4

Determinants

Short Answer Type Questions

Q. 1 $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

Sol. We have, $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = \begin{vmatrix} x^2 - 2x + 2 & x - 1 \\ 0 & x + 1 \end{vmatrix}$ $[\because C_1 \to C_1 - C_2]$

$= (x^2 - 2x + 2) \cdot (x + 1) - (x - 1) \cdot 0$

$= x^3 - 2x^2 + 2x + x^2 - 2x + 2$

$= x^3 - x^2 + 2$

Q. 2 $\begin{vmatrix} a + x & y & z \\ x & a + y & z \\ x & y & a + z \end{vmatrix}$

Sol. We have, $\begin{vmatrix} a + x & y & z \\ x & a + y & z \\ x & y & a + z \end{vmatrix} = \begin{vmatrix} a & -a & 0 \\ 0 & a & -a \\ x & y & a + z \end{vmatrix}$ $\begin{bmatrix} \because R_1 \to R_1 - R_2 \\ \text{and } R_2 \to R_2 - R_3 \end{bmatrix}$

$= \begin{vmatrix} a & 0 & 0 \\ 0 & a & -a \\ x & x + y & a + z \end{vmatrix}$ $[\because C_2 \to C_2 + C_1]$

$= a (a^2 + az + ax + ay)$

$= a^2 (a + z + x + y)$

Q. 3 $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$

Sol. We have, $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix} = x^2y^2z^2 \begin{vmatrix} 0 & x & x \\ y & 0 & y \\ z & z & 0 \end{vmatrix}$

[taking x^2, y^2 and z^2 common from C_1, C_2 and C_3, respectively]

$$= x^2y^2z^2 \begin{vmatrix} 0 & 0 & x \\ y & -y & y \\ z & z & 0 \end{vmatrix} \qquad [\because C_2 \to C_2 - C_3]$$

$$= x^2y^2z^2 \, [x\,(yz + yz)]$$

$$= x^2y^2z^2 \cdot 2xyz = 2x^3y^3z^3$$

Q. 4 $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

Sol. We have, $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

Applying, $C_1 \to C_1 + C_2 + C_3$,

$$= \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$

$$= (x+y+z) \begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$

[taking $(x+y+z)$ common from column C_1]

$$= (x+y+z) \begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & x-y \\ 0 & x-z & 2z+x \end{vmatrix}$$

$[\because R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1]$

Now, expanding along first column, we get

$$(x+y+z) \cdot 1 \, [(2y+x)(2z+x) - (x-y)(x-z)]$$

$$= (x+y+z)(4yz + 2yx + 2xz + x^2 - x^2 + xz + yx - yz)$$

$$= (x+y+z)(3yz + 3yx + 3xz)$$

$$= 3\,(x+y+z)(yz + yx + xz)$$

Q. 5 $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$

Sol. We have, $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = \begin{vmatrix} 2x+4 & 2x+4 & 2x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$ $[\because R_1 \to R_1 + R_2]$

$$= \begin{vmatrix} 2x & 2x & 2x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} + \begin{vmatrix} 4 & 4 & 0 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

[here, given determinant is expressed in sum of two determinants]

$$= 2x \begin{vmatrix} 1 & 1 & 1 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} + 4 \begin{vmatrix} 1 & 1 & 0 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

[taking $2x$ common from first row of first determinant and 4 from first row of second determinant]

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$ in first and applying $C_1 \to C_1 - C_2$ in second, we get

$$= 2x \begin{vmatrix} 0 & 0 & 1 \\ 0 & 4 & x \\ -4 & -4 & x+4 \end{vmatrix} + 4 \begin{vmatrix} 0 & 1 & 0 \\ -4 & x+4 & x \\ 0 & x & x+4 \end{vmatrix}$$

Expanding both the along first column, we get

$$2x[-4(-4)] + 4[4(x+4-0)]$$
$$= 2x \times 16 + 16(x+4)$$
$$= 32x + 16x + 64$$
$$= 16(3x+4)$$

Q. 6 $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Sol. Wehave, $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$ $[\because R_1 \to R_1 + R_2 + R_3]$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

[taking $(a+b+c)$common from the first row]

$$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & -(a+b+c) & 2b \\ (a+b+c) & (a+b+c) & (c-a-b) \end{vmatrix}$$

$[\because C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3]$

Expanding along R_1,

$$= (a+b+c)\,[1\{0+(a+b+c^2\}]$$
$$= (a+b+c)\,[(a+b+c)^2]$$
$$= (a+b+c)^3$$

Q. 7 $\begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix} = 0$

Sol. We have to prove,

$$\begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix} = 0$$

$$\therefore \quad \text{LHS} = \begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} xy^2z^2 & xyz & xy+xz \\ x^2yz^2 & xyz & yz+xy \\ x^2y^2z & xyz & xz+yz \end{vmatrix}$$

$[\because R_1 \to x R_1, R_2 \to y R_2, R_3 \to z R_3]$

$$= \frac{1}{xyz}(xyz)^2 \begin{vmatrix} yz & 1 & xy+xz \\ xz & 1 & yz+xy \\ xy & 1 & xz+yz \end{vmatrix}$$

[taking (xyz) common from C_1 and C_2]

$$= xyz \begin{vmatrix} yz & 1 & xy+yz+zx \\ xz & 1 & xy+yz+zx \\ xy & 1 & xy+yz+zx \end{vmatrix} \quad [C_3 \to C_3 + C_1]$$

$$= xyz\,(xy+yz+zx) \begin{vmatrix} yz & 1 & 1 \\ xz & 1 & 1 \\ xy & 1 & 1 \end{vmatrix}$$

[taking $(xy+yz+zx)$ common from C_3]

$= 0$ [since, C_2 and C_3 are identicals]

$=$ RHS **Hence proved.**

Q. 8 $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$

Thinking Process

First in LHS use $C_1 \to C_1 + C_2 + C_3$ and then by using $C_1 \to C_1 - C_2$ and $R_1 \to R_1 - R_3$, we can get two zeroes in column 1 and then by simplification we will get the desired result.

Sol. We have to prove,

$$\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$$

$$\therefore \quad \text{LHS} = \begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} y+z+z+y & z & y \\ z+z+x+x & z+x & x \\ y+x+x+y & x & x+y \end{vmatrix} \quad [\because C_1 \to C_1 + C_2 + C_3]$$

$$= 2\begin{vmatrix} (y+z) & z & y \\ (z+x) & z+x & x \\ (x+y) & x & x+y \end{vmatrix} \quad [\text{taking 2 common from } C_1]$$

$$= 2\begin{vmatrix} y & z & y \\ 0 & z+x & x \\ y & x & x+y \end{vmatrix} \quad [\because C_1 \to C_1 - C_2]$$

$$= 2\begin{vmatrix} 0 & z-x & -x \\ 0 & z+x & x \\ y & x & x+y \end{vmatrix} \quad [\because R_1 \to R_1 - R_3]$$

$$= 2\,[y(xz - x^2 + xz + x^2)]$$

$$= 4xyz = \text{RHS}$$

Hence proved.

