2}{2 \sin \theta/2 \cdot \cos \theta/2} \right]$ $[\because \cos \theta = 1 - 2 \sin^2 \theta]$

$= \tan^{-1} \left[\tan \dfrac{\theta}{2} \right]$

$= \dfrac{\theta}{2} = \dfrac{1}{2} \tan^{-1} x$

$\therefore \quad \dfrac{du}{dx} = \dfrac{1}{2} \dfrac{d}{dx} \tan^{-1} x = \dfrac{1}{2} \cdot \dfrac{1}{1 + x^2}$ ...(i)

and $\quad \dfrac{dv}{dx} = \dfrac{d}{dx} \tan^{-1} x = \dfrac{1}{1 + x^2}$ ...(ii)

$\therefore \quad \dfrac{du}{dv} = \dfrac{du/dx}{dv/dx}$

$= \dfrac{1/2\,(1 + x^2)}{1/(1 + x^2)} = \dfrac{(1 + x^2)}{2(1 + x^2)} = \dfrac{1}{2}$

Find $\frac{dy}{dx}$ *when x and y are connected by the relation given.*

Q. 54 $\sin(xy) + \frac{x}{y} = x^2 - y$

Sol. We have, $\sin(xy) + \frac{x}{y} = x^2 - y$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\sin xy) + \frac{d}{dx}\left(\frac{x}{y}\right) = \frac{d}{dx}x^2 - \frac{d}{dx}y$$

$$\Rightarrow \cos xy \cdot \frac{d}{dx}(xy) + \frac{y\frac{d}{dx}x - x\cdot\frac{d}{dx}y}{y^2} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \cos xy \cdot \left[x\cdot\frac{d}{dx}y + y\cdot\frac{d}{dx}\cdot x\right] + \frac{y - x\frac{dy}{dx}}{y^2} = 2x - \frac{dy}{dx}$$

$$\Rightarrow x\cos xy \cdot \frac{dy}{dx} + y\cos xy + \frac{y}{y^2} - \frac{x}{y^2}\frac{dy}{dx} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left[x\cos xy - \frac{x}{y^2} + 1\right] = 2x - y\cos xy - \frac{y}{y^2}$$

$$\therefore \frac{dy}{dx} = \left[\frac{2xy - y^2\cos xy - 1}{y}\right]\left[\frac{y^2}{xy^2\cos xy - x + y^2}\right]$$

$$= \frac{(2xy - y^2\cos xy - 1)\,y}{(xy^2\cos xy - x + y^2)}$$

Q. 55 $\sec(x + y) = xy$

Sol. We have, $\sec(x + y) = xy$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\sec(x + y) = \frac{d}{dx}(xy)$$

$$\Rightarrow \sec(x + y)\cdot\tan(x + y)\cdot\frac{d}{dx}(x + y) = x\cdot\frac{d}{dx}y + y\cdot\frac{d}{dx}x$$

$$\Rightarrow \sec(x + y)\cdot\tan(x + y)\cdot\left(1 + \frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \sec(x + y)\tan(x + y) + \sec(x + y)\cdot\tan(x + y)\,.\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow \frac{dy}{dx}[\sec(x + y)\cdot\tan(x + y) - x] = y - \sec(x + y)\cdot\tan(x + y)$$

$$\therefore \frac{dy}{dx} = \frac{y - \sec(x + y)\cdot\tan(x + y)}{\sec(x + y)\cdot\tan(x + y) - x}$$

Q. 56 $\tan^{-1}(x^2+y^2)=a$

Sol. We have, $\tan^{-1}(x^2+y^2)=a$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\tan^{-1}(x^2+y^2)=\frac{d}{dx}(a)$$

$$\Rightarrow \quad \frac{1}{1+(x^2+y^2)^2}\cdot\frac{d}{dx}(x^2+y^2)=0$$

$$\Rightarrow \quad 2x+\frac{d}{dy}y^2\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \quad 2y\cdot\frac{dy}{dx}=-2x$$

$$\therefore \quad \frac{dy}{dx}=\frac{-2x}{2y}=\frac{-x}{y}$$

Q. 57 $(x^2+y^2)^2=xy$

Sol. We have, $(x^2+y^2)^2=xy$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(x^2+y^2)^2=\frac{d}{dx}(xy)$$

$$\Rightarrow \quad 2(x^2+y^2)\cdot\frac{d}{dx}(x^2+y^2)=x\cdot\frac{d}{dx}y+y\cdot\frac{d}{dx}x$$

$$\Rightarrow \quad 2(x^2+y^2)\cdot\left(2x+2y\frac{dy}{dx}\right)=x\frac{dy}{dx}+y$$

$$\Rightarrow \quad 2x^2\cdot 2x+2x^2\cdot 2y\frac{dy}{dx}+2y^2\cdot 2x+2y^2\cdot 2y\frac{dy}{dx}=x\frac{dy}{dx}+y$$

$$\Rightarrow \quad \frac{dy}{dx}[4x^2y+4y^3-x]=y-4x^3-4xy^2$$

$$\therefore \quad \frac{dy}{dx}=\frac{(y-4x^3-4xy^2)}{(4x^2y+4y^3-x)}$$

Q. 58 If $ax^2+2hxy+by^2+2gx+2fy+c=0$, then show that $\frac{dy}{dx}\cdot\frac{dx}{dy}=1$.

Sol. We have, $ax^2+2hxy+by^2+2gx+2fy+c=0$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(ax^2)+\frac{d}{dx}(2hxy)+\frac{d}{dx}(by^2)+\frac{d}{dx}(2gx)+\frac{d}{dx}(2fy)+\frac{d}{dx}(c)=0$$

$$\Rightarrow \quad 2ax+2h\left(x\cdot\frac{dy}{dx}+y\cdot 1\right)+b\cdot 2y\frac{dy}{dx}+2g+2f\frac{dy}{dx}+0=0$$

$$\Rightarrow \quad \frac{dy}{dx}[2hx+2by+2f]=-2ax-2hy-2g$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-2(ax+hy+g)}{2(hx+by+f)}$$

$$=\frac{-(ax+hy+g)}{(hx+by+f)} \quad \text{...(ii)}$$

Now, differentiating Eq. (i) w.r.t. y, we get

$$\frac{d}{dy}(ax^2)+\frac{d}{dy}(2hxy)+\frac{d}{dy}(by^2)+\frac{d}{dy}(2gx)+\frac{d}{dy}(2fy)+\frac{d}{dy}(c)=0$$

$$\Rightarrow \quad a\cdot 2x\cdot\frac{dx}{dy}+2h\cdot\left(x\cdot\frac{d}{dy}y+y\cdot\frac{d}{dy}x\right)+b\cdot 2y+2g\cdot\frac{dx}{dy}+2f+0=0$$

$$\Rightarrow \quad \frac{dx}{dy}[2ax+2hy+2g]=-2hx-2by-2f$$

$$\Rightarrow \quad \frac{dx}{dy}=\frac{-2(hx+by+f)}{2(ax+hy+g)}=\frac{-(hx+by+f)}{(ax+hy+g)} \quad \text{... (iii)}$$

$$\therefore \quad \frac{dy}{dx}\cdot\frac{dx}{dy}=\frac{-(ax+hy+g)}{(hx+by+f)}\cdot\frac{-(hx+by+f)}{(ax+hy+g)} \quad \text{[using Eqs. (ii) and (iii)]}$$

$$=1=\text{RHS}$$

Hence proved.

