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Vector Algebra

Short Answer Type Questions

Q. 1 Find the unit vector in the direction of sum of vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{j} + \hat{k}$.

💡 **Thinking Process**

We know that, unit vector in the direction of a vector \vec{a} is $\frac{\vec{a}}{|\vec{a}|}$. So, first we will find the sum of vectors and then we will use this concept.

Sol. Let \vec{c} denote the sum of \vec{a} and \vec{b} .

We have,

$$\begin{aligned}\vec{c} &= \vec{a} + \vec{b} \\ &= 2\hat{i} - \hat{j} + \hat{k} + 2\hat{j} + \hat{k} = 2\hat{i} + \hat{j} + 2\hat{k}\end{aligned}$$

$$\therefore \text{Unit vector in the direction of } \vec{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{9}}$$

$$\hat{c} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

Q. 2 If $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$, then find the unit vector in the direction of

(i) $6\vec{b}$

(ii) $2\vec{a} - \vec{b}$

Sol. Here, $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$

(i) Since,

$$6\vec{b} = 12\hat{i} + 6\hat{j} + 12\hat{k}$$

$$\begin{aligned}\therefore \text{Unit vector in the direction of } 6\vec{b} &= \frac{6\vec{b}}{|6\vec{b}|} \\ &= \frac{12\hat{i} + 6\hat{j} + 12\hat{k}}{\sqrt{12^2 + 6^2 + 12^2}} = \frac{6(2\hat{i} + \hat{j} + 2\hat{k})}{\sqrt{324}} \\ &= \frac{6(2\hat{i} + \hat{j} + 2\hat{k})}{18} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}\end{aligned}$$

(ii) Since, $2\vec{a} - \vec{b} = 2(\hat{i} + \hat{j} + 2\hat{k}) - (2\hat{i} + \hat{j} - 2\hat{k})$
 $= 2\hat{i} + 2\hat{j} + 4\hat{k} - 2\hat{i} - \hat{j} + 2\hat{k} = \hat{j} + 6\hat{k}$

\therefore Unit vector in the direction of $2\vec{a} - \vec{b} = \frac{2\vec{a} - \vec{b}}{|2\vec{a} - \vec{b}|} = \frac{\hat{j} + 6\hat{k}}{\sqrt{1+36}} = \frac{1}{\sqrt{37}}(\hat{j} + 6\hat{k})$

Q. 3 Find a unit vector in the direction of \vec{PQ} , where P and Q have coordinates $(5, 0, 8)$ and $(3, 3, 2)$, respectively.

Sol. Since, the coordinates of P and Q are $(5, 0, 8)$ and $(3, 3, 2)$, respectively.

\therefore

$$\begin{aligned}\vec{PQ} &= \vec{OQ} - \vec{OP} \\ &= (3\hat{i} + 3\hat{j} + 2\hat{k}) - (5\hat{i} + 0\hat{j} + 8\hat{k}) \\ &= -2\hat{i} + 3\hat{j} - 6\hat{k}\end{aligned}$$

\therefore Unit vector in the direction of $\vec{PQ} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{2^2 + 3^2 + 6^2}}$
 $= \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{49}} = \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{7}$

Q. 4 If \vec{a} and \vec{b} are the position vectors of \vec{A} and \vec{B} respectively, then find the position vector of a point \vec{C} in \vec{BA} produced such that $\vec{BC} = 1.5\vec{BA}$.

Sol. Since, $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$

\therefore $\vec{BA} = \vec{OA} - \vec{OB} = \vec{a} - \vec{b}$

and $1.5\vec{BA} = 1.5(\vec{a} - \vec{b})$

Since, $\vec{BC} = 1.5\vec{BA} = 1.5(\vec{a} - \vec{b})$

$\vec{OC} - \vec{OB} = 1.5\vec{a} - 1.5\vec{b}$

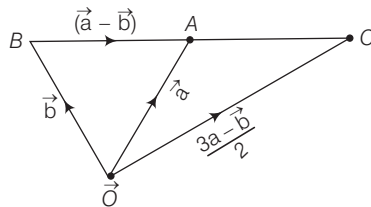
$\vec{OC} = 1.5\vec{a} - 1.5\vec{b} + \vec{b}$

$= 1.5\vec{a} - 0.5\vec{b}$

$= \frac{3\vec{a} - \vec{b}}{2}$

$[\because \vec{OB} = \vec{b}]$

Graphically, explanation of the above solution is given below

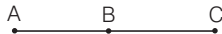


Q. 5 Using vectors, find the value of k , such that the points $(k, -10, 3)$, $(1, -1, 3)$ and $(3, 5, 3)$ are collinear.

Thinking Process

Here, use the following stepwise approach first, get the values of $|\vec{AB}|$, $|\vec{BC}|$ and $|\vec{AC}|$

and then use the concept that three points are collinear, if $|\vec{AB}| + |\vec{BC}| = |\vec{AC}|$ such that.



Sol. Let the points are $A(k, -10, 3)$, $B(1, -1, 3)$ and $C(3, 5, 3)$.

So,

$$\begin{aligned}\vec{AB} &= \vec{OB} - \vec{OA} \\ &= (\hat{i} - \hat{j} + 3\hat{k}) - (k\hat{i} - 10\hat{j} + 3\hat{k}) \\ &= (1-k)\hat{i} + (-1+10)\hat{j} + (3-3)\hat{k} \\ &= (1-k)\hat{i} + 9\hat{j} + 0\hat{k}\end{aligned}$$

$$\therefore |\vec{AB}| = \sqrt{(1-k)^2 + (9)^2 + 0} = \sqrt{(1-k)^2 + 81}$$

Similarly,

$$\begin{aligned}\vec{BC} &= \vec{OC} - \vec{OB} \\ &= (3\hat{i} + 5\hat{j} + 3\hat{k}) - (\hat{i} - \hat{j} + 3\hat{k}) \\ &= 2\hat{i} + 6\hat{j} + 0\hat{k}\end{aligned}$$

$$\therefore |\vec{BC}| = \sqrt{2^2 + 6^2 + 0} = 2\sqrt{10}$$

and

$$\begin{aligned}\vec{AC} &= \vec{OC} - \vec{OA} \\ &= (3\hat{i} + 5\hat{j} + 3\hat{k}) - (k\hat{i} - 10\hat{j} + 3\hat{k}) \\ &= (3-k)\hat{i} + 15\hat{j} + 0\hat{k}\end{aligned}$$

$$\therefore |\vec{AC}| = \sqrt{(3-k)^2 + 225}$$

If A, B and C are collinear, then sum of modulus of any two vectors will be equal to the modulus of third vectors

