11

Three Dimensional Geometry

Short Answer Type Questions

Q. 1 Find the position vector of a point A in space such that $\vec{OA}$ is inclined at 60° to OX and at 45° to OY and $|\vec{OA}| = 10$ units.

Sol. Since, $\vec{OA}$ is inclined at 60° to OX and at 45° to OY. Let $\vec{OA}$ makes angle α with OZ.

$$\therefore \quad \cos^2 60° + \cos^2 45° + \cos^2 \alpha = 1$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2 \alpha = 1 \qquad [\because l^2 + m^2 + n^2 = 1]$$

$$\Rightarrow \quad \frac{1}{4} + \frac{1}{2} + \cos^2 \alpha = 1$$

$$\Rightarrow \quad \cos^2 \alpha = 1 - \left(\frac{1}{2} + \frac{1}{4}\right)$$

$$\Rightarrow \quad \cos^2 \alpha = 1 - \left(\frac{6}{8}\right)$$

$$\Rightarrow \quad \cos^2 \alpha = \frac{1}{4}$$

$$\Rightarrow \quad \cos \alpha = \frac{1}{2} = \cos 60°$$

$$\therefore \quad \alpha = 60°$$

$$\therefore \quad \vec{OA} = |\vec{OA}|\left(\frac{1}{2}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}} + \frac{1}{2}\hat{\mathbf{k}}\right)$$

$$= 10\left(\frac{1}{2}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}} + \frac{1}{2}\hat{\mathbf{k}}\right) \qquad [\because |\vec{OA}| = 10]$$

$$= 5\hat{\mathbf{i}} + 5\sqrt{2}\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}}$$

Q. 2 Find the vector equation of the line which is parallel to the vector $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and which passes through the point (1, −2, 3).

Thinking Process

Here, we use the formula $\vec{\mathbf{r}} = \vec{\mathbf{b}} + \lambda \vec{\mathbf{a}}$, where $\vec{\mathbf{r}}$ is the equation of the line which passes through $\vec{\mathbf{b}}$ and parallel to $\vec{\mathbf{a}}$.

Sol. Let $\vec{\mathbf{a}} = 3\hat{i} - 2\hat{j} + 6\hat{k}$ and $\vec{\mathbf{b}} = \hat{i} - 2\hat{j} + 3\hat{k}$

So, vector equation of the line, which is parallel to the vector $\vec{\mathbf{a}} = 3\hat{i} - 2\hat{j} + 6\hat{k}$ and passes through the vector $\vec{\mathbf{b}} = \hat{i} - 2\hat{j} + 3\hat{k}$ is $\vec{\mathbf{r}} = \vec{\mathbf{b}} + \lambda\vec{\mathbf{a}}$.

$$\therefore \quad \vec{\mathbf{r}} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$$

$$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \lambda(3\hat{i} - 2\hat{j} + 6\hat{\mathbf{k}})$$

$$\Rightarrow \quad (x - 1)\hat{\mathbf{i}} + (y + 2)\hat{\mathbf{j}} + (z - 3)\hat{\mathbf{k}} = \lambda(3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$$

Q. 3 Show that the lines $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ and $\dfrac{x-4}{5} = \dfrac{y-1}{2} = z$ intersect. Also, find their point of intersection.

Thinking Process

If shortest distance between the lines is zero, then they intersect.

Sol. We have, $x_1 = 1, y_1 = 2, z_1 = 3$ and $a_1 = 2, b_1 = 3, c_1 = 4$

Also, $x_2 = 4, y_2 = 1, z_2 = 0$ and $a_2 = 5, b_2 = 2, c_2 = 1$

If two lines intersect, then shortest distance between them should be zero.

∴ Shortest distance between two given lines

$$= \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

$$= \frac{\begin{vmatrix} 4-1 & 1-2 & 0-3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix}}{\sqrt{(3\cdot 1 - 2\cdot 4)^2 + (4\cdot 5 - 1\cdot 2)^2 + (2\cdot 2 - 5\cdot 3)^2}}$$

$$= \frac{\begin{vmatrix} 3 & -1 & -3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix}}{\sqrt{25 + 324 + 121}}$$

$$= \frac{3(3-8) + 1(2-20) - 3(4-15)}{\sqrt{470}}$$

$$= \frac{-15 - 18 + 33}{\sqrt{470}} = \frac{0}{\sqrt{470}} = 0$$

Therefore, the given two lines are intersecting.

For finding their point of intersection for first line,

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$$

$\Rightarrow$ $x = 2\lambda + 1,\ y = 3\lambda + 2$ and $z = 4\lambda + 3$

Since, the lines are intersecting. So, let us put these values in the equation of another line.

Thus, $$\frac{2\lambda + 1 - 4}{5} = \frac{3\lambda + 2 - 1}{2} = \frac{4\lambda + 3}{1}$$

$$\Rightarrow \quad \frac{2\lambda - 3}{5} = \frac{3\lambda + 1}{2} = \frac{4\lambda + 3}{1}$$

$$\Rightarrow \quad \frac{2\lambda - 3}{5} = \frac{4\lambda + 3}{1}$$

$$\Rightarrow \quad 2\lambda - 3 = 20\lambda + 15$$

$$\Rightarrow \quad 18\lambda = -18 = -1$$

So, the required point of intersection is

$$x = 2(-1) + 1 = -1$$
$$y = 3(-1) + 2 = -1$$
$$z = 4(-1) + 3 = -1$$

Thus, the lines intersect at $(-1, -1, -1)$.

Q. 4 Find the angle between the lines

$$\vec{\mathbf{r}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}} + \lambda(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \text{ and } \vec{\mathbf{r}} = (2\hat{\mathbf{j}} - 5\hat{\mathbf{k}}) + \mu(6\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}).$$

Thinking Process

We know that, $\cos\theta = \dfrac{|\vec{\mathbf{b}}_1 \cdot \vec{\mathbf{b}}_2|}{|\vec{\mathbf{b}}_1| \cdot |\vec{\mathbf{b}}_2|}$*, where, θ is the angle between the lines* $\vec{\mathbf{a}}_1 + \lambda\vec{\mathbf{b}}_1$ *and* $\vec{\mathbf{a}}_2 + \mu\vec{\mathbf{b}}_2$.

Sol. We have, $\vec{\mathbf{r}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}} + \lambda(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

and $\vec{\mathbf{r}} = (2\hat{\mathbf{j}} - 5\hat{\mathbf{k}}) + \mu(6\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

where, $\vec{\mathbf{a}}_1 = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}},\quad \vec{\mathbf{b}}_1 = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

and $\vec{\mathbf{a}}_2 = 2\hat{\mathbf{j}} - 5\hat{\mathbf{k}},\quad \vec{\mathbf{b}}_2 = 6\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

If θ is angle between the lines, then

$$\cos\theta = \frac{|\vec{\mathbf{b}}_1 \cdot \vec{\mathbf{b}}_2|}{|\vec{\mathbf{b}}_1| \cdot |\vec{\mathbf{b}}_2|}$$

$$= \frac{|(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \cdot (6\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}})|}{|2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}||6\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}|}$$

$$= \frac{|12 + 3 + 4|}{\sqrt{9}\sqrt{49}} = \frac{19}{21}$$

$$\therefore \quad \theta = \cos^{-1}\frac{19}{21}$$

Q. 5 Prove that the line through $A(0, -1, -1)$ and $B(4, 5, 1)$ intersects the line through $C(3, 9, 4)$ and $D(-4, 4, 4)$.

Sol. We know that,the cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Hence, the cartesian equation of line passes through $A(0, -1, -1)$ and $B(4, 5, 1)$ is

$$\frac{x - 0}{4 - 0} = \frac{y + 1}{5 + 1} = \frac{z + 1}{1 + 1}$$

$$\Rightarrow \quad \frac{x}{4} = \frac{y + 1}{6} = \frac{z + 1}{2} \quad \text{...(i)}$$

and cartesian equation of the line passes through $C(3, 9, 4)$ and $D(-4, 4, 4)$ is

$$\frac{x - 3}{-4 - 3} = \frac{y - 9}{4 - 9} = \frac{z - 4}{4 - 4}$$

$$\Rightarrow \quad \frac{x - 3}{-7} = \frac{y - 9}{-5} = \frac{z - 4}{0} \quad \text{...(ii)}$$

If the lines intersect, then shortest distance between both of them should be zero.