Q. 59 If $x=e^{x/y}$, then prove that $\dfrac{dy}{dx}=\dfrac{x-y}{x\log x}$.

Sol. We have,

$$x=e^{x/y}$$

$$\therefore \quad \frac{d}{dx}x=\frac{d}{dx}e^{x/y}$$

$$\Rightarrow \quad 1=e^{x/y}\cdot\frac{d}{dx}(x/y)$$

$$\Rightarrow \quad 1=e^{x/y}\cdot\left[\frac{y\cdot 1-x\cdot dy/dx}{y^2}\right]$$

$$\Rightarrow \quad y^2=y\cdot e^{x/y}-x\cdot\frac{dy}{dx}\cdot e^{x/y}$$

$$\Rightarrow \quad x\cdot\frac{dy}{dx}\cdot e^{x/y}=ye^{x/y}-y^2$$

$$\therefore \quad \frac{dy}{dx}=\frac{y(e^{x/y}-y)}{x\cdot e^{x/y}}$$

$$=\frac{(e^{x/y}-y)}{e^{x/y}\cdot\frac{x}{y}} \quad \left[\because x=e^{x/y}\Rightarrow \log x=\frac{x}{y}\right]$$

$$=\frac{x-y}{x\cdot\log x}$$

Hence proved.

Q. 60 If $y^x=e^{y-x}$, then prove that $\dfrac{dy}{dx}=\dfrac{(1+\log y)^2}{\log y}$.

Sol. We have,

$$y^x=e^{y-x}$$

$$\Rightarrow \quad \log y^x=\log e^{y-x}$$

$$\Rightarrow \quad x\log y=y-x\cdot\log_e=(y-x) \quad [\because \log_e=1]$$

$$\Rightarrow \quad \log y=\frac{(y-x)}{x} \quad \text{...(i)}$$

Now, differentiating w.r.t. x, we get

$$\frac{d}{dy}\log y\cdot\frac{dy}{dx}=\frac{d}{dx}\frac{(y-x)}{x}$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{x\cdot\frac{d}{dx}(y-x)-(y-x)\cdot\frac{d}{dx}\cdot x}{x^2}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{x\left(\frac{dy}{dx}-1\right)-(y-x)}{x^2}$$

$$\Rightarrow \quad \frac{x^2}{y}\cdot\frac{dy}{dx} = x\frac{dy}{dx} - x - y + x$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{x^2}{y}-x\right) = -y$$

$$\therefore \quad \frac{dy}{dx} = \frac{-y^2}{x^2-xy} = \frac{-y^2}{x(x-y)}$$

$$= \frac{y^2}{x(y-x)}\cdot\frac{x}{x} = \frac{y^2}{x^2}\cdot\frac{1}{\frac{(y-x)}{x}}$$

$$= \frac{(1+\log y)^2}{\log y} \quad \left[\because \log y = \frac{y-x}{x} \; \log y = \frac{y}{x} - 1 \Rightarrow 1 + \log y = \frac{y}{x}\right]$$

Hence proved.

Q. 61 If $y = (\cos x)^{(\cos x)^{(\cos x)^{\cdots\infty}}}$, then show that $\frac{dy}{dx} = \frac{y^2 \tan x}{y \log \cos x - 1}$.

Sol. We have,

$$y = (\cos x)^{(\cos x)^{(\cos x)^{\cdots\infty}}}$$

$$\Rightarrow \quad y = (\cos x)^y$$

$$\therefore \quad \log y = \log (\cos x)^y$$

$$\Rightarrow \quad \log y = y \log \cos x$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = y\cdot\frac{d}{dx}\log\cos x + \log\cos x\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{y}{\cos x}\cdot\frac{d}{dx}\cos x + \log\cos x\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{1}{y} - \log\cos x\right] = \frac{-y\sin x}{\cos x} = -y\tan x$$

$$\therefore \quad \frac{dy}{dx} = \frac{-y^2\tan x}{(1 - y\log\cos x)}$$

$$= \frac{y^2\tan x}{y\log\cos x - 1}$$

Hence proved.

Q. 62 If $x\sin(a+y) + \sin a\cdot\cos(a+y) = 0$, then prove that

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}.$$

Sol. We have,

$$x\sin(a+y) + \sin a\cdot\cos(a+y) = 0$$

$$\Rightarrow \quad x\sin(a+y) = -\sin a\cdot\cos(a+y)$$

$$\Rightarrow \quad x = \frac{-\sin a\cdot\cos(a+y)}{\sin(a+y)}$$

$$\Rightarrow \qquad x = -\sin a \cdot \cot(a+y)$$

$$\therefore \qquad \frac{dx}{dy} = -\sin a \cdot [-\text{cosec}^2(a+y)] \cdot \frac{d}{dy}(a+y)$$

$$= \sin a \cdot \frac{1}{\sin^2(a+y)} \cdot 1$$

$$= \frac{\sin^2(a+y)}{\sin a}$$

Hence proved.

Q. 63 If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then prove that $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}$.

Sol. We have,

$$\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$$

On putting $x = \sin\alpha$ and $y = \sin\beta$, we get

$$\sqrt{1-\sin^2\alpha} + \sqrt{1-\sin^2\beta} = a(\sin\alpha - \sin\beta)$$

$$\Rightarrow \qquad \cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$$

$$\Rightarrow \qquad 2\cos\frac{\alpha+\beta}{2}.\cos\frac{\alpha-\beta}{2} = a\left(2\cos\frac{\alpha+\beta}{2}.\sin\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \qquad \cos\frac{\alpha-\beta}{2} = a\sin\frac{\alpha-\beta}{2}$$

$$\Rightarrow \qquad \cot\frac{\alpha-\beta}{2} = a$$

$$\Rightarrow \qquad \frac{\alpha-\beta}{2} = \cot^{-1}a$$

$$\Rightarrow \qquad \alpha - \beta = 2\cot^{-1}a$$

$$\Rightarrow \qquad \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a \qquad [\because x = \sin\alpha \text{ and } y = \sin\beta]$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$$\therefore \qquad \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = \sqrt{\frac{1-y^2}{1-x^2}}$$

Hence proved.

Q. 64 If $y = \tan^{-1}x$, then find $\dfrac{d^2y}{dx^2}$ in terms of y alone.

Sol. We have,

$$y = \tan^{-1}x \qquad \text{[on differentiating w.r.t. } x\text{]}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{1+x^2} \qquad \text{[again differentiating w.r.t. } x\text{]}$$

Now,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(1+x^2)^{-1}$$

$$= -1(1+x^2)^{-2}.\frac{d}{dx}(1+x^2)$$

$$= -\frac{1}{(1+x^2)^2}\cdot 2x$$

$$= \frac{-2\tan y}{(1+\tan^2 y)^2} \qquad [\because y = \tan^{-1}x \Rightarrow \tan y = x]$$

$$= \frac{-2\tan y}{(\sec^2 y)^2}$$

$$= -2\frac{\sin y}{\cos y}\cdot\cos^2 y\cdot\cos^2 y$$

$$= -\sin 2y\cdot\cos^2 y \qquad [\because \sin 2x = 2\sin x\cos x]$$

Verify the Rolle's theorem for each of the functions in following questions.

Q. 65 $f(x) = x(x-1)^2$ in $[0,1]$

Thinking Process

We know that, Rolle's theorem states that, if f be a real valued function, defined in the closed interval [a, b], such that (i) f is continuous on [a, b]. (ii) f is differentiable on]a, b[.(iii) f(a) = f(b).

Then, there exists a real number c in the open interval] a, b [, such that $f'(c) = 0$'. Here, we shall verify the Rolle's theorem for the given function.

Sol. We have, $f(x) = x(x-1)^2$ in [0, 1].

(i) Since, $f(x) = x(x-1)^2$ is a polynomial function.

So, it is continuous in [0, 1].