Q. 9 $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$

Thinking Process

Here, by using $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ in LHS, we can easily get the desired result.

Sol. We have to prove,

$$= \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

$$\therefore \quad \text{LHS} = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+2a-2a-1 & 2a+1-a-2 & 0 \\ 2a+1-3 & a+2-3 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$[\because R_1 \to R_1 - R_2 \text{ and } R_2 \to R_2 - R_3]$

$$= \begin{vmatrix} (a-1)(a+1) & (a-1) & 0 \\ 2(a-1) & (a-1) & 0 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^2 \begin{vmatrix} (a+1) & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

[taking $(a-1)$ common from R_1 and R_2 each]

$$= (a-1)^2[1(a+1) - 2] = (a-1)^3$$

$$= \text{RHS}$$

Hence proved.

Q. 10 If $A+B+C=0$, then prove that $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0.$

Thinking Process

We have, given $A+B+C=0$, so on solving the determinant by expansion, we can use $\cos(A+B)=\cos(-C)$ and similarly after simplification this expansion we will get the desired result.

Sol. We have to prove, $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$

$$\therefore \quad \text{LHS} = \begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix}$$

$$= 1(1-\cos^2 A) - \cos C(\cos C - \cos A \cdot \cos B) + \cos B(\cos C \cdot \cos A - \cos B)$$

$$= \sin^2 A - \cos^2 C + \cos A \cdot \cos B \cdot \cos C + \cos A \cdot \cos B \cdot \cos C - \cos^2 B$$

$$= \sin^2 A - \cos^2 B + 2\cos A \cdot \cos B \cdot \cos C - \cos^2 C$$

$$= -\cos(A+B) \cdot \cos(A-B) + 2\cos A \cdot \cos B \cdot \cos C - \cos^2 C$$

$$[\because \cos^2 B - \sin^2 A = \cos(A+B) \cdot \cos(A-B)]$$

$$= -\cos(-C) \cdot \cos(A-B) + \cos C(2\cos A \cdot \cos B - \cos C) \quad [\because \cos(-\theta) = \cos\theta]$$

$$= -\cos C(\cos A \cdot \cos B + \sin A \cdot \sin B - 2\cos A \cdot \cos B + \cos C)$$

$$= \cos C(\cos A \cdot \cos B - \sin A \cdot \sin B - \cos C)$$

$$= \cos C[\cos(A+B) - \cos C]$$

$$= \cos C(\cos C - \cos C) = 0 = \text{RHS}$$

Hence proved.

Q. 11 If the coordinates of the vertices of an equilateral triangle with sides of length 'a' are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}.$$

Sol. Since, we know that area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3), is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta^2 = \frac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 \quad \text{...(i)}$$

We know that, area of an equilateral triangle with side a,

$$\Delta = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)a^2 = \frac{\sqrt{3}}{4}a^2$$

$$\Rightarrow \quad \Delta^2 = \frac{3}{16}a^4 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $\frac{3}{16}a^4 = \frac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$

$\Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3}{4}a^4$ **Hence proved**.

Q. 12 Find the value of θ satisfying $\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0$

Sol. We have, $\begin{vmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 0 & 1 & \sin 3\theta \\ -7 & 3 & \cos 2\theta \\ 14 & -7 & -2 \end{vmatrix} = 0$ $[\because C_1 \to C_1 - C_2]$

$\Rightarrow 7\begin{vmatrix} 0 & 1 & \sin 3\theta \\ -1 & 3 & \cos 2\theta \\ 2 & -7 & -2 \end{vmatrix} = 0$ [taking 7 common from C_1]

$\Rightarrow 7[0 - 1(2 - 2\cos 2\theta) + \sin 3\theta(7 - 6)] = 0$ [expanding along R_1]

$\Rightarrow 7[-2(1 - \cos 2\theta) + \sin 3\theta] = 0$

$\Rightarrow -14 + 14\cos 2\theta + 7\sin 3\theta = 0$

$\Rightarrow 14\cos 2\theta + 7\sin 3\theta = 14$

$\Rightarrow 14(1 - 2\sin^2\theta) + 7(3\sin\theta - 4\sin^3\theta) = 14$

$\Rightarrow -28\sin^2\theta + 14 + 21\sin\theta - 28\sin^3\theta = 14$

$\Rightarrow -28\sin^2\theta - 28\sin^3\theta + 21\sin\theta = 0$

$\Rightarrow 28\sin^3\theta + 28\sin^2\theta - 21\sin\theta = 0$

$\Rightarrow 4\sin^3\theta + 4\sin^2\theta - 3\sin\theta = 0$

$\Rightarrow \sin\theta(4\sin^2\theta + 4\sin\theta - 3) = 0$

$\Rightarrow$ Either $\sin\theta = 0$,

$\Rightarrow \theta = n\pi$ or $4\sin^2\theta + 4\sin\theta - 3 = 0$

$\therefore \sin\theta = \frac{-4 \pm \sqrt{16 + 48}}{8} = \frac{-4 \pm \sqrt{64}}{8}$

$= \frac{-4 \pm 8}{8} = \frac{4}{8}, \frac{-12}{8}$

$\sin\theta = \frac{1}{2}, \frac{-3}{2}$

If $\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$, then

$\theta = n\pi + (-1)^n \frac{\pi}{6}$

Hence, $\sin\theta = \frac{-3}{2}$ [not possible because $-1 \le \sin\theta \le 1$]

Q. 13 If $\begin{bmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{bmatrix} = 0$, then find the value of x.

Sol. Given, $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 12+x & 12+x & 12+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$ $[\because R_1 \to R_1 + R_2 + R_3]$

$\Rightarrow (12+x)\begin{vmatrix} 1 & 1 & 1 \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$ [taking $(12+x)$ common from R_1]

$\Rightarrow (12+x)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 8 & 4+x \\ 2x & 8 & 4-x \end{vmatrix} = 0$ $[\because C_1 \to C_1 - C_3$ and $C_2 \to C_2 + C_3]$

$\Rightarrow (12+x)[1\cdot(-16x)] = 0$

$\Rightarrow (12+x)(-16x) = 0$

$\therefore x = -12, 0$

Q. 14 If $a_1, a_2, a_3, \dots, a_r$ are in GP, then prove that the determinant $\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$ is independent of r.

Thinking Process

We know that, nth term of a GP has value ar^{n-1}, where a = first term and r = common ratio. So, by using this result, we can prove the given determinant as independent of r.

