SETS, RELATIONS, FUNCTIONS & BINARY OPERATION

OPERATIONS ON SETS

1. **Symbols:** Some commonly used symbols are as follows:

Symbols	*Meaning*
$\Rightarrow$	implies
$\in$	belongs to
$A \subset B$	A is a subset of B
$\Leftrightarrow$	implies and is implied by
$\notin$	does not belong to
s.t.	such that
$\forall$	for every
$\exists$	there exists
iff	if and only if

&	and
a/b	a is a divisor of b
N	set of natural numbers
I	set of integers
R	set of real numbers
C	set of complex numbers
Q	set of rational numbers

2. **Sets:** A set (class, aggregate, ensemble) S is a well-defined collection of objects, or symbols, called *elements* or *members* of the set.

3. **Representation of Sets**

   (a) Roaster Method (Listing Method): We list all the elements and enclose them in curly brackets; e.g.,

   (i) {2, 3, 5, 7} is the set of prime number less than 10.

   (ii) {a, e, i, o, u} is a set of vowels in English alphabet.

   (iii) {1, 2, 3, 4, ...} is the set of natural numbers.

   ***(b) Set Builder Method (Property form)*:** The set of all elements of n, which satisfy a given property p (say) is represented by $\{x : p(x)\}$ or $\{x \mid p(x)\}$ where the symbols ':' or '|' stands for 'such that' and $p(x)$ means x has the property p, e.g.,

$$B = \{x : x = 2n, n \in R\} \text{ or } \{x \mid x = 2n, n \in R\}$$
$$= \{2, 4, 6, 8, ...\}$$

4. **Number Sets:** We give below number sets:
   (i) The set of natural numbers
   $N = \{1, 2, 3, 4, ...\}$
   (ii) The set of integers
   Z or $I = \{....., -3, -2, -1, 0, 1, 2, 3, ...\}$
   (iii) The set of positive integers
   Z^+ or $I^+ = \{1, 2, 3, ...\} = N$
   (iv) The set of negative integers
   Z^- or $I^- = \{-1, -2, -3, ...\}$
   (v) The set of non-negative integers
   $W \equiv$ the set of whole nos. $= \{0, 1, 2, 3, ...\}$
   (vi) The set of non-zero integers
   $Z_0 = \{\pm1, \pm2, \pm3, ...\}$
   (vii) The set of all real numbers is denoted by R.
   (viii) The set of all irrational numbers R – Q. A number which is real but not rational is called an irrational number.

   e.g. $\pi, e, \sqrt{2}, \sqrt{3}, \log 2$, etc.

5. **Subset:** If A and B are two sets such that every element of A is also element of B i.e.,

$x \in A \Rightarrow x \in B$, then we say that A is a subset of B or A is contained in B, it is denoted by $A \subseteq B$, some authors write it as $A \subset B$.

6. **Proper and Improper Subsets:** The null set ϕ is subset of every set and every set is subset of itself, i.e., $\phi \subseteq A$ and $A \subseteq A$ for every set A. They are called improper subsets of A. Thus every non-empty set has two subsets, i.e., A has two improper subset iff it is non-empty. All other subsets of A are called its proper subsets.
   Thus if $A \subseteq B$, $A \neq B$, $A \neq \phi$, then A is said to be proper subset of B.
7. **Number of Subset of a finite set:** If set A has n elements, then A has 2^n subsets.
8. **Power Set:** The family (set) of all subsets of a Set A is called the power set of A. It is denoted by P(A).
   Thus, $n(A) = p \Rightarrow n\ (P(A)) = 2^p$.

Some Important Deductions:

(i) $A \subseteq A, \forall\ A$

(ii) $\phi \subseteq A, \forall\ A$

(iii) $A \subseteq U$ (the universal set), $\forall$ A in U

(iv) $A = B \Leftrightarrow A \subseteq B.\ B \subseteq A$.

9. **Theorem:** If a finite set S has n elements, then the proper set of S has 2^n elements.

Proof: Let a set S contains n elements. The number of the subsets which have no elements at all = ${}^nC_0 = 1$. The null set is a subset of every set.

The no. of those subsets of S each of which has exactly one element = no. of elements in the set $n = {}^nC_1$.

The no. of those subsets of S each of which has exactly two elements out of n elements of S = ${}^nC_2$.

The no. of those subsets of S each of which has exactly r elements out of n elements of S = ${}^nC_r$.

Similarly, the no. of those subsets of which has exactly n elements of $S = {}^nC_n = 1$.

Therefore, total number of subsets of S

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_r + ... + {}^nC_n$$
$$= 1 + {}^nC_1 + {}^nC_2 + ... + {}^nC_r + ... + 1$$
$$= (1 + 1)^n = 2^n$$

Hence, if S is a finite set of order n, then the power set of S has 2^n elements. We can say that a set with three elements has 2^3, i.e., 8 subsets.

10. **Union of two sets:** The union of sets A and B, written $A \cup B$ is the set of all elements

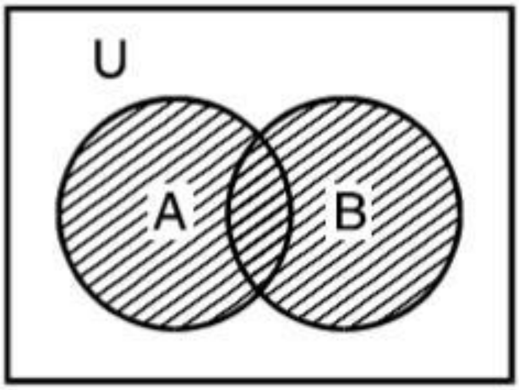
that are in A or in B or in A and B both.

Thus, $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$, and

$x \notin A \cup B \Leftrightarrow x \notin A$ or $x \notin B$.

For example, let $A = \{a, b, c\}$, $B = \{c, g, h\}$

then, $A \cup B = \{a, b, c, g, h\}$



Important Deductions:

(i) $A \cup A = A$ i.e., the union of set is indempotent.

(ii) $A \cup U = U$; where U is the universal set.

(iii) $A \cup \phi = A$; where A is any set.

(iv) $A \subseteq A \cup B$; $B \subseteq A \cup B$.

(v) $A \cup B = B \cup A$ (Commutative law)

(vi) Let A and B be two finite sets such that $n(A) = p$, $n(B) = q$, then, min.$n(A \cup B) = \max.(p, q)$ and max. $n(A \cup B) = p + q$.

(vii) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)

(viii) $A \cup A' = U$ where A is any subset of the universal set U and A' is complement.

Rule to write $A \cup B$ when A and B are both in Roaster form:

(i) Write all the elements of A.

(ii) Write all the elements of B, dropping the element which are in A.