(ii) Now,

$$f'(x) = x\cdot\frac{d}{dx}(x-1)^2 + (x-1)^2\frac{d}{dx}x$$

$$= x\cdot 2(x-1)\cdot 1 + (x-1)^2$$

$$= 2x^2 - 2x + x^2 + 1 - 2x$$

$$= 3x^2 - 4x + 1, \text{ which exists in } (0, 1).$$

So, $f(x)$ is differentiable in (0, 1).

(iii) Now, $f(0) = 0$ and $f(1) = 0 \Rightarrow f(0) = f(1)$

f satisfies the above conditions of Rolle's theorem.

Hence, by Rolle's theorem $\exists\, c \in (0, 1)$ such that

$$f'(c) = 0$$

$$\Rightarrow \quad 3c^2 - 4c + 1 = 0$$

$$\Rightarrow \quad 3c^2 - 3c - c + 1 = 0$$

$$\Rightarrow \quad 3c(c-1) - 1(c-1) = 0$$

$$\Rightarrow \quad (3c-1)(c-1) = 0$$

$$\Rightarrow \quad c = \frac{1}{3}, 1 \Rightarrow \frac{1}{3} \in (0, 1)$$

Thus, we see that there exists a real number c in the open interval (0, 1).

Hence, Rolle's theorem has been verified.

Q. 66 $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$

Sol. We have, $f(x) = \sin^4 x + \cos^4 x$ in $\left[0, \frac{\pi}{2}\right]$ ...(i)

(i) $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$

[since, $\sin^4 x$ and $\cos^4 x$ are continuous functions and we know that, if g and h be continuous functions, then $(g + h)$ is a continuous function.]

(ii)
$$f'(x) = 4(\sin x)^3 \cdot \cos x + 4(\cos x)^3 \cdot (-\sin x)$$
$$= 4\sin^3 x \cdot \cos x - 4\sin x \cdot \cos^3 x$$
$$= 4\sin x \cos x (\sin^2 x - \cos^2 x) \text{ which exists in } \left(0, \frac{\pi}{2}\right) \quad \text{...(ii)}$$

Hence, $f(x)$ is differentiable in $\left(0, \frac{\pi}{2}\right)$.

(iii) Also, $f(0) = 0 + 1 = 1$ and $f\left(\frac{\pi}{2}\right) = 1 + 0 = 1$

$\Rightarrow \quad f(0) = f\left(\frac{\pi}{2}\right)$

Conditions of Rolle's theorem are satisfied.

Hence, there exists atleast one $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

$\therefore \quad 4\sin c \cos c (\sin^2 c - \cos^2 c) = 0$

$\Rightarrow \quad 4\sin c \cos c (-\cos 2c) = 0$

$\Rightarrow \quad -2\sin 2c \cdot \cos 2c = 0$

$\Rightarrow \quad -\sin 4c = 0$

$\Rightarrow \quad \sin 4c = 0$

$\Rightarrow \quad 4c = \pi$

$\Rightarrow \quad c = \frac{\pi}{4}$

and $\quad \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

Hence, Rolle's theorem has been verified.

Q. 67 $f(x) = \log(x^2 + 2) - \log 3$ in $[-1, 1]$

Sol. We have, $\quad f(x) = \log(x^2 + 2) - \log 3.$

(i) Logarithmic functions are continuous in their domain.

Hence, $f(x) = \log(x^2 + 2) - \log 3$ is continuous in $[-1, 1]$.

(ii)
$$f'(x) = \frac{1}{x^2 + 2} \cdot 2x - 0$$
$$= \frac{2x}{x^2 + 2}, \text{ which exists in } (-1, 1).$$

Hence, $f(x)$ is differentiable in $(-1, 1)$.

(iii) $f(-1) = \log[(-1)^2 + 2] - \log 3 = \log 3 - \log 3 = 0$ and

$f(1) = \log(1^2 + 2) - \log 3 = \log 3 - \log 3 = 0$

$\Rightarrow \quad f(-1) = f(1)$

Conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that

$$f'(c) = 0.$$

$$\Rightarrow \quad \frac{2c}{c^2 + 2} = 0$$

$$\Rightarrow \quad c = 0 \in (-1, 1)$$

Hence, Rolle's theorem has been verified.

Q. 68 $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$

Sol. We have, $\quad f(x) = x(x + 3)e^{-x/2}$

(i) $f(x)$ is a continuous function. [since, it is a combination of polynomial functions $x(x + 3)$ and an exponential function $e^{-x/2}$ which are continuous functions]

So, $f(x) = x(x + 3)e^{-x/2}$ is continuous in $[-3, 0]$.

(ii) $\therefore$

$$f'(x) = (x^2 + 3x) \cdot \frac{d}{dx} e^{-x/2} + e^{-x/2} \cdot \frac{d}{dx}(x^2 + 3x)$$

$$= (x^2 + 3x) \cdot e^{-x/2} \cdot \left(-\frac{1}{2}\right) + e^{-x/2} \cdot (2x + 3)$$

$$= e^{-x/2}\left[2x + 3 - \frac{1}{2} \cdot (x^2 + 3x)\right]$$

$$= e^{-x/2}\left[\frac{4x + 6 - x^2 - 3x}{2}\right]$$

$$= e^{-x/2} \cdot \frac{1}{2}[-x^2 + x + 6]$$

$$= \frac{-1}{2} e^{-x/2}[x^2 - x - 6]$$

$$= \frac{-1}{2} e^{-x/2}[x^2 - 3x + 2x - 6]$$

$$= \frac{-1}{2} e^{-x/2}[(x + 2)(x - 3)] \text{ which exists in } (-3, 0).$$

Hence, $f(x)$ is differentiable in $(-3, 0)$.

(iii) $\therefore$

$$f(-3) = -3(-3 + 3)e^{-3/2} = 0$$

and $\quad f(0) = 0(0 + 3)e^{-0/2} = 0$

$\Rightarrow \quad f(-3) = f(0)$

Since, conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that $f'(c) = 0$

$$\Rightarrow \quad -\frac{1}{2} e^{-c/2}(c + 2)(c - 3) = 0$$

$$\Rightarrow \quad c = -2, 3, \text{ where } -2 \in (-3, 0)$$

Therefore, Rolle's theorem has been verified.

Q. 69 $f(x)=\sqrt{4-x^2}$ in $[-2, 2]$

Sol. We have, $f(x)=\sqrt{4-x^2}=(4-x^2)^{1/2}$

(i) $f(x)=\sqrt{4-x^2}$ is a continuous function.

[since every polynomial function is a continuous function]

Hence, $f(x)$ is continuous in $[-2, 2]$.

(ii) $f'(x)=\frac{1}{2}(4-x^2)^{-1/2}\cdot(-2x)$

$=-x.\frac{1}{\sqrt{4-x^2}}$, which exists everywhere except at $x=\pm 2$.

Hence, $f(x)$ is differentiable in $(-2, 2)$.

(iii) $f(-2)=\sqrt{(4-4)}=0$ and $f(2)=\sqrt{(4-4)}=0$

$\Rightarrow \quad f(-2)=f(2)$

conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that $f'(c)=0$.

$$\Rightarrow \quad -c\frac{1}{\sqrt{4-c^2}}=0$$

$$\Rightarrow \quad c=0\in(-2, 2)$$

Hence, Rolle's theorem has been verified.

Q. 70 Discuss the applicability of Rolle's theorem on the function given by

$$f(x)=\begin{cases}x^2+1, & \text{if } 0\le x\le 1\\ 3-x, & \text{if } 1\le x\le 2\end{cases}$$

Sol. We have,
$$f(x)=\begin{cases}x^2+1, \text{if } 0\le x\le 1\\ 3-x, \text{ if } 1\le x\le 2\end{cases}$$

We know that, polynomial function is everywhere continuous and differentiability.

