Application of Derivatives
Part - 2

ASSERTION AND REASON BASED MCQs (1 Mark each)

Directions : In the following questions, A statement of Assertion (A) is followed by a statement of Reason (R). Mark the correct choice as.

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true but R is NOT the correct explanation of A

(C) A is true but R is false

(D) A is false and R is True

Q. 1. The total revenue received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$ in rupees.

Assertion (A): The marginal revenue when $x = 5$ is 66.

Reason (R): Marginal revenue is the rate of change of total revenue with respect to the number of items sold at an instance.

Ans. Option (A) is correct.

Marginal revenue is the rate of change of total revenue with respect to the number of items sold at an instance. Therefore R is true.

$$R'(x) = 6x + 36$$
$$R'(5) = 66$$

$\therefore$ A is true.

R is the correct explanation of A.

Q. 2. The radius r of a right circular cylinder is increasing at the rate of 5 cm/min and its height h, is decreasing at the rate of 4 cm/min.

Assertion (A): When $r = 8$ cm and $h = 6$ cm, the rate of change of volume of the cylinder is 224π cm^3/min

Reason (R): The volume of a cylinder is $V = \frac{1}{3}\pi r^2 h$

Ans. Option (C) is correct.

Explanation: The volume of a cylinder is $V=\pi r^2 h$. So R is false.

$$\frac{dr}{dt} = 5 \text{ cm/min}, \frac{dh}{dt} = -4 \text{ cm/min}$$

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = \pi\left(r^2\frac{dh}{dt} + 2hr\frac{dr}{dt}\right)$$

$$\frac{dV}{dt} = \pi[64 \times (-4) + 2 \times 6 \times 8 \times 5]$$

$$\left.\frac{dV}{dt}\right)_{r=8,\, h=6} = 224\pi \text{ cm}^3\text{/min}$$

$\therefore$ Volume is increasing at the rate of 224π cm^3/min.

$\therefore$ A is true.

Q. 3. Assertion (A): For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then at $x = 3$ the slope of curve is decreasing at 36 units/sec.

Reason (R): The slope of the curve is $\frac{dy}{dx}$.

Ans. Option (D) is correct.

Explanation: The slope of the curve $y = f(x)$ is $\frac{dy}{dx}$. R is true.

Given curve is $y = 5x - 2x^3$

or $$\frac{dy}{dx} = 5 - 6x^2$$

or $$m = 5 - 6x^2 \quad \left[\text{slope}\, m = \frac{dy}{dx}\right]$$

$$\frac{dm}{dt} = -12x\frac{dx}{dt} = -24x$$
$$\left[\because \frac{dx}{dt} = 2 \text{ units / sec}\right]$$

$$\left.\frac{dm}{dt}\right|_{x=3} = -72$$

Rate of Change of the slope is decreasing by 72 units/s.

A is false.

Q. 4. A particle moves along the curve $6y = x^3 + 2$.

Assertion (A): The curve meets the Y axis at three points.

Reason (R): At the points $\left(2, \frac{5}{3}\right)$ and $(-2, -1)$ the ordinate changes two times as fast as the abscissa.

Ans. Option (D) is correct.

Explanation:

On Y axis, $x = 0$. The curve meets the Y axis at only one point, *i.e.*, $\left(0, \frac{1}{3}\right)$.

Hence A is false.

$$6y = x^3 + 2$$

or $$6\frac{dy}{dt} = 3x^2\frac{dx}{dt}$$

Given, $$\frac{dy}{dt} = 2\frac{dx}{dt}$$

or $$12 = 3x^2$$

or $$x = \pm 2$$

Put $x = 2$ and -2 in the given equation to get y

$\therefore$ The points are $\left(2, \frac{5}{3}\right), (-2, -1)$

R is true.

Q. 5. Assertion (A): At $x = \frac{\pi}{6}$, the curve $y = 2\cos^2(3x)$ has a vertical tangent.

Reason (R): The slope of tangent to the curve $y = 2\cos^2(3x)$ at $x = \frac{\pi}{6}$ is zero.

Ans. Option (D) is correct.