$$\text{For } |\vec{AB}| + |\vec{BC}| = |\vec{AC}|,$$

$$\begin{aligned}\sqrt{(1-k)^2 + 81} + 2\sqrt{10} &= \sqrt{(3-k)^2 + 225} \\ \Rightarrow \sqrt{(3-k)^2 + 225} - \sqrt{(1-k)^2 + 81} &= 2\sqrt{10} \\ \Rightarrow \sqrt{9 + k^2 - 6k + 225} - \sqrt{1 + k^2 - 2k + 81} &= 2\sqrt{10} \\ \Rightarrow \sqrt{k^2 - 6k + 234} - 2\sqrt{10} &= \sqrt{k^2 - 2k + 82} \\ \Rightarrow k^2 - 6k + 234 + 40 - 2\sqrt{k^2 - 6k + 234} \cdot 2\sqrt{10} &= k^2 - 2k + 82 \\ \Rightarrow k^2 - 6k + 234 + 40 - k^2 + 2k - 82 &= 4\sqrt{10}\sqrt{k^2 + 234 - 6k} \\ \Rightarrow -4k + 192 &= 4\sqrt{10}\sqrt{k^2 + 234 - 6k} \\ \Rightarrow -k + 48 &= \sqrt{10}\sqrt{k^2 + 234 - 6k}\end{aligned}$$

On squaring both sides, we get

$$\begin{aligned}48 \times 48 + k^2 - 96k &= 10(k^2 + 234 - 6k) \\ \Rightarrow k^2 - 96k - 10k^2 + 60k &= -48 \times 48 + 2340 \\ \Rightarrow -9k^2 - 36k &= -48 \times 48 + 2340\end{aligned}$$

$$\begin{aligned}
\Rightarrow & (k^2 + 4k) = + 16 \times 16 - 260 && \text{[dividing by 9 in both sides]} \\
\Rightarrow & k^2 + 4k = - 4 \\
& k^2 + 4k + 4 = 0 \\
\Rightarrow & (k + 2)^2 = 0 \\
\therefore & k = -2
\end{aligned}$$

Q. 6 A vector \vec{r} is inclined at equal angles to the three axes. If the magnitude of \vec{r} is $2\sqrt{3}$ units, then find the value of \vec{r} .

Thinking Process

If a vector \vec{r} is inclined at equal angles to the three axes, then direction cosines of vector, \vec{r} will be same and then use, $\vec{r} = \vec{r} \cdot |\vec{r}|$.

Sol. We have, $|\vec{r}| = 2\sqrt{3}$

Since, \vec{r} is equally inclined to the three axes, \vec{r} so direction cosines of the unit vector \hat{r} will be same. i.e., $l = m = n$.

We know that,

$$\begin{aligned}
& l^2 + m^2 + n^2 = 1 \\
\Rightarrow & l^2 + l^2 + l^2 = 1 \\
\Rightarrow & l^2 = \frac{1}{3}
\end{aligned}$$

$$\Rightarrow l = \pm \left(\frac{1}{\sqrt{3}} \right)$$

So, $\hat{r} = \pm \frac{1}{\sqrt{3}} \hat{i} \pm \frac{1}{\sqrt{3}} \hat{j} \pm \frac{1}{\sqrt{3}} \hat{k}$

$$\begin{aligned}
\therefore \vec{r} &= \hat{r} |\vec{r}| && \left[\because \hat{r} = \frac{\vec{r}}{|\vec{r}|} \right] \\
&= \left[\pm \frac{1}{\sqrt{3}} \hat{i} \pm \frac{1}{\sqrt{3}} \hat{j} \pm \frac{1}{\sqrt{3}} \hat{k} \right] 2\sqrt{3} && [\because |\vec{r}| = 2\sqrt{3}] \\
&= \pm 2\hat{i} \pm 2\hat{j} \pm 2\hat{k} = \pm 2(\hat{i} + \hat{j} + \hat{k})
\end{aligned}$$

Q. 7 If a vector \vec{r} has magnitude 14 and direction ratios 2, 3 and -6 . Then, find the direction cosines and components of \vec{r} , given that \vec{r} makes an acute angle with X-axis.

Sol. Here, $|\vec{r}| = 14$, $\vec{a} = 2k$, $\vec{b} = 3k$ and $\vec{c} = -6k$

\therefore Direction cosines l , m and n are

$$l = \frac{\vec{a}}{|\vec{r}|} = \frac{2k}{14} = \frac{k}{7}$$

$$m = \frac{\vec{b}}{|\vec{r}|} = \frac{3k}{14}$$

and
$$n = \frac{\vec{c}}{|\vec{r}|} = \frac{-6k}{14} = \frac{-3k}{7}$$

Also, we know that

$$\begin{aligned}
 & l^2 + m^2 + n^2 = 1 \\
 \Rightarrow & \frac{k^2}{49} + \frac{9k^2}{196} + \frac{9k^2}{49} = 1 \\
 \Rightarrow & \frac{4k^2 + 9k^2 + 36k^2}{196} = 1 \\
 \Rightarrow & k^2 = \frac{196}{49} = 4 \\
 \Rightarrow & k = \pm 2
 \end{aligned}$$

So, the direction cosines (l, m, n) are $\frac{2}{7}, \frac{3}{7}$ and $\frac{-6}{7}$.

[since, \vec{r} makes an acute angle with X-axis]

$$\begin{aligned}
 \therefore \vec{r} &= \hat{r} \cdot |\vec{r}| \\
 \therefore \vec{r} &= (l\hat{i} + m\hat{j} + n\hat{k})|\vec{r}| \\
 &= \left(\frac{+2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right) \cdot 14 \\
 &= +4\hat{i} + 6\hat{j} - 12\hat{k}
 \end{aligned}$$

Q. 8 Find a vector of magnitude 6, which is perpendicular to both the vectors $2\hat{i} - \hat{j} + 2\hat{k}$ and $4\hat{i} - \hat{j} + 3\hat{k}$.

Thinking Process

First, we will use this concept any vector perpendicular to both the vectors \vec{a} and \vec{b} is

given by $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ and then we will find the vector with magnitude 6.