So, $f(x)$ is continuous and differentiable at all points except possibly at $x=1$.

Now, check the differentiability at $x=1$,

At $x=1$,

$$\text{LDH}=\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}$$

$$=\lim_{x\to 1}\frac{(x^2+1)-(1+1)}{x-1} \qquad [\because f(x)=x^2+1, \forall\ 0\le x\le 1]$$

$$=\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x+1)(x-1)}{x-1}$$

$$=2$$

and
$$\text{RDH}=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{(3-x)\,f(1+1)}{(x-1)}$$

$$=\lim_{x\to 1}\frac{3-x-2}{x-1}=\lim_{x\to 1}\frac{-(x-1)}{x-1}=-1$$

$\therefore \quad$ LHD $\ne$ RHD

So, $f(x)$ is not differentiable at $x=1$.

Hence, polle's theorem is not applicable on the interval [0, 2].

Q. 71 Find the points on the curve $y = (\cos x - 1)$ in $[0, 2\pi]$, where the tangent is parallel to X-axis.

Thinking Process

We know that, if f be a real valued function defined in the closed interval $[a, b]$ such that it follows all the three conditions of Rolle's theorem, then $f'(c) = 0$ shows that the tangent to the curve at $x = c$ has a slope 0, i.e., it is parallel to the X-axis. So, by getting the value of c' we can get the required point.

Sol. The equation of the curve is $y = \cos x - 1$.

Now, we have to find a point on the curve in $[0, 2\pi]$,

where the tangent is parallel to X-axis *i.e.,* the tangent to the curve at $x = c$ has a slope o, where $c \in]0, 2\pi[$.

Let us apply Rolle's theorem to get the point.

(i) $y = \cos x - 1$ is a continuous function in $[0, 2\pi]$.

[since it is a combination of cosine function and a constant function]

(ii) $y' = -\sin x$, which exists in $(0, 2\pi)$.

Hence, y is differentiable in $(0, 2\pi)$.

(iii) $y(0) = \cos 0 - 1 = 0$ and $y(2\pi) = \cos 2\pi - 1 = 0$,

$\therefore \quad y(0) = y(2\pi)$

Since, conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that

$$f'(c) = 0$$

$$\Rightarrow \quad -\sin c = 0$$

$$\Rightarrow \quad c = \pi \text{ or } 0, \text{ where } \pi \in (0, 2\pi)$$

$$\Rightarrow \quad x = \pi$$

$$\therefore \quad y = \cos \pi - 1 = -2$$

Hence, the required point on the curve, where the tangent drawn is parallel to the X-axis is $(\pi, -2)$.

Q. 72 Using Rolle's theorem, find the point on the curve $y = x(x - 4)$, $x \in [0, 4]$, where the tangent is parallel to X-axis.

Sol. We have, $y = x(x - 4)$, $x \in [0, 4]$

(i) y is a continuous function since $x(x - 4)$ is a polynomial function.

Hence, $y = x(x - 4)$ is continuous in $[0, 4]$.

(ii) $y' = (x - 4)\cdot 1 + x \cdot 1 = 2x - 4$ which exists in $(0, 4)$.

Hence, y is differentiable in $(0, 4)$.

(iii) $y(0) = 0(0 - 4) = 0$

and $\quad y(4) = 4(4 - 4) = 0$

$\Rightarrow \quad y(0) = y(4)$

Sicne, conditions of Rolle's theorem are satisfied.

Hence, there exists a point c such that

$$f'(c) = 0 \text{ in } (0, 4) \qquad [\because f'(x) = y']$$

$$\Rightarrow \quad 2c - 4 = 0$$

$$\Rightarrow \quad c = 2$$

$$\Rightarrow \quad x = 2;\ y = 2(2 - 4) = -4$$

Thus, $(2, -4)$ is the point on the curve at which the tangent drawn is parallel to X-axis.

Verify mean value theorem for each of the functions.

Q. 73 $f(x)=\dfrac{1}{4x-1}$ in [1, 4]

Thinking Process

We know that, mean value theorem states that, if f be a real function such that
(i) $f(x)$ *is continuous on [a,b]*
(ii) $f(x)$ *is differentiable on]a,b[*

Then, there exists a real number $c\in]a,b[$ *such that* $f'(c)=\dfrac{f(b)-f(a)}{b-a}$, *thus we can verify it for given function.*

Sol. We have, $f(x)=\dfrac{1}{4x-1}$ in [1, 4]

(i) $f(x)$ is continuous in [1, 4].
Also, at $x=\dfrac{1}{4}$, $f(x)$ is discontinuous.
Hence, $f(x)$ is continuous in [1, 4].

(ii) $f'(x)=-\dfrac{4}{(4x-1)^2}$, which exists in (1, 4).

Since, conditions of mean value theorem are satisfied.
Hence, there exists a real number $c\in]1,4[$ such that

$$f'(c)=\frac{f(4)-f(1)}{4-1}$$

$$\Rightarrow \quad \frac{-4}{(4c-1)^2}=\frac{\frac{1}{16-1}-\frac{1}{4-1}}{4-1}=\frac{\frac{1}{15}-\frac{1}{3}}{3}$$

$$\Rightarrow \quad \frac{-4}{(4c-1)^2}=\frac{1-5}{45}=\frac{-4}{45}$$

$$\Rightarrow \quad (4c-1)^2=45$$

$$\Rightarrow \quad 4c-1=\pm 3\sqrt{5}$$

$$\Rightarrow \quad c=\frac{3\sqrt{5}+1}{4}\in(1,4) \qquad \text{[neglecting (– ve) value]}$$

Hence, mean value theorem has been verified.

Q. 74 $f(x)=x^3-2x^2-x+3$ in [0, 1]

Sol. We have, $f(x)=x^3-2x^2-x+3$ in [0, 1]

(i) Since, $f(x)$ is a polynomial function.
Hence, $f(x)$ is continuous in [0, 1].

(ii) $f'(x)=3x^2-4x-1$, which exists in (0,1).
Hence, $f(x)$ is differentiable in (0,1).
Since, conditions of mean value theorem are satisfied.
Therefore, by mean value theorem $\exists\, c\in(0,1)$, such that

$$f'(c)=\frac{f(1)-f(0)}{1-0}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = \frac{[1-2-1+3]-[0+3]}{1-0}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = \frac{-2}{1}$$

$$\Rightarrow \quad 3c^2 - 4c + 1 = 0$$

$$\Rightarrow \quad 3c^2 - 3c - c + 1 = 0$$

$$\Rightarrow \quad 3c(c-1) - 1(c-1) = 0$$

$$\Rightarrow \quad (3c-1)(c-1) = 0$$

$$\Rightarrow \quad c = 1/3, 1, \text{ where } \frac{1}{3} \in (0, 1)$$

Hence, the mean value theorem has been verified.

Q. 75 $f(x) = \sin x - \sin 2x$ in $[0, \pi]$

Sol. We have, $f(x) = \sin x - \sin 2x$ in $[0,\pi]$

(i) Since, we know that sine functions are continuous functions hence $f(x) = \sin x - \sin 2x$ is a continuous function in $[0,\pi]$.

(ii) $f'(x) = \cos x - \cos 2x \cdot 2 = \cos x - 2\cos 2x$, which exists in $(0, \pi)$.

So, $f(x)$ is differentiable in $(0, \pi)$. Conditions of mean value theorem are satisfied.