$\therefore$ Shortest distance between the lines

$$= \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

$$= \frac{\begin{vmatrix} 3 - 0 & 9 + 1 & 4 + 1 \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}}{\sqrt{(6 \cdot 0 + 10)^2 + (-14 - 0)^2 + (-20 + 42)^2}}$$

$$= \frac{\begin{vmatrix} 3 & 10 & 5 \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}}{\sqrt{100 + 196 + 484}}$$

$$= \frac{3(0 + 10) - 10(14) + 5(-20 + 42)}{\sqrt{780}}$$

$$= \frac{30 - 140 + 110}{\sqrt{780}} = 0$$

So, the given lines intersect.

Q. 6 Prove that the lines $x = py + q$, $z = ry + s$ and $x = p'y + q'$, $z = r'y + s'$ are perpendicular, if $pp' + rr' + 1 = 0$.

Sol. We have, $x = py + q \Rightarrow y = \dfrac{x - q}{P}$ ..(i)

and $z = ry + s \Rightarrow y = \dfrac{z - s}{r}$ ...(ii)

$$\Rightarrow \quad \frac{x - q}{p} = \frac{y}{1} = \frac{z - s}{r} \quad \text{[using Eqs. (i) and (ii)] ...(iii)}$$

Similarly, $$\frac{x - q'}{p'} = \frac{y}{1} = \frac{z - s'}{r'} \quad \text{...(iv)}$$

From Eqs. (iii) and (iv),

$$a_1 = p, b_1 = 1, c_1 = r$$

and $$a_2 = p', b_2 = 1, c_2 = r'$$

If these given lines are perpendicular to each other, then

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

$$\Rightarrow \quad pp' + 1 + rr' = 0$$

which is the required condition.

Q. 7 Find the equation of a plane which bisects perpendicularly the line joining the points $A(2, 3, 4)$ and $B(4, 5, 8)$ at right angles.

Sol. Since, the equation of a plane is bisecting perpendicular the line joining the points $A(2, 3, 4)$ and $B(4, 5, 8)$ at right angles.

So, mid-point of AB is $\left(\frac{2+4}{2}, \frac{3+5}{2}, \frac{4+8}{2}\right)$ *i.e.,* $(3, 4, 6)$.

Also, $$\vec{\mathbf{N}} = (4-2)\hat{\mathbf{i}} + (5-3)\hat{\mathbf{j}} + (8-4)\hat{\mathbf{k}} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

So, the required equation of the plane is $(\vec{\mathbf{r}} - \vec{\mathbf{a}}) \cdot \vec{\mathbf{N}} = 0$.

$$\Rightarrow \quad [(x-3)\hat{\mathbf{i}} + (y-4)\hat{\mathbf{j}} + (z-6)\hat{\mathbf{k}}] \cdot (2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = 0 \qquad [\because \vec{\mathbf{a}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}]$$

$$\Rightarrow \quad 2x - 6 + 2y - 8 + 4z - 24 = 0$$

$$\Rightarrow \quad 2x + 2y + 4z = 38$$

$$\therefore \quad x + y + 2z = 19$$

Q. 8 Find the equation of a plane which is at a distance $3\sqrt{3}$ units from origin and the normal to which is equally inclined to coordinate axis.

Sol. Since, normal to the plane is equally inclined to the coordinate axis.

Therefore, $$\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{\sqrt{3}}$$

So, the normal is $\vec{\mathbf{N}} = \frac{1}{\sqrt{3}}\hat{\mathbf{i}} + \frac{1}{\sqrt{3}}\hat{\mathbf{j}} + \frac{1}{\sqrt{3}}\hat{\mathbf{k}}$ and plane is at a distance of $3\sqrt{3}$ units from origin.

The equation of plane is $\vec{\mathbf{r}} \cdot \hat{\mathbf{N}} = 3\sqrt{3}$ $\quad \left[\because \hat{\mathbf{N}} = \frac{\vec{\mathbf{N}}}{|\mathbf{N}|}\right]$

[since, vector equation of the plane at a distance p from the origin is $\vec{\mathbf{r}} \cdot \hat{\mathbf{N}} = p$]

$$\Rightarrow \quad (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \frac{\left(\frac{1}{\sqrt{3}}\hat{\mathbf{i}} + \frac{1}{\sqrt{3}}\hat{\mathbf{j}} + \frac{1}{\sqrt{3}}\hat{\mathbf{k}}\right)}{1} = 3\sqrt{3}$$

$$\Rightarrow \quad \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = 3\sqrt{3}$$

$$\therefore \quad x + y + z = 3\sqrt{3} \cdot \sqrt{3} = 9$$

So, the required equation of plane is $x + y + z = 9$.

Q. 9 If the line drawn from the point (–2, –1, –3) meets a plane at right angle at the point (1, –3, 3), then find the equation of the plane.

Sol. Since, the line drawn from the point (–2, – 1, – 3) meets a plane at right angle at the point (1, – 3, 3). So, the plane passes through the point (1, – 3, 3) and normal to plane is $(-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$.

$$\Rightarrow \quad \vec{\mathbf{a}} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

and
$$\vec{\mathbf{N}} = -3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$$

So, the equation of required plane is $(\vec{\mathbf{r}} - \vec{\mathbf{a}}) \cdot \vec{\mathbf{N}} = 0$

$$\Rightarrow \quad [(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) - (\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}})] \cdot (-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}}) = 0$$

$$\Rightarrow \quad [(x-1)\hat{\mathbf{i}} + (y+3)\hat{\mathbf{j}} + (z-3)\hat{\mathbf{k}}] \cdot (-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}}) = 0$$

$$\Rightarrow \quad -3x + 3 + 2y + 6 - 6z + 18 = 0$$

$$\Rightarrow \quad -3x + 2y - 6z = -27$$

$$\therefore \quad 3x - 2y + 6z - 27 = 0$$

Q. 10 Find the equation of the plane through the points (2, 1, 0), (3, –2, –2) and (3, 1, 7).

Thinking Process

Here, apply the equation of the plane passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$

Sol. We know that, the equation of a plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x - 2 & y - 1 & z - 0 \\ 3 - 2 & -2 - 1 & -2 - 0 \\ 3 - 2 & 1 - 1 & 7 - 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x - 2 & y - 1 & z \\ 1 & -3 & -2 \\ 1 & 0 & 7 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-2)(-21+0) - (y-1)(7+2) + z(3) = 0$$

$$\Rightarrow \quad -21x + 42 - 9y + 9 + 3z = 0$$

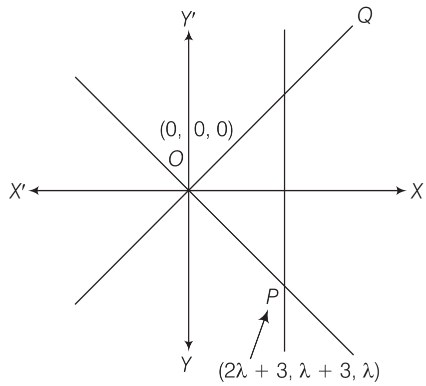
$$\Rightarrow \quad -21x - 9y + 3z = -51$$

$$\therefore \quad 7x + 3y - z = 17$$

So, the required equation of plane is $7x + 3y - z = 17$.

Q. 11 Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ at angles of $\frac{\pi}{3}$ each.

Sol. Given equation of the line is $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}=\lambda$ ...(i)



So, DR's of the line are 2, 1, 1 and DC's of the given line are $\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{16}}$.

Also, the required lines make angle $\frac{\pi}{3}$ with the given line.

From Eq. (i), $x=(2\lambda+3),\ y=(\lambda+3)$ and $z=\lambda$

$$\because \quad \cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\therefore \quad \cos\frac{\pi}{3}=\frac{(4\lambda+6)+(\lambda+3)+(\lambda)}{\sqrt{6}\sqrt{(2\lambda+3)^2+(\lambda+3)^2+\lambda^2}}$$

$$\Rightarrow \quad \frac{1}{2}=\frac{6\lambda+9}{\sqrt{6}\sqrt{(4\lambda^2+9+12\lambda+\lambda^2+9+6\lambda+\lambda^2)}}$$

$$\Rightarrow \quad \frac{\sqrt{6}}{2}=\frac{6\lambda+9}{\sqrt{6\lambda^2+18\lambda+18}}$$

$\Rightarrow \quad 6\sqrt{(\lambda^2+3\lambda+3)}=2(6\lambda+9)$

$\Rightarrow \quad 36(\lambda^2+3\lambda+3)=36(4\lambda^2+9+12\lambda)$

$\Rightarrow \quad \lambda^2+3\lambda+3=4\lambda^2+9+12\lambda$

$\Rightarrow \quad 3\lambda^2+9\lambda+6=0$

$\Rightarrow \quad \lambda^2+3\lambda+2=0$

$\Rightarrow \quad \lambda(\lambda+2)+1(\lambda+2)=0$

$\Rightarrow \quad (\lambda+1)(\lambda+2)=0$

$\therefore \quad \lambda=-1,-2$

So, the DC's are 1, 2, −1 and −1, 1, −2.

