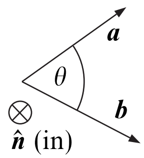
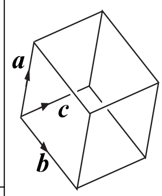
## 2.2 Vectors and matrices

### Vector algebra

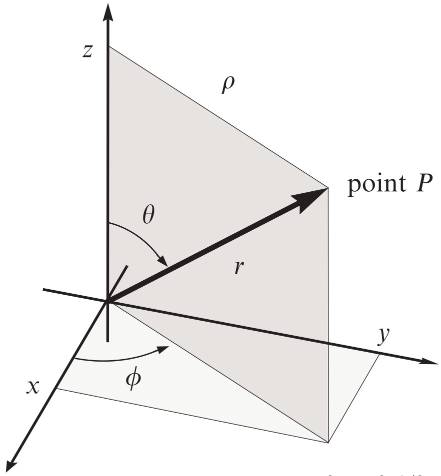
| | | |
|---|---|---|
| Scalar product[a] | $\boldsymbol{a}\cdot\boldsymbol{b}=\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\cos\theta$ | (2.1) |
| Vector product[b] | $\boldsymbol{a}\times\boldsymbol{b}=\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\sin\theta\,\hat{\boldsymbol{n}}=\begin{vmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$ | (2.2) |
| Product rules | $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{a}$ | (2.3) |
| | $\boldsymbol{a}\times\boldsymbol{b}=-\boldsymbol{b}\times\boldsymbol{a}$ | (2.4) |
| | $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{b})+(\boldsymbol{a}\cdot\boldsymbol{c})$ | (2.5) |
| | $\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c})=(\boldsymbol{a}\times\boldsymbol{b})+(\boldsymbol{a}\times\boldsymbol{c})$ | (2.6) |
| Lagrange's identity | $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d})=(\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d})-(\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})$ | (2.7) |
| Scalar triple product | $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=\begin{vmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}$ | (2.8) |
| | $=(\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{a}=(\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b}$ | (2.9) |
| | $=$ volume of parallelepiped | (2.10) |
| Vector triple product | $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}$ | (2.11) |
| | $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ | (2.12) |
| Reciprocal vectors | $\boldsymbol{a}'=(\boldsymbol{b}\times\boldsymbol{c})/[(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}]$ | (2.13) |
| | $\boldsymbol{b}'=(\boldsymbol{c}\times\boldsymbol{a})/[(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}]$ | (2.14) |
| | $\boldsymbol{c}'=(\boldsymbol{a}\times\boldsymbol{b})/[(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}]$ | (2.15) |
| | $(\boldsymbol{a}'\cdot\boldsymbol{a})=(\boldsymbol{b}'\cdot\boldsymbol{b})=(\boldsymbol{c}'\cdot\boldsymbol{c})=1$ | (2.16) |
| Vector $\boldsymbol{a}$ with respect to a nonorthogonal basis $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$[c] | $\boldsymbol{a}=(\boldsymbol{e}_1'\cdot\boldsymbol{a})\boldsymbol{e}_1+(\boldsymbol{e}_2'\cdot\boldsymbol{a})\boldsymbol{e}_2+(\boldsymbol{e}_3'\cdot\boldsymbol{a})\boldsymbol{e}_3$ | (2.17) |




[a]Also known as the "dot product" or the "inner product."
[b]Also known as the "cross-product." $\hat{\boldsymbol{n}}$ is a unit vector making a right-handed set with $\boldsymbol{a}$ and $\boldsymbol{b}$.
[c]The prime (′) denotes a reciprocal vector.

## Common three-dimensional coordinate systems




$$x = \rho\cos\phi = r\sin\theta\cos\phi \quad (2.18)$$

$$y = \rho\sin\phi = r\sin\theta\sin\phi \quad (2.19)$$

$$z = r\cos\theta \quad (2.20)$$

$$\rho = (x^2 + y^2)^{1/2} \quad (2.21)$$

$$r = (x^2 + y^2 + z^2)^{1/2} \quad (2.22)$$

$$\theta = \arccos(z/r) \quad (2.23)$$

$$\phi = \arctan(y/x) \quad (2.24)$$

| | | | |
|---|---|---|---|
| coordinate system: | rectangular | spherical polar | cylindrical polar |
| coordinates of $P$: | $(x,y,z)$ | $(r,\theta,\phi)$ | $(\rho,\phi,z)$ |
| volume element: | $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ | $r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$ | $\rho\,\mathrm{d}\rho\,\mathrm{d}z\,\mathrm{d}\phi$ |
| metric elements[a] $(h_1,h_2,h_3)$: | $(1,1,1)$ | $(1,r,r\sin\theta)$ | $(1,\rho,1)$ |

[a]In an orthogonal coordinate system (parameterised by coordinates $q_1,q_2,q_3$), the differential line element $\mathrm{d}l$ is obtained from $(\mathrm{d}l)^2 = (h_1\,\mathrm{d}q_1)^2 + (h_2\,\mathrm{d}q_2)^2 + (h_3\,\mathrm{d}q_3)^2$.