Explanation:

Given $y = 2\cos^2(3x)$

$$\frac{dy}{dx} = 2 \times 2 \times \cos(3x) \times (-\sin 3x) \times 3$$

$$\frac{dy}{dx} = -6\sin 6x$$

$$\left.\frac{dy}{dx}\right]_{x=\frac{\pi}{6}} = -6\sin\pi = -6 \times 0 = 0$$

$\therefore$ R is true.

Since the slope of tangent is zero, the tangent is parallel to the X-axis. That is the curve has a horizontal tangent at $x = \frac{\pi}{6}$. Hence A is false.

Q. 6. Assertion (A): The equation of tangent to the curve $y = \sin x$ at the point (0, 0) is $y = x$.

Reason (R): If $y = \sin x$, then $\frac{dy}{dx}$ at $x = 0$ is 1.

Ans. Option (A) is correct.

Explanation: Given $y = \sin x$

$$\frac{dy}{dx} = \cos x$$

Slope of tangent at (0, 0) $= \left[\frac{dy}{dx}\right]_{(0,0)}$

$$= \cos 0° = 1$$

$\therefore$ R is true.

Equation of tangent at (0, 0) is

$$y - 0 = 1(x - 0)$$

$\Rightarrow$ $y = x$.

Hence A is true.

R is the correct explanation of A.

Q. 7. Assertion (A): The slope of normal to the curve $x^2 + 2y + y^2 = 0$ at (–1, 2) is –3.

Reason (R): The slope of tangent to the curve $x^2 + 2y + y^2 = 0$ at (–1, 2) is $\frac{1}{3}$.

Ans. Option (A) is correct.

Explanation:

Given $x^2 + 2y + y^2 = 0$

$$2x + 2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2 + 2y) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2(1+y)} = -\frac{x}{1+y}$$

Slope of tangent at (–1, 2)

$$\left[\frac{dy}{dx}\right]_{(-1,2)} = \frac{-(-1)}{1+2} = \frac{1}{3}$$

Hence R is true.

Slope of normal at (–1, 2)

$$= \frac{-1}{\text{Slope of tangent}} = -3.$$

Hence A is true.

R is the correct explanation for A.

Q. 8. The equation of tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$.

Assertion (A): The value of a is ± 2

Reason (R): The value of b is ± 7

Ans. Option (C) is correct.

Explanation:

$$y^2 = ax^3 + b$$

Differentiate with respect to x,

$$2y\frac{dy}{dx} = 3ax^2$$

or $$\frac{dy}{dx} = \frac{3ax^2}{2y}$$

or $$\frac{dy}{dx} = \frac{3ax^2}{\pm 2\sqrt{ax^3 + b}} \quad [\because y^2 = ax^3 + b]$$

or $$\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{3a(2)^2}{\pm 2\sqrt{a(2)^3 + b}} = \frac{12a}{\pm 2\sqrt{8a + b}} = \frac{6a}{\pm\sqrt{8a + b}}$$

Since (2, 3) lies on the curve

$$y^2 = ax^3 + b$$

or $$9 = 8a + b \quad \text{...(i)}$$

Also from equation of tangent

$$y = 4x - 5$$

slope of the tangent = 4

$\therefore$ $\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{6a}{\pm\sqrt{8a + b}}$ becomes

$$4 = \frac{6a}{\pm\sqrt{9}} \quad \{\text{from (i)}\}$$

$\therefore$ $$4 = \frac{6a}{\pm 3}$$

$\therefore$ $$4 = \frac{6a}{3} \text{ or } 4 = \frac{6a}{-3}$$

either, $a = 2$ or $a = -2$

For $a = 2$,

$9 = 8(2) + b$

or $b = -7$

$\therefore$ $a = 2$ and $b = -7$

and for $a = -2$,

$9 = 8(-2) + b$

or $b = 25$

or $a = -2$ and $b = 25$

Hence A is true and R is false.

Q. 9. Assertion (A): The function $f(x) = x^3 - 3x^2 + 6x - 100$ is strictly increasing on the set of real numbers.

Reason (R): A strictly increasing function is an injective function.

Ans. Option (B) is correct.

