

Class 11

Important Formulas

Complex Numbers and Quadratic Equations

Complex Numbers:

1. $\sqrt{-1}$ is an imaginary quantity and is denoted by i which has the following properties:
 $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and, $i^{\pm n} = i^{\pm k}$, $n \in N$

where k is the remainder when n is denoted by 4.

2. For any positive real number a , $\sqrt{-a} = i\sqrt{a}$.
3. For any two real numbers a and b , we have

$$\sqrt{a}\sqrt{b} = \begin{cases} \sqrt{ab}, & \text{if at least one of } a \text{ and } b \text{ is positive} \\ -\sqrt{ab}, & \text{if } a < 0, b < 0. \end{cases}$$

4. If a, b are real numbers, then a number $z = a + ib$ is called a complex number, real number a is known as the real part of z and b is known as its imaginary part. We write $a = \operatorname{Re}(z)$, $b = \operatorname{Im}(z)$.

A complex number z is purely real iff $\operatorname{Im}(z) = 0$ and z is purely imaginary iff $\operatorname{Re}(z) = 0$

5. For any two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, we define

Addition: $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

Subtraction: $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$

Multiplication: $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

Reciprocal: $\frac{1}{z_1} = \frac{a_1}{a_1^2 + b_1^2} - i \frac{b_1}{a_1^2 + b_1^2}$

Division: $\frac{z_1}{z_2} = z_1 \left(\frac{1}{z_2} \right) = (a_1 + ib_1) \left(\frac{a_2}{a_2^2 + b_2^2} - i \frac{b_2}{a_2^2 + b_2^2} \right) = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$

Addition is commutative and associative. Complex number $0 = 0 + i0$ is the identity element for addition and every complex number $z = a + ib$ has its additive inverse $-z = -a - ib$.

Multiplication is also commutative and associative. Complex number $1 = 1 + 0i$ is the identity element for multiplication. Every non-zero complex number $z = a + ib$ has its multiplicative inverse $1/z$ (also known as reciprocal of z) such that $\frac{1}{z} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$.

6. The conjugate of a complex number $z = a + ib$ is denoted by $\bar{z}$ and is equal to $a - ib$.

For any three complex numbers z, z_1, z_2 , we have

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2 \operatorname{Re}(z)$

(iii) $z - \bar{z} = 2i \operatorname{Im}(z)$

(iv) $z = \bar{z} \Leftrightarrow z$ is purely real

(v) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary

(vi) $z \bar{z} = \{\operatorname{Re}(z)\}^2 + \{\operatorname{Im}(z)\}^2 = |z|^2$

(vii) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(viii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

$$(ix) \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$$

7. The modulus of a complex number $z = a + ib$ is denoted by $|z|$ and is defined as

$$|z| = \sqrt{a^2 + b^2} = \sqrt{\{\operatorname{Re}(z)\}^2 + \{\operatorname{Im}(z)\}^2}$$

If z, z_1, z_2 are three complex numbers, then

$$(i) |z| = 0 \Leftrightarrow z = 0 \text{ i.e. } \operatorname{Re}(z) = \operatorname{Im}(z) = 0 \quad (ii) |z| = |\bar{z}| = |-z|$$

$$(iii) -|z| \leq \operatorname{Re}(z) \leq |z|; -|z| \leq \operatorname{Im}(z) \leq |z| \quad (iv) z\bar{z} = |z|^2$$

$$(v) |\operatorname{Im}(z^n)| \leq n |\operatorname{Im}(z)| |z|^{n-1}, n \in \mathbb{N} \quad (vi) |\operatorname{Re}(z)| + |\operatorname{Im}(z)| \leq \sqrt{2} |z|$$

8. A complex number $z = x + iy$ can be represented by a point $P(x, y)$ (see Fig. 13.8) on the plane which is known as the Argand or Gaussian or Complex plane. The length of the line segment OP is called the modulus of z and is denoted by $|z|$.