## Gradient

| | | | |
|---|---|---|---|
| Rectangular coordinates | $\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$ | (2.25) | $f$ scalar field<br>$\hat{}$ unit vector |
| Cylindrical coordinates | $\nabla f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$ | (2.26) | $\rho$ distance from the $z$ axis |
| Spherical polar coordinates | $\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$ | (2.27) | |
| General orthogonal coordinates | $\nabla f = \frac{\hat{\boldsymbol{q}}_1}{h_1}\frac{\partial f}{\partial q_1} + \frac{\hat{\boldsymbol{q}}_2}{h_2}\frac{\partial f}{\partial q_2} + \frac{\hat{\boldsymbol{q}}_3}{h_3}\frac{\partial f}{\partial q_3}$ | (2.28) | $q_i$ basis<br>$h_i$ metric elements |

## Divergence

| | | | | |
|---|---|---|---|---|
| Rectangular coordinates | $\nabla\cdot\boldsymbol{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$ | (2.29) | $\boldsymbol{A}$<br>$A_i$ | vector field<br>$i$th component of $\boldsymbol{A}$ |
| Cylindrical coordinates | $\nabla\cdot\boldsymbol{A}=\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}$ | (2.30) | $\rho$ | distance from the $z$ axis |
| Spherical polar coordinates | $\nabla\cdot\boldsymbol{A}=\frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$ | (2.31) | | |
| General orthogonal coordinates | $\nabla\cdot\boldsymbol{A}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(A_1h_2h_3)+\frac{\partial}{\partial q_2}(A_2h_3h_1)+\frac{\partial}{\partial q_3}(A_3h_1h_2)\right]$ | (2.32) | $q_i$<br>$h_i$ | basis<br>metric elements |

## Curl

| | | | | |
|---|---|---|---|---|
| Rectangular coordinates | $\nabla\times\boldsymbol{A}=\begin{vmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ A_x & A_y & A_z\end{vmatrix}$ | (2.33) | $\hat{}$<br>$\boldsymbol{A}$<br>$A_i$ | unit vector<br>vector field<br>$i$th component of $\boldsymbol{A}$ |
| Cylindrical coordinates | $\nabla\times\boldsymbol{A}=\begin{vmatrix}\hat{\boldsymbol{\rho}}/\rho & \hat{\boldsymbol{\phi}} & \hat{\boldsymbol{z}}/\rho\\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z\\ A_\rho & \rho A_\phi & A_z\end{vmatrix}$ | (2.34) | $\rho$ | distance from the $z$ axis |
| Spherical polar coordinates | $\nabla\times\boldsymbol{A}=\begin{vmatrix}\hat{\boldsymbol{r}}/(r^2\sin\theta) & \hat{\boldsymbol{\theta}}/(r\sin\theta) & \hat{\boldsymbol{\phi}}/r\\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi\\ A_r & rA_\theta & rA_\phi\sin\theta\end{vmatrix}$ | (2.35) | | |
| General orthogonal coordinates | $\nabla\times\boldsymbol{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix}\hat{\boldsymbol{q}}_1h_1 & \hat{\boldsymbol{q}}_2h_2 & \hat{\boldsymbol{q}}_3h_3\\ \partial/\partial q_1 & \partial/\partial q_2 & \partial/\partial q_3\\ h_1A_1 & h_2A_2 & h_3A_3\end{vmatrix}$ | (2.36) | $q_i$<br>$h_i$ | basis<br>metric elements |

## Radial forms[a]

| | | | |
|---|---|---|---|
| $\nabla r=\frac{\boldsymbol{r}}{r}$ | (2.37) | $\nabla(1/r)=\frac{-\boldsymbol{r}}{r^3}$ | (2.41) |
| $\nabla\cdot\boldsymbol{r}=3$ | (2.38) | $\nabla\cdot(\boldsymbol{r}/r^2)=\frac{1}{r^2}$ | (2.42) |
| $\nabla r^2=2\boldsymbol{r}$ | (2.39) | $\nabla(1/r^2)=\frac{-2\boldsymbol{r}}{r^4}$ | (2.43) |
| $\nabla\cdot(r\boldsymbol{r})=4r$ | (2.40) | $\nabla\cdot(\boldsymbol{r}/r^3)=4\pi\delta(\boldsymbol{r})$ | (2.44) |

[a]Note that the curl of any purely radial function is zero. $\delta(\boldsymbol{r})$ is the Dirac delta function.