Sol. We know that, $a_{r+1} = AR^{(r+1)-1} = AR^r$

where r = r th term of a GP, A = First term of a GP and R = Common ratio of GP

We have, $\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$

$= \begin{vmatrix} AR^r & AR^{r+4} & AR^{r+8} \\ AR^{r+6} & AR^{r+10} & AR^{r+14} \\ AR^{r+10} & AR^{r+16} & AR^{r+20} \end{vmatrix}$

$= AR^r \cdot AR^{r+6} \cdot AR^{r+10} \begin{vmatrix} 1 & AR^4 & AR^8 \\ 1 & AR^4 & AR^8 \\ 1 & AR^6 & AR^{10} \end{vmatrix}$

[taking AR^r, AR^{r+6} and AR^{r+10} common from R_1, R_2 and R_3, respectively]

$= 0$ [since, R_1 and R_2 are identicals]

Q. 15 Show that the points $(a+5, a-4)$, $(a-2, a+3)$ and (a, a) do not lie on a straight line for any value of a.

Thinking Process

We know that, if three points lie in a straight line, then area formed by these points will be equal to zero. So, by showing area formed by these points other than zero, we can prove the result.

Sol. Given, the points are $(a+5, a-4)$, $(a-2, a+3)$ and (a, a).

$$\therefore \quad \Delta = \frac{1}{2}\begin{vmatrix} a+5 & a-4 & 1 \\ a-2 & a+3 & 1 \\ a & a & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 5 & -4 & 0 \\ -2 & 3 & 0 \\ a & a & 1 \end{vmatrix} \quad [\because R_1 \to R_1 - R_3 \text{ and } R_2 \to R_2 - R_3]$$

$$= \frac{1}{2}[1(15-8)]$$

$$\Rightarrow \quad = \frac{7}{2} \neq 0$$

Hence, given points form a triangle *i.e.,* points do not lie in a straight line.

Q. 16 Show that ΔABC is an isosceles triangle, if the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix} = 0.$$

Sol. We have, $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix} = 0$

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos C & \cos B - \cos C & 1+\cos C \\ \cos^2 A + \cos A - \cos^2 C - \cos C & \cos^2 B + \cos B - \cos^2 C - \cos C & \cos^2 C + \cos C \end{vmatrix} = 0$$

$[\therefore \quad C_1 \to C_1 - C_3 \text{ and } C_2 \to C_2 - C_3]$

$$\Rightarrow (\cos A - \cos C)\cdot(\cos B - \cos C)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1+\cos C \\ \cos A + \cos C + 1 & \cos B + \cos C + 1 & \cos^2 C + \cos C \end{vmatrix} = 0$$

[taking $(\cos A - \cos C)$ common from C_1 and $(\cos B - \cos C)$ common from C_2]

$$\Rightarrow (\cos A - \cos C)\cdot(\cos B - \cos C)[(\cos B + \cos C + 1) - (\cos A + \cos C + 1)] = 0$$

$$\Rightarrow (\cos A - \cos C)\cdot(\cos B - \cos C)(\cos B + \cos C + 1 - \cos A - \cos C - 1) = 0$$

$$\Rightarrow (\cos A - \cos C)\cdot(\cos B - \cos C)(\cos B - \cos A) = 0$$

i.e., $\cos A = \cos C$ or $\cos B = \cos C$ or $\cos B = \cos A$

$\Rightarrow$ $A = C$ or $B = C$ or $B = A$

Hence, ABC is an isosceles triangle.

Q. 17 Find A^{-1}, if $A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ and show that $A^{-1} = \dfrac{A^2 - 3I}{2}$.

Sol. We have, $A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$\therefore \quad A_{11} = -1, A_{12} = 1, A_{13} = 1, A_{21} = 1, A_{22} = -1, A_{23} = 1, A_{31} = 1, A_{32} = 1$ and $A_{33} = -1$

$$\therefore \quad \text{adj } A = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}^T = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

and $\quad |A| = -1(-1) + 1 \cdot 1 = 2$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{...(i)}$$

$$\text{and} \quad A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{...(ii)}$$

$$\therefore \quad \frac{A^2 - 3I}{2} = \frac{1}{2}\left\{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix}\right\} = \frac{1}{2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= A^{-1}$ [using Eq. (i)]

Hence proved.

Long Answer Type Questions

Q. 18 If $A = \begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{vmatrix}$, then find the value of A^{-1}.

Using A^{-1}, solve the system of linear equations $x - 2y = 10$, $2x - y - z = 8$ and $-2y + z = 7$.

Sol. We have, $A = \begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{vmatrix}$ ...(i)

$\therefore \quad |A| = 1(-3) - 2(-2) + 0 = 1 \neq 0$

Now, $A_{11} = -3, A_{12} = 2, A_{13} = 2, A_{21} = -2, A_{22} = 1, A_{23} = 1, A_{31} = -4, A_{32} = 2$ and $A_{33} = 3$

$$\therefore \quad \text{adj }(A) = \begin{vmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{vmatrix}^T = \begin{vmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|}$$

$$= \frac{1}{1}\begin{vmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

$$\Rightarrow \quad A^{-1} = \begin{vmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix} \quad \text{...(ii)}$$

Also, we have the system of linear equations as

$$x - 2y = 10,$$
$$2x - y - z = 8$$
and $$-2y + z = 7$$

In the form of $CX = D$,

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

where, $C = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$

We know that, $(A^T)^{-1} = (A^{-1})^T$

$$\therefore \quad C^T = \begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{vmatrix} = A \quad \text{[using Eq. (i)]}$$

$$\therefore \quad X = C^{-1} D$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}\begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -30 + 16 + 14 \\ -20 + 8 + 7 \\ -40 + 16 + 21 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}$$

$$\therefore \quad x = 0, y = -5 \text{ and } z = -3$$

Q. 19 Using matrix method, solve the system of equations $3x + 2y - 2z = 3$, $x + 2y + 3z = 6$ and $2x - y + z = 2$.

Thinking Process

We know that, for given system of equations in the matrix form, we get $AX = B \Rightarrow X = A^{-1}B$, where $A^{-1} = \frac{\text{adj}(A)}{|A|}$ and then by getting inverse of A and determinant of A, we can get the desired result.