Sol. Let $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 4\hat{i} - \hat{j} + 3\hat{k}$

So, any vector perpendicular to both the vectors \vec{a} and \vec{b} is given by

$$\begin{aligned}
 \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ 4 & -1 & 3 \end{vmatrix} \\
 &= \hat{i}(-3 + 2) - \hat{j}(6 - 8) + \hat{k}(-2 + 4) \\
 &= -\hat{i} + 2\hat{j} + 2\hat{k} = \vec{r}
 \end{aligned}$$

[say]

A vector of magnitude 6 in the direction of \vec{r}

$$\begin{aligned}
 &= \frac{\vec{r}}{|\vec{r}|} \cdot 6 = \frac{-\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}} \cdot 6 \\
 &= \frac{-6}{3}\hat{i} + \frac{12}{3}\hat{j} + \frac{12}{3}\hat{k} \\
 &= -2\hat{i} + 4\hat{j} + 4\hat{k}
 \end{aligned}$$

Q. 9 Find the angle between the vectors $2\hat{i} - \hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j} - \hat{k}$.

Thinking Process

If \vec{a} and \vec{b} are two vectors, making angle θ with each other, then $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$, using

this concept we will find θ

Sol. Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} + 4\hat{j} - \hat{k}$

We know that, angle between two vectors \vec{a} and \vec{b} is given by

$$\begin{aligned} \cos \theta &= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \\ &= \frac{(2\hat{i} - \hat{j} + \hat{k})(3\hat{i} + 4\hat{j} - \hat{k})}{\sqrt{4 + 1 + 1}\sqrt{9 + 16 + 1}} \\ &= \frac{6 - 4 - 1}{\sqrt{6}\sqrt{26}} = \frac{1}{2\sqrt{39}} \end{aligned}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{2\sqrt{39}}\right)$$

Q. 10 If $\vec{a} + \vec{b} + \vec{c} = 0$, then show that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$. Interpret the result geometrically.

Sol. Since, $\vec{a} + \vec{b} + \vec{c} = 0$

$$\Rightarrow \vec{b} = -\vec{c} - \vec{a}$$

$$\text{Now, } \vec{a} \times \vec{b} = \vec{a} \times (-\vec{c} - \vec{a})$$

$$= \vec{a} \times (-\vec{c}) + \vec{a} \times (-\vec{a}) = -\vec{a} \times \vec{c}$$

$$\Rightarrow \vec{a} \times \vec{b} = \vec{c} \times \vec{a} \quad \dots(i)$$

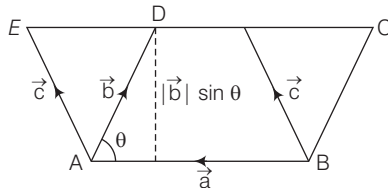
$$\text{Also, } \vec{b} \times \vec{c} = (-\vec{c} - \vec{a}) \times \vec{c}$$

$$= (-\vec{c} \times \vec{c}) + (-\vec{a} \times \vec{c}) = -\vec{a} \times \vec{c}$$

$$\Rightarrow \vec{b} \times \vec{c} = \vec{c} \times \vec{a} \quad \dots(ii)$$

$$\text{From Eqs. (i) and (ii), } \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

Geometrical interpretation of the result



If $ABCD$ is a parallelogram such that $\vec{AB} = \vec{a}$ and $\vec{AD} = \vec{b}$ and these adjacent sides are making angle θ between each other, then we say that

$$\text{Area of parallelogram } ABCD = |\vec{a}| |\vec{b}| |\sin \theta| = |\vec{a} \times \vec{b}|$$

Since, parallelogram on the same base and between the same parallels are equal in area.

We can say that,

$$|\vec{a} \times \vec{b}| = |\vec{a} \times \vec{c}| = |\vec{b} \times \vec{c}|$$

This also implies that,

$$\vec{a} \times \vec{b} = \vec{a} \times \vec{c} = \vec{b} \times \vec{c}$$

So, area of the parallelograms formed by taking any two sides represented by \vec{a} , \vec{b} and \vec{c} as adjacent are equal.

Q. 11 Find the sine of the angle between the vectors $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$.

Thinking Process

We know that, if \vec{a} and \vec{b} are in their component form, then

$$\cos \theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}}. \text{ After getting } \cos \theta \text{ we shall find the sine of the angle.}$$

Sol. Here, $a_1 = 3, a_2 = 1, a_3 = 2$ and $b_1 = 2, b_2 = -2, b_3 = 4$
We know that,

$$\begin{aligned} \cos \theta &= \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}} \\ &= \frac{3 \times 2 + 1 \times (-2) + 2 \times 4}{\sqrt{3^2 + 1^2 + 2^2} \sqrt{2^2 + (-2)^2 + 4^2}} \\ &= \frac{6 - 2 + 8}{\sqrt{14} \sqrt{24}} = \frac{12}{2\sqrt{14}\sqrt{6}} = \frac{6}{\sqrt{84}} = \frac{6}{2\sqrt{21}} = \frac{3}{\sqrt{21}} \end{aligned}$$

$$\begin{aligned} \therefore \sin \theta &= \sqrt{1 - \cos^2 \theta} \\ &= \sqrt{1 - \frac{9}{21}} = \sqrt{\frac{12}{21}} = \frac{2\sqrt{3}}{\sqrt{3}\sqrt{7}} = \frac{2}{\sqrt{7}} \end{aligned}$$

Q. 12 If A, B, C and D are the points with position vectors $\hat{i} - \hat{j} + \hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$, $2\hat{i} - 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$ respectively, then find the projection of \vec{AB} along \vec{CD} .

Thinking Process

We shall use the concept that projection of \vec{a} along \vec{b} is $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.

Sol. Here, $\vec{OA} = \hat{i} + \hat{j} - \hat{k}$, $\vec{OB} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\vec{OC} = 2\hat{i} - 3\hat{k}$ and $\vec{OD} = 3\hat{i} - 2\hat{j} + \hat{k}$

$$\begin{aligned} \therefore \vec{AB} &= \vec{OB} - \vec{OA} = (2 - 1)\hat{i} + (-1 - 1)\hat{j} + (3 + 1)\hat{k} \\ &= \hat{i} - 2\hat{j} + 4\hat{k} \end{aligned}$$

$$\begin{aligned} \text{and } \vec{CD} &= \vec{OD} - \vec{OC} = (3 - 2)\hat{i} + (-2 - 0)\hat{j} + (1 + 3)\hat{k} \\ &= \hat{i} - 2\hat{j} + 4\hat{k} \end{aligned}$$

So, the projection of \vec{AB} along $\vec{CD} = \vec{AB} \cdot \frac{\vec{CD}}{|\vec{CD}|}$

$$= \frac{(\hat{i} - 2\hat{j} + 4\hat{k}) \cdot (\hat{i} - 2\hat{j} + 4\hat{k})}{\sqrt{1^2 + 2^2 + 4^2}}$$

$$= \frac{1 + 4 + 16}{\sqrt{21}} = \frac{21}{\sqrt{21}}$$

$$= \sqrt{21} \text{ units}$$

Q.13 Using vectors, find the area of the $\triangle ABC$ with vertices $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$.