Hence, $\exists\, c \in (0, \pi)$ such that, $f'(c) = \dfrac{f(\pi) - f(0)}{\pi - 0}$

$$\Rightarrow \quad \cos c - 2\cos 2c = \frac{\sin\pi - \sin 2\pi - \sin 0 + \sin 2\cdot 0}{\pi - 0}$$

$$\Rightarrow \quad 2\cos 2c - \cos c = \frac{0}{\pi}$$

$$\Rightarrow \quad 2\cdot(2\cos^2 c - 1) - \cos c = 0$$

$$\Rightarrow \quad 4\cos^2 c - 2 - \cos c = 0$$

$$\Rightarrow \quad 4\cos^2 c - \cos c - 2 = 0$$

$$\Rightarrow \quad \cos c = \frac{1 \pm \sqrt{1+32}}{8} = \frac{1 \pm \sqrt{33}}{8}$$

$$\therefore \quad c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$$

Also, $\cos^{-1}\left(\dfrac{1 \pm \sqrt{33}}{8}\right) \in (0, \pi)$

Hence, mean value theorem has been verified.

Q. 76 $f(x) = \sqrt{25 - x^2}$ in $[1, 5]$

Sol. We have, $f(x) = \sqrt{25 - x^2}$ in $[1, 5]$

(i) Since, $f(x) = (25 - x^2)^{1/2}$, where $25 - x^2 \geq 0$

$$\Rightarrow \quad x^2 \leq \pm 5 \Rightarrow -5 \leq x \leq 5$$

Hence, $f(x)$ is continuous in $[1, 5]$.

(ii) $f'(x) = \dfrac{1}{2}(25 - x^2)^{-1/2} \cdot -2x = \dfrac{-x}{\sqrt{25 - x^2}}$, which exists in $(1, 5)$.

Hence, $f'(x)$ is differentiable in $(1, 5)$.

Since, conditions of mean value theorem are satisfied.
By mean value theorem $\exists\, c \in (1, 5)$ such that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1} \Rightarrow \frac{-c}{\sqrt{25 - c^2}} = \frac{0 - \sqrt{24}}{4}$$

$$\Rightarrow \quad \frac{c^2}{25 - c^2} = \frac{24}{16}$$

$$\Rightarrow \quad 16c^2 = 600 - 24c^2$$

$$\Rightarrow \quad c^2 = \frac{600}{40} = 15$$

$$\therefore \quad c = \pm\sqrt{15}$$

Also, $c = \sqrt{15} \in (1, 5)$

Hence, the mean value theorem has been verified.

Q. 77 **Find a point on the curve $y = (x - 3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).**

Thinking Process

We know that, if $y = f(x)$ be a function defined on [a, b] which follows mean value theorem, then there exists atleast one point c in (a, b) such that the tangent at the point $[c, f(c)]$ is parallel to the secant joining the points $[a, f(a)]$ and $[b, f(b)]$. So, we shall use this concept.

Sol. We have, $y = (x - 3)^2$, which is continuous in $x_1 = 3$ and $x_2 = 4$ *i.e.,* [3, 4].
Also, $y' = 2(x - 3) \cdot 1 = 2(x - 3)$ which exists in (3, 4).
Hence, by mean value theorem there exists a point on the curve at which tangent drawn is parallel to the chord joining the points (3,0) and (4,1).

Thus, $$f'(c) = \frac{f(4) - f(3)}{4 - 3}$$

$$\Rightarrow \quad 2(c - 3) = \frac{(4 - 3)^2 - (3 - 3)^2}{4 - 3}$$

$$\Rightarrow \quad 2c - 6 = \frac{1 - 0}{1} \Rightarrow c = \frac{7}{2}$$

For $x = \frac{7}{2}$, $$y = \left(\frac{7}{2} - 3\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

So, $\left(\frac{7}{2}, \frac{1}{4}\right)$ is the point on the curve at which tangent drawn is parallel to the chord joining the points (3, 0) and (4, 1).

Q. 78 **Using mean value theorem, prove that there is a point on the curve $y = 2x^2 - 5x + 3$ between the points $A(1, 0)$ and $B(2, 1)$, where tangent is parallel to the chord AB. Also, find that point.**

Sol. We have, $y = 2x^2 - 5x + 3$, which is continuous in [1, 2] as it is a polynomial function.
Also, $y' = 4x - 5$, which exists in (1, 2).
By mean value theorem, $\exists\, c \in (1, 2)$ at which drawn tangent is parallel to the chord *AB*, where *A* and *B* are (1, 0) and (2,1), respectively.

$$\therefore \quad f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$\Rightarrow \quad 4c - 5 = \dfrac{(8 - 10 + 3) - (2 - 5 + 3)}{1}$

$\Rightarrow \quad 4c - 5 = 1$

$\therefore \quad c = \dfrac{6}{4} = \dfrac{3}{2} \in (1, 2)$

For $x = \dfrac{3}{2}$, $\quad y = 2\left(\dfrac{3}{2}\right)^2 - 5\left(\dfrac{3}{2}\right) + 3$

$= 2 \times \dfrac{9}{4} - \dfrac{15}{2} + 3 = \dfrac{9 - 15 + 6}{2} = 0$

Hence, $\left(\dfrac{3}{2}, 0\right)$ is the point on the curve $y = 2x^2 - 5x + 3$ between the points A (1, 0) and B (2, 1), where tangent is parallel to the chord AB.

Long Answer Type Questions

Q. 79 Find the values of p and q, so that $f(x) = \begin{cases} x^2 + 3x + p, & \text{if } x \le 1 \\ qx + 2, & \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$.

Sol. We have, $f(x) = \begin{cases} x^2 + 3x + p, & \text{if } x \le 1 \\ qx + 2, & \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$.

$\therefore \quad Lf'(1) = \lim\limits_{x \to 1^-} \dfrac{f(x) - f(1)}{x - 1}$

$= \lim\limits_{x \to 1^-} \dfrac{(x^2 + 3x + p) - (1 + 3 + p)}{x - 1}$

$= \lim\limits_{h \to 0} \dfrac{[(1 - h)^2 + 3(1 - h) + p] - [1 + 3 + p]}{(1 - h) - 1}$

$= \lim\limits_{h \to 0} \dfrac{[1 + h^2 - 2h + 3 - 3h + p] - [4 + p]}{-h}$

$= \lim\limits_{h \to 0} \dfrac{[h^2 - 5h + p + 4 - 4 - p]}{-h} = \lim\limits_{h \to 0} \dfrac{h[h - 5]}{-h}$

$= \lim\limits_{h \to 0} -[h - 5] = 5$

$Rf'(1) = \lim\limits_{x \to 1^+} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1^+} \dfrac{(qx + 2) - (1 + 3 + p)}{x - 1}$

$= \lim\limits_{h \to 0} \dfrac{[q(1 + h) + 2] - (4 + p)}{1 + h - 1}$

$= \lim\limits_{h \to 0} \dfrac{[q + qh + 2 - 4 - p]}{h} = \lim\limits_{h \to 0} \dfrac{qh + (q - 2 - p)}{h}$

$\Rightarrow \quad q - 2 - p = 0 \Rightarrow p - q = -2$ ...(i)

$\Rightarrow \quad \lim\limits_{h \to 0} \dfrac{qh + 0}{h} = q$ [for existing the limit]

If $Lf'(1) = Rf'(1)$, then $5 = q$

$\Rightarrow \quad p - 5 = -2 \Rightarrow p = 3$

$\therefore \quad p = 3$ and $q = 5$

Q. 80 If $x^m \cdot y^n = (x+y)^{m+n}$, prove that

(i) $\dfrac{dy}{dx} = \dfrac{y}{x}$ and (ii) $\dfrac{d^2y}{dx^2} = 0$

Sol. We have, $x^m \cdot y^n = (x+y)^{m+n}$ ...(i)

