

7

Integrals

Short Answer Type Questions

Verify the following

Q. 1 $\int \frac{2x-1}{2x+3} dx = x - \log|(2x+3)^2| + C$

Sol. Let $I = \int \frac{2x-1}{2x+3} dx = \int \frac{2x+3-3-1}{2x+3} dx$
 $= \int 1 dx - 4 \int \frac{1}{2x+3} dx = x - \int \frac{4}{2\left(x+\frac{3}{2}\right)} dx$
 $= x - 2 \log \left| \left(x + \frac{3}{2}\right) \right| + C'$
 $= x - 2 \log|(2x+3)| + 2 \log 2 + C'$
 $= x - \log|(2x+3)^2| + C$ [$\because \log \frac{m}{n} = \log m - \log n$]
[$\because C = 2 \log 2 + C'$]

Q. 2 $\int \frac{2x+3}{x^2+3x} dx = \log|x^2+3x| + C$

Sol. Let $I = \int \frac{2x+3}{x^2+3x} dx$
Put $x^2+3x = t$
 $\Rightarrow (2x+3) dx = dt$
 $\therefore I = \int \frac{1}{t} dt = \log|t| + C$
 $= \log|(x^2+3x)| + C$

Q. 3 $\int \frac{(x^2 + 2)dx}{x + 1}$

Thinking Process

First of all divided numerator by denominator, then use the formula $\int \frac{1}{x} dx = \log |x|$ to get the solution.

Sol. Let

$$\begin{aligned} I &= \int \frac{x^2 + 2}{x + 1} dx \\ &= \int \left(x - 1 + \frac{3}{x + 1} \right) dx \\ &= \int (x - 1) dx + 3 \int \frac{1}{x + 1} dx \\ &= \frac{x^2}{2} - x + 3 \log |(x + 1)| + C \end{aligned}$$

Q. 4 $\int \frac{e^{6 \log x} - e^{5 \log x}}{e^{4 \log x} - e^{3 \log x}} dx$

Sol. Let

$$\begin{aligned} I &= \int \left(\frac{e^{6 \log x} - e^{5 \log x}}{e^{4 \log x} - e^{3 \log x}} \right) dx \\ &= \int \left(\frac{e^{\log x^6} - e^{\log x^5}}{e^{\log x^4} - e^{\log x^3}} \right) dx && [\because a \log b = \log b^a] \\ &= \int \left(\frac{x^6 - x^5}{x^4 - x^3} \right) dx && [\because e^{\log x} = x] \\ &= \int \left(\frac{x^3 - x^2}{x - 1} \right) dx = \int \frac{x^2(x - 1)}{x - 1} dx \\ &= \int x^2 dx = \frac{x^3}{3} + C \end{aligned}$$

Q. 5 $\int \frac{(1 + \cos x)}{x + \sin x} dx$

Sol. Consider that,

Let

\therefore

$$\begin{aligned} I &= \int \frac{(1 + \cos x)}{(x + \sin x)} dx \\ x + \sin x &= t \Rightarrow (1 + \cos x) dx = dt \\ I &= \int \frac{1}{t} dt = \log |t| + C \\ &= \log |(x + \sin x)| + C \end{aligned}$$

Q. 6 $\int \frac{dx}{1 + \cos x}$

Thinking Process

$\cos x = 2 \cos^2 \frac{x}{2} - 1$ and also use formula i.e., $\int \sec^2 x = \tan x + C$ to solve the above problem.

Sol. Let

$$\begin{aligned} I &= \int \frac{dx}{1 + \cos x} = \int \frac{dx}{1 + 2 \cos^2 \frac{x}{2} - 1} \\ &= \frac{1}{2} \int \frac{1}{\cos^2 \frac{x}{2}} dx = \frac{1}{2} \int \sec^2 \frac{x}{2} dx \\ &= \frac{1}{2} \cdot \tan \frac{x}{2} \cdot 2 + C = \tan \frac{x}{2} + C \quad \left[\because \int \sec^2 x dx = \tan x \right] \end{aligned}$$

Q. 7 $\int \tan^2 x \sec^4 x dx$

Thinking Process

Use the formula $\sec^2 x = 1 + \tan^2 x$ and put $\tan x = t$ to solve this problem.

Sol. Let

$$I = \int \tan^2 x \sec^4 x dx$$

Put

$$\tan x = t \Rightarrow \sec^2 x dx = dt$$

\therefore

$$\begin{aligned} I &= \int t^2 (1 + t^2) dt = \int (t^2 + t^4) dt \\ &= \frac{t^3}{3} + \frac{t^5}{5} + C = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C \end{aligned}$$

Q. 8 $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx$

Sol. Let $I = \int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx = \int \frac{(\sin x + \cos x)}{\sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x}} dx$

$$= \int \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}} dx = \int 1 dx = x + C$$

Q. 9 $\int \sqrt{1 + \sin x} dx$

Sol. Let

$$\begin{aligned} I &= \int \sqrt{1 + \sin x} dx \\ &= \int \sqrt{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}} dx \quad \left[\because \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = 1 \right] \\ &= \int \sqrt{\left(\sin \frac{x}{2} + \cos \frac{x}{2} \right)^2} dx = \int \left(\sin \frac{x}{2} + \cos \frac{x}{2} \right) dx \\ &= -\cos \frac{x}{2} \cdot 2 + \sin \frac{x}{2} \cdot 2 + C = -2 \cos \frac{x}{2} + 2 \sin \frac{x}{2} + C \end{aligned}$$

Q. 10 $\int \frac{x}{\sqrt{x+1}} dx$

Sol. Let

$$I = \int \frac{x}{\sqrt{x+1}} dx$$

Put

$$\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt$$

\Rightarrow

$$dx = 2\sqrt{x} dt$$

\therefore

$$\begin{aligned} I &= 2 \int \left(\frac{x\sqrt{x}}{t+1} \right) dt = 2 \int \frac{t^2 \cdot t}{t+1} dt = 2 \int \frac{t^3}{t+1} dt \\ &= 2 \int \frac{t^3 + 1 - 1}{t+1} dt = 2 \int \frac{(t+1)(t^2 - t + 1)}{t+1} dt - 2 \int \frac{1}{t+1} dt \\ &= 2 \int (t^2 - t + 1) dt - 2 \int \frac{1}{t+1} dt \\ &= 2 \left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log |(t+1)| \right] + C \\ &= 2 \left[\frac{x\sqrt{x}}{3} - \frac{x}{2} + \sqrt{x} - \log |(\sqrt{x}+1)| \right] + C \end{aligned}$$

Q. 11 $\int \sqrt{\frac{a+x}{a-x}} dx$

Thinking Process

Here, put $x = a \cos 2\theta$ and also use the formula i.e., $\cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$ to get the solution.

Sol. Let

$$I = \int \sqrt{\frac{a+x}{a-x}} dx$$

Put

$$x = a \cos 2\theta$$

\Rightarrow

$$dx = -a \cdot \sin 2\theta \cdot 2 \cdot d\theta$$

\therefore

$$I = -2 \int \sqrt{\frac{a + a \cos 2\theta}{a - a \cos 2\theta}} \cdot a \sin 2\theta d\theta$$

$$\left[\because \cos 2\theta = \frac{x}{a} \Rightarrow 2\theta = \cos^{-1} \frac{x}{a} \Rightarrow \theta = \frac{1}{2} \cos^{-1} \frac{x}{a} \right]$$

$$= -2a \int \frac{\sqrt{1 + \cos 2\theta}}{\sqrt{1 - \cos 2\theta}} \sin 2\theta d\theta = -2a \int \frac{\sqrt{2 \cos^2 \theta}}{\sqrt{2 \sin^2 \theta}} \sin 2\theta d\theta$$

$$= -2a \int \cot \theta \cdot \sin 2\theta d\theta = -2a \int \frac{\cos \theta}{\sin \theta} \cdot 2 \sin \theta \cdot \cos \theta d\theta$$

$$= -4a \int \cos^2 \theta d\theta = -2a \int (1 + \cos 2\theta) d\theta$$

$$= -2a \left[\theta + \frac{\sin 2\theta}{2} \right] + C$$

$$= -2a \left[\frac{1}{2} \cos^{-1} \frac{x}{a} + \frac{1}{2} \sqrt{1 - \frac{x^2}{a^2}} \right] + C$$

$$= -a \left[\cos^{-1} \left(\frac{x}{a} \right) + \sqrt{1 - \frac{x^2}{a^2}} \right] + C$$

Alternate Method

Let
$$I = \int \sqrt{\frac{a+x}{a-x}} dx = \int \sqrt{\frac{(a+x)(a+x)}{(a-x)(a+x)}} dx$$

$$= \int \frac{(a+x)}{\sqrt{a^2-x^2}} dx$$

$$I = \int \frac{a}{\sqrt{a^2-x^2}} + \int \frac{x}{\sqrt{a^2-x^2}} dx$$

$\therefore I = I_1 + I_2 \quad \dots(i)$

Now,
$$I_1 = \int \frac{a}{\sqrt{a^2-x^2}} = a \sin^{-1}\left(\frac{x}{a}\right) + C_1$$

and
$$I_2 = \int \frac{x}{\sqrt{a^2-x^2}} dx$$

Put $a^2 - x^2 = t^2 \Rightarrow -2x dx = 2t dt$

$\therefore I_2 = -\int \frac{t}{t} dt = -\int 1 dt$

$$= -t + C_2 = -\sqrt{a^2-x^2} + C_2$$

$\therefore I = a \sin^{-1}\left(\frac{x}{a}\right) + C_1 - \sqrt{a^2-x^2} + C_2 \quad [\because t^2 = a^2 - x^2]$

$$I = a \sin^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2-x^2} + C \quad [\because C = C_1 + C_2]$$

Q. 12 $\int \frac{x^{1/2}}{1+x^{3/4}} dx$

Sol. Let
$$I = \int \frac{x^{1/2}}{1+x^{3/4}} dx$$

Put $x = t^4 \Rightarrow dx = 4t^3 dt$

$\therefore I = 4 \int \frac{t^2(t^3)}{1+t^3} dt = 4 \int \left(t^2 - \frac{t^2}{1+t^3} \right) dt$

$$I = 4 \int t^2 dt - 4 \int \frac{t^2}{1+t^3} dt$$

$I = I_1 - I_2$

$$I_1 = 4 \int t^2 dt = 4 \cdot \frac{t^3}{3} + C_1 = \frac{4}{3} x^{3/4} + C_1$$

