Chapter – 7

Coordinate Geometry

Distance Formula

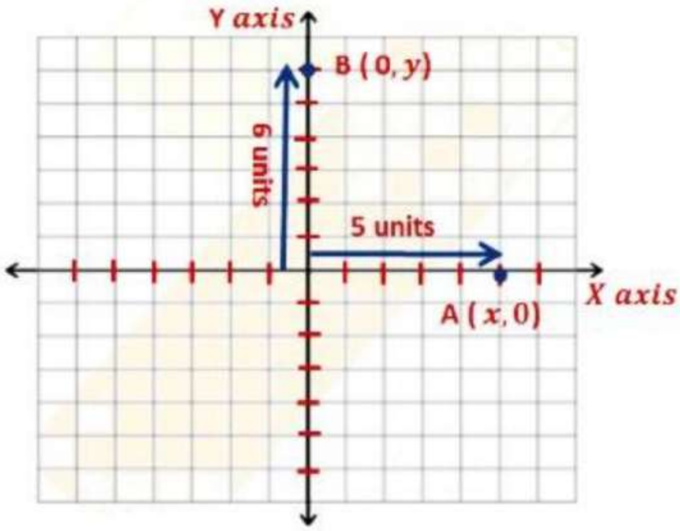
Coordinate geometry is widely applied in various fields such as
Physics, Engineering, Navigation and Art.

We can describe the position of any object in the real world with the help of coordinate geometry.

All the air traffic is controlled by traffic controller which uses coordinates to describe the current location of an aircraft.

Distance Formula

A plane is a flat surface that goes infinitely in both directions. To locate the position of a point on a plane, we require a pair of coordinate axes.



The distance of the point from the y-axis is called its x − coordinate, or abscissa. The distance of the point from the x-axis is called its y coordinate, or ordinate.

The coordinates of the point on the x-axis are of the form (x, 0) and of the point on the y-axis are of the form (0, y).

To find the distance between any two points P (x_1, y_1) and Q(x_2, y_2)

- Draw PR and QS perpendicular to the x-axis.
- A perpendicular is drawn from the point P on QS to meet it at the point T.

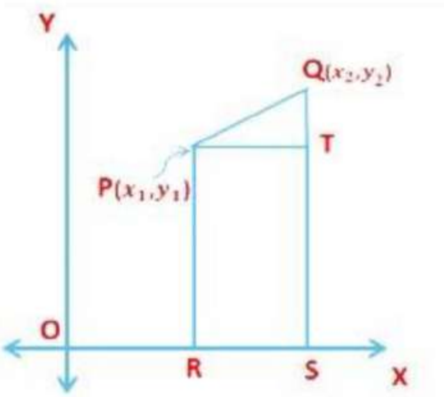
Then, OR = x_1, OS = x_2. So, RS = $x_2 - x_1$= PT SQ =y_2, ST = PR =y_1. So, QT = $y_2 - y_1$

Applying Pythagoras Triangle in Δ PTQ, we get
$PQ^2 = PT^2 + QT^2$

$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

PQ = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Or PQ = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$



We take only positive square root as the distance is always non-negative.

The distance between the points P $(x_1, y_1)$and Q (x_2, y_2) is PQ = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, which is called the distance formula.

Let P (x, y) and O (0, 0) be the origin. The distance of point P from the origin O (0, 0) is given by, OP = $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{(x)^2 + (y)^2}$

Example: Check whether (5,-2), (6, 4) and (7, -2) are the vertices of an isosceles triangle.

An isosceles triangle is a triangle with two equal sides.

Let A(5,-2), B(6, 4) and C(7, -2) be the given points.