Also, both the required lines passes through origin.

So, the equations of required lines are $\frac{x}{1}=\frac{y}{2}=\frac{z}{-1}$ and $\frac{x}{-1}=\frac{y}{1}=\frac{z}{-2}$.

Q. 12 **Find the angle between the lines whose direction cosines are given by the equation $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$.**

Sol. Eliminating n from both the equations, we have

$$l^2 + m^2 - (l - m)^2 = 0$$

$\Rightarrow$ $l^2 + m^2 - l^2 - m^2 + 2ml = 0 \Rightarrow 2lm = 0$

$\Rightarrow$ $lm = 0 \Rightarrow (-m - n)m = 0$ $[\because l = -m - n]$

$\Rightarrow$ $(m + n)m = 0$

$\Rightarrow$ $m = -n \Rightarrow m = 0$

$\Rightarrow$ $l = 0, l = -n$

Thus, Dr's two lines are proportional to $0, -n, n$ and $-n, 0, n$ *i.e.,* $0, -1, 1$ and $-1, 0, 1$.

So, the vector parallel to these given lines are $\vec{a} = -\hat{j} + \hat{k}$ and $\vec{b} = -\hat{i} + \hat{k}$

Now, $$\cos\theta = \frac{\vec{a}\,\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \Rightarrow \cos\theta = \frac{1}{2}$$

$\therefore$ $$\theta = \frac{\pi}{3}$$ $\left[\because \cos\frac{\pi}{3} = \frac{1}{2}\right]$

Q. 13 **If a variable line in two adjacent positions has direction cosines l, m, n and $l + \delta l, m + \delta m, n + \delta n$, then show that the small angle $\delta\theta$ between the two positions is given by $\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$.**

Sol. We have l, m, n and $l + \delta l, m + \delta m, n + \delta n$ as direction cosines of a variable line in two different positions.

$\therefore$ $$l^2 + m^2 + n^2 = 1 \quad \text{... (i)}$$

and $$(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1 \quad \text{...(ii)}$$

$\Rightarrow$ $l^2 + m^2 + n^2 + \delta l^2 + \delta m^2 + \delta n^2 + 2(l\,\delta l + m\,\delta m + n\,\delta n) = 1$

$\Rightarrow$ $\delta l^2 + \delta m^2 + \delta n^2 = -2(l\,\delta l + m\,\delta m + n\,\delta n)$ $[\because l^2 + m^2 + n^2 = 1]$

$\Rightarrow$ $$l\,\delta l + m\,\delta m + n\,\delta n = \frac{-1}{2}(\delta l^2 + \delta m^2 + \delta n^2) \quad \text{...(iii)}$$

Now, $\vec{a}$ and $\vec{b}$ are unit vectors along a line with direction cosines l, m, n and $(l + \delta l), (m + \delta m), (n + \delta n)$, respectively.

$\therefore$ $\vec{a} = l\hat{i} + m\hat{j} + n\hat{k}$ and $\vec{b} = (l + \delta l)\hat{i} + (m + \delta m)\hat{j} + (n + \delta n)\hat{k}$

$\Rightarrow$ $$\cos\delta\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \vec{a}\cdot\vec{b}$$ $[\because |\hat{a}| = |\hat{b}| = 1]$

$\Rightarrow$ $\cos\delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$

$= (l^2 + m^2 + n^2) + (l\,\delta l + m\,\delta m + n\,\delta n)$

$= 1 - \frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$ [using Eq. (iii)]

$\Rightarrow$ $2(1 - \cos\delta\theta) = (\delta l^2 + \delta m^2 + \delta n^2)$

$\Rightarrow$ $2 \cdot 2\sin^2\frac{\delta\theta}{2} = \delta l^2 + \delta m^2 + \delta n^2$ $\left[\because 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$

$\Rightarrow$ $4\left(\frac{\delta\theta}{2}\right)^2 = \delta l^2 + \delta m^2 + \delta n^2$ $\left[\text{since, } \frac{\delta\theta}{2} \text{ is small, then } \sin\frac{\delta\theta}{2} = \frac{\delta\theta}{2}\right]$

$\therefore$ $$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$$

Q. 14 If O is the origin and A is (a, b, c), then find the direction cosines of the line OA and the equation of plane through A at right angle to OA.

Sol. Since, DC's of line OA are $\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}$ and $\frac{c}{\sqrt{a^2 + b^2 + c^2}}$.

Also, $\vec{\mathbf{n}} = \overrightarrow{\mathbf{OA}} = \vec{\mathbf{a}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$

The equation of plane passes through (a, b, c) and perpendicular to OA is given by

$$[\vec{\mathbf{r}} - \vec{\mathbf{a}}] \cdot \vec{\mathbf{n}} = 0$$

$\Rightarrow$ $$\vec{\mathbf{r}} \cdot \vec{\mathbf{n}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{n}}$$

$\Rightarrow$ $$[(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot (a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}})] = (a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}) . (a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}})$$

$\Rightarrow$ $$ax + by + cz = a^2 + b^2 + c^2$$

Q. 15 Two systems of rectangular axis have the same origin. If a plane cuts them at distances a, b, c and a', b', c', respectively from the origin, then prove that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$.

Sol. Consider OX, OY, OZ and ox, oy, oz are two system of rectangular axes.
Let their corresponding equation of plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{...(i)}$$

and $$\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} = 1 \quad \text{...(ii)}$$

Also, the length of perpendicular from origin to Eqs. (i) and (ii) must be same.

$\therefore$ $$\frac{\frac{0}{a} + \frac{0}{b} + \frac{0}{c} - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{\frac{0}{a'} + \frac{0}{b'} + \frac{0}{c'} - 1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}}$$

$\Rightarrow$ $$\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}} = \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$$

$\Rightarrow$ $$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$$

Long Answer Type Questions

Q. 16 Find the foot of perpendicular from the point $(2, 3, -8)$ to the line $\frac{4 - x}{2} = \frac{y}{6} = \frac{1 - z}{3}$. Also, find the perpendicular distance from the given point to the line.

Sol. We have, equation of line as $\frac{4 - x}{2} = \frac{y}{6} = \frac{1 - z}{3}$

$\Rightarrow$ $$\frac{x - 4}{-2} = \frac{y}{6} = \frac{z - 1}{-3} = \lambda$$

$\Rightarrow$ $$x = -2\lambda + 4, \; y = 6\lambda \text{ and } z = -3\lambda + 1$$

Let the coordinates of L be $(4-2\lambda, 6\lambda, 1-3\lambda)$ and direction ratios of PL are proportional to $(4-2\lambda-2, 6\lambda-3, 1-3\lambda+8)$ *i.e.,* $(2-2\lambda, 6\lambda-3, 9-3\lambda)$.

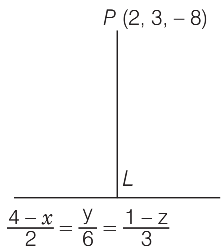
Also, direction ratios are proportional to $-2, 6, -3$. Since, PL is perpendicular to give line.

$\therefore \quad -2(2-2\lambda)+6(6\lambda-3)-3(9-3\lambda)=0$

$\Rightarrow \quad -4+4\lambda+36\lambda-18-27+9\lambda=0$

$\Rightarrow \quad 49\lambda=49 \Rightarrow \lambda=1$

So, the coordinates of L are $(4-2\lambda, 6\lambda, 1-3\lambda)$ *i.e.,* $(2, 6, -2)$.