Thinking Process

We know that,

$$\text{Area of } \triangle ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}|. \text{ So, here we shall use this concept.}$$

Sol. Here,

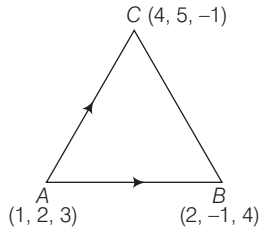
$$\vec{AB} = (2 - 1)\hat{i} + (-1 - 2)\hat{j} + (4 - 3)\hat{k}$$

$$= \hat{i} - 3\hat{j} + \hat{k}$$

and

$$\vec{AC} = (4 - 1)\hat{i} + (5 - 2)\hat{j} + (-1 - 3)\hat{k}$$

$$= 3\hat{i} + 3\hat{j} - 4\hat{k}$$



$$\therefore \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix}$$

$$= \hat{i}(12 - 3) - \hat{j}(-4 - 3) + \hat{k}(3 + 9)$$

$$= 9\hat{i} + 7\hat{j} + 12\hat{k}$$

and

$$|\vec{AB} \times \vec{AC}| = \sqrt{9^2 + 7^2 + 12^2}$$

$$= \sqrt{81 + 49 + 144}$$

$$= \sqrt{274}$$

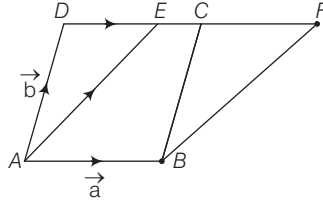
\therefore

$$\text{Area of } \triangle ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}|$$

$$= \frac{1}{2} \sqrt{274} \text{ sq units}$$

Q. 14 Using vectors, prove that the parallelogram on the same base and between the same parallels are equal in area.

Sol. Let $ABCD$ and $ABFE$ are parallelograms on the same base AB and between the same parallel lines AB and DF .
Here, $AB \parallel CD$ and $AE \parallel BF$



Let

$$\vec{AB} = \vec{a} \text{ and } \vec{AD} = \vec{b}$$

\therefore Area of parallelogram $ABCD = \vec{a} \times \vec{b}$

Now, area of parallelogram $ABFE = \vec{AB} \times \vec{AE}$

$$= \vec{AB} \times (\vec{AD} + \vec{DE})$$

$$= \vec{AB} \times (\vec{b} + k\vec{a})$$

[let $\vec{DE} = k\vec{a}$, where k is a scalar]

$$= \vec{a} \times (\vec{b} + k\vec{a})$$

$$= (\vec{a} \times \vec{b}) + (\vec{a} \times k\vec{a})$$

$$= (\vec{a} \times \vec{b}) + k(\vec{a} \times \vec{a})$$

$$= (\vec{a} \times \vec{b})$$

[$\because \vec{a} \times \vec{a} = 0$]

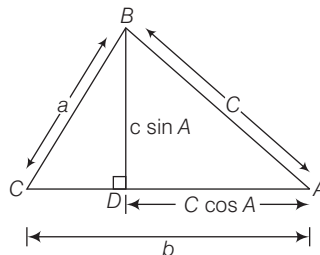
$$= \text{Area of parallelogram } ABCD$$

Hence proved.

Long Answer Type Questions

Q. 15 Prove that in any $\triangle ABC$, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, where a , b and c are the magnitudes of the sides opposite to the vertices A , B and C , respectively.

Sol. Here, components of C are $c \cos A$ and $c \sin A$ is drawn.



Since,
In $\triangle BDC$,

$$\vec{CD} = b - c \cos A$$

$$\begin{aligned} a^2 &= (b - c \cos A)^2 + (c \sin A)^2 \\ \Rightarrow a^2 &= b^2 + c^2 \cos^2 A - 2bc \cos A + c^2 \sin^2 A \\ \Rightarrow 2bc \cos A &= b^2 - a^2 + c^2 (\cos^2 A + \sin^2 A) \\ \therefore \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \end{aligned}$$

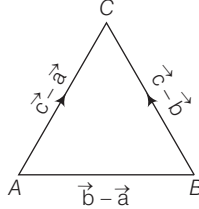
Q. 16 If \vec{a} , \vec{b} and \vec{c} determine the vertices of a triangle, show that $\frac{1}{2} [\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}]$ gives the vector area of the triangle. Hence, deduce the condition that the three points \vec{a} , \vec{b} and \vec{c} are collinear. Also, find the unit vector normal to the plane of the triangle.

Thinking Process

Here, we shall use the following two concepts.

- (i) If \vec{a} , \vec{b} and \vec{c} are collinear, then the area of the triangle formed by the vectors will be zero.
- (ii) We know that, $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin \theta \hat{n}$.

Sol. Since, \vec{a} , \vec{b} and \vec{c} are the vertices of a $\triangle ABC$ as shown.



$$\therefore \text{Area of } \triangle ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}|$$

Now,

$$\vec{AB} = \vec{b} - \vec{a} \quad \text{and} \quad \vec{AC} = \vec{c} - \vec{a}$$

\therefore

$$\begin{aligned} \text{Area of } \triangle ABC &= \frac{1}{2} |\vec{b} - \vec{a} \times \vec{c} - \vec{a}| \\ &= \frac{1}{2} |\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}| \\ &= \frac{1}{2} |\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a} + \vec{0}| \\ &= \frac{1}{2} |\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}| \end{aligned} \quad \dots (i)$$

For three points to be collinear, area of the $\triangle ABC$ should be equal to zero.

$$\Rightarrow \frac{1}{2} [\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}] = 0$$

$$\Rightarrow \vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b} = 0 \quad \dots (ii)$$

This is the required condition for collinearity of three points \vec{a} , \vec{b} and \vec{c} .

Let \hat{n} be the unit vector normal to the plane of the ΔABC .

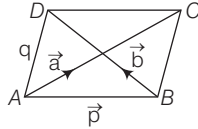
$$\begin{aligned} \therefore \hat{n} &= \frac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|} \\ &= \frac{\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}}{|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|} \end{aligned}$$

Q. 17 Show that area of the parallelogram whose diagonals are given by \vec{a} and \vec{b} is $\frac{|\vec{a} \times \vec{b}|}{2}$. Also, find the area of the parallelogram, whose diagonals are $2\hat{i} - \hat{j} + k$ and $\hat{i} + 3\hat{j} - \hat{k}$.