Using the distance formula we have,

$AB = \sqrt{(6-5)^2 + (4+2)^2} = \sqrt{(1)^2 + (6)^2} = \sqrt{1+36} = \sqrt{37}$ units

$BC = \sqrt{(7-6)^2 + (-2-4)^2} = \sqrt{(1)^2 + (-6)^2} = \sqrt{1+36} = \sqrt{37}$ units

$AC = \sqrt{(7-5)^2 + (-2+2)^2} = \sqrt{(2)^2} = \sqrt{4} = 2$ units

Now, AB = BC and so A, B and C will form the vertices of the isosceles triangle ABC.

Example: Find the point on X-axis, which is equidistant from the point (7, 6) and (-3, 4).

(REFERENCE: NCERT)

Let P(x, 0) be any point on the X-axis, which is equidistant from point Q (7, 6) and R(-3, 4).

If P(x, 0) is equidistant from Q (7, 6) and R(-3, 4) then, PQ = PR Using Distance Formula,

$PQ = \sqrt{(7-x)^2 + (6-0)^2} = \sqrt{(7-x)^2 + 36}$

$PR = \sqrt{(-3-x)^2 + (4-0)^2} = \sqrt{(-3-x)^2 + 16}$

Now, PQ = PR

$$\sqrt{(7-x)^2 + 36} = \sqrt{(-3-x)^2 + 16}$$

Squaring both sides we get,

$$(7-x)^2 + 36 = (-3-x)^2 + 16$$

$49 - 14x + x^2 + 36 = 9 + 6x + x^2 + 16 \Rightarrow 85 - 14x = 25 + 6x$

$$\Rightarrow -20x = -60 \Rightarrow x = 3$$

∴The required point is P(3,0).

Example: If the point P(x, y) is equidistant from L(4,1) and M(-1,4), then find the relation between x and y.

The point P (x, y) is equidistant from the points L(4,1) and M(-1,4).

$\therefore$ PL = PM

Using Distance Formula,

$PL = \sqrt{(4-x)^2 + (1-y)^2}$

$PM = \sqrt{(-1-x)^2 + (4-y)^2}$

Now, PL = PM

$\sqrt{(4-x)^2 + (1-y)^2} = \sqrt{(-1-x)^2 + (4-y)^2}$

Squaring both sides we get,

$(4-x)^2 + (1-y)^2 = (-1-x)^2 + (4-y)^2$

$16 - 8x + x^2 + 1 - 2y + y^2 = 1 + 2x + x^2 + 16 - 8y + y^2$

$\Rightarrow 17 - 8x - 2y = 17 + 2x - 8y \Rightarrow -8x - 2y = 2x - 8y$

$\Rightarrow -10x = -6y \Rightarrow 5x = 3y$

$\therefore$ The required relation is $5x = 3y$

Section Formula

In this topic, we will derive a formula to find the coordinates of a point which divides the line segments internally or externally in the given ratio.

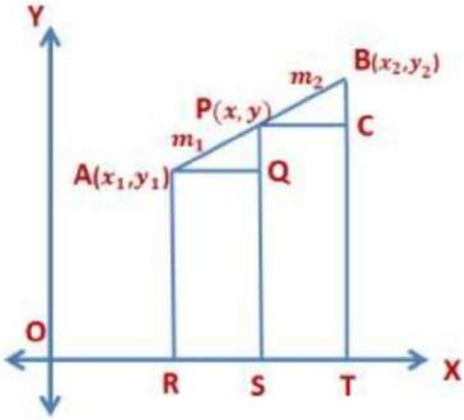
Let A (x_1, y_1) and B (x_2, y_2) be two points and assume that

P(x, y) divides AB internally in the ratio $m_1 : m_2$, i.e., $\frac{PA}{PB} = \frac{m_1}{m_2}$

Draw AR, PS and BT perpendicular to x − axis.

Draw AQ, PC parallel to x − axis.