Also, $\quad$ length of $PL=\sqrt{(2-2)^2+(6-3)^2+(-2+8)^2}$

$=\sqrt{0+9+36}=3\sqrt{5}$ units

Q. 17 **Find the distance of a point (2, 4, –1) from the line**

$$\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}.$$

Sol. We have, equation of the line as $\dfrac{x+5}{1}=\dfrac{y+3}{4}=\dfrac{z-6}{-9}=\lambda$

$\Rightarrow \quad x=\lambda-5,\ y=4\lambda-3,\ z=6-9\lambda$

Let the coordinates of L be $(\lambda-5, 4\lambda-3, 6-9\lambda)$, then Dr's of PL are $(\lambda-7, 4\lambda-7, 7-9\lambda)$.

Also, the direction ratios of given line are proportional to $1, 4, -9$.

Since, PL is perpendicular to the given line.

$\therefore \quad (\lambda-7)\cdot 1+(4\lambda-7)\cdot 4+(7-9\lambda)\cdot(-9)=0$

$\Rightarrow \quad \lambda-7+16\lambda-28+81\lambda-63=0$

$\Rightarrow \quad 98\lambda=98 \Rightarrow \lambda=1$

So, the coordinates of L are $(-4, 1, -3)$.

$\therefore \quad$ Required distance, $PL=\sqrt{(-4-2)^2+(1-4)^2+(-3+1)^2}$

$=\sqrt{36+9+4}=7$ units

Q. 18 **Find the length and the foot of perpendicular from the point $\left(1, \dfrac{3}{2}, 2\right)$ to the plane $2x-2y+4z+5=0$.**

Sol. Equation of the given plane is $2x-2y+4z+5=0$ … (i)

$\Rightarrow \quad \vec{\mathbf{n}}=2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+4\hat{\mathbf{k}}$

So, the equation of line through $\left(1, \dfrac{3}{2}, 2\right)$ and parallel to $\vec{\mathbf{n}}$ is given by

$$\frac{x-1}{2}=\frac{y-3/2}{-2}=\frac{z-2}{4}=\lambda$$

$\Rightarrow \quad x=2\lambda+1,\ y=-2\lambda+\dfrac{3}{2}$ and $z=4\lambda+2$

If this point lies on the given plane, then

$$2(2\lambda+1)-2\left(-2\lambda+\frac{3}{2}\right)+4(4\lambda+2)+5=0 \qquad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad 4\lambda+2+4\lambda-3+16\lambda+8+5=0$$

$$\Rightarrow \quad 24\lambda=-12 \Rightarrow \lambda=\frac{-1}{2}$$

$\therefore$ Required foot of perpendicular

$$=\left[2\times\left(\frac{-1}{2}\right)+1, -2\times\left(\frac{-1}{2}\right)+\frac{3}{2}, 4\times\left(\frac{-1}{2}\right)+2\right] \text{ i.e., } \left(0, \frac{5}{2}, 0\right)$$

$\therefore$ Required length of perpendicular $=\sqrt{(1-0)^2+\left(\frac{3}{2}-\frac{5}{2}\right)^2+(2-0)^2}$

$$=\sqrt{1+1+4}=\sqrt{6} \text{ units}$$

Q. 19 **Find the equation of the line passing through the point (3, 0, 1) and parallel to the planes $x+2y=0$ and $3y-z=0$.**

Sol. Equation of the two planes are $x+2y=0$ and $3y-z=0$.

Let $\vec{\mathbf{n}}_1$ and $\vec{\mathbf{n}}_2$ are the normals to the two planes, respectively.

$$\therefore \quad \vec{\mathbf{n}}_1=\hat{\mathbf{i}}+2\hat{\mathbf{j}} \quad \text{and} \quad \vec{\mathbf{n}}_2=3\hat{\mathbf{j}}-\hat{\mathbf{k}}$$

Since, required line is parallel to the given two planes.

Therefore,
$$\vec{\mathbf{b}}=\vec{\mathbf{n}}_1\times\vec{\mathbf{n}}_2=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 2 & 0\\ 0 & 3 & -1\end{vmatrix}$$

$$=\hat{\mathbf{i}}(-2)-\hat{\mathbf{j}}(-1)+\hat{\mathbf{k}}(3)$$

$$=-2\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}}$$

So, the equation of the lines through the point (3, 0, 1) and parallel to the given two planes are

$$(x-3)\hat{\mathbf{i}}+(y-0)\hat{\mathbf{j}}+(z-1)\hat{\mathbf{k}}+\lambda(-2\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}})$$

$$\Rightarrow \quad (x-3)\hat{\mathbf{i}}+y\hat{\mathbf{j}}+(z-1)\hat{\mathbf{k}}+\lambda(-2\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}})$$

Q. 20 **Find the equation of the plane through the points (2, 1, – 1), (–1, 3, 4) and perpendicular to the plane $x-2y+4z=10$.**

Sol. The equation of the plane passing through (2, 1, – 1) is

$$a(x-2)+b(y-1)+c(z+1)=0 \qquad \text{...(i)}$$

Sicne, this passes through (–1, 3, 4).

$$\therefore \quad a(-1-2)+b(3-1)+c(4+1)=0$$

$$\Rightarrow \quad -3a+2b+5c=0 \qquad \text{... (ii)}$$

Since, the plane (i) is perpendicular to the plane $x-2y+4z=10$.

$$\therefore \quad 1\cdot a-2\cdot b+4\cdot c=0$$

$$\Rightarrow \quad a-2b+4c=0 \qquad \text{...(iii)}$$

On solving Eqs. (ii) and (iii), we get

$$\frac{a}{8+10}=\frac{-b}{-17}=\frac{c}{4}=\lambda$$

$$\Rightarrow \quad a=18\lambda, b=17\lambda, c=4\lambda$$

From Eq. (i),

$$18\,\lambda(x-2)+17\,\lambda(y-1)+4\lambda(z+1)=0$$

$\Rightarrow$ $$18x-36+17y-17+4z+4=0$$

$\Rightarrow$ $$18x+17y+4z-49=0$$

$\therefore$ $$18x+17y+4z=49$$

Q. 21 Find the shortest distance between the lines gives by

$$\vec{\mathbf{r}}=(8+3\lambda)\hat{\mathbf{i}}-(9+16\lambda)\hat{\mathbf{j}}+(10+7\lambda)\hat{\mathbf{k}}$$

and $$\vec{\mathbf{r}}=15\hat{\mathbf{i}}+29\hat{\mathbf{j}}+5\hat{\mathbf{k}}+\mu(3\hat{\mathbf{i}}+8\hat{\mathbf{j}}-5\hat{\mathbf{k}}).$$

Sol. We have, $$\vec{\mathbf{r}}=(8+3\lambda)\,\hat{\mathbf{i}}-(9+16\lambda)\hat{\mathbf{j}}+(10+7\lambda)\,\hat{\mathbf{k}})$$

$$=8\hat{\mathbf{i}}-9\hat{\mathbf{j}}+10\,\hat{\mathbf{k}}+3\lambda\,\hat{\mathbf{i}}-16\lambda\,\hat{\mathbf{j}}+7\lambda\hat{\mathbf{k}}$$

$$=8\hat{\mathbf{i}}-9\hat{\mathbf{j}}+10\,\hat{\mathbf{k}}+\lambda(3\hat{\mathbf{i}}-16\hat{\mathbf{j}}+7\hat{\mathbf{k}})$$

$\Rightarrow$ $$\vec{\mathbf{a}}_1=8\hat{\mathbf{i}}-9\hat{\mathbf{j}}+10\,\hat{\mathbf{k}}\text{ and }\vec{\mathbf{b}}_1=3\hat{\mathbf{i}}-16\,\hat{\mathbf{j}}+7\hat{\mathbf{k}} \quad \text{...(i)}$$

Also $$\vec{\mathbf{r}}=15\hat{\mathbf{i}}+29\hat{\mathbf{j}}+5\,\hat{\mathbf{k}}+\mu\,(3\hat{\mathbf{i}}+8\hat{\mathbf{j}}-5\hat{\mathbf{k}})$$

$\Rightarrow$ $$\vec{\mathbf{a}}_2=15\hat{\mathbf{i}}+29\hat{\mathbf{j}}+5\,\hat{\mathbf{k}}\text{ and }\vec{\mathbf{b}}_2=3\hat{\mathbf{i}}+8\hat{\mathbf{j}}-5\hat{\mathbf{k}} \quad \text{... (ii)}$$

Now, shortest distance betwen two lines is given by $\left|\dfrac{(\vec{\mathbf{b}}_1\times\vec{\mathbf{b}}_2)\cdot(\vec{\mathbf{a}}_2-\vec{\mathbf{a}}_1)}{|\vec{\mathbf{b}}_1\times\vec{\mathbf{b}}_2|}\right|$