Thinking Process

If \vec{p} and \vec{q} are adjacent sides of a parallelogram, then the area formed by parallelogram $= |\vec{p} \times \vec{q}|$ and then we shall obtained the desired result.

Sol. Let ABCD be a parallelogram such that



$$\vec{AB} = \vec{p}, \vec{AD} = \vec{q} \Rightarrow \vec{BC} = \vec{q}$$

By triangle law of addition, we get

$$\vec{AC} = \vec{p} + \vec{q} = \vec{a} \quad [\text{say}] \dots(i)$$

Similarly,

$$\vec{BD} = -\vec{p} + \vec{q} = \vec{b} \quad [\text{say}] \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$\vec{a} + \vec{b} = 2\vec{q} \Rightarrow \vec{q} = \frac{1}{2}(\vec{a} + \vec{b})$$

On subtracting Eq. (ii) from Eq. (i), we get

$$\vec{a} - \vec{b} = 2\vec{p} \Rightarrow \vec{p} = \frac{1}{2}(\vec{a} - \vec{b})$$

Now,

$$\begin{aligned} \vec{p} \times \vec{q} &= \frac{1}{4}(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) \\ &= \frac{1}{4}(\vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}) \\ &= \frac{1}{4}[\vec{a} \times \vec{b} + \vec{a} \times \vec{b}] \\ &= \frac{1}{2}(\vec{a} \times \vec{b}) \end{aligned}$$

So, area of a parallelogram ABCD $= |\vec{p} \times \vec{q}| = \frac{1}{2}|\vec{a} \times \vec{b}|$

Now, area of a parallelogram, whose diagonals are $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j} - \hat{k}$.

$$\begin{aligned}
 &= \frac{1}{2} |(2\hat{i} - \hat{j} + \hat{k}) \times (\hat{i} + 3\hat{j} - \hat{k})| \\
 &= \frac{1}{2} \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{vmatrix} \right| \\
 &= \frac{1}{2} |[\hat{i}(1-3) - \hat{j}(-2-1) + \hat{k}(6+1)]| \\
 &= \frac{1}{2} |-2\hat{i} + 3\hat{j} + 7\hat{k}| \\
 &= \frac{1}{2} \sqrt{4+9+49} \\
 &= \frac{1}{2} \sqrt{62} \text{ sq units}
 \end{aligned}$$

Q. 18 If $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, then find a vector \vec{c} such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$.

Thinking Process

We know that, for any two vectors

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

and $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$, where $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ and $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$.

So, we shall use this concept.

Sol. Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$
 Also, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$
 For $\vec{a} \times \vec{c} = \vec{b}$,

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{j} - \hat{k}$$

$$\Rightarrow \hat{i}(z-y) - \hat{j}(z-x) + \hat{k}(y-x) = \hat{j} - \hat{k}$$

$$\therefore \begin{aligned}
 & z - y = 0 && \dots(i) \\
 & x - z = 1 && \dots(ii) \\
 & x - y = 1 && \dots(iii)
 \end{aligned}$$

Also, $\vec{a} \cdot \vec{c} = 3$

$$(\hat{i} + \hat{j} + \hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 3$$

$$\Rightarrow x + y + z = 3 \quad \dots(iv)$$

On adding Eqs. (ii) and (iii), we get

$$2x - y - z = 2 \quad \dots(v)$$

On solving Eqs. (iv) and (v), we get

$$x = \frac{5}{3}$$

$$\therefore y = \frac{5}{3} - 1 = \frac{2}{3} \text{ and } z = \frac{2}{3}$$

Now,

$$\begin{aligned} \vec{c} &= \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k} \\ &= \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k}) \end{aligned}$$

Objective Type Questions

Q. 19 The vector in the direction of the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ that has magnitude 9 is

(a) $\hat{i} - 2\hat{j} + 2\hat{k}$

(b) $\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$

(c) $3(\hat{i} - 2\hat{j} + 2\hat{k})$

(d) $9(\hat{i} - 2\hat{j} + 2\hat{k})$

Sol. (c) Let $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

Any vector in the direction of a vector \vec{a} is given by $\frac{\vec{a}}{|\vec{a}|}$.

$$= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

$$\begin{aligned} \therefore \text{Vector in the direction of } \vec{a} \text{ with magnitude } 9 &= 9 \cdot \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3} \\ &= 3(\hat{i} - 2\hat{j} + 2\hat{k}) \end{aligned}$$

Q. 20 The position vector of the point which divides the join of points $2\vec{a} - 3\vec{b}$ and $\vec{a} + \vec{b}$ in the ratio 3 : 1, is

(a) $\frac{3\vec{a} - 2\vec{b}}{2}$

(b) $\frac{7\vec{a} - 8\vec{b}}{4}$

(c) $\frac{3\vec{a}}{4}$

(d) $\frac{5\vec{a}}{4}$

Sol. (d) Let the position vector of the point R divides the join of points $2\vec{a} - 3\vec{b}$ and $\vec{a} + \vec{b}$.

$$\therefore \text{Position vector } R = \frac{3(\vec{a} + \vec{b}) + 1(2\vec{a} - 3\vec{b})}{3 + 1}$$

Since, the position vector of a point R dividing the line segment joining the points P and Q , whose position vectors are \vec{p} and \vec{q} in the ratio $m : n$ internally, is given by $\frac{m\vec{q} + n\vec{p}}{m + n}$.

$$\therefore R = \frac{5\vec{a}}{4}$$

Q. 21 The vector having initial and terminal points as $(2, 5, 0)$ and $(-3, 7, 4)$, respectively is

- (a) $-\hat{i} + 12\hat{j} + 4\hat{k}$ (b) $5\hat{i} + 2\hat{j} - 4\hat{k}$
 (c) $-5\hat{i} + 2\hat{j} + 4\hat{k}$ (d) $\hat{i} + \hat{j} + \hat{k}$

Sol. (c) Required vector = $(-3 - 2)\hat{i} + (7 - 5)\hat{j} + (4 - 0)\hat{k}$
 $= -5\hat{i} + 2\hat{j} + 4\hat{k}$

Similarly, we can say that for having initial and terminal points as

- (i) $(4, 1, 1)$ and $(3, 13, 5)$, respectively.
 (ii) $(1, 1, 9)$ and $(6, 3, 5)$, respectively.
 (iii) $(1, 2, 3)$ and $(2, 3, 4)$, respectively, we shall get (a), (b) and (d) as its correct options.