(i) Differentiating Eq. (i) w.r.t. x, we get

$$\frac{d}{dx}(x^m \cdot y^n) = \frac{d}{dx}(x+y)^{m+n}$$

$$\Rightarrow \quad x^m \cdot \frac{d}{dy} y^n \cdot \frac{dy}{dx} + y^n \cdot \frac{d}{dx} x^m = (m+n)(x+y)^{m+n-1} \frac{d}{dx}(x+y)$$

$$\Rightarrow \quad x^m \cdot ny^{n-1} \frac{dy}{dx} + y^n \cdot mx^{m-1} = (m+n)(x+y)^{m+n-1}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}[x^m \cdot ny^{n-1} - (m+n)\cdot(x+y)^{m+n-1}] = (m+n)(x+y)^{m+n-1} - y^n mx^{m-1}$$

$$\Rightarrow \quad \frac{dy}{dx}[nx^m y^{n-1} - (m+n)(x+y)^{m+n-1}] = (m+n)\cdot(x+y)^{m+n-1} - \frac{y^{n-1}\cdot y \cdot mx^m}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\dfrac{(m+n)(x+y)^{m+n}}{(x+y)} - \dfrac{y^{n-1}\cdot y\cdot mx^m}{x}}{\dfrac{n\,x^m y^n}{y} - (m+n)(x+y)^{m+n}\dfrac{1}{(x+y)}}$$

$$= \frac{\dfrac{x(m+n)(x+y)^{m+n} - (x+y)\cdot y\cdot^{n-1} y\cdot mx^m}{(x+y)\cdot x}}{\dfrac{(x+y)\,n\,x^m y^n - y(m+n)(x+y)^{m+n}}{(x+y)\cdot y}}$$

$$= \frac{\dfrac{x(m+n)\cdot x^m \cdot y^n - m(x+y)\,y^n\,x^m}{(x+y)\cdot x}}{\dfrac{(x+y)\,nx^m\cdot y^n - y(m+n)\cdot x^m \cdot y^n}{(x+y)\cdot y}} \quad [\because (x+y)^{m+n} = x^m \cdot y^n]$$

$$= \frac{x^m\, y^n[mx + nx - mx - my]\cdot(x+y)\,y}{x^m\, y^n[nx + ny - my - ny]\cdot(x+y)\cdot x}$$

$$= \frac{y}{x} \quad \text{...(ii)}$$

Hence proved.

(ii) Further, differentiating Eq. (ii) *i.e.*, $\dfrac{dy}{dx} = \dfrac{y}{x}$ on both the sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{x\cdot\dfrac{dy}{dx} - y\cdot 1}{x^2}$$

$$= \frac{x\cdot\dfrac{y}{x} - y}{x^2} \quad \left[\because \frac{dy}{dx} = \frac{y}{x}\right]$$

$$= 0$$

Hence proved.

Q. 81 If $x = \sin t$ and $y = \sin pt$, then prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0.$$

Sol. We have, $x = \sin t$ and $y = \sin pt$

$$\therefore \quad \frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = \cos pt \cdot p$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{p \cdot \cos pt}{\cos t} \quad \text{...(i)}$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{\cos t \cdot \frac{d}{dt}(p \cdot \cos pt)\frac{dt}{dx} - p\cos pt \cdot \frac{d}{dt}\cos t \cdot \frac{dt}{dx}}{\cos^2 t}$$

$$= \frac{[\cos t \cdot p \cdot (-\sin pt) \cdot p - p\cos pt \cdot (-\sin t)]\frac{dt}{dx}}{\cos^2 t}$$

$$= \frac{[-p^2 \sin pt \cdot \cos t + p\sin t \cdot \cos pt] \cdot \frac{1}{\cos t}}{\cos^2 t}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-p^2 \sin pt \cdot \cos t + p\cos pt \cdot \sin t}{\cos^3 t} \quad \text{...(ii)}$$

Since, we have to prove

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$$

$$\therefore \quad \text{LHS} = (1 - \sin^2 t)\frac{[-p^2 \sin pt \cdot \cos t + p\cos pt \cdot \sin t]}{\cos^3 t} - \sin t \cdot \frac{p\cos pt}{\cos t} + p^2 \sin pt$$

$$= \frac{1}{\cos^3 t}\begin{bmatrix}(1 - \sin^2 t)(-p^2 \sin pt \cdot \cos t + p\cos pt \cdot \sin t) \\ -p\cos pt \cdot \sin t \cdot \cos^2 t + p^2 \sin pt \cdot \cos^3 t\end{bmatrix}$$

$$= \frac{1}{\cos^3 t}\begin{bmatrix}-p^2 \sin pt \cdot \cos^3 t + p\cos pt \cdot \sin t \cdot \cos^2 t \\ -p\cos pt \cdot \sin t \cdot \cos^2 t + p^2 \sin pt \cdot \cos^3 t\end{bmatrix} \quad [\because 1 - \sin^2 t = \cos^2 t]$$

$$= \frac{1}{\cos^3 t} \cdot 0$$

$$= 0$$

Hence proved.

Q. 82 Find the value of $\frac{dy}{dx}$, if $y = x^{\tan x} + \sqrt{\frac{x^2 + 1}{2}}$.

Sol. We have,

$$y = x^{\tan x} + \sqrt{\frac{x^2 + 1}{2}} \quad \text{...(i)}$$

Taking $u = x^{\tan x}$ and $v = \sqrt{\frac{x^2 + 1}{2}}$,

$$\log u = \tan x \log x \quad \text{...(ii)}$$

and

$$v^2 = \frac{x^2 + 1}{2} \quad \text{...(iii)}$$

On, differentiating Eq. (ii) w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = \tan x\cdot\frac{1}{x} + \log x\cdot\sec^2 x$$

$$\Rightarrow \quad \frac{du}{dx} = u\left[\frac{\tan x}{x} + \log x\cdot\sec^2 x\right]$$

$$= x^{\tan x}\left[\frac{\tan x}{x} + \log x\cdot\sec^2 x\right] \quad \dots\text{(iv)}$$

Also, differentiating Eq. (iii) w.r.t. x, we get

$$2v\cdot\frac{dv}{dx} = \frac{1}{2}(2x) \Rightarrow \frac{dv}{dx} = \frac{1}{4v}\cdot(2x)$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{4\cdot\sqrt{\frac{x^2+1}{2}}}\cdot 2x = \frac{x\cdot\sqrt{2}}{2\sqrt{x^2+1}}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{x}{\sqrt{2(x^2+1)}} \quad \dots\text{(v)}$$

Now, $\quad y = u + v$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= x^{\tan x}\left[\frac{\tan x}{x} + \log x\cdot\sec^2 x\right] + \frac{x}{\sqrt{2(x^2+1)}}$$

Objective Type Questions

Q. 83 If $f(x) = 2x$ and $g(x) = \dfrac{x^2}{2} + 1$, then which of the following can be a discontinuous function?

(a) $f(x) + g(x)$ (b) $f(x) - g(x)$

(c) $f(x)\cdot g(x)$ (d) $\dfrac{g(x)}{f(x)}$

Sol. *(d)* We know that, if f and g be continuous functions, then

(a) $f + g$ is continuous **(b)** $f - g$ is continuous.

(c) fg is continuous **(d)** $\dfrac{f}{g}$ is continuous at these points, where $g(x) \neq 0$.

Here, $$\frac{g(x)}{f(x)} = \frac{\frac{x^2}{2} + 1}{2x} = \frac{x^2+2}{4x}$$

which is discontinuous at $x = 0$.