Explanation:

$$f(x) = x^3 - 3x^2 + 6x - 100$$
$$f'(x) = 3x^2 - 6x + 6$$
$$= 3[x^2 - 2x + 2]$$
$$= 3[(x-1)^2 + 1]$$

since $f'(x) > 0;\ x \in R$

$f(x)$ is strictly increasing on R.

Hence A is true.

For a strictly increasing function,

$x_1 > x_2$

$\Rightarrow$ $f(x_1) > f(x_2)$

i.e.; $x_1 = x_2$

$\Rightarrow$ $f(x_1) = f(x_2)$

Hence, a strictly increasing function is always an injective function.

So R is true.

But R is not the correct explanation of A.

Q. 10. Consider the function $f(x) = \sin^4 x + \cos^4 x$.

Assertion (A): $f(x)$ is increasing in $\left[0, \frac{\pi}{4}\right]$

Reason (R): $f(x)$ is decreasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

Ans. Option (B) is correct.

Explanation:

$$f(x) = \sin^4 x + \cos^4 x$$

or $$f'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x$$
$$= -4\sin x\cos x\,[-\sin^2 x + \cos^2 x]$$
$$= -2\sin 2x \cos 2x$$
$$= -\sin 4x$$

On equating,

$f'(x) = 0$

or $-\sin 4x = 0$

or $4x = 0, \pi, 2\pi, \ldots\ldots\ldots\ldots$

or $x = 0, \frac{\pi}{4}, \frac{\pi}{2}$.

Sub-intervals are $\left[0, \frac{\pi}{4}\right], \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

or $f'(x) < 0$ in $\left[0, \frac{\pi}{4}\right]$

or $f(x)$ is decreasing in $\left[0, \frac{\pi}{4}\right]$

and, $f'(x) > 0$ in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

$\therefore$ $f'(x)$ is increasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

Both A and R are true. But R is not the correct explanation of A.

Q. 11. Assertion (A): The function $y = [x(x-2)]^2$ is increasing in $(0, 1) \cup (2, \infty)$

Reason (R): $\frac{dy}{dx} = 0$, when $x = 0, 1, 2$.

Ans. Option (B) is correct.

Explanation:

$$y = [x(x-2)]^2$$
$$= [x^2 - 2x]^2$$

$\therefore$ $$\frac{dy}{dx} = 2(x^2 - 2x)(2x - 2)$$

or $$\frac{dy}{dx} = 4x(x-1)(x-2)$$

On equating $\frac{dy}{dx} = 0$,

$4x(x-1)(x-2) = 0 \Rightarrow x = 0, x = 1, x = 2$

$\therefore$ Intervals are $(-\infty, 0), (0,1), (1,2), (2, \infty)$

Since, $\frac{dy}{dx} > 0$ in $(0,1)$ or $(2, \infty)$

$\therefore$ $f(x)$ is increasing in $(0,1) \cup (2, \infty)$

Both A and R are true. But R is not the correct explanation of A.

Q. 12. Assertion (A): The function $y = \log(1 + x) - \frac{2x}{2+x}$ is a decreasing function of x throughout its domain.

Reason (R): The domain of the function $f(x) = \log(1 + x) - \frac{2x}{2+x}$ is $(-1, \infty)$

Ans. Option (D) is correct.

Explanation:

$\log(1 + x)$ is defined only when $x + 1 > 0$ or $x > -1$.

Hence R is true.

$$y = \log(1+x) - \frac{2x}{2+x}$$

Diff. w.r.t. 'x',

$$\frac{dy}{dx} = \frac{1}{1+x} - \frac{[(2+x)(2) - 2x]}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{[4 - 2x - 2x]}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(2+x)^2(1+x)}$$

$$= \frac{4 + x^2 + 4x - 4 - 4x}{(2+x)^2(1+x)}$$

$$= \frac{x^2}{(2+x)^2(1+x)}$$

For increasing function,

$$\frac{dy}{dx} \geq 0$$

or $$\frac{x^2}{(2+x)^2(x+1)} \geq 0$$

or $$\frac{(2+x)^2(x+1)x^2}{(2+x)^4(x+1)^2} \geq 0$$

or $$(2+x)^2(x+1)x^2 \geq 0$$

When $x > -1$,

$\frac{dy}{dx}$ is always greater than zero.