In Venn diagram, the shaded region under the boundary curve represents $A \cup B$.

Union of more than two sets: If $A_1, A_2, A_3 ..... A_n$ are the subset of U and $n \in N$, then the set

$$A_1 \cup A_2 \cup A_3 \overset{\subset}{\cup} ..... A_n = \bigcup_{i=1}^{n} A_i$$

consists of the members of U which belongs to at least of the subsets of A_i.

For example: if $A = \{a, b, c, d\}, B = \{c, d, e, f\}$

$C = \{a, c, e, g\}, D = \{b, c, d, g\}$

$A \cup B \cup C \cup D = \{a, b, c, d, e, f, g\}$

11. Intersection of two sets: The intersection of two sets A and B is the set of elements which are common to both A and B and is denoted by $A \cap B$.

This is read as 'the intersection of A and B' or simply 'A intersection B'.

Thus,

$$x \in A \cap B \Leftrightarrow x \in A \text{ and } x \in B.$$

$$\therefore \quad A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

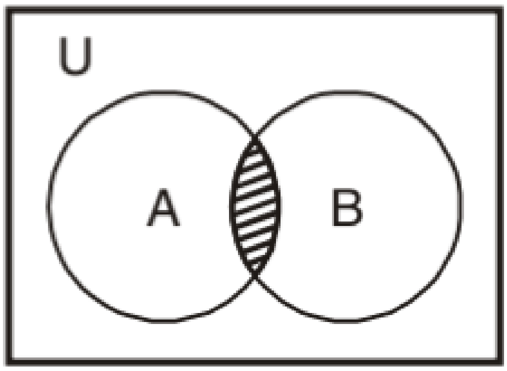
Also, $x \notin A \cap B \Leftrightarrow x \notin A$ or $x \notin B$.

For example: If $A = \{a, c, d, e, g, i, r, o, t, u\}$ and
$B = \{a, e, i, o, u\}$

then, $A \cap B = \{a, e, i, o, u\}$

If A and B are disjoint sets (i.e., if A and B have no element in common), then, $A \cap B = \phi$.

In Venn diagram, the shaded region under the thick boundary curve represents $A \cap B$.



Intersection of more than two sets: If A_1, A_2, A_3 ..... A_n are the subset of $\cup$ and $n \in N$, then the set

$$A_1 \cap A_2 \cap A_3 \cap A_4 \cap ..... \cap A_n = \bigcap_{i=1}^{n} A_i$$

$$= \{x : x \in A_i \ \forall \ i\text{'s}\}$$

Important Deductions:

(i) $A \cap \phi = \phi$

(ii) $A \cap U = A$

(iii) $A \cap A = A$; i.e., intersection of sets is indempotent.

(iv) $A \cap B \subseteq A$

(v) $A \cap B \subseteq B$

(vi) $A \cap B \subseteq A \cup B$

(vii) $A \subseteq B \Rightarrow A \cap B = A$

(viii) Let A and B be finite sets such that $n(A) = p$, $n(B) = q$.

then, min. $n(A \cap B) = 0$,

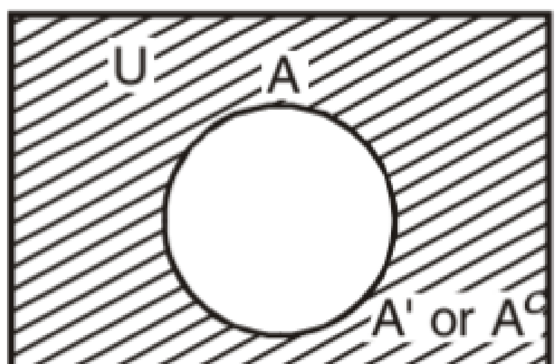
max. $n(A \cap B) = \min.(p, q)$

(ix) $A \cap B = B \cap A$ i.e., the intersection of sets is commutative

(x) $A \cap A' = \phi$

(xi) $(A \cap B) \cap C = A \cap (B \cap C)$: i.e., the intersection of sets is associative.

12. Complement: The complement of a set A is the set of all elements of the universal set U which do not belong to A and is denoted by A' (or A^c).



For Example: If $A = \{2, 4, 6, ...\}$

and $U = N = \{1, 2, 3, ...\}$, then

$A' = \{1, 3, 5, 7, ...\}$

also, $(A')' = \{2, 4, 6, ...\} = A.$

Important Deductions:

(i) $A \cap A' = \phi$

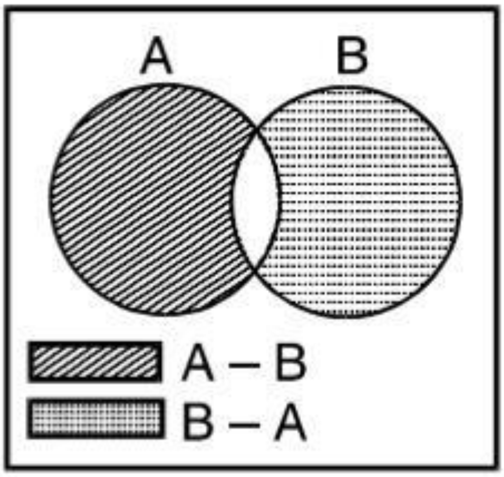
(ii) $A \cup A' = \cup$

(iii) $U' = \phi;\ \phi' = U$

(iv) $(A')' = A$

(v) $A - B = A \cap B'$

13. Difference of Sets: Difference of two sets A and B, denoted by $A - B$, is the set of all elements of A which are not in B.



Thus, $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$

and so $x \notin A - B \Leftrightarrow x \notin A$ and $x \in B$

Also, $A - B = \{x : x \in a, x \notin B\}$

For example,

(i) $A = \{1, 2, 3\}$. $B = \{3, 4, 5\}$

Then, $A - B = \{1, 2\}$

$B - A = \{4, 5\}$

(ii) $R - Q$ is the set of all irrational numbers.

Important Deductions:

(i) $A - B \subseteq A$, $B - A \subseteq B$

(ii) $A \subseteq B \Leftrightarrow A - B = \phi$

(iii) $A - B \neq B - A$

(iv) $A - B = A - (A \cap B)$

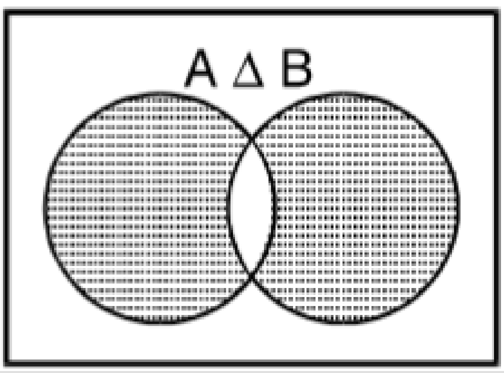
(v) $A - B$, $B - A$, $A \cap B$ are pair wise disjoint.

