BINOMIAL THEOREM

Binomial Expression: A binomial expression is an algebraic expression consisting of two terms connected by plus (+) or minus (–) sign. For example, $2x + 3y$, $3x + 9y$, $2x - 5y$ are binomial expressions in x and y.

The Factorial Function: For $n \in \mathrm{N}$, factorial of n, denoted by $n!$ is defined by

$$n! = n.(n-1)(n-2)\ \ldots\ 3.2.1.$$

It is also assumed that $0! = 1$

Remarks:

(*a*) $n! = n.(n-1)! = n\,(n-1)\,(n-2)!$ etc.

(*b*) $1/(-n)! = 0$

Binomial Theorem: For any positive Integer n
$(x + a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2\, x^{n-2} a^2 + \ldots +$
${}^nC_r\, x^{n-r}.\, a^r + \ldots + {}^nC_n\, a^n$.
where the constant ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, … ${}^nC_n$ are binomial coefficient and ${}^nC_r = \dfrac{n!}{r!(n-r)!}$

(a) ***Expansion of $(x-a)^n$:*** If we put $a = -a$ in the above theorem, we get

$$(x-a)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1}\, a + {}^nC_2\, x^{n-2}\, a^2 + \ldots + (-1)^n\, {}^nC_n\, a^n.$$

(b) ***Expansion of $(1+x)^n$:*** If we put $x = 1$ and $a = x$ then

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2\, x^2 + \ldots + {}^nC_r\, x^r + \ldots + {}^nC_n\, x^n$$

$$= 1 + nx + \frac{n(n-1)}{2!}\, x^2 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\, x^r + \ldots + x^n.$$

(c) ***Expansion of $(1-x)^n$:*** If we put $x = 1$ and $a = -x$ in the above theorem.

$$(1-x)^n = 1 - {}^nC_1\, x + {}^nC_2 x^2 - \ldots + (-1)^r\, {}^nC_r\, x^r + \ldots + (-1)^n\, x^n.$$

Important Points: For the binomial exprssion of $(x+y)^n$, where n is a positive integer.

(i) The expansion on the right hand side contains $(n + 1)$ terms which is one more than the index of the Binomial.

(ii) In each term of the expansion the sum of the exponents (power of x and y) is n (i.e., each term is of degree n).

(iii) The binomial coefficients of terms from the beginning and the end are equal since ${}^nC_r = {}^nC_{n-r}$

i.e., since ${}^nC_r = {}^nC_{n-r}$

$\therefore \quad {}^nC_0 = {}^nC_n, {}^nC_1 = {}^nC_{n-1}, {}^nC_2 = {}^nC_{n-2}$ ... etc.

Hence, the coefficients from the beginning and end are equal.

General term in the expansion of $(x + y)^n$: In the Binomial expansion of $(x + y)^n$, the $(r + 1)$th term is called the general term, and denoted by T_{r+1}. Thus $T_{r+1} = {}^nC_r\, x^{n-r} y^r$.

For example, the 5th term from the end in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^9$ is $T_{n-p+2} = T_{9-5+2}$

$$= T_C \; [\because n = 9, p = 5]$$

Middle term of terms in the expansion of $(x + y)^n$: (Case I): If n is even, then there will be only one middle term in the expansion, which is $(n/2 + 1)$th term.

$\therefore$ The middle term $= (n/2 + 1)$th term

$= {}^nC_{n/2}\, x^{n/2} y^{n/2}$.

For example, if $n = 8$, then the middle term is $(8/2 + 1) = 5$th term *i.e.*, T_5.

(Case II): If n is odd, then there will be two middle terms in the expansion which are $\frac{1}{2}(n+1)$ and $\frac{1}{2}(n+3)$th terms.

i.e., Middle terms are

$$T_{(n+1)/2} = {}^nC_{(n-1)/2}.x^{(n+1)/2}.\,y^{(n-1)/2}$$
$$T_{(n+3)/2} = {}^nC_{(n+1)/2}.\,x^{(n-1)/2}.\,y^{(n+1)/2}$$

For example, if $n = 7$, then the number of terms in the expansion of $(x + y)^7$ is $7 + 1 = 8$

$\therefore$ Middle terms are 4th and 5th terms (i.e. T_4 & T_5)

To find $(m + 1)$th term from the end: In the Binomial expansion of $(x + y)^n$, the $(m + 1)$th term from the end = $(n - m + 1)$th term from the beginning = T_{n-m+1}

Coefficient of a particular power of x in $(1 + x)^n$

Working Rule: *Step I:* Write down the general term T_{r+1} and simplify.

Step II: Equate the index of x in T_{r+1} to the given power and find r.

Step III: Substitute the value of r in the general term and get the term and its coefficient.