By AA similarity criterion, $\Delta PAQ \sim \Delta BPC$



$\therefore \frac{PA}{PB} = \frac{AQ}{PC} = \frac{PQ}{BC} \rightarrow$ Eq 1

Now, $AQ = RS = OS - OR = x - x_1$

$PC = ST = OT - OS = x_2 - x$

$PQ = PS - QS = PS - AR = y - y_1$ ($\because QS = QR$)

$BC = BT - CT = BT - CT = y_2 - y$ ($\because CT = PS$)

Substituting these values in Eq 1 we get,

$\frac{m_1}{m_2} = \frac{x - x^1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$

Taking, $\frac{m_1}{m_2} = \frac{x - x^1}{x_2 - x} \Rightarrow m_1(x_2 - x) = m_2(x - x_1)$

$\Rightarrow m_1x_2 - m_1x = m_2x - m_2x_1 \Rightarrow m_1x_2 + m_2x_1 = m_2x + m_1x$

$\Rightarrow x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$

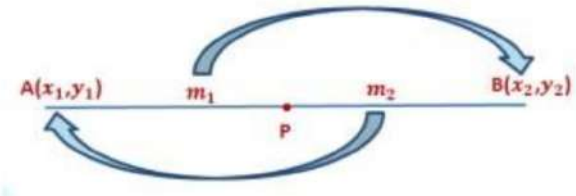
Similarly, on taking $\frac{m_1}{m_2} = \frac{y - y_1}{y_2 - y}$ we get, $y = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$

The coordinates of the point P (x, y) which divides the line segment joining the points A (x_1, y_1) and B (x_2, y_2), internally in the ratio

$m_1 : m_2$ are

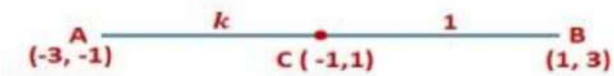
$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2} , \frac{m_1y_2 + m_2y_1}{m_1 + m_2} \right).$$

This is known as the section formula.



Example: Using section formula, show that the points A(-3, -1), B (1, 3) and C(-1, 1) are collinear.

Let C (-1,1) divides AB in the ratio k: 1



By using Section Formula ($\frac{kx_2 + x_1}{k+1}$, $\frac{ky_2 + y_1}{k+1}$) we get the coordinates of C.

Here, A (x_1, y_1) =(-3, -1) and B(x_2, y_2)= (1, 3)

Then, coordinates of C are ($\frac{k-3}{k+1}$, $\frac{3k-1}{k+1}$)

Also, coordinates of C are (-1,1)
On equating the x coordinate we get,

$$\frac{k-3}{k+1} = -1 \Rightarrow k - 3 = -1(k + 1) \Rightarrow k - 3 = -k - 1$$

$$\Rightarrow 2k = 3 - 1 \Rightarrow k = 1$$

On equating the y coordinate we get,

$$\frac{3k-1}{k+1} = 1 \Rightarrow 3k - 1 = k + 1 \Rightarrow 2k = 2 \Rightarrow k = 1$$

As the value of k is the same in both cases. So, C divides AB in the ratio 1: 1. Therefore, A, B, and C are collinear.

Example: In what ratio, does the point (1, 2) divides the line segment joining the points (3,5) and (-7, 6)?

Let the point C (1, 2) divides the line segment joining points A (3,5)

and B(-7, 6)

By using Section Formula

($\frac{kx_2 + x_1}{k+1}$, $\frac{ky_2 + y_1}{k+1}$)we get the coordinates of C.

Here, A (x_1, y_1) =(3, 1) and B (x2, y2)= (-7, 6)

Then the coordinates of C are($\frac{-7k+3}{k+1}$, $\frac{6k+3}{k+1}$)

We know the coordinates of C are (1, 2)

On equating the x coordinate we get,

$\frac{-7k+3}{k+1} = 1 \Rightarrow (-7k + 3) = k + 1 \Rightarrow 3 - 1 = k + 7k$

$\Rightarrow 8k = 2 \Rightarrow k = \frac{1}{4}$

On equating the y coordinate we get,

$\frac{6k+3}{k+1} = 2 \Rightarrow 6k + 1 = 2(k + 1) \Rightarrow 6k + 1 = 2k + 2$

$\Rightarrow 6k - 2k = 2 - 1 \Rightarrow k = \frac{1}{4}$

So, C divides AB in the ratio1: 4.