Q. 22 The angle between two vectors \vec{a} and \vec{b} with magnitudes $\sqrt{3}$ and 4, respectively and $\vec{a} \cdot \vec{b} = 2\sqrt{3}$ is

- (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{5\pi}{2}$

Sol. (b) Here, $|\vec{a}| = \sqrt{3}, |\vec{b}| = 4$ and $\vec{a} \cdot \vec{b} = 2\sqrt{3}$ [given]

We know that, $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$
 $\Rightarrow 2\sqrt{3} = \sqrt{3} \cdot 4 \cdot \cos \theta$
 $\Rightarrow \cos \theta = \frac{2\sqrt{3}}{4\sqrt{3}} = \frac{1}{2}$
 $\therefore \theta = \frac{\pi}{3}$

Q. 23 Find the value of λ such that the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ are orthogonal.

- (a) 0 (b) 1 (c) $\frac{3}{2}$ (d) $-\frac{5}{2}$

Thinking Process

Two non-zero vectors are orthogonal, if their dot product is zero. So, by using this concept, we shall get the value of λ .

Sol. (d) Since, two non-zero vectors \vec{a} and \vec{b} are orthogonal i.e., $\vec{a} \cdot \vec{b} = 0$.

$\therefore (2\hat{i} + \lambda\hat{j} + \hat{k}) \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 0$
 $\Rightarrow 2 + 2\lambda + 3 = 0$
 $\therefore \lambda = -\frac{5}{2}$

Q. 24 The value of λ for which the vectors $3\hat{i} - 6\hat{j} + \hat{k}$ and $2\hat{i} - 4\hat{j} + \lambda\hat{k}$ are parallel, is

- (a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) $\frac{2}{5}$

Sol. (a) Since, two vectors are parallel *i.e.*, angle between them is zero.

$$\therefore (3\hat{i} - 6\hat{j} + \hat{k}) \cdot (2\hat{i} - 4\hat{j} + \lambda\hat{k}) = |3\hat{i} - 6\hat{j} + \hat{k}| \cdot |2\hat{i} - 4\hat{j} + \lambda\hat{k}|$$

$$[\because \vec{a} \cdot \vec{b} = |\mathbf{a}| |\mathbf{b}| \cos 0^\circ \Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}|]$$

$$\Rightarrow 6 + 24 + \lambda = \sqrt{9 + 36 + 1} \sqrt{4 + 16 + \lambda^2}$$

$$\Rightarrow 30 + \lambda = \sqrt{46} \sqrt{20 + \lambda^2}$$

$$\Rightarrow 900 + \lambda^2 + 60\lambda = 46(20 + \lambda^2) \quad [\text{on squaring both sides}]$$

$$\Rightarrow \lambda^2 + 60\lambda - 46\lambda^2 = 920 - 900$$

$$\Rightarrow -45\lambda^2 + 60\lambda - 20 = 0$$

$$\Rightarrow -45\lambda^2 + 30\lambda + 30\lambda - 20 = 0$$

$$\Rightarrow -15\lambda(3\lambda - 2) + 10(3\lambda - 2) = 0$$

$$\Rightarrow (10 - 15\lambda)(3\lambda - 2) = 0$$

$$\therefore \lambda = \frac{2}{3}, \frac{2}{3}$$

Alternate Method

Let $\vec{a} = 3\hat{i} - 6\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 4\hat{j} + \lambda\hat{k}$

Since, $\vec{a} \parallel \vec{b}$

$$\Rightarrow \frac{3}{2} = \frac{-6}{-4} = \frac{1}{\lambda} \Rightarrow \lambda = \frac{2}{3}$$

Q. 25 The vectors from origin to the points A and B are $\vec{a} = 2\hat{i} - 3\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$ respectively, then the area of ΔOAB is equal to

- (a) 340 (b) $\sqrt{25}$
 (c) $\sqrt{229}$ (d) $\frac{1}{2}\sqrt{229}$

Sol. (d) \therefore

$$\text{Area of } \Delta OAB = \frac{1}{2} |\vec{OA} \times \vec{OB}|$$

$$= \frac{1}{2} |(2\hat{i} - 3\hat{j} + 2\hat{k}) \times (2\hat{i} + 3\hat{j} + \hat{k})|$$

$$= \frac{1}{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} |[\hat{i}(-3 - 6) - \hat{j}(2 - 4) + \hat{k}(6 + 6)]|$$

$$= \frac{1}{2} |-9\hat{i} + 2\hat{j} + 12\hat{k}|$$

$$\therefore \text{Area of } \Delta OAB = \frac{1}{2} \sqrt{(81 + 4 + 144)} = \frac{1}{2} \sqrt{229}$$

Q. 26 For any vector \vec{a} , the value of $(\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2$ is

(a) \vec{a}^2

(b) $3\vec{a}^2$

(c) $4\vec{a}^2$

(d) $2\vec{a}^2$

Sol. (d) Let

$$\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore \vec{a}^2 = x^2 + y^2 + z^2$$

$$\begin{aligned} \therefore \vec{a} \times \hat{i} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 0 & 0 \end{vmatrix} \\ &= \hat{i}[0] - \hat{j}[-z] + \hat{k}[-y] \\ &= z\hat{j} - y\hat{k} \end{aligned}$$

$$\begin{aligned} \therefore (\vec{a} \times \hat{i})^2 &= (z\hat{j} - y\hat{k})(z\hat{j} - y\hat{k}) \\ &= y^2 + z^2 \end{aligned}$$

Similarly, $(\vec{a} \times \hat{j})^2 = x^2 + z^2$

and $(\vec{a} \times \hat{k})^2 = x^2 + y^2$

$$\begin{aligned} \therefore (\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2 &= y^2 + z^2 + x^2 + z^2 + x^2 + y^2 \\ &= 2(x^2 + y^2 + z^2) = 2\vec{a}^2 \end{aligned}$$

Q. 27 If $|\vec{a}| = 10$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 12$, then the value of $|\vec{a} \times \vec{b}|$ is

(a) 5

(b) 10

(c) 14

(d) 16

Thinking Process

We know that, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta|\hat{n}$ and $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$. So, we shall use these formulae to get the value of $|\vec{a} \times \vec{b}|$.