Sol. Given system of equations is

$$3x + 2y - 2z = 3,$$
$$x + 2y + 3z = 6$$
and $$2x - y + z = 2$$

In the form of $AX = B$,

$$\begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

For A^{-1}, $\quad |A| = |3(5) - 2(1-6) + (-2)(-5)|$

$$= |15 + 10 + 10| = |35| \neq 0$$

$\therefore \quad A_{11} = 5, A_{12} = 5, A_{13} = -5, \quad A_{21} = 0, A_{22} = 7, A_{23} = 7, \quad A_{31} = 10, A_{32} = -11$ and $A_{33} = 4$

$$\therefore \quad \text{adj } A = \begin{vmatrix} 5 & 5 & -5 \\ 0 & 7 & 7 \\ 10 & -11 & 4 \end{vmatrix}^T = \begin{vmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{vmatrix}$$

Now, $$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{35}\begin{vmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{vmatrix}$$

For $X = A^{-1}B$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} 15 + 20 \\ 15 + 42 - 22 \\ -15 + 42 + 8 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 35 \\ 35 \\ 35 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore \quad x = 1, y = 1$ and $z = 1$

Q. 20 If $A = \begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix}$ and $B = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$, then find BA and use this to solve the system of equations $y + 2z = 7$, $x - y = 3$ and $2x + 3y + 4z = 17$.

Sol. We have, $$A = \begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix} \text{ and } B = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$$

$$\therefore \quad BA = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}\begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 6I$$

$$\therefore \quad B^{-1} = \frac{A}{6} = \frac{1}{6}A = \frac{1}{6}\begin{vmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{vmatrix} \quad \text{...(i)}$$

Also, $\quad x - y = 3, 2x + 3y + 4z = 17$ and $y + 2z = 7$

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} \qquad \text{[using Eq. (i)]}$$

$$= \frac{1}{6} \begin{bmatrix} 6 + 34 - 28 \\ -12 + 34 - 28 \\ 6 - 17 + 35 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$\therefore \quad x = 2, y = -1$ and $z = 4$

Q. 21 If $a + b + c \neq 0$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then prove that $a = b = c$.

Sol. Let

$$A = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a + b + c & a + b + c & a + b + c \\ b & c & a \\ c & a & b \end{vmatrix} \qquad [\because R_1 \to R_1 + R_2 + R_3]$$

$$= (a + b + c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= (a + b + c) \begin{vmatrix} 0 & 0 & 1 \\ b - a & c - a & a \\ c - b & a - b & b \end{vmatrix} \qquad [\because C_1 \to C_1 - C_3 \text{ and } C_2 \to C_2 - C_3]$$

Expanding along R_1,

$$= (a + b + c)\,[1\,(b - a)(a - b) - (c - a)(c - b)]$$

$$= (a + b + c)\,(ba - b^2 - a^2 + ab - c^2 + cb + ac - ab)$$

$$= \frac{-1}{2}(a + b + c) \times (-2)\,(-a^2 - b^2 - c^2 + ab + bc + ca)$$

$$= \frac{-1}{2}(a + b + c)[a^2 + b^2 + c^2 - 2ab - 2bc - 2ca + a^2 + b^2 + c^2]$$

$$= -\frac{1}{2}(a + b + c)[a^2 + b^2 - 2ab + b^2 + c^2 - 2bc + c^2 + a^2 - 2ac]$$

$$= \frac{-1}{2}(a + b + c)\,[(a - b)^2 + (b - c)^2 + (c - a)^2]$$

Also, $\quad A = 0$

$$= \frac{-1}{2}(a + b + c)\,[(a - b)^2 + (b - c)^2 + (c - a)^2] = 0$$

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 0 \qquad [\because a + b + c \neq 0, \text{ given}]$$

$$\Rightarrow \quad a - b = b - c = c - a = 0$$

$$a = b = c$$

Hence proved.

Q. 22 Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ is divisible by $(a+b+c)$ and find the quotient.

Sol. Let

$$\Delta = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} bc-a^2-ca+b^2 & ca-b^2-ab+c^2 & ab-c^2 \\ ca-b^2-ab+c^2 & ab-c^2-bc+a^2 & bc-a^2 \\ ab-c^2-bc+a^2 & bc-a^2-ca+b^2 & ca-b^2 \end{vmatrix}$$

[$\because C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$]

$$= \begin{vmatrix} (b-a)(a+b+c) & (c-b)(a+b+c) & ab-c^2 \\ (c-b)(a+b+c) & (a-c)(a+b+c) & bc-a^2 \\ (a-c)(a+b+c) & (b-a)(a+b+c) & ca-b^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b-a & c-b & ab-c^2 \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

[taking $(a+b+c)$ common from C_1 and C_2 each]

$$= (a+b+c)^2 \begin{vmatrix} 0 & 0 & ab+bc+ca-(a^2+b^2+c^2) \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

[$\because R_1 \to R_1 + R_2 + R_3$]

Now, expanding along R_1,

$$= (a+b+c)^2 [ab+bc+ca-(a^2+b^2+c^2)][(c-b)(b-a)-(a-c)^2]$$

$$= (a+b+c)^2 (ab+bc+ca-a^2-b^2-c^2)$$

$$(cb-ac-b^2+ab-a^2-c^2+2ac)$$

$$= (a+b+c)^2 (a^2+b^2+c^2-ab-bc-ca)$$

$$(a^2+b^2+c^2-ac-ab-bc)$$

$$= \frac{1}{2}(a+b+c)[(a+b+c)(a^2+b^2+c^2-ab-bc-ca)]$$

$$[(a-b)^2+(b-c)^2+(c-a)^2]$$

$$= \frac{1}{2}(a+b+c)(a^3+b^3+c^3-3abc)[(a-b)^2+(b-c)^2+(c-a)^2]$$

Hence, given determinant is divisible by $(a+b+c)$ and quotient is

$$(a^3+b^3+c^3-3abc)[(a-b)^2+(b-c)^2+(c-a)^2].$$

Q. 23 If $x+y+z=0$, then prove that $\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

Thinking Process

We have, given $x+y+z=0 \Rightarrow x^3+y^3+z^3=3xyz$. So, by using this in solving the given determinant from both the sides, we can equate the obtained result from both the sides to desired result.

Sol. Since, $x+y+z=0$, also we have to prove

$$\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$\therefore \quad \text{LHS} = \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix}$$

$$= xa(za \cdot ya - xb \cdot xc) - yb(yc \cdot ya - xb \cdot zb) + zc(yc \cdot xc - za \cdot zb)$$

$$= xa(a^2yz - x^2bc) - yb(y^2ac - b^2xz) + zc(c^2xy - z^2ab)$$

$$= xyza^3 - x^3abc - y^3abc + b^3xyz + c^3xyz - z^3abc$$

$$= xyz(a^3+b^3+c^3) - abc(x^3+y^3+z^3)$$

$$= xyz(a^3+b^3+c^3) - abc(3xyz)$$

$$[\because x+y+z=0 \Rightarrow x^3+y^3+z^3 - 3xyz]$$

$$= xyz(a^3+b^3+c^3-3abc) \quad \text{...(i)}$$

Now,

$$\text{RHS} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = xyz \begin{vmatrix} a+b+c & b & c \\ a+b+c & a & b \\ a+b+c & c & a \end{vmatrix} \quad [\because C_1 \to C_1 + C_2 + C_3]$$

$$= xyz(a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix} \quad [\text{taking } (a+b+c) \text{ common from } C_1]$$

$$= xyz(a+b+c) \begin{vmatrix} 0 & b-c & c-a \\ 0 & a-c & b-a \\ 1 & c & a \end{vmatrix}$$

$$[\because R_1 \to R_1 - R_3 \text{ and } R_2 \to R_2 - R_3]$$

Expanding along C_1,

$$= xyz(a+b+c)[1(b-c)(b-a) - (a-c)(c-a)]$$

$$= xyz(a+b+c)(b^2 - ab - bc + ac + a^2 + c^2 - 2ac)$$

$$= xyz(a+b+c)(a^2+b^2+c^2 - ab - bc - ca)$$

$$= xyz(a^3+b^3+c^3-3abc) \quad \text{...(ii)}$$

From Eqs. (i) and (ii),

$$\text{LHS} = \text{RHS}$$

$$\Rightarrow \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Hence proved.