$\therefore$ $$y = \log(1+x) - \frac{2x}{2+x}$$

is always increasing throughout its domain.

Hence A is false.

Q. 13. The sum of surface areas (S) of a sphere of radius 'r' and a cuboid with sides $\frac{x}{3}$, x and $2x$ is a constant.

Assertion (A): The sum of their volumes (V) is minimum when x equals three times the radius of the sphere.

Reason (R): V is minimum when $r = \sqrt{\frac{S}{54 + 4\pi}}$

Ans. Option (A) is correct.

Explanation:

Given $$S = 4\pi r^2 + 2\left[\frac{x^2}{3} + 2x^2 + \frac{2x^2}{3}\right]$$

$$S = 4\pi r^2 + 6x^2$$

or $$x^2 = \frac{S - 4\pi r^2}{6}$$

and $$V = \frac{4}{3}\pi r^3 + \frac{2x^3}{3}$$

$\therefore$ $$V = \frac{4}{3}\pi r^3 + \frac{2}{3}\left(\frac{S - 4\pi r^2}{6}\right)^{3/2}$$

$$\frac{dV}{dr} = 4\pi r^2 + \left(\frac{S - 4\pi r^2}{6}\right)^{½}\left(\frac{-8\pi r}{6}\right)$$

$$\frac{dV}{dr} = 0$$

or $$r = \sqrt{\frac{S}{54 + 4\pi}}$$

Now $$\frac{d^2V}{dr^2} = 8\pi r + \left(\frac{-8\pi}{6}\right)\left(\frac{S - 4\pi r^2}{6}\right)^{½} + \frac{1}{2}\left(\frac{S - 4\pi r^2}{6}\right)^{-½} \cdot \left(\frac{-8\pi r}{6}\right)$$

at $$r = \sqrt{\frac{S}{54 + 4\pi}}; \frac{d^2V}{dr^2} > 0$$

$\therefore$ for $r = \sqrt{\frac{S}{54 + 4\pi}}$ volume is minimum

i.e., $r^2(54 + 4\pi) = S$

or $r^2(54 + 4\pi) = 4\pi r^2 + 6x^2$

or $6x^2 = 54r^2$

or $x^2 = 9r^2$

or $x = 3r$

Hence both A and R are true.

R is the correct explanation of A.

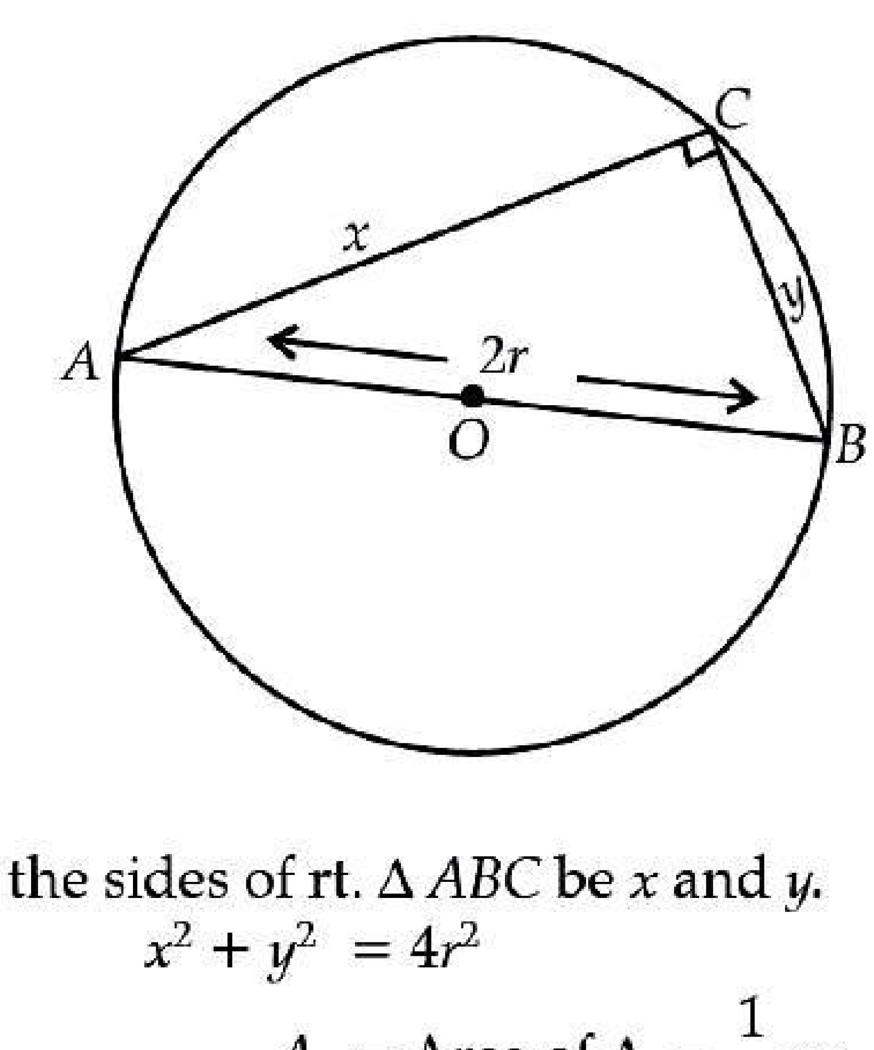
Q. 14. AB is the diameter of a circle and C is any point on the circle.