$\therefore$ $$\vec{\mathbf{b}}_1\times\vec{\mathbf{b}}_2=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 3 & -16 & 7\\ 3 & 8 & -5\end{vmatrix}$$

$$=\hat{\mathbf{i}}(80-56)-\hat{\mathbf{j}}\,(-15-21)+\hat{\mathbf{k}}\,(24+48)$$

$$=24\hat{\mathbf{i}}+36\hat{\mathbf{j}}+72\hat{\mathbf{k}}$$

Now, $$|\vec{\mathbf{b}}_1\times\vec{\mathbf{b}}_2|=\sqrt{(24)^2+(36)^2+(72)^2}$$

$$=12\sqrt{2^2+3^2+6^2}=84$$

and $$(\vec{\mathbf{a}}_2-\vec{\mathbf{a}}_1)=(15-8)\hat{\mathbf{i}}+(29+9)\hat{\mathbf{j}}+(5-10)\,\hat{\mathbf{k}}$$

$$=7\hat{\mathbf{i}}+38\hat{\mathbf{j}}-5\,\hat{\mathbf{k}}$$

$\therefore$ Shortest distance $$=\left|\frac{(24\hat{\mathbf{i}}+36\hat{\mathbf{j}}+72\hat{\mathbf{k}})\cdot(7\hat{\mathbf{i}}+38\hat{\mathbf{j}}-5\,\hat{\mathbf{k}})}{84}\right|$$

$$=\left|\frac{168+1368-360}{84}\right|=\left|\frac{1176}{84}\right|=14\text{ units}$$

Q. 22 Find the equation of the plane which is perpendicular to the plane $5x+3y+6z+8=0$ and which contains the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$.

Sol. The equation of a plane through the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$ is

$$(x+2y+3z-4)+\lambda\,(2x+y-z+5)=0$$

$\Rightarrow$ $$x\,(1+2\lambda)+y\,(2+\lambda)+z\,(-\lambda+3)-4+5\lambda=0 \quad \text{...(i)}$$

Also, this is perpendicular to the plane $5x + 3y + 6z + 8 = 0$.

$\therefore$ $5(1 + 2\lambda) + 3(2 + \lambda) + 6(3 - \lambda) = 0$ $[\because a_1 a_2 + b_1b_2 + c_1c_2 = 0]$

$\Rightarrow$ $5 + 10\lambda + 6 + 3\lambda + 18 - 6\lambda = 0$

$\therefore$ $\lambda = -29/7$

From Eq. (i),

$$x\left[1 + 2\left(\frac{-29}{7}\right)\right] + y\left(2 - \frac{29}{7}\right) + z\left(\frac{29}{7} + 3\right) - 4 + 5\left(\frac{-29}{7}\right) = 0$$

$\Rightarrow$ $x(7 - 58) + y(14 - 29) + z(29 + 21) - 28 - 145 = 0$

$\Rightarrow$ $-51x - 15y + 50z - 173 = 0$

So, the required equation of plane is $51x + 15y - 50z + 173 = 0$.

Q. 23 If the plane $ax + by = 0$ is rotated about its line of intersection with the plane $z = 0$ through an angle α, then prove that the equation of the plane in its new position is $ax + by \pm (\sqrt{a^2 + b^2}\tan\alpha)\, z = 0$.

Sol. Equation of the plane is $ax + by = 0$ ... (i)

$\therefore$ Equation of the plane after new position is

$$\frac{ax\cos\alpha}{\sqrt{a^2 + b^2}} + \frac{by\cos\alpha}{\sqrt{b^2 + a^2}} \pm z\sin\alpha = 0$$

$\Rightarrow$ $$\frac{ax}{\sqrt{a^2 + b^2}} + \frac{by}{\sqrt{b^2 + a^2}} \pm z\tan\alpha = 0$$ [on dividing by $\cos\alpha$]

$\Rightarrow$ $$ax + by \pm z\tan\alpha\sqrt{\alpha^2 + b^2} = 0$$ [on multiplying with $\sqrt{a^2 + b^2}$]

Alternate Method

Given,planes are $ax + by = 0$ ...(i)

and $z = 0$ ...(ii)

Therefore, the equation of any plane passing through the line of intersection of planes (i) and (ii) may be taken as $ax + by + k = 0$. ...(iii)

Then, direction cosines of a normal to the plane (iii) are $\frac{a}{\sqrt{a^2 + b^2 + k^2}}, \frac{b}{\sqrt{a^2 + b^2 + k^2}}, \frac{c}{\sqrt{a^2 + b^2 + k^2}}$ and direction cosines of the normal to the plane (i) are $\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}$, 0.

Since, the angle between the planes (i) and (ii) is α,

$\therefore$ $$\cos\alpha = \frac{a\cdot a + b\cdot b + k\cdot 0}{\sqrt{a^2 + b^2 + k^2}\sqrt{a^2 + b^2}}$$

$$= \sqrt{\frac{a^2 + b^2}{a^2 + b^2 + k^2}}$$

$\Rightarrow$ $$k^2\cos^2\alpha = a^2(1 - \cos^2\alpha) + b^2(1 - \cos^2\alpha)$$

$\Rightarrow$ $$k^2 = \frac{(a^2 + b^2)\sin^2\alpha}{\cos^2\alpha}$$

$$k = \pm\sqrt{a^2 + b^2}\tan\alpha$$

On putting this value in plane (iii), we get the equation of the plane as

$$ax + by + z\sqrt{a^2 + b^2}\tan\alpha = 0$$

Q. 24 Find the equation of the plane through the intersection of the planes $\vec{r} \cdot (\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from origin is unity.

Sol. We have, $\vec{n}_1 = (\hat{i} + 3\hat{j}), d_1 = 6$ and $\vec{n}_2 = (3\hat{i} - \hat{j} - 4\hat{k}), d_2 = 0$

Using the relation, $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + d_2\lambda$

$\Rightarrow \quad \vec{r} \cdot [(\hat{i} + 3\hat{j}) + \lambda(3\hat{i} - \hat{j} - 4\hat{k})] = 6 + 0 \cdot \lambda$

$\Rightarrow \quad \vec{r} \cdot [(1 + 3\lambda)\hat{i} + (3 - \lambda)\hat{j} + \hat{k}(-4\lambda)] = 6 \quad \text{...(i)}$

On dviding both sides by $\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + (-4\lambda)^2}$, we get

$$\frac{\vec{r} \cdot [(1 + 3\lambda)\hat{i} + (3 - \lambda)\hat{j} + \hat{k}(-4\lambda)]}{\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + (-4\lambda)^2}} = \frac{6}{\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + (-4\lambda)^2}}$$

Since, the perpendicular distance from origin is unity.

$$\therefore \quad \frac{6}{\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + (-4\lambda)^2}} = 1$$

$\Rightarrow \quad (1 + 3\lambda)^2 + (3 - \lambda)^2 + (-4\lambda)^2 = 36$

$\Rightarrow \quad 1 + 9\lambda^2 + 6\lambda + 9 + \lambda^2 - 6\lambda + 16\lambda^2 = 36$

$\Rightarrow \quad 26\lambda^2 + 10 = 36$

$\Rightarrow \quad \lambda^2 = 1$

$\therefore \quad \lambda = \pm 1$

Using Eq. (i), the required equation of plane is

$$\vec{r} \cdot [(1 \pm 3)\hat{i} + (3 \mp 1)\hat{j} + (\mp 4)\hat{k}] = 6$$

$\Rightarrow \quad \vec{r} \cdot [(1 + 3)\hat{i} + (3 - 1)\hat{j} + (-4)\hat{k}] = 6$

and $\quad \vec{r} \cdot [(1 - 3)\hat{i} + (3 + 1)\hat{j} + 4\hat{k}] = 6$

$\Rightarrow \quad \vec{r} \cdot (4\hat{i} + 2\hat{j} - 4\hat{k}) = 6$

and $\quad \vec{r} \,.\, (-2\hat{i} + 4\hat{j} + 4\hat{k}) = 6$

$\Rightarrow \quad 4x + 2y - 4z - 6 = 0$

and $\quad -2x + 4y + 4z - 6 = 0$

Q. 25 Show that the points $(\hat{i} - \hat{j} + 3\hat{k})$ and $3(\hat{i} + \hat{j} + \hat{k})$ are equidistant from the plane $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9 = 0$ and lies on opposite side of it.