Now,
$$I_2 = 4 \int \frac{t^2}{1+t^3} dt$$

Again, put $1+t^3 = z \Rightarrow 3t^2 dt = dz$

$\Rightarrow t^2 dt = \frac{1}{3} dz = \frac{4}{3} \int \frac{1}{z} dz$

$$= \frac{4}{3} \log|z| + C_2 = \frac{4}{3} \log|(1+t^3)| + C_2$$

$$= \frac{4}{3} \log|(1+x^{3/4})| + C_2$$

$\therefore I = \frac{4}{3} x^{3/4} + C_1 - \frac{4}{3} \log|(1+x^{3/4})| - C_2$

$$= \frac{4}{3} x^{3/4} - \log|(1+x^{3/4}) + C \quad [\because C = C_1 - C_2]$$

Q. 13 $\int \frac{\sqrt{1+x^2}}{x^4} dx$

Sol. Let

$$I = \int \frac{\sqrt{1+x^2}}{x^4} dx = \int \frac{\sqrt{1+x^2}}{x} \cdot \frac{1}{x^3} dx$$

$$= \int \sqrt{\frac{1+x^2}{x^2}} \cdot \frac{1}{x^3} dx = \int \sqrt{\frac{1}{x^2} + 1} \cdot \frac{1}{x^3} dx$$

Put $1 + \frac{1}{x^2} = t^2 \Rightarrow \frac{-2}{x^3} dx = 2t dt$

$\Rightarrow -\frac{1}{x^3} = t dt$

$\therefore I = -\int t^2 dt = -\frac{t^3}{3} + C = -\frac{1}{3} \left(1 + \frac{1}{x^2}\right)^{3/2} + C$

Q. 14 $\int \frac{dx}{\sqrt{16-9x^2}}$

Thinking Process

First of all convert the expression in form of $\frac{1}{\sqrt{a^2-x^2}}$, then use the formula,

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

Sol. Let $I = \int \frac{dx}{\sqrt{16-9x^2}} = \int \frac{dx}{\sqrt{(4)^2 - (3x)^2}} = \frac{1}{3} \sin^{-1}\left(\frac{3x}{4}\right) + C$

Q. 15 $\int \frac{dt}{\sqrt{3t-2t^2}}$

Sol. Let

$$I = \int \frac{dt}{\sqrt{3t-2t^2}} = \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{-\left(t^2 - \frac{3}{2}t\right)}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{-\left[t^2 - 2 \cdot \frac{1}{2} \cdot \frac{3}{2}t\right] + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{-\left[t - \frac{3}{4}\right]^2 - \left(\frac{3}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{\left(\frac{3}{4}\right)^2 - \left(t - \frac{3}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{t - \frac{3}{4}}{\frac{3}{4}}\right) + C = \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{4t - 3}{3}\right) + C$$

Q. 16 $\int \frac{3x-1}{\sqrt{x^2+9}} dx$

Thinking Process

First of all convert to the given integral into two parts, then by using formula i.e.,

$$\int \frac{1}{\sqrt{a^2+x^2}} = \log|x + \sqrt{a^2+x^2}| + C, \text{ get the desired result.}$$

Sol. Let

$$I = \int \frac{3x-1}{\sqrt{x^2+9}} dx$$

$$I = \int \frac{3x}{\sqrt{x^2+9}} dx - \int \frac{1}{\sqrt{x^2+9}} dx$$

$$I = I_1 - I_2$$

Now,

$$I_1 = \int \frac{3x}{\sqrt{x^2+9}}$$

Put

$$x^2 + 9 = t^2 \Rightarrow 2x dx = 2t dt \Rightarrow x dx = t dt$$

\therefore

$$I_1 = 3 \int \frac{t}{t} dt$$

$$= 3 \int dt = 3t + C_1 = 3\sqrt{x^2+9} + C_1$$

and

$$I_2 = \int \frac{1}{\sqrt{x^2+9}} dx = \int \frac{1}{\sqrt{x^2+(3)^2}} dx$$

$$= \log|x + \sqrt{x^2+9}| + C_2$$

\therefore

$$I = 3\sqrt{x^2+9} + C_1 - \log|x + \sqrt{x^2+9}| - C_2$$

$$= 3\sqrt{x^2+9} - \log|x + \sqrt{x^2+9}| + C$$

$$[\because C = C_1 - C_2]$$

Q. 17 $\int \sqrt{5-2x+x^2} dx$

Thinking Process

First of all convert the given expression into $\sqrt{x^2+a^2}$ form, then use the formula i.e.,

$$\int \sqrt{x^2+a^2} dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x + \sqrt{x^2+a^2}| + C.$$

Sol. Let

$$I = \int \sqrt{5-2x+x^2} dx = \int \sqrt{x^2-2x+1+4} dx$$

$$= \int \sqrt{(x-1)^2+(2)^2} dx = \int \sqrt{(2)^2+(x-1)^2} dx$$

$$= \frac{x-1}{2}\sqrt{2^2+(x-1)^2} + 2\log|x-1+\sqrt{2^2+(x-1)^2}| + C$$

$$= \frac{x-1}{2}\sqrt{5-2x+x^2} + 2\log|x-1+\sqrt{5-2x+x^2}| + C$$

Q. 18 $\int \frac{x}{x^4 - 1} dx$

Sol. Let $I = \int \frac{x}{x^4 - 1} dx$

Put $x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{1}{2} dt$

$$\therefore I = \frac{1}{2} \int \frac{dt}{t^2 - 1} = \frac{1}{2} \cdot \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| + C \quad \left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \right]$$

$$= \frac{1}{4} [\log |x^2 - 1| - \log |x^2 + 1|] + C$$

Q. 19 $\int \frac{x^2}{1 - x^4} dx$

Thinking Process

Here, use $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$ and $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{1+x}{1-x} \right| + C$, to solve this problem.

Sol. Let $I = \int \frac{x^2}{1 - x^4} dx$

$$= \int \frac{\left(\frac{1}{2} + \frac{x^2}{2} - \frac{1}{2} + \frac{x^2}{2} \right)}{(1 - x^2)(1 + x^2)} dx \quad [\because a^2 - b^2 = (a + b)(a - b)]$$

$$= \int \frac{\frac{1}{2}(1 + x^2) - \frac{1}{2}(1 - x^2)}{(1 - x^2)(1 + x^2)} dx$$

$$= \int \frac{\frac{1}{2}(1 + x^2)}{(1 - x^2)(1 + x^2)} dx - \frac{1}{2} \int \frac{(1 - x^2)}{(1 - x^2)(1 + x^2)} dx$$

$$= \frac{1}{2} \int \frac{1}{1 - x^2} dx - \frac{1}{2} \int \frac{1}{1 + x^2} dx = \frac{1}{2} \cdot \frac{1}{2} \log \left| \frac{1+x}{1-x} \right| + C_1 - \frac{1}{2} \tan^{-1} x + C_2$$

$$= \frac{1}{4} \log \left| \frac{1+x}{1-x} \right| - \frac{1}{2} \tan^{-1} x + C \quad [\because C = C_1 + C_2]$$

Q. 20 $\int \sqrt{2ax - x^2} dx$

Sol. Let $I = \int \sqrt{2ax - x^2} dx = \int \sqrt{-(x^2 - 2ax)} dx$

$$= \int \sqrt{-(x^2 - 2ax + a^2 - a^2)} dx = \int \sqrt{-(x-a)^2 - a^2} dx$$

$$= \int \sqrt{a^2 - (x-a)^2} dx$$

$$= \frac{x-a}{2} \sqrt{a^2 - (x-a)^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x-a}{a} \right) + C$$

$$= \frac{x-a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x-a}{a} \right) + C$$

Q. 21 $\int \frac{\sin^{-1} x}{(1-x^2)^{3/4}} dx$

Sol. Let $I = \int \frac{\sin^{-1} x}{(1-x^2)^{3/4}} dx = \int \frac{\sin^{-1} x}{(1-x^2)\sqrt{1-x^2}} dx$

Put $\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$

and $x = \sin t \Rightarrow 1-x^2 = \cos^2 t$

$\Rightarrow \cos t = \sqrt{1-x^2}$

$\therefore I = \int \frac{t}{\cos^2 t} dt = \int t \cdot \sec^2 t dt$

$= t \cdot \int \sec^2 t dt - \int \left(\frac{d}{dt} t \cdot \int \sec^2 t dt \right) dt$

$= t \cdot \tan t - \int 1 \cdot \tan t dt$

$= t \tan t + \log |\cos t| + C \quad [\because \int \tan x dx = -\log |\cos x| + C]$

$= \sin^{-1} x \cdot \frac{x}{\sqrt{1-x^2}} + \log |\sqrt{1-x^2}| + C$

Q. 22 $\int \frac{(\cos 5x + \cos 4x)}{1-2\cos 3x} dx$

Sol. Let $I = \int \frac{\cos 5x + \cos 4x}{1-2\cos 3x} dx = \int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{1-2\left(2\cos^2 \frac{3x}{2} - 1\right)} dx$

$\left[\because \cos C + \cos D = 2\cos \frac{C+D}{2} \cdot \cos \frac{C-D}{2} \text{ and } \cos 2x = 2\cos^2 x - 1 \right]$

$\therefore I = \int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{3-4\cos^2 \frac{3x}{2}} dx = -\int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{4\cos^2 \frac{3x}{2} - 3} dx$

$= -\int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{4\cos^3 \frac{3x}{2} - 3\cos \frac{3x}{2}} dx \quad \left[\text{multiply and divide by } \cos \frac{3x}{2} \right]$

$= -\int \frac{2\cos \frac{9x}{2} \cdot \cos \frac{x}{2} \cdot \cos \frac{3x}{2}}{\cos 3 \cdot \frac{3x}{2}} dx = -\int 2\cos \frac{3x}{2} \cdot \cos \frac{x}{2} dx$

$= -\int \left\{ \cos \left(\frac{3x}{2} + \frac{x}{2} \right) + \cos \left(\frac{3x}{2} - \frac{x}{2} \right) \right\} dx$

$= -\int (\cos 2x + \cos x) dx$

$= -\left[\frac{\sin 2x}{2} + \sin x \right] + C$

$= -\frac{1}{2} \sin 2x - \sin x + C$

$$\text{Q. 23 } \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$$

Thinking Process

Use $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ and $\sec^2 x = 1 + \tan^2 x$, $\operatorname{cosec}^2 x = 1 + \cot^2 x$ to solve the above problem.