Objective Type Questions

Q. 24 If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then the value of x is

(a) 3 (b) ± 3 (c) ± 6 (d) 6

Sol. *(c)* $\because$ $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$

$\Rightarrow \quad 2x^2 - 40 = 18 + 14$

$\Rightarrow \quad 2x^2 = 32 + 40$

$\Rightarrow \quad x^2 = \frac{72}{2} = 36$

$\therefore \quad x = \pm 6$

Q. 25 The value of $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$ is

(a) $a^3 + b^3 + c^3$ (b) $3bc$

(c) $a^3 + b^3 + c^3 - 3abc$ (d) None of these

Sol. *(d)* We have,

$$\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix} = \begin{vmatrix} a+c & b+c+a & a \\ b+c & c+a+b & b \\ c+b & a+b+c & c \end{vmatrix} \quad [\because C_1 \to C_1 + C_2 \text{ and } C_2 \to C_2 + C_3]$$

$$= (a+b+c)\begin{vmatrix} a+c & 1 & a \\ b+c & 1 & b \\ c+b & 1 & c \end{vmatrix} \quad [\text{taking } (a+b+c) \text{ common from } C_2]$$

$$= (a+b+c)\begin{vmatrix} a-b & 0 & a-c \\ 0 & 0 & b-c \\ c+b & 1 & c \end{vmatrix} \quad [\because R_2 \to R_2 - R_3 \text{ and } R_1 \to R_1 - R_3]$$

$$= (a+b+c)[-(b-c)\cdot(a-b)] \quad [\text{expanding along } R_2]$$

$$= (a+b+c)(c-b)(a-b)$$

Q. 26 If the area of a triangle with vertices (–3, 0), (3, 0) and (0, k) is 9 sq units. Then, the value of k will be

(a) 9 (b) 3 (c) –9 (d) 6

Sol. *(b)* We know that, area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\therefore \quad \Delta = \frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}$$

Expanding along R_1,

$$9 = \frac{1}{2}[-3(-k) - 0 + 1(3k)]$$

$$\Rightarrow \quad 18 = 3k + 3k = 6k$$

$$\therefore \quad k = \frac{18}{6} = 3$$

Q. 27 The determinant

$$\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix} \text{ equals to}$$

(a) $abc(b-c)(c-a)(a-b)$ (b) $(b-c)(c-a)(a-b)$

(c) $(a+b+c)(b-c)(c-a)(a-b)$ (d) None of these

Sol. ***(d)*** We have,

$$\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix} = \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix}$$

$$= (b-a)^2 \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

[on taking $(b-a)$ common from C_1 and C_3 each]

$$= (b-a)^2 \begin{vmatrix} b-c & b-c & c \\ a-b & a-b & b \\ c-a & c-a & a \end{vmatrix} \qquad [\because C_1 \to C_1 - C_3]$$

$$= 0$$

[since, two columns C_1 and C_2 are identical, so the value of determinant is zero]

Q. 28 The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is

(a) 0 (b) 2 (c) 1 (d) 3

Sol. ***(c)*** We have,

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$,

$$\begin{vmatrix} 2\cos x + \sin x & \cos x & \cos x \\ 2\cos x + \sin x & \sin x & \cos x \\ 2\cos x + \sin x & \cos x & \sin x \end{vmatrix} = 0$$

On taking $(2\cos x + \sin x)$ common from C_1, we get

$$\Rightarrow \quad (2\cos x + \sin x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

$$\Rightarrow \quad (2\cos x + \sin x)\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & (\sin x - \cos x) \end{vmatrix} = 0$$

[$\because R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

Expanding along C_1,

$$(2\cos x + \sin x)\,[1\cdot(\sin x - \cos x)^2] = 0$$

$$\Rightarrow \quad (2\cos x + \sin x)(\sin x - \cos x)^2 = 0$$

Either $\quad 2\cos x = -\sin x$

$$\Rightarrow \quad \cos x = -\frac{1}{2}\sin x$$

$$\Rightarrow \quad \tan x = -2 \quad \text{...(i)}$$

But here for $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$, we get $-1 \le \tan x \le 1$ so, no solution possible

and for $\quad (\sin x - \cos x)^2 = 0, \sin x = \cos x$

$$\Rightarrow \quad \tan x = 1 = \tan\frac{\pi}{4}$$

$$\therefore \quad x = \frac{\pi}{4}$$

So, only one distinct real root exist.

Q. 29 If A, B and C are angles of a triangle, then the determinant $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$ is equal to

(a) 0 (b) –1 (c) 1 (d) None of these

Sol. ***(a)*** We have, $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$

Applying $C_1 \to aC_1 + bC_2 + cC_3$,

$$\begin{vmatrix} -a + b\cos C + c\cos B & \cos C & \cos B \\ a\cos C - b + c\cos A & -1 & \cos A \\ a\cos B + b\cos A - c & \cos A & -1 \end{vmatrix}$$

Also, by projection rule in a triangle, we know that

$$a = b\cos C + c\cos B,\ b = c\cos A + a\cos C \text{ and } c = a\cos B + b\cos A$$

Using above equation in column first, we get

$$\begin{vmatrix} -a + a & \cos C & \cos B \\ b - b & -1 & \cos A \\ c - c & \cos A & -1 \end{vmatrix} = \begin{vmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{vmatrix} = 0$$

[since, determinant having all elements of any column or row gives value of determinant as zero]

Q. 30 If $f(t) = \begin{bmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{bmatrix}$, then $\lim\limits_{t \to 0} \dfrac{f(t)}{t^2}$ is equal to

(a) 0 (b) −1 (c) 2 (d) 3

Sol. *(a)* We have,

$$f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$$

Expanding along C_1,

$$= \cos t\,(t^2 - 2t^2) - 2\sin t\,(t^2 - t) + \sin t\,(2t^2 - t)$$
$$= -t^2 \cos t - (t^2 - t) 2\sin t + (2t^2 - t)\sin t$$
$$= -t^2 \cos t - t^2 \cdot 2\sin t + t \cdot 2\sin t + 2t^2 \sin t$$
$$= -t^2 \cos t + 2t \sin t$$