Illustration: Find the coefficients of x^{32} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$

Sol.: The general term in the equation is

$$T_{r+1} = {}^{15}C_r \, (x^4)^{15-r} \left[-\frac{1}{x^3}\right]^r$$

$$= (-1)^r \, {}^{15}C_r \, \frac{x^{60-4r}}{x^{3r}}$$

$$= (-1)^r \, {}^{15}C_r \, x^{60-7r}$$

It will involve x^{32} if $60 - 7r = 32 \quad \therefore r = 4$

$\therefore x^{32}$ occur in the 5th term

$$\text{Coefficients of } x^{32} = (-1)^4 \, {}^{15}C_4 = \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1}$$

$$= 1365$$

Term independent of x in $(1 + x)^n$

Working Rule: *Step I :* Write down the general term T_{r+1} and simplify.

Step II: Equate the index of x into zero and solve for r.

Step III: Substitute this value of r in the general term T_{r+1} to get term independent of x.

Illustration: Find the term independent of x in the expansion of $\left(x^2 + \frac{1}{x}\right)^{12}$.

Sol.: Let T_{r+1} be the term independent of x in $\left(x^2 + \frac{1}{x}\right)^{12}$

$$\therefore \quad T_{r+1} = {}^{12}C_r \, (x^2)^{12-r} \left(\frac{1}{x}\right)^r$$

$$= {}^{12}C_r \, x^{24-2r} \, \frac{1}{x^r}$$

$$\Rightarrow \quad T_{r+1} = {}^{12}C_r \, x^{24-3r} \qquad \text{... (i)}$$

Now x is to have power zero.

$\therefore \quad 24 - 3r = 0 \Rightarrow 3r = 24 \Rightarrow r = 8$

Putting $r = 8$ in eqn. (i) we get

$$T_{8+1} = {}^{12}C_8 \, x^0 = \frac{12!}{8! \; 4!}$$

$$= \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2}$$

$$= 495$$

Number of terms in the expansion of $(x + y + z)^n$, where n is a positive integer, is $\frac{1}{2}$ $(n + 1)(n + 2)$

Properties of Binomial Coefficients:

(*i*) $C_0 + C_1 + C_2 + ... + C_n = 2^n$

(*ii*) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

(*iii*) $C_1 + 2C_2 + 3C_3 + ... + {}^nC_n = n.\ 2^{n-1.}$

(*iv*) $C_1 - 2C_2 + 2C_3 - ... = 0.$

(*v*) $C_0 + 2C_1 + 3C_2 + ... + (n + 1)\ C_n = (n + 1)\ 2^{n-1}$

(*vi*) $C_0C_r + C_1\ C_{r-1} + ...$

$+ C_{n-r}\ C_n = (2n)!/\{(n - r)!.(n + r)!\}$

(*vii*) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = (2n)!/(n!)^2$

(*viii*) $C_0^2 - C_1^2 + C_2^2 - C_3^2 ... =$

$$\begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2}\ {}^nC_{n/2} & \text{if } n \text{ is even} \end{cases}$$

Binomial Theorem (for any index): For any rational index n $(n \neq 0)$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots$$

Remember

(*i*) If n is a negative integer or a rational fraction, then the number of terms in the above expansion is infinite.

(*ii*) If n is a negative integer or a rational fraction, then this expansion is valid for

$$|x| < 1 \text{ i.e., for } -1 < x < 1.$$

(*iii*) If n is positive integer, then the number of terms in the expansion is $(n+1)$ *i.e.*, finite, and

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

(*iv*) If n is a positive integer, then $(1+x)^{-n}$

$$= 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots$$

$$= 1 + (-1)^n\, {}^{n}C_1\, x + (-1)^2\, {}^{n+1}C_2\, x^2 + (-1)^3\, {}^{n+2}C_3\, x^2 + \ldots + (-1)^r\, {}^{n+r-1}C_r\, x^r + \ldots$$

(*v*) If n is a positive integer, then

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots$$

$$= 1 + {}^{n}C_1\, x + {}^{n+1}C_2\, x^2 + {}^{n+2}C_3\, x^3 + \ldots + {}^{n+r-1}C_r\, x^r + \ldots$$

General term

(*i*) In the expansion of $(1 + x)^n$, the general term *i.e.,* the $(r + 1)$th term

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

(*ii*) If n is a positive integer, then in the expansion of $(1 + x)^{-n}$

$$T_{r+1} = (-1)^r\ {}^{n+r-1}C_r\, x^r$$

(*iii*) If n is positive integer, then in the expansion of $(1 - x)^{-n}$

$$T_{r+1} = {}^{n+r-1}C_r\, x^r$$

Important Deductions:

(*i*) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$

(*ii*) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots + (-1)^r x^r + \ldots$

(*iii*) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... + (r+1)\, x^r + ...$

(*iv*) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... + (-1)^r (r+1)x^r + ...$

(*v*) $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^2 + ... + \dfrac{(r+1)(r+2)}{2}\, x^r + ...$

(*vi*) $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + ... + (-1)^r \dfrac{(r+1)(r+2)}{2}\, x^r + ...$