Sol. Let

$$\begin{aligned} I &= \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x \cdot \cos^2 x} dx \\ &= \int \frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)}{\sin^2 x \cdot \cos^2 x} dx \\ &= \int \frac{\sin^4 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^4 x}{\sin^2 x \cdot \cos^2 x} dx - \int \frac{\sin^2 x \cos^2 x}{\sin^2 x \cdot \cos^2 x} dx \\ &= \int \tan^2 x dx + \int \cot^2 x dx - \int 1 dx \\ &= \int (\sec^2 x - 1) dx + \int (\operatorname{cosec}^2 x - 1) dx - \int 1 dx \\ &= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx - 3 \int dx \\ I &= \tan x - \cot x - 3x + C \end{aligned}$$

$$\text{Q. 24 } \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx$$

Sol. Let

$$I = \int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} dx = \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}$$

Put

$$x^{3/2} = t \Rightarrow \frac{3}{2} x^{1/2} dx = dt$$

\therefore

$$\begin{aligned} I &= \frac{2}{3} \int \frac{dt}{\sqrt{(a^{3/2})^2 - t^2}} = \frac{2}{3} \sin^{-1} \frac{t}{a^{3/2}} + C \\ &= \frac{2}{3} \sin^{-1} \frac{x^{3/2}}{a^{3/2}} + C = \frac{2}{3} \sin^{-1} \sqrt{\frac{x^3}{a^3}} + C \end{aligned}$$

$$\text{Q. 25 } \int \frac{\cos x - \cos 2x}{1 - \cos x} dx$$

Thinking Process

Apply the formula, $\cos C - \cos D = 2 \sin \frac{C+D}{2} \cdot \sin \frac{D-C}{2}$ and $\cos x = 1 - 2 \sin^2 \frac{x}{2}$ to solve it.

Sol. Let

$$\begin{aligned} I &= \int \frac{\cos x - \cos 2x}{1 - \cos x} dx = \int \frac{2 \sin \frac{3x}{2} \cdot \sin \frac{x}{2}}{1 - 1 + 2 \sin^2 \frac{x}{2}} dx \\ &= 2 \int \frac{\sin \frac{3x}{2} \cdot \sin \frac{x}{2}}{2 \sin^2 \frac{x}{2}} dx = \int \frac{\sin \frac{3x}{2}}{\sin \frac{x}{2}} dx \end{aligned}$$

$$\begin{aligned}
 &= \int \frac{3\sin\frac{x}{2} - 4\sin^3\frac{x}{2}}{\sin\frac{x}{2}} dx && [\because \sin 3x = 3\sin x - 4\sin^3 x] \\
 &= 3\int dx - 4\int \sin^2\frac{x}{2} dx = 3\int dx - 4\int \frac{1 - \cos x}{2} dx \\
 &= 3\int dx - 2\int dx + 2\int \cos x dx \\
 &= \int dx + 2\int \cos x dx = x + 2\sin x + C = 2\sin x + x + C
 \end{aligned}$$

Q. 26 $\int \frac{dx}{x\sqrt{x^4 - 1}}$

Sol. Let $I = \int \frac{dx}{x\sqrt{x^4 - 1}}$
 Put $x^2 = \sec \theta \Rightarrow \theta = \sec^{-1} x^2$
 $\Rightarrow 2x dx = \sec \theta \cdot \tan \theta d\theta$
 $\therefore I = \frac{1}{2} \int \frac{\sec \theta \cdot \tan \theta}{\sec \theta \tan \theta} d\theta = \frac{1}{2} \int d\theta = \frac{1}{2} \theta + C$
 $= \frac{1}{2} \sec^{-1}(x^2) + C$

Q. 27 $\int_0^2 (x^2 + 3)dx$

Thinking Process

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} h [f(a) + f(a+h) + \dots + f\{a + (n-1)h\}], \text{ where } h = \frac{b-a}{n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Sol. Let $I = \int_0^2 (x^2 + 3) dx$
 Here, $a = 0, b = 2$ and $h = \frac{b-a}{n} = \frac{2-0}{n}$
 $\Rightarrow h = \frac{2}{n} \Rightarrow nh = 2 \Rightarrow f(x) = (x^2 + 3)$
 Now, $\int_0^2 (x^2 + 3) dx = \lim_{h \rightarrow 0} h [f(0) + f(0+h) + f(0+2h) + \dots + f\{0 + (n-1)h\}] \dots(i)$
 $\therefore f(0) = 3$
 $\Rightarrow f(0+h) = h^2 + 3, f(0+2h) = 4h^2 + 3 = 2^2h^2 + 3$
 $f[0 + (n-1)h] = (n^2 - 2n + 1)h + 3 = (n-1)^2h + 3$

From Eq. (i),

$$\begin{aligned}
 \int_0^2 (x^2 + 3) dx &= \lim_{h \rightarrow 0} h [3 + h^2 + 3 + 2^2h^2 + 3 + 3^2h^2 + 3 + \dots + (n-1)^2h^2 + 3] \\
 &= \lim_{h \rightarrow 0} h [3n + h^2 \{1^2 + 2^2 + \dots + (n-1)^2\}] \\
 &= \lim_{h \rightarrow 0} h \left[3n + h^2 \left(\frac{(n-1)(2n-2+1)(n-1+1)}{6} \right) \right] \left[\because \Sigma n^2 = \frac{n(n+1)(2n+1)}{6} \right] \\
 &= \lim_{h \rightarrow 0} h \left[3n + h^2 \left(\frac{(n^2 - n)(2n - 1)}{6} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} h \left[3n + \frac{h^2}{6} (2n^3 - n^2 - 2n^2 + n) \right] \\
&= \lim_{h \rightarrow 0} \left[3nh + \frac{2n^3 h^3 - 3n^2 h^2 \cdot h + nh \cdot h^2}{6} \right] \\
&= \lim_{h \rightarrow 0} \left[3 \cdot 2 + \frac{2 \cdot 8 - 3 \cdot 2^2 \cdot h + 2 \cdot h^2}{6} \right] = \lim_{h \rightarrow 0} \left[6 + \frac{16 - 12h + 2h^2}{6} \right] \\
&= 6 + \frac{16}{6} = 6 + \frac{8}{3} = \frac{26}{3}
\end{aligned}$$

Q. 28 $\int_0^2 e^x dx$

Sol. Let

$$I = \int_0^2 e^x dx$$

Here,

$$a = 0 \text{ and } b = 2$$

\therefore

$$h = \frac{b - a}{n}$$

\Rightarrow

$$nh = 2 \text{ and } f(x) = e^x$$

Now,

$$\int_0^2 e^x dx = \lim_{h \rightarrow 0} h [f(0) + f(0 + h) + f(0 + 2h) + \dots + f\{0 + (n - 1)h\}]$$

\therefore

$$I = \lim_{h \rightarrow 0} h [1 + e^h + e^{2h} + \dots + e^{(n-1)h}]$$

$$= \lim_{h \rightarrow 0} h \left[\frac{1 \cdot (e^h)^n - 1}{e^h - 1} \right] = \lim_{h \rightarrow 0} h \left(\frac{e^{nh} - 1}{e^h - 1} \right)$$

$$= \lim_{h \rightarrow 0} h \left(\frac{e^2 - 1}{e^h - 1} \right)$$

$$= e^2 \lim_{h \rightarrow 0} \frac{h}{e^h - 1} - \lim_{h \rightarrow 0} \frac{h}{e^h - 1}$$

$$\left[\because \lim_{h \rightarrow 0} \frac{h}{e^h - 1} = 1 \right]$$

$$= e^2 - 1 = e^2 - 1$$

Evaluate the following questions.

Q. 29 $\int_0^1 \frac{dx}{e^x + e^{-x}}$

Sol. Let

$$I = \int_0^1 \frac{dx}{e^x + e^{-x}} = \int_0^1 \frac{e^x}{1 + e^{2x}} dx$$

Put

$$e^x = t$$

\Rightarrow

$$e^x dx = dt$$

\therefore

$$I = \int_1^e \frac{dt}{1 + t^2} = [\tan^{-1} t]_1^e$$

$$= \tan^{-1} e - \tan^{-1} 1$$

$$= \tan^{-1} e - \frac{\pi}{4}$$

Q. 30 $\int_0^{\pi/2} \frac{\tan x}{1 + m^2 \tan^2 x} dx$

Sol. Let

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\tan x dx}{1 + m^2 \tan^2 x} \\ &= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{1 + m^2 \cdot \frac{\sin^2 x}{\cos^2 x}} dx \\ &= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{\frac{\cos^2 x + m^2 \sin^2 x}{\cos^2 x}} dx \\ &= \int_0^{\pi/2} \frac{\sin x \cos x dx}{1 - \sin^2 x + m^2 \sin^2 x} \\ &= \int_0^{\pi/2} \frac{\sin x \cos x}{1 - \sin^2 x(1 - m^2)} dx \end{aligned}$$

Put $\sin^2 x = t$

$\Rightarrow 2 \sin x \cos x dx = dt$

\therefore

$$\begin{aligned} I &= \frac{1}{2} \int_0^1 \frac{dt}{1 - t(1 - m^2)} \\ &= \frac{1}{2} \left[-\log|1 - t(1 - m^2)| \cdot \frac{1}{1 - m^2} \right]_0^1 \\ &= \frac{1}{2} \left[-\log|1 - 1 + m^2| \cdot \frac{1}{1 + m^2} + \log|1| \cdot \frac{1}{1 - m^2} \right] \\ &= \frac{1}{2} \left[-\log|m^2| \cdot \frac{1}{1 - m^2} \right] = \frac{2}{2} \cdot \frac{\log m}{(m^2 - 1)} \\ &= \log \frac{m}{m^2 - 1} \end{aligned}$$