$\therefore$

$$\lim_{t \to 0} \frac{f(t)}{t^2} = \lim_{t \to 0} \frac{(-t^2 \cos t)}{t^2} + \lim_{t \to 0} \frac{2t \sin t}{t^2}$$
$$= -\lim_{t \to 0} \cos t + 2 \cdot \lim_{t \to 0} \frac{\sin t}{t}$$
$$= -1 + 1 \qquad \left[\because \lim_{t \to 0} \frac{\sin t}{t} = 1 \text{ and } \cos 0 = 1\right]$$
$$= 0$$

Q. 31 The maximum value of

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin\theta & 1 \\ 1 + \cos\theta & 1 & 1 \end{vmatrix} \text{ is (where, } \theta \text{ is real number)}$$

(a) $\dfrac{1}{2}$ (b) $\dfrac{\sqrt{3}}{2}$ (c) $\sqrt{2}$ (d) $\dfrac{2\sqrt{3}}{4}$

Sol. *(a)* Since,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin\theta & 1 \\ 1 + \cos\theta & 1 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} 0 & 0 & 1 \\ 0 & \sin\theta & 1 \\ \cos\theta & 0 & 1 \end{vmatrix} \qquad [\because C_1 \to C_1 - C_3 \text{ and } C_2 \to C_2 - C_3]$$
$$= 1(\sin\theta \cdot \cos\theta)$$
$$= \frac{1}{2} \cdot 2\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

Since, the maximum value of $\sin 2\theta$ is 1. So, for maximum value of θ should be 45°.

$\therefore$

$$\Delta = \frac{1}{2}\sin 2 \cdot 45°$$
$$= \frac{1}{2}\sin 90° = \frac{1}{2} . 1 = \frac{1}{2}$$

Q. 32 If $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$, then

(a) $f(a)=0$ (b) $f(b)=0$ (c) $f(0)=0$ (d) $f(1)=0$

Sol. *(c)* We have,

$$f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$$

$$\Rightarrow \quad f(a) = \begin{vmatrix} 0 & 0 & a-b \\ 2a & 0 & a-c \\ a+b & a+c & 0 \end{vmatrix}$$

$$= [(a-b)\{2a \cdot (a+c)\}] \neq 0$$

$$\therefore \quad f(b) = \begin{vmatrix} 0 & b-a & 0 \\ b+a & 0 & b-c \\ 2b & b+c & 0 \end{vmatrix}$$

$$= -(b-a)[2b(b-c)]$$

$$= -2b(b-a)(b-c) \neq 0$$

$$\therefore \quad f(0) = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

$$= a(bc) - b(ac)$$

$$= abc - abc = 0$$

Q. 33 If $A = \begin{vmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{vmatrix}$, then A^{-1} exists, if

(a) $\lambda = 2$ (b) $\lambda \neq 2$

(c) $\lambda \neq -2$ (d) None of these

Sol. *(d)* We have,

$$A = \begin{vmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{vmatrix}$$

Expanding along R_1,

$$|A| = 2(6-5) - \lambda(-5) - 3(-2) = 2 + 5\lambda + 6$$

We know that, A^{-1} exists, if A is non-singular matrix *i.e.,* $|A| \neq 0$.

$$\therefore \quad 2 + 5\lambda + 6 \neq 0$$

$$\Rightarrow \quad 5\lambda \neq -8$$

$$\therefore \quad \lambda \neq \frac{-8}{5}$$

So, A^{-1} exists if and only if $\lambda \neq \frac{-8}{5}$.

Q. 34 If A and B are invertible matrices, then which of the following is not correct?

(a) $\text{adj } A = |A| \cdot A^{-1}$ (b) $\det (A)^{-1} = [\det (A)]^{-1}$

(c) $(AB)^{-1} = B^{-1} A^{-1}$ (d) $(A+B)^{-1} = B^{-1} + A^{-1}$

Sol. *(d)* Since, A and B are invertible matrices. So, we can say that

$$(AB)^{-1} = B^{-1} A^{-1} \quad \text{...(i)}$$

Also, $$A^{-1} = \frac{1}{|A|} (\text{adj } A)$$

$\Rightarrow$ $$\text{adj } A = |A| \cdot A^{-1} \quad \text{...(ii)}$$

Also, $$\det (A)^{-1} = [\det (A)]^{-1}$$

$\Rightarrow$ $$\det (A)^{-1} = \frac{1}{[\det (A)]}$$

$\Rightarrow$ $$\det (A) \cdot \det (A)^{-1} = 1 \quad \text{...(iii)}$$

which is true.

Again, $$(A+B)^{-1} = \frac{1}{|(A+B)|} \text{adj } (A+B)$$

$\Rightarrow$ $$(A+B)^{-1} \neq B^{-1} + A^{-1} \quad \text{...(iv)}$$

So, only option (d) is incorrect.

Q. 35 If x, y and z are all different from zero and $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$, then the value of $x^{-1} + y^{-1} + z^{-1}$ is

(a) xyz (b) $x^{-1}y^{-1}z^{-1}$ (c) $-x-y-z$ (d) -1

Sol. *(d)* We have, $$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$,

$\Rightarrow$ $$\begin{vmatrix} x & 0 & 1 \\ 0 & y & 1 \\ -z & -z & 1+z \end{vmatrix} = 0$$

Expanding along R_1,

$$x [y(1+z) + z] - 0 + 1(yz) = 0$$

$\Rightarrow$ $$x(y + yz + z) + yz = 0$$

$\Rightarrow$ $$xy + xyz + xz + yz = 0$$

$\Rightarrow$ $$\frac{xy}{xyz} + \frac{xyz}{xyz} + \frac{xz}{xyz} + \frac{yz}{xyz} = 0 \quad \text{[on dividing } (xyz) \text{ from both sides]}$$

$\Rightarrow$ $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 1 = 0$$

$\Rightarrow$ $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -1$$

$\therefore$ $$x^{-1} + y^{-1} + z^{-1} = -1$$

Q. 36 The value of $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ is

(a) $9x^2(x+y)$ (b) $9y^2(x+y)$
(c) $3y^2(x+y)$ (d) $7x^2(x+y)$

Sol. *(b)* We have, $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

$$= \begin{vmatrix} 3(x+y) & x+y & y \\ 3(x+y) & x & y \\ 3(x+y) & x+2y & -2y \end{vmatrix} \quad [\because C_1 \to C_1 + C_2 + C_3 \text{ and } C_3 \to C_3 - C_2]$$

$$= 3(x+y)\begin{vmatrix} 1 & (x+y) & y \\ 1 & x & y \\ 1 & (x+2y) & -2y \end{vmatrix} \quad [\text{taking } 3(x+y) \text{ common from first column}]$$

$$= 3(x+y)\begin{vmatrix} 0 & y & 0 \\ 1 & x & y \\ 1 & (x+2y) & -2y \end{vmatrix} \quad [\because R_1 \to R_1 - R_2]$$