## Laplacian (scalar)

| | | |
|---|---|---|
| Rectangular coordinates | $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ (2.45) | $f$ scalar field |
| Cylindrical coordinates | $\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$ (2.46) | $\rho$ distance from the $z$ axis |
| Spherical polar coordinates | $\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$ (2.47) | |
| General orthogonal coordinates | $\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$ (2.48) | $q_i$ basis<br>$h_i$ metric elements |

## Differential operator identities

| | | |
|---|---|---|
| $\nabla(fg) \equiv f\nabla g + g\nabla f$ | (2.49) | |
| $\nabla \cdot (f\boldsymbol{A}) \equiv f\nabla \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \nabla f$ | (2.50) | |
| $\nabla \times (f\boldsymbol{A}) \equiv f\nabla \times \boldsymbol{A} + (\nabla f) \times \boldsymbol{A}$ | (2.51) | |
| $\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) \equiv \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{B} \times (\nabla \times \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A}$ | (2.52) | |
| $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) \equiv \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$ | (2.53) | $f, g$ scalar fields |
| $\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) \equiv \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$ | (2.54) | $\boldsymbol{A}, \boldsymbol{B}$ vector fields |
| $\nabla \cdot (\nabla f) \equiv \nabla^2 f \equiv \triangle f$ | (2.55) | |
| $\nabla \times (\nabla f) \equiv \boldsymbol{0}$ | (2.56) | |
| $\nabla \cdot (\nabla \times \boldsymbol{A}) \equiv 0$ | (2.57) | |
| $\nabla \times (\nabla \times \boldsymbol{A}) \equiv \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$ | (2.58) | |

## Vector integral transformations

| | | | |
|---|---|---|---|
| Gauss's (Divergence) theorem | $\int_V (\nabla \cdot \boldsymbol{A})\,\mathrm{d}V = \oint_{S_c} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{s}$ | (2.59) | $\boldsymbol{A}$ vector field<br>$\mathrm{d}V$ volume element<br>$S_c$ closed surface<br>$V$ volume enclosed |
| Stokes's theorem | $\int_S (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{s} = \oint_L \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}$ | (2.60) | $S$ surface<br>$\mathrm{d}\boldsymbol{s}$ surface element<br>$L$ loop bounding $S$<br>$\mathrm{d}\boldsymbol{l}$ line element |
| Green's first theorem | $\oint_S (f\nabla g) \cdot \mathrm{d}\boldsymbol{s} = \int_V \nabla \cdot (f\nabla g)\,\mathrm{d}V$ | (2.61) | $f, g$ scalar fields |
| | $= \int_V [f\nabla^2 g + (\nabla f) \cdot (\nabla g)]\,\mathrm{d}V$ | (2.62) | |
| Green's second theorem | $\oint_S [f(\nabla g) - g(\nabla f)] \cdot \mathrm{d}\boldsymbol{s} = \int_V (f\nabla^2 g - g\nabla^2 f)\,\mathrm{d}V$ | (2.63) | |

## Matrix algebra[a]

| | | | |
|---|---|---|---|
| Matrix definition | $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ | (2.64) | $\mathbf{A}$ $m$ by $n$ matrix<br>$a_{ij}$ matrix elements |
| Matrix addition | $\mathbf{C}=\mathbf{A}+\mathbf{B}$ if $c_{ij}=a_{ij}+b_{ij}$ | (2.65) | |
| Matrix multiplication | $\mathbf{C}=\mathbf{AB}$ if $c_{ij}=a_{ik}b_{kj}$ | (2.66) | |
| | $(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$ | (2.67) | |
| | $\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$ | (2.68) | |
| Transpose matrix[b] | $\tilde{a}_{ij}=a_{ji}$ | (2.69) | $\tilde{a}_{ij}$ transpose matrix (sometimes $a^{\mathrm{T}}_{ij}$, or $a'_{ij}$) |
| | $(\widetilde{\mathbf{AB}\ldots\mathbf{N}})=\tilde{\mathbf{N}}\ldots\tilde{\mathbf{B}}\tilde{\mathbf{A}}$ | (2.70) | |
| Adjoint matrix (definition 1)[c] | $\mathbf{A}^\dagger=\tilde{\mathbf{A}}^*$ | (2.71) | $^*$ complex conjugate (of each component)<br>$^\dagger$ adjoint (or Hermitian conjugate) |
| | $(\mathbf{AB}\ldots\mathbf{N})^\dagger=\mathbf{N}^\dagger\ldots\mathbf{B}^\dagger\mathbf{A}^\dagger$ | (2.72) | |
| Hermitian matrix[d] | $\mathbf{H}^\dagger=\mathbf{H}$ | (2.73) | $\mathbf{H}$ Hermitian (or self-adjoint) matrix |