Q. 84 The function $f(x) = \dfrac{4 - x^2}{4x - x^3}$ is

(a) discontinuous at only one point
(b) discontinuous at exactly two points
(c) discontinuous at exactly three points
(d) None of the above

Sol. *(c)* We have,

$$f(x) = \frac{4 - x^2}{4x - x^3} = \frac{(4 - x^2)}{x\,(4 - x^2)}$$

$$= \frac{(4 - x^2)}{x\,(2^2 - x^2)} = \frac{4 - x^2}{x\,(2 + x)\,(2 - x)}$$

Clearly, $f(x)$ is discontinuous at exactly three points $x = 0$, $x = -2$ and $x = 2$.

Q. 85 The set of points where the function f given by $f(x) = |2x - 1| \sin x$ is differentiable is

(a) R (b) $R - \left(\frac{1}{2}\right)$
(c) $(0, \infty)$ (d) None of these

Sol. *(b)* We have, $f(x) = |2x - 1| \sin x$

At $x = \frac{1}{2}$, $f(x)$ is not differentiable.

Hence, $f(x)$ is differentiable in $R - \left(\frac{1}{2}\right)$.

$$\because \quad Rf'\left(\frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left|2\left(\frac{1}{2} + h\right) - 1\right| \sin\left(\frac{1}{2} + h\right) - 0}{h}$$

$$= \lim_{h \to 0} \frac{|2h| \cdot \sin\left(\frac{1 + 2h}{2}\right)}{h} = 2 \cdot \sin\frac{1}{2}$$

and

$$L f'\left(\frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2} - h\right) - f\left(\frac{1}{2}\right)}{-h}$$

$$= \lim_{h \to 0} \frac{\left|2\left(\frac{1}{2} - h\right)^{-1}\right| - \sin\left(\frac{1}{2} - h\right) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{|0 - 2h| - \sin\left(\frac{1}{2} - h\right)}{-h} = -2\sin\left(\frac{1}{2}\right)$$

$$\because \quad Rf'\left(\frac{1}{2}\right) \neq Lf'\left(\frac{1}{2}\right)$$

So, $f(x)$ is not differentiable at $x = \frac{1}{2}$.

Q. 86 The function $f(x) = \cot x$ is discontinuous on the set

(a) $\{x = n\pi : n \in Z\}$ (b) $\{x = 2n\pi : n \in Z\}$

(c) $\left\{ x = (2n+1)\frac{\pi}{2}; n \in Z \right\}$ (d) $\left\{ x = \frac{n\pi}{2}; n \in Z \right\}$

Sol. ***(a)*** We know that, $f(x) = \cot x$ is continuous in $R - \{n\pi : n \in Z\}$.

Since, $f(x) = \cot x = \frac{\cos x}{\sin x}$ [since, $\sin x = 0$ at $n\pi$, $n \in Z$]

Hence, $f(x) = \cot x$ is discontinuous on the set $\{x = n\pi : n \in Z\}$.

Q. 87 The function $f(x) = e^{|x|}$ is

(a) continuous everywhere but not differentiable at $x = 0$

(b) continuous and differentiable everywhere

(c) not continuous at $x = 0$

(d) None of the above

Sol. ***(a)*** Let $u(x) = |x|$ and $v(x) = e^x$

$\therefore \quad f(x) = vou(x) = v[u(x)]$

$= v|x| = e^{|x|}$

Since, $u(x)$ and $v(x)$ are both continuous functions.

So, $f(x)$ is also continuous function but $u(x) = |x|$ is not differentiable at $x = 0$, whereas $v(x) = e^x$ is differentiable at everywhere.

Hence, $f(x)$ is continuous everywhere but not differentiable at $x = 0$.

Q. 88 If $f(x) = x^2 \sin \frac{1}{x}$, where $x \neq 0$, then the value of the function f at $x = 0$, so that the function is continuous at $x = 0$, is

(a) 0 (b) -1

(c) 1 (d) None of these

Sol. ***(a)*** $\because \quad f(x) = x^2 \sin\left(\frac{1}{x}\right)$, where $x \neq 0$

Hence, value of the function f at $x = 0$, so that it is continuous at $x = 0$ is 0.

Q. 89 If $f(x) = \begin{cases} mx + 1, & \text{if } x \leq \frac{\pi}{2} \\ \sin x + n, & \text{if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, then

(a) $m = 1, n = 0$ (b) $m = \frac{n\pi}{2} + 1$

(c) $n = \frac{m\pi}{2}$ (d) $m = n = \frac{\pi}{2}$

Sol. ***(c)*** We have, $f(x) = \begin{cases} mx + 1, & \text{if } x \leq \frac{\pi}{2} \\ (\sin x + n), & \text{if } x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$

$\therefore$ LHL $= \lim_{x \to \frac{\pi^-}{2}} (mx + 1) = \lim_{h \to 0} \left[m\left(\frac{\pi}{2} - h\right) + 1 \right] = \frac{m\pi}{2} + 1$

and RHL $= \lim_{x \to \frac{\pi^+}{2}} (\sin x + n) = \lim_{h \to 0} \left[\sin\left(\frac{\pi}{2} + h\right) + n \right]$

$= \lim_{h \to 0} \cos h + n = 1 + n$

$\therefore$ LHL = RHL $\left[\text{to be continuous at } x = \frac{\pi}{2}\right]$

$\Rightarrow m \cdot \frac{\pi}{2} + 1 = n + 1$

$\therefore n = m \cdot \frac{\pi}{2}$

Q. 90 If $f(x) = |\sin x|$, then

(a) f is everywhere differentiable

(b) f is everywhere continuous but not differentiable at $x = n\pi, n \in Z$

(c) f is everywhere continuous but not differentiable at $x = (2n + 1)\frac{\pi}{2}, n \in Z$

(d) None of the above

Sol. *(b)* We have, $f(x) = |\sin x|$

Let $f(x) = vou\ (x) = v\ [u(x)]$ [where, $u\ (x) = \sin x$ and $v\ (x) = |x|$]

$= v\ (\sin x) = |\sin x|$

where, $u(x)$ and $v\ (x)$ are both continuous.

Hence, $f(x) = vo\ u(x)$ is also a continuous function but $v(x)$ is not differentiable at $x = 0$.

So, $f(x)$ is not differentiable where $\sin x = 0 \Rightarrow x = n\pi,\ n \in Z$

Hence, $f(x)$ is continuous everywhere but not differentiable at $x = n\pi,\ n \in Z$.

Q. 91 If $y = \log\left(\frac{1 - x^2}{1 + x^2}\right)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{4x^3}{1 - x^4}$ (b) $\frac{-4x}{1 - x^4}$ (c) $\frac{1}{4 - x^4}$ (d) $\frac{-4x^3}{1 - x^4}$

Sol. *(b)* We have, $y = \log\left(\frac{1 - x^2}{1 + x^2}\right)$

$$\therefore \frac{dy}{dx} = \frac{1}{\frac{1 - x^2}{1 + x^2}} \cdot \frac{d}{dx}\left(\frac{1 - x^2}{1 + x^2}\right)$$

$$= \frac{(1 + x^2)}{(1 - x^2)} \cdot \frac{(1 + x^2)\cdot(-2x) - (1 - x^2)\cdot 2x}{(1 + x^2)^2}$$

$$= \frac{-2x[1 + x^2 + 1 - x^2]}{(1 - x^2)\cdot(1 + x^2)} = \frac{-4x}{1 - x^4}$$

Q. 92 If $y = \sqrt{\sin x + y}$, then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{\cos x}{2y - 1}$ (b) $\dfrac{\cos x}{1 - 2y}$ (c) $\dfrac{\sin x}{1 - 2y}$ (d) $\dfrac{\sin x}{2y - 1}$