Sol. To show that these given points $(\hat{i} - \hat{j} + 3\hat{k})$ and $3(\hat{i} + \hat{j} + \hat{k})$ are equidistant from the plane $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9 = 0$, we first find out the mid- point of the points which is $2\hat{i} + \hat{j} + 3\hat{k}$.

On substituting $\vec{r}$ by the mid-point in plane, we get

$$\text{LHS} = (2\hat{i} + \hat{j} + 3\hat{k}) \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9$$
$$= 10 + 2 - 21 + 9 = 0$$
$$= \text{RHS}$$

Hence, the two points lie on opposite sides of the plane are equidistant from the plane.

Q. 26 $\vec{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\vec{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ are two vectors. The position vectors of the points *A* and *C* are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{i} + 2\hat{k}$, respectively. Find the position vector of a point *P* on the line *AB* and a point *Q* on the line *CD* such that $\vec{PQ}$ is perpendicular to $\vec{AB}$ and $\vec{CD}$ both.

Sol. We have, $\vec{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\vec{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$

Also, the position vectors of *A* and *C* are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{j} + 2\hat{k}$, respectively. Since, $\vec{PQ}$ is perpendicular to both $\vec{AB}$ and $\vec{CD}$.

So, *P* and *Q* will be foot of perpendicular to both the lines through *A* and *C*.

Now, equation of the line through A and parallel to the vector $\vec{AB}$ is,

$$\vec{r} = (6\hat{i} + 7\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - \hat{j} + \hat{k})$$

and the line through *C* and parallel to the vector $\vec{CD}$ is given by

$$\vec{r} = -9\hat{j} + 2\hat{k} + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k}) \quad \text{...(i)}$$

Let $\vec{r} = (6\hat{i} + 7\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - \hat{j} + \hat{k})$

and $\vec{r} = -9\hat{j} + 2\hat{k} + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k}) \quad \text{...(ii)}$

Let *P* $(6 + 3\lambda, 7 - \lambda, 4 + \lambda)$ is any point on the first line and *Q* be any point on second line is given by $(-3\mu, -9 + 2\mu, 2 + 4\mu)$.

$$\therefore \quad \vec{PQ} = (-3\mu - 6 - 3\lambda)\hat{i} + (-9 + 2\mu - 7 + \lambda)\hat{j} + (2 + 4\mu - 4 - \lambda)\hat{k}$$

$$= (-3\mu - 6 - 3\lambda)\hat{i} + (2\mu + \lambda - 16)\hat{j} + (4\mu - \lambda - 2)\hat{k}$$

If $\vec{PQ}$ is perpendicular to the first line, then

$$3(-3\mu - 6 - 3\lambda) - (2\mu + \lambda - 16) + (4\mu - \lambda - 2) = 0$$

$$\Rightarrow \quad -9\mu - 18 - 9\lambda - 2\mu - \lambda + 16 + 4\mu - \lambda - 2 = 0$$

$$\Rightarrow \quad -7\mu - 11\lambda - 4 = 0 \quad \text{...(iii)}$$

If $\vec{PQ}$ is perpendicular to the second line, then

$$-3(-3\mu - 6 - 3\lambda) + 2(2\mu + \lambda - 16) + 4(4\mu - \lambda - 2) = 0$$

$$\Rightarrow \quad 9\mu + 18 + 9\lambda + 4\mu + 2\lambda - 32 + 16\mu - 4\lambda - 8 = 0$$

$$\Rightarrow \quad 29\mu + 7\lambda - 22 = 0 \quad \text{...(iv)}$$

On solving Eqs. (iii) and (iv), we get

$$-49\mu - 77\lambda - 28 = 0$$

$$\Rightarrow \quad 319\mu + 77\lambda - 242 = 0$$

$$\Rightarrow \quad 270\mu - 270 = 0$$

$$\Rightarrow \quad \mu = 1$$

Using μ in Eq. (iii), we get

$$-7(1) - 11\lambda - 4 = 0$$

$$\Rightarrow \quad -7 - 11\lambda - 4 = 0$$

$$\Rightarrow \quad -11 - 11\lambda = 0$$

$$\Rightarrow \quad \lambda = -1$$

$$\therefore \quad \vec{PQ} = [-3(1) - 6 - 3(-1)]\hat{i} + [2(1) + (-1) - 16]\hat{j} + [4(1) - (-1) - 2]\hat{k}$$

$$= -6\hat{i} - 15\hat{j} + 3\hat{k}$$

Q. 27 **Show that the straight lines whose direction cosines are given by $2l + 2m - n = 0$ and $mn + nl + lm = 0$ are at right angles.**

Sol. We have, $2l + 2m - n = 0$ ...(i)

and $mn + nl + lm = 0$ ...(ii)

Eliminating m from the both equations, we get

$$m = \frac{n - 2l}{2} \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \left(\frac{n-2l}{2}\right)n + nl + l\left(\frac{n-2l}{2}\right) = 0$$

$$\Rightarrow \frac{n^2 - 2nl + 2nl + nl - 2l^2}{2} = 0$$

$$\Rightarrow n^2 + nl - 2l^2 = 0$$

$$\Rightarrow n^2 + 2nl - nl - 2l^2 = 0$$

$$\Rightarrow (n + 2l)(n - l) = 0$$

$$\Rightarrow n = -2l \text{ and } n = l$$

$$\therefore m = \frac{-2l - 2l}{2}, m = \frac{l - 2l}{2}$$

$$\Rightarrow m = -2l, m = \frac{-l}{2}$$

Thus, the direction ratios of two lines are proportional to $l, -2l, -2$ and $l, \frac{-l}{2}, l$.

$$\Rightarrow 1, -2, -2 \text{ and } 1, \frac{-1}{2}, 1$$

$$\Rightarrow 1, -2, -2 \text{ and } 2, -1, 2$$

Also, the vectors parallel to these lines are $\vec{\mathbf{a}} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, respectively.

$$\therefore \cos\theta = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}}{|\vec{\mathbf{a}}||\vec{\mathbf{b}}|} = \frac{(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) \cdot (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{3 \cdot 3}$$

$$= \frac{2 + 2 - 4}{9} = 0$$

$$\therefore \theta = \frac{\pi}{2} \quad \left[\because \cos\frac{\pi}{2} = 0\right]$$

Q. 28 **If $l_1, m_1, n_1, l_2, m_2, n_2$ and l_3, m_3, n_3 are the direction cosines of three mutually perpendicular lines, then prove that the line whose direction cosines are proportional to $l_1 + l_2 + l_3, m_1 + m_2 + m_3$ and $n_1 + n_2 + n_3$ makes equal angles with them.**

Sol. Let

$$\vec{\mathbf{a}} = l_1\hat{\mathbf{i}} + m_1\hat{\mathbf{j}} + n_1\hat{\mathbf{k}}$$

$$\vec{\mathbf{b}} = l_2\hat{\mathbf{i}} + m_2\hat{\mathbf{j}} + n_2\hat{\mathbf{k}}$$

$$\vec{\mathbf{c}} = l_3\hat{\mathbf{i}} + m_3\hat{\mathbf{j}} + n_3\hat{\mathbf{k}}$$

$$\vec{\mathbf{d}} = (l_1 + l_2 + l_3)\hat{\mathbf{i}} + (m_1 + m_2 + m_2)\hat{\mathbf{j}} + (n_1 + n_2 + n_3)\hat{\mathbf{k}}$$

Also, let α, β and γ are the angles between $\vec{\mathbf{a}}$ and $\vec{\mathbf{d}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{d}}$, $\vec{\mathbf{c}}$ and $\vec{\mathbf{d}}$.

$$\therefore \cos\alpha = l_1(l_1 + l_2 + l_3) + m_1(m_1 + m_2 + m_3) + n_1(n_1 + n_2 + n_3)$$

$$= l_1^2 + l_1 l_2 + l_1 l_3 + m_1^2 + m_1 m_2 + m_1 m_3 + n_1^2 + n_1 n_2 + n_1 n_3$$

$= (l_1^2 + m_1^2 + n_1^2) + (l_1 l_2 + l_1 l_3 + m_1 m_2 + m_1 m_3 + n_1 n_2 + n_1 n_3)$

$= 1 + 0 = 1$

$[\because l_1^2 + m_1^2 + n_1^2 = 1$ and $l_1 \perp l_2, l_1 \perp l_3, m_1 \perp m_2, m_1 \perp m_3, n_1 \perp n_2, n_1 \perp n_3]$

Similarly, $\cos\beta = l_2 (l_1 + l_2 + l_3) + m_2 (m_1 + m_2 + m_3) + n_2 (n_1 + n_2 + n_3)$

$= 1 + 0$ and $\cos \gamma = 1 + 0$

$\Rightarrow \quad \cos \alpha = \cos \beta = \cos \gamma$

$\Rightarrow \quad \alpha = \beta = \gamma$

So, the line whose direction cosines are proportional to $l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3$ makes equal angles with the three mutually perpendicular lines whose direction cosines are $l_1, m_1, n_1, l_2, m_2, n_2$ and l_3, m_3, n_3 respectively.