examples:

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \mathbf{B}=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$\tilde{\mathbf{A}}=\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \qquad \mathbf{A}+\mathbf{B}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

$$\mathbf{AB}=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{pmatrix}$$

[a]Terms are implicitly summed over repeated suffices; hence $a_{ik}b_{kj}$ equals $\sum_k a_{ik}b_{kj}$.
[b]See also Equation (2.85).
[c]Or "Hermitian conjugate matrix." The term "adjoint" is used in quantum physics for the transpose conjugate of a matrix and in linear algebra for the transpose matrix of its cofactors. These definitions are not compatible, but both are widely used [cf. Equation (2.80)].
[d]Hermitian matrices must also be square (see next table).

## Square matrices[a]

| | | | |
|---|---|---|---|
| Trace | $\operatorname{tr}\mathbf{A} = a_{ii}$ (2.74)<br>$\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$ (2.75) | $\mathbf{A}$<br>$a_{ij}$<br>$a_{ii}$ | square matrix<br>matrix elements<br>implicitly $= \sum_i a_{ii}$ |
| Determinant[b] | $\det\mathbf{A} = \epsilon_{ijk\ldots} a_{1i} a_{2j} a_{3k} \ldots$ (2.76)<br>$= (-1)^{i+1} a_{i1} M_{i1}$ (2.77)<br>$= a_{i1} C_{i1}$ (2.78)<br>$\det(\mathbf{AB}\ldots\mathbf{N}) = \det\mathbf{A}\det\mathbf{B}\ldots\det\mathbf{N}$ (2.79) | tr<br>det<br>$M_{ij}$<br>$C_{ij}$ | trace<br>determinant (or $\lvert\mathbf{A}\rvert$)<br>minor of element $a_{ij}$<br>cofactor of the element $a_{ij}$ |
| Adjoint matrix (definition 2)[c] | $\operatorname{adj}\mathbf{A} = \tilde{C}_{ij} = C_{ji}$ (2.80) | adj<br>$\sim$ | adjoint (sometimes written $\hat{\mathbf{A}}$)<br>transpose |
| Inverse matrix ($\det\mathbf{A} \neq 0$) | $a_{ij}^{-1} = \dfrac{C_{ji}}{\det\mathbf{A}} = \dfrac{\operatorname{adj}\mathbf{A}}{\det\mathbf{A}}$ (2.81)<br>$\mathbf{AA}^{-1} = \mathbf{1}$ (2.82)<br>$(\mathbf{AB}\ldots\mathbf{N})^{-1} = \mathbf{N}^{-1}\ldots\mathbf{B}^{-1}\mathbf{A}^{-1}$ (2.83) | $\mathbf{1}$ | unit matrix |
| Orthogonality condition | $a_{ij} a_{ik} = \delta_{jk}$ (2.84)<br>i.e., $\tilde{\mathbf{A}} = \mathbf{A}^{-1}$ (2.85) | $\delta_{jk}$ | Kronecker delta ($=1$ if $i=j$, $=0$ otherwise) |
| Symmetry | If $\mathbf{A} = \tilde{\mathbf{A}}$, $\mathbf{A}$ is symmetric (2.86)<br>If $\mathbf{A} = -\tilde{\mathbf{A}}$, $\mathbf{A}$ is antisymmetric (2.87) | | |
| Unitary matrix | $\mathbf{U}^{\dagger} = \mathbf{U}^{-1}$ (2.88) | $\mathbf{U}$<br>$\dagger$ | unitary matrix<br>Hermitian conjugate |

examples:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\operatorname{tr}\mathbf{A} = a_{11} + a_{22} + a_{33} \qquad \operatorname{tr}\mathbf{B} = b_{11} + b_{22}$$