Q. 31 $\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$

Thinking Process

First of all convert the given function into $\frac{1}{\sqrt{a^2 - x^2}}$ form, then apply the formula i.e.,

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C.$$

Sol. Let

$$\begin{aligned} I &= \int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}} = \int_1^2 \frac{dx}{\sqrt{2x - x^2 - 2 + x}} \\ &= \int_1^2 \frac{dx}{\sqrt{-(x^2 - 3x + 2)}} \end{aligned}$$

$$\begin{aligned}
&= \int_1^2 \frac{dx}{\sqrt{-\left[x^2 - 2 \cdot \frac{3}{2}x + \left(\frac{3}{2}\right)^2 + 2 - \frac{9}{4}\right]}} \\
&= \int_1^2 \frac{dx}{\sqrt{-\left\{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right\}}} \\
&= \int_1^2 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}} = \left[\sin^{-1} \left(\frac{x - \frac{3}{2}}{\frac{1}{2}} \right) \right]_1^2 \\
&= [\sin^{-1}(2x - 3)]_1^2 = \sin^{-1} 1 - \sin^{-1}(-1) \\
&= \frac{\pi}{2} + \frac{\pi}{2} \quad \left[\because \sin \frac{\pi}{2} = 1 \text{ and } \sin(-\theta) = -\sin \theta \right] \\
&= \pi
\end{aligned}$$

Q. 32 $\int_0^1 \frac{x}{\sqrt{1+x^2}} dx$

Sol. Let

$$I = \int_0^1 \frac{x}{\sqrt{1+x^2}} dx$$

Put

$$1 + x^2 = t^2$$

\Rightarrow

$$2x dx = 2t dt$$

\Rightarrow

$$x dx = t dt$$

\therefore

$$I = \int_1^{\sqrt{2}} \frac{t dt}{t}$$

$$= [t]_1^{\sqrt{2}} = \sqrt{2} - 1$$

Q. 33 $\int_0^\pi x \sin x \cos^2 x dx$

Thinking Process

Here, use the property i.e., $\int_0^a f(x) dx = \int_0^a (a-x) dx$ and

$$\sin(\pi - x) = \sin x, \cos(\pi - x) = \cos x.$$

Sol. Let

$$I = \int_0^\pi x \sin x \cos^2 x dx \quad \dots(i)$$

and

$$I = \int_0^\pi (\pi - x) \sin(\pi - x) \cos^2(\pi - x) dx$$

\Rightarrow

$$I = \int_0^\pi (\pi - x) \sin x \cos^2 x dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^\pi \pi \sin x \cos^2 x dx$$

Put

$$\cos x = t$$

\Rightarrow

$$-\sin x dx = dt$$

As $x \rightarrow 0$, then $t \rightarrow 1$
 and $x \rightarrow \pi$, then $t \rightarrow -1$

$$\begin{aligned} \therefore I &= -\pi \int_1^{-1} t^2 dt \Rightarrow I = -\pi \left[\frac{t^3}{3} \right]_1^{-1} \\ \Rightarrow 2I &= -\frac{\pi}{3}[-1-1] \Rightarrow 2I = \frac{2\pi}{3} \\ \therefore I &= \frac{\pi}{3} \end{aligned}$$

Q. 34 $\int_0^{1/2} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Sol. Let $I = \int_0^{1/2} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Put $x = \sin \theta$
 $\Rightarrow dx = \cos \theta d\theta$

As $x \rightarrow 0$, then $\theta \rightarrow 0$
 and $x \rightarrow \frac{1}{2}$, then $\theta \rightarrow \frac{\pi}{6}$

$$\begin{aligned} \therefore I &= \int_0^{\pi/6} \frac{\cos \theta}{(1+\sin^2 \theta)\cos \theta} d\theta = \int_0^{\pi/6} \frac{1}{1+\sin^2 \theta} d\theta \\ &= \int_0^{\pi/6} \frac{1}{\cos^2 \theta (\sec^2 \theta + \tan^2 \theta)} d\theta \\ &= \int_0^{\pi/6} \frac{\sec^2 \theta}{\sec^2 \theta + \tan^2 \theta} d\theta \\ &= \int_0^{\pi/6} \frac{\sec^2 \theta}{1 + \tan^2 \theta + \tan^2 \theta} d\theta \\ &= \int_0^{\pi/6} \frac{\sec^2 \theta}{1 + 2\tan^2 \theta} d\theta \end{aligned}$$

Again, put $\tan \theta = t$
 $\Rightarrow \sec^2 \theta d\theta = dt$

As $\theta \rightarrow 0$, then $t \rightarrow 0$
 and $\theta \rightarrow \frac{\pi}{6}$, then $t \rightarrow \frac{1}{\sqrt{3}}$

$$\begin{aligned} \therefore I &= \int_0^{1/\sqrt{3}} \frac{dt}{1+2t^2} = \frac{1}{2} \int_0^{1/\sqrt{3}} \frac{dt}{\left(\frac{1}{\sqrt{2}}\right)^2 + t^2} \\ &= \frac{1}{2} \cdot \frac{1}{1/\sqrt{2}} \left[\tan^{-1} \frac{t}{\frac{1}{\sqrt{2}}} \right]_0^{1/\sqrt{3}} = \frac{1}{\sqrt{2}} [\tan^{-1}(\sqrt{2}t)]_0^{1/\sqrt{3}} \\ &= \frac{1}{\sqrt{2}} \left[\tan^{-1} \frac{\sqrt{2}}{\sqrt{3}} - 0 \right] = \frac{1}{\sqrt{2}} \tan^{-1} \left(\sqrt{\frac{2}{3}} \right) \end{aligned}$$

Long Answer Type Questions

Q. 35 $\int \frac{x^2}{x^4 - x^2 - 12} dx$

Thinking Process

Use $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$, where $a \neq b$, then compare the coefficient of x to get the value of A and B .

Sol. Let

$$\begin{aligned} I &= \int \frac{x^2}{x^4 - x^2 - 12} dx \\ &= \int \frac{x^2}{x^4 - 4x^2 + 3x^2 - 12} dx \\ &= \int \frac{x^2 dx}{x^2(x^2 - 4) + 3(x^2 - 4)} \\ &= \int \frac{x^2 dx}{(x^2 - 4)(x^2 + 3)} \end{aligned}$$

Now,

$$\frac{x^2}{(x^2 - 4)(x^2 + 3)} \quad [\text{let } x^2 = t]$$

$$\Rightarrow \frac{t}{(t - 4)(t + 3)} = \frac{A}{t - 4} + \frac{B}{t + 3}$$

$$\Rightarrow t = A(t + 3) + B(t - 4)$$

On comparing the coefficient of t on both sides, we get

$$\Rightarrow A + B = 1 \quad \dots(i)$$

$$\Rightarrow 3A - 4B = 0 \quad \dots(ii)$$

$$\Rightarrow 3(1 - B) - 4B = 0$$

$$\Rightarrow 3 - 3B - 4B = 0$$

$$\Rightarrow 7B = 3$$

$$\Rightarrow B = \frac{3}{7}$$

If $B = \frac{3}{7}$, then $A + \frac{3}{7} = 1$

$$\Rightarrow A = 1 - \frac{3}{7} = \frac{4}{7}$$

$$\frac{x^2}{(x^2 - 4)(x^2 + 3)} = \frac{4}{7(x^2 - 4)} + \frac{3}{7(x^2 + 3)}$$

$$\begin{aligned} \therefore I &= \frac{4}{7} \int \frac{1}{x^2 - (2)^2} dx + \frac{3}{7} \int \frac{1}{x^2 + (\sqrt{3})^2} dx \\ &= \frac{4}{7} \cdot \frac{1}{2 \cdot 2} \log \left| \frac{x-2}{x+2} \right| + \frac{3}{7} \cdot \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + C \\ &= \frac{1}{7} \log \left| \frac{x-2}{x+2} \right| + \frac{\sqrt{3}}{7} \tan^{-1} \frac{x}{\sqrt{3}} + C \end{aligned}$$

Q. 36 $\int \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx$

Sol. Let

$$I = \int \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx$$

Now,

$$\begin{aligned} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} &= \frac{t}{(t + a^2)(t + b^2)} = \frac{A}{t + a^2} + \frac{B}{t + b^2} \\ t &= A(t + b^2) + B(t + a^2) \end{aligned}$$

[let $x^2 = t$]

On comparing the coefficient of t , we get

$$A + B = 1 \quad \dots(i)$$

$$b^2 A + a^2 B = 0 \quad \dots(ii)$$

$$\Rightarrow b^2(1 - B) + a^2 B = 0$$

$$\Rightarrow b^2 - b^2 B + a^2 B = 0$$

$$\Rightarrow b^2 + (a^2 - b^2) B = 0$$

$$\Rightarrow B = \frac{-b^2}{a^2 - b^2} = \frac{b^2}{b^2 - a^2}$$

From Eq. (i), $A + \frac{b^2}{b^2 - a^2} = 1$

$$\Rightarrow A = \frac{b^2 - a^2 - b^2}{b^2 - a^2} = \frac{-a^2}{b^2 - a^2}$$

$$\begin{aligned} \therefore I &= \int \frac{-a^2}{(b^2 - a^2)(x^2 + a^2)} dx + \int \frac{b^2}{b^2 - a^2} \cdot \frac{1}{x^2 + b^2} dx \\ &= \frac{-a^2}{(b^2 - a^2)} \int \frac{1}{x^2 + a^2} dx + \frac{b^2}{b^2 - a^2} \int \frac{1}{x^2 + b^2} dx \\ &= \frac{-a^2}{b^2 - a^2} \cdot \frac{1}{a} \tan^{-1} \frac{x}{a} + \frac{b^2}{b^2 - a^2} \cdot \frac{1}{b} \tan^{-1} \frac{x}{b} \\ &= \frac{1}{b^2 - a^2} \left[-a \tan^{-1} \frac{x}{a} + b \tan^{-1} \frac{x}{b} \right] \\ &= \frac{1}{a^2 - b^2} \left[a \tan^{-1} \frac{x}{a} - b \tan^{-1} \frac{x}{b} \right] \end{aligned}$$