Expanding along R_1,

$$= 3(x+y)[-y(-2y-y)]$$
$$= 3y^2 \cdot 3(x+y) = 9y^2(x+y)$$

Q. 37 If there are two values of a which makes determinant,

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86, \text{ then the sum of these number is}$$

(a) 4 (b) 5 (c) – 4 (d) 9

Sol. *(c)* We have,

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$$

$\Rightarrow$ $1(2a^2 + 4) - 2(-4a - 20) + 0 = 86$ [expanding along first column]

$\Rightarrow$ $2a^2 + 4 + 8a + 40 = 86$

$\Rightarrow$ $2a^2 + 8a + 44 - 86 = 0$

$\Rightarrow$ $a^2 + 4a - 21 = 0$

$\Rightarrow$ $a^2 + 7a - 3a - 21 = 0$

$\Rightarrow$ $(a+7)(a-3) = 0$

$a = -7$ and 3

$\therefore$ Required sum $= -7 + 3 = -4$

Fillers

Q. 38 If A is a matrix of order 3×3, then $|3A|$ is equal to ....

Sol. If A is a matrix of order 3×3, then $|3A| = 3 \times 3 \times 3|A| = 27|A|$

Q. 39 If A is invertible matrix of order 3×3, then $|A^{-1}|$ is equal to ....

Sol. If A is invertible matrix of order 3×3, then $|A^{-1}| = \dfrac{1}{|A|}$. [since, $|A| \cdot |A^{-1}| = 1$]

Q. 40 If $x, y, z \in R$, then the value of $\begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$ is

Sol. We have,

$$\begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (2 \cdot 2^x)(2 \cdot 2^{-x}) & (2^x - 2^{-x})^2 & 1 \\ (2 \cdot 3^x)(2 \cdot 3^{-x}) & (3^x - 3^{-x})^2 & 1 \\ (2 \cdot 4^x)(2 \cdot 4^{-x}) & (4^x - 4^{-x})^2 & 1 \end{vmatrix} \quad [\because (a+b)^2 - (a-b)^2 = 4ab] \quad [\because C_1 \to C_1 - C_2]$$

$$= \begin{vmatrix} 4 & (2^x - 2^{-x})^2 & 1 \\ 4 & (3^x - 3^{-x})^2 & 1 \\ 4 & (4^x - 4^{-x})^2 & 1 \end{vmatrix} = 0 \quad \text{[since, } C_1 \text{ and } C_3 \text{ are proportional to each other]}$$

Q. 41 If $\cos 2\theta = 0$, then $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$ is equal to ....

Sol. Since, $\cos 2\theta = 0$

$$\Rightarrow \quad \cos 2\theta = \cos\frac{\pi}{2} \Rightarrow 2\theta = \frac{\pi}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

$$\therefore \quad \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \begin{vmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix}^2$$

Expanding along R_1,

$$= \left[-\frac{1}{\sqrt{2}}\left(\frac{1}{2}\right) + \frac{1}{\sqrt{2}}\left(-\frac{1}{2}\right)\right]^2 = \left[\frac{-2}{2\sqrt{2}}\right]^2 = \left(\frac{-1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

Q. 42 If A is a matrix of order 3×3, then $(A^2)^{-1}$ is equal to ....

Sol. If A is a matrix of order 3×3, then $(A^2)^{-1} = (A^{-1})^2$.

Q. 43 If A is a matrix of order 3×3, then the number of minors in determinant of A are ....

Sol. If A is a matrix of order 3×3, then the number of minors in determinant of A are 9. [since, in a 3×3 matrix, there are 9 elements]

Q. 44 The sum of products of elements of any row with the cofactors of corresponding elements is equal to ....

Sol. The sum of products of elements of any row with the cofactors of corresponding elements is equal to value of the determinant.

$$\text{Let } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R_1,

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

= Sum of products of elements of R_1 with their corresponding cofactors

Q. 45 If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then other two roots are ....

Sol. Since, $$\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

Expanding along R_1,

$$x(x^2 - 12) - 3(2x - 14) + 7(12 - 7x) = 0$$

$\Rightarrow$ $$x^3 - 12x - 6x + 42 + 84 - 49x = 0$$

$\Rightarrow$ $$x^3 - 67x + 126 = 0 \quad \text{...(i)}$$

Here, $126 \times 1 = 9 \times 2 \times 7$

For $x = 2$, $2^3 - 67 \times 2 + 126 = 134 - 134 = 0$

Hence, $x = 2$ is a root.

For $x = 7$, $7^3 - 67 \times 7 + 126 = 469 - 469 = 0$

Hence, $x = 7$ is also a root.

Q. 46 $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix}$ is equal to ....

Sol. We have, $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix} = \begin{vmatrix} z-x & xyz & x-z \\ z-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix}$ $[\because C_1 \to C_1 - C_3]$

$$= (z-x)\begin{vmatrix} 1 & xyz & x-z \\ 1 & 0 & y-z \\ 1 & z-y & 0 \end{vmatrix}$$

[taking $(z-x)$ common from column 1]

Expanding along R_1,

$= (z-x)[1\cdot\{-(y-z)(z-y)\} - xyz(z-y) + (x-z)(z-y)]$

$= (z-x)(z-y)\ (-y+z-xyz+x-z)$

$= (z-x)(z-y)(x-y-xyz)$

$= (z-x)(y-z)(y-x+xyz)$

Q. 47 If $f(x) = \begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix}$

$= A + Bx + Cx^2 + \dots$, then A is equal to ....

Sol. Since,

$$f(x) = (1+x)^{17}(1+x)^{23}(1+x)^{41}\begin{vmatrix} 1 & (1+x)^2 & (1+x)^6 \\ 1 & (1+x)^6 & (1+x)^{11} \\ 1 & (1+x)^2 & (1+x)^6 \end{vmatrix} = 0$$

[since, R_1 and R_3 are identical]

$\therefore \quad A = 0$

True/False

Q. 48 $(A^3)^{-1} = (A^{-1})^3$, where A is a square matrix and $|A| \neq 0$.

Sol. ***True***

Since, $(A^n)^{-1} = (A^{-1})^n$, where $n \in N$.