Sol. *(a)* $\because \quad y = (\sin x + y)^{1/2}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(\sin x + y)^{-1/2} \cdot \frac{d}{dx}(\sin x + y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{(\sin x + y)^{1/2}} \cdot \left(\cos x + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}\left(\cos x + \frac{dy}{dx}\right) \quad [\because (\sin x + y)^{1/2} = y]$$

$$\Rightarrow \quad \frac{dy}{dx}\left(1 - \frac{1}{2y}\right) = \frac{\cos x}{2y}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos x}{2y} \cdot \frac{2y}{2y - 1} = \frac{\cos x}{2y - 1}$$

Q. 93 The derivative of $\cos^{-1}(2x^2 - 1)$ w.r.t. $\cos^{-1} x$ is

(a) 2 (b) $\dfrac{-1}{2\sqrt{1 - x^2}}$

(b) $\dfrac{2}{x}$ (d) $1 - x^2$

Sol. *(a)* Let $u = \cos^{-1}(2x^2 - 1)$ and $v = \cos^{-1} x$

$$\therefore \quad \frac{dv}{dx} = \frac{+-1}{\sqrt{1 - (2x^2 - 1)^2}} \cdot 4x = \frac{-4x}{\sqrt{1 - (4x^4 + 1 - 4x^2)}}$$

$$= \frac{-4x}{\sqrt{-4x^4 + 4x^2}} = \frac{-4x}{\sqrt{4x^2(1 - x^2)}}$$

$$= \frac{-2}{\sqrt{1 - x^2}}$$

and
$$\frac{du}{dx} = \frac{-1}{\sqrt{1 - x^2}}$$

$$\therefore \quad \frac{dx}{dv} = \frac{du/dx}{dv/dx} = \frac{-2/\sqrt{1 - x^2}}{-1/\sqrt{1 - x^2}} = 2$$

Q. 94 If $x = t^2$ and $y = t^3$, then $\dfrac{d^2y}{dx^2}$ is equal to

(a) $\dfrac{3}{2}$ (b) $\dfrac{3}{4t}$ (c) $\dfrac{3}{2t}$ (d) $\dfrac{3}{2t}$

Sol. *(b)* We have, $x = t^2$ and $y = t^3$

$$\therefore \quad \frac{dx}{dt} = 2t \text{ and } \frac{dy}{dt} = 3t^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3}{2}t$$

On further differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{3}{2} \cdot \frac{d}{dt}t \cdot \frac{dt}{dx}$$

$$= \frac{3}{2} \cdot \frac{1}{2t} \qquad \left[\because \frac{dt}{dx} = \frac{1}{2t}\right]$$

$$= \frac{3}{4t}$$

Q. 95 The value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is

(a) 1 (b) -1 (c) $\frac{3}{2}$ (d) $\frac{1}{3}$

Sol. *(a)* $\because \quad f'(c) = 0 \qquad [\because f'(x) = 3x^2 - 3]$

$\Rightarrow \quad 3c^2 - 3 = 0$

$\Rightarrow \quad c^2 = \frac{3}{3} = 1$

$\Rightarrow \quad c = \pm 1$, where $1 \in (0, \sqrt{3})$

$\therefore \quad c = 1$

Q. 96 For the function $f(x) = x + \frac{1}{x}$, $x \in [1, 3]$, the value of c for mean value theorem is

(a) 1 (b) $\sqrt{3}$

(c) 2 (d) None of these

Sol. *(b)* $\because \quad f'(c) = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow \quad 1 - \frac{1}{c^2} = \frac{\left[3 + \frac{1}{3}\right] - \left[1 + \frac{1}{1}\right]}{3 - 1} \qquad \left[\because f'(x) = 1 - \frac{1}{x^2} \text{ and } b = 3, a = 1\right]$

$\Rightarrow \quad \frac{c^2 - 1}{c^2} = \frac{\frac{10}{3} - 2}{2}$

$\Rightarrow \quad \frac{c^2 - 1}{c^2} = \frac{4}{3 \times 2} = \frac{2}{3}$

$\Rightarrow \quad 3(c^2 - 1) = 2c^2$

$\Rightarrow \quad 3c^2 - 2c^2 = 3$

$\Rightarrow \quad c^2 = 3 \Rightarrow c = \pm\sqrt{3}$

$\because \quad c = \sqrt{3} \in (1, 3)$

Fillers

Q. 97 An example of a function which is continuous everywhere but fails to be differentiable exactly at two points is ....

Sol. $|x| + |x - 1|$ is continuous everywhere but fails to be differentiable exactly at two points $x = 0$ and $x = 1$.

So, there can be more such examples of functions.

Q. 98 Derivative of x^2 w.r.t. x^3 is ....

Sol. Derivative of x^2 w.r.t. x^3 is $\frac{2}{3x}$.

Let $\quad u = x^2$ and $v = x^3$

$\therefore \quad \frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 3x^2$

$\Rightarrow \quad \frac{du}{dv} = \frac{2x}{3x^2} = \frac{2}{3x}$

Q. 99 If $f(x) = |\cos x|$, then $f'\left(\frac{\pi}{4}\right)$ is equal to ....

Sol. If $f(x) = |\cos x|$, then $f'\left(\frac{\pi}{4}\right)$

$\because \quad 0 < x < \frac{\pi}{2}, \cos x > 0.$

$f(x) = +\cos x$

$\therefore \quad f'(x) = (-\sin x)$

$\Rightarrow \quad f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}} \quad \left[\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$

Q. 100 If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{3}\right)$ is equal to ....

Sol. $\because \quad f(x) = |\cos x - \sin x|,$

$\therefore \quad f'\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}+1}{2}$

We know that, $\frac{\pi}{4} < x < \frac{\pi}{2}, \sin x > \cos x$

$\therefore \cos x - \sin x \le 0$ *i.e.*, $\quad f(x) = -(\cos x - \sin x)$

$f'(x) = -[-\sin x - \cos x]$

$\therefore \quad f'\left(\frac{\pi}{3}\right) = -\left(\frac{-\sqrt{3}}{2} - \frac{1}{2}\right) = \left(\frac{\sqrt{3}+1}{2}\right)$

Q. 101 For the curve $\sqrt{x}+\sqrt{y}=1, \frac{dy}{dx}$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$ is ....

Sol. For the curve $\sqrt{x}+\sqrt{y}=1, \frac{dy}{dx}$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$ is -1.

We have, $\sqrt{x}+\sqrt{y}=1$

$$\Rightarrow \quad \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\left(\frac{1}{4}, \frac{1}{4}\right)}=\frac{-\frac{1}{2}}{\frac{1}{2}}=-1$$

True/False

Q. 102 Rolle's theorem is applicable for the function $f(x)=|x-1|$ in $[0, 2]$.

Sol. ***False***

Hence, $f(x)=|x-1|$ in $[0, 2]$ is not differentiable at $x=1 \in (0, 2)$.

Q. 103 If f is continuous on its domain D, then $|f|$ is also continuous on D.

Sol. ***True***

Q. 104 The composition of two continuous function is a continuous function.

Sol. ***True***

Q. 105 Trigonometric and inverse trigonometric functions are differentiable in their respective domain.

Sol. ***True***

Q. 106 If $f \cdot g$ is continuous at $x=a$, then f and g are separately continuous at $x=a$.

Sol. ***False***

Let $f(x)=\sin x$ and $g(x)=\cot x$

$$\therefore \quad f(x)\cdot g(x)=\sin x \cdot \frac{\cos x}{\sin x}=\cos x$$

which is continuous at $x=0$ but $\cot x$ is not continuous at $x=0$.