Q. 37 $\int_0^\pi \frac{x}{1 + \sin x}$

Sol. Let

$$I = \int_0^\pi \frac{x}{1 + \sin x} dx \quad \dots(i)$$

and

$$I = \int_0^\pi \frac{\pi - x}{1 + \sin(\pi - x)} dx = \int_0^\pi \frac{\pi - x}{1 + \sin x} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$\begin{aligned} 2I &= \pi \int_0^\pi \frac{1}{1 + \sin x} dx \\ &= \pi \int_0^\pi \frac{(1 - \sin x) dx}{(1 + \sin x)(1 - \sin x)} \end{aligned}$$

$$\begin{aligned}
&= \pi \int_0^\pi \frac{(1 - \sin x) dx}{\cos^2 x} \\
&= \pi \int_0^\pi (\sec^2 x - \tan x \cdot \sec x) dx \\
&= \pi \int_0^\pi \sec^2 x dx - \pi \int_0^\pi \sec x \cdot \tan x dx \\
&= \pi [\tan x]_0^\pi - \pi [\sec x]_0^\pi \\
&= \pi [\tan x - \sec x]_0^\pi \\
&= \pi [\tan \pi - \sec \pi - \tan 0 - \sec 0] \\
\Rightarrow \quad 2I &= \pi [0 + 1 - 0 + 1] \\
2I &= 2\pi \\
\therefore I &= \pi
\end{aligned}$$

Q. 38 $\int \frac{2x - 1}{(x - 1)(x + 2)(x - 3)} dx$

Thinking Process

Apply $\frac{px + q}{(x - a)(x - b)(x - c)} = \frac{A}{(x - a)} + \frac{B}{(x - b)} + \frac{C}{(x - c)}$, then get the values of A, B and

C and use $\int \frac{1}{x} dx = \log |x| + C$.

Sol. Let

$$I = \int \frac{(2x - 1)}{(x - 1)(x + 2)(x - 3)} dx$$

Now, $\frac{2x - 1}{(x - 1)(x + 2)(x - 3)} = \frac{A}{(x - 1)} + \frac{B}{(x + 2)} + \frac{C}{(x - 3)}$

$\Rightarrow 2x - 1 = A(x + 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x + 2)$

Put $x = 3$, then

$$6 - 1 = C(3 - 1)(3 + 2)$$

$\Rightarrow 5 = 10C \Rightarrow C = \frac{1}{2}$

Again, put $x = 1$, then

$$2 - 1 = A(1 + 2)(1 - 3)$$

$\Rightarrow 1 = -6A \Rightarrow A = -\frac{1}{6}$

Now, put $x = -2$, then

$$-4 - 1 = B(-2 - 1)(-2 - 3)$$

$\Rightarrow -5 = 15B \Rightarrow B = -\frac{1}{3}$

$\therefore I = -\frac{1}{6} \int \frac{1}{x - 1} dx - \frac{1}{3} \int \frac{1}{x + 2} dx + \frac{1}{2} \int \frac{1}{x - 3} dx$

$$= -\frac{1}{6} \log |x - 1| - \frac{1}{3} \log |x + 2| + \frac{1}{2} \log |x - 3| + C$$

$$= -\log |(x - 1)|^{1/6} - \log |(x + 2)|^{1/3} + \log |(x - 3)|^{1/2} + C$$

$$= \log \left| \frac{\sqrt{x - 3}}{(x - 1)^{1/6} (x + 2)^{1/3}} \right| + C$$

Q. 39 $\int e^{\tan^{-1} x} \left(\frac{1+x+x^2}{1+x^2} \right) dx$

Sol. Let $I = \int e^{\tan^{-1} x} \left(\frac{1+x+x^2}{1+x^2} \right) dx$
 $= \int e^{\tan^{-1} x} \left(\frac{1+x^2}{1+x^2} + \frac{x}{1+x^2} \right) dx$
 $= \int e^{\tan^{-1} x} dx + \int \frac{x e^{\tan^{-1} x}}{1+x^2} dx$

$I = I_1 + I_2 \quad \dots(i)$

Now, $I_2 = \int \frac{x e^{\tan^{-1} x}}{1+x^2} dx$

Put $\tan^{-1} x = t \Rightarrow x = \tan t$

$\Rightarrow \frac{1}{1+x^2} dx = dt$

$\therefore I = \int \underset{\text{I}}{\tan t} \cdot \underset{\text{II}}{e^t} dt$
 $= \tan t \cdot e^t - \int \sec^2 t \cdot e^t dt + C$
 $= \tan t \cdot e^t - \int (1 + \tan^2 t) e^t dt + C \quad [\because \sec^2 \theta = 1 + \tan^2 \theta]$

$I_2 = \tan t \cdot e^t - \int (1+x^2) \frac{e^{\tan^{-1} x}}{1+x^2} dx + C$

$I_2 = \tan t \cdot e^t - \int e^{\tan^{-1} x} dx + C$

$\therefore I = \int e^{\tan^{-1} x} dx + \tan t \cdot e^t - \int e^{\tan^{-1} x} dx + C$
 $= \tan t \cdot e^t + C$
 $= x e^{\tan^{-1} x} + C$

Q. 40 $\int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$

Thinking Process

First of all put $x = a \tan^2 \theta$ and convert the given expression into two parts, then use the

formulae for integration by part i.e., $\int \text{I} \cdot \text{II} dx = \text{I} \int \text{II} dx - \int \left(\frac{d}{dx} \text{I} \int \text{II} dx \right) dx$

Sol. Let $I = \int \sin^{-1} \sqrt{\frac{x}{a+x}} dx$

Put $x = a \tan^2 \theta$

$\Rightarrow dx = 2a \tan \theta \sec^2 \theta d\theta$

$\therefore I = \int \sin^{-1} \sqrt{\frac{a \tan^2 \theta}{a + a \tan^2 \theta}} (2a \tan \theta \cdot \sec^2 \theta) d\theta$

$= 2a \int \sin^{-1} \left(\frac{\tan \theta}{\sec \theta} \right) \tan \theta \cdot \sec^2 \theta d\theta$

$= 2a \int \sin^{-1}(\sin \theta) \tan \theta \cdot \sec^2 \theta d\theta$

$$\begin{aligned}
&= 2a \int \theta \cdot \tan \theta \sec^2 \theta d\theta \\
&= 2a \left[\theta \cdot \int \tan \theta \cdot \sec^2 \theta d\theta - \int \left(\frac{d}{d\theta} \theta \cdot \int \tan \theta \cdot \sec^2 \theta d\theta \right) d\theta \right] \\
&\qquad\qquad\qquad \left[\begin{array}{l} \text{Put} \qquad \tan \theta = t \\ \Rightarrow \sec \theta \cdot \tan \theta \cdot d\theta = dt \\ \Rightarrow \int \tan \theta \sec^2 \theta d\theta = \int t dt \end{array} \right] \\
&= 2a \left[\theta \cdot \frac{\tan^2 \theta}{2} - \int \frac{\tan^2 \theta}{2} d\theta \right] \\
&= a\theta \tan^2 \theta - a \int (\sec^2 \theta - 1) d\theta \\
&= a\theta \cdot \tan^2 \theta - a \tan \theta + a\theta + C \\
&= a \left[\frac{x}{a} \tan^{-1} \sqrt{\frac{x}{a}} + \tan^{-1} \sqrt{\frac{x}{a}} \right] + C
\end{aligned}$$

Q. 41 $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1 + \cos x}}{(1 - \cos x)^{5/2}} dx$

Sol. Let

$$\begin{aligned}
I &= \int_{\pi/3}^{\pi/2} \frac{\sqrt{1 + \cos x}}{(1 - \cos x)^{5/2}} dx \\
&= \int_{\pi/3}^{\pi/2} \frac{\sqrt{1 + \cos x}}{(1 - \cos x)^2 \sqrt{1 + \cos x}} dx \\
&= \int_{\pi/3}^{\pi/2} \frac{1}{(1 - \cos^2 x)} dx = \int_{\pi/3}^{\pi/2} \frac{1}{\sin^2 x} dx \\
&= \int_{\pi/3}^{\pi/2} \operatorname{cosec}^2 x dx = [-\cot x]_{\pi/3}^{\pi/2} \\
&= - \left[\cot \frac{\pi}{2} - \cot \frac{\pi}{3} \right] = - \left[0 - \frac{1}{\sqrt{3}} \right] = + \frac{1}{\sqrt{3}}
\end{aligned}$$

Alternate Method

Let

$$\begin{aligned}
I &= \int_{\pi/3}^{\pi/2} \frac{\sqrt{1 + \cos x}}{(1 - \cos x)^{5/2}} dx = \int_{\pi/3}^{\pi/2} \frac{\left(2\cos^2 \frac{x}{2} \right)^{1/2}}{\left(2\sin^2 \frac{x}{2} \right)^{5/2}} dx \\
&= \frac{\sqrt{2}}{4\sqrt{2}} \int_{\pi/3}^{\pi/2} \frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)} dx = \frac{1}{4} \int_{\pi/3}^{\pi/2} \frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)} dx
\end{aligned}$$

Put $\sin \frac{x}{2} = t$

$$\Rightarrow \cos \frac{x}{2} \cdot \frac{1}{2} dx = dt$$

$$\Rightarrow \cos \frac{x}{2} dx = 2dt$$

As $x \rightarrow \frac{\pi}{3}$, then $t \rightarrow \frac{1}{2}$
 and $x \rightarrow \frac{\pi}{2}$, then $t \rightarrow \frac{1}{\sqrt{2}}$

$$\begin{aligned} \therefore I &= \frac{2}{4} \int_{1/2}^{1/\sqrt{2}} \frac{dt}{t^5} = \frac{1}{2} \left[\frac{t^{-5+1}}{-5+1} \right]_{1/2}^{1/\sqrt{2}} \\ &= -\frac{1}{8} \left[\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^4} - \frac{1}{\left(\frac{1}{2}\right)^4} \right] \\ &= -\frac{1}{8} (4 - 16) = \frac{12}{8} = \frac{3}{2} \end{aligned}$$

Note If we integrate the trigonometric function in different ways [using different identities] then, we can get different answers.