Assertion (A): The area of ΔABC is maximum when it is isosceles.

Reason (R): ΔABC is a right-angled triangle.

Ans. Option (A) is correct.

Explanation:



Let the sides of rt. Δ ABC be x and y.

$\therefore$ $x^2 + y^2 = 4r^2$

and A = Area of Δ = $\frac{1}{2}xy$

Let, $S = A^2$

$$= \frac{1}{4}x^2y^2$$

$$= \frac{1}{4}x^2(4r^2 - x^2)$$

$$= \frac{1}{4}(4r^2x^2 - x^4)$$

$\therefore$ $$\frac{dS}{dx} = \frac{1}{4}[8r^2x - 4x^3]$$

or $$\frac{dS}{dx} = 0$$

or $$x^2 = 2r^2 \text{ or } x = \sqrt{2}r$$

and $$y^2 = 4r^2 - 2r^2 = 2r^2$$

or $$y = \sqrt{2}r$$

i.e., $$x = y \text{ and } \frac{d^2S}{dx^2} = (2r^2 - 3x^2)$$

$$= 2r^2 - 6r^2 < 0$$

or Area is maximum, when Δ is isosceles.

Hence A is true.

Angle in a semicircle is a right angle.

$\therefore \angle C = 90°$

$\Rightarrow \Delta ABC$ is a right-angled triangle.

$\therefore$ R is true.

R is the correct explanation of A.

Q. 15. A cylinder is inscribed in a sphere of radius R.

Assertion (A): Height of the cylinder of maximum volume is $\frac{2R}{\sqrt{3}}$ units.

Reason (R): The maximum volume of the cylinder is $\frac{4\pi R^3}{\sqrt{3}}$ cubic units.

Ans. Option (C) is correct.

Explanation: Let the radius and height of cylinder be r and h respectively

$\therefore$ $$V = \pi r^2 h \quad \text{...(i)}$$

But $$r^2 = R^2 - \frac{h^2}{4}$$

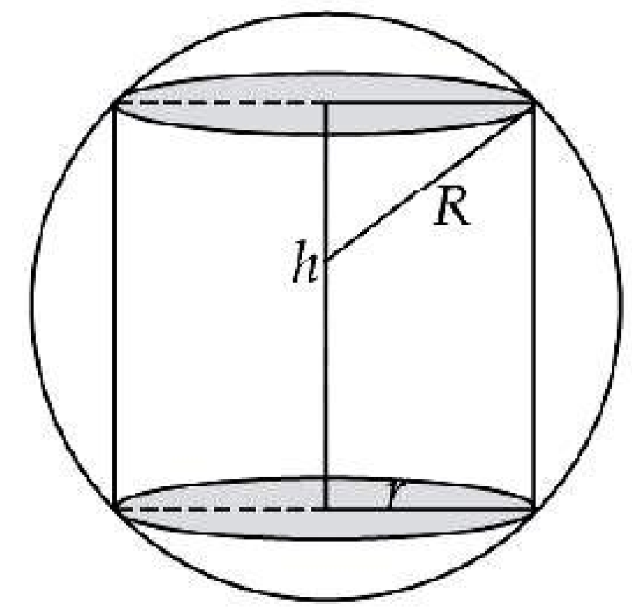
$\therefore$ $$\pi h\left(R^2 - \frac{h^2}{4}\right) = \pi\left(R^2h - \frac{h^3}{4}\right)$$