14. Symmetric Difference: It is the set of all those elements which are only in A or only in B, i.e., $(A - B) \cup (B - A)$ is called the symmetric difference of A and B. It is usually denoted by $A \Delta B$.

Also, $$A \Delta B = (A \cup B) - (A \cap B).$$

For example, if $A = \{a, e, i, o, u\}$, $B = \{b, d, e, i\}$

then,
$$A \Delta B = (A \cup B) - (A \cap B)$$
$$= \{a, b, d, e, i, o, u\} - \{e, i\}$$
$$= \{a, b, d, o, u\}$$



15. Cartesian Product of two sets: Cartesian product of sets A and B denoted by $A \times B$, is the set of all ordered pairs of which first coordinates are elements of the set A and second coordinates, the elements of set B.

Symbolically

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

$$B \times A = \{(b, a) \mid b \in B \text{ and } a \in A\}$$ is the cartesian product of B and A.

For example:

Let $A = \{a, b\}, B = \{1, 2, 3\}$

Then, $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

Important Deductions:

(i) In a ordered pair, the order of occurrence of members is of prime importance. Ordered pair $(a, 1)$ is not the same as ordered pair $(1, a)$ i.e., $(a, 1) \neq (1, a)$.

(ii) Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$, i.e.,

$$(a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d.$$

(iii) The cartesian product of two sets is not commutative

i.e., $A \times B \neq B \times A$

But, $n(A \times B) = n(B \times A)$

(*iv*) If A has p elements and B has q elements, then $A \times B$ has pq elements i.e., if $n(A) = p$, $n(B) = q \Rightarrow n\,(A \times B) = pq$.

(*v*) If $A = \phi$ or $B = \phi$, then $A \times B = \phi$

(*vi*) $B \subset C \Leftrightarrow A \times B \subset A \times C$

16. Some Important Laws:

(*i*) **Commutative laws**

(*a*) $A \cup B = B \cup A$

(*b*) $A \cap B = B \cap A$

(*c*) $A - B \neq B - A$

(*d*) $A \Delta B = B \Delta A$

(*e*) $A \times B \neq B \times A$

(*ii*) **De-Morgan's law**

(*a*) $A - (B \cup C) = (A - B) \cap (A - C)$

(*b*) $A - (B \cap C) = (A - B) \cup (A - C)$

(*c*) $(A \cup B)' = A' \cap B'$

(*d*) $(A \cup B)' = A' \cap B'$

(*iii*) **Distributive law**

(*a*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(*b*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(*c*) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(*d*) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(*e*) $A \times (B - C) = A \times B - A \times C$

(*iv*) **Associative law**

(*a*) $(A \cup B) \cup C = A \cup (B \cup C)$

(*b*) $(A \cap B) \cap C = A \cap (B \cap C)$

(c) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$
(d) $(A - B) - C \neq A - (B - C)$
(e) $(A \times B) \times C \neq A \times (B \times C)$

17. Some more results: Let A, B and C be finite sets and U be the finite universal set, then

(a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(b) $n(A \cup B) = n(A) + n(B)$; [if A and B are disjoint set]

(c) $n(A - B) = n(A) - n(A \cap B)$
i.e., $n(A) = n(A - B) + n(A \cap B)$

(d) $n(A \Delta B) = n[(A - B) \cup (B - A)]$
$= n[(A \cup A') \cup (A' \cap B)]$
$= n(A) + n(B) - 2n(A \cap B)$

(e) $n(A' \cap B') = n$ (neither A nor B)
$= n(A \cup B)'$
$= n(U) - n(A \cup B)$

(f) $n(A' \cup B') = n(A \cap B)'$
$= n(U) - n(A \cap B)$

(g) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

(h) n (set of elements in exactly two of the sets A, B, C)
$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

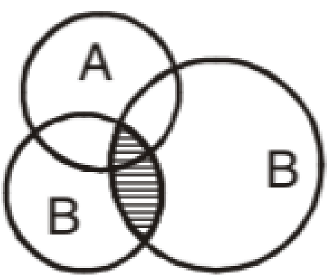
(i) n (set of elements which are in at least two of the sets A, B, C)

$$= n(A \cap B) + n(A \cap C) + n(B \cap C) - 2n(A \cap B \cap C)$$

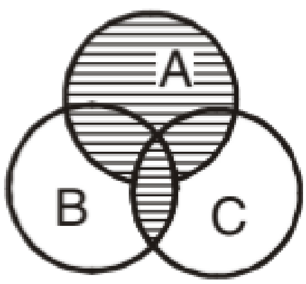
(j) n (set of elements which are in exactly one of the sets A, B, C)

$$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap C)$$

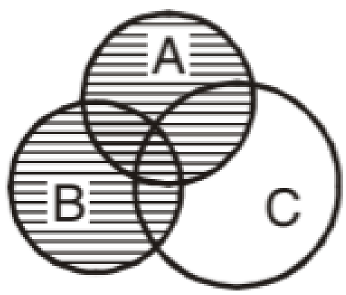
(k) If A, B, C are three sets, then the distributive law can be easily illustrated by Venn diagrams shown below:



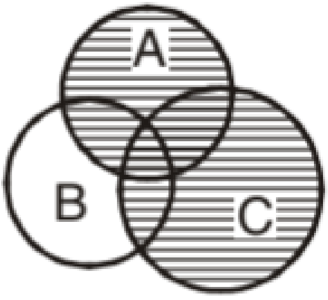
$B \cap C$



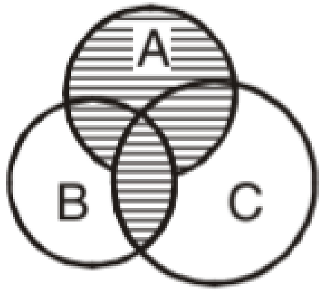
$A \cup (B \cap C)$



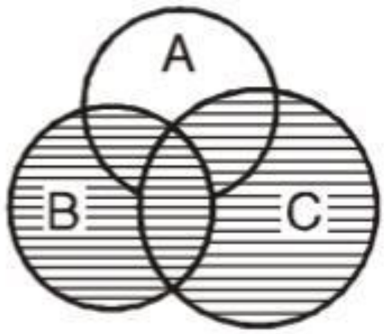
$A \cup B$



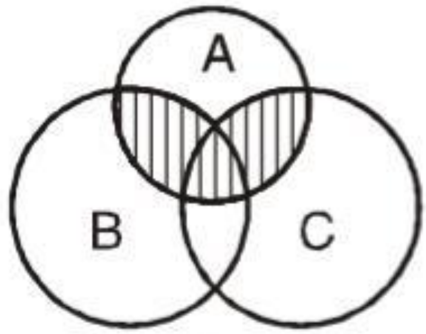
$A \cup C$



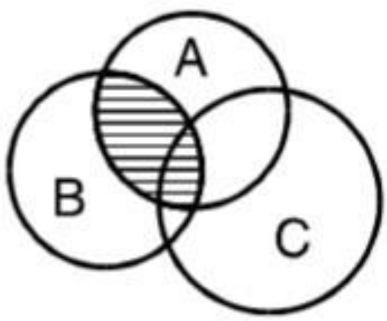
$(A \cup B) \cap (A \cup C)$



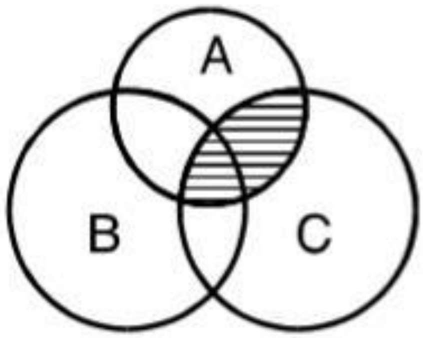
$B \cup C$



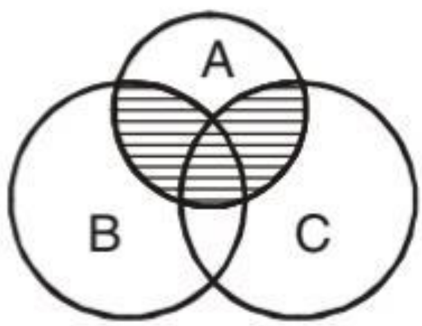
$A \cap (B \cup C)$



$A \cap B$



$A \cap C$



$(A \cap B) \cup (A \cap C)$

RELATIONS

1. If A and B are two sets, then a relation from A to B is a subset of the product $A \times B$. Symbolically if R is a relation from A to B *i.e.*, if $R \subseteq A \times B$ and $(a, b) \in R$, then we can say that a is related to b by the relation R and write it as $a\text{R}b$. If $(a, b) \notin \text{R}$ then, $a \not\text{R}\, b$.

For example:

Let $\text{A} = \{1, 2, 3, 4, 5\}$, $\text{B} = \{1, 3\}$

then, we set a relation from A to B as: $a \text{ R } b$ iff $a \leq b$; $a \in \text{A}$, $b \in \text{B}$

Then, $\text{R} = \{(1, 1), (1, 3), (2, 3)\} \subset \text{A} \times \text{B}$

2. **Domain and Range of relation:** The set of all the first elements of the ordered pairs which belong to R is called the *domain* and the set of all the second elements of that ordered pairs is called the *range*.

   Thus, Dom. $\text{R} \subseteq \text{A}$; Range $\text{R} \subseteq \text{B}$.

   For example, in the relation $\{(6, 8), (3, 7), (1, 2)\}$ the domain is $(6, 3, 1)$ and range is $(8, 7, 2)$.

3. **Composition of Relations:** If $\text{R} \subseteq \text{A} \times \text{B}$ and $\text{S} \subseteq \text{B} \times \text{C}$ be two relations then composition of the relations R and S is denoted by $\text{S}o\text{R} \subseteq \text{A} \times \text{C}$. It is defined by $(a,c) \in (\text{S } o \text{ R})$ iff $\exists\, b \in \text{B}$ such that $(a, b) \in \text{R}$ and $(b, c) \in \text{S}$.

For Example,

$$\text{Let A} = \{1, 2, 3\}, \text{B} = \{a, b, c, d\}, \text{C} = \{\alpha, \beta, \gamma\}$$

$$\text{R} (\subseteq \text{A} \times \text{B}) = \{(1, a), (1, c), (2, d)\}$$

$$S (\subseteq B \times C) = \{(a, \alpha), (a, \gamma), (c, \beta)\}$$

Then, $S \, o \, R (\subseteq A \times C) = \{(1, \alpha), (1, \gamma), (1, \beta)\}$

Note: One should be careful in computing the relation R *o* S. Actually R *o* S starts with S and S *o* R starts will R. In general $S \, o \, R \neq R \, o \, S$.

4. **Inverse Relation:** If R be a relation from A set to *a* set B, then the relation R^{-1} from set B to the set A is defined as the inverse relation R.

   Symbolically, $R^{-1} = \{(b, a) : (a, b) \in R\}$

   Hence, it is clear that

   (i) $a \, R \, b \Leftrightarrow b \, R^{-1} \, a$

   (ii) dom. R^{-1} = range R and range R^{-1} = dom. R

   (iii) $(R^{-1})^{-1} = R$.

5. **Void relation in a Set:** Consider the set A × A, then every subset of A × A is a relation in A. Again the null set ϕ is a subset of A × A, therefore, the null set ϕ is also a relation in A. This relation is called the void relation in A.

   For any ordered pair (a, b) with $a \in A$ and $b \in A$ we have $a \not R \, b$ i.e., $(a, b) \notin R$.

   For example, let A = (2, 3, 5) and let R be the relation 'divides' then, R is a void relation since $R = \phi \subset A \times A$.

6. **Reflexive Relation:** R is a reflexive relation if $(a, a) \in R, \forall a \in R$. It should be noted that if $\exists$ any $a \in A$ such that $a \not{R}\, a$, then R is not reflexive.

For example, let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 3)\}$, then R is not reflexive since $3 \in A$ but $(3, 3) \in R$.

7. **Symmetric Relation:** R is called a symmetric relation on A if $(x, y) \in R \Rightarrow (y, x) \in R$, i.e., if x is R-related to y, then y is also R-related to x. It should be noted that R is symmetric iff $R^{-1} = R$.

8. **Anti-Symmetric Relation:** R is called anti-symmetric relation if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$.

Thus, if $a \neq b$, then a may be related to b or b may be related to a, but never both.

For example, let N be the set of natural numbers. A relation $R \subseteq N \times N$ is defined by:

$$x \text{ R } y \text{ iff } x \text{ divides } y \text{ (i.e., } x/y)$$

Then, $x \text{ R } y, y \text{ R } x \Rightarrow x$ divides y, y divides x

$$\Rightarrow x = y$$

It should be noted that the set $\{(a, a) : a \in A\} = D$ is called the diagonal line of $A \times A$.

Then "the relation R in A is antisymmetric iff $R \cap R^{-1} \subseteq D$".