Example: Point P divides the line segment joining the points

A (-1,3) and B (9,8) such that $\frac{AP}{AB} = \frac{1}{3}$. If point P lies on the line

2x − y + k = 0, find the value of k

We have , $\frac{AP}{AB} = \frac{1}{3} \Rightarrow \frac{AP}{AP + BP} = \frac{1}{3} \Rightarrow$ 3AP = AP + BP ⇒ 2AP = BP

$\frac{AP}{BP} = \frac{1}{3}$

Therefore, P divides the line segment AB in the ratio 1 : 2.

By using Section Formula, ($\frac{m_1x_2 + m_2x_1}{m_1 + m_2}$, $\frac{m_1y_2 + m_2y_1}{m_1 + m_2}$) we get the coordinates of P.

Then, coordinates of P = ($\frac{1X5 + 2X2}{1 + 2}$,1 $\frac{1X(-8) + 2X1}{1 + 2}$) = ($\frac{9}{3}$, $\frac{-6}{3}$) = (3, −2)

Coordinates of P are (3,-2)

As P lies on the line 2x − y + k = 0

∴ 2 × 3 − (−2) + k = 0 ⇒ 6 + 2 + k = 0 ⇒ k = −8

Example: If P and Q are the points of trisection of the line segment joining the points A (2, -2) and B (-7, 4) such that P is nearer to A.

Find the coordinates of P and Q.

Points P and Q are the points of trisection of the line segment AB.

Then, AP : PB = 1 : 2 and AQ : QB = 2 : 1

i) AP : PB = 1 : 2∴ $\frac{m_1}{m_2} = \frac{1}{2}$

Using section formula we get the coordinates of P.

The coordinates of P are ($\frac{1X(-7)+2X2}{1+2}$, $\frac{1X(-7)+2X(-2)}{1+2}$) = ($\frac{-3}{3}$, 0) = (−1, 0)

ii) AQ : QB = 2 : 1∴ $\frac{m_1}{m_2} = \frac{2}{1}$

Using section formula we get the coordinates of Q.

The coordinates of Q are ($\frac{2X(-7)+1X2}{1+2}$, $\frac{2X(-4)+1X2}{1+2}$) = ($\frac{-12}{3}$,$\frac{6}{3}$) = (−4, 2)

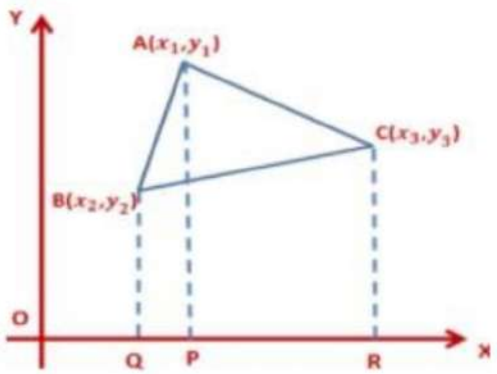
The coordinates of Pare (-1,0) and the coordinates of Q are (-4,2)

Area of a Triangle

Area of a triangle = $\frac{1}{2}$ × Base×Height

Let ABC be any triangle with vertices A(x_1, y_1), B (x_2, y_2) and C(x_3, y_3).

Draw AP, BQ and CR perpendiculars from A, B and C to the x −axis, respectively.



