INVERSE TRIGONOMETRIC FUNCTIONS

Inverse Trigonometric Functions: We know that the equation $x = \sin y$ ...(i) means that y is an angle whose sine is x or x is the sine of y.
After solving equation (i) for y, we get

$$y = \sin^{-1} x \text{ or } y = \text{arc} \sin x$$

Similarly $y = \cos^{-1} x$ if $\cos y = x$ and $y = \tan^{-1} x$ if $x = \tan y$ etc. The functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$, $\text{cosec}^{-1} x$ and $\cot^{-1} x$ are called inverse trigonometric functions. It is important to note that

(i) $\sin y$ is a number whereas $\sin^{-1} x$ is an angle

(ii) $\sin^{-1} \neq (\sin x)^{-1} = \dfrac{1}{\sin x}$

Properties of inverse trigonometric functions

I. Some angle can be expressed by different inverse trigonometric functions.

We know that

$$\sin 60° = \frac{\sqrt{3}}{2} \Rightarrow 60° = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\cos 60° = \frac{1}{2} \Rightarrow 60° = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\tan 60° = \sqrt{3} \Rightarrow 60° = \tan^{-1}\left(\sqrt{3}\right)$$

Here, $60° = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\sqrt{3} = \ldots$

II. *Inverse property:* We know that

$$x = \cos\theta, \text{ then, } \theta = \cos^{-1} x$$

$$\therefore \quad \theta = \cos^{-1}(\cos\theta) \quad (\because x = \cos\theta)$$

III. *Principle of reciprocity:* Following reciprocal relation exists between inverse trigonometric functions,

$$\text{cosec}^{-1}\frac{1}{x} = \sin^{-1} x$$

$$\sec^{-1}\frac{1}{x} = \cos^{-1} x$$

$$\cot^{-1}\frac{1}{x} = \tan^{-1} x$$

IV. Inverse trigonometric functions are odd functions within the principal values

(i) $\sin^{-1}(-x) = -\sin^{-1} x$

(ii) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x$

V. ***Some fundamental formulae***

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

(ii) $\tan^{-1}x + \cot^{-1} x = \pi/2$

(iii) $\text{cosec}^{-1} x + \sec^{-1} x = \pi/2$

(iv) $\tan^{-1} x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(v) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

(vi) $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}$

$= \cos^{-1}\frac{1-x^2}{1+x^2}$

$= \tan^{-1}\frac{2x}{1-x^2}$

VI. *To express one inverse trigonometric function in terms of other ones:*

(i) $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2} = \tan^{-1} \dfrac{x}{\sqrt{1-x^2}}$

(ii) $\cos^{-1} x = \sin^{-1} \sqrt{1-x^2} = \tan^{-1} \dfrac{\sqrt{1-x^2}}{x}$

(iii) $\text{cosec}^{-1} \dfrac{1}{x} = \sin^{-1} \dfrac{1}{\sqrt{1-x^2}}$

$$= \cot^{-1} \frac{\sqrt{1-x^2}}{x}$$

VII. *Some Important Deductions:*

(a) $\sin(\sin^{-1} x) = x \,;\, -1 \le x \le 1$

$\tan(\tan^{-1} x) = x \,;\, -\infty < x < \infty$

$\cos(\cos^{-1} x) = x \,;\, -1 \le x \le 1$

$\cot(\cot^{-1} x) = x \,;\, -\infty < x < \infty$

$\sec(\sec^{-1} x) = x \,;\, x \le -1 \text{ or } x \ge 1$

$\text{cosec}(\text{cosec}^{-1} x) = x \,;\, x \le -1 \text{ or } x \ge 1$

(b) $\sin^{-1}(\sin \theta) = \theta \,;\, -\pi/2 \le \theta \le \pi/2$

$\cos^{-1}(\cos \theta) = \theta \,;\, 0 \le \theta \le \pi$

$\tan^{-1}(\tan\theta) = \theta \; ; -\pi/2 < \theta < \pi/2$

$\cot^{-1}(\cot\theta) = \theta \; ; 0 < \theta < \pi$

$\sec^{-1}(\sec\theta) = \theta \; ; 0 \leq \theta < \pi, \theta \neq \pi/2$

$\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta \; ; -\pi/2 \leq \theta \leq \pi/2, \theta \neq 0$

(c) $\sin^{-1}(-x) = -\sin^{-1}x \; ; -1 \leq x \leq 1$

$\cos^{-1}(-x) = \pi - \cos^{-1}x \; ; -1 \leq x \leq 1$

$\tan^{-1}(-x) = -\tan^{-1}x \; ; x \in R$

$\cot^{-1}(-x) = \pi - \cot^{-1}x \; ; x \in R.$

$\sec^{-1}(-x) = \pi - \sec^{-1}x \; ; x \geq 1 \text{ or } x \leq -1$

$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x; x \geq 1 \text{ or } x \leq -1.$