9. **Transitive Relations:** A relation R in a set A is said to be transitive if a R b and b R $c \Rightarrow a$ R c i.e., if $(a, b) \Rightarrow R$ and $(b, c) \in R$ then $(a, c) \in R \ \forall \ a, b, c \in A$.

   It should be noted that if a R b, then it is not necessary that b must be related to some element, i.e., if L.H.S. of the implication (*) does not hold, the relation is automatically transitive.

   For example, consider the set A = {1, 2, 3} and the relations

   $R_1 = \{(1, 2), (1, 3)\}$; $R_2 = \{(1, 2)\}$; $R_3 = \{(1, 1)\}$
   $R_4 = \{(1, 2) (2, 1), (1, 1)\}$

   Then R_1, R_2 and R_3 are transitive while R_4 is not transitive since in R_4 $(2, 1) \in R_4$, $(1, 2) \in R_4$ but $(2, 2) \notin R_4$.

   For another example, Let A be the set of all line in a plane R be the relation in A defined by "is parallel to". Then if line a is parallel to line b and line b is parallel to line c, then a is parallel to c. Here, R is transitive.

10. **Identity Relation:** R is an identity relation if $(a, b) \in R$ iff $a = b$. In other words, every element of A is related to only itself.

It is interesting to not that every identity relation is reflexive but every reflexive relation need not be an identity relation.

It should be noted that identity relation is reflexive, symmetric and transitive.

For example,

on the set $= \{1, 2, 3\}$

$R = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on A.

11. Equivalence Relation: A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

(*i*) R is reflexive, i.e., for every $a \in A$, $(a, a) \in R$ i.e., $a \, R \, a \; \forall \; a \in R$.

(*ii*) R is symmetric i.e., $(a, b) \in R \Rightarrow (b, a) \in R$ i.e., $a \, R \, b \Rightarrow b \, R \, a \; \forall \; a, b \in A$.

(*iii*) R is transitive i.e., $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ i.e., $a \, R \, b$ and $b \, R \, c \Rightarrow a \, R \, c$, when $a, b, c \in A$.

For example,

let A be the set of all triangles in a plane and let R be defined by "is congruent to". We observe that

(*i*) R is reflexive i.e., $a \, R \, a \; \forall \; a \in A$. Since every triangle is congruent to itself.

(ii) R is symmetric i.e., $a \, R \, b \Rightarrow b \, R \, a$. Since, if triangle a is congruent to triangle b then, b is congruent to a.

(iii) R is transitive i.e., $a \, R \, b$ and $b \, R \, c \Rightarrow a \, R \, c$. Since, if triangle a is congruent to triangle b and triangle b is congruent to triangle c then, a is congruent to triangle c.

Thus, the relation R defined above is an equivalence relation.

12. **Equivalence classes of an equivalence relation:** Let R be equivalence relation in A $(\neq \phi)$. Let $a \in A$, then the equivalence class of a, denoted by $[a]$ or $\{\overline{a}\}$ is defined as the set of all those points of A which are related to a under the relation R. Thus $[a] = \{x \in A : x \, R \, a\}$.

Hence, it is easy to see that

(i) $b \in [a] \Rightarrow a \in [b]$

(ii) $b \in [a] \Rightarrow [a] = [b]$

(iii) Two equivalence classes are either disjoint or identical.

For example, we consider a very important equivalence relation.

$x \equiv y \pmod{n}$ iff n divides $(x - y)$, n is a fixed position integer. Consider $n = 5$. Then,

$[0] = \{x : x \equiv 0 \pmod 5\} = \{5p : p \in z\} = \{0, \pm 5, \pm 10, \pm 15, ...\}$

$$[1] = \{x : x \equiv 1 \pmod 5\} = \{x : x - 1 = 5k, k \in Z\}$$
$$= \{5k + 1 : k \in Z\}$$
$$= \{1, 6, 11, ..., -4, -9, ...\}$$

It should be noted that there are only 5 distinct equivalence classes viz. [0], [1], [2], [3] and [4], when $n = 5$.

FUNCTIONS (MAPPING)

A function is "a rule" or a "device" or "a mechanism" which defines some association or correspondence or relationship between the elements of two sets.

1. **Definition:** Suppose that to each element in a set A there is assigned by some rule, an unique element of a set B. Such rules are called functions or mappings. If we let f denote these rules, then we write $f : A \to B$ or $A \xrightarrow{f} B$.

   Which reads "f is a function of A into B".

   It is interesting to note that every rule works as a relation but not necessarily as a function.

An Alternative Definition: Let $R \subseteq A \times B$ i.e., R is a relation from A to B. Then R is a function if

(*i*) dom. R = A.

(*ii*) $(a, b), (a, c) \in R \Rightarrow b = c$

Terminology: Let $f : A \to B$ be a mapping. Then,

(*i*) A is called the domain of f.

(*ii*) B is called the co-domain of f.

(*iii*) If $a \in A$, then the element in B which is assigned to a, is called the image of a and is denoted by $f(a)$. Thus if $f(a) = b$, b is called the image of a and a is called a pre-image of b.

(**Note:** image of an element in the domain A uniquely exists but pre-image of an element of co-domain may or may not exist in A and if exist, it may not be unique).

(*iv*) Set of all images is called the range of f. Thus,

Range $f = \{f(a) : a \in A\} \subseteq B$.

Range of f is also written as $f(A)$.

Important Deductions: In a function $f : A \to B$

(*i*) Each element of the set A must be associated with unique element of B.

(*ii*) Two or more elements of set A may be associated with the same element of the set B.

(*iii*) There may be some elements of B which are not assigned to any element of the set A.

2. **Equal Functions:** If f and g are functions defined on the same domain A and if $f(a) = g(a)$ for every $a \in A$, then $f = g$.

For example,

$$\begin{aligned} \text{Let} \quad f &: \{1, 2\} \to \{1, 2, 3, 4\}; \\ f &= \{(1, 1), (2, 3)\} \text{ and} \\ g &: \{1, 2\} \to \{1, 3, 4, 5\}; \\ g &= \{(1, 1), (2, 3)\} \end{aligned}$$

Then, since dom. f = dom. g and $f(a) = g(a)$ for all a in the domain, $f = g$.

3. **Types of Functions:**

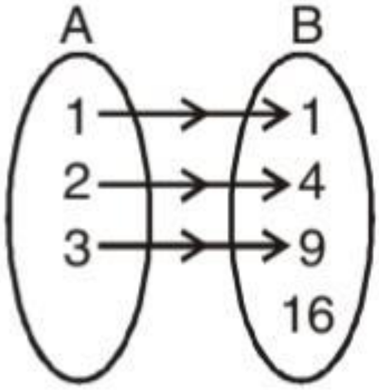
(*i*) ***Constant Function :*** Let $f : A \to B$. If $f(a) = b$ for all $a \in A$, then f is called a constant function. Thus, f is called a constant function if range f consists of only one element.