Objective Type Questions

Q. 29 Distance of the point (α, β, γ) from Y-axis is

(a) β (b) $|\beta|$

(c) $|\beta| + |\gamma|$ (d) $\sqrt{\alpha^2 + \gamma^2}$

Sol. *(d)* Required distance $= \sqrt{(\alpha - 0)^2 + (\beta - \beta)^2 + (\gamma - 0)^2} = \sqrt{\alpha^2 + \gamma^2}$

Q. 30 If the direction cosines of a line are k, k and k, then

(a) $k > 0$ (b) $0 < k < 1$

(c) $k = 1$ (d) $k = \frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$

Sol. *(d)* Since, direction cosines of a line are k, k and k.

$\therefore \quad l = k, m = k$ and $n = k$

We know that, $\quad l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad k^2 + k^2 + k^2 = 1$

$\Rightarrow \quad k^2 = \frac{1}{3}$

$\therefore \quad k = \pm \frac{1}{\sqrt{3}}$

Q. 31 The distance of the plane $\vec{r}\left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right) = 1$ from the origin is

(a) 1 (b) 7

(c) $\frac{1}{7}$ (d) None of these

Sol. *(a)* The distance of the plane $\vec{r}\left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right) = 1$ from the origin is 1.

[since, $\vec{r} \cdot \vec{n} = d$ is the form of above equation, where d represents the distance of plane from the origin *i.e.*, $d = 1$]

Q. 32 The sine of the angle between the straight line $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ and the plane $2x-2y+z=5$ is

(a) $\frac{10}{6\sqrt{5}}$ (b) $\frac{4}{5\sqrt{2}}$ (c) $\frac{2\sqrt{3}}{5}$ (d) $\frac{\sqrt{2}}{10}$

Sol. *(d)* We have, the equation of line as

$$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$$

Now, the line passes through point (2, 3, 4) and having direction ratios (3, 4, 5).
Since, the line passes through point (2, 3, 4) and parallel to the vector $(3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+5\hat{\mathbf{k}})$.

$$\therefore \quad \vec{\mathbf{b}}=3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+5\hat{\mathbf{k}}$$

Also, the cartesian form of the given plane is $2x-2y+z=5$.

$$\Rightarrow \quad (x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})=5$$

$$\therefore \quad \vec{\mathbf{n}}=(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})$$

We know that, $$\sin\theta=\frac{|\vec{\mathbf{b}}\cdot\vec{\mathbf{n}}|}{|\vec{\mathbf{b}}|\cdot|\vec{\mathbf{n}}|}=\frac{\left|(3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+5\hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}}-2\hat{\mathbf{j}}+\hat{\mathbf{k}})\right|}{\sqrt{3^2+4^2+5^2}\cdot\sqrt{4+4+1}}$$

$$=\frac{|6-8+5|}{\sqrt{50}\cdot 3}=\frac{3}{15\sqrt{2}}=\frac{1}{5\sqrt{2}}$$

$$\sin\theta=\frac{\sqrt{2}}{10}$$

Q. 33 The reflection of the point (α,β,γ) in the XY-plane is

(a) $(\alpha,\beta,0)$ (b) $(0,0,\gamma)$ (c) $(-\alpha,-\beta,\gamma)$ (d) $(\alpha,\beta,-\gamma)$

Sol. *(d)* In XY-plane, the reflection of the point (α,β,γ) is $(\alpha,\beta,-\gamma)$.

Q. 34 The area of the quadrilateral $ABCD$ where A (0,4,1), B (2, 3, −1), C (4, 5, 0), and D (2, 6, 2) is equal to

(a) 9 sq units (b) 18 sq units
(c) 27 sq units (d) 81 sq units

Sol. *(a)* We have,

$$\vec{\mathbf{AB}}=(2-0)\hat{\mathbf{i}}+(3-4)\hat{\mathbf{j}}+(-1-1)\hat{\mathbf{k}}=2\hat{\mathbf{i}}-\hat{\mathbf{j}}-2\hat{\mathbf{k}}$$

$$\vec{\mathbf{BC}}=(4-2)\hat{\mathbf{i}}+(5-3)\hat{\mathbf{j}}+(0+1)\hat{\mathbf{k}}=2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$$

$$\vec{\mathbf{CD}}=(2-4)\hat{\mathbf{i}}+(6-5)\hat{\mathbf{j}}+(2-0)\hat{\mathbf{k}}=-2\hat{\mathbf{i}}+\hat{\mathbf{j}}+2\hat{\mathbf{k}}$$

$$\vec{\mathbf{DA}}=(0-2)\hat{\mathbf{i}}+(4-6)\hat{\mathbf{j}}+(1-2)\hat{\mathbf{k}}=-2\hat{\mathbf{i}}-2\hat{\mathbf{j}}-\hat{\mathbf{k}}$$

$$\therefore \text{ Area of quadrilateral } ABCD=|\vec{\mathbf{AB}}\times\vec{\mathbf{BC}}|=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & -1 & -2\\ 2 & 2 & 1\end{vmatrix}$$

$$=\left|\hat{\mathbf{i}}(-1+4)-\hat{\mathbf{j}}(2+4)+\hat{\mathbf{k}}(4+2)\right|$$

$$=\left|3\hat{\mathbf{i}}-6\hat{\mathbf{j}}+6\hat{\mathbf{k}}\right|$$

$$=\sqrt{9+36+36}=9 \text{ sq units}$$

Q. 35 The locus represented by $xy + yz = 0$ is

(a) a pair of perpendicular lines
(b) a pair of parallel lines
(c) a pair of parallel planes
(d) a pair of perpendicular planes

Sol. *(d)* We have, $xy + yz = 0$

$\Rightarrow \quad xy = -yz$

So, a pair of perpendicular planes.

Q. 36 If the plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1} \alpha$ with X-axis, then the value of α is

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{2}}{3}$

(c) $\frac{2}{7}$ (d) $\frac{3}{7}$

Sol. *(c)* Since, $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1} \alpha$ with X-axis.

$$\vec{\mathbf{b}} = (1\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}}) \text{ and } \vec{\mathbf{n}} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$$

We know that, $$\sin \theta = \frac{|\vec{\mathbf{b}} \cdot \vec{\mathbf{n}}|}{|\vec{\mathbf{b}}| \cdot |\vec{\mathbf{n}}|}$$

$$= \frac{|(1\hat{\mathbf{i}}) \cdot (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})|}{\sqrt{1}\sqrt{4 + 9 + 36}} = \frac{2}{7}$$

Fillers

Q. 37 If a plane passes through the points (2, 0,0) (0, 3, 0) and (0, 0, 4) the equation of plane is ……… .

Sol. We know that, equation of a the plane that cut the coordinate axes at $(a, 0, 0)$ $(0, b, 0)$ and $(0, 0, c)$ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Hence, the equation of plane passes through the points (2, 0, 0), (0, 3, 0) and (0, 0, 4) is $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$.

Q. 38 The direction cosines of the vector $(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$ are ……… .

Sol. Direction cosines of $(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$ are $\frac{2}{\sqrt{4 + 4 + 1}}, \frac{2}{\sqrt{4 + 4 + 1}}, \frac{-1}{\sqrt{4 + 4 + 1}}$ *i.e.*, $\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$.