$$\det\mathbf{A} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}$$

$$\det\mathbf{B} = b_{11}b_{22} - b_{12}b_{21}$$

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}} \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & -a_{12}a_{33} + a_{13}a_{32} & a_{12}a_{23} - a_{13}a_{22} \\ -a_{21}a_{33} + a_{23}a_{31} & a_{11}a_{33} - a_{13}a_{31} & -a_{11}a_{23} + a_{13}a_{21} \\ a_{21}a_{32} - a_{22}a_{31} & -a_{11}a_{32} + a_{12}a_{31} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}$$

$$\mathbf{B}^{-1} = \frac{1}{\det\mathbf{B}} \begin{pmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{pmatrix}$$

[a] Terms are implicitly summed over repeated suffices; hence $a_{ik}b_{kj}$ equals $\sum_k a_{ik}b_{kj}$.
[b] $\epsilon_{ijk\ldots}$ is defined as the natural extension of Equation (2.443) to $n$-dimensions (see page 50). $M_{ij}$ is the determinant of the matrix $\mathbf{A}$ with the $i$th row and the $j$th column deleted. The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$.
[c] Or "adjugate matrix." See the footnote to Equation (2.71) for a discussion of the term "adjoint."

## Commutators

| | | | | |
|---|---|---|---|---|
| Commutator definition | $[\mathbf{A},\mathbf{B}] = \mathbf{AB} - \mathbf{BA} = -[\mathbf{B},\mathbf{A}]$ | (2.89) | $[\cdot,\cdot]$ | commutator |
| Adjoint | $[\mathbf{A},\mathbf{B}]^\dagger = [\mathbf{B}^\dagger,\mathbf{A}^\dagger]$ | (2.90) | $\dagger$ | adjoint |
| Distribution | $[\mathbf{A}+\mathbf{B},\mathbf{C}] = [\mathbf{A},\mathbf{C}] + [\mathbf{B},\mathbf{C}]$ | (2.91) | | |
| Association | $[\mathbf{AB},\mathbf{C}] = \mathbf{A}[\mathbf{B},\mathbf{C}] + [\mathbf{A},\mathbf{C}]\mathbf{B}$ | (2.92) | | |
| Jacobi identity | $[\mathbf{A},[\mathbf{B},\mathbf{C}]] = [\mathbf{B},[\mathbf{A},\mathbf{C}]] - [\mathbf{C},[\mathbf{A},\mathbf{B}]]$ | (2.93) | | |

## Pauli matrices

| | | | | |
|---|---|---|---|---|
| Pauli matrices | $\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}$ $\boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | (2.94) | $\boldsymbol{\sigma}_i$<br>$\mathbf{1}$<br>$\mathbf{i}$ | Pauli spin matrices<br>$2\times 2$ unit matrix<br>$\mathbf{i}^2 = -1$ |
| Anticommutation | $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j + \boldsymbol{\sigma}_j\boldsymbol{\sigma}_i = 2\delta_{ij}\mathbf{1}$ | (2.95) | $\delta_{ij}$ | Kronecker delta |
| Cyclic permutation | $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j = \mathbf{i}\boldsymbol{\sigma}_k$<br>$(\boldsymbol{\sigma}_i)^2 = \mathbf{1}$ | (2.96)<br>(2.97) | | |

## Rotation matrices[a]

| | | | | |
|---|---|---|---|---|
| Rotation about $x_1$ | $\mathbf{R}_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$ | (2.98) | $\mathbf{R}_i(\theta)$<br>$\theta$ | matrix for rotation about the $i$th axis<br>rotation angle |
| Rotation about $x_2$ | $\mathbf{R}_2(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$ | (2.99) | | |
| Rotation about $x_3$ | $\mathbf{R}_3(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | (2.100) | $\alpha$<br>$\beta$<br>$\gamma$ | rotation about $x_3$<br>rotation about $x_2'$<br>rotation about $x_3''$ |
| Euler angles | | | $\mathbf{R}$ | rotation matrix |

$$\mathbf{R}(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{pmatrix} \quad (2.101)$$

[a]Angles are in the right-handed sense for rotation of axes, or the left-handed sense for rotation of vectors. i.e., a vector $\boldsymbol{v}$ is given a right-handed rotation of $\theta$ about the $x_3$-axis using $\mathbf{R}_3(-\theta)\boldsymbol{v} \mapsto \boldsymbol{v}'$. Conventionally, $x_1 \equiv x$, $x_2 \equiv y$, and $x_3 \equiv z$.