(*ii*) ***Into function:*** Let $f : A \to B$, if there exists even a single element in B having no pre-image at all in A, then such a function is said to be an into function.

For example,

Let $A = \{1, 2, 3\}$ and $B = \{1, 4, 9, 16\}$

Let $f : A \rightarrow B; f(x) = x^2 \; \forall \; x \in A$



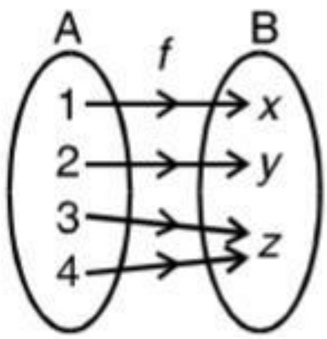
From the figure, it is clear that there exists an element, namely $16 \rightarrow B$, which has no pre-image in A. So f is an into function.

(iii) ***Onto function (Surjection):*** A function $f : A \rightarrow B$ is said to be an onto function if each element of B is the f image of at least one element of A. An onto function is also called a surjective mapping.

In onto mapping, we observe that

$\{f(x)\} = B \; \forall \; x \in A$

Thus, f is onto iff $f(A) = B$, i.e., range = co-domain



[**Note:** A function which is not onto, is called an into function].

For example, consider $f = \mathrm{R} \to \mathrm{R} : f(x) = x^2$. Then f is not an onto function since the negative numbers do not appear in the range of f i.e., no negative number is the square of a real number. So f is an into function.

Important Deduction:

(*i*) In some cases, the following method to test the onto property is useful. Consider $a \in \mathrm{A}$, $b \in \mathrm{B}$ and $f(a) = b$. Find a in terms of b. If a exists in A for every $b \in \mathrm{B}$, f is onto otherwise into.

Illustration: Consider the function

$$f : \mathrm{R} \to \mathrm{R} : f(x) = \frac{x+1}{x-2}$$

Clearly f is not a mapping in the domain R as $f(2) = \dfrac{3}{0}$ does not exist in R . So dom. $f = \mathrm{R} - \{2\}$. On this domain, consider $a \in \mathrm{R} - \{2\}$, $b \in \mathrm{R}$,

$$\begin{aligned} f(a) = b &\Rightarrow b = (a+1)/(a-2) \\ &\Rightarrow ab - 2b = a + 1 \\ &\Rightarrow a = (2b+1)/(b-1) \end{aligned}$$

Thus $a \notin (\mathrm{R} - \{-2\})$ for $b = 1$. Thus 1 is not image of any member of the domain so f is into. This method is also useful to find the range. In the above example $f = \mathrm{R} - \{1\}$.

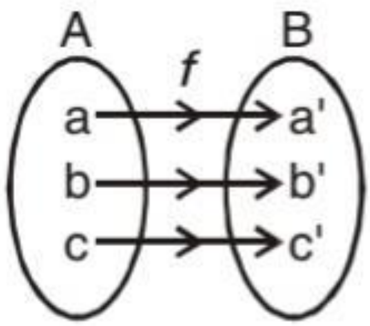
(iv) ***Constant Function:*** Let $f : \mathrm{A} \to \mathrm{B}$. If $f(a) = b$ for all $a \in \mathrm{A}$, then f is called a constant function. Thus f is called a constant function if range f consists of only one element.

(v) ***One-One Function (Injection):*** Let $f : \mathrm{A} \to \mathrm{B}$. Then f is called a one-one function if no two different elements in A have the same image i.e., different elements in A have different elements in B. Symbolically, f is one-one if

$$f(a) = f(a')$$

$$\Rightarrow \quad a = a'$$

i.e., $a \neq a' \Rightarrow f(a) \neq f(a')$



A mapping which is not one-one is called many-one.

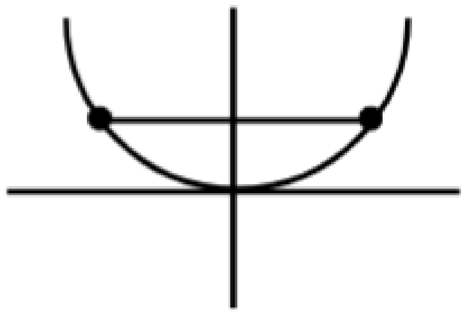
For example,

let $f : \mathrm{R} \to \mathrm{R}$:

$f(x) = x^2$. Since, $f(-1) = (1) = 1$, f is not one-one so it is many-one.

Important Deductions:

(i) If any line parallel to x-axis in the graph of the function on cartesian plane intersect the graph at two or more points, f must be many-one. For example, consider the function $y = f(x) = x^2$



Therefore f is many-one.

(ii) Let $f(x) = f(y)$. If solution of this equation has solution other than $x = y$, f is many-one. For example, let $f : \mathrm{R} \to \mathrm{R}$.

$$f(x) = \frac{x^2}{(x+1)},$$

then $f(x) = f(y)$ gives $\dfrac{x^2}{x+1} = \dfrac{y^2}{y+1}$

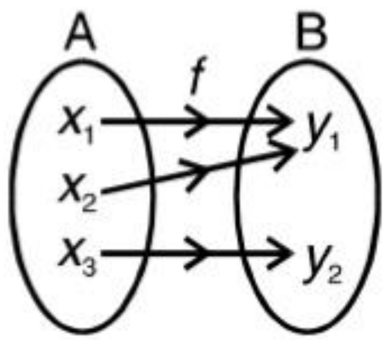
$$\Rightarrow \quad x^2y + x^2 = y^2x + y^2$$

$$\Rightarrow \quad xy\,(x - y) + (x - y)\,(x + y\,) = 0$$

$$\Rightarrow \quad (x - y)(xy + x + y) = 0$$
$$\Rightarrow \quad x = y \text{ or } xy + x + y = 0$$
$$\Rightarrow \quad x = y \text{ or } y = \frac{-x}{x+1}$$

$\therefore$ f is many-one.

(*vi*) ***Many-one function:*** $f : \mathrm{A} \to \mathrm{B}$ is said to be many-one function, if two or more elements of A have the same f image in B, i.e., if $f(x_1) = f(x_2)$, $x_1 \neq x_2$.