Sol. (d) Here,

$$|\vec{a}| = 10, |\vec{b}| = 2 \text{ and } \vec{a} \cdot \vec{b} = 12$$

[given]

$$\therefore \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$12 = 10 \times 2 \cos\theta$$

$$\Rightarrow \cos\theta = \frac{12}{20} = \frac{3}{5}$$

$$\Rightarrow \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{9}{25}}$$

$$\sin\theta = \pm \frac{4}{5}$$

$$\therefore |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

$$= 10 \times 2 \times \frac{4}{5}$$

$$= 16$$

Q. 28 The vectors $\lambda \hat{i} + \hat{j} + 2\hat{k}$, $\hat{i} + \lambda \hat{j} - \hat{k}$ and $2\hat{i} - \hat{j} + \lambda \hat{k}$ are coplanar, if

(a) $\lambda = -2$

(b) $\lambda = 0$

(c) $\lambda = 1$

(d) $\lambda = -1$

Sol. (a) Let $\vec{a} = \lambda \hat{i} + \hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} + \lambda \hat{j} - \hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + \lambda \hat{k}$

For \vec{a} , \vec{b} and \vec{c} to be coplanar,

$$\begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda^2 - 1) - 1(\lambda + 2) + 2(-1 - 2\lambda) = 0$$

$$\Rightarrow \lambda^3 - \lambda - \lambda - 2 - 2 - 4\lambda = 0$$

$$\Rightarrow \lambda^3 - 6\lambda - 4 = 0$$

$$\Rightarrow (\lambda + 2)(\lambda^2 - 2\lambda - 2) = 0$$

$$\Rightarrow \lambda = -2 \text{ or } \lambda = \frac{2 \pm \sqrt{12}}{2}$$

$$\Rightarrow \lambda = -2 \text{ or } \lambda = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

Q. 29 If \vec{a} , \vec{b} and \vec{c} are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then the value

of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ is

(a) 1

(b) 3

(c) $-\frac{3}{2}$

(d) None of these

Sol. (c) We have, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $\vec{a}^2 = 1$, $\vec{b}^2 = 1$, $\vec{c}^2 = 1$

$$\therefore (\vec{a} + \vec{b} + \vec{c})(\vec{a} + \vec{b} + \vec{c}) = 0$$

$$\Rightarrow \vec{a}^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + \vec{b}^2 + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + \vec{c}^2 = 0$$

$$\Rightarrow \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

$$[\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}, \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{b} \text{ and } \vec{c} \cdot \vec{a} = \vec{a} \cdot \vec{c}]$$

$$\Rightarrow 1 + 1 + 1 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -\frac{3}{2}$$

Q. 30 The projection vector of \vec{a} on \vec{b} is

(a) $\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} \right) \vec{b}$

(b) $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

(c) $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

(d) $\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{b}$

Sol. (a) Projection vector of \vec{a} on \vec{b} is given by $= \vec{a} \cdot \frac{\vec{b}}{|\vec{b}|} \vec{b} = \left(\vec{a} \cdot \frac{\vec{b}}{|\vec{b}|} \right) \cdot \vec{b}$

Q. 31 If \vec{a} , \vec{b} and \vec{c} are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 2$,

$|\vec{b}| = 3$ and $|\vec{c}| = 5$, then the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ is

- (a) 0 (b) 1 (c) -19 (d) 38

Sol. (c) Here, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $a^2 = 4, b^2 = 9, c^2 = 25$

$$\therefore (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{0}$$

$$\Rightarrow a^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + b^2 + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + c^2 = 0$$

$$\Rightarrow a^2 + b^2 + c^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0 \quad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$$

$$\Rightarrow 4 + 9 + 25 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = \frac{-38}{2} = -19$$

Q. 32 If $|\vec{a}| = 4$ and $-3 \leq \lambda \leq 2$, then the range of $|\lambda \vec{a}|$ is

- (a) [0, 8] (b) [-12, 8]
(c) [0, 12] (d) [8, 12]

Sol. (c) We have, $|\vec{a}| = 4$ and $-3 \leq \lambda \leq 2$

$$\therefore |\lambda \vec{a}| = |\lambda| |\vec{a}| = \lambda |4|$$

$$\Rightarrow |\lambda \vec{a}| = |-3| 4 = 12, \text{ at } \lambda = -3$$

$$|\lambda \vec{a}| = |0| 4 = 0, \text{ at } \lambda = 0$$

$$\text{and } |\lambda \vec{a}| = |2| 4 = 8, \text{ at } \lambda = 2$$

So, the range of $|\lambda \vec{a}|$ is [0, 12].

Alternate Method

$$\text{Since, } -3 \leq \lambda \leq 2$$

$$0 \leq |\lambda| \leq 3$$

$$\Rightarrow 0 \leq 4|\lambda| \leq 12$$

$$|\lambda \vec{a}| \in [0, 12]$$

Q. 33 The number of vectors of unit length perpendicular to the vectors

$\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$ is

- (a) one (b) two
(c) three (d) infinite

Sol. (b) The number of vectors of unit length perpendicular to the vectors \vec{a} and \vec{b} is \vec{c} (say)

$$\text{i.e., } \vec{c} = \pm (\vec{a} \times \vec{b}).$$

So, there will be two vectors of unit length perpendicular to the vectors \vec{a} and \vec{b} .

Fillers

Q. 34 The vector $\vec{a} + \vec{b}$ bisects the angle between the non-collinear vectors \vec{a} and \vec{b} , if..... .

Sol. If vector $\vec{a} + \vec{b}$ bisects the angle between the non-collinear vectors, then

$$\vec{a} \cdot (\vec{a} + \vec{b}) = |\vec{a}| |\vec{a} + \vec{b}| \cos \theta$$

$$\vec{a} \cdot (\vec{a} + \vec{b}) = a\sqrt{a^2 + b^2} \cos \theta$$

$$\Rightarrow \cos \theta = \frac{\vec{a} \cdot (\vec{a} + \vec{b})}{a\sqrt{a^2 + b^2}} \quad \dots(i)$$

and

$$\vec{b} \cdot (\vec{a} + \vec{b}) = |\vec{b}| \cdot |\vec{a} + \vec{b}| \cos \theta$$

$$\vec{b} \cdot (\vec{a} + \vec{b}) = b\sqrt{a^2 + b^2} \cos \theta \quad [\text{since, } \theta \text{ should be same}]$$

$$\Rightarrow \cos \theta = \frac{\vec{b} \cdot (\vec{a} + \vec{b})}{b\sqrt{a^2 + b^2}} \quad \dots(ii)$$

From Eqs. (i) and (ii),

$$\frac{\vec{a} \cdot (\vec{a} + \vec{b})}{a\sqrt{a^2 + b^2}} = \frac{\vec{b} \cdot (\vec{a} + \vec{b})}{b\sqrt{a^2 + b^2}} \Rightarrow \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{b}}{|\vec{b}|}$$

$\therefore \hat{a} = \hat{b} \Rightarrow \vec{a}$ and \vec{b} are equal vectors.