Q. 49 $(aA)^{-1} = \frac{1}{a} A^{-1}$, where a is any real number and A is a square matrix.

Sol. ***False***

Since, we know that, if A is a non-singular square matrix, then for any scalar a (non-zero), aA is invertible such that

$$(aA)\left(\frac{1}{a}A^{-1}\right) = \left(a \cdot \frac{1}{a}\right)(A \cdot A^{-1})$$
$$= I$$

i.e., (aA) is inverse of $\left(\frac{1}{a}A^{-1}\right)$ or $(aA)^{-1} = \frac{1}{a}A^{-1}$, where a is any non-zero scalar.

In the above statement a is any real number. So, we can conclude that above statement is false.

Q. 50 $|A^{-1}| \neq |A|^{-1}$, where A is a non-singular matrix.

Sol. ***False***

$|A^{-1}| = |A|^{-1}$, where A is a non-singular matrix.

Q. 51 If A and B are matrices of order 3 and $|A| = 5$, $|B| = 3$, then $|3AB| = 27 \times 5 \times 3 = 405$.

Sol. ***True***

We know that, $|AB| = |A| \cdot |B|$

$\therefore$ $|3AB| = 27\,|AB|$

$= 27\,|A| \cdot |B|$

$= 27 \times 5 \times 3 = 405$

Q. 52 If the value of a third order determinant is 12, then the value of the determinant formed by replacing each element by its cofactor will be 144.

Sol. ***True***

Let A is the determinant.

$\therefore$ $|A| = 12$

Also, we know that, if A is a square matrix of order n, then $|\text{adj } A| = |A|^{n-1}$

For $n = 3$, $|\text{adj } A| = |A|^{3-1} = |A|^2$

$= (12)^2 = 144$

Q. 53 $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$, where a, b and c are in AP.

Sol. ***True***

Since, a, b and c are in AP, then $2b = a + c$

$$\therefore \quad \begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2x+4 & 2x+6 & 2x+a+c \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0 \qquad [\because R_1 \to R_1 + R_3]$$

$$\Rightarrow \quad \begin{vmatrix} 2(x+2) & 2(x+3) & 2(x+b) \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0 \qquad [\because 2b = a + c]$$

$\Rightarrow \quad 0 = 0$ [since, R_1 and R_2 are in proportional to each other]

Hence, statement is true.

Q. 54 $|\text{adj } A| = |A|^2$, where A is a square matrix of order two.

Sol. ***False***

If A is a square matrix of order n, then

$$|\text{adj } A| = |A|^{n-1}$$

$$\Rightarrow \quad |\text{adj } A| = |A|^{2-1} = |A| \qquad [\because n = 2]$$

Q. 55 The determinant $\begin{vmatrix} \sin A & \cos A & \sin A + \cos B \\ \sin B & \cos A & \sin B + \cos B \\ \sin C & \cos A & \sin C + \cos B \end{vmatrix}$ is equal to zero.

Sol. ***True***

Since, $\begin{vmatrix} \sin A & \cos A & \sin A + \cos B \\ \sin B & \cos A & \sin B + \cos B \\ \sin C & \cos A & \sin C + \cos B \end{vmatrix} = \begin{vmatrix} \sin A & \cos A & \sin A \\ \sin B & \cos A & \sin B \\ \sin C & \cos A & \sin C \end{vmatrix} + \begin{vmatrix} \sin A & \cos A & \cos B \\ \sin B & \cos A & \cos B \\ \sin C & \cos A & \cos B \end{vmatrix}$

$$= 0 + \begin{vmatrix} \sin A & \cos A & \cos B \\ \sin B & \cos A & \cos B \\ \sin C & \cos A & \cos B \end{vmatrix}$$

[since, in first determinant C_1 and C_3 are identicals]

$$= \cos A \cdot \cos B \begin{vmatrix} \sin A & 1 & 1 \\ \sin B & 1 & 1 \\ \sin C & 1 & 1 \end{vmatrix}$$

[taking $\cos A$ common from C_2 and $\cos B$ common from C_3]

$= 0$ [since, C_2 and C_3 are identicals]

Q. 56 If the determinant $\begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$ splits into exactly k determinants of order 3, each element of which contains only one term, then the value of k is 8.

Sol. ***True***

Since,
$$\begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$$

$$= \begin{vmatrix} x & p & l \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix} + \begin{vmatrix} a & u & f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix} \qquad \text{[splitting first row]}$$

$$= \begin{vmatrix} x & p & l \\ y & q & m \\ z+c & r+m & n+h \end{vmatrix} + \begin{vmatrix} x & p & l \\ b & v & g \\ z+c & r+w & n+h \end{vmatrix}$$

$$+ \begin{vmatrix} a & u & f \\ y & q & m \\ z+c & r+w & n+h \end{vmatrix} + \begin{vmatrix} a & u & f \\ b & v & g \\ z+c & r+w & n+h \end{vmatrix} \qquad \text{[splitting second row]}$$

Similarly, we can split these 4 determinants in 8 determinants by splitting each one in two determinants further. So, given statement is true.

Q. 57 If $\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$, then $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = 32$.

Sol. ***True***

We have,
$$\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$$

and we have to prove,
$$\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = 32$$

$$\Delta_1 = \begin{vmatrix} 2p+2x+2a & a+x & a+p \\ 2q+2y+2b & b+y & b+q \\ 2r+2z+2c & c+z & c+r \end{vmatrix} \qquad [\because C_1 \to C_1 + C_2 + C_3]$$

$$= 2\begin{vmatrix} p & x-p & a+p \\ q & y-q & b+q \\ r & z-r & c+r \end{vmatrix}$$

[taking 2 common from C_1 and then $C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$]

$$= 2\left[\begin{vmatrix} p & x & a+p \\ q & y & b+q \\ r & z & c+r \end{vmatrix} - \begin{vmatrix} p & p & a+p \\ q & q & b+q \\ r & r & c+r \end{vmatrix}\right]$$

$$= 2\begin{vmatrix} p & x & a+p \\ q & y & b+q \\ r & z & c+r \end{vmatrix} - 0$$

[since, two columns C_1 and C_2 are identicals]

$$= 2\begin{vmatrix} p & x & a \\ q & y & b \\ r & z & c \end{vmatrix} + 2\begin{vmatrix} p & x & p \\ q & y & q \\ r & z & r \end{vmatrix}$$

$$= 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} + 0$$

[since, C_1 and C_3 are identical in second determinant and in first determinant, $C_1 \leftrightarrow C_2$ and then $C_1 \leftrightarrow C_3$]

$= 2 \times 16$ [$\because \Delta = 16$]

$= 32$ **Hence proved.**

Q. 58 The maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ is $\frac{1}{2}$.

Sol. ***True***

Since, $\begin{vmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \cos\theta \end{vmatrix}$ [$\because R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

On expanding along third row, we get the value of the determinant

$$= \cos\theta \cdot \sin\theta = \frac{1}{2}\sin 2\theta = \frac{1}{2}$$

[when θ is $45°$ which gives maximum value]