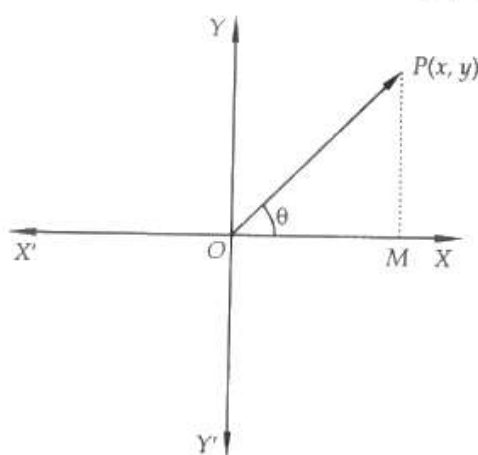


Fig. 13.8

$$\text{Clearly, } |z| = \sqrt{x^2 + y^2} = \sqrt{\{\operatorname{Re}(z)\}^2 + \{\operatorname{Im}(z)\}^2}$$

The angle θ which OP makes with the positive direction of x -axis in anti-clockwise sense is called the argument or amplitude of z and is denoted by $\arg(z)$ or $\operatorname{amp}(z)$.

$$\text{Clearly, } \tan \theta = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}.$$

Let $OP = r$ and $\angle XOP = \theta$. Then, $x = r \cos \theta$ and $y = r \sin \theta$

$$\therefore z = x + iy = r(\cos \theta + i \sin \theta)$$

This is known as the polar form of complex number z . The Euler's notations are

$$e^{\pm i\theta} = \cos \theta \pm i \sin \theta$$

$$\therefore z = r(\cos \theta + i \sin \theta)$$

or, $z = re^{i\theta}$, which is known as the Eulerian form of z .

Quadratic Equations:

1. Fundamental Theorem of Algebra: Every polynomial equation $f(x) = 0$ has at least one root, real or imaginary (complex).
2. Every polynomial equation $f(x) = 0$ of degree n has exactly n roots real or imaginary.
3. A quadratic equation cannot have more than two roots.
4. If $ax^2 + bx + c = 0$, $a \neq 0$ is a quadratic equation with real coefficients, then its roots α and β given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ or, } \alpha = \frac{-b + \sqrt{D}}{2a} \text{ and } \beta = \frac{-b - \sqrt{D}}{2a}$$

where $D = b^2 - 4ac$ is as the discriminant of the equation.

(i) If $D = 0$, then $\alpha = \beta = -\frac{b}{2a}$

So, the equation has real and equal roots each equal to $-\frac{b}{2a}$.

- (ii) If $a, b, c \in \mathbb{Q}$ and D is positive and a perfect square, then roots are rational and unequal.
- (iii) If $a, b, c \in \mathbb{R}$ and D is positive and a perfect square, then the roots are real and distinct.
- (iv) If $D > 0$ but it is not a perfect square, then roots are irrational and unequal.
- (v) If $D < 0$, then the roots are imaginary and are given by

$$\alpha = \frac{-b + i\sqrt{4ac - b^2}}{2a} \text{ and } \beta = \frac{-b - i\sqrt{4ac - b^2}}{2a}$$

- (vi) If $a \in \mathbb{Z}$, $b, c \in \mathbb{I}$ and the roots are rational numbers, then these roots must be integers.
- (vii) If a quadratic equation in x has more than two roots, then it is an identity in x that is $a = b = c = 0$.
- (viii) Complex roots of an equation with real coefficients always occur in pairs. However, this may not be true in case of equations with complex coefficients. For example, $x^2 - 2ix - 1 = 0$ has both roots equal to i .
- (ix) Surd root of an equation with rational coefficients always occur in pairs like $2 + \sqrt{3}$ and $2 - \sqrt{3}$. However, Surd roots of an equation with irrational coefficients may not occur in pairs. For example, $x^2 - 2\sqrt{3}x + 3 = 0$ has both roots equal to $\sqrt{3}$.