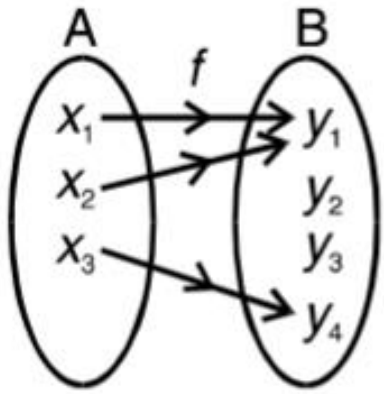


Many-one into mapping:

Let f: A $\to$ B then

(*i*) many-one

(*ii*) into, then f is called a many-one into mapping. In this mapping some elements of B will remain uncovered.

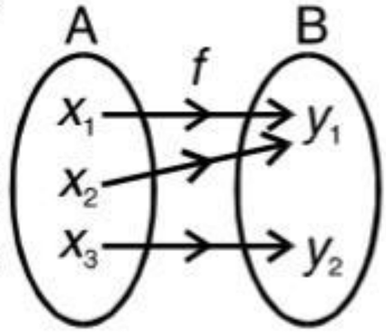


Many-one onto mapping:

Let f: A $\rightarrow$ B, then

(i) many-one

(ii) onto, then f is called a many-one onto mapping.



(vii) ***Inverse of a function:*** Let f : A $\rightarrow$ B and let $b \in$ B. Then, the inverse of b, i.e., f^{-1} (b) consists of those elements in A which are mapped onto b,

i.e., $f^{-1}(b) = \{x : x \in A, f(x) = b\}$.

Therefore, $f^{-1}(b) \subseteq A$. $f^{-1}(b)$ may be a null set or singleton. For example,

Let A = {1, 2, 3, 4} B = {a, b, c}

Define, f = A $\rightarrow$ B by $f = \{(1, a)\ (2, a)\ (3, b), (4, a)\}$.

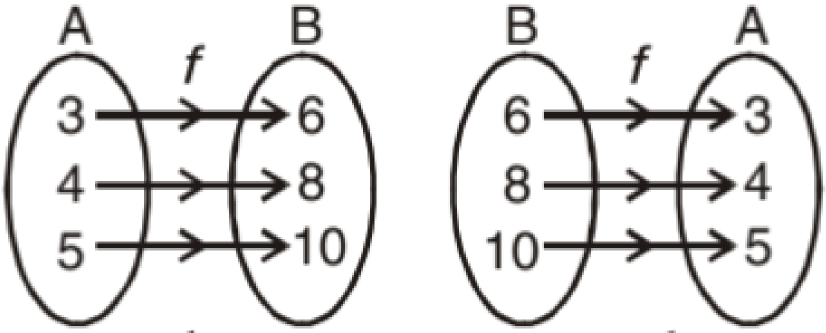
Then, $f^{-1}(a) = \{1, 2, 4\}$, $f^{-1}(b) = (3)$, $f^{-1}(c) = \phi$

In the same way, if C be a subset of B, f^{-1} (C) is defined by f^{-1} (C)

$$= \{x : x \in A, f(x) \in C\}.$$

Thus, f^{-1} is defined as a rule (relation) from B to A. But then dom. $f^{-1} \subset$ B, i.e., f^{-1} (b) may not belong to some $b \in$ B. Also, f^{-1} (b) may be unique. So f^{-1} may be a function from B to A.

(viii) ***Inverse Function:*** Let f: A $\to$ B be a one-one onto function. Then the function of f^{-1} : B $\to$ A which associates to each element $y \in$ B, the element $x \in$ A, such that $f(x) = y$ is called the inverse function of the function f : A $\to$ B



Domain of f^{-1} = B and range of f^{-1} = A.

It is easy to see that f^{-1} : B $\to$ A is also one-one onto

A = {3, 4, 5}, B = {6, 8, 10}

f : A $\to$ B, $f(x) = 2x$, $f(3) = 6$, $f(4) = 8$, $f(5) = 10$

f^{-1} : B $\to$ A, $f^{-1}(x) = x/2$, $f^{-1}(6) = 3$, $f^{-1}(8) = 4$, $f^{-1}(10) = 5$

f = (3, 6), (4, 8), (5, 10)

f^{-1} = (6, 3), (8, 4), (10, 5).

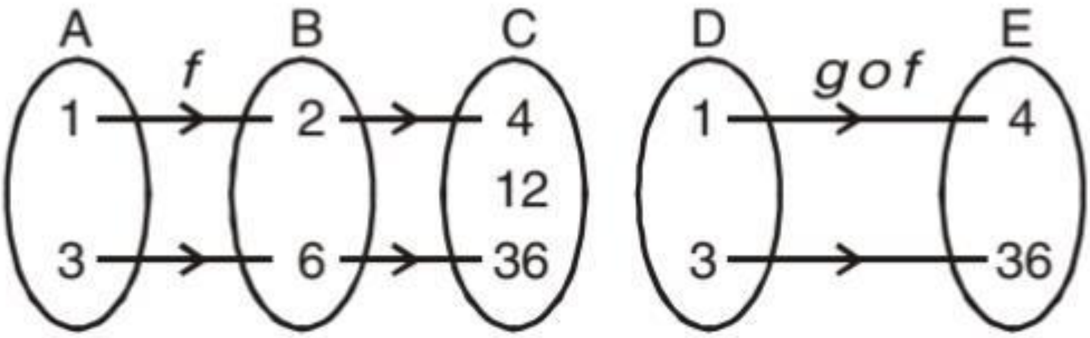
The function f^{-1}; B $\to$ A is also one-one onto function.

Uniqueness of inverse function

If f : A $\to$ B is one-one onto function, the inverse function of f is unique.

(*ix*) ***Composition Function or product function:*** Let $f : A \to B$ and $g : B \to C$ be two functions. Then the product or composition of the functions f and g, denoted by $g \ o \ f$, is a mapping of A into C given by $g \ o \ f : A \to C$, such that $(g \ o \ f)(x) = g(f(x)), \forall \ x \in A$.

Illustration: Let $f : A \to B$, $f(x) = 2x$; $g : B \to C$, $g(x) = x^2$, where $A = \{1, 3\}$, $B = \{2, 6\}$, $C = \{4, 36, 12\}$



$(g \ o \ f)(1) = g(f(1)) = g(2) = 4$

$(g \ o \ f)(3) = g(f(3)) = g(6) = 36$

$(g \ o \ f)(x) = g(f(x)) = g(2x) = 4x^2$

Thus, $g \ o \ f : A \to C$, $(g \ o \ f)(x) = 4x^2$

$f = \{(1, 2), (3, 6)\}$

$g = \{(2, 4), (6, 36)\}$

$g \ o \ f = \{(1, 4), (3, 36)\}$

Important Deductions:

(*i*) If $O(A) = m$, $O(B) = n$, then total number of mappings from A to B is n^m.