Q. 42 $\int e^{-3x} \cos^3 x dx$

Sol. Let $I = \int e^{-3x} \cos^3 x dx$

$$\begin{aligned} & \quad \quad \quad \text{II} \quad \quad \quad \text{I} \\ &= \cos^3 x \int e^{-3x} dx - \int \left(\frac{d}{dx} \cos^3 x \int e^{-3x} dx \right) dx \\ &= \cos^3 x \cdot \frac{e^{-3x}}{-3} - \int (-3\cos^2 x) \sin x \cdot \frac{e^{-3x}}{-3} dx \\ &= -\frac{1}{3} \cos^3 x e^{-3x} - \int \cos^2 x \sin x e^{-3x} dx \\ &= -\frac{1}{3} \cos^3 x e^{-3x} - \int (1 - \sin^2 x) \sin x e^{-3x} dx \\ &= -\frac{1}{3} \cos^3 x e^{-3x} - \int \sin x e^{-3x} dx + \int \sin^3 x e^{-3x} dx \\ & \quad \quad \quad \text{I} \quad \quad \quad \text{II} \\ &= -\frac{1}{3} \cos^3 x e^{-3x} - \int \sin x e^{-3x} dx + \sin^3 x \cdot \frac{e^{-3x}}{-3} - \int 3\sin^2 x \cos x \cdot \frac{e^{-3x}}{-3} dx \\ &= -\frac{1}{3} \cos^3 x e^{-3x} - \int \sin x e^{-3x} dx - \frac{1}{3} \sin^3 x e^{-3x} + \int (1 - \cos^2 x) \cos x e^{-3x} dx \\ I &= -\frac{1}{3} \cos^3 x e^{-3x} - \int \sin x e^{-3x} dx - \frac{1}{3} \sin^3 x e^{-3x} + \int \cos x e^{-3x} dx - \int \cos^3 x e^{-3x} dx \\ & \quad \quad \quad \text{I} \quad \quad \quad \text{II} \\ 2I &= \frac{e^{-3x}}{3} [\cos^3 x + \sin^3 x] - \left[\sin x \cdot \frac{e^{-3x}}{-3} - \int \cos x \cdot \frac{e^{-3x}}{-3} dx \right] + \int \cos x e^{-3x} dx \\ 2I &= \frac{e^{-3x}}{-3} [\cos^3 x + \sin^3 x] + \frac{1}{3} \sin x \cdot e^{-3x} - \frac{1}{3} \int \cos x \cdot e^{-3x} dx + \int \cos x e^{-3x} dx \\ 2I &= \frac{e^{-3x}}{-3} [\cos^3 x + \sin^3 x] + \frac{1}{3} \sin x e^{-3x} + \frac{2}{3} \int \cos x e^{-3x} dx \end{aligned}$$

Now, let

$$\begin{aligned}
 I_1 &= \int \cos x e^{-3x} dx \\
 I_1 &= \cos x \cdot \frac{e^{-3x}}{-3} - \int (-\sin x) \cdot \frac{e^{-3x}}{-3} dx \\
 I_1 &= -\frac{1}{3} \cos x \cdot e^{-3x} - \frac{1}{3} \int \sin x \cdot e^{-3x} dx \\
 &= -\frac{1}{3} \cos x \cdot e^{-3x} - \frac{1}{3} \left[\sin x \cdot \frac{e^{-3x}}{-3} - \int \cos x \cdot \frac{e^{-3x}}{-3} dx \right] \\
 &= -\frac{1}{3} \cos x \cdot e^{-3x} + \frac{1}{9} \sin x \cdot e^{-3x} - \frac{1}{9} \int \cos x \cdot e^{-3x} dx \\
 I_1 + \frac{1}{9} I_1 &= -\frac{1}{3} e^{-3x} \cdot \cos x + \frac{1}{9} \sin x \cdot e^{-3x} \\
 \left(\frac{10}{9}\right) I_1 &= -\frac{1}{3} e^{-3x} \cdot \cos x + \frac{1}{9} \sin x \cdot e^{-3x} \\
 I_1 &= \frac{-3}{10} e^{-3x} \cdot \cos x + \frac{1}{10} e^{-3x} \sin x \\
 2I &= -\frac{1}{3} e^{-3x} [\sin^3 x + \cos^3 x] + \frac{1}{3} \sin x \cdot e^{-3x} - \frac{3}{10} e^{-3x} \cdot \cos x \\
 &\quad + \frac{1}{10} e^{-3x} \cdot \sin x + C \\
 \therefore I &= -\frac{1}{6} e^{-3x} [\sin^3 x + \cos^3 x] + \frac{13}{30} e^{-3x} \cdot \sin x - \frac{3}{10} e^{-3x} \cdot \cos x + C \\
 &\quad \left[\because \sin 3x = 3\sin x - 4\sin^3 x \right. \\
 &\quad \left. \text{and } \cos 3x = 4\cos^3 x - 3\cos x \right] \\
 &= \frac{e^{-3x}}{24} [\sin 3x - \cos 3x] + \frac{3e^{-3x}}{40} [\sin x - 3\cos x] + C
 \end{aligned}$$

Q. 43 $\int \sqrt{\tan x} dx$

Sol. Let

Put

\therefore

$$\begin{aligned}
 I &= \int \sqrt{\tan x} dx \\
 \tan x &= t^2 \Rightarrow \sec^2 x dx = 2t dt \\
 I &= \int t \cdot \frac{2t}{\sec^2 x} dt = 2 \int \frac{t^2}{1+t^4} dt \\
 &= \int \frac{(t^2+1) + (t^2-1)}{(1+t^4)} dt \\
 &= \int \frac{t^2+1}{1+t^4} dt + \int \frac{t^2-1}{1+t^4} dt \\
 &= \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt + \int \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt \\
 &= \int \frac{1 - \left(\frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + 2} + \int \frac{1 + \left(\frac{1}{t^2}\right) dt}{\left(t + \frac{1}{t}\right)^2 - 2} dt
 \end{aligned}$$

Put $u = t - \frac{1}{t} \Rightarrow du = \left(1 + \frac{1}{t^2}\right) dt$

and $v = t + \frac{1}{t} \Rightarrow dv = \left(1 - \frac{1}{t^2}\right) dt$

\therefore

$$I = \int \frac{du}{u^2 + (\sqrt{2})^2} + \int \frac{dv}{v^2 - (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \log \left| \frac{v - \sqrt{2}}{v + \sqrt{2}} \right| + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2} \tan x} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{\tan x - \sqrt{2} \tan x + 1}{\tan x + \sqrt{2} \tan x + 1} \right| + C$$

Q. 44 $\int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$

Sol. Let $I = \int_0^{\pi/2} \frac{dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2}$

Divide numerator and denominator by $\cos^4 x$, we get

$$I = \int_0^{\pi/2} \frac{\sec^4 x dx}{(a^2 + b^2 \tan^2 x)^2}$$

$$= \int_0^{\pi/2} \frac{(1 + \tan^2 x) \sec^2 x dx}{(a^2 + b^2 \tan^2 x)^2}$$

Put $\tan x = t$
 $\Rightarrow \sec^2 x dx = dt$

As $x \rightarrow 0$, then $t \rightarrow 0$

and $x \rightarrow \frac{\pi}{2}$, then $t \rightarrow \infty$ $I = \int_0^{\infty} \frac{(1+t^2)}{(a^2 + b^2 t^2)^2}$

Now, $\frac{1+t^2}{(a^2 + b^2 t^2)^2}$ [let $t^2 = u$]

$$\frac{1+u}{(a^2 + b^2 u)^2} = \frac{A}{(a^2 + b^2 u)} + \frac{B}{(a^2 + b^2 u)^2}$$

$\Rightarrow 1+u = A(a^2 + b^2 u) + B$

On comparing the coefficient of x and constant term on both sides, we get

$$a^2 A + B = 1 \quad \dots (i)$$

and $b^2 A = 1 \quad \dots (ii)$

$\therefore A = \frac{1}{b^2}$

Now, $\frac{a^2}{b^2} + B = 1$

$\Rightarrow B = 1 - \frac{a^2}{b^2} = \frac{b^2 - a^2}{b^2}$

\therefore

$$I = \int_0^{\infty} \frac{(1+t^2)}{(a^2 + b^2 t^2)^2}$$

$$= \frac{1}{b^2} \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} + \frac{b^2 - a^2}{b^2} \int_0^{\infty} \frac{dt}{(a^2 + b^2 t^2)^2}$$

$$\begin{aligned}
&= \frac{1}{b^2} \int_0^\infty \frac{dt}{b^2 \left(\frac{a^2}{b^2} + t^2 \right)} + \frac{b^2 - a^2}{b^2} \int_0^\infty \frac{dt}{(a^2 + b^2 t^2)^2} \\
&= \frac{1}{ab^3} \left[\tan^{-1} \left(\frac{tb}{a} \right) \right]_0^\infty + \frac{b^2 - a^2}{b^2} \left(\frac{\pi}{4} \cdot \frac{1}{a^3 b} \right) \\
&= \frac{1}{ab^3} [\tan^{-1} \infty - \tan^{-1} 0] + \frac{\pi}{4} \cdot \frac{b^2 - a^2}{(a^3 b^3)} \\
&= \frac{\pi}{2ab^3} + \frac{\pi}{4} \cdot \frac{b^2 - a^2}{(a^3 b^3)} \\
&= \pi \left(\frac{2a^2 + b^2 - a^2}{4a^3 b^3} \right) = \frac{\pi}{4} \left(\frac{a^2 + b^2}{a^3 b^3} \right)
\end{aligned}$$

Q. 45 $\int_0^1 x \log(1 + 2x) dx$

Thinking Process

Use formula for integration by part i.e., $\int I \cdot II dx = I \int II dx - \int \left(\frac{d}{dx} I \int II dx \right) dx$ and also

use $\int \frac{1}{x} = \log|x| + C$.