Q. 35 If $\vec{r} \cdot \vec{a} = 0$, $\vec{r} \cdot \vec{b} = 0$ and $\vec{r} \cdot \vec{c} = 0$ for some non-zero vector \vec{r} , then the value of $\vec{a} \cdot (\vec{b} \times \vec{c})$ is..... .

Sol. Since, \vec{r} is a non-zero vector. So, we can say that \vec{a} , \vec{b} and \vec{c} are in a same plane.

$$\therefore \vec{a} \cdot (\vec{b} \times \vec{c}) = 0$$

[since, angle between \vec{a} , \vec{b} and \vec{c} are zero i.e., $\theta = 0$]

Q. 36 The vectors $\vec{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} - 2\hat{k}$ are the adjacent sides of a parallelogram. The angle between its diagonals is..... .

Sol. We have, $\vec{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} - 2\hat{k}$

$$\therefore \vec{a} + \vec{b} = 2\hat{i} - 2\hat{j} \text{ and } \vec{a} - \vec{b} = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

Now, let θ is the acute angle between the diagonals $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

$$\begin{aligned} \therefore \cos \theta &= \frac{(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})}{|\vec{a} + \vec{b}| |\vec{a} - \vec{b}|} \\ &= \frac{(2\hat{i} - 2\hat{j}) \cdot (4\hat{i} - 2\hat{j} + 4\hat{k})}{\sqrt{8} \sqrt{16 + 4 + 16}} = \frac{8 + 4}{2\sqrt{2} \cdot 6} = \frac{1}{\sqrt{2}} \end{aligned}$$

$$\therefore \theta = \frac{\pi}{4} \quad \left[\because \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \right]$$

Q. 37 The values of k , for which $|k\vec{a}| < |\vec{a}|$ and $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to \vec{a} holds true are

Sol. We have, $|k\vec{a}| < |\vec{a}|$ and $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to \vec{a} .

$$\begin{aligned} \therefore |k\vec{a}| < |\vec{a}| &\Rightarrow |k||\vec{a}| < |\vec{a}| \\ \Rightarrow |k| < 1 &\Rightarrow -1 < k < 1 \end{aligned}$$

Also, since $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to \vec{a} , then we see that at $k = \frac{-1}{2}$, $k\vec{a} + \frac{1}{2}\vec{a}$ becomes a null vector and then it will not be parallel to \vec{a} .

So, $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to \vec{a} holds true when $k \in]-1, 1 [$ [$k \neq \frac{-1}{2}$].

Q. 38 The value of the expression $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2$ is

Sol.

$$\begin{aligned} |\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 &= |\vec{a}|^2 |\vec{b}|^2 \sin^2 \theta + (\vec{a} \cdot \vec{b})^2 \\ &= |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2 \theta) + (\vec{a} \cdot \vec{b})^2 \\ &= |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2 \theta + (\vec{a} \cdot \vec{b})^2 \\ &= |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 + (\vec{a} \cdot \vec{b})^2 \\ |\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 &= |\vec{a}|^2 |\vec{b}|^2 \end{aligned}$$

Q. 39 If $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 144$ and $|\vec{a}| = 4$, then $|\vec{b}|$ is equal to

Thinking Process

We know that, $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2$. So, we shall use this concept here to find the value of $|\vec{b}|$.

Sol.

$$\begin{aligned} \therefore |\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 &= 144 = |\vec{a}|^2 \cdot |\vec{b}|^2 \\ \Rightarrow |\vec{a}|^2 |\vec{b}|^2 &= 144 \\ \Rightarrow |\vec{b}|^2 &= \frac{144}{|\vec{a}|^2} = \frac{144}{16} = 9 \\ \therefore |\vec{b}| &= 3 \end{aligned}$$

Q. 40 If \vec{a} is any non-zero vector, then $(\vec{a} \cdot \hat{i}) \cdot \hat{i} + (\vec{a} \cdot \hat{j}) \cdot \hat{j} + (\vec{a} \cdot \hat{k}) \cdot \hat{k}$ is equal to

Sol. Let

$$\begin{aligned} \vec{a} &= a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \\ \therefore \vec{a} \cdot \hat{i} &= a_1, \vec{a} \cdot \hat{j} = a_2 \text{ and } \vec{a} \cdot \hat{k} = a_3 \\ \therefore (\vec{a} \cdot \hat{i}) \hat{i} + (\vec{a} \cdot \hat{j}) \hat{j} + (\vec{a} \cdot \hat{k}) \hat{k} &= a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} = \vec{a} \end{aligned}$$

True/False

Q. 41 If $|\vec{a}| = |\vec{b}|$, then necessarily it implies $\vec{a} = \pm \vec{b}$.

Sol. True

$$\text{If } |\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \pm \vec{b}$$

So, it is a true statement.

Q. 42 Position vector of a point \vec{P} is a vector whose initial point is origin.

Sol. True

Since, $\vec{P} = \vec{OP}$ = displacement of vector \vec{P} from origin

Q. 43 If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, then the vectors \vec{a} and \vec{b} are orthogonal

Sol. True

$$\begin{aligned} \text{Since, } & |\vec{a} + \vec{b}| = |\vec{a} - \vec{b}| \\ \Rightarrow & |\vec{a} + \vec{b}|^2 = |\vec{a} - \vec{b}|^2 \\ \Rightarrow & 2|\vec{a}||\vec{b}| = -2|\vec{a}||\vec{b}| \\ \Rightarrow & 4|\vec{a}||\vec{b}| = 0 \\ \Rightarrow & |\vec{a}||\vec{b}| = 0 \end{aligned}$$

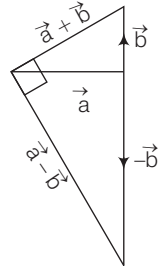
Hence, \vec{a} and \vec{b} are orthogonal.

$$[\because \vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos 90^\circ = 0]$$

Q. 44 The formula $(\vec{a} + \vec{b})^2 = \vec{a}^2 + \vec{b}^2 + 2\vec{a} \times \vec{b}$ is valid for non-zero vectors \vec{a} and \vec{b} .

Sol. False

$$\begin{aligned} (\vec{a} + \vec{b})^2 &= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) \\ &= \vec{a}^2 + \vec{b}^2 + 2\vec{a} \cdot \vec{b} \end{aligned}$$



Q. 45 If \vec{a} and \vec{b} are adjacent sides of a rhombus, then $\vec{a} \cdot \vec{b} = 0$.

Sol. False

$$\text{If } \vec{a} \cdot \vec{b} = 0, \text{ then } \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos 90^\circ$$

Hence, angle between \vec{a} and \vec{b} is 90° , which is not possible in a rhombus. Since, angle between adjacent sides in a rhombus is not equal to 90° .