Area of Δ ABC = Area of trapezium ABQP + Area of trapezium APRC − Area of trapezium BQRC

We know, Area of trapezium = $\frac{1}{2}$ × (Sum of parallel sides) × (distance between them)

Area of Δ ABC = $\frac{1}{2}$ ×(BQ+AP) QP + $\frac{1}{2}$ ×(AP + CR) PR - $\frac{1}{2}$ ×(BQ + CR) QR

$= \frac{1}{2} \times (y_2 + y_1)(x_1 - x_2) + \frac{1}{2} \times (y_1 + y_3)(x_3 - x_1) - \frac{1}{2} (y_2 + y_3)(x_3 - x_2)$

$= \frac{1}{2} \times [x_1y_2 - x_2y_2 + x_1y_1 - x_2y_1 + x_3y_1 - x_1y_1 + x_3y_3 - x_1y_3 - x_3y_2 + x_2y_2 - x_3y_3 + x_2y_3]$

$= \frac{1}{2} \times [x_1y_2 - x_2y_1 + x_3y_1 - x_1y_3 - x_3y_2 + x_2y_3]$

$= \frac{1}{2} \times [x_1y_2 + x_2y_3 + x_3y_1 - (x_1y_3 + x_2y_1 + x_3y_2)]$

$= \frac{1}{2} \times [(x_1y_2 - x_1y_3) + (x_2y_3 - x_2y_1) + (x_3y_1 - x_3y_2)]$

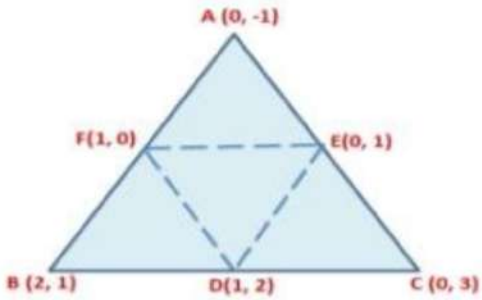
$= \frac{1}{2} \times [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

Area of Triangle ABC = $\frac{1}{2} \times [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

Example: Find the area of the triangle formed by joining the midpoints of the sides of the triangle whose vertices are (0, -1), (2, 1) and (0, 3). Find the ratio of the area of the triangle formed to the area of the given triangle.

Let A (0, -1), B (2, 1) and C (0, 3) be the vertices of the triangle ABC.

Let P, Q, and R be the midpoints of sides BC, CA and AB respectively.



The coordinates of P are $\left(\frac{2+0}{2}, \frac{1+3}{2}\right) = (1, 2)$

Similarly, the coordinates of Q and R are (0, 1) and (1, 0) respectively.

Area of Δ ABC $= \frac{1}{2} \times [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

Here, A $(x_1y_1) = (0, -1)$, B $(x_2y_2) = (2, 1)$ and $C(x_3y_3) = (0, 3)$

Area of Δ ABC $= \frac{1}{2} \times [0(1 - 3) + 2(3 + 1) + 0(-1 - 1)] = \frac{1}{2} \times 8 = 4$ sq. units

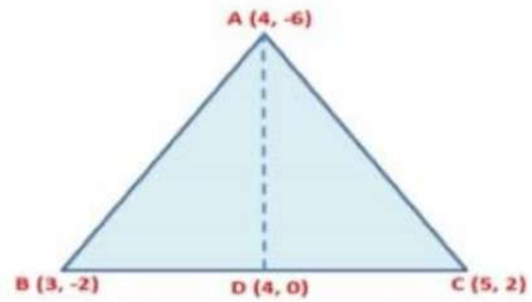
Area of ΔPQR $= \frac{1}{2} \times [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

Here, $P(x_1y_1) = (1, 2)$, $Q(x_2y_2) = (0, 1)$ and $R(x_3y_3) = (1, 0)$

Area of ΔPQR $= \frac{1}{2} \times [1(1 - 0) + 0(1 - 2) + 1(2 - 1)] = \frac{1}{2} \times (1 + 1) = 1$ sq. units

Area of ΔPQR: Area of ΔABC = 1 : 4

Example: If A (4, -6), B (3, -2) and C (5, 2) are the vertices of Δ ABC, then verify the fact that a median of ΔABC divides it into two triangles of equal areas.