Sol. Let

$$\begin{aligned}
I &= \int_0^1 x \log(1 + 2x) dx \\
&= \left[\log(1 + 2x) \frac{x^2}{2} \right]_0^1 - \int_0^1 \frac{1}{1 + 2x} \cdot 2 \cdot \frac{x^2}{2} dx \\
&= \frac{1}{2} [x^2 \log(1 + 2x)]_0^1 - \int_0^1 \frac{x^2}{1 + 2x} dx \\
&= \frac{1}{2} [1 \log 3 - 0] - \left[\int_0^1 \left(\frac{x}{2} - \frac{\frac{x}{2}}{1 + 2x} \right) dx \right] \\
&= \frac{1}{2} \log 3 - \frac{1}{2} \int_0^1 x dx + \frac{1}{2} \int_0^1 \frac{x}{1 + 2x} dx \\
&= \frac{1}{2} \log 3 - \frac{1}{2} \left[\frac{x^2}{2} \right]_0^1 + \frac{1}{2} \int_0^1 \frac{\frac{1}{2}(2x + 1 - 1)}{(2x + 1)} dx \\
&= \frac{1}{2} \log 3 - \frac{1}{2} \left[\frac{1}{2} - 0 \right] + \frac{1}{4} \int_0^1 dx - \frac{1}{4} \int_0^1 \frac{1}{1 + 2x} dx \\
&= \frac{1}{2} \log 3 - \frac{1}{4} + \frac{1}{4} [x]_0^1 - \frac{1}{8} [\log |(1 + 2x)|]_0^1 \\
&= \frac{1}{2} \log 3 - \frac{1}{4} + \frac{1}{4} - \frac{1}{8} [\log 3 - \log 1] \\
&= \frac{1}{2} \log 3 - \frac{1}{8} \log 3 \\
&= \frac{3}{8} \log 3
\end{aligned}$$

Q. 46 $\int_0^\pi x \log \sin x \, dx$

Thinking Process

First of all use property of definite integral i.e., $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$, then use $\int_0^{2a} f(x) \, dx = 2 \int_0^a f(x) \, dx$.

Sol. Let

$$I = \int_0^\pi x \log \sin x \, dx \quad \dots(i)$$

$$\begin{aligned} I &= \int_0^\pi (\pi - x) \log \sin(\pi - x) \, dx \\ &= \int_0^\pi (\pi - x) \log \sin x \, dx \quad \dots(ii) \end{aligned}$$

$$2I = \pi \int_0^\pi \log \sin x \, dx \quad \dots(iii)$$

$$2I = 2\pi \int_0^{\pi/2} \log \sin x \, dx \quad \left[\because \int_0^{2a} f(x) \, dx = 2 \int_0^a f(x) \, dx \right]$$

$$I = \pi \int_0^{\pi/2} \log \sin x \, dx \quad \dots(iv)$$

Now, $I = \pi \int_0^{\pi/2} \log \sin(\pi/2 - x) \, dx \quad \dots(v)$

On adding Eqs. (iv) and (v), we get

$$2I = \pi \int_0^{\pi/2} (\log \sin x + \log \cos x) \, dx$$

$$\begin{aligned} 2I &= \pi \int_0^{\pi/2} \log \sin x \cos x \, dx \\ &= \pi \int_0^{\pi/2} \log \frac{2 \sin x \cos x}{2} \, dx \end{aligned}$$

$$2I = \pi \int_0^{\pi/2} (\log \sin 2x - \log 2) \, dx$$

$$2I = \pi \int_0^{\pi/2} \log \sin 2x \, dx - \pi \int_0^{\pi/2} \log 2 \, dx$$

Put $2x = t \Rightarrow dx = \frac{1}{2} dt$

As $x \rightarrow 0$, then $t \rightarrow 0$

and $x \rightarrow \frac{\pi}{2}$, then $t \rightarrow \pi$

$$\therefore 2I = \frac{\pi}{2} \int_0^\pi \log \sin t \, dt - \frac{\pi^2}{2} \log 2$$

$$\Rightarrow 2I = \frac{\pi}{2} \int_0^\pi \log \sin x \, dx - \frac{\pi^2}{2} \log 2$$

$$\Rightarrow 2I = I - \frac{\pi^2}{2} \log 2 \quad \text{[from Eq. (iii)]}$$

$$\therefore I = -\frac{\pi^2}{2} \log 2 = \frac{\pi^2}{2} \log \left(\frac{1}{2} \right)$$

Q. 47 $\int_{\pi/4}^{\pi/4} \log(\sin x + \cos x) dx$

Sol. Let

$$I = \int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x) dx \quad \dots(i)$$

$$\begin{aligned} I &= \int_{-\pi/4}^{\pi/4} \log \left\{ \sin \left(\frac{\pi}{4} - \frac{\pi}{4} - x \right) + \cos \left(\frac{\pi}{4} - \frac{\pi}{4} - x \right) \right\} dx \\ &= \int_{-\pi/4}^{\pi/4} \log \{ \sin(-x) + \cos(-x) \} dx \end{aligned}$$

and

$$I = \int_{-\pi/4}^{\pi/4} \log(\cos x - \sin x) dx \quad \dots(ii)$$

From Eqs. (i) and (ii),

$$2I = \int_{-\pi/4}^{\pi/4} \log \cos 2x dx$$

$$2I = \int_0^{\pi/4} \log \cos 2x dx \quad \dots(iii)$$

$$\left[\because \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx, \text{ if } f(-x) = f(x) \right]$$

Put $2x = t \Rightarrow dx = \frac{dt}{2}$

As $x \rightarrow 0$, then $t \rightarrow 0$

and $x \rightarrow \frac{\pi}{4}$, then $t \rightarrow \frac{\pi}{2}$

$$2I = \frac{1}{2} \int_0^{\pi/2} \log \cos t dt \quad \dots(iv)$$

$$2I = \frac{1}{2} \int_0^{\pi/2} \log \cos \left(\frac{\pi}{2} - t \right) dt \quad \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$\Rightarrow 2I = \frac{1}{2} \int_0^{\pi/2} \log \sin t dt \quad \dots(v)$$

On adding Eqs. (iv) and (v), we get

$$4I = \frac{1}{2} \int_0^{\pi/2} \log \sin t \cos t dt$$

$$\Rightarrow 4I = \frac{1}{2} \int_0^{\pi/2} \log \frac{\sin 2t}{2} dt$$

$$\Rightarrow 4I = \frac{1}{2} \int_0^{\pi/2} \log \sin 2x dx - \frac{1}{2} \int_0^{\pi/2} \log 2 dx$$

$$\Rightarrow 4I = \frac{1}{2} \int_0^{\pi/2} \log \sin \left(\frac{\pi}{2} - 2x \right) dx - \log 2 \cdot \frac{\pi}{4}$$

$$\Rightarrow 4I = \frac{1}{2} \int_0^{\pi/2} \log \cos 2x dx - \frac{\pi}{4} \log 2$$

$$\Rightarrow 4I = \int_0^{\pi/4} \log \cos 2x dx - \frac{\pi}{4} \log 2 \quad \left[\because \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx \right]$$

$$\Rightarrow 4I = 2I - \frac{\pi}{4} \log 2 \quad \text{[from Eq. (iii)]}$$

$$\therefore I = -\frac{\pi}{8} \log 2 = \frac{\pi}{8} \log \left(\frac{1}{2} \right)$$

Q. 50 $\int \tan^{-1} \sqrt{x} \, dx$ is equal to

(a) $(x + 1) \tan^{-1} \sqrt{x} - \sqrt{x} + C$

(b) $x \tan^{-1} \sqrt{x} - \sqrt{x} + C$

(c) $\sqrt{x} - x \tan^{-1} \sqrt{x} + C$

(d) $\sqrt{x} - (x + 1) \tan^{-1} \sqrt{x} + C$

Thinking Process

Use formula for integration by part i.e., $\int I \cdot II \, dx = I \int II \, dx - \int \left(\frac{d}{dx} I \int II \, dx \right) dx$

Sol. (a) Let

$$\begin{aligned} I &= \int 1 \cdot \tan^{-1} \sqrt{x} \, dx \\ &= \tan^{-1} \sqrt{x} \cdot x - \frac{1}{2} \int \frac{1}{(1+x)} \cdot \frac{2}{\sqrt{x}} \, dx \\ &= x \tan^{-1} \sqrt{x} - \frac{1}{2} \int \frac{2}{\sqrt{x}(1+x)} \, dx \end{aligned}$$

Put $x = t^2 \Rightarrow dx = 2t \, dt$

$$\begin{aligned} \therefore I &= x \tan^{-1} \sqrt{x} - \int \frac{t}{t(1+t^2)} \, dt \\ &= x \tan^{-1} \sqrt{x} - \int \frac{t^2}{1+t^2} \, dt \\ &= x \tan^{-1} \sqrt{x} - \int \left(1 - \frac{1}{1+t^2} \right) \, dt \\ &= x \tan^{-1} \sqrt{x} - \sqrt{x} + \tan^{-1} t + C \\ &= x \tan^{-1} \sqrt{x} - \sqrt{x} + \tan^{-1} \sqrt{x} + C \\ &= (x + 1) \tan^{-1} \sqrt{x} - \sqrt{x} + C \end{aligned}$$

Q. 51 $\int \frac{x^9}{(4x^2 + 1)^6} \, dx$ is equal to

(a) $\frac{1}{5x} \left(4 + \frac{1}{x^2} \right)^{-5} + C$

(b) $\frac{1}{5} \left(4 + \frac{1}{x^2} \right)^{-5} + C$

(c) $\frac{1}{10x} (1 + 4)^{-5} + C$

(d) $\frac{1}{10} \left(\frac{1}{x^2} + 4 \right)^{-5} + C$

Sol. (d) Let

$$\begin{aligned} I &= \int \frac{x^9}{(4x^2 + 1)^6} \, dx = \int \frac{x^9}{x^{12} \left(4 + \frac{1}{x^2} \right)^6} \, dx \\ &= \int \frac{dx}{x^3 \left(4 + \frac{1}{x^2} \right)^6} \end{aligned}$$

Put $4 + \frac{1}{x^2} = t \Rightarrow \frac{-2}{x^3} dx = dt$

$\Rightarrow \frac{1}{x^3} dx = -\frac{1}{2} dt$

$$\begin{aligned} \therefore I &= -\frac{1}{2} \int \frac{dt}{t^6} = -\frac{1}{2} \left[\frac{t^{-6+1}}{-6+1} \right] + C \\ &= \frac{1}{10} \left[\frac{1}{t^5} \right] + C = \frac{1}{10} \left(4 + \frac{1}{x^2} \right)^{-5} + C \end{aligned}$$

Q. 52 If $\int \frac{dx}{(x+2)(x^2+1)} = a \log|1+x^2| + b \tan^{-1} x + \frac{1}{5} \log|x+2| + C$, then

(a) $a = \frac{-1}{10}, b = \frac{-2}{5}$

(b) $a = \frac{1}{10}, b = -\frac{2}{5}$

(c) $a = \frac{-1}{10}, b = \frac{2}{5}$

(d) $a = \frac{1}{10}, b = \frac{2}{5}$

Thinking Process

Use method of partial fraction i.e., $\frac{1}{(x-a)(x^2+bx+c)} = \frac{A}{(x-a)} + \frac{Bx+C}{(x^2+bx+c)}$

to solve the above problem.