or $$\frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right)$$

For maximum or minimum

$\therefore$ $$\frac{dV}{dh} = 0 \text{ or } h^2 = \frac{4R^2}{3}$$

or $$h = \frac{2R}{\sqrt{3}}$$



and $$\frac{d^2V}{dh^2} = \pi\left(-\frac{6h}{4}\right) < 0$$

$$\text{Maximum volume} = \pi\cdot\left[R^2\cdot\frac{2R}{\sqrt{3}} - \frac{1}{4}\left(\frac{2R}{\sqrt{3}}\right)^3\right]$$

$$= \frac{4\pi R^3}{3\sqrt{3}} \text{ cubic units}$$

Hence A is true and R is false.

Q. 16. Assertion (A): The altitude of the cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Reason (R): The maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.

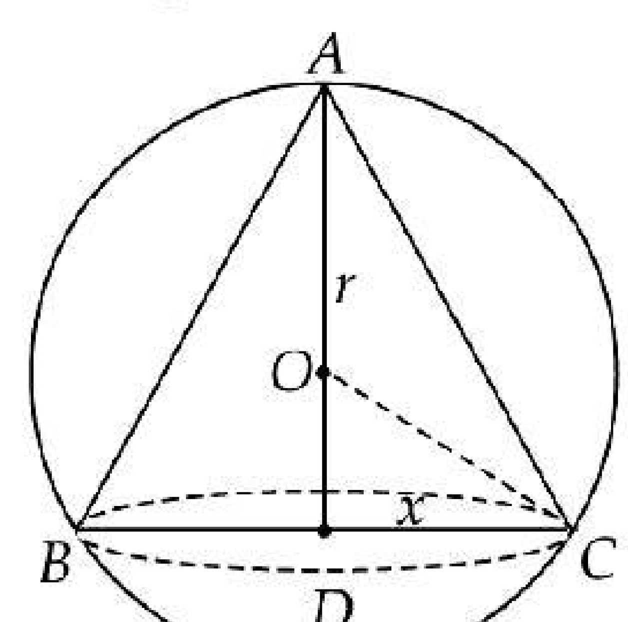
Ans. Option (B) is correct.

Explanation: Let radius of cone be x and its height be h.

$\therefore$ $$OD = (h - r)$$

Volume of cone

$$(V) = \frac{1}{3}\pi x^2 h \quad \text{...(i)}$$



In ΔOCD, $x^2 + (h - r)^2 = r^2$ or $x^2 = r^2 - (h - r)^2$

$\therefore$ $$V = \frac{1}{3}\pi h\{r^2 - (h - r)^2\}$$

$$= \frac{1}{3}\pi(-h^3 + 2h^2r)$$

or $$\frac{dV}{dh} = \frac{\pi}{3}(-3h^2 + 4hr)$$

$\therefore$ $$\frac{dV}{dh} = 0 \text{ or } h = \frac{4r}{3}$$

$$\frac{d^2V}{dh^2} = \frac{\pi}{3}(-6h+4r)$$

$$= \frac{\pi}{3}\left(-6\left(\frac{4r}{3}\right)+4r\right)$$

$$= -\frac{4\pi r}{3} < 0$$

$\therefore$ at $h = \frac{4r}{3}$, Volume is maximum

Maximum volume

$$= \frac{1}{3}\pi\cdot\left\{-\left(\frac{4r}{3}\right)^3 + 2\left(\frac{4r}{3}\right)^2 r\right\}$$

$$= \frac{8}{27}\cdot\left(\frac{4}{3}\pi r^3\right)$$

$$= \frac{8}{27} \text{ (volume of sphere)}$$

Hence both A and R are true.

R is not the correct explanation of A.