D is the midpoint of line segment BC

∴The coordinates of D are $\left(\frac{3+5}{2}, \frac{-2+2}{2}\right) = (4, 0)$

Area of a Triangle$= \frac{1}{2} \times [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

Here, $A(x_1y_1) = (4, -6)$, $B\ (x_2y_2) = (3, -2)$ and $C(x_3y_3) = (5, 2)$

Area of Δ ABC $= \frac{1}{2} \times [4(-2-2) + 3(2+6) + 5(-6+2)] = \frac{1}{2} \times (-16 + 24 - 20)$

$= \frac{1}{2} \times (-12) = -6$

The area of a triangle cannot be a negative value; therefore, we consider only the positive value.

Area of Δ ABC $= 6$ sq. units

In Δ ABD we have, $A(x_1y_1) = (4, -6)$, $B\ (x_2y_2) = (3, -2)$ and $D(x_3y_3) = (4, 0)$

Similarly, Area of Δ ABD $= \frac{1}{2} \times [4(-2-0) + 3(0+6) + 4(-6+2)]$

$= \frac{1}{2} \times (-8 + 18 - 16) = -3$

Now, Area of Δ ABD $= 3$ sq. units

$\frac{Area of \triangle ABC}{Area of \triangle ABD} = \frac{6}{3} = 2$

Area of Δ ABC = 2(Area of Δ ABD)

Example: For what value of x will the points$(x, -1)$, $(2, 1)$ and $(4, 5)$ lie on a line? (REFERENCE: NCERT)

Now, the given points are collinear if the area of the triangle formed by them is zero.

Let A $(x, -1)$, B $(2, 1)$ and C$(4, 5)$ be the given points.

Here, $A(x_1y_1) = (x, -1)$, B $(x_2y_2) = (2, 1)$ and $C(x_3y_3) = (4, 5)$

Area of Δ ABC $= 12\times [x1(y2 - y3) + x2(y3 - y1) + x3(y1 - y2)]$

$= \frac{1}{2}\times [x(1 - 5) + 2(5 + 1) + 4(-1 - 1)] = \frac{1}{2} \times (x - 5x + 12 - 8)$

$= \frac{1}{2} \times (-4x + 4) = -2x + 2$

As the given points are collinear, area of Δ ABC $= 0$

$0 = -2x + 2 \Rightarrow x = 1$

The given points lie on a line if $x = 1$

Example: Find k so that the point P (-4, 6) lies on the line segment joining A (k, 10) and B (3, -8). Also find the ratio in which P divides AB.

If P lies on the line segment AB then P, A and B are collinear and the area of the triangle formed by them is 0.

Here, $A(x_1y_1) = (k, 10)$, B $(x_2y_2) = (3, -8)$ and $P(x_3y_3) = (-4, 6)$

Area of Δ APB $= \frac{1}{2} \times [k(-8 - 6) + 3(6 - 10) - 4(10 + 8)]$

$0 = \frac{1}{2} \times (-14k - 12 - 72) \Longrightarrow -14k - 84 = 0 \Longrightarrow k = -6$

Now, coordinates of A and B are (-6, 10) and (3, -8) respectively

Let P divides AB in the ratio of m : 1. Then, the coordinates of P are

$$\left(\frac{3m-6}{m+1}, \frac{-8m+10}{m+1}\right)$$

But, coordinates of P are (-4, 6)

$$\therefore \frac{3m-6}{m+1} = -a \ and \ \frac{-8m+10}{m+1}$$

$$\Rightarrow 3m - 6 = -4(m + 1) \Rightarrow 3m - 6 = -4m - 4 \Rightarrow 7m = 2 \Rightarrow m = \frac{2}{7}$$

$$\Rightarrow -8m + 10 = 6(m + 1) \Rightarrow -8m + 10 = 6m + 6 \Rightarrow -14m = -4 \Rightarrow m = \frac{2}{7}$$

Hence, P divides AB in the ratio of 2 : 7.