Sol. (c) Given that, $\int \frac{dx}{(x+2)(x^2+1)} = a \log|1+x^2| + b \tan^{-1} x + \frac{1}{5} \log|x+2| + C$

Now, $I = \int \frac{dx}{(x+2)(x^2+1)}$

$$\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$\Rightarrow 1 = A(x^2+1) + (Bx+C)(x+2)$

$\Rightarrow 1 = Ax^2 + A + Bx^2 + 2Bx + Cx + 2C$

$\Rightarrow 1 = (A+B)x^2 + (2B+C)x + A+2C$

$\Rightarrow A+B=0, A+2C=1, 2B+C=0$

We have, $A = \frac{1}{5}, B = -\frac{1}{5}$ and $C = \frac{2}{5}$

$$\begin{aligned} \therefore \int \frac{dx}{(x+2)(x^2+1)} &= \frac{1}{5} \int \frac{1}{x+2} dx + \int \frac{-\frac{1}{5}x + \frac{2}{5}}{x^2+1} dx \\ &= \frac{1}{5} \int \frac{1}{x+2} dx - \frac{1}{5} \int \frac{x}{1+x^2} dx + \frac{1}{5} \int \frac{2}{1+x^2} dx \\ &= \frac{1}{5} \log|x+2| - \frac{1}{10} \log|1+x^2| + \frac{2}{5} \tan^{-1} x + C \end{aligned}$$

$\therefore b = \frac{2}{5}$ and $a = \frac{-1}{10}$

Q. 53 $\int \frac{x^3}{x+1}$ is equal to

(a) $x + \frac{x^2}{2} + \frac{x^3}{3} - \log|1-x| + C$

(b) $x + \frac{x^2}{2} - \frac{x^3}{3} - \log|1-x| + C$

(c) $x - \frac{x^2}{2} - \frac{x^3}{3} - \log|1+x| + C$

(d) $x - \frac{x^2}{2} + \frac{x^3}{3} - \log|1+x| + C$

Sol. (d) Let $I = \int \frac{x^3}{x+1} dx$

$$= \int \left((x^2 - x + 1) - \frac{1}{(x+1)} \right) dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1| + C$$

Q. 54 $\int \frac{x + \sin x}{1 + \cos x} dx$ is equal to

(a) $\log|1 + \cos x| + C$

(b) $\log|x + \sin x| + C$

(c) $x - \tan \frac{x}{2} + C$

(d) $x \cdot \tan \frac{x}{2} + C$

Sol. (d) Let

$$\begin{aligned} I &= \int \frac{x + \sin x}{1 + \cos x} dx \\ &= \int \frac{x}{1 + \cos x} dx + \int \frac{\sin x}{1 + \cos x} dx \\ &= \int \frac{x}{2\cos^2 x/2} dx + \int \frac{2\sin x/2\cos x/2}{2\cos^2 x/2} dx \\ &= \frac{1}{2} \int x \sec^2 x/2 dx + \int \tan x/2 dx \\ &= \frac{1}{2} \left[x \cdot \tan x/2 \cdot 2 - \int \tan \frac{x}{2} \cdot 2 dx \right] + \int \tan \frac{x}{2} dx \\ &= x \cdot \tan \frac{x}{2} + C \end{aligned}$$

Q. 55 If $\frac{x^3 dx}{\sqrt{1+x^2}} = a(1+x^2)^{3/2} + b\sqrt{1+x^2} + C$, then

(a) $a = \frac{1}{3}, b = 1$

(b) $a = \frac{-1}{3}, b = 1$

(c) $a = \frac{-1}{3}, b = -1$

(d) $a = \frac{1}{3}, b = -1$

Sol. (d) Let $I = \int \frac{x^3}{\sqrt{1+x^2}} dx = a(1+x^2)^{3/2} + b\sqrt{1+x^2} + C$

$$\therefore I = \int \frac{x^3}{\sqrt{1+x^2}} dx = \int \frac{x^2 \cdot x}{\sqrt{1+x^2}} dx$$

Put $1+x^2 = t^2$

$\Rightarrow 2x dx = 2t dt$

$$\therefore I = \int \frac{t(t^2-1)}{t} dt = \frac{t^3}{3} - t + C$$

$$= \frac{1}{3}(1+x^2)^{3/2} - \sqrt{1+x^2} + C$$

$$\therefore a = \frac{1}{3} \text{ and } b = -1$$

Q. 56 $\int_{-\pi/4}^{\pi/4} \frac{dx}{1 + \cos 2x}$ is equal to

(a) 1

(b) 2

(c) 3

(d) 4

Sol. (a) Let

$$\begin{aligned} I &= \int_{-\pi/4}^{\pi/4} \frac{dx}{1 + \cos 2x} = \int_{-\pi/4}^{\pi/4} \frac{dx}{2\cos^2 x} \\ &= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \sec^2 x dx = \int_0^{\pi/4} \sec^2 x dx = [\tan x]_0^{\pi/4} = 1 \end{aligned}$$

Q. 57 $\int_0^{\pi/2} \sqrt{1 - \sin 2x} \, dx$ is equal to

(a) $2\sqrt{2}$

(b) $2(\sqrt{2} + 1)$

(c) 2

(d) $2(\sqrt{2} - 1)$

Sol. (d) Let

$$\begin{aligned} I &= \int_0^{\pi/2} \sqrt{1 - \sin 2x} \, dx \\ &= \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2} \, dx + \int_{\pi/4}^{\pi/2} \sqrt{(\sin x - \cos x)^2} \, dx \\ &= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2} \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\ &= 2\sqrt{2} - 2 = 2(\sqrt{2} - 1) \end{aligned}$$

Q. 58 $\int_0^{\pi/4} \cos x e^{\sin x} \, dx$ is equal to

(a) $e + 1$

(b) $e - 1$

(c) e

(d) $-e$

Sol. (b) Let

$$I = \int_0^{\pi/2} \cos x e^{\sin x} \, dx$$

Put

$$\sin x = t \Rightarrow \cos x \, dx = dt$$

As $x \rightarrow 0$, then $t \rightarrow 0$

and $x \rightarrow \pi/2$, then $t \rightarrow 1$

\therefore

$$\begin{aligned} I &= \int_0^1 e^t \, dt = [e^t]_0^1 \\ &= e^1 - e^0 = e - 1 \end{aligned}$$

Q. 59 $\int \frac{x+3}{(x+4)^2} e^x \, dx$ is equal to

(a) $e^x \left(\frac{1}{x+4} \right) + C$

(b) $e^{-x} \left(\frac{1}{x+4} \right) + C$

(c) $e^{-x} \left(\frac{1}{x-4} \right) + C$

(d) $e^{2x} \left(\frac{1}{x-4} \right) + C$

Sol. (a) Let

$$\begin{aligned} I &= \int \frac{x+3}{(x+4)^2} e^x \, dx \\ &= \int \frac{e^x}{(x+4)} - \int \frac{e^x}{(x+4)^2} \, dx \\ &= \int e^x \left(\frac{1}{(x+4)} - \frac{1}{(x+4)^2} \right) \, dx \\ &= e^x \left(\frac{1}{x+4} \right) + C \quad [\because \int e^x \{f(x) + f'(x)\} \, dx = e^x f(x) + C] \end{aligned}$$

Fillers

Q. 60 If $\int_0^a \frac{1}{1+4x^2} dx = \frac{\pi}{8}$, then $a = \dots\dots\dots$

Sol. Let $I = \int_0^a \frac{1}{1+4x^2} dx = \frac{\pi}{8}$

Now,
$$\int_0^a \frac{1}{4\left(\frac{1}{4} + x^2\right)} dx = \frac{2}{4} [\tan^{-1} 2x]_0^a$$

$$= \frac{1}{2} \tan^{-1} 2a - 0 = \pi/8$$

$$\frac{1}{2} \tan^{-1} 2a = \frac{\pi}{8}$$

$$\Rightarrow \tan^{-1} 2a = \pi/4$$

$$\Rightarrow 2a = 1$$

$$\therefore a = \frac{1}{2}$$

Q. 61 $\int \frac{\sin x}{3+4\cos^2 x} dx = \dots\dots\dots$

Sol. Let $I = \int \frac{\sin x}{3+4\cos^2 x} dx$

Put $\cos x = t \Rightarrow -\sin x dx = dt$

$$\therefore I = -\int \frac{dt}{3+4t^2} = -\frac{1}{4} \int \frac{dt}{\left(\frac{\sqrt{3}}{2}\right)^2 + t^2}$$

$$= -\frac{1}{4} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \frac{2t}{\sqrt{3}} + C$$

$$= -\frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2\cos x}{\sqrt{3}} \right) + C$$

Q. 62 The value of $\int_{-\pi}^{\pi} \sin^3 x \cos^2 x dx$ is $\dots\dots\dots$

Sol. We have, $f(x) = \int_{-\pi}^{\pi} \sin^3 x \cos^2 x dx$

$$f(-x) = \int_{-\pi}^{\pi} \sin^3(-x) \cos^2(-x) dx$$

$$= -f(x)$$

Since, $f(x)$ is an odd function.

$$\therefore \int_{-\pi}^{\pi} \sin^3 x \cos^2 x dx = 0$$