(ii) If A and B are finite sets and $O(A) = m$, $O(B) = n, m \leq n$. Then number of injection (one-one) from A to B is ${}^{n}P_{m} = n!/(n-m)!$

(iii) If $f : A \to B$ is injective (one-one), then $O(A) \leq O(B)$.

(iv) If $f : A \to B$ is surjective (onto), then $O(A) \geq O(B)$.

(v) $f : A \to B$ is bijective (one-one onto), then $O(A) = O(b)$.

(vi) Let $f : A \to B$ and $O(A) = O(B)$. Then f is one-one $\Leftrightarrow$ it is onto.

(vii) Let $f : A \to B$ and $X_1, X_2 \subseteq A$. Then f is one-one if $f(X_1 \cap X_2) = f(X_1) \cap f(X_2)$.

(viii) Let $f : A \to B$ and $X \subseteq A$, $Y \subseteq B$. Then in general $f^{-1}(f(x)) \subseteq X$, $f(f^{-1}(Y)) \subseteq Y$. If f is one-one onto, $f^{-1}(f(x)) = X$, $f(f^{-1}(Y)) = Y$.

(ix) In general $g \, o \, f \neq f \, o \, g$.

(x) $f : A \to B$, be one-one, onto, then $f^{-1} \, o \, f = I_A$ and $f \, o \, f^{-1} = I_B$

(xi) $f : A \to B$, $g : B \to C$, $h : C \to D$. Then $(h \, o \, g) \, o \, f = h \, o \, (g \, o \, f)$.

(xii) $f : A \to B$, $g : B \to C$ be one-one and onto, then $g \, o \, f : A \to C$ is also one-one onto and $(g \, o \, f)^{-1} = f^{-1} \, o \, g^{-1}$.

(xiii) Let $f : A \to B$, then $I_B \, o \, f = f$ and $f \, o \, I_B = f$. It should be noted here that $f o I_B$ is not

defined since for $(f \, o \, I_B)(x) = f o \{I_B(x)\} = f(x)$, $I_B(x)$ exist when $x \in B$ and $f(x)$ exist when $x \in A$.

(xiv) $f: A \to B$, $g: B \to C$ are both one-one, then $g \, o \, f: A \to C$ is also one-one but converse is not true i.e., though $g \, o \, f$ is one-one, both f and g need not be one-one. It should be noted that for $g \, o \, f$ to be one-one, f must be one-one.

(xv) If $f: A \to B$, $g: B \to C$ are both onto then $g \, o \, f$ must be onto. However, the converse is not true. But for $g \, o \, f$ to be onto g must be onto.

(xvi) The domain of the function $(f + g)(x) = f(x) + g(x)$, $(f - g)(x) = f(x) - g(x)$, $(fg)(x) = f(x).\, g(x)$ is given by (dom. f) $\cap$ (dom. g). While domain of the function $(f/g)(x) = f(x)/g(x)$ is given by (dom. f) $\cap$ (dom. g) $- \{x : g(x) = 0\}$.

OPERATION

An operation over a set is a rule which combines any two elements of the set.

Binary Operation: An operation O is called a binary operation on a set A if $a \, O \, b \in A$, $a, b \in A$.

If O is a binary operation on the set A, then A is said to be closed with respect to the operation O.

For Example:

(*i*) If we consider the composition of multiplication in the set of odd integers, then it is binary composition, since the multiplication of two odd integers is an odd integer i.e.,

$5 \times 3 = 15, \quad 9 \times 5 = 45$

(*ii*) Division is not a Binary operation on N, because 2/3 $\in$N. Thus, N is not closed with respect to division operation.

(*iii*) Subtraction is a binary operation on the set of integers I, because $a - b \in$ I, for all $a, b \in$ I.

Laws of Binary Operation

(*i*) ***Commutative Composition:*** A composition in a set A is used to be commutative if

$$x \, o \, y = y \, o \, x, \; \forall \, x, y \in A$$

For example,
$$5 + 4 = 4 + 5$$
$$5 \times 4 = 4 \times 5$$

This shows that addition and multiplication composition in the set of integers are commutative. It should be noted that subtraction is not commutative composition on the set of real numbers as

$$a - b \neq b - a \qquad \forall\, a, b \in \mathrm{R}$$

e.g., $8 - 2 \neq 2 - 8$

(*ii*) ***Associative Composition:*** A composition in a set of elements A is said to be association if

$$(x \ o\ y)\ o\ z = x\ o (y\ o\ z), \forall\, x, y, z \in \mathrm{A}$$

For example, addition in an associative composition in the set of natural numbers, as we have

$$(5 + 2) + 4 = 5 + (2 + 4) \text{ etc.}$$

In general $(x + y) + z = x + (y + z)$

$$\forall\, x, y, z, \in \mathrm{N}$$

Multiplication is an associative composition in the set of integers, since

$$(5 \times 2) \times 4 = 5 \times (2 \times 4)$$

In general $(x \times y) \times z = x \times (y \times z)$

$$\forall\, x, y, z \in \mathrm{I}$$

(*iii*) ***Distributive Composition:*** If *o* and * are two binary composition defined on a set A, then

(*a*) * is said to be left distributive over *o* if

$$(y \; o \; z) * x = (y * x) \; o \; (z * x) \; \forall \; x, y, z \in \text{I}$$

(*b*) $*$ is said to be right distributive over o if.

$$x * (y \; o \; z) = (x * y) \; o \; (x * z), \; \forall \; x, y, z, \in \text{A}$$

If both (*a*) & (*b*) hold then we say $*$ is distributive over o.

For example, multiplication is left distributive over addition in the set of real numbers if

$$(b + c) \times a = b \times a + c \times a, \; \forall \; a, b, c \in \text{R}$$

Multiplication is right distributive over addition in the set of real numbers if

$$a \times (b + c) = a \times b + a \times c, \; \forall \; a, b, c, \in \text{R}.$$

(*iv*) The binary operation O is said to be indempotent on a set of element A, if for every $a \in \text{A}$.

$$a \; o \; a = a$$

(*v*) An element e in a set A is said to be a unit element with respect to the Binary operation o on A if for every $a \in \text{A}$.

$$a \; o \; e = e \; o \; a = a$$

(*vi*) An element b in a set A is said to be the inverse element of an element $a \in \text{A}$ with respect to the binary operation o if

$$a \; o \; b = b \; o \; a = e \; \{\text{If } e \text{ exists in A}\}$$

Operation Table: When the set A being considered has a small number of elements then the result of applying the binary operation *o* to its elements may be represented in the table known as operation table. We write the elements of A in the same order both vertically horizontally. The result $a\ o\ b$ then appears in the body of table at the intersection of row headed by a and the column headed by b.

o	a	b	c
a	b	c	b
b	a	c	b
c	c	b	a