Q. 39 The vector equation of the line $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$ is ....

Sol. We have, $\vec{\mathbf{a}} = 5\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = 3\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

So, the vector equation will be

$$\vec{\mathbf{r}} = (5\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) + \lambda(3\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

$$\Rightarrow \quad (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) - (5\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) = \lambda(3\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

$$\Rightarrow \quad (x-5)\hat{\mathbf{i}} + (y+4)\hat{\mathbf{j}} + (z-6)\hat{\mathbf{k}} = \lambda(3\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

Q. 40 The vector equation of the line through the points (3, 4, −7) and (1, −1, 6) is ....

Sol. We know that, vector equation of a line passes through two points is represented by $\vec{\mathbf{r}} = \vec{\mathbf{a}} + \lambda(\vec{\mathbf{b}} - \vec{\mathbf{a}})$

Here, $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 3\hat{\mathbf{k}}, \vec{\mathbf{a}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$

and $\vec{\mathbf{b}} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

$$\Rightarrow \quad (\vec{\mathbf{b}} - \vec{\mathbf{a}}) = -2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 13\hat{\mathbf{k}}$$

So, the required equation is

$$x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 7\hat{\mathbf{k}} + \lambda(-2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 13\hat{\mathbf{k}})$$

$$\Rightarrow \quad (x-3)\hat{\mathbf{i}} + (y-4)\hat{\mathbf{j}} + (z+7)\hat{\mathbf{k}} = \lambda(-2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 13\hat{\mathbf{k}})$$

Q. 41 The cartesian equation of the plane $\vec{\mathbf{r}} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}) = 2$ is ....

Sol. We have, $\vec{\mathbf{r}} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}) = 2$

$$\Rightarrow \quad (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}) = 2$$

$$\Rightarrow \quad x + y - z = 2$$

which is the required form

True/False

Q. 42 The unit vector normal to the plane $x + 2y + 3z - 6 = 0$ is

$$\frac{1}{\sqrt{14}}\hat{\mathbf{i}} + \frac{2}{\sqrt{14}}\hat{\mathbf{j}} + \frac{3}{\sqrt{14}}\hat{\mathbf{k}}.$$

Sol. ***True***

We have, $\vec{\mathbf{n}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

$$\therefore \quad \hat{\mathbf{n}} = \frac{\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{\hat{\mathbf{i}}}{\sqrt{14}} + \frac{2\hat{\mathbf{j}}}{\sqrt{14}} + \frac{3\hat{\mathbf{k}}}{\sqrt{14}}$$

Q. 43 The intercepts made by the plane $2x - 3y + 5z + 4 = 0$ on the coordinate axis are $-2, \frac{4}{3}$ and $-\frac{4}{5}$.

Sol. ***True***

We have, $2x - 3y + 5z + 4 = 0$

$\Rightarrow \quad 2x - 3y + 5z = -4$

$\Rightarrow \quad \frac{2x}{-4} - \frac{3y}{-4} + \frac{5z}{-4} = 1$

$\Rightarrow \quad \frac{x}{-2} + \frac{y}{\frac{4}{3}} - \frac{z}{\frac{4}{5}} = 1$

$\Rightarrow \quad \frac{x}{-2} + \frac{y}{\frac{4}{3}} + \frac{z}{\left(-\frac{4}{5}\right)} = 1$

So, the intercepts are $-2, \frac{4}{3}$ and $-\frac{4}{5}$.

Q. 44 The angle between the line $\vec{r} = (5\hat{i} - \hat{j} - 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and the plane $\vec{r}\,(3\hat{i} - 4\hat{j} - \hat{k}) + 5 = 0$ is $\sin^{-1}\left(\frac{5}{2\sqrt{91}}\right)$.

Sol. ***False***

We have, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{n} = 3\hat{i} - 4\hat{j} - \hat{k}$

Let θ is the angle between line and plane.

Then, $\sin\theta = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}| \cdot |\vec{n}|} = \frac{|(2\hat{i} - \hat{j} + \hat{k}) \cdot (3\hat{i} - 4\hat{j} - \hat{k})|}{\sqrt{6} \cdot \sqrt{26}}$

$= \frac{|6 + 4 - 1|}{\sqrt{156}} = \frac{9}{2\sqrt{39}}$

$\therefore \quad \theta = \sin^{-1} \frac{9}{2\sqrt{39}}$

Q. 45 The angle between the planes $\vec{r}\,(2\hat{i} - 3\hat{j} + \hat{k}) = 1$ and $\vec{r}\,(\hat{i} - \hat{j}) = 4$ is $\cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$.

Sol. ***False***

We know that, the angle between two planes is given by $\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}|\,|\vec{n_2}|}$

Here, $\vec{n_1} = (2\hat{i} - 3\hat{j} + \hat{k})$ and $\vec{n_2} = (\hat{i} - \hat{j})$

$\therefore \quad \cos\theta = \frac{|(2\hat{i} - 3\hat{j} + \hat{k})\ (\hat{i} - \hat{j})|}{\sqrt{4 + 9 + 1}\ \sqrt{1 + 1}}$

$\Rightarrow \quad \cos\theta = \frac{|2 + 3|}{\sqrt{14} \cdot \sqrt{2}} = \frac{5}{2\sqrt{7}}$

$\therefore \quad \theta = \cos^{-1}\left(\frac{5}{2\sqrt{7}}\right)$

Q. 46 The line $\vec{r} = 2\hat{i} - 3\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$ lies in the plane $\vec{r}(3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$.

Sol. ***False***

We have, $\vec{r} = 2\hat{i} - 3\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$

$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) = \hat{i}(2+\lambda) + \hat{j}(-3-\lambda) + \hat{k}(-1+2\lambda)$

Since, $x = (2+\lambda)$, $y = (-3-\lambda)$ and $z = (-1+2\lambda)$ are coordinates of general point which should satisfy the equation of the given plane.

$\therefore \quad [(2+\lambda)\hat{i} + (-3-\lambda)\hat{j} + (2\lambda-1)\hat{k}]\cdot[\hat{i} + \hat{j} - \hat{k}] = 2$

$\Rightarrow \quad (2+\lambda) - 3 - \lambda - 2\lambda + 1 = 2$

$\Rightarrow \quad -2\lambda = 2$

$\Rightarrow \quad \lambda = -1$

$\therefore \quad \vec{r} = (2-1)\hat{i} + (-3+1)\hat{j} + (-2-1)\hat{k}$

$= \hat{i} - 2\hat{j} - 3\hat{k}$

Again, from the equation of the plane

$\vec{r}\cdot(3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$

$\Rightarrow \quad (\hat{i} - 2\hat{j} - 3\hat{k})\ (3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$

$\Rightarrow \quad (3 - 2 + 3) + 2 = 0$

$\Rightarrow \quad 6 \neq 0$

which is not true.

So, the line $\vec{r} = 2\hat{i} - 3\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$ does not lie in a plane.

Q. 47 The vector equation of the line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$ is

$$\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$$

Sol. ***True***

We have, $x = 5,\ y = -4,\ z = 6$

and $a = 3,\ b = 7,\ c = 2$

$\therefore \quad \vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

Q. 48 The equation of a line, which is parallel to $2\hat{i} + \hat{j} + 3\hat{k}$ and which passes through the point $(5, -2, 4)$ is $\frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3}$.

Sol. ***False***

Here, $x_1 = 5,\ y_1 = -2,\ z_1 = 4$

and $a = 2,\ b = 1,\ c = 3$

$$\Rightarrow \quad \frac{x-5}{2} = \frac{y+2}{1} = \frac{z-4}{3}$$

Q. 49 If the foot of perpendicular drawn from the origin to a plane is $(5, -3, -2)$, then the equation of plane is $\vec{\mathbf{r}}\ (5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) = 38$.

Sol. ***True***

Since, the required plane passes through the point $P(5, -3, -2)$ and is perpendicular to $\overrightarrow{\mathbf{OP}}$.

$$\therefore \qquad \vec{\mathbf{a}} = 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

$$\text{and} \qquad \vec{\mathbf{n}} = \overrightarrow{\mathbf{OP}} = 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

Now, the equation of the plane is

$$(\vec{\mathbf{r}} - \vec{\mathbf{a}}) \cdot \vec{\mathbf{n}} = 0$$

$$\Rightarrow \qquad \vec{\mathbf{r}} \cdot \vec{\mathbf{n}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{n}}$$

$$\Rightarrow \qquad \vec{\mathbf{r}} \cdot (5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) = (5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) \cdot (5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$$

$$\Rightarrow \qquad \vec{\mathbf{r}} . (5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) = 25 + 9 + 4$$

$$\Rightarrow \qquad \vec{\mathbf{r}} \cdot (5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) = 38$$